Voting rules as Group Decision Making Models

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Chapter 2

Aims

- Study decision problems in which a group has to take a decision among several alternatives
- Analyze a number of properties of electoral systems
- Present a few elements of the classical theory: Social choice theory
- Parameters to take into account:
  - nature of the decision
  - size of the group
  - nature of the group
- Many (deep) results
  - Economics, Political Science, Applied Mathematics, Operation Research
  - Two Nobel Prizes: Kenneth J. Arrow, Amartya Sen

Problem

Study election problems in which a society has to take a decision among several candidates

Election of one candidate

- Common sense:
  - the choice of the candidate will affect all members of the society
  - the choice of the candidate should take the opinion of all members of society into account
- Intuition:
  - Democracy $\Longrightarrow$ Elections $\Longrightarrow$ Majority

Political problems

- direct or indirect democracy?
- rôle of parties?
- who can vote? (age, sex, nationality, paying taxes,...)
- who can be candidate?
- what type of mandate?
- how to organize the campaign?
- rôle of polls?

Technical problems

- Majority: When there are only two candidates
  - elect the one receiving the more votes
- Majority: When there are more than two candidates
  - many ways to extend this simple idea
  - not equivalent
  - sometimes leading to unwanted results

Typology of elections

- Two main criteria
  1. type of ballots admitted
     - one name
     - ranking of all candidates
     - other types (acceptable candidates, grading candidates, etc.)
  2. method for organizing the election and for tallying ballots
- Consequences:
  - many possible types of elections
  - many have been proposed
  - many have have been used in practice

Two hypotheses

1. All voters are able to rank order the set of all candidates (ties admitted)
   - e.g. each voter has a weak order on the set of all candidates:

$$a \succ b \succ c \sim d \succ e$$

2. Voters are sincere
   - if I have to vote for one candidate, I vote for a

Plurality voting

Rules

- one round of voting
- ballots with one name
- “first past the post”

Remark

- ties are neglected (unlikely)
    - one voter has special power (the Queen chooses in case of a tie)
    - one candidate receives special treatment (the older candidate is elected)
    - random tie breaking rule

Plurality voting

Example

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, . . .)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Which candidate is elected ?

Plurality voting

Example

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, $\ldots$)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Which candidate is elected ?

Results

- $a : 10 \quad b : 6 \quad c : 5$
- a is elected.
- but an absolute majority of voters $11/21$ prefer all loosing candidates to the elected one!

Plurality voting

Remarks

- Problems are expected as soon as there are more than 2 candidates
- A system based on an idea of "majority" may well violate the will of a majority of voters
- Sincerity hypothesis is heroic!

Plurality with runoff

Rules

- Ballots with one name
- First round
  - the candidate with most votes is elected if he receives more than 50% of votes
  - otherwise go to the second round
- Second round
  - keep the two candidates having received more votes
  - apply plurality voting

Plurality with runoff

Example (Previous Example)

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, . . .)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Which candidate is elected ?

Plurality with runoff

Example (Previous Example)

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$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Which candidate is elected ?

Results

- $a : 10 \quad b : 6 \quad c : 5$
- absolute majority: $21/2 = 11$ voters.
- go to the second round with a and b: $a : 10 \quad b : 11$
- b is elected
- no candidate is preferred to b by a majority of voters

Plurality with runoff

Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

Which candidate is elected ?

Plurality with runoff

Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, . . .)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

Which candidate is elected ?

Results

- b is elected
- absolute majority: $21/2 = 11$ voters.
- an absolute majority of voters $(11/21)$ prefer a and d to b

Plurality with runoff

Plurality vs plurality with runoff

- The French system does only a little better than the UK one
- Preferences used in the above example are not bizarre.

Plurality with runoff : manipulation

Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

b is elected

Non-sincere voting

- Suppose that the 6 voters for which $c \succ a \succ d \succ b$ vote as if their preferences were $a \succ c \succ d \succ b$
- Result:
  - a is elected at the first round (11/21)
  - profitable to the six manipulating voters (for them $a \succ b$)

Manipulable voting rules

Definition

- A voting rule is manipulable if it may happen that some voters may have an interest to vote in a non-sincere way

Remarks

- Plurality with runoff is manipulable

Plurality with runoff: monotonicity

Before campaign

- 3 candidates $\{a, b, c\}$
- 17 voters

$$\begin{array}{ll} 6 \text{ voters:} & a \succ b \succ c \\ 5 \text{ voters:} & c \succ a \succ b \\ 4 \text{ voters:} & b \succ c \succ a \\ 2 \text{ voters:} & b \succ a \succ c \end{array}$$

Which candidate is elected?

