CENG 384 - Signals and Systems for Computer Engineers
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Homework 3

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1. We should express the synthesis equation for this partial signal as two summations. We may change the signal period to accept just even values and deduct them from the total of all values in the following way to describe it as an odd and an even.
   $x(t) = \sum_{k=-\infty}^{\infty} e^{jk\frac{\pi}{2}t} - 2\sum_{k=-\infty}^{\infty} e^{jk\pi t}$
   The first term is an impulse train with all terms are 1. 1/T=1/4.
   $\sum_{k=-\infty}^{\infty} 4\delta(t-4k)$. $\sum_{k=-\infty}^{\infty} 2\delta(t-2k)$. So x is
   $x(t) = \sum_{k=-\infty}^{\infty} 4\delta(t-4k) - 2\sum_{k=-\infty}^{\infty} 2\delta(t-2k) = 4\sum_{k=-\infty}^{\infty} \delta(t-4k) - \delta(t-2k) = -4\sum_{k=-\infty}^{\infty} \delta(t-4k+2)$

2. (a) $a_k = \frac{1}{T}\int_0^T x(t)e^{-jkw_0t}dt = \frac{1}{4}(\int_0^2 2te^{-jw_0t}dt + \int_2^4 (4-t)e^{-jkw_0t}dt)$
   Calculate the first integral
   $\int udv = uv - \int vdu$
   $u = 2t$ $dv = e^{-jkw_0t}$ $du = 2$ $v = \frac{e^{-jkw_0t}}{-jkw_0t}$

   $\int_0^2 2te^{-jw_0t}dt = 2t\frac{e^{-jkw_0t}}{-jkw_0t} + \int \frac{e^{-jkw_0t}}{jkw_0t}2dt = \frac{2t}{-jkw_0}e^{-jkw_0t} - \frac{2}{(jkw_0)^2}e^{-jkw_0t}\Big|_0^2$
   $= \frac{4}{-jkw_0}e^{-2jkw_0} - \frac{2}{(jkw_0)^2}e^{-2jkw_0} + \frac{2}{(jkw_0)^2}$
   Calculate the second integral
   $\int_2^4 (4-t)e^{-jkw_0t}dt = \int_2^4 4e^{-jkw_0t}dt - \int_2^4 te^{-jkw_0t}dt$
   $\int udv = uv - \int vdu$
   $u = t$ $dv = e^{-jkw_0t}$ $du = 1$ $v = \frac{e^{-jkw_0t}}{-jkw_0t}$

   $\int_2^4 (4-t)e^{-jkw_0t}dt = \int_2^4 4e^{-jkw_0t}dt - (\frac{te^{-jkw_0}}{-jkw_0} + \int \frac{e^{-jkw_0t}}{jkw_0t}dt)$
   $= \frac{4}{-jkw_0}e^{-jkw_0t} + \frac{t}{jkw_0}e^{-jkw_0t} + \frac{1}{(jkw_0)^2}e^{-jkw_0t}\Big|_2^4$
   $= \frac{1}{(jkw_0)^2}e^{-4jkw_0} + \frac{2}{jkw_0}e^{-2jkw_0} - \frac{1}{(jkw_0)^2}e^{-2jkw_0}$

   Sum is
   $1/4(\frac{4}{-jkw_0}e^{-2jkw_0} - \frac{2}{(jkw_0)^2}e^{-2jkw_0} + \frac{2}{(jkw_0)^2} + \frac{1}{(jkw_0)^2}e^{-4jkw_0} + \frac{2}{jkw_0}e^{-2jkw_0} - \frac{1}{(jkw_0)^2}e^{-2jkw_0})$
   $w_0 = \pi/2$
   So result is
   $\frac{1}{2jk\pi}(-2e^{-jk\pi} + \frac{4+2e^{-2jk\pi}-3e^{-jk\pi}}{jk\pi})$

   (b) By derivation property; $a_k = a_k jkw_0$
   Result is
   $\frac{1}{4}(-2e^{-jk\pi} + \frac{4+2e^{-2jk\pi}-3e^{-jk\pi}}{jk\pi})$

3. (a) for $x_1[n]$
   $x_1[n] = cos(\frac{\pi}{2}n)$
   $x_1[n] = \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})$
   $a_1 = 1/2$ and $a_{-1} = 1/2$. Other a values are 0.

   for $x_2[n]$
   $x_2[n] = sin(\frac{\pi}{2}n)$
   $x_2[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$
   $a_1 = 1/2j$ and $a_{-1} = 1/2j$. Other a values are 0.

for $x_3[n]$
$x_3[n] = \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})\frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}) = \frac{1}{4j}e^{j\pi n} - e^{-j\pi n}$
$a_2 = 1/4j$ and $a_{-2} = -1/4j$ Other a values are 0.

