Math 291
Torsion and Non-Metricity: An Introduction to Metric-Affine Gravity

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Outline

Directional Derivative on $\mathbb{R}^n$

Directional Derivative

Let $X \in \mathfrak{X}(\mathbb{R}^n)$, and let $v \in T_p\mathbb{R}^n$. The directional derivative of X at p in the direction of v is denoted by $\nabla_v X$ and defined as

$$\nabla_v X := \lim_{t \to 0} \frac{X(p+tv) - X(p)}{t}.$$

$$
\begin{aligned}
\overline{\nabla}_v X = \lim_{t\to 0}\frac{X(p+tv)-X(p)}{t} &= \left(\lim_{t\to 0}\frac{X^i(p+tv)-X^i(p)}{t}\right)\partial_i\Big|_p \\
&= \left(\frac{d}{dt}\Big|_{t=0} X^i(p+tv)\right)\partial_i\Big|_p \\
&= \left(\frac{\partial X^i}{\partial x^j}\Big|_p \frac{dx^j}{dt}\Big|_{t=0}\right)\partial_i\Big|_p \quad \text{(chain rule)} \\
&= \left(\frac{\partial X^i}{\partial x^j}\Big|_p v^j\right)\partial_i\Big|_p \\
&= \left(\left(v^j\partial_j\big|_p\right)X^i\right)\partial_i\Big|_p \\
&= \left(v(X^i)\right)\partial_i\Big|_p .
\end{aligned} \tag{1}
$$

Motivated by this calculation, for $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, define a new vector field $\overline{\nabla}_X Y$ by

$$(\overline{\nabla}_X Y)_p := \overline{\nabla}_{X_p} Y = (X_p(Y^i))\, \partial_i \big|_p$$

for all $p \in \mathbb{R}^n$. As p runs over $\mathbb{R}^n$, one has

$$\overline{\nabla}_X Y = X(Y^i)\, \partial_i. \tag{2}$$

Proposition

Let $X, Y \in \mathfrak{X}(\mathbb{R}^n)$. Then,
$i)$ $\overline{\nabla}_X Y$ is $C^\infty(\mathbb{R}^n)$-linear in X.
$ii)$ $\overline{\nabla}_X Y$ is $\mathbb{R}$-linear in Y.
$iii)$ For $f \in C^\infty(\mathbb{R}^n)$,

$$\overline{\nabla}_X(fY) = (Xf)Y + f\overline{\nabla}_X Y.$$

Proof

Use (2) and the fact that a vector field on $\mathbb{R}^n$ is a derivation of $C^\infty(\mathbb{R}^n)$.

Connections on Smooth Manifolds

Definition

Let M be a smooth manifold. A connection on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

written $(X, Y) \mapsto \nabla_X Y$, such that
$(i)\, \nabla_X Y$ is linear over $C^\infty(M)$ in X : for $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$:

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$

(ii) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y : for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \mathfrak{X}(M)$,

$$\nabla_X(a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2.$$

(iii) ∇ satisfies the following product rule: for $f \in C^\infty(M)$,

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y.$$

$\nabla_X Y$ is called the covariant derivative of Y in the direction of X.

For $f \in C^\infty(M)$, we set $\nabla_X f := Xf$.

Restriction of a Connection

Let ∇ be a connection on M. For every open subset $U \subseteq M$, there is a unique connection $\nabla^U : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ such that

$$\nabla^U_{(X|_U)} Y|_U = (\nabla_X Y)|_U.$$

Corollary

$\nabla_X Y\big|_p$ depends only on the value of X at p and the values of Y in a neighborhood of p.

Let $U \subseteq M$ be an open subset with $p \in U$. Then, for $Y \in \mathfrak{X}(U)$, choose a vector field X defined on a neighborhood of p with $X_p = v$ and set $\nabla_v Y := \nabla_X Y\big|_p$.

Connection Coefficients

Let $(U,(x^i))$ be a coordinate chart. We can now write

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\,\partial_k,$$

where $\Gamma_{ij}^k : U \to \mathbb{R}$ are called connection coefficients of ∇ with respect to (∂_i).

Proposition

Let $(U,(x^i))$ be a coordinate chart for M. For $X, Y \in \mathfrak{X}(U)$, written in terms of the basis as $X = X^i\partial_i$ and $Y = Y^j\partial_j$, one has

$$\nabla_X Y = (X(Y^k) + X^iY^j\Gamma_{ij}^k)\partial_k$$

Proof

We simply use the definition.