Plurality with runoff: monotonicity

Before campaign

- 3 candidates $\{a, b, c\}$
- 17 voters

$$\begin{array}{ll} 6 \text{ voters:} & a \succ b \succ c \\ 5 \text{ voters:} & c \succ a \succ b \\ 4 \text{ voters:} & b \succ c \succ a \\ 2 \text{ voters:} & b \succ a \succ c \end{array}$$

Results

- a is elected!
- a gets more money to campaign against b

Plurality with runoff: monotonicity

Before campaign

- 3 candidates $\{a, b, c\}$
- 17 voters

$$\begin{array}{ll} 6 \text{ voters:} & a \succ b \succ c \\ 5 \text{ voters:} & c \succ a \succ b \\ 4 \text{ voters:} & b \succ c \succ a \\ 2 \text{ voters:} & b \succ a \succ c \end{array}$$

- Suppose that last 2 voters ($b \succ a \succ c$) change their minds in favor of a
- Their new preferences are $a \succ b \succ c$

Which candidate is elected?

Plurality with runoff: monotonicity

Before campaign

- 3 candidates $\{a, b, c\}$
- 17 voters

$$\begin{array}{ll} 6 \text{ voters:} & a \succ b \succ c \\ 5 \text{ voters:} & c \succ a \succ b \\ 4 \text{ voters:} & b \succ c \succ a \\ 2 \text{ voters:} & b \succ a \succ c \end{array}$$

- Suppose that last 2 voters ($b \succ a \succ c$) change their minds in favor of a
- Their new preferences are $a \succ b \succ c$

Results

- c is elected!
- the good campaign of a is fatal to him/her
- non-monotonic method: increasing possibilities of manipulation

Condorcet voting rule (1785)

Principles

- compare all candidates by pair
- declare that *a* is "socially preferred" to *b* if (strictly) more voters prefer *a* to *b* (social indifference in case of a tie)
- Condorcet's principle: if one candidate is preferred to all other candidates, it should be elected. This candidate is called a Condorcet Winner
- Condorcet Winner (CW: must be unique)

Remarks

- Plurality rule and Plurality with runoff violate Condorcet's principle
- Condorcet's principle does not solve the "dictature of the majority" difficulty
- a Condorcet winner is not necessarily "ranked high" by voters

Condorcet voting rule

Example

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Is there a Condorcet winner?

Condorcet voting rule

Example

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Results

- b is the Condorcet winner
  - b beats a (11/21)
  - b beats c (16/21)
- a is the plurality winner
- a is the Condorcet looser

Condorcet voting rule

Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

Is there a Condorcet winner?

Condorcet voting rule

Example

- 4 candidates $\{a, b, c, d\}$
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$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

Results

- a is the Condorcet winner
  - a beats b (11/21)
  - a beats c (15/21)
  - a beats d (21/21)
- b is the plurality with runoff winner (beats c in the second round)

Condorcet's Paradox

Example

- 3 candidates $\{a, b, c\}$
- 3 voters

$$\begin{array}{ll} 1 \text{ voters:} & a \succ b \succ c \\ 1 \text{ voters:} & b \succ c \succ a \\ 1 \text{ voters:} & c \succ a \succ b \end{array}$$

the social strict preference relation may have circuits!

Electing the CW

- attractive but not always effective!

Borda voting rule (1783)

Principles

- Each ballot is an ordered list of candidates (exclude ties for simplicity)
- On each ballot compute the rank of the candidates in the list
- Rank order the candidates according to the decreasing sum of their ranks

Remarks

- simple
- efficient: always lead to a result
- separable, monotonic, participation incentive

Borda voting rule

Example

- 4 candidates $\{a, b, c, d\}$
- 3 voters

$$\begin{array}{ll} \text{2 voters:} & b \succ a \succ c \succ d \\ \text{1 voters:} & a \succ c \succ d \succ b \end{array}$$

Which candidate is elected by using a Borda procedure?