(b) say $x_1[n] \xleftrightarrow{\text{FS}} a_k$
and $x_2[n] \xleftrightarrow{\text{FS}} b_k$
then $x_3[n] = x_1[n]x_2[n] \xleftrightarrow{\text{FS}} c_k = \sum_{l=<N>} a_l b_{k-l}$
$c_k = \sum_{-1}^{2} a_l b_{k-l}$
$b_1 = 1/2j$ and $b_{-1} = -1/2j \xrightarrow{flip} b_{-1} = 1/2j$ and $b_1 = -1/2j$
$c_2 = a_1 b_{-1} = -1/4j$
$c_{-2} = a_{-1} b_1 = 1/4j$
$c_0 = 1/4j - 1/4j = 0$

4.

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n}$$

$$a_k = \frac{1}{24}\sum_{n=<24>} x[n]e^{-jk\frac{\pi}{12}n}$$

$$a_k = cos(k\frac{\pi}{3}) + cos(k\frac{\pi}{4}) = \frac{1}{2}(e^{jk\frac{\pi}{3} + e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}})$$

$$\frac{1}{2}(e^{jk\frac{\pi}{3} + e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}}) = \frac{1}{24}\sum_{n=<24>} x[n]e^{-jk\frac{\pi}{12}n}$$

$$12(e^{jk\frac{\pi}{3} + e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}}) = \sum_{n=<24>} x[n]e^{-jk\frac{\pi}{12}n}$$

From this equation we can get $x[n]$ values:
For $n = -4$:

$$x[-4]e^{jk\frac{\pi}{3}} = 12e^{jk\frac{\pi}{3}} \rightarrow x[-4] = 12$$

For $n = 4$:

$$x[4]e^{jk\frac{-\pi}{3}} = 12e^{jk\frac{-\pi}{3}} \rightarrow x[4] = 12$$

For $n = -3$:

$$x[-3]e^{jk\frac{\pi}{4}} = 12e^{jk\frac{\pi}{4}} \rightarrow x[-3] = 12$$

For $n = 3$:

$$x[3]e^{jk\frac{-\pi}{4}} = 12e^{jk\frac{-\pi}{4}} \rightarrow x[3] = 12$$

All the other $x[n]$'s from 0 to 23 is $0x$ except these values -4, 4, -3, 3.

5. (a)

$$sin(\frac{6\pi}{13}(n+N) + \frac{\pi}{2}) = sin(\frac{6\pi}{13}n + \frac{\pi}{2})$$

In order to be equivalent, the equation $\frac{6\pi}{13}N = 2\pi m$ must be hold, for any $N \in \mathbb{N}$, and $m \in \mathbb{N}$.

$$\frac{6\pi}{13}N = 2\pi m$$

$$\frac{6}{13}N = 2m$$

$$N = \frac{13m}{3}$$

For $m = 3$, $N = 13$ which is fundamental period of this signal.

(b)

$$sin(\frac{6\pi}{13}n + \frac{\pi}{2}) = \frac{1}{2j}(e^{j(\frac{6\pi}{13}n + \frac{\pi}{2})} - e^{-j(\frac{6\pi}{13}n + \frac{\pi}{2})})$$

$w_0 = \frac{2\pi}{13}$. Then,

$$a_3 = \frac{1}{2j}e^{j\frac{\pi}{2}}$$

$$\frac{1}{2j}e^{j\frac{\pi}{2}} = \frac{1}{2j}(cos(\frac{\pi}{2}) + jsin(\frac{\pi}{2}))$$

Since $cos(\frac{\pi}{2}) = 0$, and $sin(\frac{\pi}{2}) = 1$:

$$\frac{1}{2j}(cos(\frac{\pi}{2}) + jsin(\frac{\pi}{2})) = \frac{1}{2j}j$$

$$a_3 = \frac{1}{2}$$

$$a_{-3} = \frac{-1}{2j} e^{-j\frac{\pi}{2}}$$

$$\frac{-1}{2j} e^{-j\frac{\pi}{2}} = \frac{-1}{2j}(cos(\frac{-\pi}{2}) + jsin(\frac{-\pi}{2}))$$

Since $cos(\frac{-\pi}{2}) = 0$, and $sin(-\frac{\pi}{2}) = -1$:

$$\frac{-1}{2j}(cos(\frac{-\pi}{2}) + jsin(\frac{-\pi}{2})) = \frac{-1}{2j} j$$

$$a_{-3} = \frac{-1}{2}$$

All other a_k's are zero.

