Diophantine Approximation

Emirhan Duysak & Yusuf Emre Baran

Bilkent University

12 May 2024

Outline

- What is Diophantine Approximation?
- Dirichlet's Theorem
- Continued Fractions
- Algebraic & Transcendental Numbers
- Roth's Theorem
- Applications

What is Diophantine Approximation?

- From $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{Q}$.
- $\pi \approx \frac{22}{7}$

What is Diophantine Approximation?

- From $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{Q}$.
- $\pi \approx \frac{22}{7}$

Dirichlet's Theorem

Let $\alpha \in \mathbb{R}$ and $Q \in \mathbb{Z}^+$. Then, there exists a rational number $\frac{p}{q}$ such that $0 < q \leq Q$, and

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q(Q+1)}$$

Proof

Pigeonhole Principle.

Continued Fractions

Finite Continued Fractions

Let $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $a_1, a_2, \ldots, a_n > 0$. An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

is said to be a finite continued fraction and denoted $[a_0, a_1, \ldots, a_n]$. A finite continued fraction is called simple if $a_0, a_1, \ldots, a_n \in \mathbb{Z}$.

Example

$$\frac{7}{10} = 0 + \frac{7}{10}$$

Example

$$\frac{7}{10} = 0 + \frac{7}{10}$$
$$= 0 + \frac{1}{\frac{10}{7}}$$

Example

$$
\begin{aligned}
\frac{7}{10} &= 0 + \frac{7}{10} \\
&= 0 + \frac{1}{\frac{10}{7}} \\
&= 0 + \frac{1}{1 + \frac{3}{7}}
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{7}{10} &= 0 + \frac{7}{10} \\
&= 0 + \frac{1}{\frac{10}{7}} \\
&= 0 + \frac{1}{1 + \frac{3}{7}} \\
&= 0 + \frac{1}{1 + \frac{1}{\frac{7}{3}}}
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{7}{10} &= 0 + \frac{7}{10} \\
&= 0 + \frac{1}{\frac{10}{7}} \\
&= 0 + \frac{1}{1 + \frac{3}{7}} \\
&= 0 + \frac{1}{1 + \frac{1}{\frac{7}{3}}} \\
&= 0 + \frac{1}{1 + \frac{1}{2+\frac{1}{3}}}
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{7}{10} &= 0 + \frac{7}{10} \\
&= 0 + \frac{1}{\frac{10}{7}} \\
&= 0 + \frac{1}{1 + \frac{3}{7}} \\
&= 0 + \frac{1}{1 + \frac{1}{\frac{7}{3}}} \\
&= 0 + \frac{1}{1 + \frac{1}{2+\frac{1}{3}}} \;=\; [0, 1, 2, 3].
\end{aligned}
$$

Proposition

Let $\alpha \in \mathbb{R}$. Then, $\alpha \in \mathbb{Q}$ if and only if α is expressible as a finite simple continued fraction.

Proof

($\Longrightarrow$) Division algorithm
($\Longleftarrow$) Induction on n, where $\alpha = [a_0, a_1, \ldots, a_n]$.

Convergents

Let $\alpha = [a_0, a_1, \ldots, a_n]$ be a finite continued fraction. The i^{th} convergent C_i of α is the finite continued fraction $C_i = [a_0, a_1, \ldots, a_i]$.

Example

The 0^{th} convergent of $\frac{7}{10}$ is the finite continued fraction $[0] = 0$.
The 1^{st} convergent of $\frac{7}{10}$ is the finite continued fraction $[0, 1] = 1$.
The 2^{nd} convergent of $\frac{7}{10}$ is the finite continued fraction $[0, 1, 2] = \frac{2}{3}$.
The 3^{rd} convergent of $\frac{7}{10}$ is the finite continued fraction $[0, 1, 2, 3] = \frac{7}{10}$.

Proposition

Let $\alpha = [a_0, a_1, \dots, a_n]$ be a finite continued fraction. Then,

$$C_i = \frac{p_i}{q_i}$$

where $p_i = a_i p_{i-1} + p_{i-2}$ and $q_i = a_i q_{i-1} + q_{i-2}$ with $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, $q_1 = 1$, $q_1 = a_1$. Moreover, $p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1}$.

Proof

Induction on i.

Corollaries

(1) $(p_i, q_i) = 1.$

(2) $C_i - C_{i-1} = \dfrac{(-1)^{i-1}}{q_i q_{i-1}},\ 1 \leq i \leq n.$

(3) $C_i - C_{i-2} = \dfrac{(-1)^{i}}{q_i q_{i-2}},\ 2 \leq i \leq n.$

Proposition

$C_0 < C_2 < C_4 < \cdots < C_5 < C_3 < C_1.$

Proof

Application of corollaries (2) and (3).

Infinite Continued Fractions

Lemma

$q_i \geq i.$

Proof

Induction on i.

Proposition

Let $a_0, a_1, a_2, \dots \in \mathbb{Z}$ with $a_1, a_2, \dots > 0$ and $C_i = [a_0, a_1, a_2, \dots, a_i]$. Then,

$$\lim_{i \to \infty} C_i$$

exists.

Proof

The limits $\lim_{i\to\infty} C_{2i}$ and $\lim_{i\to\infty} C_{2i+1}$ exist by the monotone convergence theorem. Hence,

$$\lim_{i\to\infty}(C_{2i+1} - C_{2i}) = \lim_{i\to\infty} C_{2i+1} - \lim_{i\to\infty} C_{2i}$$

It remains to show that $\lim_{i\to\infty}(C_{2i+1} - C_{2i}) = 0$, which is a simple application of the lemma.

Definition

$\lim_{i\to\infty} C_i$ is said to be the value of the infinite simple continued fraction $[a_0, a_1, a_2, \ldots]$.

