CENG 382 - Analysis of Dynamic Systems
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Homework 2

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1.

$$V(x_1, x_2) = (x_1 + 1)^2 + x_2^2$$

First, we compute the time derivative of the Lyapunov function V:

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

We need to find the partial derivatives of V:

$$\frac{\partial V}{\partial x_1} = 2(x_1 + 1)$$
$$\frac{\partial V}{\partial x_2} = 2x_2$$

Now, substituting $\dot{x}_1$ and $\dot{x}_2$:

$$\dot{V}(x_1, x_2) = 2(x_1 + 1)(2x_2^2 - x_1 - 1) + 2x_2(-2x_1x_2 - x_2)$$

Combining like terms, last version of the equation:

$$\dot{V}(x_1, x_2) = -2(x_1 + 1)^2 + 2x_2^2$$

It is not always negative for all values. In fact, it is not always negative around fixed point also. If x_2 is very close to zero but not exactly zero and x_1 is -1, then it is positive. So we can not prove that the fixed point is stable from this Lyapunov function.

Let's try linearization method:

To linearize the system around the fixed point $\tilde{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, we first find the Jacobian matrix of the system.

The Jacobian matrix J is given by:

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 4x_2 \\ -2x_2 & -2x_1 - 1 \end{bmatrix}$$

Substituting $\tilde{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$:

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues of this Jacobian matrix are -1 and 1. Since one of the eigenvalues is positive, the fixed point $\tilde{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is unstable

Using the Lyapunov function method, we can not prove that the fixed point is stable. However, the linearization method revealed that the fixed point is actually unstable due to the presence of a positive eigenvalue in the Jacobian matrix. Therefore, the fixed point $\tilde{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is unstable.

2. In order to show the fixed point is stable, the Lyapunov function should hold two conditions:
1. $V(x_1, x_2) > 0$ for all x except for the fixed points. Here we have one fixed point which is:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For other values, the Lyapunov function outputs a positive value. $V(x_1, x_2) > 0$
2. Second condition is:

$$\Delta V(x) = V(x_1(k+1), x_2(k+1), x_3(k+1), x_4(k+1)) - V(x_1(k), x_2(k), x_3(k), x_4(k)) < 0$$

$$(\frac{1}{3}x_1 + \frac{1}{3}x_4)^2 + (\frac{1}{3}x_2 + \frac{1}{3}x_3)^2 + (\frac{1}{3}x_1 - \frac{1}{3}x_4)^2 + (\frac{1}{3}x_2 - \frac{1}{3}x_3)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2$$

To show that, we should simplify the equation:

$$-\frac{7}{9}x_1^2 - \frac{7}{9}x_2^2 - \frac{7}{9}x_3^2 - \frac{7}{9}x_4^2 < 0$$

For all values, $\Delta V(x) < 0$ except for $x = 0$.

Since both two conditions are hold, $x = 0$ is a stable fixed point.

3. **Stable Points**
We have to check two conditions: 1.Our Lyapunov function gives a positive value except for fixed points. Here there is one fixed point which is:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For other values, $V(x) > 0$.
2. In order to show that fixed point is stable, we have to check for which values of A $\Delta V(x) < 0$:

$$\begin{aligned} \Delta V(x) &= 2x_1(-4x_2) + 2x_2(Ax_2 + 4x_1 - 3x_2^3) \\ &= 2x_2^2(A - 3x_2^2) \end{aligned}$$

For $A < 0$, $\Delta V(x) < 0$. Therefore, if $A < 0$, the fixed point is stable.
Unstable Points
Since we can not determine the unstability of the fixed point by examining just one Lyapunov function, we can prove that fixed point is unstable by using linearization method:

$$Df(x) = \begin{bmatrix} 0 & -4 \\ 4 & -9x_2^2 + A \end{bmatrix}$$

Putting fixed point to the jacobian:

$$Df(0) = \begin{bmatrix} 0 & -4 \\ 4 & A \end{bmatrix}$$

characteristic equation of resulting matrix:

$$\lambda^2 - \lambda A + 16 = 0$$

If A is a positive real value, then we have two positive eigenvalues by using multiplication and addition properties of second order inequalities. Since we have two positive eigenvalues, the fixed point is unstable for $A > 0$.

After proving that fixed point is unstable, we can further analyze the behaviour of $x(t)$ by using Poincare-Bendixson theorem.
$\Delta V(x)$:

$$\begin{aligned} \Delta V(x) &= 2x_1(-4x_2) + 2x_2(Ax_2 + 4x_1 - 3x_2^3) \\ &= 2x_2^2(A - 3x_2^2) \end{aligned}$$

For $A > 0$, $\Delta V(x) > 0$ to a value determined by A. After that value, $\Delta V(x) < 0$. By Poincare-Bendixson theorem, as $t \to \infty$, x(t) tending to a periodic orbit.