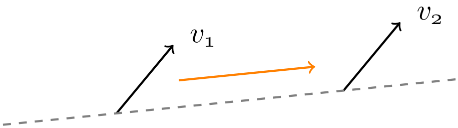
$$\begin{aligned}\nabla_X Y &= \nabla_X (Y^j \partial_j) \\ &= Y^j \nabla_X \partial_j + X(Y^j)\partial_j \\ &= Y^j \nabla_{X^i \partial_i} \partial_j + X(Y^j)\partial_j \\ &= X^i Y^j \nabla_{\partial_i} \partial_j + X(Y^j)\partial_j \\ &= X^i Y^j \Gamma^k_{ij} \partial_k + X(Y^j)\partial_j\end{aligned}$$

Renaming the dummy index in the second term, one obtains

$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma^k_{ij})\partial_k.$$

Covariant Derivative Along Curves

Q: How can we determine whether $v_1 = v_2$?
A:



Tangent Bundle

Let M be a smooth manifold. The tangent bundle TM of M is defined by

$$TM = \bigcup_{p \in M} \{p\} \times T_pM.$$

Vector Fields Along Curves

Let $\gamma : I \to M$ be a smooth curve. A smooth vector field along γ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$.

The set of smooth vector fields along γ is denoted by $\mathfrak{X}(\gamma)$.

Example

Let $(U, (x^i))$ be a chart. Suppose $\gamma : I \to M$ is a smooth curve with $\gamma(I) \subseteq U$. Define a map $V : I \to TM$ by

$$V(t) = \partial_i\big|_{\gamma(t)}.$$

Extending Vector Fields Along Curves

Let V be a vector field along $\gamma : I \to M$. We say V is extendible if there exists a vector field $\tilde{V}$ on a neighborhood of the image of γ such that $V = \tilde{V} \circ \gamma$.

Theorem

Let ∇ be a connection on M. For each $\gamma : I \to M$, there exists a unique operator $D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, called the covariant derivative along γ, such that
(i) D_t is $\mathbb{R}$-linear.
(ii) $D_t(fV) = \dot{f}V + fD_tV$ for all $f \in C^\infty(I)$.
(iii) If V is extendible, $D_tV = \nabla_{\dot{\gamma}(t)}\tilde{V}$ for every extension $\tilde{V}$.

Proof

Set $p := \gamma(t)$. Let $V(t) = V^j \partial_j\big|_p$. Then,

$$D_tV(t) = D_t\left(V^j(t)\partial_j\big|_p\right) = \dot{V}^j(t)\,\partial_j\big|_p + V^j(t)D_t\left(\partial_j\big|_p\right).$$

Let Y be a vector field on a neighborhood of p with $Y_p = \dot{\gamma}(t)$.

$$
\begin{aligned}
D_t V(t) &= \dot{V}^j(t)\,\partial_j\big|_p + V^j(t)\nabla_Y \partial_j\big|_p \\
&= \dot{V}^j(t)\,\partial_j\big|_p + V^j(t)\Gamma_{ij}^k(p)\,Y^i(p)\,\partial_k\big|_p \\
&= (\dot{V}^k(t) + \Gamma_{ij}^k(p)\dot{\gamma}^i(t)V^j(t))\,\partial_k\big|_p .
\end{aligned}
$$

Parallel Vector Fields

Let ∇ be a connection on M. A vector field V along γ is said to be parallel along γ if $D_t V \equiv 0$.

Parallel Transport

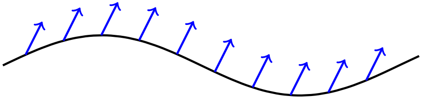


Figure: A parallel vector field along a curve (adapted from [6])

Theorem

Let ∇ be a connection on M. Suppose $\gamma : I \to M$ is a smooth curve with $t_0 \in I$. For $v \in T_{\gamma(t_o)}M$, there exists a unique parallel vector field along γ such that $V(t_o) = v$.

Proof

Existence, uniqueness, and smoothness of ODEs.

Torsion

Definition

Let ∇ be a connection on M. The map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

where $[X, Y] = XY - YX$, is called the torsion tensor of ∇.

The connection ∇ is said to be symmetric if its torsion tensor vanishes.

