MATH-583 Lecture Notes

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Contents

1 Review of GR

GR is based on a 4-dimensional pseudo-Riemannian manifold, which we call spacetime. Every point on spacetime is described by coordinates x^μ as

$$x^\mu = (x^0, x^1, x^2, x^3), \quad \mu = \overline{0,3}$$

Example 1.1. Cartesian coordinates described by $x^\mu = (ct, x, y, z)$.

Example 1.2. Spherical coordinates given by $x^\mu = (ct, r, \theta, \phi)$, where $r \in [0, \infty),\ \theta \in [0, 2\pi]$, and $\phi \in [0, 2\pi)$.

Remark. In general, the whole spacetime cannot be covered with a single coordinate system.

Exercise 1.3. Provide an example for the above remark and justify your example.

1.1 Coordinate Transformations

Under a general coordinate transformation from a coordinate system x^μ to another coordinate system x'^μ, one has the following transformation rules.

(i) Scalars do not change under coordinate transformations, or we say that scalars are invariant under coordinate transformations. This is expressed by

$$f(x) = f'(x')$$

In the above, $x = x^\mu(p)$ and $x' = x'^\mu(p)$ are the coordinates of $p \in M$ in two different charts, where M is our manifold. The functions f and f' are the component expressions of the same scalar field in different coordinate systems. This is because x and x' are coordinates for the same point under different charts, and functions defined in terms of those coordinates should have different symbolic expressions.

(ii) Contravariant vectors transform as

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu = \Lambda^\mu{}_\nu A^\nu$$

(iii) Covariant vectors transform as

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu = (\Lambda^{-1})^\nu{}_\mu A_\nu$$

(iv) Covariant tensors of rank $(0, n)$ transform as

$$T'_{\mu_1\mu_2\ldots\mu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}} T_{\alpha_1\alpha_2\ldots\alpha_n}$$

(v) Contravariant tensors of rank $(m, 0)$ transform as

$$T'^{\mu_1\mu_2\ldots\mu_m} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{\mu_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{\mu_m}}{\partial x^{\alpha_m}} T^{\alpha_1\alpha_2\ldots\alpha_m}$$

(vi) Mixed tensors of rank (m, n) transform as

$$T'^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_n}}{\partial x'^{\nu_n}} T^{\alpha_1\ldots\alpha_m}{}_{\nu_1\ldots\nu_n}$$

1.2 Metric Tensor

All the information about a gravitational field is encoded in a symmetric $(0,2)-$rank tensor called the metric tensor. Using the metric tensor, one can compute the line element ds^2, which represents the square of the spacetime interval. Precisely, we have

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu := g_{\mu\nu}dx^\mu \otimes dx^\nu,$$

where the juxtaposition $dx^\mu dx^\nu$ is the symmetric product of dx^μ and dx^ν, defined by

$$\text{Sym}(dx^\mu \otimes dx^\nu) := dx^\mu dx^\nu := \frac{1}{2}\left(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu\right)$$

Exercise 1.4. Show that $ds'^2 = ds^2$.

Example 1.5. The Minkowski metric describes the flat spacetime as in special relativity and is given by

$$g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1,1,1,1)$$

Then, the line element is

$$\begin{aligned} ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu &= \eta_{00}dx^0dx^0 + \eta_{11}dx^1dx^1 + \eta_{22}dx^2dx^2 + \eta_{33}dx^3dx^3 \\ &= \eta_{00}(dx^0)^2 + \eta_{11}(dx^1)^2 + \eta_{22}(dx^2)^2 + \eta_{33}(dx^3)^2 \\ &= -c^2(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= -c^2dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \end{aligned}$$

Exercise 1.6. Use the coordinate transformation from Cartesian coordinates to spherical coordinates to write the line element of flat Minkowski space in spherical coordinates as

$$\begin{aligned} ds^2 &= -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \\ &= -c^2dt^2 + dr^2 + r^2d\Omega^2 \end{aligned}$$

The inverse metric tensor is defined uniquely by

$$g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha = \begin{cases} 1, & \mu = \alpha \\ 0, & \mu \neq \alpha \end{cases}$$

We can raise or lower indices of tensors using the inverse metric and the metric, respectively.

$$\begin{aligned} T^{\mu\nu} &= g^{\alpha\mu}g^{\nu\beta}T_{\alpha\beta} \\ T_{\mu\nu} &= g_{\alpha\mu}g_{\nu\beta}T^{\alpha\beta} \end{aligned}$$

1.3 Covariant Differentiation

Covariant differentiation is a generalization of partial differentiation to curved spaces.

(i) For scalar fields, the covariant derivative reduces to the ordinary partial derivative.

$$\nabla_\mu\phi = \partial_\mu\phi$$

(ii) For contravariant vectors, we define the covariant derivative as

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\lambda} A^\lambda,$$

where $\Gamma^\nu{}_{\mu\lambda}$ are called the connection coefficients, which encode how the basis vectors change when we move along other basis vectors. More explicitly, the covariant derivative of the basis vector ∂_λ is

$$\nabla_\mu \partial_\lambda = \Gamma^\nu{}_{\mu\lambda} \partial_\nu$$

This means that as we move infinitesimally along ∂_μ, the basis vector ∂_λ transforms into a linear combination of basis vectors, each weighted by $\Gamma^\nu{}_{\mu\lambda}$.

(iii) Using the covariant differentiation formula for contravariant vectors, one can derive the formula for the covariant derivative of a covariant vector:

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda{}_{\mu\nu} A_\lambda$$

Exercise 1.7. Derive the above formula.

(iv) For a mixed tensor of rank (m, n), the covariant derivative is

$$\begin{aligned}\nabla_\mu T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l} = \partial_\mu T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l} + \Gamma^{\alpha_1}{}_{\mu\lambda} T^{\lambda\alpha_2 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l} + \cdots + \Gamma^{\alpha_k}{}_{\mu\lambda} T^{\alpha_1 \ldots \alpha_{k-1}\lambda}{}_{\beta_1 \ldots \beta_l} \\ - \Gamma^\lambda{}_{\mu\beta_1} T^{\alpha_1 \ldots \alpha_k}{}_{\lambda\beta_2 \ldots \beta_l} - \cdots - \Gamma^\lambda{}_{\mu\beta_l} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_{l-1}\lambda}\end{aligned}$$

1.3.1 Levi-Civita Connection

So far, the connection has a general notion connecting the changes of our basis vectors from one point (tangent space) to another (different tangent space). In GR, we have the following conditions on our connection:

(i) Symmetry: $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$

(ii) Metric compatibility: $\nabla_\alpha g_{\mu\nu} = 0$

Exercise 1.8. Show that

- $\nabla_\alpha g^{\mu\nu} = 0$
- $g_{\mu\lambda} \nabla_\rho v^\lambda = \nabla_\rho v_\mu$ and $g^{\alpha\mu} \nabla_\nu A_\mu = \nabla_\nu A^\alpha$
- $\nabla_\alpha \epsilon_{\mu\nu\rho\sigma} = 0$, where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor given by

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{|g|} \tilde{\epsilon}_{\mu\nu\rho\sigma},$$

where

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1, & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 1.9. Show that the unique connection which is torsion-free and metric compatible has the following form

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

called the Christoffel or Levi-Civita connection.

Exercise 1.10. Show that under a coordinate transformation from x^μ to x'^μ, Christoffelt connection transforms as

$$\Gamma'^{\lambda}{}_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\alpha} \frac{\partial x^\lambda}{\partial x'^\beta} \Gamma^{\rho}{}_{\sigma\lambda} + \frac{\partial^2 x^\sigma}{\partial x'^\alpha \partial x'^\beta} \frac{\partial x'^\mu}{\partial x^\sigma}$$

Hence, $\Gamma^{\mu}{}_{\alpha\beta}$ is not a tensor.

1.4 Local Inertial Frames (Coordinates)

At any point p on a manifold, one can define a coordinate system such that $\Gamma^{\alpha}{}_{\mu\nu}$ vanishes,i.e.,

$$\Gamma^{\alpha}{}_{\mu\nu}(x)\Big|_p = 0$$

Exercise 1.11. Show that in local inertial frames, the first derivative of the metric vanishes at p. Then, show that by a further linear transformation of coordinates, the metric can be reduced to a flat Minkowski metric.

$$\Gamma^{\alpha}{}_{\mu\nu}(x)\Big|_p = 0 \overset{\partial_\alpha g_{\mu\nu}=0}{\Longleftrightarrow} g_{\mu\nu}(x)\Big|_p = \eta_{\mu\nu} \;\; \text{(Equivalence principle)}$$

Exercise 1.12. Consider the 2-dimensional plane in polar coordinates with the line element

$$ds^2 = dr^2 + r^2 d\theta^2$$

(i) Write down the metric components.

(ii) Show that the non-zero components of the inverse metric are

$$g^{rr} = 1, \;\; g^{\theta\theta} = \frac{1}{r^2}$$

(iii) Show that the Christoffel connection coefficients read as

$$\Gamma^{r}{}_{rr} = 0, \;\; \Gamma^{r}{}_{\theta\theta} = -r, \;\; \Gamma^{r}{}_{\theta r} = \Gamma^{r}{}_{r\theta} = 0$$

$$\Gamma^{\theta}{}_{rr} = 0, \;\; \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}, \;\; \Gamma^{\theta}{}_{\theta\theta} = 0$$

1.5 Parallel Transport and Geodesic Equation

1.5.1 Parallel Transport

The concept of moving a vector along a path keeping it constant all the while is known as parallel transport. The difference between flat and curved space is that in curved space, the result of parallel transport is path-dependent. [INSERT FIGURE]

In flat space, given a curve $x^\mu(\lambda)$, to have a tensor to be parallel transported, we need

$$\frac{d}{d\lambda}T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = 0 \iff \frac{dx^\alpha}{d\lambda}\partial_\alpha T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = 0 \text{ (Directional derivative)}$$

Let $u^\mu = \frac{dx^\mu}{d\lambda}$. Then, we generalize to curved space as follows:

$$\frac{d}{d\lambda} = u^\alpha\partial_\alpha \longrightarrow \frac{D}{d\lambda} = u^\alpha\nabla_\alpha \text{ (Directional covariant derivative)}$$

Hence, for a tensor to be parallel transported in curved space, we need

$$\frac{D}{d\lambda}T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = u^\alpha\nabla_\alpha T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = 0$$

For vectors, we have

$$\frac{D}{d\lambda}v^\mu = u^\alpha\nabla_\alpha v^\mu = 0$$

An observation is that the notion of parallel transport is connection-dependent. However, since we have the Christoffel connection in GR, the metric is parallel transported with respect to it, i.e., we have

$$\frac{D}{d\lambda}g_{\mu\nu} = u^\alpha\nabla_\alpha g_{\mu\nu} = 0$$

Exercise 1.13. Show that the inner product of two parallel-transported vectors is preserved for a metric-compatible connection.

1.5.2 Geodesics

A geodesic is a generalization of the shortest path in flat space to curved spaces. In flat space, a straight line is the path of shortest distance between two points, but there is a better definition: A straight line is the path whose tangent vector is parallel transported [INSERT FIGURE]. Accordingly, the generalization of the definition of a geodesic in flat space to curved space is done by the notion of parallel transport. More precisely, we have

$$\begin{aligned}\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 &= u^\alpha\nabla_\alpha u^\mu \\ &= u^\alpha(\partial_\alpha u^\mu + \Gamma^\mu{}_{\alpha\nu}u^\nu) \\ &= \frac{dx^\alpha}{d\lambda}\frac{\partial u^\mu}{\partial x^\alpha} + \Gamma^\mu{}_{\alpha\nu}u^\alpha u^\nu \\ &= \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\nu}u^\alpha u^\nu\end{aligned}$$

Thus, we obtain the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + {\Gamma^\mu}_{\alpha\nu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Note that ${\Gamma^\mu}_{\alpha\nu}$ here is not necessarily the Christoffel connection.

Example 1.14. Consider the flat space whose connection vanishes everywhere, i.e., ${\Gamma^\alpha}_{\mu\nu} = 0$. Then, the geodesic equation reads as

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \implies x^\mu = a\lambda + b, \quad a, b \in \mathbb{R}$$

Exercise 1.15. Show that the geodesic equation is invariant under the change of parameter

$$\lambda \longrightarrow \tau = a\lambda + b, \quad a, b \in \mathbb{R}$$

Here, τ is known as an affine parameter.

Exercise 1.16. Show that for some parameter $\alpha(\lambda)$, the geodesic equation transforms as

$$\frac{d^2x^\mu}{d\alpha^2} + {\Gamma^\mu}_{\nu\sigma}\frac{dx^\nu}{d\alpha}\frac{dx^\sigma}{d\alpha} = f(\alpha)\frac{dx^\mu}{d\alpha},$$

where $f(\alpha) \coloneqq -\left(\frac{d^2\alpha}{d\lambda^2}\right)\left(\frac{d\alpha}{d\lambda}\right)^{-2}$.

The geodesic equation can also be obtained by extremizing the length functional. Given the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, the length functional is

$$S = \int ds = \int \sqrt{g_{\mu\nu}dx^\mu dx^\nu} = \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

Letting $f = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$, we have

$$\delta S = \int \delta\sqrt{f}\, d\lambda = \frac{1}{2}\int f^{-\frac{1}{2}}\delta f\, d\lambda$$

Observe that $I \coloneqq \frac{1}{2}\int f\, d\lambda$ has the same stationary points as S. By variation, we have

$$x^\mu \longrightarrow x^\mu + \delta x^\mu, \quad \delta(dx^\mu) = d(\delta x^\mu)$$

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x) + \delta g_{\mu\nu} = g_{\mu\nu}(x) + (\partial_\sigma g_{\mu\nu})\delta x^\sigma$$

Then,

$$\begin{aligned}
\delta I &= \frac{1}{2}\int \left[\delta g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\delta\left(\frac{dx^\mu}{d\lambda}\right)\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\delta\left(\frac{dx^\nu}{d\lambda}\right)\right] d\lambda \\
&= \frac{1}{2}\int \left[\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta x^\sigma + \underbrace{g_{\mu\nu}\frac{dx^\nu}{d\lambda}\frac{d(\delta x^\mu)}{d\lambda}}_{\mathrm{I}} + \underbrace{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d(\delta x^\nu)}{d\lambda}}_{\mathrm{II}}\right] d\lambda
\end{aligned}$$

Integrating by parts,

$$\mathrm{I} = \int g_{\mu\nu}\frac{dx^\nu}{d\lambda}\frac{d}{d\lambda}(\delta x^\mu)\, d\lambda = g_{\mu\nu}\frac{dx^\nu}{d\lambda}\delta x^\mu\Big|_{p_1}^{p_2} - \int \left(\frac{dg_{\mu\nu}}{d\lambda}\frac{dx^\nu}{d\lambda}\delta x^\mu + g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2}\delta x^\mu\right) d\lambda$$

Since the variation vanishes on the boundary, we have

$$\mathrm{I} = -\int \left(\partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\lambda} \frac{dx^\nu}{d\lambda} + g_{\mu\nu} \frac{d^2 x^\nu}{d\lambda^2} \right) \delta x^\mu \, d\lambda$$

Similarly,

$$\mathrm{II} = -\int \left(\partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\lambda} \frac{dx^\mu}{d\lambda} + g_{\mu\nu} \frac{d^2 x^\mu}{d\lambda^2} \right) \delta x^\nu \, d\lambda$$

Then,

$$\begin{aligned}
\delta \mathrm{I} &= \frac{1}{2} \int \left[\partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta x^\sigma - \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\lambda} \frac{dx^\mu}{d\lambda} \delta x^\nu - \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\lambda} \frac{dx^\nu}{d\lambda} \delta x^\mu \right] d\lambda \\
&\quad + \frac{1}{2} \int \left[-g_{\mu\nu} \frac{d^2 x^\nu}{d\lambda^2} \delta x^\mu - g_{\mu\nu} \frac{d^2 x^\mu}{d\lambda^2} \delta x^\nu \right] d\lambda \\
&= \frac{1}{2} \int \left[\partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta x^\sigma - \partial_\nu g_{\mu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} \delta x^\sigma - \partial_\mu g_{\sigma\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta x^\sigma - 2 g_{\mu\sigma} \frac{d^2 x^\mu}{d\lambda^2} \delta x^\sigma \right] d\lambda \\
&= -\frac{1}{2} \int \left[2 g_{\mu\sigma} \frac{d^2 x^\mu}{d\lambda^2} + \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right] \delta x^\sigma \, d\lambda
\end{aligned}$$

$\delta \mathrm{I}$ should vanish for any variation. Hence,

$$2 g_{\mu\sigma} \frac{d^2 x^\mu}{d\lambda^2} + \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

Contracting with $g^{\alpha\sigma}$,

$$\frac{d^2 x^\alpha}{d\lambda^2} + \frac{1}{2} g^{\alpha\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

It follows that to have compatible equations for GR, we must have

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})$$

Remark. (i) Geodesic equation implies paths by unaccelerated test particles

$$\frac{D}{d\tau} u^\mu = 0, \quad u^\mu = \text{velocity vector}$$

In other words, the geodesic equation is a generalization of $\vec{f} = m\vec{a}$ for $\vec{f} = 0$ to curved spaces.

