# CENG 384 - Signals and Systems for Computer Engineers
# Spring 2024
# Homework 4

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1. (a) Differential equation of the system is:

$$\frac{d^2y(t)}{d^2t} + 5\frac{dy(t)}{dt} + 6y(t) = 4\frac{dx(t)}{dt} + x$$

(b) Using the linearity and differentiation properties, we can convert it to frequency domain:

$$(j^2w^2 + 5jw + 6)Y(jw) = (4jw + 1)X(jw)$$

Frequency response of the system is:

$$H(jw) = \frac{Y(jw)}{X(jw)}$$

$$\frac{Y(jw)}{X(jw)} = \frac{4jw + 1}{j^2w^2 + 5jw + 6}$$

(c) We should modify the equation to find proper relation from the look-up table:

$$\frac{a}{wj + 2} + \frac{b}{wj + 3} = \frac{4jw + 1}{j^2w^2 + 5jw + 6}$$

After solving the equation, we get:

$$a = -7$$

$$b = 11$$

So the final version is:

$$H(jw) = \frac{-7}{wj + 2} + \frac{11}{wj + 3}$$

We know that:

$$F\{H(jw)\} = h(t)$$

From look-up table, we get:

$$h(t) = (-7e^{-2t} + 11e^{-3t})$$

(d) We know that:

$$y(t) = x(t) * h(t)$$

From the multiplication property:

$$Y(jw) = X(jw)H(jw)$$

We should find F.T. of $x(t)$:

$$x(t) = \frac{1}{4}e^{\frac{-t}{4}}u(t)$$

From the look-up table we get:

$$X(jw) = \frac{\frac{1}{4}}{jw + \frac{1}{4}}$$

$$Y(jw) = X(jw)H(jw)$$

$$Y(jw) = \frac{\frac{1}{4}}{jw + \frac{1}{4}}\left(\frac{-7}{jw + 2} + \frac{11}{jw + 3}\right)$$

$$= \frac{\frac{-7}{4}}{(jw + \frac{1}{4})(jw + 2)} + \frac{\frac{11}{4}}{(jw + \frac{1}{4})(jw + 3)}$$

$$\frac{\frac{-7}{4}}{(jw+\frac{1}{4})(jw+2)} = \frac{a}{jw+\frac{1}{4}} + \frac{b}{jw+2}$$

After solving the equation we get:

$$a = -1$$
$$b = 1$$

$$\frac{\frac{11}{4}}{(jw+\frac{1}{4})(jw+3)} = \frac{c}{jw+\frac{1}{4}} + \frac{d}{jw+3}$$

After solving the equation we get:

$$c = 1$$
$$d = -1$$

Summation of the equation:

$$\frac{a}{jw+\frac{1}{4}} + \frac{b}{jw+2} + \frac{c}{jw+\frac{1}{4}} + \frac{d}{jw+3}$$
$$Y(jw) = \frac{1}{jw+2} + \frac{-1}{jw+3}$$

From look-up table, we find the equation:

$$y(t) = (e^{-2t} - e^{-3t})u(t)$$

2. (a)

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{jw+4}{6-w^2+5jw}$$

$$F^{-1}(Y(jw)(6+(jw)^2+5jw)) = F^{-1}(X(jw)(4+jw))$$

Equation is:

$$\frac{d^2y(t)}{dt} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

(b)

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{jw+4}{6-w^2+5jw} = \frac{2}{2+jw} - \frac{1}{3+jw}$$

$$h(t) = F^{-1}(H(jw) = F^{-1}(\frac{2}{2+jw} - \frac{1}{3+jw})$$

from table we have

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

(c)

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t)$$

from table we have

$$X(jw) = \frac{1}{4+jw} - \frac{1}{(4+jw)^2} = \frac{3+jw}{(4+jw)^2}$$
$$Y(jw) = X(jw)H(jw) = \frac{3+jw}{(4+jw)^2}\frac{jw+4}{-w^2+5jw+6} = \frac{jw+3}{jw+4}\frac{1}{(jw+2)(jw+3)}$$
$$Y(jw) = \frac{1}{(4+jw)(2+jw)} = \frac{1/2}{2+jw} - \frac{1/2}{4+jw}$$