Proposition

Let $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$. Define $a_0, a_1, \ldots$ and $\alpha_1, \alpha_2, \ldots$ by

$$a_i = \lfloor \alpha_i \rfloor, \quad \alpha_{i+1} = \frac{1}{\alpha_i - a_i}$$

Then, $\alpha_0 = \lim_{i \to \infty} C_i = [a_0, a_1, a_2, \ldots]$.

Example

Let $\alpha_0 = \pi$. Then, $a_0 = \lfloor \pi \rfloor = 3$ and $\alpha_1 = \frac{1}{\pi - 3}$, yielding $a_1 = 7$.
Therefore, $\pi \approx [3, 7] = 3 + \frac{1}{7} = \frac{22}{7}$.

Example

Let $\alpha_0 = \sqrt{2} = 1.41421\ldots$ Then,

$$a_0 = \lfloor \alpha_0 \rfloor = 1 \implies \alpha_1 = \frac{1}{\alpha_0 - a_0} = 2.41421\ldots$$

$$a_1 = \lfloor \alpha_1 \rfloor = 2 \implies \alpha_2 = \frac{1}{\alpha_1 - a_1} = 2.41421\ldots$$

$$a_2 = \lfloor \alpha_2 \rfloor = 2 \implies \alpha_3 = \frac{1}{\alpha_2 - a_2} = 2.41421\ldots$$

$$a_3 = \lfloor \alpha_3 \rfloor = 2 \implies \alpha_4 = \frac{1}{\alpha_3 - a_3} = 2.41421\ldots$$

Proposition

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{p_i}{q_i}$ be the convergents of the infinite simple continued fraction expansion of α. Let $a, b \in \mathbb{Z}$ such that $1 \leq b \leq q_{i+1}$, then

$$|q_i\alpha - p_i| \leq |b\alpha - a|$$

Corollary

If $1 \leq b \leq q_i$, then

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \left|\alpha - \frac{a}{b}\right|$$

Example: $\alpha_0 = e$.

$$a_0 = \lfloor \alpha_0 \rfloor = 2 \implies \alpha_1 = \frac{1}{\alpha_0 - a_0} = 1.39221\ldots$$

$$a_1 = \lfloor \alpha_1 \rfloor = 1 \implies \alpha_2 = \frac{1}{\alpha_1 - a_1} = 2.54964\ldots$$

$$a_2 = \lfloor \alpha_2 \rfloor = 2 \implies \alpha_3 = \frac{1}{\alpha_2 - a_2} = 1.81935\ldots$$

$$a_3 = \lfloor \alpha_3 \rfloor = 1 \implies \alpha_4 = \frac{1}{\alpha_3 - a_3} = 1.22047\ldots$$

The 3^{rd} convergent of e is $[2, 1, 2, 1] = \frac{11}{4}$. By corollary, no rational number with a denominator less than or equal to 4 can not be closer to e than $\frac{11}{4}$.

Algebraic & Transcendental Numbers

Definition

- $\alpha \in \mathbb{R}$ is said to be an algebraic number if it is a root of a polynomial of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \in \mathbb{Z}$ and $a_n \neq 0$. It is the *irreducible polynomial* associated with α, and the degree of α is n.
- $\alpha \in \mathbb{R}$ is said to be transcendental if it is not algebraic. (Liouville's Theorem)

Examples

- $\sqrt{3}$ is algebraic since it is a root of $f(x) = x^2 - 3$. Notice $f(x)$ is irreducible, so the degree of $\sqrt{3}$ is 2.
- Let $\alpha = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001\ldots$ Then, α is transcendental.

Bonus

The answers to the following first two questions are still unknown.

- Is $e + \pi$ algebraic or transcendental?
- Is $e + \pi$ rational or irrational?
- In the example, such α's called *Liouville numbers*. All *Liouville numbers* are transcendental. What should be the definition of a *Liouville number*?

Roth's Theorem

Definition

Let α be an *algebraic number* of degree $d \geq 2$ and let $\varepsilon > 0$. Then, there exists a constant $c(\alpha, \varepsilon) > 0$ such that for all $\frac{p}{q}$,

$$\frac{c(\alpha, \varepsilon)}{q^{2+\varepsilon}} < \left|\alpha - \frac{p}{q}\right|.$$

Theorem

Let α be an *algebraic number* of degree $d \geq 2$ and let $\varepsilon > 0$. Then, there are only finitely many solutions $\frac{p}{q}$ to the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$.

Example

Baker's Theorem

For all rational numbers $\frac{p}{q}$,

$$\frac{10^{-6}}{q^{2.955}} < \left| \sqrt[3]{2} - \frac{p}{q} \right|.$$

Using Baker's result, we can show that there are only finitely many pairs of integers (x, y) which satisfy the *diophatine equation*

$$x^3 - 2y^3 = m,$$

for any fixed integer m.

Corollary

Let $a_0, a_1, a_2, \dots$ be the sequence of integers defined by $a_0 = 0$, and for all $N \geq 1$,

$$a_N = \prod_{n=0}^{N-1} (1 + a_n).$$

If $\alpha = [a_0, a_1, a_2, \dots]$, then α is a *transcendental number*.

Applications

- Given any fixed integers m and n with $n > 0$ and $\sqrt[3]{n} \notin \mathbb{Q}$, the *diophantine equation*
$$x^3 - ny^3 = m$$
has only finitely many integer solutions (x, y).
- A *diophantine equation* of the form $x^2 - ny^2 = 1$, where n is a positive non-square integer, is called Pell's Equation.
The solutions of Pell's Equations are connected with *continued fractions* and give an accessible introduction to some ideas from algebraic number theory.