(ii) The proper time is given by

$$\tau = \frac{1}{c} \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda$$

Then, $\delta\tau = 0$ gives geodesics and null geodesics minimize proper time, whereas time-like geodesics maximize proper time.

(iii) If the moving body is charged and there are electromagnetic fields, then the tangent vector will not be parallel transported along the path, i.e., $\frac{Du^\mu}{d\lambda} = u^\alpha \nabla_\alpha u^\mu \neq 0$. So, the RHS of the geodesic equation should be modified.

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = \frac{q}{m} F_\nu{}^\mu \frac{dx^\nu}{d\lambda}$$

(iv) For timelike geodesics, we can choose $\lambda = \tau$, where τ is the proper time. Then, $u^\alpha \nabla_\alpha u^\mu = 0$ and using 4-momentum $p^\mu = mu^\mu = (E, \vec{p})$, we have

$$p^\alpha \nabla_\alpha p^\mu = 0$$

$$mu^\alpha (\partial_\alpha p^\mu + \Gamma^\mu{}_{\alpha\lambda} p^\lambda) = 0$$

$$m\frac{dp^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\lambda} p^\alpha p^\lambda = 0$$

(v) Newtonian limit (non-relativistic limit) Consider the line element

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2),$$

where $\phi \ll 1$ and $\phi = \phi(x, y, z)$. The slow motion condition reads as $E \gg |\vec{p}|, E \approx m$ and we also have the normalization $p^\alpha p_\alpha = -m^2$. Then,

$$m\frac{dp^\mu}{d\tau} + \underbrace{\Gamma^\mu{}_{\alpha\beta}\, p^\alpha p^\beta}_{\text{dominant term}} = 0$$

Since $E \gg |\vec{p}|, \Gamma^\mu{}_{\alpha\beta}\, p^\alpha p^\beta \approx \Gamma^\mu{}_{00} p^0 p^0 \approx m^2 \Gamma^\mu{}_{00}$. Hence,

$$m\frac{dp^\beta}{d\tau} \approx -m^2 \Gamma^\beta{}_{00} \implies \frac{dp^\beta}{d\tau} \approx -m\Gamma^\beta{}_{00}$$

Time component ($\beta = 0$) gives

$$\Gamma^0{}_{00} = \frac{1}{2} g^{0\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) = -\frac{1}{2} g^{0\lambda} \partial_\lambda g_{00} = -\frac{1}{2} g^{00} \partial_0 g_{00} = 0$$

$$\implies \frac{dp^0}{d\tau} \approx 0 \implies E = \text{constant}$$

Spatial components ($\beta = i$) give

$$\begin{aligned} \Gamma^i{}_{00} = \frac{1}{2} g^{i\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) = -\frac{1}{2} g^{i\lambda} \partial_\lambda g_{00} &= -\frac{1}{2} g^{ii} \partial_i g_{00} \\ &= -\frac{1}{2}(1+2\phi)^{-1} \delta^{ij} \partial_j (-2\phi) \end{aligned}$$

$$\implies \Gamma^i{}_{00} = \delta^{ij} \partial_j \phi + O(\phi^2)$$

Hence,

$$\frac{dp^i}{d\tau} \approx -m\delta^{ij} \partial_j \phi = -m\nabla\phi$$

If $v/c \ll 1,\ t \approx \tau$ and we have

$$\frac{dp^i}{dt} \approx -m\delta^{ij} \partial_j \phi = -m\nabla\phi$$

which is just Newton's law.

1.6 Curvature

In flat space, parallel transport of a vector from one point to another does not change the vector.[INSERT FIGURE] In curved space, parallel transport depends on the paths. [INSERT FIGURE]

Intuitively, we expect that

(i) $\delta v^\rho \propto v^\rho, A^\mu, B^\nu$

(ii) δv^ρ changes its sign if the order of paths in a closed curve is changed.

The two requirements can be written more succinctly as

$$\delta v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma A^\mu B^\nu$$
$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$

where $R^\rho{}_{\sigma\mu\nu}$ is to be determined.

$\nabla_\mu v^\rho$ represents the change of v^ρ with respect to the case when it is parallel transported along A^μ ($A^\mu \nabla_\mu v^\rho = 0$), wheras $\nabla_\nu \nabla_\mu v^\rho$ represents the change of v^ρ with respect to the case when it is parallel transported first along A^μ then B^ν. Similarly, $\nabla_\mu \nabla_\nu v^\rho$ represents the change of v^ρ with respect to the case when it is parallel transported first along B^ν then A^μ. Then,

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] v^\rho &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v^\rho \\
&= \nabla_\mu \nabla_\nu v^\rho - \nabla_\nu \nabla_\mu v^\rho \\
&= \partial_\mu (\nabla_\nu v^\rho) - \Gamma^\lambda{}_{\mu\nu} \nabla_\lambda v^\rho + \Gamma^\rho{}_{\mu\sigma} \nabla_\nu v^\sigma - \partial_\nu (\nabla_\mu v^\rho) + \Gamma^\lambda{}_{\nu\mu} \nabla_\lambda v^\rho - \Gamma^\rho{}_{\nu\sigma} \nabla_\mu v^\sigma \\
&= \partial_\mu \partial_\nu v^\rho + \partial_\mu (\Gamma^\rho{}_{\nu\sigma}) v^\sigma + \Gamma^\rho{}_{\nu\sigma} \partial_\mu v^\sigma - \Gamma^\lambda{}_{\mu\nu} \nabla_\lambda v^\rho + \Gamma^\rho{}_{\mu\sigma} \partial_\nu v^\sigma + \Gamma^\rho{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\lambda} v^\lambda \\
&\quad - \partial_\nu \partial_\mu v^\rho - \partial_\nu (\Gamma^\rho{}_{\mu\sigma}) v^\sigma - \Gamma^\rho{}_{\mu\sigma} \partial_\nu v^\sigma + \Gamma^\lambda{}_{\nu\mu} \nabla_\lambda v^\rho - \Gamma^\rho{}_{\nu\sigma} \partial_\mu v^\sigma - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\lambda} v^\lambda \\
&= (\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma}) v^\sigma + (\Gamma^\rho{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\lambda}) v^\lambda + (\Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}) \nabla_\lambda v^\rho \\
&= (\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma}) v^\sigma + (\Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}) \nabla_\lambda v^\rho \\
&= R^\rho{}_{\sigma\mu\nu} v^\sigma + T^\lambda{}_{\nu\mu} \nabla_\lambda v^\rho
\end{aligned}$$

$R^\rho{}_{\sigma\mu\nu}$ is called the Riemann tensor and $T^\lambda{}_{\nu\mu}$ is called the torsion tensor. Since we use the Levi-Civita connection in GR, $T^\lambda{}_{\mu\nu} \equiv 0 \implies [\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma$.

Exercise 1.17. (i) Show that $T^\lambda{}_{\mu\nu}$ is a tensor.

(ii) Show that $R^\rho{}_{\mu\nu\sigma}$ is a tensor.

(iii) $R_{\sigma\mu\nu\lambda} = -R_{\sigma\mu\lambda\nu} = -R_{\mu\sigma\nu\lambda}$

(iv) $R_{\sigma\mu\nu\lambda} = R_{\nu\lambda\sigma\mu}$

(v) $R_{\sigma[\mu\nu\lambda]} = 0$

(vi) (Bianchi identity)

$$\nabla_\rho R^\sigma{}_{\mu\nu\lambda} + \nabla_\lambda R^\sigma{}_{\mu\rho\nu} + \nabla_\nu R^\sigma{}_{\mu\lambda\rho} = 0$$

Equivalently, $\nabla_{[\rho} R_{\nu\lambda]\sigma\mu} = 0$.

The Ricci tensor and the Ricci scalar are defined by

$$R_{\mu\nu} := g^{\lambda\sigma} R_{\sigma\mu\lambda\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$
$$R := R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

Exercise 1.18. (i) $\nabla^{\mu} R_{\mu\rho} = \frac{1}{2}\nabla_{\rho} R = \frac{1}{2}\partial_{\rho} R$

(ii) $\Gamma^{\lambda}{}_{\mu\lambda} = \partial_{\mu}(\ln(\sqrt{-g}))$, where $g = \det(g_{\mu\nu})$.

(iii) $R_{\mu\nu} = \partial_{\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\sigma}{}_{\lambda\sigma} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma}$

(iv) $R_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_{\lambda}(\sqrt{-g}\Gamma^{\lambda}{}_{\mu\nu}) - \partial_{\mu}\partial_{\nu}\ln(\sqrt{-g}) - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma}$

The Einstein tensor is defined by

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

Exercise 1.19. Show that

$$\nabla_{\mu} G^{\mu\nu} = 0 \quad \text{and} \quad \nabla^{\mu} G_{\mu\nu} = 0$$

1.7 Einstein Field Equations

We know that Maxwell's equations govern how the electromagnetic field responds to the presence of charges and currents. Einstein field equations (EFEs or EFE for short) govern how the metric responds to energy and momentum. These equations can be postulated and then tested against experiments by plausible arguments. Derivation of EFEs will be done by

(i) Some informal reasoning by analogy and using the weak field limit, which is the approach adopted by Einstein himself.

(ii) Variational Calculus

For the informal arguments, we begin with the Poisson equation (as the Newtonian limit)

$$\nabla^2 \phi = 4\pi G \rho,$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$.

Example 1.20. Pointlike mass

$$\nabla^2 \phi = 0 \implies \phi = -\frac{GM}{r}$$

Regarding the Poisson equation, there are two important observations that will guide us to EFEs:

(i) The Poisson equation is a second-order differential equation.

(ii) The right-hand side is a measure of mass distribution. It can be represented by the energy-momentum tensor, denoted by $T_{\mu\nu}$. Let us try to quantify further:

- Suppose that $\Box g_{\mu\nu} \propto T_{\mu\nu}$, where $\Box = \nabla^\alpha \nabla_\alpha$. Then, $\Box g_{\mu\nu} = \nabla^\alpha \nabla_\alpha g_{\mu\nu} = 0$ by metric compatibility. So, this approach fails.

- Suppose that $R_{\mu\nu} = \kappa T_{\mu\nu}$, where κ is a constant. Imposing energy-momentum conservation, which is described mathematically by $\nabla_\mu T^{\mu\nu} = 0$, we get

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu \left(\frac{1}{\kappa} R^{\mu\nu}\right) = \frac{1}{\kappa} \nabla_\mu \left(G^{\mu\nu} + \frac{1}{2} R g^{\mu\nu}\right) = \frac{1}{2\kappa} g^{\mu\nu} \nabla_\mu R = \frac{1}{2\kappa} \nabla^\nu R$$

  However, $\nabla^\nu R \neq 0$ in general. This contradiction shows that this approach fails as well.

- Suppose that $G_{\mu\nu} = \kappa T_{\mu\nu}$, where κ is a constant. $G_{\mu\nu}$ is symmetric and obeys covariant conservation: $\nabla_\mu G^{\mu\nu} = 0$. Let us consider a perfect fluid to investigate this situation further. The energy-momentum tensor of a perfect fluid is given by $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$. In the Newtonian limit, the gravitational field is weak, the metric can be considered stationary (time-independent), and we have slow-moving bodies so that $p \approx 0$ and the energy-momentum tensor reduces to $T_{\mu\nu} = \rho u_\mu u_\nu$. In the rest frame of the fluid,

$$u_\mu = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) = (1, 0, 0, 0)$$

  Hence, the only non-zero component of the energy-momentum tensor is $T_{00} = \rho$. Assume that

$$g_{00} = -1 + h_{00} \implies g^{00} = -1 - h^{00},$$

  where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$. Then,

$$T = g^{\mu\nu} T_{\mu\nu} = g^{00} T_{00} \approx -\rho$$

  Contracting $G_{\mu\nu} = \kappa T_{\mu\nu}$, we get

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu}$$

$$R - \frac{1}{2} R(4) = \kappa T \implies R = -\kappa T$$

  Consider $\mu = \nu = 0$:

$$R_{00} - \frac{1}{2} R g_{00} = \kappa T_{00}$$

$$R_{00} + \frac{1}{2} \kappa T(-1 + h_{00}) = \kappa \rho$$

  Since $|h_{\mu\nu}| \ll 1$, we can ignore h_{00} and get

$$R_{00} = \kappa \rho + \frac{1}{2} \kappa T = \kappa \rho - \frac{1}{2} \kappa \rho = \frac{\kappa \rho}{2}$$

On the other hand,

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = R^{0}{}_{000} + R^{i}{}_{0i0} = R^{i}{}_{0i0}$$

since $R_{0000} = -R_{0000} = 0$ by the symmetry of the Riemann tensor. Note that

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}{}_{00} - \partial_0 \Gamma^{i}{}_{j0} + \Gamma^{i}{}_{j\lambda}\Gamma^{\lambda}{}_{00} - \Gamma^{i}{}_{0\lambda}\Gamma^{\lambda}{}_{j0}$$

The second term is zero since the metric is stationary and the last two terms are zero since they are second-order in $h_{\mu\nu}$. Thus,

$$R^{i}{}_{0j0} \approx \partial_j \Gamma^{i}{}_{00}$$

Accordingly, we obtain

$$R_{00} = R^{i}{}_{0i0} \approx \partial_i \Gamma^{i}{}_{00} = \partial_i \left(\frac{1}{2} g^{i\lambda} (\partial_0 g_{0\lambda} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \right) = -\frac{1}{2} \partial_i (g^{i\lambda} \partial_\lambda g_{00})$$

Using $g_{00} \approx h_{00}$, we have

$$R_{00} = -\frac{1}{2} \partial_i (g^{i\lambda} \partial_\lambda h_{00}) = -\frac{1}{2} \delta^{ij} \partial_i \partial_j h_{00}$$

$$\implies R_{00} = -\frac{1}{2} \nabla^2 h_{00}$$

Combining with $R_{00} = \frac{1}{2}\kappa\rho$, we finally obtain

$$\nabla^2 h_{00} = -\kappa\rho$$

Taking $h_{00} = -2\phi$, we find that $\kappa = 8\pi G$ gives the Poisson equation.

Lightlike, timelike, and spacelike vectors

(The metric signature is $(-, +, +, +)$)

(i) v^μ is said to be a lightlike (null) vector if $g_{\mu\nu} v^\mu v^\nu = 0$

(ii) v^μ is said to be a timelike vector if $g_{\mu\nu} v^\mu v^\nu < 0$. Timelike vectors are associated with trajectories of massive objects.

(iii) v^μ is said to be a spacelike (null) vector if $g_{\mu\nu} v^\mu v^\nu > 0$. Spacelike vectors are associated with bodies moving faster than light.

[INSERT FIGURE]

Remark. If $g_{\mu\nu} v^\mu w^\nu = 0$, then

(i) If v^μ is a timelike vector, then w^ν is a spacelike vector and vice versa.

(ii) If v^μ is a null vector, then there exists w^ν which is also a null vector.

1.8 Killing Vectors and Symmetries

Finding a metric describing a physical situation with perfect precision is impossible. Usually, one makes use of symmetries or approximation techniques appropriate for the problem at hand. Let us make this clearer with an example.

Example 1.21. Geometry outside a star

(i) $T_{\mu\nu} = 0 \implies G_{\mu\nu} = 0$

(ii) 10 second-order nonlinear field equations for metric

To overcome this complexity, we use spherical symmetry and the weak field limit.

1.8.1 Isometries

A manifold M has a symmetry if the geometry is invariant under a certain transformation mapping M to itself. In other words, the metric remains the same from one point to another under such transformations.

Definition 1.22. Symmetries of the metric are called isometries.

Example 1.23. Flat Minkowski space

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

The metric does not change under

(i) Translations: $x^\mu \longrightarrow x^\mu + a^\mu, \quad a^\mu =$ constant vector.

(ii) Lorentz transformations (Lorentz boosts): $x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$, where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix.

(iii) In Cartesian coordinates, $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Then, $\partial_\alpha \eta_{\mu\nu} = 0$.

Exercise 1.24. Show that the Minkowski metric does not change under Lorentz transformations.