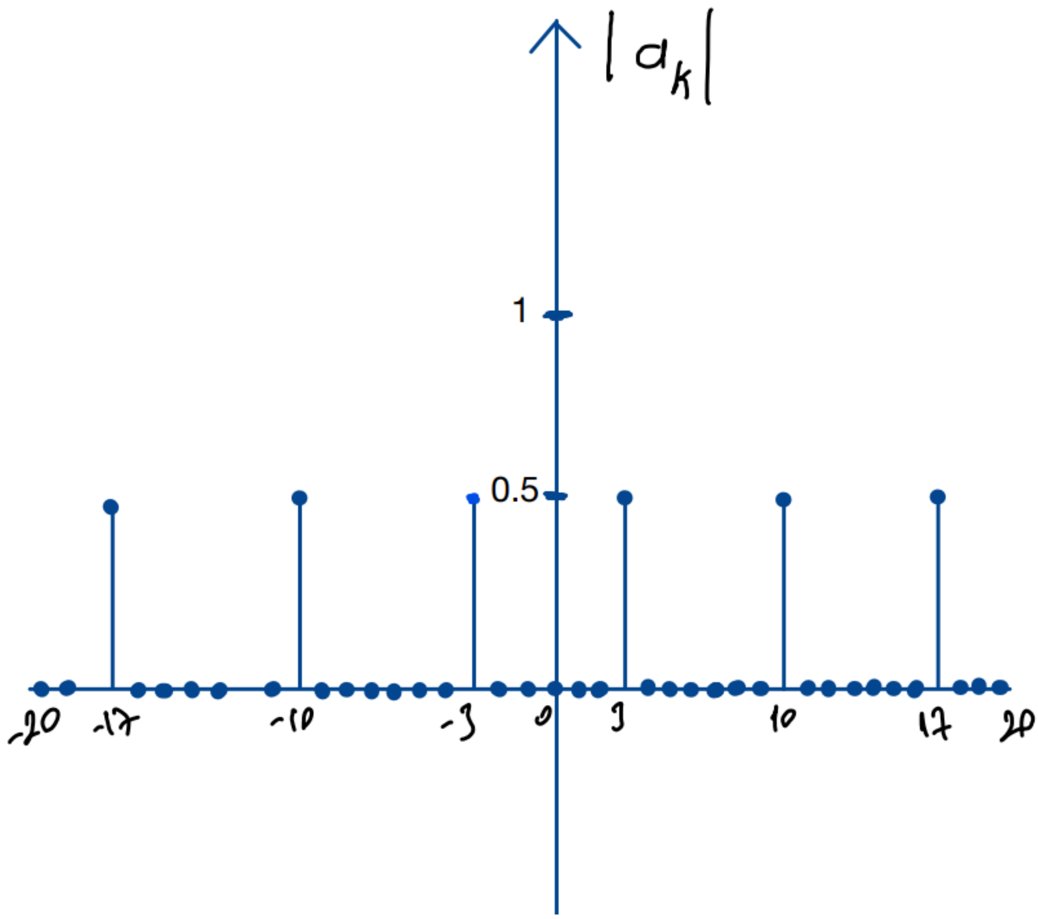

Figure 1: Magnitude Figure

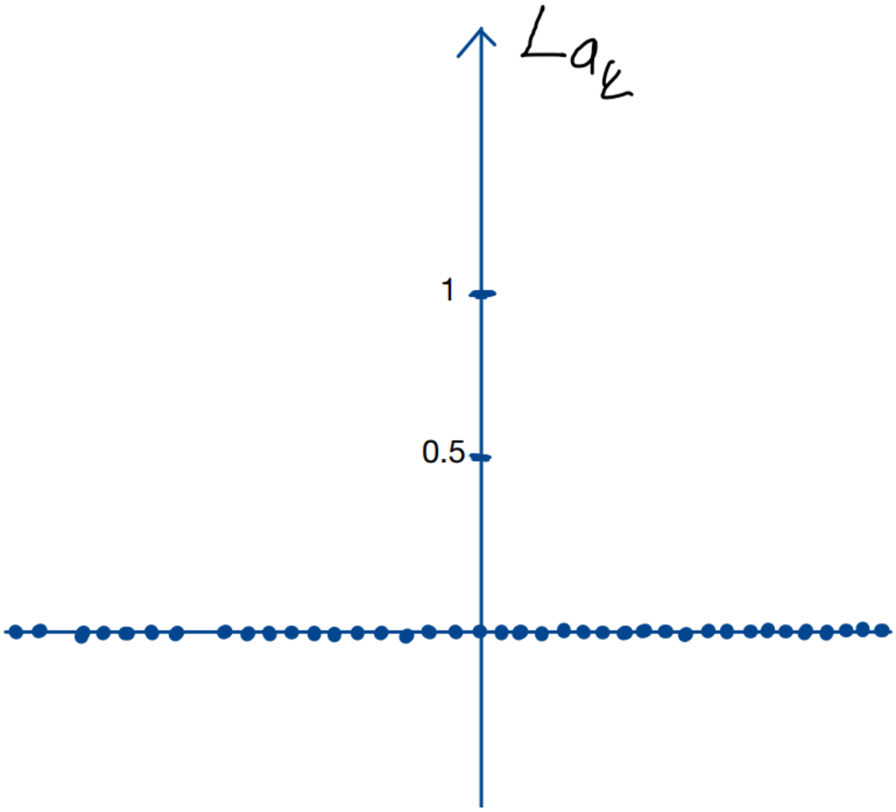

Figure 2: Phase Figure

6. (a) From the lookup table we have:

$$e^{-at}u(t), Re\{a\} > 0 \overset{\text{FT}}{\longleftrightarrow} \frac{1}{a+jw}$$

$$e^{-3t}u(t) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{3+jw}$$

From the time scaling property of fourier transform:

$$x(at) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{a}X(\frac{jw}{a})$$

$$e^{-\frac{3t}{4}}u(t) \overset{\text{FT}}{\longleftrightarrow} 4\frac{1}{4jw+\frac{3}{4}}$$

If we multiply a function by any constant then we must multiply the Fourier transform by the same constant. Multiply with $\frac{1}{4}$:

$$\frac{1}{4}e^{-\frac{3t}{4}}u(t) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{4jw+\frac{3}{4}}$$

Therefore, the result is:

$$h(t) = \frac{1}{4}e^{-\frac{3t}{4}}u(t)$$

(b) From the lookup table:

$$y(t) = (e^{-5t} - e^{-10t})u(t)$$

$$Y(jw) = \frac{1}{5+jw} - \frac{1}{10+jw} = \frac{5}{(5+jw)(10+jw)}$$

We know that:

$$Y(jw) = X(jw)H(jw) = \frac{5}{(5+jw)(10+jw)}$$

where $H(jw) = \frac{1}{4jw+3}$

$$X(jw) = \frac{5(4jw+3)}{(5+jw)(10+jw)}$$

To convert it to time domain we can transform it:

$$\frac{c_1}{5+jw} + \frac{c_2}{10+jw} = \frac{10c_1 + 5c_2 + jw(c_1+c_2)}{(5+jw)(10+jw)} = \frac{5(4jw+3)}{(5+jw)(10+jw)}$$

From this equation we get:

$$10c_1 + 5c_2 = 15$$
$$c_1 + c_2 = 20$$

From these equations we found that:

$$c_1 = -17$$
$$c_2 = -37$$
$$X(jw) = \frac{-17}{5+jw} + \frac{37}{10+jw}$$

