Massive Graph Management & Analytics

Preliminaries

Nacéra Seghouani

Computer Science Department, CentraleSupélec
Laboratoire Interdisciplinaire des Sciences du Numérique, LISN
nacera.seghouani@centralesupelec.fr

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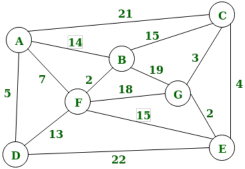
Graph Theory Preliminaries

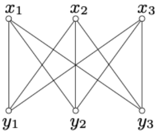
Graph Typology

☞ $\mathcal{G}(V, E)$, V set of vertices, $E = \{(v_i, v_j) | v_i, v_j \in V\}$ set of edges, $|V| = n, |E| = m$

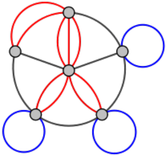
- ✓ **Undirected** - edge: symmetric pair of vertices - **Directed** - edge: asymmetric pair of vertices -
- ✓ **weighted** vertice $w_v : V \to \mathbb{R}$ or edge $w_e : E \to \mathbb{R}$
- ✓ **labeled** vertice $w_v : V \to \mathbb{L}$ or edge $w_e : E \to \mathbb{L}$
- ✓ **Bipartite** - $V = V_1 \cup V_2$, $E = \{(v_i, v_j) | v_i \in V_1, v_j \in V_2\}$ - Generalization to ***k*-partite**
- ✓ **Multigraph** or **Multidigraph** - $r : E \to v_i, v_j \in V$ where r assigns to each $e \in E$ a pair of vertices -
- ✓ **Hypergraph** - edge: relates a subset of vertices -
- ✓ **Complete graph**: $\forall (v_i, v_j) \in VxV, (v_i, v_j) \in E$



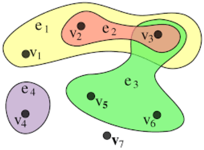
weighted graph



bipartite graph



multigraph



hypergraph

Graph Properties

- Let $\mathcal{G}(V, E)$ a directed graph, d_i^+, d_i^- denote resp. the number of edges coming out and coming to v_i. The degree of v_i:
$$d_i = d_i^+ + d_i^-$$
- $\mathcal{N}_i^+, \mathcal{N}_i^-$ denote resp. the set of the successors and predecessors of v_i. The set of the neighbors of v_i:
$$\mathcal{N}_i = \mathcal{N}_i^+ \cup \mathcal{N}_i^-$$
- A (directed) path $(v_i \rightsquigarrow v_j)$ is a sequence of vertices in the graph $(v_i, v_k, .., v_j)$ where each consecutive vertices pair $\in E$
- A (directed) cycle is $(v_i \rightsquigarrow v_j = v_i)$
- The length of a path $(v_i \rightsquigarrow v_j)$ is the number of the edges in $(v_i \rightsquigarrow v_j)$.
- A distance between (v_i, v_j) is the shortest path length between (v_i, v_j)

$$dist(v_i, v_j) = Min_{v_i \rightsquigarrow v_j} length(v_i \rightsquigarrow v_j)$$

Graph Properties

☞ The eccentricity *ecc* of v: the greatest distance between v and any other vertex;

$$ecc(v) = \max_{s \in V} dist(v, s)$$

☞ The diameter of $\mathcal{G}$ is

$$\max_{v,s \in V} dist(v, s)$$

It is also the maximum eccentricity of any v in $\mathcal{G}$

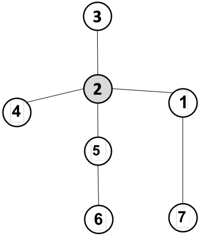
☞ The radius of $\mathcal{G}$ is the minimum eccentricity of any vertex

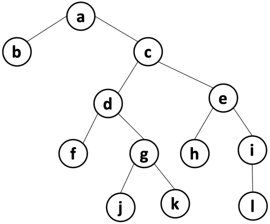
$$\min_{v \in V} ecc(v)$$

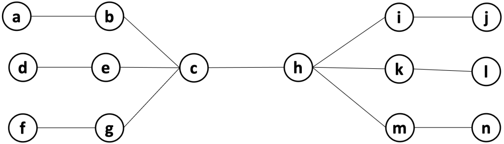
☞ The center of a graph is the set of all vertices of minimum eccentricity, equal to the graph's radius.

