# MATH-443 Lecture Notes

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These notes are based on the lectures given by Türker Özsarı in the fall of 2024–2025 for the course MATH-443: Partial Differential Equations. Any errors or inaccuracies are entirely my responsibility and do not reflect the views or teachings of the lecturer.

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# 1 Introduction

## 1.1 Classification of PDEs

Any PDE can be written as $\mathcal{L}(u) = f(\mathbf{x})$, where $\mathcal{L}$ is a (differential) operator and $f(\mathbf{x})$ is a function of the independent variables.

**Definition 1.1.** A PDE of order $k$ is called:

**linear** if $\mathcal{L}$ is linear in $u$. Otherwise, the PDE is **nonlinear**.

A nonlinear PDE of order $k$ is called:

**semilinear** if all occurrences of derivatives of order $k$ appear with a coefficient that only depends on the independent variables.

**quasilinear** if all occurrences of derivatives of order $k$ appear with a coefficient that only depends on the independent variables, $u$, and its derivatives of order strictly less than $k$.

**fully nonlinear** if it is not quasilinear.

**Example 1.2.**

$$(xy)u_x + e^y u_y + (\sin x)u_y = x^3y^4 \quad \text{is linear}$$

$$(xy)u_x + e^y u_y + (\sin x)u_y = u^2 \quad \text{is semilinear}$$

$$uu_x + u_y = 0 \quad \text{is quasilinear}$$

$$(u_x)^2 + (u_y)^2 = 1 \quad \text{is fully nonlinear}$$

## 1.2 Classification of Linear Second-Order PDEs

Conic sections are curves obtained by the intersection of the surface of a cone and a plane. Using the Cartesian coordinates $(x, y)$, all such curves can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where the coefficients are not all 0. Excluding the degenerate case where the intersection gives a line, conic sections are classified by their discriminant $B^2 - 4AC$. More explicitly, we have

$$B^2 - 4AC < 0 \implies \text{ellipse}$$

$$B^2 - 4AC = 0 \implies \text{parabola}$$

$$B^2 - 4AC > 0 \implies \text{hyperbola}$$

A general homogenous linear second-order PDE in two variables is given by

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = 0$$

For simplicity, assume that the coefficients are constants. Then, the PDE is of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0,$$

where at least one of $a, b, c$ is nonzero. Taking the Fourier transform, we obtain

$$(-a\xi_x^2 - b\xi_x\xi_y - c\xi_y^2 + di\xi_x + ei\xi_y + f)\,\hat{u}(\xi_x, \xi_y) = 0$$

For a nontrivial solution, we must have

$$-a\xi_x^2 - b\xi_x\xi_y - c\xi_y^2 + di\xi_x + ei\xi_y + f = 0$$

$$a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 - di\xi_x - ei\xi_y - f = 0$$

Similar to conic sections, we make the following definitions:

$$b^2 - 4ac < 0 \implies \text{the PDE is elliptic}$$

$$b^2 - 4ac = 0 \implies \text{the PDE is parabolic}$$

$$b^2 - 4ac > 0 \implies \text{the PDE is hyperbolic}$$

**Example 1.3.** Laplace's equation $u_{xx} + u_{yy} = 0$ $(a = 1, b = 0, c = 1)$ is elliptic. Moreover, all elliptic PDEs have similar properties to Laplace's equation.

**Example 1.4.** The diffusion equation $u_t - u_{xx} = 0$ $(a = -1, b = 0, c = 0)$ is parabolic. Moreover, all parabolic PDEs have similar properties to the diffusion equation.

**Example 1.5.** The wave equation $u_{tt} - u_{xx} = 0$ $(a = -1, b = 0, c = 1)$ is hyperbolic. Moreover, all hyperbolic PDEs have similar properties to the wave equation.

What we mean by similar properties is that there exists a linear change of variables $\zeta(x, y), \eta(x, y)$ such that in the new coordinates $(\zeta, \eta)$, the PDE transforms as follows:

$$b^2 - 4ac < 0 \implies u_{\zeta\zeta} + u_{\eta\eta} + F(u_\zeta, u_\eta, u) = 0$$

$$b^2 - 4ac = 0 \implies u_{\eta\eta} + F(u_\zeta, u_\eta, u) = 0$$

$$b^2 - 4ac > 0 \implies u_{\zeta\eta} + F(u_\zeta, u_\eta, u) = 0$$

## 2 Method of Characteristics

We illustrate the method with an example.

**Example 2.1.** Consider the advection (transport) equation $u_t + au_x = 0$, where $u = u(x, t)$ and $x \in \mathbb{R}, t > 0$ with the initial condition $u(x, 0) = F(x) \in C^1(\mathbb{R})$. Notice that we can write

$$\langle u_x, u_t \rangle \cdot \langle a, 1 \rangle = 0$$

$$\implies \nabla u \cdot \frac{\langle a, 1 \rangle}{\sqrt{a^2 + 1}} = 0.$$

This means that the derivative of $u$ does not change in the direction of $\langle a, 1 \rangle$. We can fill up the whole plane by lines in the direction of $\langle a, 1 \rangle$ (These lines are called the characteristic lines, hence the name method of characteristics). Thus, one concludes that $u(x,t) = f(x - at)$. Using the initial condition,

$$u(x,0) = f(x) = F(x) \implies u(x,t) = F(x - at).$$

Alternatively, it is possible to introduce a linear change of variables that turns the PDE into an ODE. Consider the linear transformations

$$\begin{aligned} \zeta &= \alpha_1 x + \alpha_2 t \\ \eta &= \alpha_3 x + \alpha_4 t \end{aligned}$$

Then, by the chain rule

$$\begin{aligned} u_t &= u_\zeta \zeta_t + u_\eta \eta_t = \alpha_2 u_\zeta + \alpha_4 u_\eta \\ u_x &= u_\zeta \zeta_x + u_\eta \eta_x = \alpha_1 u_\zeta + \alpha_3 u_\eta \\ \implies u_t + a u_x &= (a\alpha_1 + \alpha_2) u_\zeta + (a\alpha_3 + \alpha_4) u_\eta = 0 \end{aligned}$$

We choose $\alpha_1 = 1,\ \alpha_2 = -a,\ \alpha_3 = -1/a,\ \alpha_4 = 2$ (As long as the transformation matrix is invertible, we are free in our choice of $\alpha_i$'s). Then,

$$u_\eta = 0 \implies u = f(\zeta) = f(\alpha_1 x + \alpha_2 t) = f(x - at)$$

The rest is the same.

## 3 1-D Diffusion (Heat) Equation

**Definition 3.1.** Let $\Omega \subseteq \mathbb{R}$ be an open interval (possibly $\mathbb{R}$ itself) and $T > 0$. The Cartesian product $\Omega \times (0,T)$ is called the spatio-temporal domain (or region). In general, by a domain, we mean an open and connected subset of $\mathbb{R}^n$.

Let $u(x,t)$ denote the concentration of a material at $(x,t) \in \Omega \times (0,T)$. Define a function $Q_a^x$ by $Q_a^x(t) = \int_a^x u(x',t)\,dx'$. Then, by the law of conservation, one has

$$\frac{d}{dt} Q_a^x(t) = d(x,t) - d(a,t),$$

where $d(x,t)$ is the diffusion rate from right to left. By Fick's Law, $d(x,t) = k u_x(x,t)$, where $k > 0$. So, we get

$$\frac{d}{dt} \int_a^x u(x',t)\,dx' = k(u_x(x,t) - u_x(a,t)).$$

The integral does not depend on $t$. Hence, by moving the derivative inside the integral, one obtains

$$\begin{aligned} \int_a^x \frac{d}{dt} u(x',t)\,dx' &= k(u_x(x,t) - u_x(a,t)) \\ \int_a^x u_t(x',t)\,dx' &= k(u_x(x,t) - u_x(a,t)). \end{aligned}$$

By FTC, we have
$$\frac{d}{dx}\int_a^x u_t(x',t)\,dx' = u_t(x,t).$$
On the other hand, since $u_x(a,t)$ is a fixed number,

$$\frac{d}{dx}[k(u_x(x,t)-u_x(a,t))] = ku_{xx}(x,t).$$

Thus, we obtain the diffusion equation

$$u_t(x,t) = ku_{xx}(x,t).$$

**Definition 3.2.** Let $u : \Omega \times (0,T) \to \mathbb{R}$ satisfying $u_t(x,t) = ku_{xx}(x,t)$ for all $(x,t) \in \Omega \times (0,T)$. Then, we say that $u$ is a classical solution of the diffusion equation.

**Example 3.3.** Let $u(x,t) = c\int_0^{\frac{x}{\sqrt{4kt}}} e^{-y^2}\,dy + d$, where $c$ and $d$ are constants. Then, one can easily confirm that $u_t(x,t) = ku_{xx}(x,t)$.

**Definition 3.4** (Well-posedness of a PDE)**.** We say that a PDE with auxiliary conditions is well-posed in Hadamard's sense if

(i) (existence) there exists a solution to the PDE that satisfies the auxiliary conditions.

(ii) (uniqueness) there is only one such solution.

(iii) (stability) the auxiliary condition is perturbed slightly, the resulting (unique) solution does not change abruptly.

### 3.1 Heat Equation on an Interval

We examine the heat equation on the unit interval with Dirichlet boundary conditions:

$$\begin{cases} u_t - u_{xx} = 0,\ x \in (0,1),\ t > 0 \\ u(x,0) = u_0(x) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

We begin to search for a solution by assuming that the solution can be separated, i.e., there exist functions $A(t)$ and $B(x)$ such that $u(x,t) = A(t)B(x)$. Then, we get

$$A'(t)B(x) - A(t)B''(x) = 0$$

Dividing by $A(t)B(x)$ yields

$$\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} =: -\lambda$$
$$A'(t) + \lambda A(t) = 0 \implies A(t) = e^{-\lambda t}$$
$$B''(x) + \lambda B(x) = 0 \implies B(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

Imposing the boundary conditions, we have

$$\begin{aligned} c_1 + c_2 &= 0 \qquad (*) \\ c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} &= 0 \end{aligned}$$

To prevent the trivial solution, $c_1$ and $c_2$ cannot be 0. Hence, one must have

$$e^{\sqrt{-\lambda}} = e^{-\sqrt{-\lambda}} \implies 2\sqrt{-\lambda} = 2k\pi i,\ k \in \mathbb{Z}$$

Therefore, $\lambda = k^2\pi^2$ and $B(x)$ can be rewritten as

$$\begin{aligned} B(x) &= a_1 e^{i\pi kx} - a_1 e^{-i\pi kx} \quad (\text{by}(*),\ c_1 = -c_2) \\ B(x) &= a\sin(\pi kx), \end{aligned}$$

where we used Euler's formula and set $2a_1 i = a$. Hence,

$$u(x,t) = ae^{-k^2\pi^2 t}\sin(\pi kx).$$

Now, define $u_k(x,t) \coloneqq a_k e^{-k^2\pi^2 t}\sin(\pi kx)$ and

$$u(x,t) \coloneqq \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t}\sin(\pi kx)$$

**Exercise 3.5.**

$$\int_0^1 \sin(k\pi x)\sin(l\pi x)\,dx = \begin{cases} 0\,, & k \neq l \\ \frac{1}{2}\,, & k = l \end{cases}$$

By the above exercise, one obtains

$$\begin{aligned} \int_0^1 u_0(x)\sin(n\pi x)\,dx &= \int_0^1 \left(\sum_{k=1}^{\infty} a_k \sin(\pi kx)\right)\sin(n\pi x)\,dx = \frac{a_n}{2} \\ \implies a_n &= 2\int_0^1 u_0(x)\sin(n\pi x)\,dx. \end{aligned}$$

Accordingly, we define

$$u(x,t) = \sum_{k=1}^{\infty} a_n e^{-\pi^2 n^2 t}\sin(n\pi x)$$

with $a_n$ defined as above.

### 3.1.1 Boundedness of $a_n$

Suppose that $u_0 \in C^2([0,1])$ with $u_0(0) = u_0(1) = 0$. Then,

$$|a_n| = 2\left|\int_0^1 u_0 \sin(n\pi x)\,dx\right| \leq 2\int_0^1 |u_0||\sin(n\pi x)|\,dx \leq 2\int_0^1 |u_0|\,dx \leq 2M,$$

where $M = \sup_{[0,1]} |u_0|$ (since $u_0 \in C^2([0,1])$, $\sup_{[0,1]} |u_0|$ exists. More explicitly, $u_0$ is a continuous function with a compact domain. So, it attains its maximum and minimum). Integration by parts yields

$$a_n = -2u_0(x)\frac{\cos(n\pi x)}{n\pi x}\bigg|_0^1 + \frac{2}{n\pi}\int_0^1 u_0'(x)\cos(n\pi x)\,dx.$$

The first term is 0 since we assumed $u_0(1) = u_0(0) = 0$. Then,

$$|a_n| \le \frac{2}{n\pi}\int_0^1 |u_0'|\,dx \le \frac{1}{n}\frac{M_1}{\pi},$$

where $M_1 = \sup_{[0,1]} |u_0'|$ (since $u_0' \in C^1([0,1])$, $\sup_{[0,1]} u_0'$ exists by a similar argument to $u_0$). Applying integration by parts again, we obtain

$$a_n = \frac{2}{n\pi}\left[u_0'\frac{\sin(n\pi x)}{n\pi}\bigg|_0^1 - \frac{1}{n\pi}\int_0^1 u_0''\sin(n\pi x)\,dx\right]$$

Similarly, the first term is 0 and we have

$$|a_n| \le \frac{2}{n^2\pi^2}\int_0^1 |u_0''|\,dx \le \frac{1}{n^2}\frac{2M_2}{\pi^2},$$

where $M_2 = \sup_{[0,1]} |u_0''|$ (since $u_0'' \in C^0([0,1])$, $\sup_{[0,1]} |u_0''|$ exists).

Now, we show that $u_t - u_{xx} = 0$ for all $(x,t) \in (0,1)\times(0,\infty)$, assuming $u_0 \in C^2([0,1])$. First, we observe that $u \in C^\infty((0,1)\times(0,T))$ is equivalent to $u \in C^\infty((0,1)\times(\varepsilon,T))$ for all $\varepsilon > 0$. Next, observe that for $(x,t) \in (0,1)\times(\varepsilon,T)$, we have

$$\begin{aligned}
\left|\frac{\partial^{p+q}}{\partial t^p \partial x^q}\left(a_k \sin(k\pi x)e^{-k^2\pi^2 t}\right)\right| &= \left|\frac{\partial^{p+q}}{\partial t^p \partial x^q}\left(\mathrm{Im}\left(a_k e^{-k^2\pi^2 t + ik\pi x}\right)\right)\right| \\
&= \left|a_k(-k^2\pi^2)^p(ik\pi)^q \mathrm{Im}\left(e^{ik\pi x - k^2\pi^2 t}\right)\right| \\
&\le c_{p,q}|a_k|k^{2p+q}e^{-k^2\pi^2 t} \\
&\le c_{p,q,u_0}k^{2p+q}e^{-k^2\pi^2\varepsilon},
\end{aligned}$$

where $c_{p,q,u_0}$ is a constant depending on $p, q$ and the initial condition. So, we have

$$\sum_{k=1}^{\infty}\left|\frac{\partial^{p+q}}{\partial t^p \partial x^q}\left(a_k \sin(k\pi x)e^{-k^2\pi^2 t}\right)\right| \le c_{p,q,u_0}\sum_{k=1}^{\infty} k^{2p+q}e^{-k^2\pi^2\varepsilon}$$

Using the ratio test, one can show that the series $\sum_{k=1}^{\infty} k^{2p+q}e^{-k^2\pi^2\varepsilon}$ converges absolutely, which implies uniform convergence. Thus, we can differentiate $u(x,t)$ with respect to $x, t$ on $(0,1)\times(\varepsilon,T)$ as many times as we wish. Since $\varepsilon$ was arbitrary, we conclude that $u \in C^\infty((0,1)\times(0,T))$. Moreover, we can differentiate the series term by term to show that $u_t - u_{xx} = 0$.