Characterization of Symmetric Connections

∇ is symmetric if and only if its connection coefficients with respect to every coordinate basis are symmetric: $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Proof

Let (∂_i) be a coordinate basis. Writing $X = X^i\partial_i$ and $Y = Y^j\partial_j$, we have

$$
\begin{aligned}
[X, Y] &= (X^i\partial_i(Y^j\partial_j) - Y^j\partial_j(X^i\partial_i)) \\
&= X^i(\partial_i Y^j)\partial_j + X^i Y^j \partial_i\partial_j - Y^j(\partial_j X^i)\partial_i - Y^j X^i \partial_j \partial_i \\
&= (X^i\partial_i Y^j - Y^i\partial_i X^j)\partial_j.
\end{aligned}
$$

One also has

$$
\begin{aligned}
\nabla_X Y - \nabla_Y X &= (X(Y^k) + X^i Y^j \Gamma^k_{ij})\partial_k - (Y(X^k) + Y^i X^j \Gamma^k_{ij})\partial_k \\
&= (X^a\partial_a Y^k + X^i Y^j \Gamma^k_{ij} - Y^a \partial_a X^k - Y^i X^j \Gamma^k_{ij})\partial_k.
\end{aligned}
$$

Then,

$$T(X,Y) = (X^i Y^j \Gamma^k_{ij} - Y^i X^j \Gamma^k_{ij})\partial_k$$
$$= (\Gamma^k_{ij} - \Gamma^k_{ji}) X^i Y^j \partial_k.$$

It is clear that $\Gamma^k_{ij} = \Gamma^k_{ji}$ implies the symmetry of ∇. Conversely, taking $X^i = Y^j = 1$, the vanishing of the torsion tensor implies $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Geometrical Meaning of Torsion

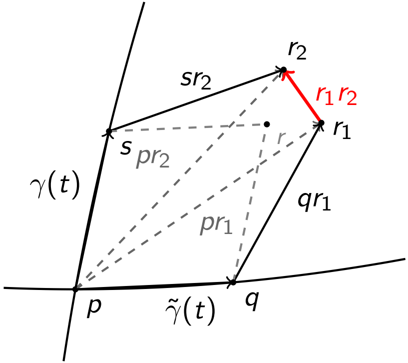


Figure: Torsion as the closure of infinitesimal parallelograms (adapted from [8])

Let $\gamma : I \to M$ and $\tilde{\gamma} : \tilde{I} \to M$ such that $\gamma(t_0) = \tilde{\gamma}(t_0) =: p$ for some $t_o \in I \cap \tilde{I}$. Suppose $\epsilon > 0$ is small such that $(t_0, t_0 + \epsilon) \in I \cap \tilde{I}$. Set $s := \gamma(t_0 + \epsilon)$ and $q := \tilde{\gamma}(t_0 + \epsilon)$.
There is a unique parallel vector field $\tilde{W}$ along $\tilde{\gamma}$ such that $\tilde{W}(t_0) = \dot{\gamma}(t_0)$ given by

$$\dot{\tilde{W}}^k(t) = -\Gamma^k_{ij}(\tilde{\gamma}(t))\, \tilde{W}^i(t)\, \dot{\tilde{\gamma}}^j(t).$$

To first order, one has

$$\tilde{W}^k(t_0 + \epsilon) = \tilde{W}^k(t_0) + \epsilon \dot{\tilde{W}}^k(t_0) + O(\epsilon^2).$$

Combining,

$$\tilde{W}^k(t_0 + \epsilon) = \dot{\gamma}^k(t_0) - \epsilon\, \Gamma^k_{ij}(p)\, \dot{\gamma}^i(t_0)\, \dot{\tilde{\gamma}}^j(t_0) + O(\epsilon^2).$$

Similarly, there is a unique parallel vector field W along γ such that $W(t_0) = \dot{\tilde{\gamma}}(t_0)$. Analogous calculations give

$$W^k(t_0 + \epsilon) = \dot{\tilde{\gamma}}^k(t_0) - \epsilon\, \Gamma^k_{ij}(p)\, \dot{\tilde{\gamma}}^i(t_0)\, \dot{\gamma}^j(t_0) + O(\epsilon^2).$$

Then,

$$\begin{aligned} r_1 r_2 = pr_2 - pr_1 &= (W^k(t_0 + \epsilon) + \dot{\gamma}^k(t_0)) - (\tilde{W}^k(t_0 + \epsilon) + \dot{\tilde{\gamma}}^k(t_0)) \\ &= \epsilon\, \Gamma^k_{ij}(p)\, \dot{\gamma}^i(t_0)\, \dot{\tilde{\gamma}}^j(t_0) - \epsilon \Gamma^k_{ij}(p)\, \dot{\tilde{\gamma}}^i(t_0)\, \dot{\gamma}^j(t_0) \\ &= \epsilon\, (\Gamma^k_{ij}(p) - \Gamma^k_{ji}(p))\, \dot{\gamma}^i(t_0)\, \dot{\tilde{\gamma}}^j(t_0). \end{aligned}$$

Riemannian Manifolds

Definition (Riemannian metrics)

Let M be a smooth manifold. A Riemannian metric on M is a 2-tensor field g such that $g(p) =: g_p$ is an inner product on T_pM for all $p \in M$. The pair (M, g) is called a Riemannian manifold.

