CENG 382 - Analysis of Dynamic Systems
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Homework 1

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1. (a)

$$y_1(t) = y(t)$$
$$y_2(t) = y'(t)$$

The system of first order equation for part (a) is:

$$y_2'(t) - 5y_2(t) + 6y_1(t)$$

(b)

$$y_1(k) = y(k)$$
$$y_2(k) = y(k+1)$$
$$y_3(k) = y(k+2)$$

and,

$$y_2(k) = y_1(k+1)$$
$$y_3(k) = y_2(k+1)$$

The system of first order equation for part (b) is:

$$y_3(k+1) = y_2(k) + y_1(k)$$

2. (a)

$$x(k+1) = x(k)$$

So we can put the values starting from x_0:

$$x(2) = x(1) = x(0) = 3$$

Therefore, $x(k) = 3$

(b)

$$x(k+1) = 0.5x(k) - 1$$
$$x(1) = 0 - 1$$
$$x(2) = -0.5 - 1$$
$$x(3) = -0.5^2 - 0.5 - 1$$

So the pattern is:

$$x(k) = -\frac{(0.5)^k - 1}{0.5 - 1} = 2(0.5)^k - 2$$

(c)

$$x(k+1) = -x(k) + 1$$
$$x(1) = -7 + 1 = -6$$
$$x(2) = +6 + 1 = 7$$
$$x(3) = -6$$
$$x(4) = 7$$

So the system is:
$x(k) = 7$ if k is even
$x(k) = -6$ if k is odd

3. (a) It doesn't approach to infinity. It is always fixed at 3.
   (b) It doesn't approach to infinity. It approaches to fixed point at -2.
   (c) It doesn't approach to infinity. It oscillates between 7 and -6.

4. (a)

$$x'(t) = x$$
$$\frac{dx}{dt} = x$$
$$\frac{dx}{x} = dt$$

   Integrate both sides:

$$lnx = t + c$$
$$x = ce^t$$

   Since x_0 is 1, c is also 1. Therefore,

$$x(t) = e^t$$

   (b)

$$x'(t) = 1$$
$$\frac{dx}{dt} = 1$$
$$dx = dt$$

   Integrate both sides:

$$x = t + c$$

   Since x_0 is 0, c is also 0. Therefore,

$$x(t) = t$$

   (c)

$$x'(t) = -x(t) + 2$$

   Homogeneous solution is:

$$x_h(t) = ce^{-t}$$

   Particular solution is:

$$x_p(t) = D$$
$$0 = -D + 2$$
$$D = 2$$
$$x(t) = x_h(t) + x_p(t)$$
$$x(t) = ce^{-t} + 2$$

   Since x_0 is 3:

$$x(0) = c + 2 = 3$$

   C is 1, therefore overall equation is:

$$x(t) = e^{-t} + 2$$

5. (a) It approaches to infinity. There is no fixed point. There isn't any t which satisfies $e_t = t$
   (b) It approaches to infinity. There are infinite numbers of fixed points, and every t is a fixed point. Therefore it cannot approach to another fixed point.
   (c) It does not go to infinity. It converges to 2. It has a fixed point at the solution of $e^- t + 2 = t$. It approaches that point by oscilating.

6. To find the state transition matrix $\Phi(k,l)$ for the given system $x(k+1) = \begin{pmatrix} \frac{k+2}{k+1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} x(k)$, we can start by observing the equation $x(k+1) = A(k)x(k)$, where $A(k) = \begin{pmatrix} \frac{k+2}{k+1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

   To compute the state transition matrix, we can recursively apply the system equation:

$$\Phi(k,l) = A(k-1) \cdot A(k-2) \cdot \ldots \cdot A(l)$$

   Let's calculate this:

$$\Phi(k,l) = A(k-1) \cdot A(k-2) \cdot \ldots \cdot A(l)$$

$$= \begin{pmatrix} \frac{k+1}{k} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{k}{k-1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{k-1}{k-2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \dots \cdot \begin{pmatrix} \frac{l+1}{l} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{l+2}{l+1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k+1}{k} \cdot \frac{k}{k-1} \cdot \frac{k-1}{k-2} \cdot \dots \cdot \frac{l+1}{l} \cdot \frac{l+2}{l+1} & 0 \\ 0 & \left(\frac{1}{2}\right)^{k-l} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k+1}{l+1} & 0 \\ 0 & \left(\frac{1}{2}\right)^{k-l} \end{pmatrix}$$

Now, as $k \to \infty$, the elements of the state transition matrix become:

$$\lim_{k\to\infty} \Phi(k,l) = \begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix}$$

So, as $k \to \infty$, the state transition matrix diverges, with the first element $\Phi_{11}(k,l) = \frac{k+1}{l+1}$ approaching infinity, and the second element $\Phi_{22}(k,l) = \left(\frac{1}{2}\right)^{k-l}$ approaching zero.

This means that as $k \to \infty$, the behavior of the system is such that the first state variable grows unbounded, while the second state variable tends to zero.

7.

8. (a) The eigenvalues λ are found $\lambda_1 = -1$, $\lambda_2 = -2$ by solving characteristic equation of system.

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The general solution must be:

$$x(k) = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$$

We can use $x(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and from here we find that:

$$c_1 = -3$$

$$c_2 = 2$$

So the overall answer is:

$$x(k) = \begin{pmatrix} -(-2)^k \\ 2(-2)^k \end{pmatrix}$$

(b) Both parts of the $|x(k)|$ goes infinity. But they oscillate because of their - sign.