Graph Properties

Examples:







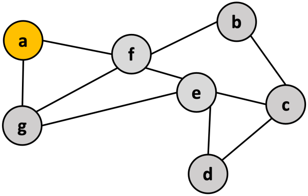
Graph Properties

- ☞ $\mathcal{G}\prime(V, E\prime \subset E)$ is a partial graph of $\mathcal{G}(V, E)$
- ☞ $\mathcal{G}\prime(V\prime \subset V, E\prime \subset E)$ is a subgraph of $\mathcal{G}(V, E)$
- ☞ $\mathcal{G}(V, E)$ is a connected graph $\iff \forall(v_i, v_j) \in V \exists(v_i \rightsquigarrow v_j)$
- ☞ (strongly) connected component of $\mathcal{G}(V, E)$ is a subgraph - $\mathcal{G}_{cc}(V_{cc}, E_{cc})$ where $\exists(v_i \rightsquigarrow v_j)$, a (directed) path between each v_i and $v_j \in V_{cc}$,
- ☞ $\mathcal{G}(V, E)$ is a tree $\Leftrightarrow$ $\mathcal{G}$ is a connected graph without cycle $\Rightarrow$ graph with $m = n - 1$ edges
- ☞ $\mathcal{G}(V, E)$ is a forest $\iff$ each connected component is a tree

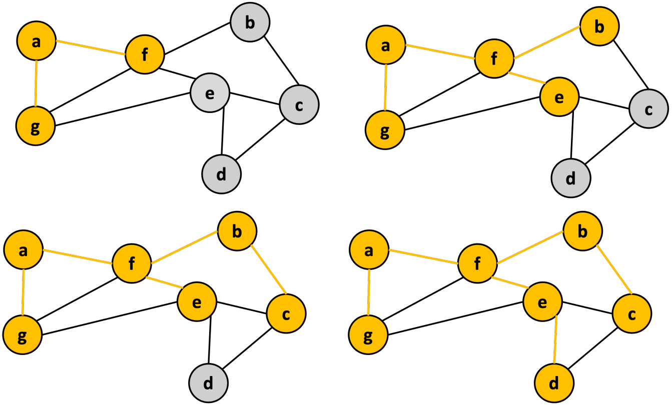
Breadth First Search (BFS)

☞ Queue data structure: an element first added in the list first removed out the list FIFO (First In First Out)

procedure BFS($\mathcal{G}(V, E), r$)
 $Q \leftarrow \emptyset$, $enqueue(Q, r)$,
 $r.label = true$
 while $Q \neq \emptyset$ **do**
  $v \leftarrow dequeue(Q)$
  for $w \in \mathcal{N}_v$ **do**
   if $\neg w.label$ **then**
    $enqueue(Q, w)$
    $w.label = true$
   end if
  end for
 end while
end procedure



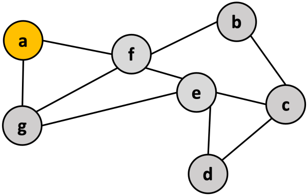
Breadth First Search (BFS)



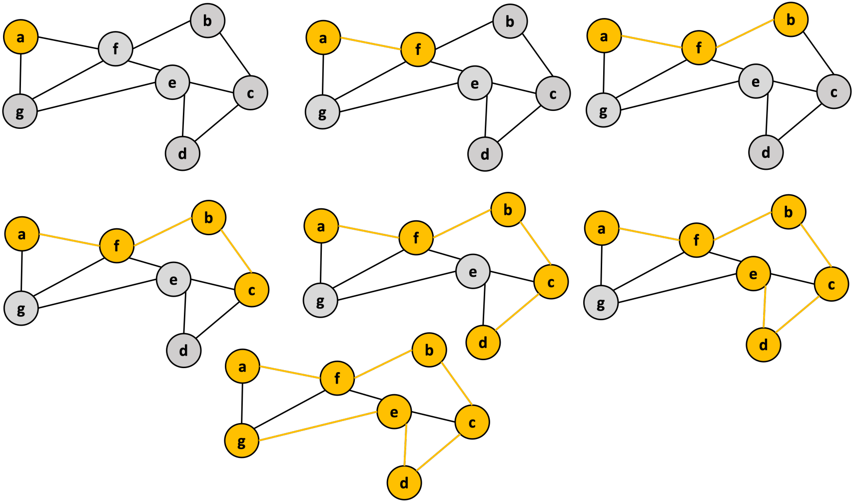
Depth First Search (DFS)

