Preferences as binary relations

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Chapter 1

Definition

- A subset of ordered pairs of a set X is called a binary relation.
- Formally, R is a binary relation on X if $R \subseteq X \times X$.
- Usually we write $x \; R \; y$ if $(x, y) \in R$

Relation as Directed Graphs

Let be R a relation on a set A. A direct graph representation of relation R is $G = (A, E)$ where A is the set of nodes and E the set of direct edges where

$$(a, b) \in R \Longleftrightarrow (a, b) \in E (\text{an arrow from } a \text{ to } b)$$

Relation as Matrices

Let be R a relation on a set A. The matrix representation of relation R is $M_R = [m_{ab}]_{(a,b) \in R}$ where

$$\begin{cases} m_{ab} = 1 & \text{if } (a, b) \in R \\ m_{ab} = 0 & \text{if } (a, b) \notin R \end{cases}$$

Definitions

A binary relation R on X is said to be:

- Reflexive if for every $x \in X$, $x \; R \; x$;
- Irreflexive if for every $x \in X$, $\text{not}(x \; R \; x)$
- Complete if for every $x, y \in X$, $x \; R \; y$ or $y \; R \; x$ (possibly both);
- Weakly complete if for every $x, y \in X$, $x \neq y \Longrightarrow [x \; R \; y$ or $y \; R \; x]$ (possibly both);
- Symmetric if for every $x, y \in X$, $[x \; R \; y \Longrightarrow y \; R \; x]$;
- Asymmetric if for every $x, y \in X$, $[x \; R \; y \Longrightarrow \text{not}(y \; R \; x)]$;
- Antisymmetric if for every $x, y \in X$, $[x \; R \; y \text{ and } y \; R \; x \Longrightarrow x = y]$;
- Transitive if for every $x, y, z \in X$, $[x \; R \; y \text{ and } y \; R \; z \Longrightarrow x \; R \; z]$;
- Negatively transitive if for every $x, y, z \in X$,
  $[\text{not}(x \; R \; y) \text{ and } \text{not}(y \; R \; z) \Longrightarrow \text{not}(x \; R \; z)]$;
- Semi-transitive if for every $x, y, z, t \in X$,
  $[(x \; R \; y) \text{ and } (y \; R \; z)] \Longrightarrow [(x \; R \; t) \text{ or } (t \; R \; z)]$

Properties

For all $x, y \in X$,

- There is a path from x to y if there exist $x_1, x_2, \ldots, x_n$ such that $x = x_1 \ R \ x_2 \ R \ \ldots \ R \ x_{n-1} \ R \ x_n = y$;
- There is a cycle from x to x if there is a path from x to x.

Relations P and I from R

For a binary relation R on X, we define a symmetric part I and an asymmetric part P as follows: for all $x, y \in X$

- $x\ I\ y$ if $[x\ R\ y$ and $y\ R\ x]$
- $x\ P\ y$ if $[x\ R\ y$ and $\mathrm{not}(y\ R\ x)]$

Concatenation of two binary relations

Let be $\mathcal{R}$ and $\mathcal{R}'$ two binary relations on X. For all $x, y \in X$

$$x\ \mathcal{R} \bullet \mathcal{R}'\ y \iff \text{there exists } z \in X \text{ s.t. } [x\ \mathcal{R}\ z \text{ and } z\ \mathcal{R}'\ y]$$

Proposition

Let be $\mathcal{R}$ a binary relation on X.

1. $\mathcal{R}$ *transitive* $\Longrightarrow \mathcal{R} \bullet \mathcal{R} \subseteq \mathcal{R}$ (*i.e.* $\mathcal{R}^2 \subseteq \mathcal{R}$)
2. $\mathcal{R}$ *asymmetric* $\Longrightarrow \mathcal{R}$ *irreflexive*
3. $\mathcal{R}$ *complete* $\iff \mathcal{R}$ *reflexive and weakly complete*
4. $\mathcal{R}$ *asymmetric and negative transitive* $\Longrightarrow \mathcal{R}$ *transitive*
5. $\mathcal{R}$ *complete and transitive* $\Longrightarrow \mathcal{R}$ *negative transitive*

Definition

- A binary relation R on X that is reflexive, symmetric and transitive is called an equivalence relation.
- A binary relation R on X is a preorder if R is reflexive and transitive.
- A binary relation R on X is a weak order or a complete preorder if R is complete and transitive.
- A binary relation R on X is a total order or a linear order if R is complete, antisymmetric and transitive.