1.8.2 Killing Equation

If a displacement ϵk^μ, where $|\epsilon| \ll 1$, on a manifold generates an isometry, then k^μ is called a Killing vector. Under this isometry, we have

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + \epsilon k^\mu \\ g_{\mu\nu}(x) &\longrightarrow g'_{\mu\nu}(x') \end{aligned}$$

Then,

$$\begin{aligned}
g_{\mu\nu}(x) &= g'_{\mu\nu}(x') \\
&= \frac{\partial(x^\alpha + \epsilon k^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta + \epsilon k^\beta)}{\partial x^\nu} g_{\alpha\beta}(x+\epsilon k) \\
&= (\delta^\alpha_\mu + \epsilon\partial_\mu k^\alpha)(\delta^\beta_\nu + \epsilon\partial_\nu k^\beta)(g_{\alpha\beta}(x) + \epsilon k^\sigma \partial_\sigma g_{\alpha\beta}(x)) \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta}(x) + \epsilon\partial_\mu k^\alpha \delta^\beta_\nu g_{\alpha\beta}(x) + \epsilon\delta^\alpha_\mu \partial_\nu k^\beta g_{\alpha\beta}(x) + \epsilon\delta^\alpha_\mu\delta^\beta_\nu k^\sigma\partial_\sigma g_{\alpha\beta} + O(\epsilon^2) \\
&= g_{\mu\nu}(x) + \epsilon g_{\alpha\nu}(x)\partial_\mu k^\alpha + \epsilon g_{\mu\beta}\partial_\nu k^\beta + \epsilon k^\sigma\partial_\sigma g_{\mu\nu}(x) + O(\epsilon^2)
\end{aligned}$$

Ignoring higher-order terms (namely $O(\epsilon^2)$), we obtain

$$k^\sigma\partial_\sigma g_{\mu\nu} + g_{\alpha\nu}\partial_\mu k^\alpha + g_{\mu\beta}\partial_\nu k^\beta = 0$$

Definition 1.25. $\mathcal{L}_k g_{\mu\nu} \coloneqq k^\sigma\partial_\sigma g_{\mu\nu} + g_{\alpha\nu}\partial_\mu k^\alpha + g_{\mu\beta}\partial_\nu k^\beta$ is called the Lie derivative of the metric along the vector k.

Then, an isometry can be equivalently defined as a transformation that is infinitesimally generated by a Killing vector field, which is a vector field that Lie transports the metric.

Lie derivative represents the change of a tensor along a flow defined by a vector field:

$$\begin{aligned}
\mathcal{L}_X f &= X^\alpha\partial_\alpha f = X[f] \\
\mathcal{L}_X Y^\mu &= [X,Y]^\mu = X^\alpha\partial_\alpha Y^\mu + Y^\alpha\partial_\alpha X^\mu \\
\mathcal{L}_X T_{\mu\nu} &= X^\alpha\partial_\alpha T_{\mu\nu} + T_{\mu\alpha}\partial_\nu X^\alpha + T_{\alpha\nu}\partial_\mu X^\alpha
\end{aligned}$$

Exercise 1.26. Show that for the Levi-Civita connection, the Killing equation reads as

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0,$$

or equivalently

$$\partial_\mu k_\nu + \partial_\nu k_\mu - 2\Gamma^\lambda_{\mu\nu} k_\lambda = 0$$

Example 1.27. Let the metric be time independent, i.e., $\partial_0 g_{\mu\nu} = 0$. Show that $k^\mu = (1,0,0,0)$ is a Killing vector.

Solution. We have $k_\mu = g_{\mu\nu}k^\nu = g_{\mu 0}$. Then,

$$\begin{aligned}
\nabla_\mu k_\nu + \nabla_\nu k_\mu &= \partial_\mu k_\nu + \partial_\nu k_\mu - 2k_\sigma\Gamma^\sigma_{\mu\nu} \\
&= \partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - 2\frac{1}{2}k_\sigma g^{\sigma\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\
&= \partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - k^0(\partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}) \\
&= 0
\end{aligned}$$

Exercise 1.28. Show that the commutator of two linearly independent Killing vectors $k^{(1)}$ and $k^{(2)}$ is also a Killing vector:

$$X^\mu \coloneqq [k^{(1)}, k^{(2)}]^\mu = k^{(2)\alpha}\nabla_\alpha k^{(1)\mu} - k^{(1)\alpha}\nabla_\alpha k^{(2)\mu}$$

We showed that $\partial_0 g_{\mu\nu} = 0 \implies k^\mu = (1,0,0,0) = \partial_t$: timelike Killing vector.

Definition 1.29. A spacetime having a Killing vector ∂_t is called stationary.

Exercise 1.30. Suppose $\partial_i g_{\mu\nu} = 0$, where $i = \overline{0,3}$. Show that there exists a Killing vector k^μ having the only non-zero i-th entry.

Consequence of Exercise 1.28: Suppose that there are only n linearly independent Killing vectors $k^{(i)\mu}$, $i = \overline{1,n}$. Then, the commutator of these is also a Killing vector, and it must be a linear combination of these n Killing vectors. In coordinate independent form, we can write

$$[k^{(i)}, k^{(j)}] = \sum_{l=1}^{n} a^{ij}{}_l k^{(l)}, \quad i,j = \overline{1,n}$$

or equivalently

$$k^{(j)\alpha} \nabla_\alpha k^{(i)\mu} - k^{(i)\alpha} \nabla_\alpha k^{(j)\mu} = \sum_{l=1}^{n} a^{ij}{}_l k^{(l)\mu}, \quad i,j = \overline{1,n}$$

Remark. Let $k^{(1)}$ and $k^{(2)}$ be Killing vector fields. Then,

(i) A linear combination, i.e. $ak^{(1)} + bk^{(2)}$ with $a, b \in \mathbb{R}$ is a Killing vector.

(ii) The Lie bracket of $k^{(1)}$ and $k^{(2)}$ is also a Killing vector. Hence, all Killing vector fields form a Lie algebra of symmetry operations on a manifold.

Exercise 1.31. Let $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ be the standard line element of 2-sphere $\mathbb{S}^2$. Show that Killing equations read as

(i) $\partial_\theta k_\theta + \partial_\theta k_\theta = 0$

(ii) $\partial_\phi k_\phi + \partial_\phi k_\phi + 2\sin\theta\cos\theta\, k_\theta = 0$

(iii) $\partial_\theta k_\phi + \partial_\phi k_\theta - 2\cot\theta\, k_\phi = 0$

Then, solve them to obtain the Killing vectors.

Exercise 1.32. Find the Killing vectors of $(\mathbb{R}^2, \delta)$, i.e., 2-dimensional Euclidean space. Show that two of these correspond to translations, and the third one represents a rotation.

Exercise 1.33. Find the Killing vectors of $(\mathbb{R}^4, \eta)$, i.e., 4-dimensional flat Minkowski spacetime.

Definition 1.34. A space of dimension m that admits $m(m+1)/2$ Killing vectors is called maximally symmetric.

Example 1.35. 4-dimensional Minkowski spacetime: 4 translations, 3 boosts, 3 rotations. For m-dimensional Minkowski spacetime, we have

(i) m translations

(ii) $(m-1)$ boosts

(iii) $(m-1)(m-2)/2$ rotations

Example 1.36. Let $\partial_\gamma g_{\alpha\lambda} = 0$, which means we have a Killing vector k^γ. How does $p^\gamma k_\gamma$ evolve along a geodesic wordline, namely $\frac{D}{d\tau}(p^\alpha k_\alpha) =$? We first write the geodesic equation in a different form. We start by contracting $p^\alpha \nabla_\alpha p^\mu = 0$ with $g_{\mu\gamma}$, which yields

$$\begin{aligned} p^\alpha \nabla_\alpha p_\gamma &= 0 \\ p^\alpha(\partial_\alpha p_\gamma - \Gamma^\lambda{}_{\alpha\gamma} p_\lambda) &= 0 \\ p^\alpha \partial_\alpha p_\gamma - \Gamma^\lambda{}_{\alpha\gamma} p^\alpha p_\lambda &= 0 \\ m\frac{dp_\gamma}{d\tau} = \frac{1}{2} g^{\lambda\sigma}(\partial_\alpha g_{\sigma\gamma} + \partial_\gamma g_{\alpha\sigma} - \partial_\sigma g_{\alpha\gamma}) p^\alpha p_\lambda = \frac{1}{2}\partial_\gamma g_{\lambda\sigma} p^\alpha p^\sigma &= 0 \\ \implies \frac{dp_\gamma}{d\tau} &= 0 \end{aligned}$$

Then,

$$m\frac{D}{d\tau}(p^\gamma k_\gamma) = \left(m\frac{D}{d\tau}p^\gamma\right) k_\gamma + mp^\gamma \frac{D}{d\tau} k_\gamma$$

The first term is 0 by geodesic equation. Hence, we get

$$\begin{aligned} m\frac{D}{d\tau}(p^\gamma k_\gamma) &= mp^\gamma \frac{D}{d\tau} k_\gamma \\ &= mp^\gamma u^\alpha \nabla_\alpha k_\gamma \\ &= p^\gamma p^\alpha \nabla_\alpha k_\gamma \\ &= p^\gamma p^\alpha (\nabla_{(\alpha} k_{\gamma)} + \nabla_{[\alpha} k_{\gamma]}) \\ &= p^\gamma p^\alpha \nabla_{(\alpha} k_{\gamma)} \\ &= 0 \quad \text{by Killing equation} \end{aligned}$$

Thus, we obtain

$$p^\gamma k_\gamma = \text{constant}$$

1.9 Tensor Densities

Tensor densities are objects that transform almost like a tensor under coordinate transformations, off by a factor of the determinant of the coordinate transformation matrix to some power called the weight of the tensor density. More precisely, a set of quantities $T^{\nu_1 \ldots \nu_k}_{\mu_1 \ldots \mu_l}$ is said to be a tensor density of weight W if it transforms as

$$T'^{\nu_1 \ldots \nu_k}_{\mu_1 \ldots \mu_l} = \left|\frac{\partial x'}{\partial x}\right|^W \frac{\partial x'^{\nu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\nu_k}}{\partial x^{\alpha_k}} \frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\beta_l}}{\partial x'^{\mu_l}} T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l}$$

In this course, important tensor densities are

(i) Levi-Civita symbol

$$\tilde{\epsilon}_{\alpha\beta\gamma\delta} = \begin{cases} +1, & \text{even permutations of 0123} \\ -1, & \text{odd permutations of 0123} \\ 0, & \text{otherwise} \end{cases}$$

It can be shown that for any 4×4 matrix $M^\alpha{}_\mu$,

$$\tilde{\epsilon}_{\alpha\beta\gamma\delta} M^\alpha{}_\mu M^\beta{}_\nu M^\gamma{}_\rho M^\delta{}_\sigma = \tilde{\epsilon}'_{\mu\nu\rho\sigma} |M|$$

where $|M|$ is the determinant of M. Let $M^\alpha{}_\mu$ be the transformation matrix. Then,

$$M^\alpha{}_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}, \quad (M^\alpha{}_\mu)^{-1} = \frac{\partial x'^\mu}{\partial x^\alpha}$$

$$\begin{aligned} \tilde{\epsilon}'_{\mu\nu\rho\sigma} &= \frac{1}{|M|} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\rho} \frac{\partial x^\delta}{\partial x'^\sigma} \tilde{\epsilon}_{\alpha\beta\gamma\delta} \\ &= J \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\rho} \frac{\partial x^\delta}{\partial x'^\sigma} \tilde{\epsilon}_{\alpha\beta\gamma\delta} \end{aligned}$$

So, the Levi-Civita symbol is a tensor density of weight +1.

(ii) Metric determinant

$$\begin{gathered} g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \\ \det\left(g'_{\alpha\beta}\right) = \det\left(\frac{\partial x^\mu}{\partial x'^\alpha}\right) \det\left(\frac{\partial x^\nu}{\partial x'^\beta}\right) \det(g_{\mu\nu}) = J^{-2} \det(g_{\mu\nu}) \\ \implies g' = J^{-2} g \implies \sqrt{-g'} = J^{-1} \sqrt{-g} \end{gathered}$$

(iii) Volume element In n-dimensions, the volume element is given by

$$\begin{aligned} d^n x &= dx^0 \wedge \cdots \wedge dx^{n-1} \\ &= \frac{1}{n!} \tilde{\epsilon}_{\mu_0 \ldots \mu_{n-1}} dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_{n-1}} \\ &= \tilde{\epsilon}_{\mu_0 \ldots \mu_{n-1}} \frac{\partial x^{\mu_0}}{\partial x'^{\nu_0}} \cdots \frac{\partial x^{\mu_{n-1}}}{\partial x'^{\nu_{n-1}}} dx'^{\nu_0} \wedge \cdots \wedge dx'^{\nu_{n-1}} \\ &= J^{-1} d^n x' \end{aligned}$$

Hence, $d^n x' = J d^n x$ and the volume element is a tensor density of weight +1.

Combining $\sqrt{-g}$ and $d^n x$, we can introduce an invariant volume element: $dV = \sqrt{-g} d^n x$.

$$dV' = \sqrt{-g'} d^n x' = \sqrt{-g'} dx'^0 \wedge \cdots \wedge dx'^{n-1} = (J^{-1}\sqrt{-g})(J d^n x) = \sqrt{-g}\, d^n x = dV$$

Example 1.37. Consider Euclidean space in Cartesian and spherical coordinates

$$\begin{gathered} ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \\ g_{\mu\nu} = \text{diag}(1,1,1), \; g'_{\mu\nu} = \text{diag}(1, r^2, r^2 \sin^2\theta) \\ dV = dx\, dy\, dz, \; dV' = r^2 \sin\theta\, dr\, d\theta\, d\phi \end{gathered}$$

Example 1.38. Minkowski space

$$\begin{gathered} ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \\ g_{\mu\nu} = \text{diag}(-1,1,1,1), \; g'_{\mu\nu} = \text{diag}(-1, 1, r^2, r^2 \sin^2\theta) \\ dV = dt\, dx\, dy\, dz, \; dV' = r^2 \sin\theta\, dt\, dr\, d\theta\, d\phi \end{gathered}$$

Exercise 1.39. Show that $\int S\sqrt{-g}\, d^n x$ is invariant under coordinate transformation, where S is a scalar.

1.10 Gauss's Theorem in Curved Spaces

In a flat space, one has [INSERT FIGURE]

$$\int \partial_\mu B^\mu \, dV = \oint_{\partial V} B^\mu \, dS_\mu = \oint_{\partial V} B^\mu n_\mu \, dS$$

Exercise 1.40. Show that the Christoffel connection ${\Gamma^\mu}_{\mu\beta}$ can be written as

$${\Gamma^\mu}_{\mu\beta} = \frac{1}{2} g^{\mu\nu} \partial_\beta g_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\beta \sqrt{|g|} = \partial_\beta \ln\left(\sqrt{|g|}\right)$$

By the above exercise,

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + {\Gamma^\mu}_{\mu\alpha} A^\alpha = \partial_\mu A^\mu + \frac{1}{\sqrt{-g}} (\partial_\alpha \sqrt{-g}) A^\alpha = \frac{1}{\sqrt{-g}} \partial_\mu (A^\mu \sqrt{-g})$$

This gives

$$\int \nabla_\mu A^\mu \, dV = \int \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu) \sqrt{-g} \, d^n x = \int \partial_\mu (\sqrt{-g} A^\mu) \, d^n x = \oint_{\partial V} A^\mu \, dS_\mu,$$

which represents the flux of the vector field across the boundary, where $dS_\mu = n_\mu \sqrt{h} \, dS$ and h is the determinant of the metric of the boundary $h_{\mu\nu}$. If the flux is 0, one has

$$\int_V \partial_\mu (A^\mu \sqrt{-g}) \, d^n x = 0$$

$$\int_V \partial_0 (A^0 \sqrt{-g}) \, d^n x = - \int_V \partial_i (A^i \sqrt{-g}) \, d^n x$$

Setting $n = 4$, we obtain (?)

$$\int \partial_0 (A^0 \sqrt{-g}) \, d^4 x = - \int \partial_i (A^i \sqrt{-g}) \, d^4 x$$

$$\partial_0 \int A^0 \sqrt{-g} \, d^3 x = - \int \partial_i (A^i \sqrt{-g}) \, d^3 x = - \oint A^i n_i \, dS^{(2)}$$

If the flux across the 2-D boundary is 0,

$$\partial_0 \int A^0 \sqrt{-g} \, d^3 x = 0 \implies \int A^0 \sqrt{-g} \, d^3 x = \text{constant}$$

The other way to show this conservation is to start with $\nabla_\mu A^\mu = 0$ and integrate over the 3-D volume.