☞ Recursive DFS

```
procedure DFS*(𝒢(V, E), r)
    r.label = true
    for v ∈ 𝒩 do
        if ¬v.label then
            DFS*(𝒢(V, E), v)
        end if
    end for
end procedure
```

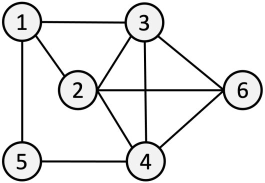


Breadth First Search (BFS)



Depth and Breadth First Search

Another example



Graph Representation using Matrices

☞ $\mathcal{G}(E, V)$ with n vertices and m edges can be encoded using:

- ✓ Adjacency Matrix $\mathbf{A}(n \times n)$, $n = |V|$

$$\mathbf{A}_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

  Symmetric matrix if $\mathcal{G}$ is an undirected (without loops)

- ✓ Adjacency list $\mathbf{L}$ each vertex holds a list of its neighbours

$$\forall v_i \in V,\ \mathbf{L}_i = \{v_j | (v_i, v_j) \in E\}$$

  If $\mathcal{G}$ is directed the choice of the direction depends on analytic needs

- ✓ Incidence matrix $\mathbf{B}$, $n \times m$

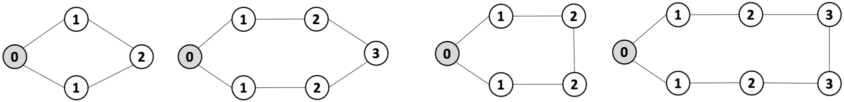
$$\mathbf{B}_{ij} = \begin{cases} 1 & \text{if } e_j = (v_i, v_k) \in E, \\ 0 & \text{otherwise} \end{cases}$$

Graph Theory Preliminaries

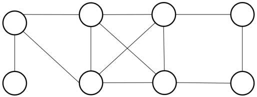
Some Exercises

Exercise: Breadth-First Search and Bipartite graphs

1) Using graph traversal algorithms, propose an algorithm that computes the number of edges between a given vertex and all other vertices.
2) Given the following cycles with even and odd length (with the distances or depths from the grey vertex), what do you think about the case of graphs with an **odd** cycle (in number of edges)? Is this a characteristic property? State the general case.



3) Propose an algorithm that determines if a graph contains an odd cycle.
4) In a bipartite graph, can there be a cycle with an odd number of edges? Is this a characteristic property? Justify your answer.
5) Propose an algorithm that allows to determine if a graph is bipartite. Test your algorithm on the following graph. Is it bipartite? Justify your answer



Exercise: Breadth-First Search and Bipartite graphs

1)

procedure BFS_DISTANCE($\mathcal{G}(V, E), s$)
    $depth \leftarrow \{s : 0\}$
    $next \leftarrow [s]$
    while $len(next) > 0$ **do**
        $n \leftarrow pop(next)$
        for $v \in \mathcal{N}_n$ **do**
            if $v \notin next$ **then**
                $push(v, next)$
                $depth[v] \leftarrow depth[n] + 1$
            end if
        end for
    end while
    return depth ***works only for connected graphs***
end procedure

2)

Lemma: A graph contains a cycle C with an odd number of edges iff: $\exists (x, y) \in E \mid depth(x) = depth(y)$
First we know that all edges connect vertices of "neighboring" depths: $\forall (x, y) \in E\ |depth(x) - depth(y)| \leq 1$

- $\Rightarrow$ by contrapositive, we suppose that $\forall (x, y) \in C\ depth(x) \neq depth(y)$, then $\forall (x, y) \in C\ depth(x) = depth(y) \pm 1$, we will have along the cycle a node of even depth, followed by a node of odd depth and so on and we will not be able to close a cycle of odd size!
- $\Leftarrow$ if there exists an edge $(x, y) \in E$ for which $depth(x) = depth(y)$. We consider the path tree that was used to annotate the depths. In this tree x and y have a first ancestor z in common (possibly the root) from which we can form an odd cycle of size $2(depth(x) - depth(z)) + 1$ by adding the edge (x, y) to this subtree starting at z.

Exercise: Breadth-First Search and Bipartite graphs

3)

```
procedure HASODDCYCLE((G(V ≠ ∅, E)))
    depth ← BFS_depth(G, V[0])
    for (n1, n2) ∈ E do
        if depth[n1] == depth[n2] then
            return true
        end if
    end for
    return false
end procedure
```

4)

$\Rightarrow$ If the graph is bipartite, any path alternates between each vertex of each partition to create a cycle ending by the initial vertex. This means an even number of edges. In a bipartite graph all cycles are pair.