(d)

$$y(t) = F^{-1}(Y(jw)) = F^{-1}(\frac{1/2}{2+jw} - \frac{1/2}{4+jw})$$

from table

$$y(t) = 1/2e^{-2t}u(t) - 1/2e^{-4t}u(t)$$

3. (a)

$$x[n] = (2/3)^n u[n]$$
$$y[n] = 2/3n(2/3)^n u[n]$$

From table, we have

$$F(x[n]) = \frac{1}{1 - 2/3e^{-jw}} = X(e^{jw}) = \frac{3e^{jw}}{3e^{jw} - 2}$$
$$y[n] = 2/3nx[n]$$

Differentiation in frequency property:

$$F(y[n] = 2/3j\frac{dX(e^{jw})}{dw}$$
$$Y(e^{jw}) = 2/3j\frac{-6je^{jw}}{(3e^{jw} - 2)^2} = \frac{4e^{jw}}{(3e^{jw} - 2)^2}$$
$$H(e^{jw}) = \frac{4e^{jw}}{(3e^{jw} - 2)^2}\frac{3e^{jw} - 2}{3e^{jw}}$$
$$H(e^{jw}) = \frac{4}{9e^{jw} - 6}$$

(b) We can rewrite H as like

$$H(e^{jw}) = \frac{-2}{3 - 9/2e^{jw}} = -2/3\frac{1}{1 - 1/(\frac{2}{3}e^{-jw})}$$

Say $r = \frac{2}{3}e^{-jw}$ then

$$H(e^{jw}) = \frac{2}{3}\frac{r}{1 - r}$$
$$H(e^{jw}) = \frac{4}{9}e^{-jw}\frac{1}{1 - \frac{2}{3}e^{-jw}} = \frac{4}{9}e^{-jw}X(e^{jw})$$

We see that $H(e^{jw})$ is a time shifted version of x[n] .

$$F^{-1}(H(e^{jw})) = \frac{4}{9}x[n-1]$$
$$h[n] = \frac{4}{9}x[n-1] = (\frac{2}{3})^{n+1}u[n-1]$$

(c)

$$H(e^{jw}) = \frac{4}{9e^{jw} - 6} = \frac{Y(e^{jw})}{X(e^{jw})}$$
$$-6Y(e^{jw}) + 9e^{jw}Y(e^{jw}) = 4X(e^{jw})$$
$$-6y[n] + 9y[n+1] = 4x[n]$$
$$9y[n] = 4x[n-1] + 6y[n-1]$$
$$y[n] = \frac{4}{9}x[n-1] + \frac{2}{3}y[n-1]$$

(d) Block diagram representation of the system is:

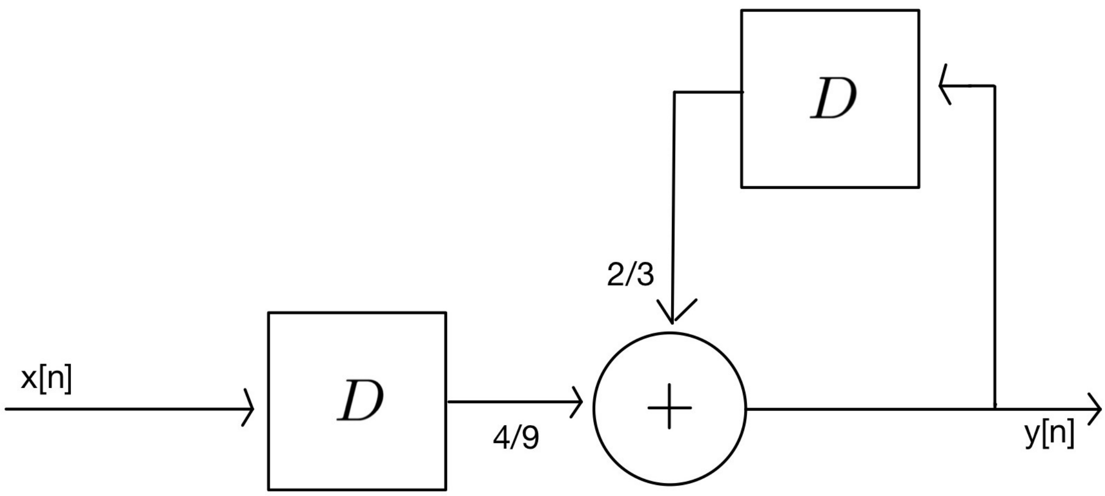


Figure 1: Block Diagram Representation of The System

4. (a) Difference equation of the system is:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(b) Using time shifting and linearity property, we can transform the equation:

$$Y(e^{jw})(\frac{-3}{4}e^{-jw} + \frac{1}{8}e^{-2jw} + 1) = 2X(e^{jw})$$

We know that:

$$H(jw) = \frac{Y(e^{jw})}{X(e^{jw})}$$
$$= \frac{16}{e^{-2jw} - 6e^{-jw} + 8}$$

(c)

$$\frac{16}{e^{-2jw} - 6e^{-jw} + 8} = \frac{a}{e^{-jw} - 4} + \frac{b}{e^{-jw} - 2}$$

After solving the equation, we get:

$$a = 8$$
$$b = -8$$

We should convert the equation to proper version:

$$H(jw) = \frac{-2}{1 - \frac{e^{-jw}}{4}} + \frac{4}{1 - \frac{e^{-jw}}{2}}$$

From look-up table, and linearity property, we get:

$$h[n] = (-2(\frac{1}{4})^n + 4(\frac{1}{2})^n)u[n]$$

(d)

$$x[n] = (\frac{1}{4})^n[n]$$

From look-up table, we get:

$$X(e^{jw}) = \frac{1}{1 - \frac{1}{4}e^{-jw}}$$

We know that:

$$Y(e^{jw}) = X(e^{jw})H(e^{jw})$$
$$= \frac{4}{4 - e^{-jw}}(\frac{-8}{4 - e^{-jw}} + \frac{8}{2 - e^{-jw}})$$
$$\frac{-32}{(4 - e^{-jw})^2} + \frac{32}{(4 - e^{-jw})(2 - e^{-jw})}$$
$$\frac{32}{(4 - e^{-jw})(2 - e^{-jw})} = \frac{a}{4 - e^{-jw}} + \frac{b}{2 - e^{-jw}}$$

After solving above equation, we get:

$$a = -16$$
$$b = 16$$

Overall modified equation is:

$$Y(e^{jw}) = \frac{-2}{(1 - \frac{e^{-jw}}{4})^2} - \frac{4}{1 - \frac{e^{-jw}}{4}} + \frac{8}{1 - \frac{e^{-jw}}{2}}$$

From look-up table, we get:

$$y[n] = (-2(n+1)(\frac{1}{4})^n - 4(\frac{1}{4})^n + 8(\frac{1}{2})^n)u[n]$$

5. The impulse response of this system is:

$$h[n] = h_1[n] + h_2[n]$$

Taking the Fourier Transform of both sides of the equation, we get:

$$H(e^{jw}) = \frac{5e^{-jw} - 12}{e^{-2jw} - 7e^{-jw} + 12} = H_1(e^{jw}) + H_2(e^{jw})$$

Fourier transform of $h_1[n]$ is:

$$H_1(e^{jw}) = \sum_{n=0}^{\infty}(\frac{1}{3})^n e^{-jwn} = \sum_{n=0}^{\infty}(\frac{1}{3}e^{-jw})^n$$

$$H_1(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}} = \frac{3}{3-e^{-jw}}$$

