Student Information

Full Name : Yusuf Sami Lök
Id Number : 2521748

Answer 1

$P(n) = 6^{2n} - 1 \quad \text{for } n \in N^+$

Base Step:
$P(1) = 6^2 - 1 = 35$ $P(1)$ is divisible by both 5 and 7.

Inductive Step:
Assume $6^{2n} - 1$ is divisible both 5 and 7 for $n = k$, $k \in N^+$.
So, $6^{2k} - 1 = 35m$, where m is some positive integer. (1)

If we set $n = k + 1$, $P(k + 1) = 6^{2k+2} - 1 = 36 \cdot 6^{2k} - 1$ (2)
Multiply (1) by 36 $\rightarrow 36 \cdot 6^{2k} - 36 = 36 \cdot 35m$
Modify the equation and get:
$(36 \cdot 6^{2k} - 1) - 35 = 36 \cdot 35m$
$(36 \cdot 6^{2k} - 1) = 36 \cdot 35(m + 1)$
Since $(36 \cdot 6^{2k} - 1)$ is divisible by 35, $P(k + 1)$ is divisible by both 5 and 7. So, it is true for both k and $(k + 1)$. Hence, from the principle of mathematical induction, $6^{2n} - 1$ is divisible by both 5 and 7 for $n \in N^+$.

Answer 2

Let $P(n)$ be the proposition that $H_n \leq 9^n$ for any integer $n \geq 3$.
Base Step:
$H_3 = 8H_2 + 8H_1 + 9H_0 = 105$
$9^3 = 729$ So, $H_3 \leq 9^3$.

Inductive Step:
Assume $P(j)$ is true for all integers j with $3 \leq j \leq k$. (1)
So, $H_k \leq 9^k$.
Set $n = k + 1$:
$H_{k+1} = 8H_k + 8H_{k-1} + 9H_{k-2}$.
$8H_k + 8H_{k-1} + 9H_{k-2} \leq 9 \cdot 9^k$ (2)
$H_{k-1} \leq 9^{k-1}$ from (1). Since both sides are positive, we can multiply both sides by 8:
$8H_{k-1} \leq 8 \cdot 9^{k-1}$
$H_{k-2} \leq 9^{k-2}$ from (1). Since both sides are positive, we can multiply both sides by 9:
$9H_{k-2} \leq 9 \cdot 9^{k-2}$

If we add the two inequalities, we get $8H_{k-1} + 9H_{k-2} \leq 8 \cdot 9^{k-1} + 8 \cdot 9^{k-1}$
, which is $8H_{k-1} + 9H_{k-2} \leq 9^k$
If we subtract this from (2):
$8H_k \leq 8 \cdot 9^k$, which is $H_k \leq 9^k$. Since $H_k \leq 9^k$ is correct from the first assumption, $P(k+1)$ is also true. Therefore, $H_n \leq 9^n$ for any integer $n \geq 3$, by strong induction.

Answer 3

Let $T(n)$ is the number of bit strings of length n contain 4 consecutive 0s.
If we examine it starting from the last part of string:
If the last bit is '1', then the rest is just a string that contain '0000' from size n-1.
If the last bits is '10', then the rest is just a string that contain '0000' from size n-2.
If the last bits is '100', then the rest is just a string that contain '0000' from size n-3.
If the last bits is '0000', then it doesn't have to be in the rest and the number of all possibilities is 2^{n-4}.
Then we have the recursion formula:
$T(n) = T(n-1) + T(n-2) + T(n-3) + 2^{n-4}$
With the initial conditions:
$T(1) = 0$
$T(2) = 0$
$T(3) = 0$
$T(4) = 1$
We can find $T(8)$ recursively:
$T(5) = 3$
$T(6) = 8$
$T(7) = 20$
$T(8) = 48$
Let $F(n)$ is the number of bit strings of length n contain 4 consecutive 1s.
If we examine it starting from the last part of string:
If the last bit is '0', then the rest is just a string that contain '1111' from size n-1.
If the last bits is '01', then the rest is just a string that contain '1111' from size n-2.
If the last bits is '011', then the rest is just a string that contain '1111' from size n-3.
If the last bits is '1111', then it doesn't have to be in the rest and the number of all possibilities is 2^{n-4}.
Then we have the recursion formula:
$F(n) = F(n-1) + F(n-2) + F(n-3) + 2^{n-4}$
With the initial conditions:
$F(1) = 0$
$F(2) = 0$
$F(3) = 0$
$F(4) = 1$
We can find $F(8)$ recursively:
$F(5) = 3$

$F(6) = 8$
$F(7) = 20$
$F(8) = 48$
So, if we define:
A = bit-string of size 8 that contain '0000'
B = bit-string of size 8 that contain '1111'
Answer is $|A| + |B| - |A \cap B|$ because we counted the intersection of two sets twice. To fix this, we subtract the number of elements in the intersection set.
$|A| = 48$ and $|B| = 48$.
$|A \cap B| = 2$, which are '00001111' and '11110000'.
Hence, the answer is $96 - 2 = 94$.

Answer 4

Choosing a star:
$\binom{10}{1}$
Choosing 2 habitable planets:
$\binom{10}{1} \cdot \binom{20}{2}$
Choosing 8 non-habitable planets:
$\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8}$
When placing planets:
There are 3 cases in total if there are 6 planets between habitable planets.
There are 2 cases in total if there are 7 planets between habitable planets.
There are 1 case in total if there are 8 planets between habitable planets.
The number of total options is 6.
Additionally, nonhabitable planets have 8! different sequences.
Also, there are 2! option to place habitable planets.
So the result is:
$\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8} \cdot 2! \cdot 8! \cdot 6.$

Answer 5

a)
We can create a recurrence relation by examining the last jump. Let $T(n)$ is the number of ways to land n cells away.
If the last jump is 1 cells away. The total number of different ways is $T(n-1)$.
If the last jump is 2 cells away. The total number of different ways is $T(n-2)$.
If the last jump is 3 cells away. The total number of different ways is $T(n-3)$.
Then $T(n) = T(n-1) + T(n-2) + T(n-3)$ for any integer $n \geq 4$

b) $T(n)$ **is the number of ways to land n cells away.**
$T(1) = 1$
$T(2) = 2$
$T(3) = 4$

c) $T(1) = 1$
$T(2) = 2$
$T(3) = 4$
The recurrence relation is $T(n) = T(n-1) + T(n-2) + T(n-3)$ **so:**
$T(4) = 1 + 2 + 4 = 7$
$T(5) = 2 + 4 + 7 = 13$
$T(6) = 4 + 7 + 13 = 24$
$T(7) = 7 + 13 + 24 = 44$
$T(8) = 13 + 24 + 44 = 81$
$T(9) = 24 + 44 + 81 = 149$ **Hence, the answer is 149.**