$\Leftarrow$ We just consider the partition of vertices with even depth V_1 on one side and the partition of vertices with odd depth on the other V_2. We assume that there is no odd cycle then (from question 2): $\forall (x, y) \in E, depth(x) = depth(y) \pm 1$ the all the edges are in $V_1 \times V_2$. The graph is obviously bipartite.

5)

From the previous questions

Bipartite $\Leftrightarrow_{question4}$ no odd cycle

$\Leftrightarrow_{question2}$ no pair of connected vertices have the same depth

The proposed algorithm selects randomly a vertex to compute the depths. The graph is bipartie iff no pair of connected vertices have the same depth

Exercise: Depth-First Search and 2-colorable graphs

Graph coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. A 2-colorable graph is a graph that can be colored with only 2 colors.

1) What is the link with the previous exercise? Justify your answer.
2) We want to write an algorithm, inspired by DFS search which takes as input a graph $\mathcal{G}(V, E)$ and which returns a pair `(result, color)` where *result* is *true* if the graph is colorable, *false* otherwise and *color* is a dictionary associating a color 0 or 1 to each vertex. This algorithm should *stop as soon as possible* when the graph is not 2-colorable. Propose an **iterative** version or a **recursive** version.

Exercise: Depth-First Search and 2-colorable graphs

1)

Theorem : a graph is 2-colorable if and only if it is bipartite.

$\Longrightarrow$ we partition V into $V_1 \cup V_2$ such that V_1 gathers all white nodes and V_2 all black nodes ($V_1 \cap V_2 = \emptyset$). Any edge connects a white node (in $V1$) to a black node (in V_2).

$\Longleftarrow$ it is enough to choose the white for V_1 and the black for V_2 and as all the edges are in $V_1 \times V_2$, we are done.

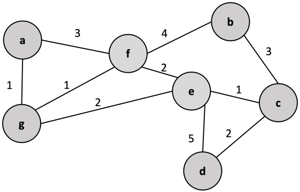
2)

```
procedure COLORING2ITER((G, s = V[0]))
    color ← {s : 0}
    next ← [s]
    while len(next) > 0 do
        n ← pop(next)
        for v ∈ N_n do
            if v ∉ color then
                push(v, next)
                color[v] ← 1 − color[n]
            else
                if color[v] == color[n] then
                    return false, color
                end if
            end if
        end for
    end while
    return true, color
end procedure
```

Exercise: Shortest path

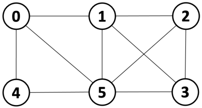
Compute the shortest path using Dijkstra algorithm

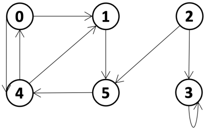
procedure $\textsc{Dijkstra}(\mathcal{G}(V, E, W), s)$
    $dist \leftarrow \{s : 0\}$
    $P \leftarrow \emptyset$
    for $v \in V \wedge v \neq s$ **do**
        $dist[v] \leftarrow +\infty$
    end for
    $w \leftarrow select(v \in V - P \wedge dist[v] = min_v\, dist[v]$
    $P \leftarrow P \cup \{w\}$
    for $v \in \mathcal{N}_w \wedge v \notin P$ **do**
        if $dist[w] + w_{(v,w)} < dist[v]$ **then**
            $predecessor(v) \leftarrow w$
            $dist[v] \leftarrow w_{(v,w)} + dist[w]$
        end if
    end for
end procedure



Exercise: Matrix Multiplication & Power

1) Give the different representations of these graphs.
2) Compute $\mathbf{A}^2$, $\mathbf{A}^3$. What $\mathbf{A}^r_{ij}$ represents?
3) What is the complexity of $\mathbf{A}^r$, Is it possible to reduce it?



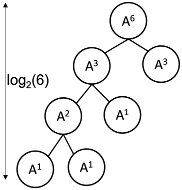


Exercise: Matrix Multiplication & Power

Using the adjacency matrix **A**, the number of r- length paths:

$$\mathbf{A}^r = \mathbf{A} x \mathbf{A}^{r-1} \forall r > 1$$

$\mathbf{A}^r_{ij} = \sum_k \mathbf{A}^{r-1}_{ik} \mathbf{A}_{kj}$, the number of r-length paths between nodes i and j
Computing $\mathbf{A}^r$ is $\mathcal{O}(rn^3)$: requires r multiplications, each one $\mathcal{O}(n^3)$.
If r is pair $\mathbf{A}^r = (\mathbf{A}^{\frac{r}{2}})^2$, otherwise $\mathbf{A}^r = \mathbf{A}(\mathbf{A}^{\frac{r-1}{2}})^2$. Computing $\mathbf{A}^r$ requires $\mathcal{O}(log_2(r))n^3$ multiplications.