In conclusion, if A is negative, then the fixed point is stable. If A is positive, then the fixed point is unstable and $x(t)$ is tending to a periodic orbit.

4. (a) To find fixed points of the system, we should find values of x which satisfies $f(x) = x$:

$$-3x^2 + 4 = x$$

$$(3x + 4)(x - 1) = 0$$

So the fixed values are 1, and $-\frac{4}{3}$.
In order to check whether the points are stable or unstable, we can use linearization method:

$$f'(x) = -6x$$

We should apply above function to the fixed points:

$$f'(1) = -6$$

Since $|f'(1)| > 1$, 1 is unstable fixed point.

$$f'(-\frac{4}{3}) = 8$$

Since $|f'(-\frac{4}{3})| > 1$, $-\frac{4}{3}$ is unstable fixed point.

(b) We have already found periodic points of prime period 1, since they are fixed points. From part (A), they are 1, and $-\frac{4}{3}$.
For the periodic points of prime period 2, we should find values of x which satisfies the equation $f^2(x) = x$.

$$f^2(x) = -3(-3x^2 + 4) + 4$$

$$-27x^4 + 72x^2 - x - 44$$

We found 4 roots of the above equation which are $-\frac{4}{3}$, 1, -0.951, 1.285. The two number are fixed points so they are not periodic points of prime period 2. The other two values which are -0.951, 1.285 are periodic points of prime period 2.

(c) To determine whether the system tends to the periodic points with prime period 2, we need to analyze the stability of these periodic points. To check the stability of these periodic points, we need to analyze the derivative of the function f at these points. For a periodic point x_p with prime period 2, we need to evaluate the magnitude of the derivative of the composed function f^2 at x_p.

$$(f^2)'(x) = f'(f(x)) \cdot f'(x) = [-6(-3x^2 + 4)] \cdot (-6x) = 36x(-3x^2 + 4)$$

We need to evaluate $(f^2)'(x)$ at the periodic points $x \approx -0.951$ and $x \approx 1.285$:

$$(f^2)'(-0.951) = 36(-0.951)(-3(-0.951)^2 + 4)$$
$$(f^2)'(-0.951) \approx -44.1$$
$$(f^2)'(1.285) = 36(1.285)(-3(1.651225) + 4)$$
$$(f^2)'(1.285) \approx -44.1$$

Since $|(f^2)'(x)|$ for both periodic points is greater than 1, both periodic points $x \approx -0.951$ and $x \approx 1.285$ are unstable.
Therefore, the system does not tend to the periodic points with prime period 2 because they are unstable.

5. (a) We can show that the system is controllable by using controllability matrix:

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, and B = $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

If the columns of controllability matrix is linearly independent, the system is controllable. We can check linearly independence by converting the matrix to row reduced echelon matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the columns of controllability matrix is linearly independent, the rank of this matrix is 3. Therefore, the system is controllable.

(b)

$$x(1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(0) = \begin{bmatrix} u(0) \\ u(0)+1 \\ 2 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(1) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(1) = \begin{bmatrix} u(0)+u(1) \\ 2u(0)+u(1)+1 \\ u(0)+3 \end{bmatrix}$$

$$x(3) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(2) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(2) = \begin{bmatrix} u(0)+u(1)+u(2) \\ 3u(0)+2u(1)+u(2)+1 \\ 3u(0)+u(1)+4 \end{bmatrix}$$

$$\begin{bmatrix} u(0)+u(1)+u(2) \\ 3u(0)+2u(1)+u(2)+1 \\ 3u(0)+u(1)+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

Let $x = u(0)$, $y = u(1)$, $z = u(2)$
The system of equations is:

$$x + y + z = 6 \tag{1}$$

$$3x + 2y + z = 5 \tag{2}$$

$$3x + y = 2 \tag{3}$$

Eliminate 'z' from (1) and (2):

$$(3x + 2y + z) - (x + y + z) = 5 - 6$$
$$2x + y = -1$$

Now we have:

$$2x + y = -1$$
$$3x + y = 2$$

Eliminate 'y':

$$(3x + y) - (2x + y) = 2 - (-1)$$
$$x = 3$$

Substitute $x = 3$ into:

$$2(3) + y = -1$$
$$y = -7$$

Substitute $x = 3$ and $y = -7$ into (1):

$$3 + (-7) + z = 6$$
$$z = 10$$

Therefore: $u(0) = 3$, $u(1) = -7$, $u(2) = 10$

6. We can show that the system is observable by using observability matrix:

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, and C = $\begin{bmatrix} 1 & \frac{1}{2} & 1 \end{bmatrix}$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 1 & \frac{5}{2} & 3 \\ 1 & \frac{9}{2} & 8 \end{bmatrix}$$