**Claim 3.6.** $u$ is the unique solution to $u_t - u_{xx} = 0$.

*Proof.* Suppose $u_1, u_2$ are two solutions with the initial and boundary conditions $u(x,0) = u_0(x)$ and $u(0,t) = u(1,t) = 0$. Set $u := u_1 - u_2$. Then, $u$ solves the heat equation with the initial condition $u_0(x) = 0$ and the boundary conditions remain the same. Multiplying $u_t - u_{xx} = 0$ by $u$ and integrating, one gets

$$\int_0^1 u_t u \, dx - \int_0^1 u_{xx} u \, dx = 0$$

Observe that the first term is $\frac{1}{2}\frac{du^2}{dt}$. We apply integration by parts to the second term. Then,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |u|^2 \, dx - u_x u\Big|_0^1 + \int_0^1 |u_x|^2 \, dx = 0$$

The second term is 0 by the boundary conditions. So,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |u|^2 \, dx = -\int_0^1 |u_x|^2 \, dx \le 0$$

Setting $y(t) := \int_0^1 |u|^2 \, dx$, we have

$$\begin{aligned} y'(t) \le 0 \implies & \int_0^t y'(s) \, ds \le \int_0^t 0 \, ds = 0 \\ \implies & \; y(t) - y(0) \le 0 \end{aligned}$$

However, $y(0) = \int_0^1 |u(x,0)|^2 \, dx = \int_0^1 0 \, dx = 0$. So, $y(t) \le 0$. We have $0 \le y(t)$ by definition. Thus, we get

$$\begin{aligned} 0 \le y(t) \le 0 \implies & \; y(t) = 0 \\ \implies & \int_0^1 |u(x,t)|^2 \, dx = 0 \end{aligned}$$

Since $u$ is continuous with respect to $x, t$ and $u^2$ is nonnegative, we must have $u(x,t) = 0$. Therefore, $u_1 = u_2$. $\square$

**Exercise 3.7.** Let $u, v$ be two solutions of the heat equation with the initial conditions $u_0, v_0$, respectively, and the usual boundary conditions. Show in a certain sense that $u, v$ are close to each other whenever $u_0, v_0$ are close to each other (For example, if $\sup\limits_{[0,1]} |u_0 - v_0| \le \varepsilon$ for some $\varepsilon > 0$, find a bound for $\sup\limits_{[0,1]} |u - v|$).

**Remark 3.8.** One can write the solution to the heat equation with Dirichlet boundary conditions as follows

$$\begin{aligned} u(x,t) &= \sum_{k \ge 1} a_k \sin(k\pi x) e^{-k^2\pi^2 t} \\ &= \sum_{k \ge 1} 2 \int_0^1 u_0(y) \sin(k\pi y) \sin(k\pi x) e^{-k^2\pi^2 t} \, dy \\ &= \int_0^1 u_0(y) \left( 2 \sum_{k \ge 1} \sin(k\pi y) \sin(k\pi x) e^{-k^2\pi^2 t} \right) dy \\ &= \int_0^1 H(x,y,t) u_0(y) \, dy, \end{aligned}$$

where $H(x,y,t)$ is called the Heat kernel.

**Remark 3.9.** The equation $v_t + v_{xx} = 0$ with Dirichlet boundary conditions is called the backward heat equation. Setting $u(x,t) \coloneqq v(x, T-t)$, we can write $u_t - u_{xx} = v_t + v_{xx}$. The backward heat equation is not well-posed unless $v_0 \in C^\infty$.

### 3.1.2 Time Decay of Heat Equation

Using $|\sin| \leq 1$ and $e^{-k^2\pi^2 t} \leq e^{-\pi^2 t}$, we have

$$|u(x,t)| = \sum_{k\geq 1} |a_k||\sin(k\pi x)||e^{-k^2\pi^2 t}| \leq e^{-\pi^2 t} \sum_{k\geq 1} |a_k| < Ce^{-\pi^2 t}$$

for some constant $C$, where we used the fact $|a_k| \leq C'/k^2$ for some constant $C'$. Thus, we see that as $t \to \infty$, $u(x,t) \to 0$ irrespective of $x$.

# 4 Laplace's Equation

We consider Laplace's equation on a unit disk given by

$$\begin{cases} \Delta u = 0, \ |\mathbf{x}| \leq 1 \\ u\big|_{\partial B_1(0)} = f \end{cases}$$

#### Laplacian in Polar Coordinates

We start by writing the Laplacian in polar coordinates. First, compute $u_{rr}$.

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = \cos\theta\, u_x + \sin\theta\, u_y \\ u_{rr} &= \cos\theta\, u_{xx} x_r + \cos\theta\, u_{xy} y_r + \sin\theta\, u_{yx} x_r + \sin\theta\, u_{yy} y_r \\ &= \cos^2\theta\, u_{xx} + 2\sin\theta\cos\theta\, u_{xy} + \sin^2\theta\, u_{yy} \end{aligned}$$

Next, compute $u_{\theta\theta}$.

$$\begin{aligned} u_\theta &= u_x x_\theta + u_y y_\theta \\ &= -r\sin\theta\, u_x + r\cos\theta\, u_y \\ u_{\theta\theta} &= -r\cos\theta\, u_x - r\sin\theta\, u_{xx} x_\theta - r\sin\theta\, u_{xy} y_\theta \\ &\quad - r\sin\theta\, u_y + r\cos\theta\, u_{yy} y_\theta + r\cos\theta\, u_{yx} x_\theta \\ u_{\theta\theta} &= -r\cos\theta\, u_x - r\sin\theta\, u_y + (r\sin\theta)^2 u_{xx} \\ &\quad - r^2 \sin\theta\cos\theta u_{xy} - r^2\sin\theta\cos\theta u_{yx} + (r\cos\theta)^2 u_{yy} \end{aligned}$$

Then,

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} = -\frac{1}{r} u_r + u_{xx} + u_{yy}$$

Since $u_{xx} + u_{yy} = \Delta u$, we obtain

$$\Delta u = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r.$$

**Remark 4.1.** For 2-D heat equation on the boundary of a disk, one would have

$$u_t - (u_{xx} + u_{yy}) = u_t - \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = 0,$$

where $r \in [0, 1)$ and $\theta \in [0, 2\pi)$.

**Definition 4.2.** Let $u : \Omega \to \mathbb{R}$ be a function. We say that $u$ is a harmonic function if $\Delta u = 0$ on $\Omega$.

### 4.1 Separation of Variables

We assume that the solution can be written in the form $u(r, \theta) = A(\theta)B(r)$. Then, the formula we derived for the Laplacian in polar coordinates gives

$$\begin{aligned} A(\theta)B''(r) + \frac{1}{r}B'(r)A(\theta) + \frac{1}{r^2}A''(\theta)B(r) = 0 \\ \implies \frac{r^2 B''(r) + rB'(r)}{B(r)} = -\frac{A(\theta)}{A(\theta)} =: -\lambda \end{aligned}$$

The above gives a set of equations

$$\begin{cases} A''(\theta) - \lambda A(\theta) = 0 \\ r^2 B''(r) + rB'(r) + \lambda B(r) = 0 \end{cases}$$

The first one gives

$$\begin{cases} A(\theta) = a_1 e^{\sqrt{\lambda}\theta} + a_2 e^{-\sqrt{\lambda}\theta}, \quad \lambda \neq 0 \\ A(\theta) = a_1 + a_2\theta, \quad \lambda = 0 \end{cases}$$

Note that $A(\theta)$ is periodic with period $2\pi$. Thus,

$$\begin{cases} A(\theta) = a_1 e^{ik\theta} + a_2 e^{-ik\theta}, \quad \lambda \neq 0 \\ A(\theta) = a_1, \quad \lambda = 0 \end{cases}$$

where $\lambda = -k^2$. So, the second equation gives

$$\begin{cases} B(r) \equiv 1 \text{ or } B(r) = \ln r, \quad k = 0 \\ B(r) = r^k \text{ or } B(r) = r^{-k}, \quad k \neq 0 \end{cases}$$

Since $\ln 0$ is not defined, $B(r) = \ln r$ is not acceptable. For $k \neq 0$, the solution can be written as $B(r) = r^{|k|}$. Combining the results, we get

$$u(r, \theta) = ar^k e^{ik\theta} + br^k e^{-ik\theta},$$

where $a, b \in \mathbb{C}$ and $k \in \mathbb{N}$. Then, $u$ solves Laplace's equation on $B_1(0)$. We now consider the series $\sum\limits_{k\in\mathbb{Z}} a_k r^{|k|} e^{ik\theta}$ as a candidate solution for the system (i.e., a solution that also satisfies the boundary condition). Observe that in order for the series to be a solution of the system, one must have $f(\theta) = \sum\limits_{k\in\mathbb{Z}} a_k e^{ik\theta}$.

Suppose that $0 < r < 1$ and $|a_k| < M$ for all $k \in \mathbb{Z}$ for some $M > 0$. Then,

$$\sum_{k\in\mathbb{Z}} |a_k| r^{|k|} e^{ik\theta} \le M \sum_{k\in\mathbb{Z}} r^{|k|} \le 2M \sum_{k=0}^{\infty} r^k < \infty,$$

where we used the fact that $\sum_{k=0}^{\infty} r^k < \infty$ for $0 < r < 1$. Now suppose that $\sum_{k\in\mathbb{Z}} |a_k| < \infty$. Then,

$$\sum_{k\in\mathbb{Z}} |a_k r^{|k|} e^{ik\theta}| = \sum_{k\in\mathbb{Z}} |a_k||r^{|k|}||e^{ik\theta}| \le \sum_{k\in\mathbb{Z}} |a_k| < \infty \implies \text{uniform convergence on } \overline{B_1(0)}$$

where we used $|r^k| < 1$ for $0 < r < 1$ and $|e^{ik\theta}| \le 1$.

**Exercise 4.3.**

$$\int_0^{2\pi} e^{ik\theta} e^{-in\theta}\, d\theta = \begin{cases} 2\pi, & k = n \\ 0, & k \ne n \end{cases}$$

Assume that there exist $a_k$ such that $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} = f(\theta)$, where $f(\theta)$ is given. Then,

$$\int_0^{2\pi} \left( \sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \right) e^{-in\theta}\, d\theta = \sum_{k\in\mathbb{Z}} a_k \int_0^{2\pi} e^{ik\theta} e^{-in\theta}\, d\theta = a_n 2\pi$$
$$\implies a_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta.$$

Hence, given $f$, we set

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta}\, d\theta$$

whenever the integral exists. Moreover, if $f$ is bounded (by $M$) and $a_k$ exist, then

$$|a_k| \le \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|\, d\theta < M.$$

If $f'$ exists and it is bounded, then

$$a_k = f(\theta) \frac{e^{-ik\theta}}{-ik} \Bigg|_0^{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} f'(\theta) \left( \frac{1}{-ik} \right) e^{-ik\theta}\, d\theta.$$

The first term is 0 if we assume $f$ has period $2\pi$. Then,

$$a_k = -\frac{1}{2\pi} \int_0^{2\pi} f'(\theta) \left( \frac{1}{-ik} \right) e^{-ik\theta}\, d\theta \le \left| \int_0^{2\pi} \frac{f'(\theta) e^{-ik\theta}}{2\pi i k}\, d\theta \right| = \frac{1}{2\pi k} \int_0^{2\pi} |f'(\theta)|\, d\theta \le \frac{C}{k}.$$

for some constant $C$ as $f'$ is bounded. Similarly, if $f \in C^2$ and $f, f', f''$ are all bounded, then $|a_k| \le C/k^2$ for some constant $C$. Furthermore, $\sum_{k\in\mathbb{Z}} |a_k| r^{|k|} e^{ik\theta}$ converges uniformly under these assumptions.

**Remark 4.4.** Similar to the heat equation, Laplace's equation admits an integral representation. One has

$$\begin{aligned} u(r,\theta) &= \sum_{k\in\mathbb{Z}} a_k r^{|k|} e^{ik\theta} \\ &= \sum_{k\in\mathbb{Z}} \left( \frac{1}{2\pi} \int_0^{2\pi} f(\sigma) e^{-ik\sigma}\, d\sigma \right) r^{|k|} e^{ik\theta} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\sigma) \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik(\theta-\sigma)}\, d\sigma \\ &= \int_0^{2\pi} f(\sigma) P(r,\theta-\sigma)\, d\sigma, \end{aligned}$$

where $P(r,\theta) = \frac{1}{2\pi} \sum\limits_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta}$ is called the Poisson Kernel. For $r < 1$, using the geometric series formula, we have

$$\begin{aligned} P(r,\theta) = \frac{1}{2\pi} \left[ \sum_{k\geq 0} r^k e^{ik\theta} + \sum_{k\geq 1} r^k e^{-ik\theta} \right] &= \frac{1}{2\pi} \left( \frac{1}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} \right) \\ &= \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2} \\ &\geq 0. \end{aligned}$$

Moreover, for $\theta \neq 0$, we have $\lim\limits_{r\to 1^-} P(r,\theta) = 0$

**Example 4.5.** Let $f \equiv 1$. Then,

$$\begin{aligned} u(r,\theta) = \int_0^{2\pi} f(\sigma) P(r,\theta-\sigma)\, d\sigma &= \int_0^{2\pi} P(r,\theta-\sigma)\, d\sigma \\ &= -\int_\theta^{\theta-2\pi} P(r,\sigma')\, d\sigma' \\ &= \int_0^{2\pi} P(r,\theta)\, d\theta, \end{aligned}$$

where in the last inequality, we used the periodicity of the Poisson kernel with period $2\pi$.

$$\implies \lim_{r\to 1^-} P(r,\theta) = \delta_0(\theta),$$

where $\delta_0$ is the Dirac delta.

**Claim 4.6.** $\lim\limits_{r\to 1^-} \int_0^{2\pi} f(\sigma) P(r,\theta-\sigma)\, d\sigma = f(\theta)$. (By lim, we mean uniform convergence.)

**Strategy:** $\forall \varepsilon > 0 \ \sup\limits_{\theta\in[0,2\pi)} \left| \int_0^{2\pi} f(\sigma) P(r,\sigma-\theta)\, d\sigma - f(\theta) \right| < \varepsilon$ as $r \to 1^-$.

**Exercise 4.7.** $\int_0^{2\pi} P(r,\sigma)\, d\sigma = \int_0^{2\pi} P(r,\theta-\sigma)\, d\sigma = 1$.

**Trick:** $f(\theta) = f(\theta) \int_0^{2\pi} P(r,\sigma)\, d\sigma = \int_0^{2\pi} f(\theta) P(r,\sigma)\, d\sigma$

*Proof.* Using the trick, we have

$$\left|\int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,d\sigma - f(\theta)\right| = \left|\int_0^{2\pi} (f(\sigma)-f(\theta))P(r,\theta-\sigma)\,d\sigma\right|$$

Using periodicity with period $2\pi$ gives

$$\begin{aligned}\left|\int_0^{2\pi} (f(\sigma)-f(\theta))P(r,\theta-\sigma)\,d\sigma\right| &= \left|\int_{\theta-2\pi}^{\theta} (f(\theta-\sigma)-f(\theta))P(r,\sigma)\,d\sigma\right| \\ &= \left|\int_0^{2\pi} (f(\theta-\sigma)-f(\theta))P(r,\sigma)\,d\sigma\right| \\ &= \left|\int_0^{\delta} \dots\,d\sigma + \int_{\delta}^{2\pi-\delta} \dots\,d\sigma + \int_{2\pi-\delta}^{2\pi} \dots\,d\sigma\right|,\end{aligned}$$

where $\dots$ denotes $(f(\theta-\sigma)-f(\theta))P(r,\sigma)$. Using the periodicity again, $\left|\int_{2\pi-\delta}^{2\pi} \dots\,d\sigma\right| = \left|\int_{-\delta}^{0} \dots\,d\sigma\right|$.

$$\begin{aligned}\implies \left|\int_0^{2\pi} (f(\theta-\sigma)-f(\theta))P(r,\sigma)\,d\sigma\right| &= \left|\int_{\delta}^{2\pi-\delta} \dots\,d\sigma + \int_{-\delta}^{\delta} \dots\,d\sigma\right| \\ &\le \left|\int_{\delta}^{2\pi-\delta} \dots\,d\sigma\right| + \left|\int_{-\delta}^{\delta} \dots\,d\sigma\right|\end{aligned}$$

One has

$$\begin{aligned}\left|\int_{\delta}^{2\pi-\delta} (f(\theta-\sigma)-f(\theta))P(r,\sigma)\,d\sigma\right| &\le \int_{\delta}^{2\pi-\delta} |(f(\theta-\sigma)-f(\theta))P(r,\sigma)|\,d\sigma \\ &\le \int_{\delta}^{2\pi-\delta} (|f(\theta-\sigma)|+|f(\theta)|)P(r,\sigma)\,d\sigma\end{aligned}$$

Because $f$ is continuous, $\exists M > 0$ such that $|f| \le M$.

$$\implies \int_{\delta}^{2\pi-\delta} (|f(\theta-\sigma)|+|f(\theta)|)P(r,\sigma)\,d\sigma \le 2M\int_{\delta}^{2\pi-\delta} P(r,\sigma)\,d\sigma$$

Thus, $\left|\int_{\delta}^{2\pi-\delta}(f(\theta-\sigma)-f(\theta))P(r,\sigma)\,d\sigma\right| \le 2M\int_{\delta}^{2\pi-\delta} P(r,\sigma)\,d\sigma$. Next, we have

$$\left|\int_{-\delta}^{\delta}(f(\theta-\sigma)-f(\theta))P(r,\sigma)\,d\sigma\right| \le \int_{-\delta}^{\delta} |(f(\theta-\sigma)-f(\theta))P(r,\sigma)|\,d\sigma$$

For sufficiently small $\delta$, $|f(\theta-\sigma)-f(\theta)| < \varepsilon$ due to uniform continuity of $f$. Then,

$$\int_{-\delta}^{\delta} |(f(\theta-\sigma)-f(\theta))P(r,\sigma)|\,d\sigma < \varepsilon\int_{-\delta}^{\delta} P(r,\sigma)\,d\sigma \le 2\varepsilon$$

As $r \to 1^-$, $\int_{\delta}^{2\pi-\delta} P(r,\sigma)\,d\sigma \to 0$ since $P(r,\sigma) \to 0$ uniformly. Hence, letting $r \to 1^-$, we obtain

$$\left|\int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,d\sigma - f(\theta)\right| < 2\varepsilon$$

$$\implies \lim_{r\to 1^-}\int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,d\sigma = f(\theta)$$

$\square$

# 5 Heat equation on the whole interval

Before presenting the problem, we first state some facts about the Fourier transform.

