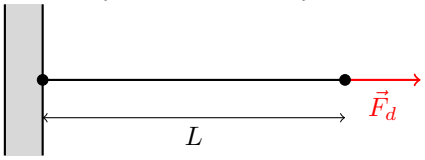
1 Introduction

2 Preliminary data setting work

2.1 Analytical solution of Fig1. problem

As a simple $1D$ problem, we only consider the reaction on the axis x, which is along the bar and the force applied on its right extremity $\vec{F}_d$. The positive sense is defined as $+x$, from left to right. We firstly search the analytical solution.



As the system is in static equilibrium, we apply the principle of linear momentum conservation:

$$\nabla \cdot \boldsymbol{\sigma} + \vec{f_v} = 0 \tag{1}$$
$$\sigma_{xx,x} + 0 = 0 \tag{2}$$
$$\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}x} = 0 \tag{3}$$
$$\sigma_{xx} = C_1 \tag{4}$$
$$\tag{5}$$

Then we check the boundary condition on forces, with cross-sectional area A which is the surface through which the forces are transmitted:

$$\sigma_{xx}(x = L) = \frac{F_d}{A} = C_1$$
$$\sigma_{xx}(x) = \frac{F_d}{A}$$

The kinematical admissibility gives:

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$$
$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\mathrm{d}u_x}{\mathrm{d}x}$$

In an $1D$ problem, the constitutive law of linear elastic isotropic material gives:

$$\sigma_{xx} = E\epsilon_{xx}$$
$$\frac{\mathrm{d}u_x}{\mathrm{d}x} = \epsilon_{xx} = \frac{F_d}{EA}$$

We integrate both sides of the equation:

$$u_x(x) = \frac{F_d}{EA}x + C_2$$

Check the boundary condition for displacements:

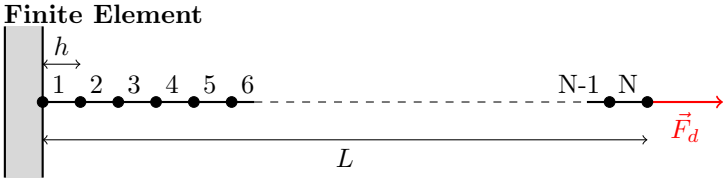
$$\begin{aligned} u_x(x=0) &= 0 \\ \frac{F_d}{EA} \times 0 + C_2 &= 0 \\ C_2 &= 0 \end{aligned}$$

We obtain finally the analytical solution of this problem:

$$\begin{aligned} \sigma_{xx}(x) &= \frac{F_d}{A} \\ \epsilon_{xx}(x) &= \frac{F_d}{EA} \\ u_x(x) &= \frac{F_d}{EA}x \end{aligned}$$

2.2 Carry out a finite element bar code

Finite Element



Now we discretize the bar with N elements, which has $n = N + 1$ nodes, and an identical distance $h = \frac{L}{N}$ between each pair of neighbor nodes.

Let's approximate the admissible displacement, noted $w(x)$, within an element having 2 nodes x_1, x_2 whose nodal displacements are d_1, d_2. Assuming that the displacement is linear, we have:

$$w(x) = \alpha_0 + \alpha_1 x$$

Thus :

$$\begin{aligned} w(x_1) &= \alpha_0 + \alpha_1 x_1 = d_1 \\ w(x_2) &= \alpha_0 + \alpha_1 x_2 = d_2 \end{aligned}$$

Resolving this system, we have the values of α_0, α_1 :

$$w(x) = \frac{x_2 - x}{x_2 - x_1} \cdot d_1 + \frac{x - x_1}{x_2 - x_1} \cdot d_2$$

Knowing that an element's length is $h = x_2 - x_1$, we have the equivalence :

$$w(x) = (1 - \frac{x}{h}) \cdot d_1 + \frac{x}{h} \cdot d_2$$

Then, in the 1D case, strain is simply defined as:

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \vec{w(x)} + (\nabla \vec{w(x)})^T) = \frac{1}{2}(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \frac{\mathrm{d}w(x)}{\mathrm{d}x}) = \frac{\mathrm{d}w(x)}{\mathrm{d}x} = \frac{d_2 - d_1}{h}$$