Let us try to derive a similar conservation law for tensors. We have

$$\nabla_\mu A^{\mu\nu} = \partial_\mu A^{\mu\nu} + {\Gamma^\mu}_{\mu\lambda} A^{\lambda\nu} + {\Gamma^\nu}_{\mu\lambda} A^{\mu\lambda} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^{\mu\nu}) + {\Gamma^\nu}_{\mu\lambda} A^{\mu\lambda}$$

Then,

$$\begin{aligned} \nabla_\mu A^{\mu\nu} &= \int \frac{1}{\sqrt{-g}} \partial_\mu (A^{\mu\nu} \sqrt{-g}) \sqrt{-g} \, d^n x + \int {\Gamma^\nu}_{\mu\lambda} A^{\mu\lambda} \sqrt{-g} \, d^n x \\ &= \oint A^{\mu\nu} n_\nu \, dS^{(n-1)} + \int {\Gamma^\nu}_{\mu\lambda} A^{\mu\lambda} \sqrt{-g} \, d^n x \end{aligned}$$

We think of the first term as the flux of $A^{\mu\nu}$. If $\nabla_\mu A^{\mu\nu} = 0$, one has (?)

$$\begin{aligned}
\int \partial_\mu (A^{\mu\nu}\sqrt{-g})\, d^n x &= -\int \Gamma^\nu{}_{\mu\lambda} A^{\mu\lambda}\sqrt{-g}\, d^n x \\
\int \partial_0 (A^{0\nu}\sqrt{-g})\, d^n x &= -\int \partial_i (A^{i\nu}\sqrt{-g})\, d^n x - \int \Gamma^\nu{}_{\mu\lambda} A^{\mu\lambda}\sqrt{-g}\, d^n x \\
\partial_0 \int A^{0\nu}\sqrt{-g}\, d^n x &= -\int \partial_i (A^{i\nu}\sqrt{-g})\, d^n x - \partial_0 \int \Gamma^\nu{}_{\mu\lambda} A^{\mu\lambda}\sqrt{-g}\, d^n x \\
\partial_0 \int A^{0\nu}\sqrt{-g}\, d^n x &= -\oint A^{i\nu}\, dS_\nu - \partial_0 \int \Gamma^\nu{}_{\mu\lambda} A^{\mu\lambda}\sqrt{-g}\, d^n x
\end{aligned}$$

In 4-D (?)

$$\partial_0 \int A^{0\nu}\sqrt{-g}\, d^3 x = -\int \Gamma^\nu{}_{\mu\lambda} A^{\mu\lambda}\sqrt{-g}\, d^3 x$$

Remark.

(i) If $A^{\mu\lambda}$ is antisymmetric (like the Maxwell tensor), then we have a kind of conservation law.

(ii) If $A^{\mu\lambda}$ is symmetric (like the energy-momentum tensor), the extra term does not vanish, and there should be a term for the Einstein tensor to cancel out this extra term to have a conservation.

(iii) If $A^{\mu\lambda}$ has no special symmetry, then the extra term remains and there will be no conservation.

1.11 Action Principle in GR

Consider the action

$$S = \int \mathcal{L}\, d^4 x = \int L_m \sqrt{-g}\, d^4 x,$$

where $\mathcal{L}$ is called the Lagrangian density and L_m is called the Lagrangian. Since L_m is a scalar, the action is invariant under coordinate transformations.

Now, consider a local field theory (2nd and higher-order derivatives of the field do not exist). Then, the action can be written as

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4 x$$

Our aim is to find the equations of motion for the scalar field ϕ. We demand that the action is invariant under the small variations of the field. This can be expressed as follows:

$$\begin{aligned}
\phi(x) &\longrightarrow \tilde{\phi}(x) = \phi(x) + \delta\phi(x) \\
\partial_\mu \phi(x) &\longrightarrow \partial_\mu \tilde{\phi}(x) = \partial_\mu \phi(x) + \delta(\partial_\mu \phi(x))
\end{aligned}$$

Using these relations, we can easily show that δ and ∂ commute:

$$\delta(\partial_\mu \phi) = \partial_\mu \tilde{\phi} - \partial_\mu \phi = \partial_\mu (\tilde{\phi} - \phi) = \partial_\mu (\delta\phi)$$

Then, the variation of the action is

$$\begin{aligned}
\delta S &= \int_V \delta\mathcal{L}\, d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta \phi) \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi\, d^4x + \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi\, d^4x + \oint \frac{\mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\, dS_\mu
\end{aligned}$$

Demanding that $\delta\phi\big|_{\partial V} = 0$ and $\delta S = 0$ for every variation, we obtain

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

The above equation is the Euler-Lagrange equations for a scalar field.

Exercise 1.41. Show that $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ and $\mathcal{L}' = \mathcal{L}(\phi, \partial_\mu\phi) + \partial_\mu Q^\mu(\phi)$ give the same equations of motion.

1.11.1 Einstein-Hilbert Action

In GR, the dynamical field is the metric itself (GR is a background-independent theory). Hence, the Lagrangian is a scalar function of the metric and its derivatives. Thus, one looks for a scalar that is derived from the metric. The Ricci scalar is the simplest non-trivial scalar in the metric and its derivatives (1st and 2nd order). Accordingly, the Einstein-Hilbert action is given by

$$S_{EH} = \int R\sqrt{-g}\, d^4x$$

Exercise 1.42. Show that S_{EH} is invariant under coordinate transformations.

We have

$$\begin{aligned}
\delta S_{EH} &= \int \left(\delta R\sqrt{-g} + R\delta\sqrt{-g} \right) d^4x \\
&= \int \left(\delta(g^{\mu\nu}R_{\mu\nu})\sqrt{-g} + R\delta\sqrt{-g} \right) d^4x \\
&= \int \left((\delta g^{\mu\nu})R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}(\delta R_{\mu\nu})\sqrt{-g} + R\delta\sqrt{-g} \right) d^4x \\
&= \delta S_1 + \delta S_2 + \delta S_3
\end{aligned}$$

The variation of the metric is defined by

$$\begin{aligned}
g_{\mu\nu} &\longrightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \\
\partial_\alpha g_{\mu\nu} &\longrightarrow \partial_\alpha \tilde{g}_{\mu\nu} = \partial_\alpha g_{\mu\nu} + \delta(\partial_\alpha g_{\mu\nu})
\end{aligned}$$

Then,

$$\delta(\partial_\alpha g_{\mu\nu}) = \partial_\alpha \tilde{g}_{\mu\nu} - \partial_\alpha g_{\mu\nu} = \partial_\alpha(\tilde{g}_{\mu\nu} - g_{\mu\nu}) = \partial_\alpha(\delta g_{\mu\nu})$$

The variation of the inverse metric can be written in terms of the metric by using $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$:

$$\begin{gathered} \delta(g^{\mu\rho}g_{\rho\nu}) = \delta(\delta^\mu_\nu) = 0 \\ (\delta g^{\mu\rho})g_{\rho\nu} + g^{\mu\rho}\delta g_{\rho\nu} = 0 \\ (\delta g^{\mu\rho})g_{\rho\nu} = -g^{\mu\rho}\delta g_{\rho\nu} \\ (\delta g^{\mu\rho})g_{\rho\nu}g^{\nu\lambda} = -g^{\nu\lambda}g^{\mu\rho}\delta g_{\rho\nu} \\ \delta g^{\mu\lambda} = -g^{\nu\lambda}g^{\mu\rho}\delta g_{\rho\nu} \\ \implies \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \end{gathered}$$

Then,

$$\delta S_1 = \int -(g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma})R_{\mu\nu}\sqrt{-g}\, d^4x = -\int R^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x$$

To find δS_2, we work in an inertial coordinate system. Then,

$$\begin{gathered} \Gamma^\sigma{}_{\mu\nu}(p) = 0 \\ R^\sigma{}_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma{}_{\mu\rho} - \partial_\rho \Gamma^\sigma{}_{\mu\nu} \\ \delta R^\sigma{}_{\mu\nu\rho} = \delta(\partial_\nu\Gamma^\sigma{}_{\mu\rho}) - \delta(\partial_\rho\Gamma^\sigma{}_{\mu\nu}) = \partial_\nu(\delta\Gamma^\sigma{}_{\mu\rho}) - \partial_\rho(\delta\Gamma^\sigma{}_{\mu\nu}) = \nabla_\nu(\delta\Gamma^\sigma{}_{\mu\rho}) - \nabla_\rho(\delta\Gamma^\sigma{}_{\mu\nu}) \end{gathered}$$

Exercise 1.43. Show that $\delta\Gamma^\sigma{}_{\mu\nu}$ is a tensor, where the variation is defined by

$$\Gamma^\sigma{}_{\mu\nu} \longrightarrow \tilde{\Gamma}^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu} + \delta\Gamma^\sigma{}_{\mu\nu}.$$

By the above exercise, we have a tensorial equation that is applicable for any point and any coordinate system.

$$\delta R^\sigma{}_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^\sigma{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\sigma{}_{\mu\rho})$$

Setting $\sigma = \rho$, we obtain the variation of the Ricci tensor:

$$\delta R_{\mu\nu} = \nabla_\rho(\delta\Gamma^\rho{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho{}_{\mu\rho})$$

Then,

$$\begin{aligned} \delta S_2 &= \int g^{\mu\nu}\left(\nabla_\rho(\delta\Gamma^\rho{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho{}_{\mu\rho})\right)\sqrt{-g}\, d^4x \\ &= \int \left(\nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu}) - g^{\mu\nu}\nabla_\nu(\delta\Gamma^\sigma{}_{\mu\sigma})\right)\sqrt{-g}\, d^4x \\ &= \int \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\sigma{}_{\mu\sigma})\sqrt{-g}\, d^4x \\ &= \oint (g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\sigma{}_{\mu\sigma})\, dS_\rho \end{aligned}$$

Using $\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$, one has $\delta\Gamma^\rho{}_{\mu\nu} = 0$ due to the following variations

$$\begin{gathered} \delta g^{\mu\lambda}\big|_{\partial V} = 0 \\ \delta(\partial_\lambda g_{\mu\nu})\big|_{\partial V} = 0 \end{gathered}$$

Thus, we have $\delta S_2 = 0$.

Exercise 1.44. Show that for any invertible matrix M, $\ln(\det(M)) = \mathrm{tr}(\ln(M))$.

Using the exercise, we have

$$\delta \ln(\det(M)) = \frac{\delta(\det(M))}{\det(M)}$$
$$\delta\, \mathrm{tr}(\ln(M)) = \mathrm{tr}(\delta \ln(M)) = \mathrm{tr}\left(M^{-1}\delta M\right)$$

where we used

$$\delta M = \delta\left(e^{\ln(M)}\right) = e^{\ln(M)}\delta(\ln(M)) = M\delta(\ln(M))$$
$$\delta(\ln(M)) = M^{-1}\delta M$$

Now, letting $\det(M) =: g$, we obtain

$$\frac{\delta g}{g} = \mathrm{tr}\left(g^{\mu\rho}\delta g_{\rho\nu}\right) = \delta^{\nu}_{\mu} g^{\mu\rho}\delta g_{\rho\nu} = g^{\nu\rho}\delta g_{\rho\nu}$$
$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}$$
$$\delta\sqrt{-g} = -\frac{1}{2}\frac{\delta g}{\sqrt{-g}} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$$

Hence,

$$\delta S_3 = \frac{1}{2}\int R g^{\mu\nu}(\delta g_{\mu\nu})\sqrt{-g}\, d^4x$$

Combining our results, we have

$$\delta S = -\int\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu}\sqrt{-g}\, d^4x = 0$$
$$\implies R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 0$$

In the presence of matter fields, the total action is of the form

$$S = \frac{1}{2\kappa}S_{EH} + S_M$$

Then,

$$\begin{aligned}\delta S &= \frac{1}{2\kappa}\delta S_{EH} + \delta S_M \\ &= -\frac{1}{2\kappa}\int G^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x + \delta S_M\end{aligned}$$

Let the matter action be

$$S_M = \int L_M\sqrt{-g}\, d^4x = \int \mathcal{L}\, d^4x$$

Then, variation with respect to the metric reads as

$$\begin{aligned}\delta S_M &= \int \delta(L_M\sqrt{-g})\, d^4x \\ &= \int\left(\frac{\partial L_M}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\sqrt{-g} + L_M\delta\left(\sqrt{-g}\right)\right)\, d^4x \\ &= \int\left(\frac{\partial L_M}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\sqrt{-g} + L_M\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}\right)\, d^4x \\ &= \frac{1}{2}\int\left(2\frac{\partial L_M}{\partial g_{\mu\nu}} + L_M g^{\mu\nu}\right)\delta g_{\mu\nu}\sqrt{-g}\, d^4x\end{aligned}$$

We define $T^{\mu\nu} := 2\frac{\partial L_M}{\partial g_{\mu\nu}} + L_M g^{\mu\nu}$, called the energy-momentum tensor. Observe that this tensor is symmetric by definition. Then,

$$\delta S_M = \frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\,d^4x$$

Hence,

$$\delta S = \int \left(-\frac{1}{2\kappa}G^{\mu\nu} + \frac{1}{2}T^{\mu\nu}\right)\delta g_{\mu\nu}\sqrt{-g}\,d^4x$$
$$\delta S = 0 \implies G^{\mu\nu} = \kappa T^{\mu\nu}$$

One finds by the weak field approximation that the constant κ is equal to $\frac{8\pi G}{c^4}$.

One can define the energy-momentum tensor using the Lagrangian density as follows

$$\begin{aligned}\delta S_M &= \int \mathcal{L}\,d^4x \\ &= \int \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\,d^4x \\ &= \frac{1}{2}\int \frac{2}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\sqrt{-g}\,d^4x \\ &= \frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\,d^4x\end{aligned}$$

So, we see that $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}$.

Since the Einstein tensor is covariantly conserved ($\nabla_\mu G^{\mu\nu} = 0$), the energy-momentum tensor must also be covariantly conserved. This can be shown by the invariance of the action under coordinate transformations. Let $x'^\mu = x^\mu + k^\mu$, where k^μ is an infinitesimal smooth vector. Invariance under coordinate transformations implies that $\delta S_M = 0$. Then,

$$\begin{aligned}g_{\mu\nu}(x) &= \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}g'_{\alpha\beta}(x') \\ &= (\delta^\alpha_\mu + \partial_\mu k^\alpha)(\delta^\beta_\nu + \partial_\nu k^\beta)g'_{\alpha\beta}(x') \\ &= g'_{\mu\nu}(x') + g'_{\mu\alpha}(x')\partial_\nu k^\alpha + g'_{\alpha\nu}(x')\partial_\mu k^\alpha + O(k^2)\end{aligned}$$

This gives

$$\begin{aligned}\delta g_{\mu\nu} &= g_{\mu\nu}(x) - g'_{\mu\nu}(x) \\ &= (g_{\mu\nu}(x) - g'_{\mu\nu}(x')) - (\overbrace{g'_{\mu\nu}(x)}^{g'_{\mu\nu}(x'-k)} - g'_{\mu\nu}(x')) \\ &= (g_{\mu\nu}(x) - g'_{\mu\nu}(x')) - (g'_{\mu\nu}(x') - k^\alpha\partial_\alpha g'_{\mu\nu}(x') - g'_{\mu\nu}(x') + O(k^2)) \\ &= (g_{\mu\nu}(x) - g'_{\mu\nu}(x')) + k^\alpha\partial_\alpha g'_{\mu\nu}(x') + O(k^2) \\ &= g'_{\mu\alpha}(x')\partial_\nu k^\alpha + g'_{\alpha\nu}(x')\partial_\mu k^\alpha + k^\alpha\partial_\alpha g'_{\mu\nu}(x') + O(k^2) \\ &= \nabla_\mu k_\nu + \nabla_\nu k_\mu + O(k^2)\end{aligned}$$

Then,

$$\begin{aligned}
\delta S_M &= \frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\,d^4x \\
&= \frac{1}{2}\int T^{\mu\nu}(\nabla_\mu k_\nu + \nabla_\nu k_\mu)\sqrt{-g}\,d^4x \\
&= \int (T^{\mu\nu}\nabla_\mu k_\nu)\sqrt{-g}\,d^4x \\
&= \int \left(\nabla_\mu(T^{\mu\nu}k_\nu) - (\nabla_\mu T^{\mu\nu})k_\nu\right)\sqrt{-g}\,d^4x \\
&= \int \nabla_\mu(T^{\mu\nu}k_\nu)\sqrt{-g}\,d^4x - \int (\nabla_\mu T^{\mu\nu})k_\nu\sqrt{-g}\,d^4x \\
&= \oint T^{\mu\nu}k_\nu\,dS_\mu - \int (\nabla_\mu T^{\mu\nu})k_\nu\sqrt{-g}\,d^4x
\end{aligned}$$

Since $\delta x^\mu\big|_{\partial V} = 0$, we have $k_\nu\big|_{\partial V} = 0$. This gives

$$\delta S_M = -\int \left(\nabla_\mu T^{\mu\nu}\right)k_\nu\sqrt{-g}\,d^4x = 0$$

As k_ν is arbitrary, we must have

$$\nabla_\mu T^{\mu\nu} = 0$$

Exercise 1.45. Using $S = \int \mathcal{L}\,d^4x = \int \mathcal{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\beta\partial_\alpha g_{\mu\nu})\,d^4x$, show that the Euler-Lagrange equations for vacuum, which are equivalent to $G^{\mu\nu} = 0$, read as

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} - \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma g_{\mu\nu})}\right) + \partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\partial_\sigma g_{\mu\nu})}\right) = 0$$

Exercise 1.46. Let

$$S_M = \int\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)\sqrt{-g}\,d^4x$$

be the action for a scalar field ϕ.