Let V be a real inner product space and (e_i) be a basis for V. Then, for $v, w \in V$,

$$\langle v, w\rangle = \langle v^i e_i, w^j e_j\rangle = v^i w^j \langle e_i, e_j\rangle = g_{ij}\, v^i w^j,$$

where $g_{ij} := \langle e_i, e_j\rangle$. Suppose that (ε^i) is the basis for V^* dual to (e_i). Then,

$$\langle v, w\rangle = g_{ij}\, \varepsilon^i(v)\varepsilon^j(w) = g_{ij}\, (\varepsilon^i \otimes \varepsilon^j)(v, w)$$

The above calculations show that any inner product can be written as

$$\langle\ ,\ \rangle = g_{ij}\, \varepsilon^i \otimes \varepsilon^j.$$

In any local coordinates (x^i) for an open subset $U \subseteq M$, we have $\varepsilon^i = dx^i$. Hence, a Riemannian metric can be written as

$$\langle , \rangle = g = g_{ij}\, dx^i \otimes dx^j,$$

where (g_{ij}) is a symmetric positive definite matrix of smooth functions.

For $X, Y \in \mathfrak{X}(M)$, $\langle X, Y\rangle \in C^\infty(M)$ is defined by

$$\langle X, Y\rangle(p) := \langle X_p, Y_p\rangle.$$

Definition

Let (M, g) be a Riemannian manifold. A connection ∇ on M is said to be compatible with g if

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for all $X, Y, Z \in \mathfrak{X}(M)$. We also say that ∇ satisfies the metricity condition.

Geometrical Meaning of Metric-Compatibility

Characterization of Metric Compatibility

Let (M, g) be a Riemannian manifold with connection ∇ and $\gamma : I \to M$ be a smooth curve. If V and W are parallel vector fields along γ, then ∇ is compatible with g if and only if $\langle V, W \rangle$ is constant along γ.

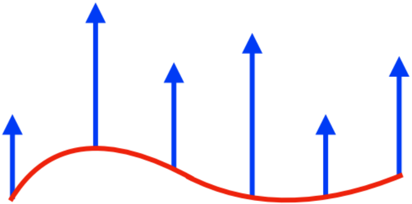


Figure: Non-metricity causes lengths of parallel transported vectors to change [4].

Fundamental Theorem of Riemannian Geometry

Let (M, g) be a Riemannian manifold. There is a unique connection on M that is metric-compatible and symmetric, called the Levi-Civita connection of g.

Proof

We first show uniqueness. Using the compatibility equation and symmetry, we have

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$$\nabla_Y \langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$

$$\nabla_Z \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle \quad (1)$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle \quad (2)$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle \quad (3)$$

Solving (1) + (2) − (3) for $\langle \nabla_X Y, Z \rangle$ yields

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\ - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle).$$

In local coordinates, take $X = \partial_i, Y = \partial_j, Z = \partial_k$. Then,

$$\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \frac{1}{2}(\partial_i \langle \partial_j, \partial_k \rangle + \partial_j \langle \partial_k, \partial_i \rangle - \partial_k \langle \partial_i, \partial_j \rangle)$$

This gives

$$\Gamma^l_{ij} g_{lk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Multiplying both sides by the inverse metric g^{km} and using the relation $g_{kl}g^{km} = \delta_l^m$,

$$\Gamma_{ij}^m = \frac{1}{2}g^{km}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

General Relativity

The gravitational action yielding the equations of motion in vacuum is given by

$$S = \frac{1}{16\pi} \int_V R\sqrt{|g|}\, d^4x.$$

The principle of least action:

$$R_{ab} - \frac{1}{2} R g_{ab} = 0.$$

Metric-Affine Gravity

In Metric-Affine Gravity, the gravitational action in vacuum is given by

$$S = \int_V L(g_{ij}, \varepsilon^i, Q_{ij}, T^i, \Omega_i{}^j).$$

The principle of least action:

$$D\left(\frac{\partial L}{\partial Q_{ij}}\right) + \frac{\partial L}{\partial g_{ij}} = 0$$

$$D\left(\frac{\partial L}{\partial T^i}\right) + \frac{\partial L}{\partial \varepsilon^i} = 0$$

$$D\left(\frac{\partial L}{\partial \Omega_i{}^j}\right) + \frac{\partial L}{\partial \omega_i{}^j} = 0.$$

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