Using constitutive law, we have the stress:

$$\boldsymbol{\sigma} = E \cdot \boldsymbol{\epsilon} = E \cdot \frac{d_2 - d_1}{h}$$

Potential energy of the element corresponding to an admissible displacement w(x):

$$J(w) = \frac{1}{2} \cdot \int_\Omega \boldsymbol{\epsilon} : E : \boldsymbol{\epsilon} \mathrm{d}\Omega - \int_\Omega f_d \cdot w \mathrm{d}\Omega - \int_{\delta_2 \Omega} F_d \cdot w \mathrm{d}\Gamma$$

Let's plug in the approximation and apply to 1D case, knowing that $f_d = 0$:

$$J(w) = \frac{1}{2} \cdot \int_\Omega \boldsymbol{\epsilon} : E : \boldsymbol{\epsilon} \mathrm{d}\Omega - \int_{\delta_2 \Omega} F_d \cdot w \mathrm{d}\Gamma$$

- For elements 1 to n-1:
$$J(w) = \frac{1}{2} \cdot \int_{x_1}^{x_2} E \boldsymbol{\epsilon}^2 A \mathrm{d}x$$

- For the element n:
$$J(w) = \frac{1}{2} \cdot \int_{x_1}^{x_2} E \boldsymbol{\epsilon}^2 A \mathrm{d}x - F_d \cdot w(x = x_2)$$

  Indeed, since F_d is on the second node x_2 of the last element N:

$$\int_{\delta_2 \Omega} F_d \cdot w \mathrm{d}\Gamma = \int_{x_1}^{x_2} F_d \cdot w(x) \cdot \delta(x - x_2) \mathrm{d}x = F_d \cdot \int_{x_1}^{x_2} w(x) \cdot \delta(x - x_2) \mathrm{d}x$$

  Where δ is the Dirac delta function, having the property:

$$\int_{x_1}^{x_2} f(x) \cdot \delta(x - x_2) \mathrm{d}x = f(x_2)$$

  We have:

$$\int_{\delta_2 \Omega} F_d \cdot w \mathrm{d}\Gamma = F_d \cdot \int_{\delta_2 \Omega} w \mathrm{d}\Gamma = F_d \cdot w(x = x_2)$$

In matrix form, define the vector of nodal displacements $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, and the shape function $\boldsymbol{N} = \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix}$. So $w(\vec{x}) = \boldsymbol{N}\vec{d}$, $\boldsymbol{\epsilon} = \boldsymbol{B}\vec{d}$, $\boldsymbol{\sigma} = E\boldsymbol{B}\vec{d}$. Then:

- For elements 1 to n-1:
$$J_{1\ldots n-1}(\vec{d}) = \frac{1}{2}\vec{d}^T[\int_{x_1}^{x_2} \boldsymbol{B}^T E\boldsymbol{B} \cdot A\mathrm{d}x]\vec{d}$$
- For the element n:
$$J_n(\vec{d}) = \frac{1}{2}\vec{d}^T[\int_{x_1}^{x_2} \boldsymbol{B}^T E\boldsymbol{B} \cdot A\mathrm{d}x]\vec{d} - F_d \cdot d_2$$

Let's see what the matrix is for a 1D linear element :

$$\boldsymbol{K} = \int_{x_1}^{x_2} \boldsymbol{B}^T E\boldsymbol{B} \cdot A\mathrm{d}x$$

Recall that $\mathbf{B} = \frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}x} = \frac{1}{x_2 - x_1}[-1 \quad 1]$. Hence :

$$\mathbf{B}^T E\mathbf{B}A = \left(\frac{1}{x_2 - x_1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) E \left(\frac{1}{x_2 - x_1}[-1 \quad 1]\right) A$$

This becomes

$$\mathbf{B}^T E\mathbf{B}A = \frac{EA}{(x_2 - x_1)^2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}[-1 \quad 1] = \frac{EA}{(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assuming E and A are constant. Thus:

$$\boldsymbol{K} = \int_{x_1}^{x_2} \boldsymbol{B}^T E\boldsymbol{B} \cdot A\mathrm{d}x = \boldsymbol{B}^T E\boldsymbol{B}A \int_{x_1}^{x_2} \mathrm{d}x = \boldsymbol{B}^T E\boldsymbol{B}A = \frac{EA}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(x_2 - x_1) = \frac{EA}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} h$$