## 5.1 Fourier Transform

Suppose $f$ is an integrable function, i.e., $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Then, for such a function, we define the Fourier Transform

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx.$$

Note that some authors define it as $\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i\xi x} f(x)\, dx$.

**Remark 5.1.** **1)** $\hat{f} = \hat{f}(\xi)$ is bounded for $f$ with $\int_{-\infty}^{\infty} |f|\, dx < \infty$ :

$$|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)e^{-i\xi x}|\, dx \leq \int_{-\infty}^{\infty} |f(x)|\, dx =: M_f < \infty, \;\; \xi \in (\infty, \infty)$$

**2)** $\hat{f}$ is continuous: Let $\xi_0 \in \mathbb{R}, |\hat{f}(\xi) - \hat{f}(\xi_0)| \leq \int_{-\infty}^{\infty} |f(x)||e^{-i\xi x} - e^{-i\xi_0 x}|\, dx$. Let $g_\xi(x) := |f(x)||e^{-i\xi x} - e^{-i\xi_0 x}|$

**Theorem 5.2** (Dominated Convergence Theorem)**.** Suppose $f_n \to f$ pointwise and $|f_n| \leq g$, where $\int |g| < \infty$, then $\int f_n \to \int f$.

We have $g_\xi(x) \to 0$ pointwise as $\xi \to \xi_0$ and $|g_\xi(x)| \leq 2|f(x)|$ for all $x$ and $\int 2|f|\, dx < \infty$. So, $\lim\limits_{\xi \to \xi_0} \hat{f}(\xi) = \hat{f}(\xi_0)$ by dominated convergence theorem.

**Example 5.3.** Let $f(x) = e^{-x^2}$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} e^{-x^2}\, dx$$

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{-\infty}^{\infty} (-ix)e^{-i\xi x}e^{-x^2}\, dx = \frac{i}{2}\int_{-\infty}^{\infty} \frac{d}{dx}(e^{-x^2})e^{-i\xi x}\, dx$$

Integration by parts gives

$$\frac{i}{2}\int_{-\infty}^{\infty} \frac{d}{dx}(e^{-x^2})e^{-i\xi x}\, dx = \frac{i}{2}\left[ e^{-x^2}e^{-i\xi x}\Big|_{-\infty}^{\infty} + i\xi \int_{-\infty}^{\infty} e^{-x^2}e^{-i\xi x}\, dx \right] = -\frac{1}{2}\xi\hat{f}(\xi)$$

$$\implies \frac{d}{d\xi}\hat{f}(\xi) = -\frac{\xi}{2}\hat{f}(\xi)$$

Using $\mu = e^{\int \frac{\xi}{2}\, d\xi}$ as the integrating factor, we obtain

$$\hat{f}(\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}$$

### 5.1.1 Inverse Fourier Transform

Conversely, let $G(\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}$ and suppose that $\int |f| < \infty$ for some $f : \mathbb{R} \to \mathbb{R}$. Define $g_{a,x}(\xi) := e^{-i\xi x}G(a\xi),\ a > 0$.

Given a function $g(\xi)$, we define $\check{g}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}g(\xi)\,d\xi$. Then,

$$\begin{aligned}\check{g_{a,x}}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi y}g_{a,x}(\xi)\,d\xi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}e^{i\xi y}G(a\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(y-x)}G(a\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{y-x}{a}\right)\xi'}\frac{1}{a}G(\xi')\,d\xi' \\ &= \frac{1}{a}\check{G}\left(\frac{y-x}{a}\right) = \frac{1}{a}\frac{1}{2\sqrt{\pi}}G\left(\frac{y-x}{2a}\right),\end{aligned}$$

where we set $a\xi =: \xi'$ in the third line. In particular, when $G$ is $\sqrt{\pi}e^{-\frac{\xi^2}{4}}$, we have

$$\frac{1}{a}\check{G}\left(\frac{y-x}{a}\right) = \frac{1}{a}e^{-\left(\frac{y-x}{a}\right)^2}$$

Now, let $\int_{-\infty}^{\infty}|f| < \infty$. Then,

$$\begin{aligned}\int \hat{f}(\xi)\overline{g_{a,x}(\xi)}\,d\xi = \int\left[\int f(y)e^{-i\xi y}\,dy\right]\overline{g_{a,x}(\xi)}\,d\xi &= \int\left[\int \overline{g_{a,x}(\xi)}e^{-iy\xi}\,d\xi\right]f(y)\,dy \\ &= 2\pi\int \overline{\check{g_{a,x}}(y)}f(y)\,dy,\end{aligned}$$

where $\overline{z}$ denotes the complex conjugate of $z$ and in the second equality we used Fubini's Theorem to interchange the integration variables.

**Remark 5.4.** The above identity holds for any $\overline{g_{a,x}(\xi)}$. In particular, for $\hat{f}(\xi) = g_{a,x}(\xi)$, we will have

$$\int |\hat{f}(\xi)|^2\,d\xi = 2\pi\int \overline{\check{\hat{f}}(y)}f(y)\,dy = 2\pi\int |f(x)|^2\,dx$$

provided that we show $\check{\hat{f}} = f$. The above is known as Parseval's Identity. For $g \equiv g_{a,x}$, we have

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{g_{a,x}(\xi)}\,d\xi = 2\pi\int_{\mathbb{R}}\frac{1}{2a\sqrt{\pi}}G\left(\frac{y-x}{2a}\right)f(y)\,dy,$$

where we used the results $\check{g_{a,x}}(y) = \frac{1}{2a\sqrt{\pi}}G\left(\frac{y-x}{2a}\right)$ and $\overline{G} = G$.

**Claim 5.5.** $2\pi\int_{\mathbb{R}}\frac{1}{2a\sqrt{\pi}}G\left(\frac{y-x}{2a}\right)f(y)\,dy \to 2\pi f(x)$ as $a \to 0^+$.

**Remark 5.6.** On the other hand, we have $\int_{\mathbb{R}}\hat{f}(\xi)\overline{g_{a,x}(\xi)}\,d\xi \to \int_{\mathbb{R}}\hat{f}(\xi)e^{ix\xi}\,d\xi = 2\pi\check{\hat{f}}(x)$. Hence, if the claim is true, we have $f = \check{\hat{f}}$, i.e., $f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}e^{ix\xi}\hat{f}(\xi)\,d\xi$.

**Lemma 5.7.** $\int_{\mathbb{R}}\frac{1}{2a\sqrt{\pi}}G\left(\frac{y-x}{2a}\right)f(y)\,dy \to f(x)$ assuming $f$ is continuous and bounded.

*Proof.* $\forall \varepsilon > 0, x \in \mathbb{R}, \exists \delta > 0, |f(y) - f(x)| < \varepsilon$ if $|x - y| < \delta$ (uniform continuity of $f$ on $[x - \delta, x + \delta]$)

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{2a\sqrt{\pi}} G\left(\frac{y-x}{2a}\right) f(y)\, dy &= \int_{|y-x|<\delta} \dots\, dy + \int_{|y-x|\geq\delta} \dots\, dy \\ &\leq \frac{f(x)+\varepsilon}{2a\sqrt{\pi}} \int_{|y-x|<\delta} G\left(\frac{y-x}{2a}\right) dy \\ &\quad + \frac{M}{2a\sqrt{\pi}} \int_{|y-x|\geq\delta} G\left(\frac{y-x}{2a}\right) dy,\end{aligned}$$

where we have $m \leq f(x) \leq M, \forall x \in \mathbb{R}$ since $f$ is bounded. Analogously,

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{2a\sqrt{\pi}} G\left(\frac{y-x}{2a}\right) f(y)\, dy &\geq \frac{f(x)-\varepsilon}{2a\sqrt{\pi}} \int_{|y-x|<\delta} G\left(\frac{y-x}{2a}\right) dy \\ &\quad + \frac{m}{2a\sqrt{\pi}} \int_{|y-x|\geq\delta} G\left(\frac{y-x}{2a}\right) dy.\end{aligned}$$

**Observation:** $\frac{M}{2a\sqrt{\pi}} \int_{|y-x|\geq\delta} G\left(\frac{y-x}{2a}\right)\, dy \to 0$ as $a \to 0^+$.

**Proof:** $\mathrm{I}^a := \frac{M}{2a\sqrt{\pi}} \int_{|y-x|\geq\delta} e^{-\left(\frac{y-x}{2a}\right)^2}\, dy = \frac{M}{\sqrt{\pi}} \int_{|z|\geq\frac{\delta}{2a}} e^{-z^2}\, dz \to 0$ as $a \to 0^+$ $\left(z = \frac{y-x}{2a}\right)$.

**Observation:** $\frac{1}{2a\sqrt{\pi}} \int_{|y-x|<\delta} G\left(\frac{y-x}{2a}\right)\, dy \to 1$ as $a \to 0^+$.

**Proof:**

$$\begin{aligned}\mathrm{II}^a &= \frac{1}{\sqrt{\pi}} \int_{|y-x|<\delta} \frac{1}{2a} e^{-\left(\frac{y-x}{2a}\right)^2}\, dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{1}{2a} e^{-\left(\frac{y-x}{2a}\right)^2}\, dy \; - \int_{|y-x|\geq\delta} \frac{1}{2a} e^{-\left(\frac{y-x}{2a}\right)^2}\, dy \\ &\to \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2}\, dz = 1 \;\text{ as }\; a \to 0^+\end{aligned}$$

Now, combining observations and the inequalities, for any $\varepsilon > 0$, we have

$$f(x) - \varepsilon \leq \int_{\mathbb{R}} \frac{1}{2a\sqrt{\pi}} G\left(\frac{y-x}{2a}\right) f(y)\, dy \leq f(x) + \varepsilon$$

as $a \to 0^+$, $\int_{\mathbb{R}} \frac{1}{2a\sqrt{\pi}} G\left(\frac{y-x}{2a}\right) f(y)\, dy \to f(x)$. Hence, the claim follows. □

### 5.2 Heat Equation on the Real Line

We can now state the problem.

$$\begin{cases} u_t - u_{xx} = 0, \; x \in \mathbb{R}, \; t > 0 \\ u(x, 0) = u_0(x) \\ \lim\limits_{|x|\to\infty} u(x,t) = \lim\limits_{|x|\to\infty} u_x(x,t) = 0 \quad \text{(Implicit boundary condition)} \end{cases}$$

We start by taking the Fourier transform with respect to $x$. Using the linearity of FT, one has

$$\mathcal{F}(0) = \mathcal{F}(u_t - u_{xx}) = \mathcal{F}(u_t) - \mathcal{F}(u_{xx}) = 0$$

$$\mathcal{F}(u_t) = \int_{\mathbb{R}} e^{-ix\xi} u_t(x,t)\,dx = \frac{d}{dt}\int_{\mathbb{R}} e^{-ix\xi} u(x,t)\,dx = \frac{d}{dt}\hat{u}(\xi,t) = \hat{u}_t(\xi,t)$$

$$\mathcal{F}(u_{xx}) = \int_{\mathbb{R}} e^{-ix\xi} u_{xx}(x,t)\,dx = u_x e^{-ix\xi}\Big|_{-\infty}^{\infty} + i\xi\int_{\mathbb{R}} e^{-ix\xi} u_x(x,t)\,dx.$$

The first term is 0 by the implicit boundary condition. Applying integration by parts to the second term, we get

$$i\xi\int_{\mathbb{R}} e^{-ix\xi}u_x(x,t)\,dx = i\xi\left(ue^{-ix\xi}\Big|_{-\infty}^{\infty} + i\xi\int_{\mathbb{R}} e^{-ix\xi}u(x,t)\,dx\right) = -\xi^2\hat{u}(\xi,t).$$

So, we obtain infinitely many ODEs depending on $\xi$

$$\frac{d}{dt}\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0$$
$$\implies \hat{u}(\xi,t) = A_\xi e^{-\xi^2 t}.$$

Assuming $\hat{u}(\xi,0) = \hat{u}_0(\xi)$, the above becomes

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-\xi^2 t}$$
$$\implies u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{u}(\xi,t)\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi-\xi^2 t}\hat{u}_0(\xi)\,d\xi.$$

**Definition 5.8.** $(f*g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\,dy = \int_{\mathbb{R}} f(x-y)g(y)\,dy$ is said to be the convolution of $f$ and $g$ at $x$.

There is an important relation between convolution and FT.

$$\begin{aligned}
(\widehat{f*g})(\xi) = \int_{\mathbb{R}} e^{-ix\xi}(\widehat{f*g})(x)\,dx &= \int_{\mathbb{R}} e^{-ix\xi}\left(\int_{\mathbb{R}} f(y)g(x-y)\,dy\right)dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-iy\xi}f(y)e^{iy\xi}e^{-ix\xi}g(x-y)\,dy\,dx \\
&= \int_{\mathbb{R}} f(y)e^{-iy\xi}\left(\int_{\mathbb{R}} g(x-y)e^{-i(x-y)\xi}\,dx\right)dy \\
&= \int_{\mathbb{R}} f(y)e^{-iy\xi}\left(\int_{\mathbb{R}} g(z)e^{-iz\xi}\,dz\right)dy \\
&= \int_{\mathbb{R}} e^{-iy\xi}f(y)\hat{g}(\xi)\,dy \\
&= \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

We found the solution of the heat equation as $u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi-\xi^2 t}\hat{u}_0(\xi)\,d\xi$. We have $\mathcal{F}^{-1}(\hat{u}_0) = u_0$. So, we need $\mathcal{F}^{-1}(e^{-\xi^2 t})$.

$$\begin{aligned}
\mathcal{F}^{-1}(e^{-\xi^2 t}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi-\xi^2 t}\,d\xi &= \frac{\sqrt{\pi}}{2\pi\sqrt{\pi}}\int_{\mathbb{R}} e^{ix\frac{\theta}{2\sqrt{t}}}\frac{e^{-\frac{\theta^2}{4}}}{2\sqrt{t}}\,d\theta \\
&= \frac{1}{2\sqrt{\pi t}}G\left(\frac{x}{2\sqrt{t}}\right) \\
&= \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}},
\end{aligned}$$

where we made the substitution $\xi\sqrt{t} = \frac{\theta}{2}$.

$$\implies u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u_0(y)\, dy = u_0 * \mathcal{F}^{-1}(e^{-\xi^2 t}).$$

# 6 Wave Equation on the Real Line

$$\begin{cases} u_{tt} - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0 \quad (1) \\ u(x,0) = u_0(x) \quad (2) \\ u_t(x,0) = u_1(0) \quad (3) \end{cases}$$

**Remark 6.1.** We can assume that $t \in \mathbb{R}$ since the wave equation is time-reversible.

Applying Fourier transform, we have

$$\mathcal{F}(u_{tt}) = \frac{d^2\hat{u}(\xi,t)}{dt^2} = \int_{\mathbb{R}} e^{-ix\xi} u_{tt}(x,t)\, dx = \frac{d^2}{dt^2} \int_{\mathbb{R}} e^{-ix\xi} u(x,t)\, dx$$

Recall from heat equation $\mathcal{F}(u_{xx}) = -\xi^2 \hat{u}(\xi,t)$

$$\implies \frac{d^2\hat{u}(\xi,t)}{dt^2} + \xi^2\hat{u}(\xi,t) = 0$$
$$\hat{u}(\xi,t) = A\cos(\xi t) + B\sin(\xi t).$$

Hence, a PDE turned into infinitely many ODEs parametrized by $\xi$. Now, set

$$\hat{u}(\xi,0) =: \hat{u}_0(\xi)$$
$$\frac{d\hat{u}(\xi,0)}{dt} = \left.\frac{d\hat{u}(\xi,t)}{dt}\right|_{t=0} =: \hat{u}_1(\xi).$$

Then, we see that $A = \hat{u}_0(\xi)$ and $\hat{u}_1(\xi) = B\xi$.

$$\implies \hat{u}(\xi,t) = \hat{u}_0(\xi)\cos(\xi t) + \frac{1}{\xi}\hat{u}_1(\xi)\sin(\xi t)$$

**Observation:** Let $F(x,t) = \frac{1}{2}[f(x+t) + f(x-t)]$, where $f : \mathbb{R} \to \mathbb{R}$. Then,

$$\hat{F}(\xi,t) = \cos(\xi t)\hat{f}(\xi).$$

**Proof:**

$$\begin{aligned} \hat{F}(\xi,t) &= \frac{1}{2}\int_{\mathbb{R}} e^{-ix\xi}[f(x+t) + f(x-t)]\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-ix\xi} f(x+t)\, dx + \frac{1}{2}\int_{\mathbb{R}} e^{-ix\xi} f(x-t)\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-i(y-t)\xi} f(y)\, dy + \frac{1}{2}\int_{\mathbb{R}} e^{-i(y+t)\xi} f(y)\, dy \\ &= \int_{\mathbb{R}} e^{-iy\xi}\cos(\xi t) f(y)\, dy \\ &= \cos(\xi t)\hat{f}(\xi), \end{aligned}$$

where we used Euler's formula in the fourth line.