9. (a) Given the system $x' = Ax + b$, where

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

To find an exact formula for $x(t)$, we can solve the system of differential equations. The solution is given by:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} b d\tau$$

First, let's find e^{At}:

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2t & 2t \\ 5t & -t \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 14t^2 & 2t \\ 3t & -7t^2 \end{pmatrix} + \dots$$

$$e^{At} = \begin{pmatrix} 1 + 2t + 7t^2/2 & 2t \\ 5t + 3t^2/2 & 1 - t/2 \end{pmatrix}$$

Now, we plug in e^{At} and x_0 into the formula:

$$x(t) = \begin{pmatrix} 1+2t+7t^2/2 & 2t \\ 5t+3t^2/2 & 1-t/2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 1+2(\tau)+7(\tau)^2/2 & 2(\tau) \\ 5(\tau)+3(\tau)^2/2 & 1-(\tau)/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} d\tau$$

After solving the integral, we find:

$$x(t) = \begin{pmatrix} 1+7t^2/2+2t \\ 6t+3t^2/2+5 \end{pmatrix}$$

(b) As $t \to \infty$, the behavior of $x(t)$ depends on the eigenvalues of matrix A.
The eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$:

$$\det\left(\begin{pmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{pmatrix}\right) = (2-\lambda)(-1-\lambda) - 10 = \lambda^2 - \lambda - 12 = 0$$

Solving this quadratic equation gives eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -3$.
Since both eigenvalues have negative real parts, the system is stable, and $x(t)$ will tend to a steady-state as $t \to \infty$.

10. (a) Let λ_1 and λ_2 be the eigenvalues of A, and let v_1 and v_2 be the corresponding eigenvectors.
Given matrix A:

$$A = \begin{pmatrix} 1/2 & 1/16 \\ -1 & 0 \end{pmatrix}$$

To find the eigenvalues λ, we solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 1/2 & 1/16 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\left(\begin{pmatrix} 1/2-\lambda & 1/16 \\ -1 & -\lambda \end{pmatrix}\right) = 0$$

$$(1/2-\lambda)(-\lambda) - \frac{1}{16}(-1) = 0$$

$$\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16} = 0$$

Solving this quadratic equation, we get two eigenvalues:

$$\lambda_1 = \frac{1}{4} \quad \text{and} \quad \lambda_2 = \frac{1}{4}$$

In order to diagonalize, we must find two linearly independent eigenvectors. However, the only eigenvalue that we get from characteristic equation of A is $\frac{1}{4}$. Therefore, we cannot get two eigenvectors with that one eigenvalue.

(b) To analyze the behavior of A^k as k approaches infinity, we can decompose matrix A into its eigenvalues and eigenvectors. Now, let's find the corresponding eigenvectors.
For $\lambda_1 = \frac{1}{4}$:

$$(A - \lambda_1 I)v_1 = 0$$

$$\left(\begin{pmatrix} 1/2 & 1/16 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v_1 = 0$$

$$\begin{pmatrix} 1/4 & 1/16 \\ -1 & -1/4 \end{pmatrix} v_1 = 0$$

$$\begin{pmatrix} 1 & 1/4 \\ -1 & -1/4 \end{pmatrix} v_1 = 0$$

Solving this system of linear equations, we get:

$$v_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

For $\lambda_2 = \frac{1}{4}$:

$$(A - \lambda_2 I)v_2 = 0$$

$$\left(\begin{pmatrix} 1/2 & 1/16 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v_2 = 0$$

$$\begin{pmatrix} 1/4 & 1/16 \\ -1 & -1/4 \end{pmatrix} v_2 = 0$$

$$\begin{pmatrix} 1 & 1/4 \\ -1 & -1/4 \end{pmatrix} v_2 = 0$$

Solving this system of linear equations, we get:

$$v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

v_1 and v_2 are the same, indicating that there is only one linearly independent eigenvector, and thus, A is not diagonalizable. This means the Jordan normal form of A should be considered.
The Jordan normal form of A is:

$$J = P^{-1}AP$$

where P is the matrix of eigenvectors and J is the Jordan form.
However, since we only have one linearly independent eigenvector, we cannot fully diagonalize A. The Jordan form J would have one eigenvalue on the diagonal and one above it.

$$J = \begin{pmatrix} 1/4 & 1 \\ 0 & 1/4 \end{pmatrix}$$

Therefore, as k approaches infinity, A^k will approach:

$$\lim_{k\to\infty} A^k = P \begin{pmatrix} 0 & 1/4 \\ 0 & 0 \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1/4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} -1/4 & -1/16 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/4 \\ 0 & 0 \end{pmatrix}$$

This implies that as k approaches infinity, A^k approaches a matrix with zeros everywhere except for the top right corner, where it has $\frac{1}{4}$.
So, the behavior of A^k as k approaches infinity indicates that the system represented by $x(k+1) = Ax(k)$ will approach a state where the first component grows without bound while the second component remains bounded.

11. (a) $\begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0.1 & 0 & 0.2 & 0 & 0.7 & 0 \end{pmatrix}$

    (b) The s2 does not have any connection to another state. Every state goes to s2 or a state that goes to s2 with higher probability.