Let's apply the principle of minimum potential energy:

- $$\frac{\partial J_{1..N-1}(\vec{d})}{\partial \vec{d}} = \boldsymbol{K}\vec{d} = 0$$
- $$\frac{\partial J_N(\vec{d})}{\partial \vec{d}} = \boldsymbol{K}\vec{d} - \begin{bmatrix} 0 \\ F_d \end{bmatrix} = 0$$

  Considering the reaction, noted f_1, at node x_1 at the 1st element, we have :

- Element 1:
$$\boldsymbol{K}\vec{d} = 0 + \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$$

- Element 2 to $N-1$:

$$\boldsymbol{K}\vec{d} = 0$$

- Element N:

$$\boldsymbol{K}\vec{d} = \begin{bmatrix} 0 \\ F_d \end{bmatrix}$$

Assembly
The L-length bar contains N finite elements. A finite element has 2 nodes and the length of an element is h, the total number of nodes is n. Let's consider the two neighbor nodes of an element. We work in 1 dimension (x):
Thus, the elementary stiffness matrix for a given element e is :

$$K^e = \frac{EA}{h} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The system's size of the bar is $(DOF \times n) \times (DOF \times n) = (1 \times n) \times (1 \times n)$, so the global stiffness matrix for the bar reads :

$$K = \frac{EA}{h} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{n \times n}$$

Displacements:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_{n-1} \\ u_n \end{bmatrix}_{n \times 1}$$

Nodal forces:

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ f_{n-1} \\ f_n \end{bmatrix}_{n \times 1}$$

After considering limit conditions, the displacement of the first node of the first element is blocked, so it is equal to 0, and we solicit a force F_d on the second node of the last element, therefore:

$$\frac{EA}{h}\cdot\begin{bmatrix}1&-1&0&0&0&0&0\\-1&2&\cdot&0&0&0&0\\0&\cdot&\cdot&\cdot&0&0&0\\0&0&\cdot&\cdot&\cdot&0&0\\0&0&0&\cdot&\cdot&-1&0\\0&0&0&0&-1&2&-1\\0&0&0&0&0&-1&1\end{bmatrix}_{n\times n}\cdot\begin{bmatrix}0\\u_2\\\cdot\\u_{n-1}\\u_n\end{bmatrix}=\begin{bmatrix}f_1\\0\\\cdot\\0\\F_d\end{bmatrix}$$

Direct elimination of DOF
We can simply reduce the system by eliminating the corresponding line and column where $u_1 = 0$. Thus:

$$\frac{EA}{h}\cdot\begin{bmatrix}2&-1&0&0&0&0\\-1&\cdot&\cdot&0&0&0\\0&\cdot&\cdot&\cdot&0&0\\0&0&\cdot&\cdot&-1&0\\0&0&0&-1&2&-1\\0&0&0&0&-1&1\end{bmatrix}_{(n-1)\times(n-1)}\cdot\begin{bmatrix}u_2\\\cdot\\u_{n-1}\\u_n\end{bmatrix}_{(n-1)\times 1}=\begin{bmatrix}0\\\cdot\\0\\F_d\end{bmatrix}_{(n-1)\times 1}$$

With an inverse iterative process, we can easily find the numerical solution:

Penalty method
We introduce a penalty term g into the system, initially, we set f_1 as 1:

$$\frac{EA}{h}\cdot\begin{bmatrix}g+1&-1&0&0&0&0&0\\-1&2&\cdot&0&0&0&0\\0&\cdot&\cdot&\cdot&0&0&0\\0&0&\cdot&\cdot&\cdot&0&0\\0&0&0&\cdot&\cdot&-1&0\\0&0&0&0&-1&2&-1\\0&0&0&0&0&-1&1\end{bmatrix}_{n\times n}\cdot\begin{bmatrix}u_1\\u_2\\\cdot\\u_{n-1}\\u_n\end{bmatrix}=\begin{bmatrix}f_1+g\times\frac{EA}{h}\times u_1\\0\\\cdot\\0\\F_d\end{bmatrix}$$

Notice that the penalty term value should be quite dominant in fixing the first node more or less at the initial position, let's say $g = 10^4$. Then by setting the term $f_1 + g \times \frac{EA}{h} \times u_1 = 0$, we can solve the approximate system and obtain $u_{1,approx}$ to $u_{n,approx}$, which are quite close to the real displacements' solution, the final solution of the force f_1 is $-g \times \frac{EA}{h} \times u_{1,approx}$.