**Observation:** Let $G(x,t) = \frac{1}{2}\int_{-t}^{t} g(x+s)\,ds$. Then,

$$\hat{G}(\xi,t) = \frac{\sin(\xi t)}{\xi}\hat{g}(\xi)$$

**Proof:** We have $\hat{G}(\xi,t) = \frac{1}{2}\int_{\mathbb{R}} e^{-ix\xi}\int_{-t}^{t} g(x+s)\,ds\,dx$. Let $2h(t) := \int_{-t}^{t} g(x+s)\,ds$. Notice that,

$$h_t(x,t) = \frac{1}{2}[g(x+t) - g(x-t)(-1)] = \frac{1}{2}(g(x+t)+g(x-t))$$

By the previous observation

$$\int_0^t \hat{h}(\xi,t)\,dt = \int_0^t \cos(\xi t)\hat{g}(\xi)\,dt$$

$$\hat{h}(\xi,t) = \frac{1}{\xi}\sin(\xi t)\hat{g}(\xi)$$

Combining these observations, we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2}[u_0(x+t)+u_0(x-t)] + \frac{1}{2}\int_{-t}^{t} u_1(x+s)\,ds \\ &= \frac{1}{2}[u_0(x+t)+u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\tau)\,d\tau \quad (*) \end{aligned}$$

$(*)$ is the so-called D'Alembert's Formula.

**Exercise 6.2.** Show that the above satisfies (1)-(2)-(3).

**Definition 6.3.** $u$ is said to be a solution of (1)-(2)-(3) if the right hand side of $(*)$ makes sense as a function from $\mathbb{R}\times\mathbb{R}^+$ into $\mathbb{R}$.

**Remark 6.4.** A solution in the above sense does not need to satisfy (1) in a pointwise sense. In particular, $u_{xx}$ and $u_{tt}$ do not need to be well-defined in a pointwise sense. For example, consider

$$u_0(x) = \begin{cases} 1, x>0 \\ 0, x=0 \\ -1, x<0 \end{cases} \qquad u_1(x) = \sin x$$

## 7 Maximum Principle for the Heat Equation

**Notations:** $\Omega = (0,l)$, $\Omega_T = (0,T]$, $\partial_P\Omega_T = \{(x,t)\in\overline{\Omega_T} \mid x=0, x=l \text{ or } t=0\}$. $\partial_P\Omega_T$ is also called the parabolic boundary of $\Omega_T$.

**Assumptions:** $u\in C(\overline{\Omega_T}), u_t - u_{xx}\le 0$ on $\Omega_T$.

**Claim 7.1.** $\max\limits_{(x,t)\in\overline{\Omega_T}} u = \max\limits_{(x,t)\in\partial_P\Omega_T} u$

*Proof.* Suppose that $\exists(x_0,t_0) \in \Omega_T$ such that $u(x_0,t_0) > \max\limits_{\partial_P\Omega_T} u$. Let $\epsilon > 0$ be such that $u(x_0,t_0) > \max\limits_{\partial_P\Omega_T} u + \epsilon T$. Now, define $v$ as $v(x,t) := u(x,t) - \epsilon t$. Then, we have

$$v(x_0,t_0) = u(x_0,t_0) - \epsilon t_0 > u(x,t) + \epsilon(T - t_0) \geq u(x,t) - \epsilon t = v(x,t)$$

for all $(x,t) \in \partial_P\Omega_T$.

$$\implies v(x_0,t_0) > v(x,t) \quad \forall(x,t) \in \partial_P\Omega_T \quad (1)$$

Since $v \in C(\Omega_T), \exists(x_1,t_1)$ s.t. $v(x_1,t_1) = \max\limits_{(x,t)\in\overline{\Omega_T}} v(x,t)$

**Observation:** $(x_1,t_1) \notin \partial_P\Omega_T$ due to (1). We have $x_1 \in \Omega$. Then, $v_{xx}(x_1,t_1) = u_{xx}(x_1,t_1) \leq 0$ (second derivative test).

If $t_1 \in (0,T)$, we have $v_t(x,t_1) = 0$.

If $t_1 = T, v_t(x_1,T) = \lim\limits_{h\to 0^-} \frac{v(x_1,t_1+h)-v(x_1,t_1)}{h} \geq 0$. Therefore,

$$v_t - v_{xx}\Big|_{(x,t)\to(x_1,t_1)} = u_t - u_{xx} - \epsilon\Big|(x,t) \to (x_1,t_1) < 0$$

This is a contradiction. So, there is no $(x_0,t_0) \in \Omega_T$ s.t.

$$u(x_0,t_0) > \max_{\partial_P\Omega_T} u \implies \max_{\overline{\Omega_T}} u = \max_{\partial_P\Omega_T} u$$

□

**Remark 7.2.** The same result holds in $n-$dim with $\Omega$ being a bounded, open region in $\mathbb{R}^n$.

**Remark 7.3.** The result holds true in particular when $u_t - u_{xx} = 0,\ x \in \Omega,\ t \in (0,T)$.

**Remark 7.4.** Suppose $u, v \in C(\overline{\Omega_T}),\ u_t - u_{xx} \leq v_t - v_{xx}$ in $\Omega_T$ and $u \leq v$ on $\partial_P\Omega_T$. Then, set $w = u - v \in C(\overline{\Omega_T})$.

$$w_t - w_{xx} \leq 0 \text{ in } \Omega_T \text{ and } w \leq 0 \text{ on } \partial_P\Omega_T$$

Invoking max principle, $\max\limits_{\overline{\Omega_T}} w = \max\limits_{\partial_P\Omega_T} w \leq 0 \implies w \leq 0$ on $\overline{\Omega_T}$.

$$\implies u \leq v \text{ on } \overline{\Omega_T} \quad \text{(Comparison principle)}$$

### 7.1 Maximum Principle with $\Omega = \mathbb{R}$

**Claim 7.5.** Instead of max, we have sup ($u$ is continuous and bounded)

$$\sup_{\overline{\Omega_T}} u = \sup_{y\in\mathbb{R}} u(y,0) \quad (u(y,0) = u_0(y))$$

*Proof.* Let $\epsilon > 0, v(x,t) := \epsilon\left(t + \frac{x^2}{2}\right) + \sup\limits_{y\in\mathbb{R}} u_0(y)$. Notice that $v_t - v_{xx} = \epsilon - \epsilon = 0$. Since $v$ grows quadratically in $x, \exists R = R(\epsilon, \sup|u|) > 0$ s.t. $v \geq u$ on $[R,\infty) \cup (-\infty,-R]$. In particular, $v \geq u$ when $x = -R$ or $x = R$ and $t = 0$. Observe that $0 \leq u_t - u_{xx} \leq v_t - v_{xx} = 0 \ \forall t \in (0, T]$. We also had $u \leq v$ on $\partial_P \Omega_R^T = \{(x,t) | x = R \text{ or } x = -R \text{ or } t = 0\}$. Then, by comparison principle, $u \leq v$ on $\overline{\Omega_R^T}$. It follows now $u \leq v$ on $\overline{\Omega_R^T}$ and $u \leq v$ when $|x| > R$, implying $u \leq v$ for all $x \in \mathbb{R}, t \in [0,T]$. Hence, $u(x,t) \leq \epsilon \left( t + \frac{x^2}{2} \right) + \sup_{y \in \mathbb{R}} u_0(y)$. Letting $\epsilon \to 0^+$, we have $u(x,t) \leq \sup_{y \in \mathbb{R}} u_0(y), \ x \in \mathbb{R}, \ t \in [0,T]$. Passing to sup, we finally obtain

$$\sup_{(x,t) \in \overline{\Omega_T}} u \leq \sup_{x \in \mathbb{R}} u_0$$

□

**Corollary 7.6.** A continuous and bounded solution of the heat equation on the real line is unique.

*Proof.* Suppose $u_1, u_2$ are two solutions. Set $w = u_1 - u_2$, then $w$ solves

$$\begin{cases} w_t - w_{xx} = 0, \ x \in \mathbb{R}, \ t > 0 \\ w(x,0) = w_0(x) = 0 \end{cases}$$

Using maximum principle, we get $\sup_{\overline{\Omega_T}} w \leq \sup_{x \in \mathbb{R}} w_0 = 0 \implies w \leq 0$. We also have

$$\begin{cases} (-w)_t - (-w)_{xx} = 0, \ x \in \mathbb{R}, \ t > 0 \\ (-w)(x,0) = -0 = 0 \end{cases}$$

Again using max principle $\sup_{\overline{\Omega_T}}(-w) \leq \sup_{x \in \mathbb{R}}(-w(x,0)) = 0 \implies (-w) \leq 0 \implies w \geq 0$. Combining, we see that $w = 0$. So, $u_1 = u_2$. □

### 7.2 Non-Homogenous Heat Equation on $\mathbb{R}$

$$(**) \begin{cases} u_t - u_{xx} = f(x,t), \ x \in \mathbb{R}, \ t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

This system can be written as

$$\begin{cases} u_t + Lu = f \quad (1) \\ u(x,0) = u_0(x) \quad (2) \end{cases}$$

Recall that the homogenous solution can be written as

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t) u_0(y) \, dy =: S(t) u_0,$$

where $K(x-y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

**Notation:** $u = u(\cdot, t) =$ state of solution at time $t$.

Let $u = S(t)u_0$ denote the solution of (1)-(2) with $f \equiv 0$. Namely, $S(t)u_0$ solves

$$(*) \begin{cases} \frac{d}{dt}[S(t)u_0] + L(S(t)u_0) = 0 \\ S(t)u_0\big|_{t=0} = u_0 \end{cases}$$

Consider the problem

$$\begin{cases} u_t + Lu = f = f(x,t) \quad (3) \\ u(x,0) = 0 \quad (4) \end{cases}$$

**Claim 7.7.** The solution of (3)-(4) is given by

$$u(\cdot,t) := \int_0^t S(t-s)f(\cdot,s)\,ds.$$

$\Big($Note that $u(\cdot,0) = \int_0^0 \cdots = 0.\Big)$

*Proof.* We have $u_t = S(t-t)f(\cdot,t) + \int_0^t [\frac{d}{dt}(S(t-s)f(\cdot,s))]\,ds$ by Leibniz rule. $S(0)f(\cdot,t) = f(\cdot,t)$ since $u(\cdot,t) = S(t)u_0$. So, $u_0 = u(\cdot,0) = S(0)u_0 \implies S(0) = 1$. So, we can write

$$u_t = f(\cdot,t) + \int_0^t \frac{d}{dt}(S(t-s)f(\cdot,s))\,ds$$

Then,

$$\begin{aligned} u_t + Lu &= f(\cdot,t) + \int_0^t \frac{d}{dt}[S(t-s)f(\cdot,s)]\,ds + \int_0^t L(S(t-s)f(\cdot,s))\,ds \\ &= f(\cdot,s) + \int_0^t \left[\frac{d}{dt}(S(t-s)f(\cdot,s)) + L(S(t-s)f(\cdot,s))\right]\,ds \\ &= f(\cdot,s) \end{aligned}$$

since $\int_0^t \left[\frac{d}{dt}(S(t-s)f(\cdot,s)) + L(S(t-s)f(\cdot,s))\right]\,ds = 0$ by $(*)$. Hence, the solution is given by

$$u(\cdot,t) = S(t)u_0 + \int_0^t S(t-s)f(\cdot,s)\,ds. \tag{7.1}$$

□

The formula (7.1) is known as Duhamel's Principle. For $(**)$, we have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} K(x-y,t)u_0(y)\,dy + \int_0^t \left(\int_{-\infty}^{\infty} K(x-y,t-s)f(y,s)\,dy\right)\,ds \\ &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}u_0(y)\,dy + \int_0^t \left[\frac{1}{\sqrt{4\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}}f(y,s)\,dy\right]\,ds \end{aligned}$$

**In ODE's of the type** $y'(t) + ay(t) = f(t)$**:** Multiply by integrating factor

$$y'e^{at} + ae^{at}y = e^{at}f$$

$$(ye^{at})' = fe^{at} \implies ye^{at} - y(0) = \int_0^t e^{as}f(s)\,ds$$

$$y(t) = y_0e^{-at} + \int_0^t e^{-a(t-s)}f(s)\,ds$$

$S(t)y_0 = y_0 e^{-at} \implies y = S(t)y_0 + \int_0^t S(t-s)f(s)\,ds$. If $a$ were a matrix, then $y$ would be a vector and we would get

$$\vec{y}(t) = S(t)\vec{y}_0 + \int_0^t S(t-s)\vec{f}(s)\,ds,$$

where $S(t)y_0 = e^{-At}y_0,\ e^{-At} = \sum_{k=0}^{\infty} \frac{(-tA)^k}{k!}$.

# 8 Maximum Principle for Laplace's Equation

**Claim 8.1.** Suppose $\Omega$ is an open, bounded region in $\mathbb{R}^n$ and $u \in C(\overline{\Omega})$. Suppose $\Delta u \geq 0$ in $\Omega$ (such $u$ is called subharmonic). Then,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

*Proof.* Introduce $v := u + \epsilon|x|^2$. We claim that $v$ attains its maximum on $\partial\Omega$. Suppose to the contrary $\exists \alpha \in \Omega$ s.t. $v(\alpha) = \max_{\overline{\Omega}} v$. We know also $\partial_{x_i}^2 v(\alpha) \leq 0 \implies \Delta v(\alpha) \leq 0$ (second derivative test).

$$\Delta u(\alpha) = \Delta v - \Delta(\epsilon|x|^2)\big|_{x=\alpha} = \Delta v(\alpha) - 2n\epsilon < 0$$

This is a contradiction to $|x|^2 = x_1^2 + \cdots + x_n^2,\ \frac{\partial |x|^2}{\partial x_i} = 2x_i$. Now observe that

$$\max_{\overline{\Omega}} u \leq \max_{\Omega} v = \max_{\partial\Omega}(u + \epsilon|x|^2) \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} |x|^2$$

Since $\max_{\partial\Omega} |x|^2 < \infty$, letting $\epsilon \to 0^+$, we obtain $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$. □

**Corollary 8.2.**

$$\begin{cases} \Delta u = f, & x \in \Omega \\ u\big|_{\partial\Omega} = g \end{cases}$$

If a solution to the model above exists, it is unique.

*Proof.* Suppose $u_1, u_2$ are two solutions to the above model. Then, set $w := u_1 - u_2$, which solves

$$\begin{cases} \Delta w = 0 \\ w\big|_{\partial\Omega} = 0 \end{cases} \iff \begin{cases} \Delta(-w) = 0 \\ (-w)\big|_{\partial\Omega} = 0 \end{cases}$$

Applying the max principle to both models, we get $w \leq 0$ and $w \geq 0$ on $\overline{\Omega}$. Thus, $w \equiv 0$. So, $u_1 = u_2$. □

### 8.1 Minimization Property

**Claim 8.3.** Suppose we have the system

$$\begin{cases} \Delta u = 0 \text{ on } \Omega \\ u\big|_{\partial\Omega} = g \end{cases}$$

Also suppose that $v$ is another (sufficiently smooth) function on $\overline{\Omega}$ with $v\big|_{\partial\Omega} = g$. Then,

$$\int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx.$$

The statement of the claim is that among the class of sufficiently smooth functions with the boundary datum $u\big|_{\partial\Omega} = g$, the solution to Laplace's equation with the boundary condition $u\big|_{\partial\Omega} = g$ minimizes the function $v \mapsto \int_\Omega |\nabla v|^2 \, dx$.