If the columns of observability matrix is linearly independent, the system is observable. We can check linearly independence by converting the matrix to echelon matrix:

$$\text{Observability matrix:} \quad \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 1 & \frac{5}{2} & 3 \\ 1 & \frac{9}{2} & 8 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - R_1, \quad R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & \frac{2}{2} & 2 \\ 0 & \frac{8}{2} & 7 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 2 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$R_3 \leftarrow -R_3$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Here it is the echelon form of the observability matrix. Since the rows of observability matrix are linearly independent, the system is observable.

7. (a) First of all, we can find the fixed points by solving the equation $3x = x^2 + 3x - 4$:

$$3x = x^2 + 3x - 4$$

$$x^2 = 4$$

So we have two fixed points which are -2, and 2. Using linearization method we can find that fixed points are stable or not:

$$f'(x) = \frac{1}{3}(2x + 3)$$

Apply this function to the fixed point -2:

$$f'(-2) = -\frac{1}{3}$$

Since $|f'(-2)| < 1$, -2 is a stable fixed point. Apply the function to the fixed point 2:

$$f'(2) = \frac{7}{3}$$

Since $|f'(2)| > 1$, 2 is a unstable fixed point.

(b) First of all, we can find the fixed points by solving the equation $\frac{1}{3}x^3 - 3x^2 + 9x - 9 = 0$:

$$\frac{1}{3}x^3 - 3x^2 + 9x - 9 = 0$$

$$\frac{1}{3}(x^3 - 9x^2 + 27x - 27)$$

$$\frac{1}{3}(x - 3)^3$$

So we have one fixed point which is 3.
Checking whether linearization fails or not:

$$f'(x) = \frac{1}{3}(3x^2 - 18x + 27) = \frac{1}{9}(x - 3)^2$$

Apply this function to the fixed point 3:

$$f'(3) = 0$$

Since $f'(3) = 0$, linearization fails. We can analyze the system $\frac{1}{9}(x - 3)^2$. If the values are less than 3, it goes to value 3. If the values are more than 3, it goes to ∞. So it is not stable point. Since the values less than, and more than 3 are both increasing, 3 is a semistable fixed point.

8. First of all, we can find the fixed points solving the equation:

$$2x_1^2 + x_1x_2 - 6$$

$$x_1 + x_2 = 0$$

We can write $-x_1$ to the first equation insted of x_2, and we get:

$$2x_1^2 - x_1^2 - 6 = 0$$

$$x_1^2 = 6$$

So we have two fixed points which are $\begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \end{bmatrix}$, and $\begin{bmatrix} -\sqrt{6} \\ \sqrt{6} \end{bmatrix}$

We can apply linearization method to find whether the system is stable or unstable:

$$Df(x) = \begin{bmatrix} 4x_1 + x_2 & x_1 \\ 1 & 1 \end{bmatrix}$$

Putting the fixed value $\begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \end{bmatrix}$ to the $Df(x)$:

$$Df(\begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \end{bmatrix}) = \begin{bmatrix} 3\sqrt{6} & \sqrt{6} \\ 1 & 1 \end{bmatrix}$$

Finding the eigenvalues:

$$(3\sqrt{6} - \lambda)(1 - \lambda) - \sqrt{6} = 0$$
$$\lambda^2 - \lambda(3\sqrt{6} + 1) + 2\sqrt{6} = 0$$

Since the discriminant of the equation is positive, the eigenvalues are real values. Also summation of the eigenvalues are positive, and multiplication of the eigenvalues are positive, eigenvalues are positive and real numbers. Since eigenvalues are positive and real numbers, the fixed point $\begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \end{bmatrix}$ is unstable fixed point.

Putting the fixed value $\begin{bmatrix} -\sqrt{6} \\ \sqrt{6} \end{bmatrix}$ to the $Df(x)$:

$$Df(\begin{bmatrix} -\sqrt{6} \\ \sqrt{6} \end{bmatrix}) = \begin{bmatrix} -3\sqrt{6} & -\sqrt{6} \\ 1 & 1 \end{bmatrix}$$

Finding the eigenvalues:

$$(-3\sqrt{6} - \lambda)(1 - \lambda) + \sqrt{6} = 0$$
$$\lambda^2 + \lambda(3\sqrt{6} - 1) - 2\sqrt{6} = 0$$

Since the discriminant of the equation is positive, the eigenvalues are real values. Also summation of the eigenvalues are negative, and multiplication of the eigenvalues are negative, one eigenvalue is positive real number, and the other is negative real number. Since the one eigenvalue is positive real number, the point $\begin{bmatrix} -\sqrt{6} \\ \sqrt{6} \end{bmatrix}$ is unstable fixed point.