Exercise 1

Let be $\mathcal{B}$ a binary relation on a set $X = \{a, b, c, d, e, f\}$ defined by:

$$a\ \mathcal{B}\ a,\ a\ \mathcal{B}\ b,\ a\ \mathcal{B}\ c,\ a\ \mathcal{B}\ d,\ a\ \mathcal{B}\ e,\ a\ \mathcal{B}\ f$$

$$b\ \mathcal{B}\ b,\ b\ \mathcal{B}\ c,\ b\ \mathcal{B}\ d,\ b\ \mathcal{B}\ e,\ b\ \mathcal{B}\ f$$

$$c\ \mathcal{B}\ c,\ c\ \mathcal{B}\ d,\ c\ \mathcal{B}\ e,\ c\ \mathcal{B}\ f$$

$$d\ \mathcal{B}\ b,\ d\ \mathcal{B}\ c,\ d\ \mathcal{B}\ d,\ d\ \mathcal{B}\ e$$

$$e\ \mathcal{B}\ d,\ e\ \mathcal{B}\ e,\ e\ \mathcal{B}\ f$$

$$f\ \mathcal{B}\ e,\ f\ \mathcal{B}\ f$$

1. Give a matrix and a graphical representation of $\mathcal{B}$
2. Is $\mathcal{B}$ reflexive? symmetric? asymmetric? transitive? negative transitive? semi-transitive?

Exercise 2

Let be $\mathcal{B}$ and $\mathcal{B}'$ two equivalence relations on a set X:

1. Prove that $\mathcal{B} \cap \mathcal{B}'$ is an equivalence relation.

$$x \ (\mathcal{B} \cap \mathcal{B}') \ y \Longleftrightarrow [x \ \mathcal{B} \ y \text{ and } x \ \mathcal{B}' \ y], \qquad \text{For all } x, y \in X$$

2. Is $\mathcal{B} \cup \mathcal{B}'$ an equivalence relation ?

$$x \ (\mathcal{B} \cup \mathcal{B}') \ y \Longleftrightarrow [x \ \mathcal{B} \ y \text{ or } x \ \mathcal{B}' \ y], \qquad \text{For all } x, y \in X$$

3. Could we have the same conclusions if $\mathcal{B}$ and $\mathcal{B}'$ are two complete preorders on a set X ?

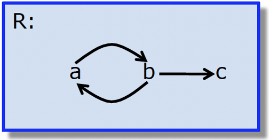
Reflexive closure

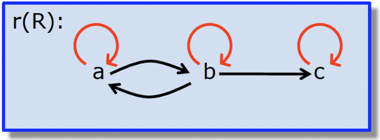
Given a relation R, we want to add to it just enough "edges" to make the resulting relation satisfy the reflexive property.

Reflexive Closure of R is $r(R) = R \cup Eq$, where Eq is the reflexive relation.

Example:

r(R) = R ∪ Eq = {(a,b),(b,a),(b,c),(a,a),(b,b),(c,c)}





Symmetric closure

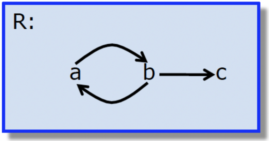
Given a relation R, we want to add to it just enough "edges" to make the resulting relation satisfy the symmetric property.