(i) Show that variation with respect to ϕ gives the following Euler-Lagrange equations

$$\frac{\partial L_M}{\partial\phi} - \nabla_\mu\left(\frac{\partial L_M}{\partial(\nabla_\mu\phi)}\right) = 0$$

By inserting the given form of L_M, show that the above reduces to

$$-\frac{dV}{d\phi} - \nabla_\mu(g^{\mu\nu}\nabla_\nu\phi) = 0$$

or

$$\Box\phi + \frac{dV}{d\phi} = 0,$$

where $\Box = \nabla^\mu\nabla_\mu = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

(ii) Show that the common choice of $V = \frac{1}{2}m^2\phi^2$ gives the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi = 0,$$

where m is the mass associated with the scalar field.

(iii) Show that the corresponding energy-momentum tensor is

$$T^{\phi}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi) \right)$$

Exercise 1.47. The action of electromagnetic theory in vacuum is given by

$$\begin{aligned} S_{EM} &= -\frac{1}{4\mu_0} \int g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} \sqrt{-g}\, d^4x \\ &= -\frac{1}{4\mu_0} \int g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} F^{\mu\nu} \sqrt{-g}\, d^4x \\ &= -\frac{1}{4\mu_0} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x \end{aligned}$$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. Show that variation with respect to the metric gives

$$T^{EM}_{\mu\nu} = -\frac{1}{\mu_0} \left(F_{\alpha\mu} F^{\alpha}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

So, EFEs with electromagnetic source read as

$$G_{\mu\nu} = -\frac{\kappa}{\mu_0} \left(F_{\alpha\mu} F^{\alpha}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

Exercise 1.48. Let the total action be

$$S = \int \left(\frac{1}{2\kappa} (R - 2\Lambda) + L_M \right) \sqrt{-g}\, d^4x,$$

where Λ is the cosmological constant. Show that variation with respect to the metric (or its inverse) gives the EFEs with cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Exercise 1.49. Let the geometric action be given by

$$S = \int f(R) \sqrt{-g}\, d^4x$$

Obtain the field equations by varying S with respect to the metric.

2 Cosmology

2.1 Hypersurfaces

In 4-D spacetime, a hypersurface is a 3-D submanifold that can be either timelike, spacelike, or null. A particular hypersurface Σ is selected by putting a restriction on the coordinates, i.e., $\phi(x^\alpha) = 0$, or by giving parametric equations of the form $x^\alpha = x^\alpha(y^i)$, where $i = 1, 2, 3$ and y^i are the intrinsic coordinates to Σ. [INSERT FIGURE]

Example 2.1. A 2-sphere in 3-D flat space is a hypersurface defined as

$$\phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0,$$

where R is the radius. Equivalently,

$$\begin{aligned} x &= R \sin\theta \cos\phi \\ y &= R \sin\theta \sin\phi \\ z &= R \cos\theta \end{aligned}$$

where θ, ϕ are intrinsic coordinates to Σ.

2.1.1 Normal Vectors to Hypersurfaces

The vector $\partial_\alpha \phi$ is normal to ϕ because the value of ϕ does not change over Σ. More generally, a unit normal to Σ can be defined as

$$n_\alpha n^\alpha = \varepsilon = \begin{cases} -1, & \Sigma \text{ is spacelike} \\ +1, & \Sigma \text{ is timelike} \end{cases}$$

We demand that ϕ increase in the direction of n_α. Namely, we require that $n^\alpha \partial_\alpha \phi > 0$.

$$n_\alpha = \frac{\partial_\alpha \phi}{\sqrt{g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi}} = \frac{\partial_\alpha \phi}{\sqrt{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}}$$

2.1.2 Induced Metrics on Hypersurfaces

The metric intrinsic to Σ is obtained by restricting the line element to displacements confined to Σ.

$$\begin{aligned} ds^2_\Sigma &= g_{\alpha\beta}\, dx^\alpha dx^\beta, \quad x^\alpha = x^\alpha(y^i) \\ &= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^i} dy^i \right) \left(\frac{\partial x^\beta}{\partial y^i} dy^j \right) \\ &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} dy^i dy^j \\ &= h_{ij}\, dy^i dy^j \end{aligned}$$

h_{ij} is the induced metric on Σ, known as the first fundamental form.

Exercise 2.2. Show that the induced metric h_{ij} is a

(i) scalar under transformations $x^\alpha \longrightarrow x'^\alpha$

(ii) tensor under transformations $y^i \longrightarrow y'^i$

2.2 Friedmann-Lemaître-Robertson-Walker (FLRW) Geometry

As an application of GR, we try to model our universe using the FLRW geometry. Just like the Schwarzschild or Kerr metrics, we have to consider some symmetries in order to determine the full metric describing the FLRW geometry. We first list some scales in the Universe to get a feeling for what we are dealing with.

(i) Solar system: $10^3 \sim 10^5$ AU

(ii) Galaxy (Milky Way): $\sim 10^{11}$ AU

(iii) Cluster of galaxies: $\sim 10^{12} - 10^{13}$ AU

(1 AU = average distance between the Earth and the Sun $\approx 1.496 \times 10^8$ km)

On local scales, matter distribution is irregular, i.e., inhomogeneous and anisotropic:

$$\rho(t, r, \theta\, \phi) = \rho(t, r', \theta', \phi')$$

On very large scales (much larger than the scale of galaxy clusters), there is good observational evidence for homogeneity and isotropy. At any given time t, we have

$$\rho(r, \theta, \phi) = \rho(r', \theta', \phi')$$

2.3 Cosmological Principles

The standard model of cosmology is based on two simplifying principles:

(i) Homogeneity: Galaxies are distributed uniformly across the Universe. Hence, the matter distribution is independent of the position. This is described by $\rho(r) \approx \rho(r')$. Mathematically, this means that we have a translational Killing vector: $\mathcal{L}_X g_{\alpha\beta} = 0$.

(ii) Isotropy: The distribution of galaxies is independent of the direction. In other words, there is no preferred direction in the Universe. Mathematically, this means that we have a rotational Killing vector: $\mathcal{L}_R g_{\alpha\beta} = 0$.

Remark. The Universe can be homogeneous but not isotropic, and vice versa. [INSERT FIGURE]

Remark. If there is perfect isotropy, then the Universe must be homogeneous as well. Consider two observers A and B seeing a perfect isotropy around them. [INSERT FIGURE]

2.3.1 Mathematical Formulation of Cosmological Principles

In Newtonian gravity, we have the notion of absolute time, and in SR, we have the well-defined notion of proper time for inertial frames. However, in GR, there is no notion of time that is valid globally. Relatively moving observers see the Universe differently. For this reason, according to which observer isotropy and homogeneity are meant should be made precise. To achieve this goal, we

(i) slice up the spacetime by introducing a series of nonintersecting spacelike hypersurfaces labeled by a parameter t. [INSERT FIGURE]

(ii) introduce the idealized notion of fundamental observers, observers lying on spacelike hypersurfaces and not moving relative to the cosmological fluid (also known as no peculiar motion or no peculiar velocity). As postulated by Weyl, the worldlines of these observers never cross except for singularities, which are divided into two types: past singularities (divergence) and future singularities (convergence). This is known as threading the spacetime. [INSERT FIGURE]

2.3.2 Synchronous and Comoving Coordinates

(i) The parameter t can be taken to be the proper time of galaxies, called synchronous time.

(ii) Consider spatial coordinates (x^1, x^2, x^3) on each Σ_t. Along any worldline of galaxies, these coordinates are fixed, called comoving coordinates. [INSERT FIGURE]

Hence, by this foliation and adopting synchronous coordinates, the spacetime metric can be written as

$$ds^2 = -c^2dt^2 + g_{ij}\, dx^i dx^j, \quad i, j = \overline{1, 3}$$

where $g_{ij} = g_{ij}(t, x^1, x^2, x^3)$.

Let us show that the above form of the metric incorporates the mentioned properties for synchronous coordinates.

(i) Let $x^\mu(\tau)$ be the worldline of a galaxy.

$$\begin{aligned} x^\mu(t) &= (x^0, x^1, x^2, x^3) \\ &= (\tau, x^1 = \text{constant}, x^2 = \text{constant}, x^3 = \text{constant}) \end{aligned}$$

Hence,

$$ds^2 = -c^2d\tau^2 = -c^2dt^2 \implies t = \tau$$

So, we see that t can be taken as the proper time of the galaxy.

(ii) Four velocity u^μ has the form

$$u^\mu = \frac{dx^\mu}{d\tau} = (1, 0, 0, 0) = \delta_0^\mu$$

Let a^μ be any vector along Σ_t. Then, $a^\mu = (0, a^1, a^2, a^3)$ and we have

$$g_{\mu\nu} u^\mu a^\nu = u^\mu a_\mu = 0$$

So, we see that the worldline is normal to the hypersurface Σ_t.

(iii) Let $x^\mu(\tau)$ be the worldline of a galaxy again. Then,

$$\begin{aligned}
x^\mu(\tau) = (\tau, x^1 = \text{constant}, x^2 = \text{constant}, x^3 = \text{constant}) \\
\frac{d^2x^\mu}{d\tau^2} = (0,0,0,0) \\
\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \iff \dot{u}^\mu + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = \Gamma^\mu_{00}u^0u^0 &= \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00})u^0u^0 \\
&= \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0})u^0u^0 \\
&= \frac{1}{2}g^{\mu 0}(\partial_0 g_{00} + \partial_0 g_{00}) \\
&= 0
\end{aligned}$$

So, we see that $x^\mu(\tau)$ is a geodesic. Thus, the galaxy is only under the influence of the gravitational field.

Finally, one needs to incorporate the homogeneity and isotropy of spacetime according to the cosmological principle and Weyl's postulate. This can be achieved as follows:

On each hypersurface, the spatial separation between two nearby points $A(x^1, x^2, x^3)$ and $B(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ is given by [INSERT FIGURE]

$$d\sigma^2 = g_{ij}\,dx^i dx^j$$

First, note that by homogeneity, if there is a scaling in the length, it should be independent of the position of the triangle in 3-D space. This is expressed mathematically by

$$\left.\frac{AB}{AC}\right|_{t_0} = \left.\frac{AB}{AC}\right|_{t} \implies d\sigma^2 = \underbrace{S^2(t)}_{\text{scale factor}}\ \underbrace{h_{ij}(x^1, x^2, x^3)\,dx^i dx^j}_{\text{stationary}}$$

Next, by isotropy, the angles should be the same. Hence, $\Delta(ABC)\big|_{t_0}$ must be similar to $\Delta(ABC)\big|_{t}$.

Exercise 2.3. Show that the most general form of a stationary isotropic metric is

$$\begin{aligned}
dl^2 &= h_{ij}(x^1, x^2, x^3)\,dx^i dx^j, \quad i,j = \overline{1,3} \\
&= B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)
\end{aligned}$$

(Hint: Hobson 9.1)

By the above exercise, the 4-D metric reads as

$$\begin{aligned}
ds^2 &= -c^2dt^2 + S^2(t)h_{ij}(x^1, x^2, x^3)\,dx^i dx^j \\
&= -c^2dt^2 + S^2(t)[B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)]
\end{aligned}$$

2.3.3 Maximally Symmetric Spaces

We begin by observing that since the above metric is spatially homogeneous and isotropic, it has the maximal number of symmetries. Luckily, the Riemann tensor for such spaces has a simple form in terms of a number K and the metric h_{ij}. More explicitly, one has

$$^{(3)}R_{ijkl} = K\left({}^{(3)}h_{ik}{}^{(3)}h_{jl} - {}^{(3)}h_{il}{}^{(3)}h_{jk}\right)$$

By contraction, we find

$$\begin{aligned} {}^{(3)}h^{ik\,(3)}R_{ijkl} &= K^{(3)}h^{ik}\left({}^{(3)}h_{ik}{}^{(3)}h_{jl} - {}^{(3)}h_{il}{}^{(3)}h_{jk}\right) \\ {}^{(3)}R_{jl} &= K\left({}^{(3)}h_{jl} - {}^{(3)}h_{jl}\right) \\ &= 2K^{(3)}h_{jl} \end{aligned}$$

Contracting once more

$$\begin{aligned} {}^{(3)}h^{ij\,(3)}R_{ij} &= 2K^{(3)}h^{ij\,(3)}h_{ij} = 6K \\ K &= \frac{1}{6}{}^{(3)}R \end{aligned}$$

One can also obtain the Christoffel symbols and the components of the Ricci tensor from the metric.

$$\begin{aligned} {}^{(3)}R_{rr} &= \frac{1}{rB}\frac{dB}{dr} \\ {}^{(3)}R_{\theta\theta} &= 1 + \frac{r}{2B^2}\frac{dB}{dr} - \frac{1}{B} \\ {}^{(3)}R_{\phi\phi} &= {}^{(3)}R_{\theta\theta}\sin^2\theta \end{aligned}$$

Comparing with the above form of the Ricci tensor in terms of ${}^{(3)}h_{ij}$, we obtain

$$^{(3)}R_{rr} = 2K^{(3)}h_{rr} = 2KB(r) = \frac{1}{rB}\frac{dB}{dr} \tag{2.1}$$

$$^{(3)}R_{\theta\theta} = 2K^{(3)}h_{\theta\theta} = 2Kr^2 = 1 + \frac{r}{2B^2}\frac{dB}{dr} - \frac{1}{B} \tag{2.2}$$

(2.1) yields

$$B(r) = \frac{1}{c - Kr^2}$$

Inserting into (2.2), we have

$$\begin{aligned} 1 + \frac{r}{2B^2}2KrB^2 - \frac{1}{B} &= 2Kr^2 \\ 1 - \frac{1}{B} &= Kr^2 \\ 1 - (c - Kr^2) &= Kr^2 \end{aligned}$$

2.4 FLRW Metric

Hence, $c = 1$ and the metric for 4-D spacetime takes the form

$$ds^2 = -c^2 dt^2 + S^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

Since K is an arbitrary number, it is useful to normalize it. Set

$$k := \frac{K}{|K|}$$

Then, define a new radial coordinate by

$$\tilde{r} := \sqrt{|K|}\, r$$

Observe that $\tilde{r}$ is dimensionless since K has units of L^{-2}. To ensure dimensional consistency, define a rescaled scale factor as

$$R(t) := \frac{S(t)}{|K|^{\frac{1}{2}}}$$

which has dimensions of L. Then, the metric reads as

$$\begin{aligned} ds^2 &= -c^2 dt^2 + \frac{S^2(t)}{|K|}\left(\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \\ &= -c^2 dt^2 + R^2(t)\left(\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \end{aligned}$$

Dropping the tildes, we obtain the FLRW metric in canonical form

$$ds^2 = -c^2 dt^2 + R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

where

$$k = \begin{cases} 0: & \text{flat 3-space} \\ +1: & \text{open 3-space (hyperbolic)} \\ -1: & \text{closed 3-space (sphere)} \end{cases}$$

2.4.1 Properties of Maximally Symmetric Hypersurfaces in the FLRW Geometry

Geometric properties of Σ_t for a given t depend on the value of k. Accordingly, we inspect the geometric properties of hypersurfaces case by case.

