CSE 321 HW3

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1) a) $T(n) = 27T(n/3) + n^2$

from master theorem;

$a * (\frac{n}{b}) + f(n)$

if $a > b^d \Rightarrow \Theta(n^{\log_b a})$

$27 > 3^2$

$27 > 9$ ✓ so $\Theta(n^{\log_3 27}) = \underline{\Theta(n^3)}$

b) $T(n) = 9T(n/4) + n$

from master theorem

$9 > 4^1 \Rightarrow$ so $\Theta(n^{\log_4 9}) = \underline{\Theta(n^{\log_4 9})}$

c) $T(n) = 2T(n/4) + \sqrt{n}$

from master theorem

$2 = 4^{1/2} \Rightarrow$ so $\Theta(\sqrt{n} \cdot \log n)$

d) $T(n) = 2T(\sqrt{n}) + 1$

$A(x) = T(2^x) \quad (2^x = n)$

$A(x) = 2T(2^{\frac{x}{2}}) + 1$

$A(x) = 2A(\frac{x}{2}) + 1$

→ In here we can find Θ notation with master theorem

$2 > 2^0$

so $\Theta(x^{\log_2 2}) = \Theta(x)$

and we know

$n = 2^x$

$x = \log n \Rightarrow \underline{\Theta(\log n)}$

e) $T(n) = 2T(n-2)$, $T(0) = 0$, $T(1) = 1$

$r^2 = 2$

$r^2 - 2 = 0$ (characteristic equation)

$r = +\sqrt{2}$ $-\sqrt{2}$

$a \cdot (+\sqrt{2})^n + \beta \cdot (-\sqrt{2})^n = 0$

For $T(0) = 1 \Rightarrow a + b = 1$

for $T(1) = 1 \Rightarrow \sqrt{2}a - \sqrt{2}b = 1$

$2\sqrt{2}a = \sqrt{2} + 1$

$a = \frac{\sqrt{2}+1}{2\sqrt{2}}$

$b = \frac{\sqrt{2}-1}{2\sqrt{2}}$

So $\underline{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)(-\sqrt{2})^n}$

f) $T(n) = 4T(n/2) + n \quad T(1) = 1$

From master theorem

$a > 2^1 \Rightarrow$ so $\Rightarrow \Theta(n^{\log_b a}) \Rightarrow \Theta(n^{\log_2 4}) = \Theta(n^2)$

g) $T(n) = 2T(\sqrt[3]{n}) + 1 \quad T(2) = 1$

$A(x) = T(2^x)$ $\quad n = 2^x$

$x = \lg n$

$= 2T(2^{\frac{x}{3}}) + 1$

$A(x) = 2A(\frac{x}{3}) + 1$

from master theorem

$\log_3^2 > 0$ so $\Theta(k^{\log_3 2})$

we know $k = \lg n$, so $\Theta(\lg n^{(\log_3 2)})$

2) $T(n) = n \cdot T(n/2) + 1$

↙ for loop ↓ n/2 recursive call ↘ 1 print

$n > 2^0$ so $\Theta(n \log n)$

$2^k > 2^0$

∝ k must be ≥ 0.

If it is 0 the will not be any recursive call.

3) first if line is constant time $\Rightarrow +1$

∝ 3 recursive call

∝ Recursive calls divides array 2/3 .

$T(n) = 3T(2/3\, n) + 1$

from master theorem,

$3 > \left(\frac{3}{2}\right)^0$ so $\Theta\left(n^{\log_{3/2} 2}\right)$

4) Averge Case Quick Sort;

The algorithm can choose any element as a pivot after first iteration. (rearrange)

$A(n)$ = operations in rearrange + recursive calls

⇓

high - low + 2

Pivot element can be placed any element so, every element has probability $1/n$.

$A(n) = (high - low + 2) + \sum_{x} [T_2 \text{ for } x] \cdot \frac{1}{n}$ (x = position of pivot)

$$A(n) = n+1 + \sum_{i=1}^{n} [A[i-1] + A(n-i)] \cdot \frac{1}{n}$$ → probability

this part represents left and right part of the pivot element

$A(n) = (n+1) \cdot \frac{2}{n} [A(0) + A(1) + \dots + A(n-1)]$

(if we convert

$A(n)$ to $\frac{A(n)}{n+1} - \frac{A(n-1)}{n}$ we obtain $\frac{2}{n+1}$

We have;

$$\frac{A(n)}{n+1} - \frac{A(n-1)}{n} = \frac{2}{n+1}$$

$T(n) = \frac{A(n)}{n+1}$ ⇒ $T(n) = T(n-1) + \frac{2}{n+1}$

initial condition

$T(0) = 0$

Lets apply backward substitution for solving this recurrence relation;

$T(0) = \frac{A(0)}{1} = 0$

$$T(n) = \sum_{i=2}^{n} \frac{2}{i+1} = 2 \cdot H(n+1) - 3$$

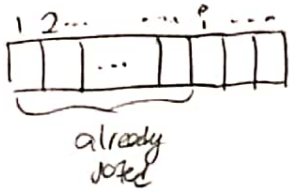
harmonic series

Therefore ⇒ $A(n) = T(n) \cdot (n+1) = 2 \cdot (n+1) \cdot \underbrace{H(n+1)}_{\ln(n+1)} - 3 \cdot (n+1) \in \Theta(n \lg n)$

Averge case of quicksort $\Theta(n \lg n)$

Insertion Sort average case;

Insertion sort tries to sort the input in to reverse order.



So algorithm tries to put i^{th} element from $i-1^{th}$ index to 0.

We need to sum all operations for every index.

$$T = T_1 + T_2 + \dots + T_n = \sum_{i=1}^{n-1} T_i$$

When we compare 2 element there is a 2 probability. if i^{th} index is smaller than $i-1^{th}$ index or not. Both of the case has 1/2 probability.

We have n this operation in average and we need to apply it for every element until array sorted.

Also for every element we have 1/2 probability to swap operation.

$$O(n) * O(n/2) = O(n^2/2) = O(n^2)$$

When we run the python code for size of 100 elements.

Avege quicksort swap operation = 387.2
Averge insertion swap operation = 2420.8) This matchs with our average case analysis.

7) a) 5 sub problems (5 recursive calls)

1/3 size $(n/3)$

Quadratic time (n^2)

$$T(n) = 5T(n/3) + n^2$$

From master theorem

$5 < 3^2$

$5 < 9$ so $\Theta(n^2)$

b) $T(n) = 2T(n/2) + n^2$

From master theorem;

$2 < 2^2$

$2 < 4$ so $\Theta(n^2)$

c) $T(n) = T(n-1) + n$ ← combine

$T(n-1) = T(n-2) + n-1$

$T(n-2) = T(n-3) + n-3$

$\vdots$

$T(1) = T(0) + 1$

$T(n) = T(0) + (1+2+\dots n)$

$T(n) = T(0) + \left(\frac{n \cdot n+1}{2}\right) = \frac{n^2+n}{2}$

$$\Theta\left(\frac{n^2+n}{2}\right) = \Theta(n^2)$$