Put $H_1(e^{jw})$ to the equation to find $h_2[n]$:

$$H_2(e^{jw}) = H(e^{jw}) - H_1(e^{jw}) = \frac{5e^{-jw}-12}{e^{-2jw}-7e^{-jw}+12} = H_1(e^{jw}) + H_2(e^{jw}) - \frac{3}{3-e^{-jw}}$$

$$= \frac{2e^{-2jw}+24e^{-jw}-72}{e^{-3jw}-4e^{-2jw}-9e^{-jw}+36}$$

$$H_2(e^{jw}) = -2\frac{1}{1-\frac{1}{4}e^{-jw}}$$

We can find the inverse of the Fourier Transform of $H_2(e^{jw})$, we get:

$$h_2[n] = -2(\frac{1}{4})^n u[n]$$

6.
```
import numpy as np
import matplotlib.pyplot as plt

def x_n(n):
    return (1/2)**np.abs(n)

n_min = -50
n_max = 50
n = np.arange(n_min, n_max + 1)

omega = np.linspace(-20, 20, 1000)

X_omega = np.zeros(len(omega), dtype=complex)
for i, w in enumerate(omega):
    X_omega[i] = np.sum(x_n(n) * np.exp(-1j * w * n))

magnitude_X_omega = np.abs(X_omega)
phase_X_omega = np.angle(X_omega)

threshold = 1e-10
phase_X_omega[np.abs(phase_X_omega) < threshold] = 0

plt.figure(figsize=(14, 6))

plt.subplot(1, 2, 1)
plt.plot(omega, magnitude_X_omega)
plt.title('Magnitude of DTFT of $x[n] = (1/2)^{|n|}$')
plt.xlabel('Frequency ()')
plt.ylabel('Magnitude |X(e^{j})|')
plt.grid(True)

plt.subplot(1, 2, 2)
plt.plot(omega, phase_X_omega)
plt.title('Phase of DTFT of $x[n] = (1/2)^{|n|}$')
plt.xlabel('Frequency ()')
plt.ylabel('Phase X(e^{j})')
plt.grid(True)

plt.tight_layout()
plt.show()
```

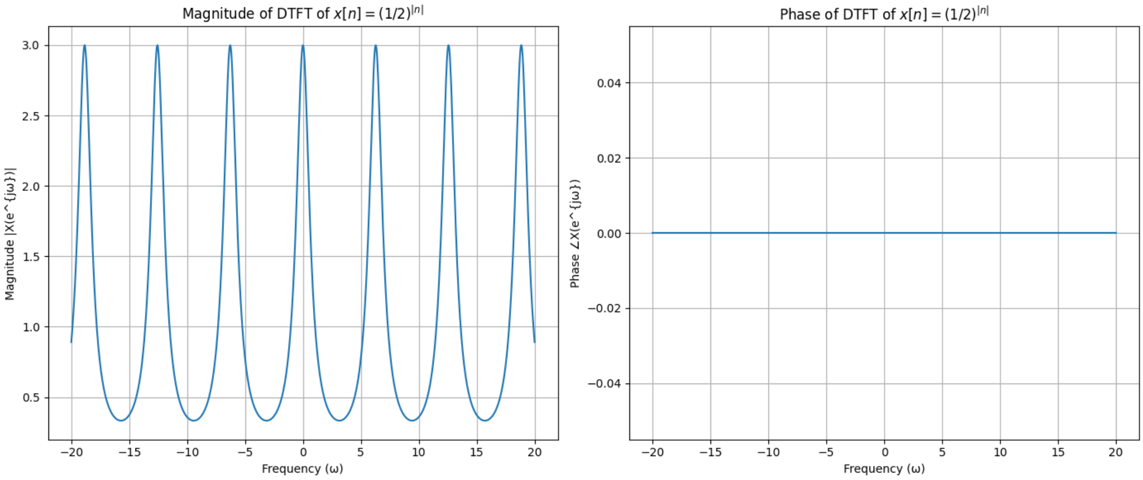


Figure 2: Plots Figure