(i) $k = 1$: Positively curved

$$d\sigma^2 = R^2(t)\left(\frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

We see that $r = 1$ is a (coordinate) singularity that can be removed by introducing a new coordinate ψ

$$r = \sin\psi,\ \ \psi \in [0, \pi] \implies dr = \cos\psi\, d\psi = (1 - r^2)^{\frac{1}{2}}\, d\psi$$

Then, the line element takes the form

$$d\sigma^2 = R^2(t)(d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2))$$

Area:

$$\begin{aligned} A &= \int (R(t)\sin\psi\, d\theta)\,(R(t)\sin\theta\sin\psi\, d\phi) \\ &= \int_0^{2\pi}\int_0^{\pi} R^2(t)\sin^2\psi\sin\theta\, d\theta\, d\phi \\ &= 4\pi R^2(t)\sin^2\psi. \end{aligned}$$

So, the area is changing with ψ.

$$A_{\text{max}} \text{ occurs when } \psi = \frac{\pi}{2}$$
$$A_{\text{min}} \text{ occurs when } \psi = 0, \pi$$

Volume:

$$\begin{aligned} V &= \int (R(t)\, d\psi)\,(R(t)\sin\psi\, d\theta)\,(R(t)\sin\psi\sin\theta\, d\phi) \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} R^3(t)\sin^2\psi\sin\theta\, d\psi\, d\theta\, d\phi \\ &= 2\pi^2 R^3(t) \end{aligned}$$

which is greater than the volume of $\mathbb{S}^2$ in $\mathbb{R}^3$. ($R(t)$ here refers to the radius of the Universe.)

(ii) $k = 0$: Flat space

$$d\sigma^2 = R^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2))$$

For notational consistency, let $r = \psi$. Then,

$$d\sigma^2 = R^2(t)(d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2))$$

There is not much to discuss in this case. We have the usual 3D Euclidean space.

(iii) $k = -1$: Negatively curved

$$d\sigma^2 = R^2(t)\left(\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

This time, we introduce the hyperbolic sine.

$$r = \sinh\psi,\ \ \psi \in [0, \infty) \implies dr = \cosh\psi\, d\psi = (1+r^2)^{\frac{1}{2}}\, d\psi$$

Then, the line element takes the form

$$d\sigma^2 = R^2(t)(d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2))$$

In this case, the 3D space cannot be embedded in 4D Euclidean space. Instead, we have the 4D Lorentzian space.

$$\begin{cases} w = R\cosh\psi \\ x = R\sinh\psi\sin\theta\cos\phi \\ y = R\sinh\psi\sin\theta\sin\phi \\ z = R\sinh\psi\cos\theta \\ w^2 - x^2 - y^2 - z^2 = R^2 \end{cases}$$

Consequently, the final form of the line element is the metric of 4D Lorentzian space

$$d\sigma^2 = dw^2 - dx^2 - dy^2 - dz^2.$$

Area:

$$\begin{aligned} A &= \int_0^{\pi}\int_0^{2\pi} \left(R(t)\sinh\psi\, d\theta\right)\left(R(t)\sinh\psi\sin\theta\, d\phi\right) \\ &= 4\pi R^2(t)\sinh^2\psi \end{aligned}$$

which is not bounded.

Volume:

$$V = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \left(R(t)\, d\psi\right)\left(R(t)\sinh\psi\, d\theta\right)\left(R(t)\sinh\psi\sin\theta\, d\phi\right) = \infty$$

The three cases can be covered in a single line element

$$ds^2 = -c^2dt^2 + R^2(t)\left[d\psi^2 + S_k^2(\psi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right],$$

where

$$S_k(\psi) = \begin{cases} \psi, \;\; k = 0 \\ \sin\psi, \;\; k = 1 \\ \sinh\psi, \;\; k = -1 \end{cases}$$

2.4.2 Geodesics in FLRW Background

We have seen that comoving observers have a four-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = (1, 0, 0, 0).$$

We can also study the trajectories of timelike bodies or photons over Σ_t. For this, we solve the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \;\text{ or }\; \dot{u}^\mu + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0,$$

where λ is an affine parameter.

Claim 2.4.

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0 \iff \dot{u}^\mu = \frac{1}{2}(\partial_\mu g_{\alpha\beta})\, u^\alpha u^\beta$$

Proof.

$$\begin{aligned} g_{\mu\nu}\dot{u}^\mu &= \frac{d}{d\lambda}(g_{\mu\nu}u^\mu) - u^\mu \frac{d}{d\lambda} g_{\mu\nu} \\ &= \dot{u}_\nu - u^\mu u^\alpha \partial_\alpha g_{\mu\nu} \end{aligned}$$

Hence,

$$\dot{u}_\nu = g_{\mu\nu}\dot{u}^\mu + u^\mu u^\alpha \partial_\alpha g_{\mu\nu}$$

On the other hand, the geodesic equation gives

$$g_{\mu\nu}\dot{u}^\mu = -g_{\mu\nu}\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$$

Then,

$$\begin{aligned} \dot{u}_\nu - u^\mu u^\alpha \partial_\alpha g_{\mu\nu} &= -g_{\mu\nu}\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \\ &= -g_{\mu\nu}\frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) u^\alpha u^\beta \\ &= -\frac{1}{2}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) u^\alpha u^\beta \\ &= -\partial_\alpha g_{\beta\nu} u^\alpha u^\beta + \frac{1}{2}\partial_\nu g_{\alpha\beta} u^\alpha u^\beta \end{aligned}$$

Finally,

$$\begin{aligned} \dot{u}_\nu &= \partial_\alpha g_{\beta\nu} u^\alpha u^\beta - \partial_\alpha g_{\beta\nu} u^\alpha u^\beta + \frac{1}{2}\partial_\nu g_{\alpha\beta} u^\alpha u^\beta \\ &= \frac{1}{2}\partial_\nu g_{\alpha\beta}\, u^\alpha u^\beta \end{aligned}$$

□

1) If $\partial_\mu g_{\alpha\beta} = 0$, then we have an isometry.

$$\dot{u}_\nu = 0 \implies u_\mu = \text{constant over the geodesic (at every point)}$$

(I) $\mu = 3$: Since $\partial_3 g_{\alpha\beta} = 0,\ u_3 =$ constant over the geodesic. More precisely,

$$u_3 = g_{3\mu}u^\mu = g_{33}u^3 = R^2(t)\, S_k^2(\psi) \sin^2\theta\, u^3 = g_{33}u^3 = R^2(t)\, S_k^2(\psi) \sin^2\theta\, \dot{\phi}$$

2) Since Σ_t is isotropic and homogeneous, there is no preferred point on this hypersurface. This means that we can take any point as the origin of our coordinate system. Taking the origin at some point P over the geodesic, we have

$$S_k(\psi)\big|_P = 0 \implies u_3\big|_P = S_k^2(\psi)\big|_P (\dots) = 0$$

This implies

$$u^3\big|_P = g^{3\mu}u_\mu\big|_P = g^{33}u_3\big|_P = 0$$

Combining with **(I)**, we obtain

$$u_3 = u^3 = 0$$

at every point on the geodesic.

(II) $\mu = 2$: We have

$$\dot{u}_2 = \frac{1}{2}\partial_2 g_{\alpha\beta}\, u^\alpha u^\beta = \frac{1}{2}\partial_2 g_{33}\, u^3 u^3 = 0$$

Hence,

$$u_2 = g_{2\mu} u^\mu = g_{22}\, u^2 = R^2(t)\, S_k^2(\psi)\, u^2 = \text{constant}$$

Since $S_k(\psi)\big|_P = 0,\ u_2 = 0$ and

$$u^2 = g^{2\mu}\, u_\mu = g^{22} u_2 = 0 \implies \theta = \text{constant}$$

at every point over the geodesic. So, we have the following picture [INSERT FIGURE]. In other words, geodesics only move in radial directions.

(III) $\mu = 1$: Similarly,

$$\dot{u}_1 = \frac{1}{2}\partial_1 g_{\alpha\beta}\, u^\alpha u^\beta = \frac{1}{2}(\partial_1 g_{22}\, u^2 u^2 + \partial_1 g_{33}\, u^3 u^3) = 0$$

Hence,

$$u_1 = g_{1\mu}\, u^\mu = g_{11}\, u^1 = R^2(t)\, \dot{\psi} = \text{constant}$$

(IV) $\mu = 0$: Geodesic equation gives

$$\begin{cases} \dot{u}_0 = \frac{1}{2}\partial_0 g_{\alpha\beta}\, u^\alpha u^\beta \\ \dot{u}^0 = \Gamma^0_{\alpha\beta}\, u^\alpha u^\beta \end{cases}$$

Instead of these, we can use

$$u_\mu u^\mu = -c^2 \text{ for massive particles}$$
$$u_\mu u^\mu = 0 \text{ for massless particles}$$

Timelike Geodesics

$$\begin{gathered} u_\mu u^\mu = u_0 u^0 + u_1 u^1 + u_2 u^2 + u_3 u^3 = u_0 u^0 + u_1 u^1 = -c^2 \\ g_{00}(u^0)^2 + g_{11}(u^1)^2 = -c^2 \\ -c^2 \dot{t}^2 + R^2(t)\dot{\psi}^2 = -c^2 \\ \implies \dot{t}^2 = 1 + \frac{R^2(t)\dot{\psi}^2}{c^2} \end{gathered}$$

Null Geodesics

$$\begin{gathered} u_\mu u^\mu = u_0 u^0 + u_1 u^1 + u_2 u^2 + u_3 u^3 = u_0 u^0 + u_1 u^1 = 0 \\ g_{00}(u^0)^2 + g_{11}(u^1)^2 = 0 \\ -c^2 \dot{t}^2 + R^2(t)\dot{\psi}^2 = 0 \\ \implies \dot{t}^2 = \frac{R^2(t)\dot{\psi}^2}{c^2} \end{gathered}$$

2.5 Kinematic Properties of the FLRW Metric

Kinematic properties refer to the quantities that describe the motion (evolution of the metric (universe)) that is independent of the specific matter content. These properties are

(i) Cosmological redshift

(ii) Hubble and deceleration parameters

(iii) Distances over this geometry

(iv) Volume and number densities over this geometry

On the other hand, the dynamics of the geometry are entirely characterized by the scale factor that can be determined by solving EFEs.

2.5.1 The Cosmological Redshift

[INSERT FIGURE]

We showed that θ and ϕ are constants over a geodesic. This means that $d\theta = d\phi = 0$ and the line element becomes

$$ds^2 = -c^2 dt^2 + R^2(t)\, d\psi^2 = 0 \text{ (null geodesic)}$$

Then,

$$\int_{t_E}^{t_R} \frac{c}{R(t)}\, dt = \int_0^{\psi_E} d\psi$$

If the emitter sends another light ray at time $t_E + \delta t_E$, then the receiver will receive at time $t_R + \delta t_R$. Then,

$$\begin{gathered}
\int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{c}{R(t)}\, dt = \int_0^{\psi_E} d\psi \\
\int_{t_E+\delta t_E}^{t_R} \frac{c}{R(t)}\, dt + \int_{t_R}^{t_R+\delta t_R} \frac{c}{R(t)}\, dt = \int_{t_E}^{t_R} \frac{c}{R(t)}\, dt \\
-\left(\int_{t_R}^{t_E} \frac{c}{R(t)}\, dt + \int_{t_E}^{t_E+\delta t_E} \frac{c}{R(t)}\, dt\right) + \int_{t_R}^{t_R+\delta t_R} \frac{c}{R(t)}\, dt = \int_{t_E}^{t_R} \frac{c}{R(t)}\, dt \\
\implies \int_{t_E}^{t_E+\delta t_E} \frac{c}{R(t)}\, dt = \int_{t_R}^{t_R+\delta t_R} \frac{c}{R(t)}\, dt
\end{gathered}$$

If δt_E and δt_R are very small, then $R(t)$ will be almost constant over the ranges. It follows that

$$\frac{\delta t_E}{R(t_E)} = \frac{\delta t_R}{R(t_R)}$$

Now set

$$\begin{aligned}
\delta t_E &= \text{the period of light emitted} \\
\delta t_R &= \text{the period of light received}
\end{aligned}$$

Then,

$$\frac{\delta t_E}{\delta t_R} = \frac{R(t_E)}{R(t_R)} \implies \frac{\nu_R}{\nu_E} = \frac{R(t_E)}{R(t_R)}$$

Remark. The frequency of the beam is changing, subject to the change of the scale factor.

For photons, $c = \lambda\nu$. Then, using the redshift formula

$$z = \frac{\lambda_R - \lambda_E}{\lambda_E} = \frac{\lambda_R}{\lambda_E} - 1 = \frac{R(t_R)}{R(t_E)} - 1 \implies 1 + z = \frac{R(t_R)}{R(t_E)}$$

If $z = 0, R(t_R) = R(t_E) \implies$ static universe
If $z > 0, R(t_R) > R(t_E) \implies$ expanding universe
If $z < 0, R(t_R) < R(t_E) \implies$ contracting universe

Exercise 2.5. (i) Show that for an emitter and receiver with fixed spatial coordinates in a geometry with metric $g_{\mu\nu}$, the frequency shift of the photon is generally given by

$$\frac{\nu_R}{\nu_E} = \frac{p_0(R)}{p_0(E)}\sqrt{\frac{g_{00}(E)}{g_{00}(R)}}, \qquad (*)$$

where p_0 is the 0-th component of the photon's four-momentum

(ii) Using $(*)$ for FLRW metric, show that

$$1 + z = \frac{\nu_E}{\nu_R} = \frac{R(t_R)}{R(t_E)}$$

(Hint: Hobson, Appendix 9A)

2.5.2 The Hubble and Deceleration Parameters

As a common notation, we introduce
t_0 : present time
$R(t_0) = R_0$: present value of the scale factor

If a galaxy emitted a photon at cosmic time t, then we have

$$t_0 - t = \delta_t,\ \ \delta_t \ll t_0$$

We can do a Taylor expansion for the scale factor $R(t)$ around the present time

$$\begin{aligned} R(t) &= R(t_0 - (t_0 - t)) \\ &= R(t_0) - (t_0 - t)\dot{R}(t_0) + \frac{1}{2}(t_0 - t)^2\ddot{R}(t_0) + \dots \end{aligned}$$

Definition 2.6. $H(t) := \dot{R}(t)/R(t)$ is called the Hubble parameter. For the present time, we write $H(t_0) = \dot{R}(t_0)/R(t_0) =: H_0$

Definition 2.7. $q(t) := -\ddot{R}(t)R(t)/\dot{R}^2(t)$

Then, we have

$$\begin{aligned} R(t) &= R(t_0)\left[1-(t_0-t)\frac{\dot R(t_0)}{R(t_0)}+\frac{1}{2}(t_0-t)^2\frac{\ddot R(t_0)R(t_0)}{\dot R^2(t_0)}\frac{\dot R^2(t_0)}{R^2(t_0)}\right] \\ &= R(t_0)\left(1-(t_0-t)H_0-\frac{1}{2}(t_0-t)^2\, q_0H_0^2+\dots\right) \end{aligned}$$

We can obtain the look-back time $t_0 - t$ in terms of the redshift z

$$z = \frac{R(t_0)}{R(t)} - 1 = \frac{R_0}{R_0\left(1-(t_0-t)H_0-\frac{1}{2}(t_0-t)^2\, q_0H_0^2+\dots\right)} - 1$$

Recall that $t_0 - t = \delta_t \ll t_0$ and $\frac{1}{1+\epsilon} = 1-\epsilon+\epsilon^2+\dots$ Then,

$$z \approx (t_0-t)H_0 + (t_0-t)^2\left(1+\frac{1}{2}q_0\right)H_0^2+\dots$$

Exercise 2.8. Show that the look-back time is

$$t_0 - t \approx \frac{z}{H_0}\left(1-\left(1+\frac{1}{2}q_0\right)z\right)+O(z^3) \qquad (*)$$

for nearby galaxies ($z \ll 1$).

Remark. The look-back time is given in terms of the present value of q and H. This means that we do not need the whole evaluation history of $R(t)$ for the Universe.