Lagrangian multiplier method
We introduce a Lagrangian multiplier λ. The system becomes:

$$\frac{EA}{h} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(n+1)\times(n+1)} \cdot \begin{bmatrix} 0 \\ u_2 \\ \cdot \\ u_{n-1} \\ u_n \\ \lambda \end{bmatrix}_{(n+1)\times 1} = \begin{bmatrix} f_1 + \lambda \\ 0 \\ \cdot \\ 0 \\ F_d \\ 0 \end{bmatrix}_{(n+1)\times 1}$$

By setting $f_1 + \lambda = 0$, we can solve directly the system with a totally known right side, and the solution of the unknown f_1 will be actually $-\lambda$.

2.3 Domain decomposition method

The L-length bar is decomposed into substructures S. A substructure of length H contains $N_s = \frac{N}{S}$ finite elements. A finite element of length h has 2 nodes. There therefore $n_s = N_s + 1$ nodes pertaining to a substructure.
Thus, the element stiffness matrix for a substructure (s) reads:

$$K^{(s)} = \frac{EA}{h} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{n_s \cdot n_s}$$

Displacements:

$$u^{(s)} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_{n_s-1} \\ u_{n_s} \end{bmatrix}$$

Nodal forces:

$$f^{(s)} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ f_{n_s-1} \\ f_{n_s} \end{bmatrix}$$

Let's reorder the terms so that we have :

$$k^{(s)} \cdot u^{(s)} = \begin{bmatrix} k_{ii}^{(s)} & k_{ib}^{(s)} \\ k_{bi}^{(s)} & k_{bb}^{(s)} \end{bmatrix} \cdot \begin{bmatrix} u_i^{(s)} \\ u_b^{(s)} \end{bmatrix} = \begin{bmatrix} f_i^{(s)} \\ f_b^{(s)} \end{bmatrix} = f^{(s)}$$

Operators for substructure (1):
Displacements of the 1st substructure:

$$u^{(1)} = \begin{bmatrix} 0 \\ u_2 \\ \cdot \\ u_{n_s-1} \\ u_{n_s} \end{bmatrix}$$

Nodal forces of the 1st substructure:

$$f^{(1)} = \begin{bmatrix} f_1 \\ 0 \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

Firstly, the interface is defined by the last local boundary node(ie: n_s), so we have the stiffness matrix $K^{(1)}$:

$$\frac{EA}{h}\cdot\begin{array}{c}\begin{matrix}1 & 2 & \dots & n_s-2 & n_s-1 & n_s\end{matrix}\\ \begin{pmatrix}1 & -1 & 0 & 0 & 0 & 0\\ -1 & 1+1 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 1\end{pmatrix}\end{array}\begin{matrix}1\\2\\\dots\\n_s-2\\n_s-1\\n_s\end{matrix}$$

which gives:

$$K^{(1)}=\frac{EA}{h}\cdot\begin{pmatrix}1 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & \cdot & 0 & 0 & 0\\ 0 & \cdot & \cdot & \cdot & 0 & 0\\ 0 & 0 & \cdot & \cdot & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 1\end{pmatrix}$$

With :

$$k_{ii}^{(1)}=\frac{EA}{h}\cdot\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & \cdots & 0\\ 0 & \cdots & \cdots & -1\\ 0 & 0 & -1 & 2\end{bmatrix}_{(n_s-1)\cdot(n_s-1)}$$

$$k_{ib}^{(1)}=\frac{EA}{h}\cdot\begin{bmatrix}0\\ \cdots\\ -1\end{bmatrix}_{(n_s-1)\cdot 1}$$

$$k_{bi}^{(1)}=\frac{EA}{h}\cdot\begin{bmatrix}0 & \cdots & -1\end{bmatrix}_{1\dot{(}n_s-1)}$$

$$k_{bb}^{(1)}=\frac{EA}{h}\cdot 1$$

And :

$$u^{(1)}=\begin{bmatrix}u_i^{(1)}\\ u_b^{(1)}\end{bmatrix}=\begin{bmatrix}u_1^{(1)}\\ u_2^{(1)}\\ \cdot\\ u_{n_s-1}^{(1)}\\ u_{n_s}^{(1)}\end{bmatrix}=\begin{bmatrix}0\\ u_2\\ \cdot\\ u_{n_s-1}\\ u_{n_s}\end{bmatrix}$$