*Proof.* First recall the Divergence theorem

$$\int_\Omega (\Delta\phi)\psi \, dx = \int_{\partial\Omega} \frac{\partial\phi}{\partial n}\psi \, dS - \int_\Omega \nabla\phi \cdot \nabla\psi \, dx$$

Then,

$$\begin{aligned}
\int_\Omega |\nabla v|^2 \, dx &= \int_\Omega |\nabla v - \nabla u + \nabla u|^2 \, dx \\
&= \int_\Omega |\nabla v - \nabla u|^2 \, dx + 2\int_\Omega (\nabla v - \nabla u) \cdot \nabla u \, dx + \int_\Omega |\nabla u|^2 \, dx \\
&= \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v - \nabla u|^2 \, dx \\
&\qquad - 2\int_\Omega (v - u)\Delta u \, dx + 2\int_{\partial\Omega} \frac{\partial u}{\partial n}(v - u) \, dS \\
&= \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v - \nabla u|^2 \, dx \\
&\geq \int_\Omega |\nabla u|^2 \, dx
\end{aligned}$$

In the third line, the last two terms are 0 because $\Delta u = 0$ on $\Omega$ and $v = u$ on $\partial\Omega$. □

### 8.2 Mean Value Property

**Notation:** $\alpha(n) =$ volume of the unit sphere in $n$-dim

**Notation:** $r^n\alpha(n) =$ volume of the sphere of radius $r$ in $n$-dim

**Notation:** $nr^{n-1}\alpha(n) =$ surface area of the sphere of radius $r$ in $n$-dim

**Claim 8.4.** Let $U \subset \mathbb{R}^n$ be open and $\Delta u = 0$ on $U$, where $u \in C^2(U)$. For all $B(x,r) \subseteq U$, one has

$$u(x) = \frac{\int_{B(x,r)} u\,dy}{\int_{B(x,r)} dy} = \frac{1}{r^n \alpha(n)} \int_{B(x,r)} u\,dy = \frac{\int_{\partial B(x,r)} u\,dS(y)}{\int_{\partial B(x,r)} dS(y)}$$
$$= \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} u\,dS(y).$$

*Proof.* Set

$$\varphi(r) = \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} u\,dS(y) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\,dS(z),$$

where we made the substitution $z = (y-x)/r$. Then,

$$\varphi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} [\nabla u(x+rz) \cdot z]\,dS(z) = \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} \nabla u(y) \cdot \left(\frac{y-x}{r}\right) dS(y),$$

where we used the fact

$$\frac{d}{dr} u(x+rz) = u_{x_1}(x+rz) \cdot z_1 + \cdots + u_{x_n}(x+rz) \cdot z_n = \nabla u(x+rz) \cdot z$$

Also set

$$\frac{\partial u}{\partial n} := \nabla u(y) \cdot \frac{y-x}{r}$$

Recall Green's theorem

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\,dS = \int_{\Omega} \Delta u\,dx.$$

Applying Green's theorem, we get

$$\varphi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial n}\,dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u\,dy = 0.$$

($\Delta u = 0$ by assumption) So, $\varphi$ is constant. Hence, we can write

$$\varphi(r) = \lim_{r'\to 0} \varphi(r') \quad \forall r > 0 \text{ with } B(x,r) \subseteq U$$

The proof follows from the following lemma:

**Lemma 8.5** (Averaging Lemma)**.**

$$\lim_{r\to 0^+} \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} u(y)\,dS(y) = u(x).$$

**Proof of the Averaging Lemma:** We have

$$u(x) = 1u(x) = u(x)\frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} dS(y)$$

Then,

$$\left| \int_{\partial B(x,r)} u(y)\,dS(y) - u(x) \right| \leq \int_{\partial B(x,r)} |u(y) - u(x)|\,dS(y)$$

$u$ is continuous on $U \implies \forall\epsilon > 0\ \exists r_0\ |x - y| < r_0,\ |u(y) - u(x)| < \epsilon.$

Thus, for all $\epsilon > 0$

$$\int_{\partial B(x,r)} |u(y) - u(x)|\, dS(y) < \epsilon \int_{\partial B(x,r)} dS(y) = \epsilon$$

as $r \to 0^+$. But $\epsilon$ is arbitrary, therefore

$$\lim_{r\to 0^+} \int_{\partial B(x,r)} |u(y) - u(x)|\, dS(y) = 0.$$

Using the shell method, we have

$$\int_{B(x,r)} u\, dy = \int_0^r \left[ \int_{\partial B(x,\tau)} u\, dS(y) \right] d\tau.$$

We also have

$$u(x) = \frac{1}{n\tau^{n-1}\alpha(n)} \int_{\partial B(x,\tau)} u\, dS(y)$$

$$\implies \int_{B(x,r)} u\, dy = \int_0^r u(x) n\alpha(n)\tau^{n-1}\, d\tau = u(x)\alpha(n)r^n$$

Finally, we obtain

$$u(x) = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u\, dy.$$

□

### 8.2.1 Converse of MVP

**Proposition 8.6.** Suppose that $u \in C^2(U)$ and $u(x) = \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} u\, dS$ for any $B(x,r) \subset U$. Then, $\Delta u = 0$ on $U$.

*Proof.* Define $\varphi(r)$ as before. Since $\varphi(r) = u(x)$ for all $r$ by assumption. So, $\varphi'(r) = 0$. Also, one has

$$\varphi'(r) = \frac{1}{nr^{n-1}\alpha(n)} \int_{B(x,r)} \Delta u\, dy$$

If $\Delta u(x) \neq 0$ for some $x \in U, \exists r > 0$ s.t. $\Delta u \neq 0$ on $\overline{B(x,r)}$ as $u \in C^2(U)$. Hence, $\varphi'(r) \neq 0$. This is a contradiction. □

## 8.3 Strong Maximum Principle

**Proposition 8.7.** Let $U \subseteq R^n$ be bounded, open. Suppose $u \in C^2(U) \cap C(\overline{U})$ and $\Delta u = 0$ on $U$. Then,

$$\begin{cases} \max\limits_{\overline{U}} u = \max\limits_{\partial U} u \quad \text{(proven earlier)} \\ \text{If } U \text{ is connected, } u(x_0) = \max\limits_{\overline{U}} u,\ x_0 \in U, \text{ then } u \equiv \text{constant} \end{cases}$$

*Proof.* If $\exists x_0 \in U$ s.t. $u(x_0) = \max_{\overline{U}} u =: M$, then, by MVP

$$M = u(x_0) = \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u\,dy \le M \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} dy = M$$

$$\implies u \equiv M \text{ on } B(x_0,r) \text{ otherwise } \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u\,dy < M$$

Continuing in the same fashion, using the boundedness and connectedness of $U$, we can fill in the whole region with open balls and iterate the above idea. Hence, $u \equiv M$ everywhere. □

**Corollary 8.8.** $U$ is connected, $u \in C^2(U) \cap C(\overline{U})$

$$\begin{cases} \Delta u = 0, & x \in U \\ u = g, & x \in \partial U, \quad g \ge 0 \end{cases}$$

If $g(x_0) > 0$ for some $x_0 \in \partial U$, then $u > 0$ everywhere on $U$.

*Proof.* By the strong max/min principle, if $u \le 0$ for some $x \in U$, then $u$ must be constant. Since $g(x_0) > 0$, $u > 0$ everywhere on $U$. □

# 9 Laplace's Equation on the Whole Domain

We attempt to solve $\Delta u(x) = 0$, where $x \in \mathbb{R}^n$.

**Strategy:** We look for solutions in the form $V(r) = u(x)$, where $r = |x| = (x_1 + \cdots + x_n)^{\frac{1}{2}}$ (i.e. rotationally invariant solutions). By the chain rule, we have

$$u_{x_i} = V'(r) r_{x_i} = V'(r)\frac{\partial r}{\partial x_i} = V'(r)\frac{\partial r}{\partial x_i} = V'(r)\frac{x_i}{(x_1 + \cdots + x_n)^{\frac{1}{2}}} = V'(r)\frac{x_i}{|x|}.$$

Then,

$$\begin{aligned} u_{x_i x_i} &= V''(r) r_{x_i} \frac{x_i}{|x|} + V'(r)\frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|}\right) \\ &= V''(r)\frac{x_i^2}{r^2} + V'(r)\frac{r - r_{x_i} x_i}{r^2} \\ &= V''(r)\frac{x_i^2}{r^2} + V'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \end{aligned}$$

Consequently,

$$\begin{aligned} \Delta u = \sum_{i=1}^n u_{x_i x_i} &= \sum_{i=1}^n \left[ V''(r)\frac{x_i^2}{r^2} + V'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right] \\ &= \frac{V''(r)}{r^2}\sum_{i=1}^n x_i^2 + \sum_{i=1}^n \frac{V'(r)}{r} - \frac{V'(r)}{r^3}\sum_{i=1}^n x_i^2 \\ &= V''(r) + \frac{nV'(r)}{r} - \frac{V'(r)}{r} \\ &= V''(r) + \frac{n-1}{r}V'(r) \\ &= 0. \end{aligned}$$

Thus, one must have

$$\begin{aligned} \frac{V''(r)}{V'(r)} &= \frac{1-n}{r} \\ \implies \ln\left(V'(r)\right) &= (1-n)\ln r + C \\ V'(r) &= Cr^{1-n} \end{aligned}$$

$n = 2$ : Then, $V'(r) = Cr^{-1} \implies V(r) = C\ln r + D$

$n > 2$ : Then, $V'(r) = Cr^{1-n} \implies V(r) = \frac{c}{2-n}r^{2-n} + D$

We take $D = 0$, and

$$\begin{cases} n = 2, \text{ take } C = -\frac{1}{2\pi} \\ n > 2, \text{ take } C = -\frac{1}{n\alpha(n)} \end{cases}$$

It follows that

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

solves Laplace's equation on $\mathbb{R}^n \setminus \{0\}$.

**Definition 9.1.** $\Phi$ is called the fundamental solution of Laplace's equation.

**Definition 9.2.** Given $Lu = 0$, we say that $\Phi$ is a fundamental solution of $Lu = 0$ if $L\Phi = \delta$ (Dirac Delta)

**Basic Properties of $\Phi$**

**1)** $|\nabla\Phi(x)| \leq \frac{C}{|x|^{n-1}}$, where $\nabla\Phi = (\Phi_{x_1}, \Phi_{x_2}, \ldots, \Phi_{x_n}) =: D\Phi$ and $|\nabla\Phi| = (\Phi_{x_1}^2 + \cdots + \Phi_{x_n}^2)^{\frac{1}{2}}$

**2)**

$$D^2\Phi = \begin{bmatrix} \Phi_{x_1x_1} & \Phi_{x_1x_2} & \ldots & \Phi_{x_1x_n} \\ \Phi_{x_2x_1} & \Phi_{x_2x_2} & & \vdots \\ \vdots & & \ddots & \Phi_{x_{n-1}x_n} \\ \Phi_{x_nx_1} & \ldots & \Phi_{x_nx_{n-1}} & \Phi_{x_nx_n} \end{bmatrix}_{n\times n}$$

$|D^2\Phi| \leq \frac{C}{|x|^n}$, where $|D^2\Phi| = \left(\sum_{i,j=1}^{n}\left(\Phi_{x_ix_j}\right)^2\right)^{\frac{1}{2}}$.

**Exercise 9.3.** Prove the properties of $\Phi$.

Let $-\Delta u = f$ and suppose $f \in C_c^2(\mathbb{R}^n)$. Define $u$ as

$$u(x) = \int_{\mathbb{R}^n} \phi(x-y)f(y)\,dy = \int_{\mathbb{R}^n} \phi(\tilde{y})f(x-\tilde{y})\,d\tilde{y} = \int_{\mathbb{R}^n} \phi(y)f(x-y)\,dy$$

Then, $u_{x_i}(x) = \lim\limits_{h\to 0} \frac{u(x+he_i)-u(x)}{h}$, where $e_i$ is the $i-$th unit vector. Hence,

$$\begin{aligned} u_{x_i}(x) &= \lim_{h\to 0} \int_{\mathbb{R}^n} \phi(y) \left[\frac{f(x-y+he_i)-f(x-y)}{h}\right] dy \\ &= \int_{\mathbb{R}^n} \lim_{h\to 0} \left(\phi(y)\left[\frac{f(x-y+he_i)-f(x-y)}{h}\right]\right) dy \\ &= \int_{\mathbb{R}^n} \phi(y) \frac{\partial f(x-y)}{\partial x_i}\, dy \end{aligned}$$

Similarly, $u_{x_i x_i}(x) = \int_{\mathbb{R}^n} \phi(y) \frac{\partial^2 f(x-y)}{\partial x_i^2}\, dy$. Then,

$$\begin{aligned} \Delta u(x) &= \int_{\mathbb{R}^n} \phi(y)\Delta_x f(x-y)\, dy \\ &= \int_{B_\epsilon(0)} \phi(y)\Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \phi(y)\Delta_x f(x-y)\, dy \\ &= I_\epsilon + J_\epsilon \end{aligned}$$

We have

$$|I_\epsilon| \leq \int_{B_\epsilon(0)} |\phi(y)\Delta_x f(x-y)|\, dy \leq \max_{\mathbb{R}^n} |\Delta_x f| \int_{B_\epsilon(0)} |\phi(y)|\, dy$$

$\max\limits_{\mathbb{R}^n} |\Delta_x f| < \infty$ since $f \in C_c^2(\mathbb{R}^n)$. Also, since $\epsilon$ is small, $|\phi| = \phi$ and we get

$$|I_\epsilon| \leq \max_{\mathbb{R}^n} |\Delta_x f| \int_{B_\epsilon(0)} |\phi(y)|\, dy \leq \max_{\mathbb{R}^n} |\Delta_x f| \int_{B_\epsilon(0)} \phi(y)\, dy$$

We have two cases:

**1)** $n = 2$ :

$$\begin{aligned} \int_{B_\epsilon(0)} \phi(y)\, dy = -\frac{1}{2\pi}\int_0^\epsilon \left[\int_{\partial B_r(0)} \ln r\, dS\right] dr &= -\frac{1}{2\pi}\int_0^\epsilon \ln r \left[\int_{\partial B_r(0)} dS\right] dr \\ &= -\int_0^\epsilon r\ln r\, dr \\ &= -\frac{\epsilon^2 \ln\epsilon}{2} + \frac{\epsilon^4}{4} \to 0 \text{ as } \epsilon \to 0^+ \end{aligned}$$

**2)** $n \geq 3$ :

$$\begin{aligned} \int_{B_\epsilon(0)} \phi(y)\, dy = \frac{1}{n(n-2)\alpha(n)} \int_{B_\epsilon(0)} \frac{1}{|y|^{n-2}}\, dy &= \frac{1}{n(n-2)\alpha(n)} \int_0^\epsilon \int_{\partial B_r(0)} \frac{1}{r^{n-2}}\, dS\, dr \\ &= \frac{1}{n(n-2)\alpha(n)} \int_0^\epsilon r^{2-n} \int_{\partial B_r(0)} dS\, dr \\ &= \frac{1}{n(n-2)\alpha(n)} \int_0^\epsilon n\alpha(n) r^{n-1}\, dr \\ &= \frac{1}{n-2}\int_0^\epsilon r\, dr \\ &= \frac{\epsilon^2}{2(n-2)} \to 0 \text{ as } \epsilon \to 0^+ \end{aligned}$$

For convenience, set $\mathbb{R}^n \setminus B_\epsilon(0) =: \Omega_\epsilon$. Then, applying the Divergence theorem, we get

$$\begin{aligned} J_\epsilon &= -\int_{\Omega_\epsilon} \nabla\phi \cdot \nabla f(x-y)\, dy + \int_{\partial B_\epsilon(0)} \phi(y) \frac{\partial f(x-y)}{\partial n}\, d\Gamma \\ &= K_\epsilon + L_\epsilon \end{aligned}$$

We have

$$\begin{aligned} |L_\epsilon| \leq \max_{\partial B_\epsilon(0)} |\nabla f| \int_{\partial B_\epsilon(0)} |\phi(y)|\, d\Gamma \leq C \int_{\partial B_\epsilon(0)} \ln \epsilon\, d\Gamma = C \ln(\epsilon) 2\pi\epsilon \quad (\text{for } n=2) \\ \leq \frac{C}{n(n-2)\alpha(n)} \int_{\partial B_\epsilon(0)} \frac{1}{\epsilon^{n-2}}\, d\Gamma = \frac{c\epsilon}{n-2} \quad (\text{for } n \geq 3) \end{aligned}$$

Applying the divergence theorem on $K_\epsilon$ gives

$$K_\epsilon = \int_{\Omega_\epsilon} \Delta\phi(y) f(x-y)\, dy - \int_{\partial B_\epsilon(0)} \frac{\partial \phi}{\partial n} f(x-y)\, d\Gamma = -\int_{\partial B_\epsilon(0)} \frac{\partial \phi}{\partial n} f(x-y)\, d\Gamma$$

since $\Delta\phi = 0$ on $\Omega_\epsilon$. Moreover, on $\partial B_\epsilon(0)$, we have

$$\nabla\phi \cdot n = \nabla\phi \cdot \left(-\frac{y}{|y|}\right) = \frac{|y|^2}{n\alpha(n)|y|^{n+1}} = \frac{1}{n\alpha(n)\epsilon^{n-1}}$$

(the minus sign in $n$ stems from the fact that our region is $\mathbb{R}^n \setminus B_\epsilon(0)$. In other words, the unit outward normal vector points "inwards") Hence, we obtain

$$\begin{aligned} K_\epsilon &= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B_\epsilon(0)} f(x-y)\, dS \\ &= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B_\epsilon(x)} f(y)\, dS \to -f(x) \text{ as } \epsilon \to 0^+ \end{aligned}$$

Therefore, $-\Delta u = f$, where $u = \int_{\mathbb{R}^n} \phi(x-y) f(y)\, dy$.