9. (a) To determine if the system is observable, we need to check if we can uniquely determine the initial state $x(0)$ from the output sequence $y(k)$.
   Starting with $y(0)$:

$$y(0) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = -2x_2(0) - 4x_3(0)$$

   Next, we compute $y(1)$:

$$x(1) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ -2x_1(0) - x_3(0) \\ x_2(0) \end{bmatrix}$$

$$y(1) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_3(0) \\ -2x_1(0) - x_3(0) \\ x_2(0) \end{bmatrix} = -2(-2x_1(0) - x_3(0)) - 4x_2(0)$$

$$y(1) = 4x_1(0) - 4x_2(0) + 2x_3(0)$$

   Then, we compute $y(2)$:

$$x(2) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_3(0) \\ -2x_1(0) - x_3(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_2(0) \\ -2x_3(0) - x_2(0) \\ -2x_1(0) - x_3(0) \end{bmatrix}$$

$$y(2) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_2(0) \\ -2x_3(0) - x_2(0) \\ -2x_1(0) - x_3(0) \end{bmatrix} = -2(-2x_3(0) - x_2(0)) - 4(-2x_1(0) - x_3(0))$$

$$y(2) = 8x_1(0) + 2x_2(0) + 8x_3(0)$$

We now have the following system of equations:

$$\begin{aligned} -2x_2(0) - 4x_3(0) &= y(0) \\ 4x_1(0) - 4x_2(0) + 2x_3(0) &= y(1) \\ 8x_1(0) + 2x_2(0) + 8x_3(0) &= y(2) \end{aligned}$$

We can write these equations in matrix form as:

$$\begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}$$

To determine if the matrix is invertible, we calculate its determinant:

$$\det \begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix}$$

Substituting subsequent determinants into the determinant calculation:

$$\det \begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix} = 0 \cdot (-36) - (-2) \cdot 16 + (-4) \cdot 40 = 0 + 32 - 160 = -128$$

Since the determinant is non-zero ($\det = -128$), the matrix is invertible. This implies that the initial state $x(0)$ can be uniquely determined from the output sequence $y(k)$.
Therefore, the system is observable.

(b) To determine the observability of the system, we construct the observability matrix $\mathcal{O}$:

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

Given:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix}$$

First, compute CA:

$$CA = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} (0 \cdot 0) + (-2 \cdot -2) + (-4 \cdot 0) & (0 \cdot 0) + (-2 \cdot 0) + (-4 \cdot 1) & (0 \cdot 1) + (-2 \cdot -1) + (-4 \cdot 0) \end{bmatrix}$$

$$CA = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix}$$

Next, compute CA^2:

$$CA^2 = CA \cdot A = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} (4 \cdot 0) + (-4 \cdot -2) + (2 \cdot 0) & (4 \cdot 0) + (-4 \cdot 0) + (2 \cdot 1) & (4 \cdot 1) + (-4 \cdot -1) + (2 \cdot 0) \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 8 & 2 & 8 \end{bmatrix}$$

Form the observability matrix M:

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix}$$

To check observability, we need to determine if $\mathcal{O}$ is full rank by calculating its determinant:

$$\det(M) = \begin{vmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{vmatrix}$$

The determinant is calculated as follows:

$$\det(M) = 0 \cdot \begin{vmatrix} -4 & 2 \\ 2 & 8 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 4 & 2 \\ 8 & 8 \end{vmatrix} + (-4) \cdot \begin{vmatrix} 4 & -4 \\ 8 & 2 \end{vmatrix}$$

Calculate the individual 2x2 determinants:

$$\begin{vmatrix} -4 & 2 \\ 2 & 8 \end{vmatrix} = (-4)(8) - (2)(2) = -32 - 4 = -36$$

$$\begin{vmatrix} 4 & 2 \\ 8 & 8 \end{vmatrix} = (4)(8) - (2)(8) = 32 - 16 = 16$$

$$\begin{vmatrix} 4 & -4 \\ 8 & 2 \end{vmatrix} = (4)(2) - (-4)(8) = 8 + 32 = 40$$

Substitute these into the determinant expression:

$$\det(M) = 0 \cdot (-36) - (-2) \cdot 16 + (-4) \cdot 40$$

$$\det(M) = 0 + 32 - 160 = -128$$

Since the determinant is non-zero ($\det = -128$), the matrix M is invertible. This implies that the system is observable.

Therefore, the system is observable as verified by the observability matrix.