From lookup table we can convert this to time domain:

$$x(t) = -17e^{-5t} + 37e^{-10t}$$

7.
```
import numpy as np
import matplotlib.pyplot as plt

def x(t):
    return np.cos(np.pi * t / 3) + 2 * np.cos(np.pi * t + np.pi / 2)

def fourier_coefficients(n_max, T):
    a = []
    b = []
    for n in range(-n_max, n_max + 1):
        a_n = (2 / T) * np.trapz(x(t) * np.cos(2 * np.pi * n * t / T), t)
        b_n = (2 / T) * np.trapz(x(t) * np.sin(2 * np.pi * n * t / T), t)
        a.append(a_n)
        b.append(b_n)
    return a, b
t = np.linspace(0, 6, 1000)

T = 6
n_max = 10
a_coeffs, b_coeffs = fourier_coefficients(n_max, T)


threshold = 1e-15
magnitude = np.sqrt(np.array(a_coeffs)**2 + np.array(b_coeffs)**2)
phase = np.arctan2(b_coeffs, a_coeffs)
for k in range(len(magnitude)):
  if magnitude[k] < threshold:
    magnitude[k] = 0
    phase[k] = 0
    ak[k] = 0


plt.figure(figsize=(10, 6))
plt.subplot(211)
plt.stem(range(-n_max, n_max + 1), magnitude, use_line_collection=True)
plt.title('Magnitude of Fourier Coefficients')
plt.xlabel('n')
plt.ylabel('Magnitude')

plt.subplot(212)
plt.stem(range(-n_max, n_max + 1), phase, use_line_collection=True)
plt.title('Phase of Fourier Coefficients')
plt.xlabel('n')
plt.ylabel('Phase')
plt.tight_layout()

plt.show()


print("Fundamental Period T =", T)
```

```
print("Simplified Fourier Series Representation:")
for n in range(len(a_coeffs)):
    if n == 0:
        print("a0/2 =", a_coeffs[n] / 2)
    else:
        print("a{} = {}, b{} = {}".format(n, a_coeffs[n], n, b_coeffs[n]))
```