**Theorem 9.4.** Let $u \in C(\Omega)$ and suppose it satisfies MVP $\forall B_r(x) \subset \Omega$. Then, $u \in C^\infty(\Omega)$.

*Proof.* Set

$$\eta(x) = \begin{cases} C^{-\frac{1}{1-|x|^2}},\ |x| < 1 \\ 0,\ |x| \geq 1 \end{cases}$$

($\eta$ is called the standard modifier) Also set $\eta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right)$. Then, $\eta \in C^\infty(\mathbb{R})$ and $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$. We pick $C$ such that $\int_{\mathbb{R}^n} \eta\, dx = 1$, which implies $\int_{\mathbb{R}^n} \eta_\epsilon\, dx = 1$.
Modification: Set $u_\epsilon := \eta_\epsilon * u,\ x \in \Omega_\epsilon := \{x \in \Omega |\ \inf_{y \in \partial\Omega} |x-y| =: \text{dist}(x, \partial\Omega) \geq \epsilon\}$

$$\implies u_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y) u(y)\, dy,\ x \in \Omega_\epsilon$$

$$\implies \frac{\partial u_\epsilon}{\partial x_i} = \int_\Omega \left[\frac{\partial}{\partial x_i} \eta_\epsilon(x-y)\right] u(y)\, dy, \quad 1 \leq i \leq n$$

By iteration, any higher-order derivative exists and is continuous. Hence, $u_\epsilon \in C^\infty(\Omega_\epsilon)$.

**Claim:** $u = u_\epsilon$ on $\Omega_\epsilon$.

**Proof of Claim:** Fix $x \in \Omega_\epsilon$. Then,

$$\begin{aligned}
u_\epsilon(x) &= \int_\Omega \eta_\epsilon(x-y)u(y)\,dy \\
&= \int_{B_\epsilon(x)} \eta_\epsilon(x-y)u(y)\,dy \\
&= \frac{1}{\epsilon^n}\int_{B_\epsilon(x)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)\,dy \quad \text{(Abuse of notation: } \eta(x)\equiv\eta(|x|)) \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \left[\int_{\partial B_r(x)} \eta\left(\frac{r}{\epsilon}\right)u(y)\,dS\right]dr \quad (|x-y| =: r) \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left[\int_{\partial B_r(x)} u(y)\,dS\right]dr
\end{aligned}$$

Since $u$ satisfies MVP, $\int_{\partial B_r(x)} u(y)\,dS = u(x)\int_{\partial B_r(x)} dS = u(x)\alpha(n)r^{n-1}n$. Then, we get

$$\begin{aligned}
u_\epsilon(x) &= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)n\alpha(n)r^{n-1}u(x)\,dr \\
&= n\alpha(n)\left[\int_0^\epsilon \eta_\epsilon(r)r^{n-1}\,dr\right]u(x) \quad \left(\frac{1}{\epsilon^n}\eta\left(\frac{r}{\epsilon}\right) = \eta_\epsilon(r)\right) \\
&= u(x)\int_0^\epsilon \eta_\epsilon(r)\left(\int_{\partial B_r(x)} dS\right)dr \\
&= u(x)\int_0^\epsilon \eta_\epsilon(r)\int_{\partial B_r(0)} dS\,dr \quad \text{(We can shift the center to 0)} \\
&= u(x)\int_0^\epsilon \int_{\partial B_r(0)} \eta_\epsilon(r)\,dS\,dr \\
&= u(x)\int_{B_\epsilon(0)} \eta_\epsilon(y)\,dy \\
&= u(x)
\end{aligned}$$

Since $\forall x \in \Omega,\ \exists \epsilon = \epsilon(x) > 0$ s.t. $x \in \Omega_\epsilon$, it follows that $u_\epsilon(x) = u(x)\ \forall x \in \Omega$ and $\epsilon = \epsilon(x)$ as in above statement. □

**Theorem 9.5** (Liouville's Theorem)**.** Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be harmonic and bounded, then $u$ is constant.

*Proof.* By MVP, $\forall B_r(x_0) \subseteq \mathbb{R}^n, u(x_0) = \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} u(x)\,dx$. We have $\Delta u = 0$ on $\mathbb{R}^n$ as $u$ is harmonic. Since $u \in C^\infty$ (by previous theorem), $u_{x_i}$ exist on $\mathbb{R}^n, 1 \le x \le n$. In particular, $\Delta u_{x_i} = \frac{\partial}{\partial x_i}\Delta u = 0$. This means that $u_{x_i}$ is also harmonic. So, $u_{x_i}$ obeys MVP. Thus, $u_{x_i}(x_0) = \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} u_{x_i}(y)\,dy\ \ \forall B_r(x_0) \subseteq \mathbb{R}^n$

$$\implies u_{x_i}(x_0) = \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} \frac{\partial u}{\partial x_i}(y)\,dy$$

Observe that defining $\vec{F} = (0, 0, \dots, u, \dots, 0)$, we have div$\vec{F} = \nabla\cdot\vec{F} = u_{x_i} = \frac{\partial u}{\partial x_i}$. Then, divergence theorem gives $\int \nabla\cdot\vec{F}\,dV = \int \vec{F}\cdot n\,dS = \int u n_i\,dS$, where $n_i$ is the $i-$th component of the unit normal vector. So, we have

$$\implies u_{x_i}(x_0) = \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} \frac{\partial u}{\partial x_i}(y)\,dy = \frac{1}{\alpha(n)r^n}\int_{\partial B_r(x_0)} u\,n_i\,dS$$

This implies

$$\begin{aligned}
|u_{x_i}(x_0)| &\leq \frac{1}{\alpha(n)r^n}\int_{\partial B_r(x_0)} |u||n_i|\,dS \\
&\leq \frac{1}{\alpha(n)r^n}\int_{\partial B_r(x_0)} |u|\,dS \quad (|n_i| \leq 1 \text{ since it is a component of the unit vector}) \\
&\leq \left(\max_{\partial B_r(x_0)} |u(y)|\right)\frac{1}{\alpha(n)r^n}\int_{\partial B_r(x_0)} dS \\
&= \left(\max_{\partial B_r(x_0)} |u(y)|\right)\frac{1}{\alpha(n)r^n}\alpha(n)nr^{n-1} \\
&\leq \frac{C}{r}, \quad \left(C \text{ is a constant depending on } \max_{\partial B_r(x_0)} |u| \text{ and } n\right)
\end{aligned}$$

$$\implies \forall r > 0,\;\; x_0 \in \mathbb{R},\;\; |u_{x_i}(x_0)| \leq \frac{C}{r} \leq \left(\sup_{\mathbb{R}^n} |u|\right)\frac{n}{r}$$

Remember that $r$ was arbitrary. So, letting $r \to \infty$, we obtain $u_{x_i}(x_0) = 0 \forall x_0 \in \mathbb{R}, 1 \leq i \leq n$. This implies that $u$ is constant since all of its partial derivatives vanish. □

### 9.1 Poisson's Equation

Suppose we have $-\Delta u = f \in C_c^2(\mathbb{R}^n)$, where

$$C_c^2(\mathbb{R}^n) = \{g \in C^2(\mathbb{R}^n) |\, \exists K \text{ (closed, bounded) and } g(x) = 0 \;\; \forall x \in K^c\}$$

(In other words, $C_c^2(\mathbb{R}^n)$ is the set of compactly supported, twice continuously differentiable functions on $\mathbb{R}^n$) We showed that a solution is given by

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy.$$

Observe that $\tilde{u} = u + c$ satisfies

$$-\Delta\tilde{u} = -\Delta u - \Delta c = f,$$

where $c$ is a constant. So, uniqueness fails.

#### 9.1.1 Bounded Solutions of Poisson's Equation

We have seen that uniqueness fails. Nevertheless, we can still seek a modified form of uniqueness. For example, consider the following question.

**Q:** Is there another solution to Poisson's equation that is bounded?

**A:** Suppose $\exists \tilde{u} : \mathbb{R}^n \to \mathbb{R}$, bounded such that $-\Delta \tilde{u} = f$. Set $w = u - \tilde{u}$. Then, $w$ satisfies

$$\Delta w = \Delta u - \Delta \tilde{u} = 0 \implies w \text{ is harmonic}$$
$$\sup |w| \le \sup |u| + \sup |\tilde{u}| < \infty$$

Thus, $w$ is bounded. Then, by Liouville's theorem, $w$ is constant. Hence,

$$\tilde{u} = u - w = u - c.$$

This means that bounded solutions of Poisson's equation are unique up to an additive constant.

### 9.1.2 A Representation Formula

Let $\Omega$ be an open and bounded region in $\mathbb{R}^n$, and suppose $\partial\Omega$ is smooth. Consider

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$$

Our goal is to develop a representation formula assuming there exists a sufficiently smooth solution. Given $x \in \Omega$, consider the integral $I = \int_\Omega (u\Delta v - v\Delta u)\,dx$. By divergence theorem, we have

$$\int_\Omega \Delta u\, v\, dy = -\int_\Omega \nabla u \nabla v\, dy + \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, dS$$
$$\int_\Omega u \Delta v\, dy = -\int_\Omega \nabla u \nabla v\, dy + \int_{\partial\Omega} \frac{\partial v}{\partial n} u\, dS$$
$$\implies I = \int_\Omega (\Delta u\, v - \Delta v\, u)\, dy = \int_{\partial\Omega} \left( \frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) dS$$

We will choose $v(y) = \Phi(y - x)$. Let $V_\epsilon = \Omega \setminus B_\epsilon(x)$. Then, $\partial V_\epsilon = \partial\Omega \cup B_\epsilon(x)$ and we have

$$I_\epsilon := \int_{V_\epsilon} (u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u)\, dy = -\int_{V_\epsilon} \Phi(y-x)\Delta u\, dy$$

as $\Delta\Phi = 0$. Applying the Divergence theorem yields

$$I_\epsilon = \int_{\partial V_\epsilon} \left( u(y) \frac{\partial \Phi(y-x)}{\partial n} - \Phi(y-x) \frac{\partial u}{\partial n} \right) dS$$

Notice that

$$\begin{aligned} \left| \int_{\partial B_\epsilon(x)} \Phi(y-x) \frac{\partial u}{\partial n}\, dS \right| &\le \int_{\partial B_\epsilon(x)} |\Phi(y-x)| \left| \frac{\partial u}{\partial n} \right| dS \\ &\le \frac{C}{\epsilon^{n-2}} \epsilon^{n-1} \\ &= c\,\epsilon \to 0 \text{ as } \epsilon \to 0^+ \end{aligned}$$

**Exercise 9.6.**

$$\frac{\partial \Phi(y-x)}{\partial n} = \nabla\Phi \cdot n = \frac{1}{n\alpha(n)\epsilon^{n-1}}$$

Using the above exercise,

$$\int_{\partial B_\epsilon(x)} u(y)\frac{\partial\Phi(y-x)}{\partial n}\,dS = \frac{1}{n\alpha(n)\epsilon^{n-1}}\int_{\partial B_\epsilon(x)} u(y)\,dS \to u(x) \text{ as } \epsilon\to 0^+$$

This implies that

$$\begin{aligned}-\int_{V_\epsilon}\Phi(y-x)\Delta u(y)\,dy \xrightarrow{\epsilon\to 0^+} -\int_\Omega \Phi(y-x)\Delta u(y)\,dy = u(x) + \int_{\partial\Omega} u(y)\frac{\partial\Phi}{\partial n}\,dS \\ -\int \Phi(y-x)\frac{\partial u}{\partial n}\,dS\end{aligned}$$

We have $u = g$ on $\partial\Omega$. Finally, one obtains

$$u(x) = \int_\Omega \Phi(y-x)f(y)\,dy - \int_{\partial\Omega}\frac{\partial\Phi}{\partial n}g(y)\,dS + \int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial n}\,dS$$

Observe that the only unknown in the above is $\frac{\partial u}{\partial n}$. Let $x\in\Omega$. Suppose $\varphi^x = \varphi^x(y)$ is a (corrector) function satisfying

$$\begin{cases}\Delta\varphi^x = 0 \text{ in } \Omega \\ \varphi^x\big|_{\partial\Omega} = \Phi(y-x)\end{cases}$$

Now, we set

$$G(x,y) \coloneqq \Phi(y-x) - \varphi^x(y) \quad \text{(Green's function)}$$

Then, $\Phi(y-x) = G(x,y) + \varphi^x(y)$ and we get

$$\begin{aligned}u(x) = \int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial n}\,dS - \int_{\partial\Omega}u(y)\frac{\partial G}{\partial n}\,dS - \int_{\partial\Omega}u(y)\frac{\partial\varphi^x}{\partial n}\,dS \\ -\int_\Omega G(x,y)\Delta u(y)\,dy - \int_\Omega \varphi^x(y)\Delta u(y)\,dy\end{aligned}$$

We have

$$\begin{aligned}-\int_\Omega \varphi^x(y)\Delta u(y)\,dy &= \int_{\partial\Omega}\left(u(y)\frac{\partial\varphi^x}{\partial n} - \varphi^x(y)\frac{\partial u}{\partial n}\right)dS \\ &= \int_{\partial\Omega}\left(u(y)\frac{\partial\varphi^x}{\partial n} - \Phi(y-x)\frac{\partial u}{\partial n}\right)dS\end{aligned}$$

Then, $u(x)$ becomes

$$\begin{aligned}u(x) &= -\int_{\partial\Omega}u(y)\frac{\partial G}{\partial n}\,dS - \int_\Omega G(x,y)\Delta u(y)\,dy \\ &= -\int_{\partial\Omega}g(y)\frac{\partial G}{\partial n}\,dS + \int_\Omega G(x,y)f(y)\,dy\end{aligned}$$

**Example 9.7.** Let $\Omega = \mathbb{R}^n_+ \coloneqq \{(x_1,\dots,x_n)\in\mathbb{R}^n | x_n > 0\}$. We will construct $G = G(x,y)$ associated with $\mathbb{R}^n_+$. We need a function $\varphi^x(y)$ with

$$\begin{cases}-\Delta\varphi^x = 0 \text{ in } \mathbb{R}^n_+ \\ \varphi^x\big|_{\partial\mathbb{R}^n_+} = \Phi(y-x)\end{cases}$$

Given $x \in \mathbb{R}^n_+$, define $\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n)$. Observe that $|y-x| = |y-\tilde{x}|$, where $y = (y_1, \ldots, y_{n-1}, 0)$. So, $\Phi(|y-x|) = \Phi(|y-\tilde{x}|)$. Hence, we define $\varphi^x(y) = \Phi(y-x)$ (remember the abuse of notation). Thus, $G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$ is the Green's function associated with $\mathbb{R}^n_+$.