Our next goal is to evaluate the radial comoving coordinate ψ of a galaxy. Since the motion is radial, we have

$$\int_t^{t_0}\frac{c}{R(t)}\,dt = \int_0^{\psi}d\psi = \psi$$

$$\implies \psi = \int_t^{t_0}\frac{c}{R_0(1-(t_0-t)H_0+O((t_0-t)^2))}\,dt \qquad (**)$$

Since $t_0 - t \ll t_0$,

$$\psi = \frac{c}{R_0}\left((t_0-t)-\frac{1}{2}(t_0-t)^2H_0+O((t_0-t)^2)\right)$$

In terms of redshift, we find by $(*)$ and $(**)$

$$\psi \approx \frac{c}{H_0R_0}\left(z-\left(1+\frac{1}{2}q_0\right)z^2+O(z^3)\right)$$

Remark. In Hobson, the second term in parantheses is $-\frac{1}{2}(1+q_0)z^2$

We can derive the Hubble law for nearby galaxies ($z \ll 1$) as follows: for the FLRW metric, we know the proper distance (instantaneous physical distance) is given as $d = R(t)\psi$. Then, for nearby galaxies at the present time, one has

$$\begin{aligned} d &= R_0\psi \\ &= R_0\frac{c}{R_0H_0}\left(z-\left(1+\frac{1}{2}q_0\right)z^2+O(z^3)\right) \\ &= \frac{c}{H_0}z+O(z^2) \\ &\implies cz = dH_0,\ \text{Hubble law} \end{aligned}$$

Remark. (i) $d = R(t)\psi \implies \dot{d} = (\dot{R}/R)\, R\psi = Hd \implies v = Hd$

(ii) Classical Doppler effect

$$\begin{aligned} z &= \frac{\lambda_R - \lambda_E}{\lambda_E} \\ &= \frac{\lambda_E \left(\frac{c}{c-v}\right) - \lambda_E}{\lambda_E} \\ &= \frac{v}{c-v} \\ &= \frac{1}{c}\frac{v}{1-\frac{v}{c}} \\ &= \frac{v}{c} + O\left(\left(\frac{v}{c}\right)^2\right) \\ &\implies z \approx \frac{v}{c} \end{aligned}$$

So far, we have considered the nearby galaxy limit where $z \ll 1$. However, we can obtain the look-back time and ψ coordinate for an arbitrary redshift z as follows:

$$\begin{aligned} z = \frac{R(t_0)}{R(t)} - 1 &\implies z + 1 = \frac{R(t_0)}{R(t)} \\ dz = d(z+1) = d\left(\frac{R_0}{R(t)}\right) &= -\frac{R_0}{R^2(t)}\dot{R}(t)\, dt = -\frac{R_0}{R(t)}\frac{\dot{R}(t)}{R(t)}\, dt \\ dz &= -(1+z)H(t)\, dt \\ t - t_0 &= \int_0^z \frac{1}{(1+z)H(z)}\, dz \end{aligned}$$

ψ coordinate can be obtained similarly.

$$\psi = \int_t^{t_0} \frac{c}{R(t)}\, dt = \int_0^z \frac{c}{(1+z)H(z)R(z)}\, dz = \frac{c}{R_0}\int_0^z \frac{1}{H(z)}\, dz$$

To evaluate these for an arbitrary z, we need the time evolution of $R(t)$, and hence $H(t)$ (or $H(z)$), which is provided by solving the EFEs for some given matter sources.

2.5.3 Distances in FLRW Geometry

Our measurements refer to events in our past light cone. In other words, $d = R(t)\,\psi$ is the distance over a spatial hypersurface, hence not measurable.

We have two main ways to measure distance in the FLRW geometry:

(i) Luminosity distance

(ii) Angular diameter distance

Luminosity Distance

F : flux received (W/m^2) (the energy per unit time per unit area of some detector)
d_L : Distance to the source
L : Source's absolute luminosity (W = J/s) Euclidean space: $F = L/A = L/4\pi d_L^2$, where $d_L = (L/4\pi f)^{\frac{1}{2}}$.

To generalize this to the case of FLRW geometry, we have
t_E : time of the photon emitted
t_0 : time of the photon observed
$L(t)$: absolute luminosity measured at time t
$A(t) = 4\pi R^2(t) S_k^2(\psi)$: proper area of the sphere in cosmic time t

At the present time t_0 :

$$F(t_0) = \frac{L(t_0)}{A(t_0)} = \frac{L(t_0)}{4\pi R^2(t_0) S_k^2(\psi)}$$

The present value of the absolute luminosity $L(t_0)$ is affected by

(i) Photon's energy reduction due to the expansion (cosmic redshift)

$$\nu_0 = \frac{\nu_E}{1+z}$$

(ii) Photon arrival rate reduction because of the expansion that is by another factor $1/(1+z)$

Hence, $L(t_0) = L_E/(1+z)^2$. So, the flux observed will be

$$F(t_0) = \frac{L_E}{4\pi (R(t_0) S_k(\psi))^2 (1+z)^2}$$

Hence, the luminosity distance is

$$d_L = R(t_0) S_k(\psi)(1+z)$$

Remark. Since ψ is not observable, we have to remove it from the above relation. This can be done by the relation

$$\psi = \int_t^{t_0} \frac{c}{R(t)}\, dt = \frac{c}{R(t_0)} \int_0^z \frac{1}{H(z)}\, dz,$$

where we can evaluate $H(z)$ using the EFEs in the presence of a matter source.

Angular Diameter Distance

This distance is based on a standard-length "rod", where the angular diameter can be observed.

In Euclidean space, we have

$$dA = \frac{l}{\Delta\theta},\ l : \text{proper diameter of the source seen at the angle } \theta$$

In FLRW space, we assume that $\phi =$ constant along the geodesic. Then,

$$l = R(t_E)S_k(\psi)\Delta\theta$$
$$d_A = \frac{R(t_E)S_k(\psi)\Delta\theta}{\Delta\theta} = R(t_E)S_k(\psi)$$

Remark. Again, since ψ is not measurable, $S_k(\psi)$ is not observable. We replace it with observable quantities

$$d_A = R(t_0)\frac{R(t_E)}{R(t_0)}S_k(\psi)$$
$$= \frac{R(t_0)S_k(\psi)}{1+z},$$

where $\psi = \int_{t_E}^{t_0} c/R(t)\,dt = c/R(t_0)\int_0^z 1/H(z)\,dz$.

Remark.

$$d_L = R_0 S_k(\psi)(1+z)$$
$$d_A = \frac{R(t_0)S_k(\psi)}{1+z}$$
$$d_L = (1+z)^2\,dA$$

For small redshifts (nearby sources), the measures give the same result. They differ for large redshifts (far away sources).

3 The Cosmological Field Equations

So far, we have discussed the geometric and kinematic properties of the FLRW metric. In order to study its dynamic properties that are determined by the scale factor $R(t)$ only, we have to solve the EFEs in the presence of matter fields. Of course, this requires a model for the energy-momentum source of the field equations. Moreover, the model has to be consistent with the cosmological principles (homogeneity, isotropy). Hence, the only dependence will be time dependence in the components of this tensor.

We consider a perfect fluid with density $\rho(t)$ and pressure $p(t)$ at each point of the following form

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu + \rho g^{\mu\nu}$$

In comoving coordinates $x^\mu = (t, r, \theta, \phi)$, we have

$$ds^2 = -c^2dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]$$

u^μ is the four-velocity of the galaxy (test particle)

$$u^\mu = (1,0,0,0) = \delta_0^\mu$$
$$u_\mu = g_{\mu\nu}u^\nu = g_{\mu\nu}\delta_0^\nu = g_{\mu 0} = -c^2\delta_\mu^0$$
$$\implies u^\mu u_\mu = -c^2\delta_0^\mu\delta_\mu^0 = -c^2$$

Multiplying by the metric twice, we find

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p\, g_{\mu\nu} = (\rho c^2 + p)c^2 \delta^0_\mu \delta^0_\nu + p\, g_{\mu\nu}$$

Accordingly, the trace is

$$T = g^{\mu\nu} T_{\mu\nu} = g_{\mu\nu} \left(\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p\, g^{\mu\nu}\right) = (-\rho c^2 - p + 4p) = -(\rho c^2 - 3p)$$

Recall the EFEs with cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4}$$

Multiply by $g^{\mu\nu}$

$$R - 2R + 4\Lambda = \kappa T \implies R = -\kappa T + 4\Lambda$$

Then,

$$\begin{aligned}
R_{\mu\nu} &= \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu} \\
&= \kappa \left(\left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu} - \frac{1}{2}(-\rho c^2 + 3p) g_{\mu\nu}\right) + \Lambda g_{\mu\nu} \\
&= \kappa \left[\left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - \frac{1}{2}(-\rho c^2 + p) g_{\mu\nu}\right] + \Lambda g_{\mu\nu}
\end{aligned}$$

We know that $u^\mu = \delta^\mu_0$, $u_\mu = -c^2 \delta^0_\mu$ and the metric components are

$$g_{00} = -c^2, \; g_{11} = \frac{R^2(t)}{1 - kr^2}, \; g_{22} = R^2(t)\, r^2, \; g_{33} = R^2(t)\, r^2 \sin^2\theta$$

Consequently, the components of the Ricci tensor can be obtained as

$$\begin{aligned}
R_{00} &= \kappa \left[\left(\rho + \frac{p}{c^2}\right) (u_0)^2 - \frac{1}{2}(-\rho c^2 + p)\, g_{00}\right] + \Lambda g_{00} \\
&= \frac{\kappa}{2}(\rho c^2 + 3p) c^2 - \Lambda c^2 \\
R_{11} &= \kappa \left[\left(\rho + \frac{p}{c^2}\right) (u_1)^2 - \frac{1}{2}(-\rho c^2 + p)\, g_{11}\right] + \Lambda g_{11} \\
&= \frac{R^2(t)}{1 - kr^2} \left(-\frac{1}{2}\kappa(-\rho c^2 + p) + \Lambda\right) \\
R_{22} &= \kappa \left[\left(\rho + \frac{p}{c^2}\right) (u_2)^2 - \frac{1}{2}(-\rho c^2 + p)\, g_{22}\right] + \Lambda g_{22} \\
&= r^2 R^2(t) \left(-\frac{1}{2}\kappa(-\rho c^2 + p) + \Lambda\right) \\
R_{33} &= \kappa \left[\left(\rho + \frac{p}{c^2}\right) (u_3)^2 - \frac{1}{2}(-\rho c^2 + p)\, g_{33}\right] + \Lambda g_{33} \\
&= r^2 R^2(t) \sin^2\theta \left(-\frac{1}{2}\kappa(-\rho c^2 + p) + \Lambda\right)
\end{aligned}$$

We can also derive these components from Christoffel symbols and metric as follows:

Exercise 3.1. Show that for the FLRW metric, we have nonzero components of the Christoffel symbols as

$$\Gamma^0_{11} = \frac{R\dot{R}}{c^2(1-kr^2)},\ \Gamma^0_{22} = \frac{R\dot{R}r^2}{c^2},\ \Gamma^0_{33} = \frac{R\dot{R}r^2\sin^2\theta}{c^2},\ \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{R}}{R},\ \Gamma^1_{11} = \frac{kr}{1-kr^2}$$
$$\Gamma^1_{22} = -r(1-kr^2),\ \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta,\ \Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{R}}{R},\ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{R}}{R},\ \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\ \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

Then, using $R_{\mu\nu} = \partial_\nu\Gamma^\sigma_{\mu\sigma} - \partial_\sigma\Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\sigma\rho}$, one finds

$$R_{00} = -\frac{3\ddot{R}}{R},\ R_{11} = \frac{1}{c^2(1-kr^2)}(R\ddot{R} + 2\dot{R}^2 + 2c^2\kappa)$$
$$R_{22} = \frac{r^2}{c^2}(\ddot{R}R + 2\dot{R}^2 + 2c^2\kappa),\ R_{33} = \frac{r^2\sin^2\theta}{c^2}(\ddot{R} + 2\dot{R}^2 + 2c^2\kappa)$$

Thus, the full field equations are (k and κ are probably mixed up)

$$-3\frac{\ddot{R}}{R} = \frac{\kappa}{2}(\rho c^2 + 3p)c^2 - \Lambda c^2$$
$$R\ddot{R} + 2\dot{R}^2 + 2c^2k = c^2R^2\left(-\frac{1}{2}\kappa(-\rho c^2 + p) + \Lambda\right)$$

Solving these equations (from the first equation, obtain an expression for $\ddot{R}$ and insert into the second equation), one obtains

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{3.1}$$
$$H^2 + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} \tag{3.2}$$

These are known as Friedmann equations for $\Lambda = 0$ and Friedmann-Lemaître equations for $\Lambda \neq 0$.

In these equations, we have three unknowns: R, ρ, and p; which are all functions of t. In order to solve this system, we need a constraint or equation of state for the matter source.

We consider the following linear equation of state

$$p = \omega\rho,$$

where $\omega = \omega(t)$ in general. In this course, $\omega =$ constant.

(i) $\omega = 0$: dust field

(ii) $\omega = 1/3$: radiation

(iii) $\omega = -1$: cosmological constant (dark energy)

(iv) $-1 < \omega < -1/3$: exotic fluid

3.1 Equation of Motion for Cosmological Fluid

From the Bianchi identity, and hence, covariant conservation of energy-momentum tensor, we have $\nabla_\mu T^{\mu\nu} = 0$.

$\nu = 0$: Then,

$$\nabla_\mu T^{\mu 0} = \nabla_\mu \left(\left(\rho + \frac{p}{c^2} \right) u^\mu u^0 + p g^{\mu 0} \right) = 0$$

Since $u^0 = \delta^0_0 = 1$, one finds

$$\begin{aligned}
\nabla_\mu T^{\mu 0} &= \nabla_\mu \left(\left(\rho + \frac{p}{c^2} \right) u^\mu \right) + (\nabla_\mu p)\, g^{\mu 0} \\
&= \nabla_\mu(\rho\, u^\mu) + \left(\nabla_\mu \frac{p}{c^2} \right) u^\mu + \frac{p}{c^2} \nabla_\mu u^\mu + (\nabla_\mu p) g^{\mu 0} \\
&= \nabla_\mu(\rho\, u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu
\end{aligned}$$

Hence, we obtain the continuity equation

$$\nabla_\mu(\rho\, u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0$$

$\nu = 1$: Then,

$$\nabla_\mu T^{\mu 1} = \nabla_\mu \left(\left(\rho + \frac{p}{c^2} \right) u^\mu u^1 + p g^{\mu 1} \right) = (\nabla_\mu p) g^{\mu 1}$$

Expanding $\nabla_\mu T^{\mu\nu} = 0$, we have

$$\begin{aligned}
\nabla_\mu \left(\left(\rho + \frac{p}{c^2} \right) u^\mu u^\nu + p g^{\mu\nu} \right) &= \left(\nabla_\mu \left(\rho + \frac{p}{c^2} \right) \right) u^\mu u^\nu + \left(\rho + \frac{p}{c^2} \right) (\nabla_\mu u^\mu)\, u^\nu \\
&\quad + \left(\rho + \frac{p}{c^2} \right) u^\mu \nabla_\mu u^\nu + (\nabla_\mu p) g^{\mu\nu} \\
&= 0
\end{aligned}$$

Then,

$$\begin{aligned}
\left(\rho + \frac{p}{c^2} \right) u^\mu \nabla_\mu u^\nu &= -(\nabla_\mu p) g^{\mu\nu} - \nabla_\mu \left(\rho + \frac{p}{c^2} \right) u^\mu u^\nu - \left(\rho + \frac{p}{c^2} \right) u^\nu \nabla_\mu u^\mu \\
&= -(\nabla_\mu p) g^{\mu\nu} - \nabla_\mu(\rho\, u^\mu) u^\nu - \left(\nabla_\mu \frac{p}{c^2} \right) u^\mu u^\nu - \frac{p}{c^2} u^\nu \nabla_\mu u^\mu \\
&= -(\nabla_\mu p) \left(g^{\mu\nu} + \frac{u^\mu u^\nu}{c^2} \right),
\end{aligned}$$

where we have used the continuity equation to obtain the last line. Moreover, the left-hand side is 0 since galaxies (fluid's particles) are moving over geodesics ($u^\mu \nabla_\mu u^\nu = 0$). Hence, there will be no pressure divergence, which is what is required by homogeneity and isotropy.

[Since $u^\mu = \delta^\mu_0$, $g^{\mu\nu} + u^\mu u^\nu / c^2 = 0$ for $\mu = \nu = 0$. Then, we must have $\nabla_\mu p = 0$ for $\mu \neq 0$ since $g^{\mu\nu} \neq 0$ for $\mu \neq 0, \nu \neq 0$]

Let us expand the continuity equation by expanding the covariant derivative:

$$\begin{gathered}
\nabla_\mu(\rho\, u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = (\nabla_\mu \rho)\, u^\mu + \rho(\nabla_\mu u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0 \\
(\partial_\mu \rho) u^\mu + \left(\rho + \frac{p}{c^2} \right) (\partial_\mu u^\mu + \Gamma^\mu_{\mu\nu} u^\nu) = 0, \;\; u^\mu = \delta^\mu_0 = (1, 0, 0, 0)
\end{gathered}$$

We have

$$\Gamma^{\mu}_{\mu\nu}u^{\nu} = \Gamma^{\mu}_{\mu 0}u^{0} = \Gamma^{\mu}_{\mu 0} = \Gamma^{0}_{00} + \Gamma^{1}_{10} + \Gamma^{2}_{20} + \Gamma^{3}_{30} = 3\Gamma^{1}_{10} = 3\frac{\dot{R}}{R}$$

Then,

$$\partial_0 \rho + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} = 0$$

So, we obtain another form of the continuity equation

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} = 0$$

Exercise 3.2. We derived the continuity equation from the covariant conservation of $T_{\mu\nu}$. Obtain it from the Friedmann equations.

