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Answer 1

a) Assume any negative real number($y \in \mathbb{R}^-$). There is no x such that $f_1(x) = y$. Say $y = -4$, then $x^2 = -4$. Since there is no $x \in \mathbb{R}$ satisfies $x^2 = -4$, function f_1 is not surjective.

For $y \in \mathbb{R}$,$x \in \mathbb{R}^+$ and $f_1(x) = y$, there are more than one x variable in real numbers corresponding to this function. For $a, b \in \mathbb{R}$, assume $f_1(a) = 4$ and $f_1(b) = 4$. The equation satisfies if $a = 2, b = -2$. Since $a \neq b$ and $f_1(a) = f_1(b)$, f_1 is not injective.

b) Assume any negative real number($y \in \mathbb{R}^-$). There is no x such that $f_1(x) = y$. Say $y = -4$, then $x^2 = -4$. Since there is no non-negative real number x ($x \in \mathbb{R}^+$ or $x = 0$) satisfies $x^2 = -4$, function f_2 is not surjective.

For all non-negative real number, $f_2(x) = x^2$. Say $y = x^2$ and y is non-negative. Since the number y always non-negative, there exist $\sqrt{y} = x \in \mathbb{R}$. For all $a, b \in R^+$, if $f_2(a) = f_2(b)$ then $a^2 = b^2$. Since the domain of the function is non-negative real numbers, a and b both should be non-negative and equal. Since $a = b$, the function f_2 is injective.

c) Consider an arbitrary element $y \in \mathbb{R}^+$. For $x \in \mathbb{R}$, $y = x^2$. Since y is always non-negative, there exists $\sqrt{y} \in \mathbb{R}$ for all $y \in \mathbb{R}^+$. Therefore, the function f_3 is surjective.

For $y \in \mathbb{R}^+$,$x \in \mathbb{R}^+$ and $f_3(x) = y$, there are more than one x variable in real numbers corresponding to this function. For $a, b \in \mathbb{R}$, assume $f_3(a) = 4$ and $f_3(b) = 4$. The equation satisfies if $a = 2, b = -2$. Since $a \neq b$ and $f_3(a) = f_3(b)$, f_3 is not injective

d) Consider an arbitrary element $y \in \mathbb{R}^+$. For $x \in \mathbb{R}^+$, $y = x^2$. Since y is always non-negative, there exists $\sqrt{y} \in \mathbb{R}^+$ for all $y \in \mathbb{R}^+$. Therefore, the function f_4 is surjective.

For all non-negative real number, $f_4(x) = x^2$. Say $y = x^2$ and y is non-negative. Since the number y always non-negative, there exist $\sqrt{y} \in \mathbb{R}$. For all $a, b \in \mathbb{R}^+$, if $f_4(a) = f_4(b)$ then $a^2 = b^2$. Since the domain of the function is non-negative real numbers, a and b both should be non-negative and equal. Since $a = b$, the function f_4 is injective.

Answer 2

a)For all function $f : A \subset \mathbb{Z} \to \mathbb{R}$ and $x \in A$, if $x = x_0$ then $\|f(x) - f(x_0)\| = 0$ and $\|x - x_0\| = 0$. Because of that, the expression $(\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \epsilon)$ is true always for some $\delta \in \mathbb{R}^+$ and for all $\epsilon \in \mathbb{R}^+$(Left and right side of the expression are always true i.e. T $\to$ T). If $x \neq x_0$, $\|x - x_0\| > 1$ then if we choose arbitrary $\delta = 0.3$, the expression $\|x - x_0\| < \delta$ is always false for arbitrary δ. Therefore, whole expression $(\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \epsilon)$ is always true(Since left side of the expression is false, whole expression is always true for $\exists\delta \in \mathbb{R}^+$). Since the expression $(\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \epsilon)$ is true for some δ and for all ϵ, the function f is continous.

b)Since the codomain of the function is $\mathbb{Z}$, there is two possibilities of $\|f(x) - f(x_0)\|$. It is 0 or $\|f(x) - f(x_0)\| \geq 1$. If it is the case $\|f(x) - f(x_0)\| = 0$, the right side of the expression is true. Therefore, the whole expression $(\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \epsilon)$ is true always for $\exists\delta \in \mathbb{R}^+$ and $\forall\epsilon \in \mathbb{R}^+$. However, if $\|f(x) - f(x_0)\| \geq 1$, then the right side of the equation $\|f(x) - f(x_0)\| < \epsilon)$ may be false for $\exists\epsilon \in \mathbb{R}^+$. If it is the case the whole expression $(\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \epsilon)$ is false and this prevents the function from being continuous. Therefore, $\|f(x) - f(x_0)\|$ should be zero. If it is zero, it should be constant function because there should be no change in the codomain of x for all $x \in \mathbb{R}$.