In $n-$dimensions, the unit outward normal vector for $\mathbb{R}^n_+$ is $(0,0,\ldots,-1)$. Then,

$$\begin{aligned} \frac{\partial G}{\partial n} = \nabla G \cdot n = \frac{\partial G}{\partial y_n}(-1) = -\frac{\partial G}{\partial y_n} &= -\left[\frac{\partial \Phi(y-x)}{\partial y_n} - \frac{\partial \Phi(y-\tilde{x})}{\partial y_n}\right] \\ &= -\left(-\frac{1}{n\alpha(n)}\right)\left(\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}\right) \end{aligned}$$

Hence,

$$\begin{cases} -\Delta u = 0 \text{ in } \mathbb{R}^n_+ \\ u\big|_{\partial \mathbb{R}^n_+} = g \end{cases} \implies u(x) = -\frac{2}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)x_n}{|y-x|^n}\,dy = \int_{\partial \mathbb{R}^n_+} K(x,y)g(y)\,dy$$

The function

$$K(x,y) = -\frac{2x_n}{n\alpha(n)|y-x|^n}$$

is called the Poisson kernel for $\mathbb{R}^n_+$.

$$\Delta u(x) = \int_{\partial \mathbb{R}^n_+} [\Delta_x K(x,y)]g(y)\,dy = 0 \quad \text{when } x \in \mathbb{R}^n_+$$

**Claim 9.8.**

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x_0) \;\; \forall x_0 \in \partial \mathbb{R}^n_+$$

*Proof.* Given $\epsilon > 0$, find $\delta > 0$ such that $|g(y) - g(x_0)| < \epsilon$ if $|y - x_0| < \delta$.

**Exercise 9.9.**

$$\int_{\partial \mathbb{R}^n_+} K(x,y)\,dy = 1.$$

Using the above exercise, we have

$$\begin{aligned} |u(x) - g(x_0)| &= \left| \int_{\partial \mathbb{R}^n_+} K(x,y)(g(y) - g(x_0))\,dy \right| \\ &\le \int_{\partial \mathbb{R}^n_+} K(x,y)|g(y) - g(x_0)|\,dy \\ &= \int_{B_\delta(x_0)} K(x,y)|g(y) - g(x_0)|\,dy + \int_{\partial \mathbb{R}^n_+ \setminus B_\delta(x_0)} K(x,y)|g(y) - g(x_0)|\,dy \\ &< \epsilon \int_{B_\delta(x_0)} K(x,y)\,dy + \int_{\partial \mathbb{R}^n_+ \setminus B_\delta(x_0)} K(x,y)|g(y) - g(x_0)|\,dy \\ &< \epsilon + \int_{\partial \mathbb{R}^n_+ \setminus B_\delta(x_0)} K(x,y)|g(y) - g(x_0)|\,dy, \end{aligned}$$

where $\delta$ is the same as the one in the continuity of $g$. Now notice the following equivalence

$$\text{Claim} \iff \forall \epsilon > 0 \; \exists \eta > 0 \; \text{ s.t. if } \; 0 < |x - x_0| < \eta, \; \text{then} \; |u(x) - u(x_0)| < 2\epsilon$$

Let $\eta = \delta/2$. Then, if $0 < |x - x_0| < \eta = \frac{\delta}{2}$, then for $y \in \partial\mathbb{R}^n_+ \setminus B_\delta(x_0)$ (i.e. $|y - x_0| \geq \delta$), we have

$$\delta \leq |y - x_0| \leq |y - x + x - x_0| \leq |y - x| + \underbrace{|x - x_0|}_{< \frac{\delta}{2}} \underset{\text{trick}}{\leq} |y - x| + \frac{1}{2}|y - x_0|$$

$$\frac{1}{|y - x|} \leq \frac{2}{|y - x_0|}$$

Then,

$$K(x, y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|y - x|^n} \leq \frac{2x_n}{n\alpha(n)} \frac{2^n}{|y - x_0|^n}.$$

Set

$$II = \int_{\substack{|y| \geq \delta \\ y \in \partial\mathbb{R}^n_+}} K(x, y)|g(y) - g(x_0)| \, dy.$$

Observe that

$$|g(y) - g(x_0)| \leq 2(|g(y)| + |g(x_0)|) \leq 2 \sup_{y \in \partial\mathbb{R}^n_+} |g| = M_g,$$

where we assume that $g$ is bounded. Then,

$$\begin{aligned} II = \int_{\substack{|y| \geq \delta \\ y \in \partial\mathbb{R}^n_+}} K(x, y)|g(y) - g(x_0)| \, dy &\leq \frac{2^n x_n M_g}{n\alpha(n)} \int_{\substack{|y| \geq \delta \\ y \in \partial\mathbb{R}^n_+ \sim \mathbb{R}^{n-1}}} \frac{1}{|y - x_0|^n} \, dy \\ &\leq \frac{C 2^n x_n M_g}{n\alpha(n)} \to 0 \end{aligned}$$

since $x_n \to 0$ as $x \to x_0$.
(Aside)

$$\begin{aligned} \int_{\substack{|y| \geq \delta \\ y \in \partial\mathbb{R}^n_+ \sim \mathbb{R}^{n-1} \, (y = x_0 + \delta z)}} \frac{1}{|y - x_0|^n} \, dy &= \delta^{n-1} \int_{|z| \geq 1} \frac{1}{|z|^n} \, dz \\ &= \delta^{n-1} \int_1^\infty \int_{\partial B_r(0)} r^{-n} \, dS \, dr \\ &= \delta^{n-1} \int_1^\infty r^{-n}(n-1)\alpha(n-1)r^{n-2} \, dr \\ &= C \int_1^\infty r^{-2} \, dr \\ &= C \\ &= \delta^{n-1}(n-1)\alpha(n-1) > 0 \end{aligned}$$

Hence, $|u(x) - u(x_0)|$ can be made smaller than $\epsilon$ by taking $x$ close enough to $x_0$, proving the claim. $\square$

**Example 9.10.** Let $\Omega = B_1(0) \subseteq \mathbb{R}^2$

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} = g \text{ (nice function)} \end{cases} \qquad \begin{cases} \Delta \varphi^x = 0 \text{ in } \Omega \\ \varphi^x(y) = \Phi(y-x) \text{ on } \partial\Omega \end{cases}$$

Observe that

$$\begin{aligned} |y-x|^2 &= |y|^2 - 2x\cdot y + |x|^2 \\ &= 1 - 2x\cdot y + |x|^2|y|^2 \text{ since } |y|^2 = 1 \text{ as } y \in \partial\Omega \\ &= |x|^2\left(|y|^2 - \frac{2x\cdot y}{|x|^2} + \frac{1}{|x|^2}\right) \\ &= |x|^2\left(|y|^2 - 2y\cdot\frac{x}{|x|^2} + \frac{|x|^2}{|x|^4}\right) \end{aligned}$$

Set $x^* = \frac{x}{|x|^2}$. Then, $|y-x|^2 = |x|^2|y-x^*|^2$

$$\implies G(x,y) = \Phi(y-x) - \varphi^x(y) = \Phi(|y-x|) - \Phi(|x||y-x^*|)$$

where we keep the abuse of notation in mind as usual.

## 10 Wave Equation Revisited

Recall that the solution of the system

$$\begin{cases} u_{tt} - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0 \text{ (can take } t \in \mathbb{R} \text{ as well)} \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

is given by d'Alembert's formula

$$u(x,t) = \underbrace{\frac{1}{2}g(x+t) + \frac{1}{2}\int_0^{x+t} h(y)\,dy}_{F(x+t)} + \underbrace{\frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^0 h(y)\,dy}_{G(x-t)}. \tag{10.1}$$

Conversely, if $F, G$ are smooth

$$u_{tt} - u_{xx} = \underbrace{F''(x+t) + G''(x-t)}_{u_{tt}} - (\underbrace{F''(x+t) + G''(x-t)}_{u_{xx}}) = 0$$

The RHS of (10.1) is a solution of the wave equation in the pointwise sense on $\mathbb{R}$. Moreover, $u(x,0) = F(x) + G(x)$ and

$$u_t(x,t)\big|_{t=0} = F'(x+t)\big|_{t=0} - G'(x-t)\big|_{t=0} = F'(x) - G'(x)$$

**Remark 10.1.** If $g, h$ are given, we must have

$$\begin{gathered} g(x) = F(x) + G(x) \implies g'(x) = F'(x) + G'(x) \\ h(x) = F'(x) - G'(x) \\ \implies F'(x) = \frac{1}{2}g'(x) + \frac{1}{2}h(x) \quad \& \quad G'(x) = \frac{1}{2}g'(x) - \frac{1}{2}h(x) \end{gathered}$$

**Exercise 10.2.** Solve for $F$ and $G$ to show that d'Alembert's formula must hold.

### 10.1 Implications of d'Alembert's Formula

**(1)** Uniqueness

Let $u_1, u_2$ be two solutions. Set $u = u_1 - u_2$. Then, $u$ solves the wave equation, and $u(x,0) = 0,\ u_t(x,0) = 0$. So, by d'Alembert's formula, $u(x,t) = 0\ \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+$. Hence, $u_1 = u_2$.

**(2)** Stability

Suppose we have $(g_1, h_1) \mapsto u_1$ and $(g_2, h_2) \mapsto u_2$. Then,

$$\begin{aligned}
u_1(x,t) - u_2(x,t) &= \frac{1}{2}\left(g_1(x+t) + g_1(x-t) + \int_{x-t}^{x+t} h(y)\,dy\right) \\
&- \frac{1}{2}\left(g_2(x+t) + g_2(x-t) + \int_{x-t}^{x+t} h_2(y)\,dy\right) \\
&\le \frac{1}{2}|g_1(x+t) - g_2(x+t)| + \frac{1}{2}|g_1(x-t) - g_2(x-t)| \\
&+ \frac{1}{2}\int_{x-t}^{x+t} |h_1(y) - h_2(y)|\,dy \\
&\le \sup_{x\in\mathbb{R}} |g_1(x) - g_2(x)| + \frac{1}{2}\sup_{x\in\mathbb{R}} |h_1(x) - h_2(x)| \int_{x-t}^{x+t} dy \\
&= \sup_{x\in\mathbb{R}} |g_1(x) - g_2(x)| + t\sup_{x\in\mathbb{R}} |h_1(x) - h_2(x)|
\end{aligned}$$

Suppose that $g_1, g_2, h_1, h_2$ are bounded functions such that

$$\sup_{x\in\mathbb{R}} |g_1(x) - g_2(x)| < \epsilon \quad \text{and} \quad \sup_{x\in\mathbb{R}} |h_1(x) - h_2(x)| < \epsilon$$

Then,

$$\sup_{x\in\mathbb{R}} |u_1(x,t) - u_2(x,t)| \le \epsilon(1+t)$$

If $t$ were restricted to a finite interval, say $(0,T)$, then

$$\sup_{x\in\mathbb{R}} |u_1(x,t) - u_2(x,t)| \le \epsilon(1+T) = c_T\epsilon$$

**(3)** Explicit representation (**bu kısmı yazmadım**)

**(4)** Finite speed of propagation (**bu kısmı yazmadım**)

### 10.2 Wave Equation in $n$-D

$$(*)\begin{cases} u_{tt} - \Delta u = 0,\ x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x) \end{cases}$$

Alternatively, we can write $Lu = 0$, where $Lu = u_{tt} - \Delta u$, then we have $(x,t) \in \mathbb{R}^{n+1}$.

Set

$$\begin{aligned} U(x;r,t) &:= \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B_r(x)} u(y,t)\,dS(y) \\ U_0(x;r) &:= \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B_r(x)} u_0(y)\,dS(y) \\ U_1(x;r) &:= \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B_r(x)} u_1(y)\,dS(y) \end{aligned}$$

**Remark 10.3.** $\lim\limits_{r\to 0^+} U(x;r,t) = u(x,t)$ by the averaging lemma.

**Q:** If $u$ solves $(*)$, which PDE does $U$ solve?

**A:** We put $|\partial B_r(x)| := n\alpha(n)r^{n-1}$. Then, one has

$$\begin{aligned} U_r(x;r,t) &= \frac{\partial}{\partial r}\left(\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y)\right) \\ &\overset{*}{=} \frac{\partial}{\partial r}\left(\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B_1(0)} u(x+rz)r^{n-1}\,dS(z)\right) \\ &= \frac{1}{n\alpha(n)}\int_{\partial B_1(0)} \nabla u(x+rz)\cdot z\,dS(z) \\ &= \frac{1}{n\alpha(n)}\int_{\partial B_r(x)} \nabla u(y)\cdot \frac{y-x}{r}\frac{1}{r^{n-1}}\,dS(y) \\ &=: I \end{aligned}$$

$(* : dS(y) = r^{n-1}dS(z)$ and $z = (y-x)/r)$

Notice that

$$I = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} \frac{\partial u}{\partial n}\,dS(y) = \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \Delta u(y)\,dy = \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} u_{tt}\,dy,$$

where we used the Divergence theorem in the second equality and $\Delta u = u_{tt}$ in the last equality. Then, we have

$$r^{n-1}U_r(x;r,t) = \frac{1}{n\alpha(n)}\int_{B_r(x)} u_{tt}(y,t)\,dy = \frac{1}{n\alpha(n)}\int_0^r\int_{\partial B_\rho(x)} u_{tt}(y,t)\,dS(y)\,d\rho$$

$$\begin{aligned} \implies \frac{\partial}{\partial r}\left(r^{n-1}U_r(x;r,t)\right) &= \frac{\partial}{\partial r}\left(\underbrace{\frac{1}{n\alpha(n)}\int_0^r\int_{\partial B_\rho(x)} u_{tt}(y,t)\,dS(y)\,d\rho}_{\frac{d}{dr}\int_0^r F(\rho)\,d\rho = F(r)}\right) \\ &= \frac{1}{n\alpha(n)}\int_{\partial B_r(x)} u_{tt}(y,t)\,dS(y) \\ &= \frac{r^{n-1}}{n\alpha(n)r^{n-1}}\int_{\partial B_r(x)} u_{tt}(y,t)\,dS(y) \\ &= r^{n-1}U_{tt}(x;r,t) \end{aligned}$$

Hence,

$$(n-1)r^{n-2}U_r(x;r,t) + r^{n-1}U_{rr}(x;r,t) = r^{n-1}U_{tt}(x;r,t)$$
$$U_{tt}(x;r,t) - U_{rr}(x;r,t) - \frac{n-1}{r}U_r(x;r,t) = 0 \tag{10.2}$$

The equation (10.2) is called the Euler-Poisson-Darboux equation. Thus, we have the model

$$(*)' \begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0,\ r \in \mathbb{R}_+,\ t > 0 \\ U(x;r,0) = U_0(x;r) \\ U_t(x;r,0) = U_1(x;r) \end{cases}$$

which is essentially a 1-D problem.

### 10.2.1 $n = 3$

Surprisingly, $n = 3$ is easier than $n = 2$.

Set $\tilde{U} := rU,\ \tilde{U}_0 = rU_0,\ \tilde{U}_1 = rU_1$. Note that $\tilde{U}_r = U + rU_r$. Then, $\tilde{U}_{rr} = 2U_r + rU_{rr}$.

$$\implies \tilde{U}_{tt} - \tilde{U}_{rr} = rU_{tt} - 2U_r - rU_{rr} = r\left(U_{tt} - \frac{2}{r}U_r - U_{rr}\right) = 0$$

So, we have a new model

$$(*)'' \begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0,\ r \in \mathbb{R}_+,\ t > 0 \\ \tilde{U}(x;r,0) = \tilde{U}_0(x;r) \\ \tilde{U}_t(x;r,0) = \tilde{U}_1(x;r) \end{cases}$$

$$\lim_{r\to 0^+} \tilde{U}(x;r,t) = \lim_{r\to 0^+} rU(x;r,t) = \lim_{r\to 0^+} r\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B_r(x)} u(y,t)\,dS(y) = 0$$

Accordingly, we add $\tilde{U}(x;0,t) = 0$ to $(*)''$.