And :

$$f^{(1)}=\begin{bmatrix}f_i^{(1)}\\ f_b^{(1)}\end{bmatrix}=\begin{bmatrix}f_1^{(1)}\\ f_2^{(1)}\\ \cdot\\ f_{n_s-1}^{(1)}\\ f_{n_s}^{(1)}\end{bmatrix}=\begin{bmatrix}f_1\\ 0\\ \cdot\\ 0\\ 0\end{bmatrix}$$

By direct elimination of 1st row and column, we are led to solve $k^{(1)}_{trim} \cdot u^{(1)}_{trim} = f^{(1)}_{trim}$:

$$K^{(1)}_{trim} = \frac{EA}{h} \cdot \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u^{(1)}_{trim} = \begin{bmatrix} u^{(1)}_2 \\ \cdot \\ u^{(1)}_{n_s-1} \\ u^{(1)}_{n_s} \end{bmatrix}$$

$$f^{(1)}_{trim} = \begin{bmatrix} 0 \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

We can therefore compute the Primal Schur Complement $Sp^{(s)}$ using trimmed matrix:

$$S^{(s)}_p = k^{(s)}_{bb} - k^{(s)}_{bi} {k^{(s)}_{ii}}^{-1} k^{(s)}_{ib}$$

$$S^{(1)}_p = k^{(1)}_{bb} - k^{(1)}_{bi} {k^{(1)}_{ii}}^{-1} k^{(1)}_{ib}$$

$$S^{(1)}_p = \frac{EA}{h} - \frac{EA}{h} \cdot \begin{bmatrix} 0 & \cdots & -1 \end{bmatrix} \cdot [\frac{EA}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}]^{-1} \cdot \frac{EA}{h} \cdot \begin{bmatrix} 0 \\ \cdots \\ -1 \end{bmatrix}$$

NB: We can compute S_p directly thanks to this paper
The Dual Schur Complement $S^{(s)}_d$ is the generalized invers of $S^{(s)}_p$:

$$S^{(s)}_d = {S^{(s)}_p}^{+}$$

$$S^{(1)}_d = {S^{(1)}_p}^{+}$$

The Primal RHS $bp^{(s)}$:

$$b^{(s)}_p = f^{(s)}_b - k^{(s)}_{bi} {k^{(s)}_{ii}}^{-1} f^{(s)}_i$$

$$b^{(1)}_p = f^{(1)}_b - k^{(1)}_{bi} {k^{(1)}_{ii}}^{-1} f^{(1)}_i$$

$$b^{(1)}_p = 0 - k^{(1)}_{bi} {k^{(1)}_{ii}}^{-1} \vec{0} = 0$$

The rigid body modes $R^{(s)}_b$:

$$S^{(s)}_p R^{(s)}_b = 0$$

$$S_p^{(1)} R_b^{(1)} = 0$$

However the scalar $Sp^{(1)} \neq 0$, so $R_b^{(1)}$ has to be 0, but the kernel cannot be 0. There are consequently no rigid body modes for subdomain (s=1).

Operators for substructures (2..S-1):
Displacements of the 2nd to before last substructure:

$$u^{(2\ldots S-1)} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_{n_s-1} \\ u_{n_s} \end{bmatrix}$$

Nodal forces of the 2nd to the before last substructure:

$$f^{(2\ldots S-1)} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

Firstly, the interfaces are defined by the 1st and last boundaries nodes(ie: $1, n_s$), so we have the stiffness matrix $K^{(2\ldots S-1)}$:

$$\frac{EA}{h} \cdot \begin{pmatrix} 2 & 3 & \ldots & n_s-2 & n_s-1 & 1 & n_s & \\ 1+1 & -1 & 0 & 0 & 0 & -1 & 0 & 2 \\ -1 & 1+1 & -1 & 0 & 0 & 0 & 0 & 3 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & n_s-2 \\ 0 & 0 & 0 & -1 & 1+1 & 0 & -1 & n_s-1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & n_s \end{pmatrix}$$

which gives:

$$K^{(2\ldots S-1)} = \frac{EA}{h} \cdot \left(\begin{array}{ccccc|cc} 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right)$$