Overall, we have three ODEs, two of which are independent:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{3.3}$$

$$H^2 + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} \tag{3.4}$$

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} = 0 \tag{3.5}$$

In order to integrate this system with three unknowns $R(t), \rho(t), p(t)$, one needs an equation of state for the matter source.

We consider a linear barotropic ($p = p(\rho)$) equation of state given by

$$p = \omega c^2 \rho,$$

where $\omega = \omega(t)$ in general. Here, we just consider the case $\omega =$ constant.

Exercise 3.3. Are there any other types of equations of state? What are their origin and applications?

We can classify types of matter as follows

$$\text{gravitationally attractive (normal)} \begin{cases} \omega = 1 : \text{stiff matter} \\ \omega = 1/3 : \text{radiation} \\ \omega = 0 : \text{matter (dust)} \\ -1/3 < \omega < 0 \text{ is also included} \end{cases}$$

$$\text{gravitationally repulsive (exotic)} \begin{cases} \omega = -1 : \text{dark energy (cosmo. const. or vacuum)} \\ -1 < \omega < -1/3 : \text{Quintessence matter} \\ \omega < -1 : \text{phantom matter} \end{cases}$$

Using $p = \omega c^2 \rho$ and the continuity equation, we can solve for $\rho(t)$ in terms of $R(t)$.

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} = 0$$
$$\dot{\rho} + 3\rho\,(1+\omega)\frac{\dot{R}}{R} = 0$$
$$\int_t^{t_0} \frac{1}{\rho}\, d\rho = -3(1+\omega)\int_t^{t_0} \frac{1}{R}\, dR$$
$$\frac{\rho}{\rho_0} = \left(\frac{R(t)}{R_0}\right)^{-3(1+\omega)}, \text{where } \rho_0 = \rho(t_0) \text{ and } R_0 = R(t_0)$$
$$\implies \rho(t) = \rho_0 \left(\frac{R(t)}{R_0}\right)^{-3(1+\omega)} \qquad (*)$$

Hence, we only need to determine $R(t)$ by solving the Friedmann equations.

Definition 3.4. The normalized scale factor is defined by $a(t) \coloneqq R(t)/R_0$

We can write $(*)$ more succinctly using the normalized scale factor.

$$\rho(t) = \rho_0\, a(t)^{-3(1+\omega)}$$

3.2 Multi-Component Cosmological Fluid

As we have seen, cosmological fluid consists of dust, radiation, and dark energy in general. In some eras, one of these dominates the Universe. For this reason, we decompose the energy-momentum tensor as follows.

$$\begin{aligned} T^{\mu\nu} &= \sum_i T_i^{\mu\nu},\ i = r, m, \Lambda \\ &= \sum_i \left(\rho_i + \frac{p_i}{c^2}\right) u^\mu u^\nu + p_i g^{\mu\nu} \text{ (each component is a perfect fluid)} \end{aligned}$$

Let $\rho_{\text{eff}} = \sum_i \rho_i$ and $p_{\text{eff}} = \sum_i p_i$. Then,

$$T^{\mu\nu} = \left(\rho_{\text{eff}} + \frac{p_{\text{eff}}}{c^2}\right) u^\mu u^\nu + p_{\text{eff}}\, g^{\mu\nu}$$

We see that the total source is a perfect fluid with effective density and pressure. We also suppose that these components are non-interacting. Then,

$$\nabla_\mu T^{\mu\nu} = 0 \implies \nabla_\mu T_i^{\mu\nu} = 0 \implies \rho_i(t) = \rho_{i_0} a(t)^{-3(1+\omega_i)}$$

3.3 Cosmological Models

The total density is given by $\rho(t) = \rho_r(t) + \rho_m(t) + \rho_\Lambda(t)$, where each component has the equation of state

$$p_i = \omega_i c^2 \rho_i$$

There are some constraints on these components due to energy conditions. For example, by the weak energy condition (WEC), one has

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq 0 \tag{3.6}$$

for all timelike vectors t^{μ}. For perfect fluids, it can be shown that (3.6) implies

$$\rho \geq 0,\ \ \rho c^2 + p \geq 0$$

Hence, WEC forces $\omega \geq -1$, which means that the phantom field violates the weak energy condition.

Exercise 3.5. What are the physical energy conditions? What are their motivation and applications? (Check Hobson 8.6-8.8)

3.3.1 Matter Components of the Universe

consists of two components:

(i) Baryonic matter: Matter of particles we know their physics (particles belonging to the standard model of particle physics). This part contributes about 5% to the total energy/mass of the Universe.

(ii) Non-Baryonic Dark Matter (no electromagnetic interaction to be visible): Particles not belonging to the SMPP. The contribution of this part is around 27% and has a gravitationally attractive effect.

$$\text{Dark Matter (DM)} \begin{cases} \text{Cold Dark Matter (CDM): non-relativistic particles} \\ \text{Hot Dark Matter: relativistic particles} \end{cases}$$

Some Candidates

(i) Primordial Black Holes: formed in the early Universe (of stellar mass)

(ii) Sterile neutrinos

(iii) Axions related to CP violation in QCD

(iv) Neutralinos from supersymmetry

(v) WIMP: Weakly Interacting Massive Particle

We can further decompose the matter density into baryonic and dark matter parts.

$$\rho_m(t) = \rho_{\text{b}}(t) + \rho_{\text{dm}}(t)$$

These components have a thermal energy much less than their rest mass energy, thus considered as dust with an equation of state with $\omega = 0$.

$$\rho_m = \rho_{m_0} a(t)^{-3} \overset{1+z=R_0/R=1/a}{\Longrightarrow} \rho_m(z) = \rho_{m_0}(1+z)^3$$

3.3.2 Radiation Components of the Universe

$$\rho_r(t) = \rho_\gamma(t) + \rho_\nu(t),$$

where γ stands for photons and ν stands for neutrinos. As equation of state parameter is $\omega = 1/3$, we have

$$\rho_r(t) = \rho_{r_0} a(t)^{-4}, \ \rho_r(z) = \rho_{r_0}(1+z)^4$$

From the Cosmic Microwave Background (CMB), we know that the dominant component is $\rho_\gamma(t)$.

3.3.3 Vacuum or Dark Energy Component of the Universe

It has a perfect fluid form, and its equation of state is $p = -c^2\rho$.

$$\rho_\Lambda(t) = \rho_0 = \frac{\Lambda c^2}{8\pi G} = \text{constant}$$

The Total Source

Combining all the contributions, we have

$$\begin{aligned} \rho(t) &= \rho_r(t) + \rho_m(t) + \rho_\Lambda \\ &= \rho_{r_0} a(t)^{-4} + \rho_{m_0} a^{-3}(t) + \rho_\Lambda \end{aligned}$$

Hence,

(i) As $t \to 0$ (the early Universe), radiation is dominant (radiation-dominated era)

(ii) In the middle era, matter is dominant (matter-dominated era)

(iii) As $t \to \infty$ (late times), vacuum (or cosmological constant or dark energy) is dominant.

3.4 Cosmological Parameters

We have obtained the form of densities $\rho_i(t)$ in terms of time. If there is a specific time t^* for which we can fix the value of $\rho_i(t)$, then the whole evaluation of $\rho_i(t)$ is determined. We choose $t^* = t_0$, the present time in which we are able to make observations.

Next, we define the dimensionless density parameters.

$$\Omega_i(t) := \frac{8\pi G}{3H^2}\rho_i(t), \ i = r, m, \Lambda$$

Remark. Notice that although ρ_Λ is constant, Ω_Λ is time varying.

The present-day values of $\Omega_i(t)$ are

(For reference, $H_0 \approx 70$ km s^{-1}Mpc$^{-1} \approx 2.27 \times 10^{-18}$ s^{-1}. This is an extremely small value, but over the cosmological time scales, billions of years, it leads to observational effects.)

$$\Omega_{m_0} \approx 0.3 \implies \Omega_{\mathrm{b}_0} \approx 0.05,\ \Omega_{\mathrm{dm}_0} \approx 0.25$$
$$\Omega_{r_0} \approx 5 \times 10^{-5} \implies \Omega_{\gamma_0} \approx 5 \times 10^{-5},\ \Omega_{\nu_0} \approx 0$$
$$\Omega_{\Lambda_0} \approx 0.70$$

Thus, observations tell us that we live in a dark energy dominated era.

From the second Friedmann equation, we have

$$H^2 + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}(\rho_r + \rho_m + \rho_\Lambda)$$
$$1 = \frac{8\pi G}{3H^2}(\rho_r + \rho_m + \rho_\Lambda) - \frac{kc^2}{H^2R^2}$$

Motivated by the above equation, we define the curvature density as

$$\Omega_k(t) \coloneqq -\frac{kc^2}{H^2R^2},\ k = 0, \pm 1$$

Then, we can write the second Friedmann equation compactly

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k.$$

Remark. In the above, one has $\Omega_r + \Omega_m > 0$ and depending on Λ and k values, $\Omega_\Lambda + \Omega_k$ can be either zero, positive, or negative.

We can determine the spatial curvature (or curvature density) as follows:

(i) $\Omega_r + \Omega_m + \Omega_\Lambda < 1 \implies k < 0\ (k = -1)$, open universe

(ii) $\Omega_r + \Omega_m + \Omega_\Lambda = 1 \implies k = 0$, flat universe

(iii) $\Omega_r + \Omega_m + \Omega_\Lambda > 1 \implies k > 0\ (k = 1)$, closed universe

Let us play with the Friedmann equations again. We have

$$\begin{aligned}\Omega_i(t) = \frac{8\pi G}{3H^2}\rho_i(t) &= \frac{8\pi G}{3H_0^2}\rho_{i_0}\left(\frac{H_0}{H}\right)^2 a^{-3(1+\omega_i)},\ i = r, m, \Lambda \\ &= \Omega_{i_0}\left(\frac{H_0}{H}\right)^2 a^{-3(1+\omega_i)} \\ &= \Omega_{i_0}\left(\frac{H_0}{H}\right)^2 (1+z)^{3(1+\omega_i)},\end{aligned}$$

where $(8\pi G/3H_0^2)\rho_{i_0} \coloneqq \Omega_{i_0}$. Similarly,

$$\begin{aligned}\Omega_k(t) = -\frac{kc^2}{H^2R^2} &= -\frac{kc^2}{H_0^2R^2}\left(\frac{H_0}{H}\right)^2 a^{-2} \\ &= \Omega_{k_0}\left(\frac{H_0}{H}\right)^2 a^{-2} \\ &= \Omega_{k_0}\left(\frac{H_0}{H}\right)^2 (1+z)^2,\end{aligned}$$

We had $\Omega_r(t) + \Omega_m(t) + \Omega_\Lambda(t) + \Omega_k(t) = 1$ for all t. The same holds for the present time:

$$\Omega_{r_0} + \Omega_{m_0} + \Omega_{\Lambda_0} + \Omega_{k_0} = 1.$$

Next, we write the second Friedmann equation in terms of redshift.

$$\begin{aligned}
H^2 &= \frac{8\pi G}{3}\sum_i \rho_i - \frac{kc^2}{R^2} \\
&= \frac{8\pi G}{3}(\rho_{r_0} a^{-4} + \rho_{m_0} a^{-3} + \rho_\Lambda) - \frac{kc^2}{R^2} \\
&= \frac{8\pi G}{3}\left(\frac{3H_0^2\Omega_{r_0}}{8\pi G}a^{-4} + \frac{3H_0^2\Omega_{m_0}}{8\pi G}a^{-3} + \frac{3H_0^2\Omega_{\Lambda_0}}{8\pi G}\right) + \Omega_{k_0}H_0^2 a^{-2} \\
&= H_0^2(\Omega_{r_0}a^{-4} + \Omega_{m_0}a^{-3} + \Omega_{\Lambda_0} + \Omega_{k_0}a^{-2}) \\
&= H_0^2(\Omega_{r_0}(1+z)^4 + \Omega_{m_0}(1+z)^3 + \Omega_{\Lambda_0} + \Omega_{k_0}(1+z)^2)
\end{aligned}$$

For the first Friedmann equation, one has

$$\ddot{R} = -\frac{4\pi G}{3}\sum_j \left(\rho_j + \frac{p_j}{c^2}\right) R + \frac{\Lambda c^2}{3} R$$

Recall that $p_i = \omega_i c^2 \rho_i$ and $p_\Lambda = -c^2\rho_\Lambda$. Then,

$$\begin{aligned}
\frac{\ddot{R}R}{\dot{R}^2} &= -\frac{4\pi G}{3}\sum_j (1+3\omega_j)\rho_j H^{-2} + \frac{\Lambda c^2}{3}H^{-2} \\
&= -\frac{4\pi G}{3H^2}\sum_j \rho_j(1+3\omega_j), \;\; j = r, m, \Lambda \\
&= -\frac{4\pi G}{3H^2}(2\rho_r + \rho_m - 2\rho_\Lambda)
\end{aligned}$$

Recall that the deceleration parameter is given

$$q(t) = -\frac{\ddot{R}R}{\dot{R}^2}$$

Then, the first Friedmann equation becomes

$$q = \frac{1}{2}(2\Omega_r + \Omega_m - 2\Omega_\Lambda)$$

4 Analytical Solutions to Cosmological Field Equations

4.1 Friedmann Models ($\Lambda = 0$)

All Friedmann Models have a Big Bang singularity in their past time

$$t \to 0 \implies \begin{cases} a(t) \to 0 \\ \rho(t) \to \infty \\ p(t) \to \infty \end{cases}$$

For all Friedmann models, the age of the Universe t_0 is less than the Hubble time, i.e., $t_0 < 1/H_0$.

We obtained

$$H^2 = H_0^2 \left(\Omega_{r_0} a^{-4} + \Omega_{m_0} a^{-3} + \Omega_{k_0} a^{-2} + \Omega_{\Lambda_0}\right),$$

where $H = \dot{a}/a$. Then,

$$\begin{aligned}
\dot{a}^2 &= H_0^2(\Omega_{r_0} a^{-2} + \Omega_{m_0} a^{-1} + \Omega_{k_0} + \Omega_{\Lambda_0} a^2) \\
\dot{a} &= H_0(\Omega_{r_0} a^{-2} + \Omega_{m_0} a^{-1} + \Omega_{k_0} + \Omega_{\Lambda_0} a^2)^{\frac{1}{2}}
\end{aligned}$$

which is a first-order nonlinear ODE. Integrating,

$$t = \int_0^t dt = \frac{1}{H_0}\int_0^a \frac{da}{(\Omega_{r_0} a^{-2} + \Omega_{m_0} a^{-1} + \Omega_{k_0} + \Omega_{\Lambda_0} a^2)^{\frac{1}{2}}},$$

where we keep in mind that $\Omega_{r_0} + \Omega_{m_0} + \Omega_{k_0} + \Omega_{\Lambda_0} = 1$. We solve this integral for some specific cases:

4.1.1 Matter (Dust) Dominated Universe ($\Omega_{\Lambda_0} = 0, \Omega_{r_0} = 0$)

For this case, the Friedmann equation takes the form

$$\dot{a}^2 = H_0^2(\Omega_{m_0} a^{-1} + \Omega_{k_0}) = H_0^2(\Omega_{m_0} a^{-1} + 1 - \Omega_{m_0})$$

Then,

$$\begin{aligned}
t = \int_0^t dt &= \frac{1}{H_0}\int_0^a \frac{da}{(\Omega_{m_0} a^{-1} + 1 - \Omega_{m_0})^{1/2}} \\
&= \frac{1}{H_0}\int_0^a \left(\frac{a}{\Omega_{m_0} + (1-\Omega_{m_0})a}\right)^{1/2} da
\end{aligned}$$

We separate this into three cases:

1) $\Omega_{m_0} = 1 \;\; (\Omega_{k_0} = 0 \implies k = 0$, flat universe)

In this case, we have

$$\begin{aligned}
t = \frac{1}{H_0}\int_0^a a^{1/2}\, da &= \frac{2}{3H_0} a^{3/2} \\
\implies a(t) &= \left(\frac{3H_0}{2} t\right)^{2/3}
\end{aligned}$$

This case is known as the Einstein - de Sitter (EdS) model

2) $\Omega_{m_0} > 1 \;\; (k = 1$, closed universe)

In this case, we have

$$t = \frac{1}{H_0 \Omega_{m_0}}$$