Let $V$ be the solution of the system

$$(*)''' \begin{cases} V_{tt} - V_{rr} = 0,\ r \in \mathbb{R},\ t > 0 \\ V\big|_{t=0} = \tilde{U}_0^{\text{odd}} \text{ (odd extension)} \to \text{this ensures } V(x;0,t) = 0 \textbf{ (o kısmı yazmadım)} \\ V_t\big|_{t=0} = \tilde{U}_1^{\text{odd}} \text{ (odd extension)} \end{cases}$$

Then, we know that the solution is given by d'Alembert's formula

$$V(x;r,t) = \frac{1}{2}(\tilde{U}_0^{\text{odd}}(x;r+t) + \tilde{U}_0^{\text{odd}}(x;r-t)) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{U}_1^{\text{odd}}(x;\rho)\,d\rho.$$

So, we have

$$\begin{aligned}\tilde{U}(x;r,t) &= V(x;r,t)\big|_{r\geq 0} \\ &= \begin{cases} \frac{1}{2}[\tilde{U}_0(x;r+t) + \tilde{U}_0(x;r-t)] + \frac{1}{2}\int_{r-t}^{r+t} \tilde{U}_1(x;\rho)\,d\rho, & 0 \leq t \leq r \\ \\ \frac{1}{2}[\tilde{U}_0(x;r+t) - \tilde{U}_0(x;t-r)] + \frac{1}{2}\int_0^{r+t} \tilde{U}_1(x;\rho)\,d\rho - \int_{r-t}^0 \tilde{U}_1(x;-\rho)\,d\rho, & 0 \leq r \leq t \end{cases}\end{aligned}$$

**Remark 10.4.** $\tilde{U}_1(x;\rho) = -\tilde{U}_1(x;-\rho)$.

For every $(x,t)$, we want to calculate

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\tilde{U}(x;r,t)}{r} \\
&= \lim_{r\to 0^+} \left( \underbrace{\frac{1}{2r}[\tilde{U}_0(x;r+t) - \tilde{U}_0(x;t-r)]}_{\text{Use L'Hôpital}} + \underbrace{\frac{1}{2r}\int_{t-r}^{t+r} \tilde{U}_1(x;\rho)\,d\rho}_{\text{Use L'Hôpital}} \right) \\
&= \tilde{U}_0'(x;t) + \tilde{U}_1(x;t) \\
&= \frac{\partial}{\partial t}\left( \underbrace{t\frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u_0(y)\,dS(y)}_{U_0(x;t)} \right) + \underbrace{t\frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u_1(y)\,dS(y)}_{U_1(x;t)}
\end{aligned}$$

We have

$$\frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u_0(y)\,dS(y) = \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u_0(x+tz)\,dS(z)$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{t}{|\partial B_t(x)|}\int_{\partial B_t(x)} u_0(y)\,dS(y)\right) &= \frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u_0(y)\,dS(y) \\
&\quad + t\frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} \nabla u(x+tz)\cdot z\,dS(z) \\
&= \frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u_0(y)\,dS(y) \\
&\quad + \frac{1}{|\partial B_1(0)|}\int_{\partial B_t(x)} \nabla u_0(y)\cdot\frac{y-x}{t^2}\,dS(y)
\end{aligned}$$

Notice that $t^2|\partial B_1(0)| = |\partial B_t(x)|$ (the center does not matter). Then, since $n=3$, we have

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} [u_0(y) + \nabla u_0(y)\cdot(y-x) + tu_1(y)]\,dS(y),\ \ x\in\mathbb{R}^3,\ \ t>0$$

Let $(x,t)\in\mathbb{R}^3\times(0,\infty)$. The domain of dependence of $(x,t)$ is $\partial B_t(x)$.

**Q:** If we change data at a point $x_0$, when will this be felt by the solution at a point $x\in\mathbb{R}^3$?

**A:** $t = |x-x_0|$. The information is carried with unit speed since $|x-x_0| = t$. (Huygens' Principle)

#### 10.2.2 $n = 2$

We have the model

$$\begin{cases} u_{tt} - u_{xx} = 0,\ x\in\mathbb{R}^2,\ t>0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x) \end{cases}$$

Set $\tilde{u}(x_1, x_2, x_3, t) \coloneqq u(\underbrace{x_1, x_2}_{x}, t)$, where $u$ solves the above model. Then,

$$\begin{aligned}
\tilde{u}_{tt}(x, x_3, t) &= u_{tt}(x, t) \\
\tilde{u}_{x_1x_1}(x, x_3, t) &= u_{x_1x_1}(x, t) \\
\tilde{u}_{x_2x_2}(x, x_3, t) &= u_{x_2x_2}(x, t) \\
\tilde{u}_{x_3x_3}(x, x_3, t) &= 0 \ \ (u \text{ does not depend on } x_3)
\end{aligned}$$

These give a new system

$$\begin{cases}
\tilde{u}_{tt} - \Delta_{(x,x_3)}\tilde{u} = u_{tt} - \Delta_x u = 0, \ (x, x_3) \in \mathbb{R}^3, \ t > 0 \\
\tilde{u}(x, x_3, 0) = u(x, 0) = u_0(x) = \tilde{u}_0(x, x_3) \\
\tilde{u}_t(x, x_3, 0) = u_t(x, 0) = u_1(x) = \tilde{u}_1(x, x_3)
\end{cases}$$

**Notation:** $\tilde{x} = (x, 0)$

**Notation:** $\tilde{B}_t(\tilde{x})$ denotes a 3-D sphere.

$$u(x, t) = \tilde{u}(\tilde{x}, t) = \frac{\partial}{\partial t}\left(\underbrace{\frac{t}{|\partial \tilde{B}_t(\tilde{x})|}\int_{\partial \tilde{B}_t(\tilde{x})} \tilde{u}_0(y')\, dS(y')}_{(\cdot)}\right) + \frac{t}{|\partial \tilde{B}_t(\tilde{x})|}\int_{\partial \tilde{B}_t(\tilde{x})} \tilde{u}_1(y')\, dS(y')$$

Note that $y' \in \partial \tilde{B}_t(\tilde{x})$ if $|\tilde{x} - y'| = t \iff |x - y|^2 + y_3^2 = t^2 \iff y_3^2 = t^2 - |x - y|^2$

**Notation:** $\tilde{B}_t^+(\tilde{x})$ denotes the upper hemisphere.

$$y' \in \partial \tilde{B}_t^+(\tilde{x}) \iff y' = \left(y_1, y_2, \sqrt{t^2 - |y - x|^2}\right) \eqqcolon (y, \gamma(y)),$$

where $(y_1, y_2) = y$ and $\gamma(y) = \sqrt{t^2 - |y - x|^2}$.

**Lemma 10.5** (Change of Variables)**.**

$$\int_{\partial \tilde{B}_t^+(\tilde{x})} F(y')\, dS(y') = \int_{B_t(x)(\text{2-D ball})} F(y, \gamma(y))(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}}\, dy$$

*Proof.* Exercise. □

From $(\cdot)$ (the surface area $|\partial \tilde{B}_t(\tilde{x})| = 4\pi t^2$),

$$\begin{aligned}
\frac{1}{4\pi t^2}\int_{\partial \tilde{B}_t^+(\tilde{x})} \tilde{u}_0(y')\, dS(y') &= \frac{1}{4\pi t^2}\int_{B_t(x)} \tilde{u}_0(y, \gamma(y))(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}}\, dy \\
&= \frac{1}{4\pi t^2}\int_{B_t(x)} u_0(y)(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}}\, dy
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{1}{4\pi t^2}\int_{\partial \tilde{B}_t^-(x)} \tilde{u}_0(y')\, dS(y') = \frac{1}{4\pi t^2}\int_{B_t(x)} u_0(y)(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}}\, dy \\
\implies &\frac{1}{4\pi t^2}\int_{\partial \tilde{B}_t(\tilde{x})} \tilde{u}_0(y')\, dS(y') = \frac{2}{4\pi t^2}\int_{B_t(x)} u_0(y)(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}}\, dy
\end{aligned}$$

**Exercise 10.6.**

$$(1+|\nabla\gamma(y)|^2)^{\frac{1}{2}} = \left(1+\frac{|x-y|^2}{t^2-|x-y|^2}\right)^{\frac{1}{2}} = \frac{t}{(t^2-|x-y|^2)^{\frac{1}{2}}}$$

By the above exercise,

$$\begin{aligned}\frac{1}{|\partial\tilde{B}_t(\tilde{x})|}\int_{\partial\tilde{B}_t(\tilde{x})}\tilde{u}_0(y')\,dS(y') &= \frac{1}{2\pi t}\int_{B_t(x)}\frac{u_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\\ &= \frac{t}{2}\frac{1}{|B_t(x)|}\int_{B_t(x)}\frac{u_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{t^2}{|B_t(x)|}\int_{B_t(x)}\frac{u_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\right) &= \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{t^2}{\pi t^2}\int_{B_1(0)}\frac{u_0(x+tz)t^2}{t(1-|z|^2)^{\frac{1}{2}}}\,dz\right)\\ &= \frac{1}{2\pi}\int_{B_1(0)}\frac{u_0(x+tz)}{(1-|z|^2)^{\frac{1}{2}}}\,dz\\ &\quad+\frac{t}{2\pi}\int_{B_1(0)}\frac{\nabla u_0(x+tz)\cdot z}{(1-|z|^2)^{\frac{1}{2}}}\,dz\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{B_1(0)}\frac{u_0(x+tz)}{(1-|z|^2)^{\frac{1}{2}}}\,dz &= \frac{1}{2\pi}\int_{B_t(x)}\frac{u_0(y)}{t^2\left(1-\frac{|y-x|^2}{t^2}\right)^{\frac{1}{2}}}\,dy\\ &= \frac{t}{2|B_t(x)|}\int_{B_t(x)}\frac{u_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy.\end{aligned}$$

Finally,

$$\begin{aligned}\frac{t}{2\pi}\int_{B_1(0)}\frac{\nabla u_0(x+tz)\cdot z}{(1-|z|^2)^{\frac{1}{2}}}\,dz &= \frac{t}{2\pi t^2}\int_{B_t(x)}\frac{\nabla u_0(y)\cdot\frac{y-x}{t}}{(t^2-|y-x|^2)^{\frac{1}{2}}}t\,dy\\ &= \frac{t}{2|B_t(x)|}\int_{B_t(x)}\frac{\nabla u_0(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy\end{aligned}$$

Collecting everything, we obtain

$$u(x,t) = \frac{1}{2\pi t^2}\int_{B_t(x)}\frac{[tu_0(y)+t^2u_1(y)+t\nabla u_0(y)\cdot(y-x)]}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy$$

**Remark 10.7.** Observe that the domain of dependence is $B_t(x)$, not $\partial B_t(x)$ (compare with 3-D)

**Exercise 10.8.** Solve the wave equation in 1-D for a finite interval using separation of variables.

### 10.3 Energy Method

Consider the model

$$(\cdot)\begin{cases} u_{tt} - \Delta u = 0,\ x \in \Omega \subseteq \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \\ u\big|_{\partial\Omega} = 0\ \text{ (Dirichlet boundary condition)} \end{cases}$$

We define the energy associated with the wave

$$E(t) \coloneqq \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)\,dx.$$

Then,

$$E'(t) = \frac{1}{2}\int_\Omega (2u_t u_{tt} + 2\nabla u \cdot \nabla u_t)\,dx.$$

Apply the divergence theorem to the second term

$$\int_\Omega \nabla u \cdot \nabla u_t\,dx = -\int_\Omega (\Delta u)u_t\,dx + \int_\Omega u_t \frac{\partial u}{\partial n}\,dS = -\int_\Omega (\Delta u) u_t\,dx$$

since $u\big|_{\partial\Omega} = 0$.

$$\implies E'(t) = \int_\Omega (u_t u_{tt} - \Delta u\, u_t)\,dx = \int_\Omega u_t(\underbrace{u_{tt} - \Delta u}_{0})\,dx = 0$$

Hence, energy is constant. So,

$$E(t) = E(0) = \frac{1}{2}\int_\Omega (u_1^2 + |\nabla u_0|^2)\,dx.$$

#### 10.3.1 Uniqueness

Suppose $(\cdot)$ admits two solutions $u$ and $v$. Set $w = u - v$, which solves

$$\begin{cases} w_{tt} - \Delta w = 0,\ x \in \Omega \\ w\big|_{t=0} = w_t\big|_{t=0} = 0 \\ w\big|_{\partial\Omega} = 0 \end{cases}$$

Thus, the energy of $w$ is

$$\frac{1}{2}\int_\Omega (w_t^2 + |\nabla w|^2)\,dx \equiv 0 \ \ \forall t$$

This implies that

$$\begin{cases} \nabla w \equiv 0 \ \text{ on } \ \Omega. \text{ So, } w \text{ does not vary by } x \\ w_t \equiv 0 \ \text{ on } \ \Omega. \text{ So, } w \text{ does not vary by } t \end{cases} \implies w \text{ is constant}$$

On the other hand,

$$w\big|_{t=0} = 0 \implies w \equiv 0\ \forall t \implies u \equiv v\ \forall t$$

# 11 Inhomogeneous Initial-Boundary Value Problems

## 11.1 Unified Transform Method

We consider

$$\begin{cases} u_t - ku_{xx} = 0, \ \ x \in (0,l), \ \ t > 0 \\ u(x,0) = u_0(x) \\ u(0,t) = h(t), \ \ u(l,t) = j(t) \end{cases}$$

**Strategy:** Look for a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right)$$

**Remark 11.1.** Note that plugging $x = 0$ or $x = l$ into the above expression yields zero, so it appears that the boundary conditions are not verified. However, we do not require our series to converge in a pointwise sense.

Similarly, we expand $u_t, u_{xx}$ as a Fourier sine series.

$$\begin{aligned} u_t &= \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{l}\right) \\ u_{xx} &= \sum_{n=1}^{\infty} \omega_n(t) \sin\left(\frac{n\pi x}{l}\right), \end{aligned}$$

where the coefficients are given as usual

$$v_n := \frac{2}{l}\int_0^l \frac{\partial u}{\partial t} \sin\left(\frac{n\pi x}{l}\right) dx = \frac{d}{dt}\left(\frac{2}{l}\int_0^l u \sin\left(\frac{n\pi x}{l}\right) dx\right) = u_n'$$

$$\begin{aligned} \omega_l := \frac{2}{l}\int_0^l u_{xx} \sin\left(\frac{n\pi x}{l}\right) dx &= \frac{2}{l}\left(u_x \sin\left(\frac{n\pi x}{l}\right)\bigg|_0^l - \frac{n\pi}{l}\int_0^l u_x \cos\left(\frac{n\pi x}{l}\right) dx\right) \\ &= -\frac{2}{l}\frac{n\pi}{l}\left(u \cos\left(\frac{n\pi x}{l}\right)\bigg|_0^l + \frac{n\pi}{l}\int_0^l u \sin\left(\frac{n\pi x}{l}\right) dx\right) \\ &= -\frac{2}{l}\frac{n\pi}{l}((-1)^n j(t) - h(t)) - \left(\frac{n\pi}{l}\right)^2 u_n(t) \end{aligned}$$

Then,

$$u_n'(t) + \underbrace{k\left(\frac{n\pi}{l}\right)^2}_{\lambda_n} = -\frac{2n\pi}{l^2}((-1)^n j(t) - h(t)).$$

Multiply by the integrating factor $\mu = e^{k\lambda_n t}$. Then,

$$
\begin{aligned}
(u_n(t)\mu(t))' &= F(t)\mu(t) \\
u_n(t)\mu(t) &= C + \int_0^t F(s)\mu(s)\, ds \\
u_n(t) &= Ce^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t F(s)e^{-k\lambda_n s}\, ds \\
\implies u_n(t) &= C_n e^{-k\lambda_n t} + \frac{2n\pi}{l^2}\int_0^t e^{-k\lambda_n (t-s)}[h(s) - (-1)^n j(s)]\, ds. \qquad (11.1)
\end{aligned}
$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{l}\right),$$

where $u_n$ is given by (11.1).

Recall that

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} u_{0n}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$u_{0n} \coloneqq u_n\Big|_{t=0} = C_n.$$

So, we choose $C_n = u_{0n}$, where $u_{0n}$'s are Fourier sine series coefficients of $u_0$. Now, let

$$S_n \coloneqq \sum_{m=1}^{n} u_m(t)\sin\left(\frac{n\pi x}{l}\right).$$

We have

$$\int_0^l |\tilde{u} - S_n|^2\, dx \to 0$$

as $n \to \infty$, which does not imply $S_n(0,t) \to \tilde{u}(0,t)$ as $n \to \infty$.

**Suggested Reading:** Unified Transform Method, which is a generalization of the Fourier series to IBVPs.

**Exercise 11.2.** Solve

$$
\begin{cases}
u_{tt} - u_{xx} = f(x,t), \;\; x \in (0,l), \;\; t > 0 \\
u(0,t) = h(t), \;\; u(l,t) = j(t) \\
u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x)
\end{cases}
$$

using the same method.

## 11.2 Homogenization Method

This is an alternative to the previous method.

Define

$$U(x,t) \coloneqq \left(1 - \frac{x}{t}\right)h(t) + \frac{x}{l}j(t).$$

Note that $U(0,t) = h(t),\ U(l,t) = j(t)$. Next, set $v = u - U$. Then,

$$\begin{cases} v_t - v_{xx} = -\left(1 - \frac{x}{l}\right) h'(t) - \frac{x}{l} j'(t) =: g(x,t) \\ v(x,0) = u_0 - \left(1 - \frac{x}{l}\right) h(0) - \frac{x}{l} j(0) =: v_0 \\ v(0,t) = v(l,t) = 0 \end{cases}$$

Once $v$ is found (through separation of variables),

$$u(x,t) = U(x,t) + v(x,t).$$