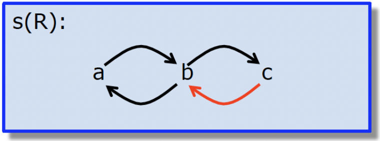
Symmetric Closure of R is $s(R) = R \cup R^c$ where R^c is the converse relation.

$$x\ R^c\ y \Longleftrightarrow y\ R\ x$$

Example:

$s(R) = R \cup R^c = \{(a,b),(b,a),(b,c),(c,b)\}$





Transitive closure

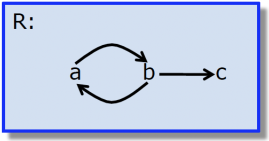
Given a relation R, we want to add to it just enough "edges" to make the resulting relation satisfy the transitivity property.

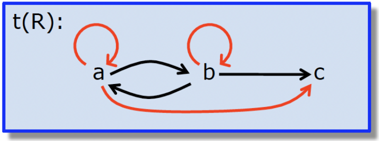
$$\text{Transitive Closure of } R \text{ is } t(R) = R \cup R^2 \cup R^3 \ldots .$$

Note: If the number of elements is n (finite) then $t(R) = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$

Example:

t(R) = R ∪ R^2 ∪ R^3 = {(a,b),(b,a),(b,c),(a,a),(b,b),(a,c)}





If there is a path from x to y, then add an edge directly from x to y.

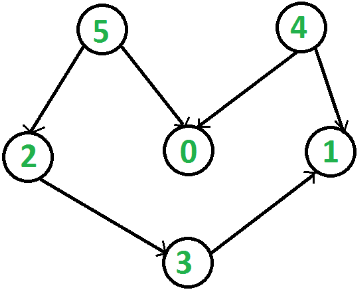
How to extend a partial pre-order to a complete preorder?

By applying a topological sorting when there is no cycle in the preferences.

Topological Sorting

- Topological sorting for Directed Acyclic Graph (DAG) is a linear ordering of vertices such that for every directed edge uv, vertex u comes before v in the ordering.
- Topological Sorting for a graph is not possible if the graph is not a DAG.

Topological Sorting



- A topological sorting of the previous graph is 542310.
- There can be more than one topological sorting for a graph. For example, another topological sorting of the previous graph is 452310. The first vertex in topological sorting is always a vertex with in-degree as 0 (a vertex with no incoming edges).

Idea of the numerical representation

We try to construct a binary relation $\succsim$ on X such that there exists a numerical function $f : X \longrightarrow \mathbb{R}$ satisfying the property:

$$x \succsim y \iff f(x) \geq f(y)$$

In general, $\succsim$ is assumed to be a preorder.

- $x \succsim y$ means x is at least as good as y
- $\succ$ is the asymmetric part of $\succsim$
- $\sim$ is the symmetric part of $\succsim$

Theorem (Cantor, 1895)

Let be X a countable set (finite or infinite countable). Let be $\succsim$ a binary relation on X.

$$\left[\exists f : X \longrightarrow \mathbb{R} \; s.t. \; \forall x, y \in X, \; x \succsim y \Longleftrightarrow f(x) \geq f(y)\right]$$

$$\Updownarrow$$

$$\succsim \textit{ is a complete preorder on } X$$

Proposition

Let be $\succsim$ a preorder (complete) on X representable by a function $f : X \longrightarrow \mathbb{R}$, i.e., $\forall x, y \in X, x \succsim y \iff f(x) \geq f(y)$

The following two properties are equivalent:

1. $v : X \longrightarrow \mathbb{R}$ *is a function representing* $\succsim$
2. *There exists a strictly increasing function* $\varphi : f(X) \longrightarrow \mathbb{R}$ *such that* $v = \varphi \circ f$

Remark

f is an ordinal scale (See Chapter 2).

Reference

- D. Bouyssou and Ph. Vincke, Binary Relations and Preference Modeling, in "Concept and Methods for decision aiding", Ch. 2, 2009
- S. MORETTI, M. OZTÜRK and A. TSOUKIAS. Preference Modelling. In J. Figueira, S. Greco, and M. Ehrgott, editors, Multiple Criteria Decision Analysis: State of the Art Surveys, (2nd edition 2016). Available at http://www.lamsade.dauphine.fr/~ozturk/publicationsbis.html