With :

$$k_{ii}^{(2\ldots S-1)} = \frac{EA}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}_{(n_s-2)\cdot(n_s-2)}$$

$$k_{ib}^{(2...S-1)} = \frac{EA}{h} \cdot \begin{bmatrix} -1 & 0 \\ \cdots & \cdots \\ 0 & -1 \end{bmatrix}_{(n_s-2)\cdot 2}$$

$$k_{bi}^{(2...S-1)} = \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \\ 0 & \cdots & -1 \end{bmatrix}_{2\cdot(n_s-2)}$$

$$k_{bb}^{(2...S-1)} = \frac{EA}{h} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\cdot 2}$$

And :

$$u^{(2...S-1)} = \begin{bmatrix} u_i^{(2...S-1)} \\ u_b^{(2...S-1)} \end{bmatrix} = \begin{bmatrix} u_2^{(2...S-1)} \\ . \\ u_{n_s-1}^{(2...S-1)} \\ u_1^{(2...S-1)} \\ u_{n_s}^{(2...S-1)} \end{bmatrix} = \begin{bmatrix} u_2 \\ . \\ u_{n_s-1} \\ u_1 \\ u_{n_s} \end{bmatrix}$$

And :

$$f^{(2...S-1)} = \begin{bmatrix} f_i^{(2...S-1)} \\ f_b^{(2...S-1)} \end{bmatrix} = \begin{bmatrix} f_2^{(2...S-1)} \\ . \\ f_{n_s-1}^{(2...S-1)} \\ f_1^{(2...S-1)} \\ f_{n_s}^{(2...S-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can therefore compute the Primal Schur Complement $Sp^{(s)}$:

$$S_p^{(s)} = k_{bb}^{(s)} - k_{bi}^{(s)} {k_{ii}^{(s)}}^{-1} k_{ib}^{(s)}$$

$$S_p^{(2...S-1)} = k_{bb}^{(2...S-1)} - k_{bi}^{(2...S-1)} {k_{ii}^{(2...S-1)}}^{-1} k_{ib}^{(2...S-1)}$$

$$S_p^{(2...S-1)} = \frac{EA}{h} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \\ 0 & \cdots & -1 \end{bmatrix} \cdot [\frac{EA}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}]^{-1} \cdot \frac{EA}{h} \cdot \begin{bmatrix} -1 & 0 \\ \cdots & \cdots \\ 0 & -1 \end{bmatrix}$$

The Dual Schur Complement $S_d^{(s)}$ is the generalized invers of $S_p^{(s)}$:

$$S_d^{(s)} = {S_p^{(s)}}^+$$

$$S_d^{(2...S-1)} = {S_p^{(2...S-1)}}^+$$

The Primal RHS $b_p^{(s)}$:

$$b_p^{(s)} = f_b^{(s)} - k_{bi}^{(s)} {k_{ii}^{(s)}}^{-1} f_i^{(s)}$$

$$b_p^{(2...S-1)} = f_b^{(2...S-1)} - k_{bi}^{(2...S-1)} {k_{ii}^{(2...S-1)}}^{-1} f_i^{(2...S-1)}$$

$$b_p^{(2\ldots S-1)} = \vec{0} - k_{bi}^{(2\ldots S-1)} {k_{ii}^{(2\ldots S-1)}}^{-1} \vec{0} = \vec{0}$$

Finding rigid body modes $R_b^{(s)}$ consists to solve:

$$S_p^{(s)} R_b^{(s)} = 0$$

$$S_p^{(2\ldots S-1)} R_b^{(2\ldots S-1)} = 0$$

Operators for substructure (S):
Displacements of the last substructure:

$$u^{(2\ldots S)} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_{n_s-1} \\ u_{n_s} \end{bmatrix}$$

Nodal forces of the last substructure:

$$f^{(S)} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ F_d \end{bmatrix}$$