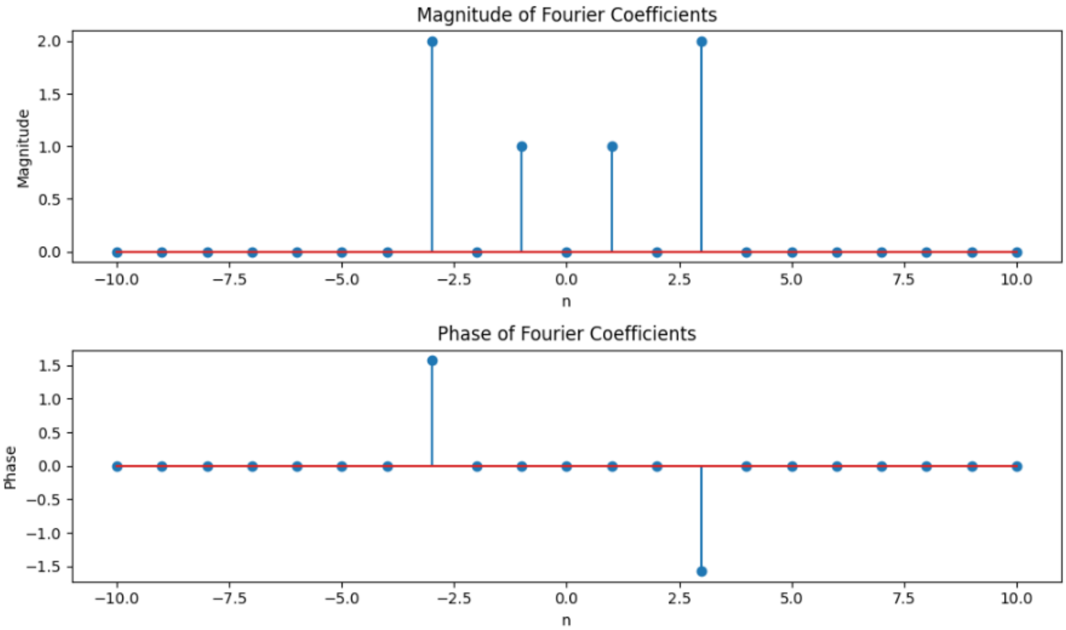

Figure 3: Plots Figure

```
Fundamental Period T = 6
Simplified Fourier Series Representation:
a0/2 = 1.4340380734741604e-16
a1 = -1.6653345369377348e-16, b1 = -1.5265566588595902e-16
a2 = -5.551115123125783e-17, b2 = 1.0177044392397268e-16
a3 = 1.850371707708594e-17, b3 = 4.625929269271485e-17
a4 = -3.700743415417188e-17, b4 = -7.401486830834377e-17
a5 = -1.1102230246251565e-16, b5 = 1.2027416100105862e-16
a6 = -2.9605947323337506e-16, b6 = 2.5905203907920316e-16
a7 = 9.25185853854297e-17, b7 = 2.0
a8 = 0.0, b8 = -7.401486830834377e-17
a9 = 1.0, b9 = -1.850371707708594e-17
a10 = -7.401486830834377e-17, b10 = 0.0
a11 = 1.0, b11 = 1.850371707708594e-17
a12 = 0.0, b12 = 7.401486830834377e-17
a13 = 9.25185853854297e-17, b13 = -2.0
a14 = -2.9605947323337506e-16, b14 = -2.5905203907920316e-16
a15 = -1.1102230246251565e-16, b15 = -1.2027416100105862e-16
a16 = -3.700743415417188e-17, b16 = 7.401486830834377e-17
a17 = 1.850371707708594e-17, b17 = -4.625929269271485e-17
a18 = -5.551115123125783e-17, b18 = -1.0177044392397268e-16
a19 = -1.6653345369377348e-16, b19 = 1.5265566588595902e-16
a20 = 2.8680761469483207e-16, b20 = -2.1279274638648832e-16
```

Figure 4: simplified Fourier series representation Figure