Answer 3

a) For the cartesian product, we can create a table.
$A_{11}, A_{21}, A_{31}, A_{41}$......
$A_{12}, A_{22}, A_{32}, A_{42}$......
$A_{13}, A_{23}, A_{33}, A_{43}$......
....
We can map it with $\mathbb{Z}^+$ like that:
$1 = A_{11}$
$2 = A_{21}$
$3 = A_{12}$
$4 = A_{13}$
$5 = A_{22}$
....
....
Since we can map the Cartesian product of countable sets with the natural numbers, it is countable infinite.

b)Assume that an infinite countable product of the set X = 0, 1 with itself is countable.
$a_1 = (a_{11}000000000.......)$
$a_2 = (0a_{22}00000001.......)$
$a_3 = (00a_{33}0000011.......)$
....
b is a tuple with 0,1 and also, b_i is the i^{th} index of the b. If $a_{ii} = 0$ then $b_i = 1$, otherwise $b_i = 0$. So b is different from a_{11},
b is different from a_{22},
b is different from a_{33},
....
In this way, we can create such b such that it is different from all sets. So infinite countable product of the set X = 0, 1 with itself is uncountable.

Answer 4

a) We will compare the functions $log(n)^2$ **and** $\sqrt{n} \cdot log(n)$
We should check the $\lim_{n\to+\infty} \dfrac{log(n)^2}{\sqrt{n} \cdot log(n)} = \lim_{n\to+\infty} \dfrac{log(n)}{\sqrt{n}}$

Apply L'Hospital's rule $\rightarrow \lim_{n\to+\infty} \dfrac{\dfrac{1}{n \cdot ln(10)}}{\dfrac{1}{\sqrt{n}}}$

$=\lim_{n\to+\infty} \dfrac{1}{\sqrt{n \ln(10)}}$ **= 0 Therefore,** $log(n)^2$ **is** $O(\sqrt{n} \cdot log(n))$

b) We will compare the functions $\sqrt{n} \cdot log(n)$ **and** n^{50}
We should check the $\lim_{n\to+\infty} \dfrac{\sqrt{n} \cdot log(n)}{n^{50}} = \lim_{n\to+\infty} \dfrac{log(n)}{n^{\frac{99}{2}}}$ **By L'Hospital's rule it is**

$\lim_{n\to+\infty} \dfrac{\dfrac{1}{n \cdot ln(10)}}{49.5 \cdot n^{\frac{97}{2}}} = \lim_{n\to+\infty} \dfrac{1}{49.5 \cdot n^{\frac{95}{2}}}$ **= 0 Therefore,** $\sqrt{n} \cdot log(n)$ **is** $O(n^{50})$**.**

c)We will compare the functions n^{50} **and** $n^{51} + n^{49}$
Check the limit $\lim_{n\to+\infty} \dfrac{n^{50}}{n^{51} + n^{49}} = \lim_{n\to+\infty} \dfrac{1}{n^1 + \dfrac{1}{n}}$

$= \lim_{n\to+\infty} \dfrac{1}{n^1 + 0}$ **= 0**
Therefore, n^{50} **is** $O(n^{51} + n^{49})$

d)We will compare the functions $(n^{51}+n^{49})$ **and** 2^n**.**

$\lim_{n\to+\infty}\dfrac{n^{51}+n^{49}\cdot\dfrac{1}{n^2}}{2^n}$ $\rightarrow$ **51 times applying L'Hospital rule**$\rightarrow$

$\lim_{n\to+\infty}\dfrac{(51)!+(49)!}{(ln(2))^{51}\cdot 2^n}=0$ **Since it is 0,** $(n^{51}+n^{49})$ **is** $O(2^n)$**.**

e)We will compare 5^n **and** 2^n**.**

$\lim_{n\to+\infty}(\dfrac{2^n}{5^n})=\lim_{n\to+\infty}(\dfrac{2}{5})^n$ **It goes to 0 because** $\dfrac{2}{5}<1$. **Since it is 0** 2^n **is** $O(5^n)$**.**

f)We will compare the functions $(n!)^2$ **and** 5^n **by ratio test.**

$\lim_{n\to+\infty}\dfrac{\dfrac{5^n+1}{((n+1)!)^2}}{\dfrac{5^n}{(n!)^2}}=\lim_{n\to+\infty}\dfrac{5}{(n+1)^2}=0$ **Since it goes to zero,** 5^n **is** $O(((n+1)!)^2)$ **by ratio test.**

Answer 5

a) gcd(94,134) $\rightarrow 94*(1)+40$
gcd(94,40) $\rightarrow 40*(2)+14$
gcd(40,14) $\rightarrow 14*(2)+12$
gcd(14,12) $\rightarrow 12*(1)+2$
gcd(12,2) $\rightarrow$ **So the highest common factor is 2. gcd(94,134) = 2.**

b)Assume $m>5$ **is an integer. We should consider two cases.**
The case if m is even:
Then m = 2s for $m\geq 3$**.**
Since $m-2=2s-2=2(s-1)$, $m-2$ **is even. Applying Goldbach's conjecture:**
$2s-2=x+y\rightarrow$ **sum of two primes x and y. Therefore,** $2m=x+y+2$ **is a sum of three prime numbers. The case if m is odd:**
$m=2s+1$ **for** $n\geq 3$
$m-3=2s-2=2(s-1)\rightarrow$ **so** $m-3$ **is even**
Applying Goldbach's conjecture:
$m-3=x+y$ **is a sum of two prime numbers** x,y**.**
Therefore, $2s+1=x+y+3$ **is a sum of three prime numbers.**
As a consequence of this, the Goldbach conjecture that every even integer greater than 2 is the sum of two prime numbers is equivalent to the statement that every integer greater than 5 is the sum of three prime numbers.