Firstly, the interface is defined by the last local boundary node(ie: 1), so we have the stiffness matrix $K^{(S)}$:

$$\frac{EA}{h} \cdot \begin{array}{c} \begin{matrix} 2 & 3 & \ldots & n_s-1 & n_s & 1 \end{matrix} \\ \begin{pmatrix} 1+1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1+1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} 2 \\ 3 \\ \ldots \\ n_s-1 \\ n_s \\ 1 \end{matrix}$$

which gives:

$$K^{(S)} = \frac{EA}{h} \cdot \left(\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

With :

$$k_{ii}^{(S)} = \frac{EA}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdots & \cdots & 0 \\ 0 & \cdots & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}_{(n_s-1)\cdot(n_s-1)}$$

$$k_{ib}^{(S)} = \frac{EA}{h} \cdot \begin{bmatrix} -1 \\ \cdots \\ 0 \end{bmatrix}_{(n_s-1)\cdot 1}$$

$$k_{bi}^{(S)} = \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \end{bmatrix}_{1\cdot(n_s-1)}$$

$$k_{bb}^{(S)} = \frac{EA}{h} \cdot 1$$

And :

$$u^{(S)} = \begin{bmatrix} u_i^{(S)} \\ u_b^{(S)} \end{bmatrix} = \begin{bmatrix} u_2^{(S)} \\ \cdot \\ u_{n_s-1}^{(S)} \\ u_{n_s}^{(S)} \\ u_1^{(S)} \end{bmatrix} = \begin{bmatrix} u_2 \\ \cdot \\ u_{n_s-1} \\ u_{n_s} \\ u_1 \end{bmatrix}$$

And :

$$f^{(S)} = \begin{bmatrix} f_i^{(S)} \\ f_b^{(S)} \end{bmatrix} = \begin{bmatrix} f_2^{(S)} \\ \cdot \\ f_{n_s-1}^{(S)} \\ f_{n_s}^{(S)} \\ f_1^{(S)} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ 0 \\ F_d \\ 0 \end{bmatrix}$$

We can therefore compute the Primal Schur Complement $Sp^{(s)}$ using trimmed matrix:

$$S_p^{(s)} = k_{bb}^{(s)} - k_{bi}^{(s)} {k_{ii}^{(s)}}^{-1} k_{ib}^{(s)}$$

$$S_p^{(S)} = k_{bb}^{(S)} - k_{bi}^{(S)} {k_{ii}^{(S)}}^{-1} k_{ib}^{(S)}$$

$$S_p^{(S)} = \frac{EA}{h} - \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \end{bmatrix} \cdot [\frac{EA}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdots & \cdots & 0 \\ 0 & \cdots & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}]^{-1} \cdot \frac{EA}{h} \cdot \begin{bmatrix} -1 \\ \cdots \\ 0 \end{bmatrix}$$

$$S_p^{(S)} = \frac{EA}{h} - \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \end{bmatrix} \cdot [\frac{h}{EA} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \end{bmatrix}] \cdot \frac{EA}{h} \cdot \begin{bmatrix} -1 \\ \cdots \\ 0 \end{bmatrix}$$

$$S_p^{(S)} = \frac{EA}{h} - \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \cdots \\ 0 \end{bmatrix}$$

$$S_p^{(S)} = 0$$

The Dual Schur Complement $S_d^{(s)}$ is the generalized invers of $S_p^{(s)}$:

$$S_d^{(s)} = {S_p^{(s)}}^+$$

$$S_d^{(S)} = {S_p^{(S)}}^+ = 0^+ = 0$$

The Primal RHS $b_p^{(s)}$:

$$b_p^{(s)} = f_b^{(s)} - k_{bi}^{(s)} {k_{ii}^{(s)}}^{-1} f_i^{(s)}$$

$$b_p^{(S)} = f_b^{(S)} - k_{bi}^{(S)} {k_{ii}^{(S)}}^{-1} f_i^{(S)}$$

$$b_p^{(S)} = 0 - \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \end{bmatrix} \cdot [\frac{EA}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \cdots & \cdots & 0 \\ 0 & \cdots & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}]^{-1} \begin{bmatrix} 0 \\ \cdot \\ 0 \\ F_d \end{bmatrix}$$

$$b_p^{(S)} = 0 - \frac{EA}{h} \cdot \begin{bmatrix} -1 & \cdots & 0 \end{bmatrix} \cdot [\frac{h}{EA} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \end{bmatrix}] \begin{bmatrix} 0 \\ \cdot \\ 0 \\ F_d \end{bmatrix}$$