The distance between v_i and v_j is the smallest d such that $\mathbf{A}^d_{ij} \neq 0$
Useful to compute shortest paths, triangles, ...

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Linear Algebra Preliminaries

Vector Norms

- A function f that measures the size of a vector is called a norm.
- A vector norm has to satisfy the following:

$$f(\mathbf{x}) = 0 \rightarrow \mathbf{x} = \mathbf{0}$$

$$f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}) \text{ triangle inequality}$$

$$\forall \alpha \in \mathbb{R} \; f(\alpha\mathbf{x}) = |\alpha| f(\mathbf{x})$$

- The general form of vector norm: $||\mathbf{x}||_p = \sqrt[p]{\sum_i |x_i|^p}$
- The most commonly used is euclidian norm $||\mathbf{x}||_2 = \sqrt{\sum_i |x_i|^2}$

Dot and Cross Vector Product

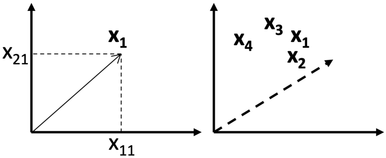
- **A dot product or scalar product of two vectors** is a scalar quantity
  - algebraic formula: $||\mathbf{x}||_2||\mathbf{y}||_2 cos(\theta)$ or $\sum_i \mathbf{x}_i \mathbf{y}_i$
  - how much two vectors points in same directions. The dot product of two orthogonal vectors is 0.
- **A cross or vector product of two vectors** is a vector orthogonal to to both the vectors.
  - algebraic formula: $||\mathbf{x}||_2||\mathbf{y}||_2 sin(\theta)\mathbf{1}$ or use cofactor matrix
  - how much two vectors points in different directions. The cross product of two linear vectors is 0.

Vectors and Matrices

- Let $\mathbf{x}_i$ $i \in [1..m]$ be m vectors of n elements:
  $\mathbf{x}_1 = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1} \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} x_{12} & x_{22} & \dots & x_{n2} \end{bmatrix}$, ... $\mathbf{x}_m = \begin{bmatrix} x_{1m} & x_{2m} & \dots & x_{nm} \end{bmatrix}$
- Matrix representation of size $n \times m$, as a row vector or column vector (transpose)

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_m \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \dots \\ \mathbf{x}_m \end{bmatrix}^T$$

- Each $\mathbf{x}_{i \in [1..m]}$ is n-dimensional point



Matrix Transpose

- Transpose of a $n \times m$ matrix $\mathbf{X}$ is a $m \times n$ matrix $\mathbf{X}^\mathsf{T}$: $x_{ij}^\mathsf{T} = x_{ji}$

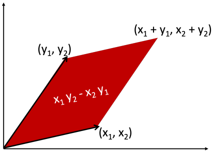
$$\begin{bmatrix} 1 & 2 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} ; \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} ; \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} ; \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 5 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

- If $\mathbf{X}$ is symmetric $x_{ij} = x_{ji} \Leftrightarrow \mathbf{X} = \mathbf{X}^\mathsf{T}$
- Let $\mathbf{X}$ and $\mathbf{Y}$ be matrices and c be a scalar, some properties:
  $\left(\mathbf{X}^\mathsf{T}\right)^\mathsf{T} = \mathbf{X}$; $(\mathbf{X} + \mathbf{Y})^\mathsf{T} = \mathbf{X}^\mathsf{T} + \mathbf{Y}^\mathsf{T}$; $(\mathbf{X}\mathbf{Y})^\mathsf{T} = \mathbf{Y}^\mathsf{T}\mathbf{X}^\mathsf{T}$; $(c\mathbf{X})^\mathsf{T} = c\mathbf{X}^\mathsf{T}$

Matrix Determinant

☞ Let a 2×2 square matrix $\begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$, its determinant is a scalar value which characterises some properties of what $(\mathbf{x}, \mathbf{y})$ represents.