Borda voting rule

Example

- 4 candidates $\{a, b, c, d\}$
- 3 voters

$$\begin{array}{ll} \text{2 voters:} & b \succ a \succ c \succ d \\ \text{1 voters:} & a \succ c \succ d \succ b \end{array}$$

Results

- Borda scores

$$a : 5 \quad b : 6 \quad c : 8 \quad d : 11$$

- a is elected
- b is the obvious Condorcet Winner

Summary

Example

- 4 candidates $\{a, b, c, d\}$
- 27 voters (may be also 27 000 000 or 54 000 000, ...)

$$\begin{array}{ll}
5 \text{ votants}: & a \succ b \succ c \succ d \\
4 \text{ votants}: & a \succ c \succ b \succ d \\
2 \text{ votants}: & d \succ b \succ a \succ c \\
6 \text{ votants}: & d \succ b \succ c \succ a \\
8 \text{ votants}: & c \succ b \succ a \succ d \\
2 \text{ votants}: & d \succ c \succ b \succ a
\end{array}$$

Determine the candidate elected by using the plurality, plurality with runoff, Condorcet principle and Borda principle.

Summary

Example

- 4 candidates $\{a, b, c, d\}$
- 27 voters (may be also 27 000 000 or 54 000 000, ...)

$$\begin{array}{rl} 5 \text{ voters}: & a \succ b \succ c \succ d \\ 4 \text{ voters}: & a \succ c \succ b \succ d \\ 2 \text{ voters}: & d \succ b \succ a \succ c \\ 6 \text{ voters}: & d \succ b \succ c \succ a \\ 8 \text{ voters}: & c \succ b \succ a \succ d \\ 2 \text{ voters}: & d \succ c \succ b \succ a \end{array}$$

Results

- a is the plurality with runoff winner
- d is the plurality winner
- b is the Borda winner
- c is the Condorcet Winner

What are we looking for?

Democratic method

- always giving a result like Borda
- always electing the Condorcet winner
- consistent w.r.t. withdrawals
- monotonic, separable, incentive to participate, not manipulable etc.

Arrow

Framework

- $n \geq 3$ candidates (otherwise use plurality)
- m voters ($m \geq 2$ and finite)
- ballots: ordered list of candidates
- A voting profile is denoted by $(\succsim_i)_{i=1,\ldots,m}$ where $\succsim_i$ is an individual preferences of the voter i.
- The result (collective preference) of the voting is denoted by $\succsim$.

Problem

- find all electoral methods respecting a small number of "desirable" principles

Arrow

Principles

- Universality
  - the method should be able to deal with any configuration of ordered lists, i.e, there is no restriction about the expression of a voter.
- Transitivity
  - the result of the method should be an ordered list of candidates
- Unanimity
  - the method should respect a unanimous preference of the voters

$$\forall x, y,\ [x \succsim_i y \quad \forall i = 1, \dots, m] \Longrightarrow x \succsim y$$

Arrow

Principles

- Absence of dictator
  - the method should not allow for dictators

$$\exists i_0, \forall x, y \; [x \succsim_{i_0} y \Longrightarrow x \succsim y]$$

- Independence of irrelevent alternatives
  - the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters

$$\forall x, y, \; [\forall i = 1, \ldots, m, \; x \succsim_i y \Longleftrightarrow x \succsim_i' y] \Longrightarrow [x \succsim y \Longleftrightarrow x \succsim' y]$$

Arrow's theorem (1951)

Theorem

There is no method respecting the five principles

Borda

- universal, transitive, unanimous with no dictator
- cannot be independent

Condorcet

- universal, independent, unanimous with no dictator
- cannot be transitive

Exercise

We consider the following profile (9 voters and 4 candidates) where the preferences of the last voter are unknown:

$$\begin{array}{rl} 4 \text{ voters:} & c \succ d \succ a \succ b \\ 2 \text{ voters:} & a \succ b \succ d \succ c \\ 2 \text{ voters:} & b \succ a \succ c \succ d \\ 1 \text{ voter:} & ? \succ ? \succ ? \succ ? \end{array}$$

Do we necessarily know the preferences of the last voter, in order to determine the result of the elections in a UK system (plurality) and French system (plurality with runoff)? If yes, gives these preferences and the results of these elections.