$$b_p^{(S)} = 0 + \begin{bmatrix} 1 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} 0 \\ \cdot \\ 0 \\ F_d \end{bmatrix}$$

$$b_p^{(S)} = F_d$$

The rigid body modes $R_b^{(s)}$:

$$S_p^{(s)} R_b^{(s)} = 0$$

$$S_p^{(S)} R_b^{(S)} = 0$$

Knowing that $S_p^{(S)} = 0$, then $R_b^{(S)} \in \mathbb{R}^*$, we choose $R_b^{(S)} = 1$

Primal assembly operators and concatenated operators
The primal assembly operators for each substructure are given by :

$$A^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(S-1)*1}$$

$$A^{(s=2...S-1)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ A_{s-1,1}=1 & 0 \\ 0 & A_{s,2}=1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}_{(S-1)*2}$$

$$A^{(S)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{(S-1)*1}$$

The lines represent interfaces, we totally have S-1 interfaces, the columns represent interface nodes of a subdomain, for the subdomains (s=1 and s=S) there is only 1 interface node, for the subdomains (s=1...S-1) there are 2 interface nodes. So the concatenated primal assembly operator is :

$$\mathbb{A}^{\diamond} = [A^{(1)} \quad A^{(2...S-1)} \quad A^{(S)}]_{(S-1)*(2S-2)}$$

Indeed, we totally have $1 + 2(S-2) + 1 = 2S - 2$ interface nodes for all subdomains.
The concatenated primal complements are defined by :

$$S_p^{\diamond} = \begin{bmatrix} S_p^1 & 0 & 0 \\ 0 & S_p^{2...S-1} & 0 \\ 0 & 0 & S_p^S \end{bmatrix}_{(2S-2)*(2S-2)}$$

The concatenated primal right-hand sides are :

$$b_p^{\diamond} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F_d \end{bmatrix}_{(2S-2)*1}$$

Thus we have :

$$\mathbb{S}_p = \mathbb{A}^{\diamond} S_p^{\diamond} \mathbb{A}^{\diamond T}$$

$$\mathbb{B}_p = \mathbb{A}^{\diamond} b_p^{\diamond}$$

Finally we solve :

$$\mathbb{S}_p U_p = \mathbb{B}_p$$

Dual assembly operators and concatenated operators
The dual assembly operators for each substructure are given by :

$$\underline{A}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(S-1)*1}$$

$$\underline{A}^{(s=2...S-1)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \underline{A}_{s-1,1} = -1 & 0 \\ 0 & \underline{A}_{s,2} = 1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}_{(S-1)*2}$$

$$\underline{A}^{(S)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}_{(S-1)*1}$$

The concatenated primal assembly operator is :

$$\underline{\mathbb{A}}^{\diamond} = [\underline{A}^{(1)} \quad \underline{A}^{(2...S-1)} \quad \underline{A}^{(S)}]_{(S-1)*(2S-2)}$$

The concatenated dual complements are defined by :

$$S_d^{\diamond} = \begin{bmatrix} S_d^1 & 0 & 0 \\ 0 & S_d^{2...S-1} & 0 \\ 0 & 0 & S_d^S \end{bmatrix}_{(2S-2)*(2S-2)} = \begin{bmatrix} S_p^{1+} & 0 & 0 \\ 0 & S_p^{2...S-1+} & 0 \\ 0 & 0 & S_p^{S+} \end{bmatrix}_{(2S-2)*(2S-2)}$$

The concatenated dual right-hand sides is the same :

$$b_p^{\diamond} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F_d \end{bmatrix}_{(2S-2)*1}$$

The matrix of concatenated rigid body modes reads :

$$R_b^{\diamond} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ R_b^{(j=2)} & 0 & \vdots & \vdots \\ \vdots & R_b^{(j=3...S-1)} & \vdots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & R_b^{(S)} = 1 \end{bmatrix}_{(2S-2)*(S-1)}$$

The columns represent the rigid body modes of a subdomain, so totally we have 1*S-1 columns since the 1st subdomain doesn't have a rigid body mode. The lines represent interfaces, we totally have S-1 interfaces.