$$\det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \text{ or } \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - y_1 x_2 = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \sin\theta$$



✓ The absolute value of the determinant is equal to the area of the parallelogram defined by $\mathbf{x}, \mathbf{y}$

✓ The determinant is $= 0$ if and only if $\mathbf{x}, \mathbf{y}$ are co-linear (line-parallelogram)

☞ In the case of 3×3 matrix $(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The absolute value of the determinant is the volume of the parallelogram defined by $\mathbf{x}, \mathbf{y}, \mathbf{z} = 0$ if they are on the same plan (plat parallelogram)

$$\det \begin{bmatrix} \mathbf{x}, \mathbf{y}, \mathbf{z} \end{bmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

☞ Generalisation to vectorial space of dimension n on $\mathbb{R}$ where the vectors define a parallelotope.

Matrix Determinant

- Many uses of matrix determinant for ex.:
  - Coefficients in a system of linear equations $\mathbf{Y} = \mathbf{AX}$, can be used to solve these equations (Cramer's rule), other methods computationally much more efficient.
  - Characteristic polynomial of $\mathbf{X}$, whose roots are the eigenvalues.
- Some properties:

$$\left|\mathbf{I}\right| = 1 \text{ where } \mathbf{I} \text{ identity matrix (1 on the diagonal and 0 elsewhere);}$$

$$\text{If } \left|\mathbf{X}\right| = 0 \ \mathbf{X} \text{ is a singular matrix;}$$

$$\left|\mathbf{XY}\right| = \left|\mathbf{X}\right| \left|\mathbf{Y}\right| \ ; \ \left|\mathbf{X}^{\mathrm{T}}\right| = \left|\mathbf{X}\right| \ ; \ \left|\mathbf{A^n}\right| = \left|\mathbf{A}\right|^{n}$$

Invertible Matrix

- A square matrix $\mathbf{X}$ is invertible (non-singular or non-degenerate), its inverse denoted $\mathbf{X}^{-1}$, if $\exists \mathbf{Y}$ such that $\mathbf{XY} = \mathbf{YX} = \mathbf{I}$
  $\Leftrightarrow$ Its vectors are linearly independent.
  $\Leftrightarrow$ $|\mathbf{X}| \neq 0$
  $\Leftrightarrow$ $\mathbf{X}^{\mathsf{T}}$ is invertible
  $\Leftrightarrow$ 0 is not an eigenvalue (see next)
- Other properties:

$$\left(\mathbf{X}^{-1}\right)^{-1} = \mathbf{X} \; ; \; \left(\mathbf{X}^{\mathsf{T}}\right)^{-1} = \left(\mathbf{X}^{-1}\right)^{\mathsf{T}} \; ; \; (\mathbf{XY})^{-1} = \mathbf{Y}^{-1}\mathbf{X}^{-1} \; ; \; (c\mathbf{X})^{-1} = \frac{1}{c}\mathbf{X}^{-1} \text{ with } c \neq 0 \; ; \; \left|\mathbf{X}^{-1}\right| = \frac{1}{|\mathbf{X}|} \; ; \; \left|\mathbf{X}^{\mathsf{T}}\right| = |\mathbf{X}|$$

- Analytic inversion of 3×3 matrix using the comatrix transpose

$$\mathbf{X}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{|\mathbf{X}|}\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{\mathsf{T}} = \frac{1}{|\mathbf{X}|}\begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}$$

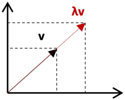
$A = (ei - fh)$ $\quad B = -(di - fg)$ $\quad C = (dh - eg)$ $\quad D = -(bi - ch)$ $\quad E = (ai - cg)$

$F = -(ah - bg)$ $\quad G = (bf - ce)$ $\quad H = -(af - cd)$ $\quad I = (ae - bd)$

- Different methods of matrix inversion: Gaussian elimination, Newton's method, Cayley–Hamilton method Cholesky decomposition and also Eigen decomposition

Eigenvectors and eigenvalues

- An eigenvector (or characteristic vector) of a linear transformation T is a nonzero vector that changes at most by a scalar factor λ (eigenvalue): $T(\mathbf{v}) = \lambda\mathbf{v}$.



- There is a direct correspondence between $n \times n$ square matrices and linear transformations of an n-dimensional vector space into itself, given any vector space basis.
  The T representation of $\mathbf{A}$ $n \times n$ matrix: $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.
- Finding all eigenvalues: Solving a polynomial function of λ called the characteristic polynomial of $\mathbf{A}$: $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$ has a nonzero solution $\mathbf{v}$ iff $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$
- Example: $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2 = (\lambda - 1)(\lambda - 3) = 0$

$$v_1 + v_2 = 0 \text{ if } \lambda_1 = 1 \text{ and } -v_1 + v_2 = 0 \text{ if } \lambda_2 = 3$$

  The eigenvectors define *eigenSpace*(λ_1) and *eigenSpace*(λ_2) $(t \in \mathbb{R}^*)$: $\mathbf{v}_{\lambda_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t, \quad \mathbf{v}_{\lambda_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$
- An infinity of collinear eigenvectors (cross product null) for λ_1 and also for λ_2.
  Digression: Strictly speaking, we talk about the eigenvector associated with one given eigenvalue

  Note that not all matrices have eigenvalues.

Eigenvectors and eigenvalues

- Algebraic multiplicity t_i of eigenvalue λ_i

$$(\lambda-\lambda_1)^{t_1}(\lambda-\lambda_2)^{t_2}\ldots(\lambda-\lambda_k)^{t_k}=0$$

  $\sum_i t_i = n$ where $t_i \in \mathbb{N}^*$ satisfying the algebraic multiplicity of λ_i
- **A** can have at most n distinct eigenvalues (complex or real).
- If eigenvalues of **A** are distinct values, then the corresponding eigenvectors are linearly independent (non collinear).

Eigenvectors and eigenvalues

☞ Example: $\begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$(\lambda-1)^2(\lambda-2)=0$ has two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -2$ with algebraic multiplicities 2 and 1, resp. and $\mathbf{v}_{\lambda_1}$ is defined by $-2v_1 + v_2 = 0$ and $v_1 + v_3 = 0$

the *eigenSpace*(λ_1) is $\mathbf{v}_{\lambda_1} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$ $\forall t \in \mathbb{R}^*$. This set has dimension 1.

☞ Example: $\begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}$ $(\lambda-1)^2(\lambda+2)=0$

$\begin{bmatrix} 2t \\ -t \\ u \end{bmatrix}$ $\forall (t,u) \in \mathbb{R}^*$ *eigenSpace*($\lambda = 1$) dimension 2. $\begin{bmatrix} -t \\ t \\ t \end{bmatrix}$ $\forall t \in \mathbb{R}^*$ *eigenSpace*($\lambda = -2$) dimension 1.

☞ The dimension of *eigenSpace*(λ_i) is referred to as the geometric multiplicity of λ_i. The geometric multiplicity of an eigenvalue is at most its algebraic multiplicity.

Eigenvectors and eigenvalues

What about:

- ☞ $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
- ☞ $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$
- ☞ $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$
- ☞ $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Eigen Decomposition

- Another representation of eigenvalues and eigenvectors

$$\mathbf{AX} = \mathbf{X}\boldsymbol{\Lambda}$$

where $\mathbf{X} = \left[\mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_n\right]$ (each $\mathbf{x}_i$ is an eigenvector) and $\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 \\ \boldsymbol{\Lambda}_2 \\ \ldots \\ \boldsymbol{\Lambda}_n \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 0 & \ldots & 0 \\ 0 & \lambda_{22} & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_{nn} \end{bmatrix}$ a diagonal matrix (each λ_{ii} is an eigenvalue).

- A matrix **A** is diagonalizable if there exist *n* linearly independent eigenvectors, i.e., if the matrix **X** is invertible:

$$\mathbf{X}^{-1}\mathbf{AX} = \boldsymbol{\Lambda}$$

- Leading to the eigen-decomposition of the matrix

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$$

Orthogonal Matrix

- A real matrix **U** is orthogonal
  $\mathbf{U}^T\mathbf{U} = \mathbf{U}^T\mathbf{U} = \mathbf{I}$
  $\Leftrightarrow$ $\mathbf{U}^T$ is orthogonal
  $\Leftrightarrow$ $\mathbf{U}^T = \mathbf{U}^{-1}$
  $\Leftrightarrow$ $\left|\mathbf{U}\right| = +1$ or -1
  $\Leftrightarrow$ **U** 's eigenvectors are orthogonal, the pairwise dot product is 0), with a norm = 1
  $\Leftrightarrow$ **U** is diagonalizable ...
- Example

$$\mathbf{I} \text{ (is orthogonal)}, \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \text{(Permutation of coordinates)},$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{(rotation)}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \text{(reflection)}$$

- For complex square matrix we talk about unitary matrix (and conjugate transpose)

Positive (semi-)definite matrices

- **A** is said to be positive semi-definite when it can be obtained as the product of a matrix by its transpose: $\mathbf{A} = \mathbf{X}\mathbf{X}^{\mathsf{T}}$.
- A positive semi-definite matrix is always symmetric $\mathbf{A}^{\mathsf{T}} = (\mathbf{X}\mathbf{X}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$. A symmetric matrix **A** is said positive semi-definite if all its eigenvalues are non negative.
- A positive semi-definite matrix implies:
  $\Rightarrow$ $0 \leq \lambda_1 \leq \lambda_2 .. \leq \lambda_n$, and its eigenvectors are pairwise orthogonal when their eigenvalues are different.
  $\Rightarrow$The eigenvectors are also composed of real values.
  $\Rightarrow$The multiplicity of an eigenvalue λ is the dimension of the space of its eigenvectors *eigenspace*(λ).
- Because eigenvectors are orthogonal, it is possible to store all the eigenvectors in an orthogonal matrix. This implies $\mathbf{U}^{-1} = \mathbf{U}^{\mathsf{T}}$ where $\mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}$ are the normalized eigenvectors ; if they are not normalized then it is a diagonal matrix.
- Therefore, the eigen-decomposition of a positive semi-definite matrix **A** could be: $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathsf{T}}$
- Example: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix}$
- As a consequence, the eigen-decomposition of a positive semi-definite matrix is often referred to as its diagonalization. $\mathbf{\Lambda} = \mathbf{U}\mathbf{A}\mathbf{U}^{\mathsf{T}}$

Positive (semi-)definite matrices - Another definition

☞ A matrix A is said to be positive semi-definite if we observe the following relationship for any non-zero vector $\mathbf{x}$: $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0 \; \forall \mathbf{x}$.
When > 0 the matrix is positive definite.
When ≤ 0 the matrix is negative semi-definite.

☞ **A matrix rank** is the dimension of the vector space generated (or spanned) by its columns. This corresponds to the maximal number of linearly independent columns of A. A matrix whose rank is equal to its size is called a full rank matrix. Only full rank matrices have an inverse.

☞ The sum of the eigenvalues of a matrix is the sum of the elements of its main diagonal

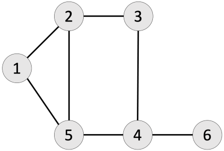
☞ The product of the eigenvalues is equal to the determinant of the matrix.

Laplacian Matrix

☞ Laplacian Matrix for undirected graph :

$$\mathbf{L}_{ij} = \begin{cases} -1, (v_i, v_j) \in E \\ 0, (v_i, v_j) \notin E \\ d_i, i = j \end{cases}$$

or equivalently $\mathbf{L} = \mathbf{D} - \mathbf{A}$ where $\mathbf{D}$ is the degree matrix of $\mathbf{A}$ where where $\mathbf{D}_{ii} = \sum_j A_{ij}$



$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Linear Algebra Preliminaries

Some Exercises

Some Exercises

- what could you say about these matrices: $\begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & \frac{3}{2} \\ \frac{2}{3} & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Show that $\mathbf{A}^n = \mathbf{X}\mathbf{\Lambda}^n\mathbf{X}^{-1}$
- Find the eigenvalues and unit eigenvectors of $\mathbf{A}^\mathsf{T}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\mathsf{T}$ with $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ Fibonacci matrix
- Without multiplying $\mathbf{S} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ find the determinant, the eigenvalues and eigenvectors, why **S** is positive definite
- For what numbers c and d such that **S** and **T** are positive definite $\mathbf{S} = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$ $\mathbf{T} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{pmatrix}$
- Show if $\lambda_1, \lambda_2, ...\lambda_n$ are the eigenvalues of a matrix **A**, then $\mathbf{A}^m$ has a Eigenvalues $\lambda_1^m, \lambda_2^m, ...\lambda_n^m$
- What is the determinant of any orthogonal matrix ?
- For an undirected graph both the adjacency matrix and the Laplacian matrix are symmetric. Show that Laplacian is positive semi-definite matrix. Show that Laplacian has 0 is an eigenvalue (the smallest one).

First mini-Project (notebook)

☞ Provide a python notebook to compute the main properties of a square matrix. Also answer the questions (formalisation+program)