Preferences Aggregation: the Outranking approach

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Chapter 4

Outline

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Qualitative approach: "Compare then aggregate"

$$x \succsim y \iff \left|\{i \in N : x_i \succsim_i y_i\}\right| \rhd \left|\{i \in N : y_i \succsim_i x_i\}\right|$$

Qualitative approach: Principles

Compare alternatives criterion by criterion and then aggregate these comparisons

$$x \succsim y \iff U(c(x_1, y_1), c(x_2, y_2), \ldots, c(x_n, y_n)) \geq 0$$

Outranking approach: Objectives

The objective of outranking methods is to build a relation on the alternatives, called the outranking relation, and to exploit it in order to solve one of the MCDA problems

Remark

- One of the particularities of outranking methods is that the relation built on the set X allows three types of comparisons of alternatives, namely *preference, indifference and incomparability*.
- They allow to represent hesitations of the DM which may result from phenomena like *uncertainty, conflicts or contradictions.*

Similarities with Social Choice Theory

Compare alternatives criterion by criterion and then aggregate these comparisons

$$x \succsim y \iff U(c(x_1, y_1), c(x_2, y_2), \ldots, c(x_n, y_n)) \geq 0$$

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Framework

- A is a set of alternatives evaluated on n real-valued criteria $g_i : A \to \mathbb{R}$, $i \in N = \{1, \ldots, n\}$.
- w_i is nonnegative weight assigned to each criterion i (w.l.o.g. we suppose $\sum_{i=1}^{n} w_i = 1$)
- $p_i \geq 0$ is a nonnegative preference threshold assigned to each criterion i
- If the value $g_i(a) - g_i(b)$ is positive but less than p_i, it is supposed that this difference is not significant, given the way g_i has been built. Hence, on this criterion, the two alternatives should be considered indifferent.

Partial Concordance index

$c_i : A \times A \rightarrow [0, 1]$ is defined on each criterion $i \in N$ by

$$c_i(a, b) = \begin{cases} 1 \text{ if } g_i(b) - g_i(a) \leq p_i \\ 0 \text{ if } g_i(b) - g_i(a) > p_i \end{cases}$$

Concordance index

$c : A \times A \rightarrow \mathbb{R}$ is an aggregation of partial concordance indices c_i

$$c(a, b) = \sum_{i=1}^{n} w_i c_i(a, b)$$

Outranking relation

An outranking relation $\mathcal{S}_\lambda$ on A is a binary relation defined by:

$$a \, \mathcal{S}_\lambda \, b \text{ iff } c(a,b) \geq \lambda$$

where $\lambda \in [0,1]$ is a cutting level (usually called a threshold and taken above $\frac{1}{2}$).

Interpretation

- Concordance: Enough reasons to say that a is at least as good as b

  An alternative $a \in A$ outranks an alternative $b \in A$ if it can be considered at "least as good" as the latter (i.e., a is not worse than b), given the values (performances) of a and b at the n criteria.
- However, if there are some criteria where a is worse than b, then a may outrank b or not, depending on the relative importance of those criteria and the differences in the evaluations (small differences might be ignored).

Outranking relation

From $\mathcal{S}_\lambda$ the following three binary relations can be derived:

- "Strictly better than" relation:

$$a \, \mathcal{P}_\lambda \, b \text{ iff } [a \, \mathcal{S}_\lambda \, b \text{ and } \mathrm{not}(b \, \mathcal{S}_\lambda \, a)]$$

- "Indifferent to" relation:

$$a \, \mathcal{I}_\lambda \, b \text{ iff } [a \, \mathcal{S}_\lambda \, b \text{ and } (b \, \mathcal{S}_\lambda \, a)]$$

- "Incomparable to" relation:

$$a \, \mathcal{U}_\lambda \, b \text{ iff } [\mathrm{not}(a \, \mathcal{S}_\lambda \, b) \text{ and } \mathrm{not}(b \, \mathcal{S}_\lambda \, a)]$$

Remark

- An outranking relation is not always transitive
- An outranking relation could use qualitative data
- One can introduce also the Discordance threshold: No important reason to say that a is worse than b

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Preliminaries

- Let us consider r ordered categories $C^1, C^2, \ldots, C^r$
- C^1 is the worst one and C^r is the best one
- The category C^k is modeled by using limiting profiles
- The lower limiting profile of C^k is π^k
- The upper limiting profile of C^k is π^{k+1}
- π^{k+1} strictly dominates π^k

$$\text{For all } a \in A, \quad a \; \mathcal{P}_\lambda \; \pi^1 \text{ and } \pi^{r+1} \; \mathcal{P}_\lambda \; a.$$

ELECTRE TRI is a MultiCriteria Decision Aid method using limiting profiles.

Definition (Pessimistic version of ELECTRE TRI)

For each $a \in A$,

- Decrease k from $r+1$ until the first value k such that $a\ \mathcal{S}_\lambda\ \pi^k$
- Assign alternative a to C^k.

The Pessimistic version of ELECTRE TRI assigns an alternative a to the unique category C^k such that a is at least as good as to the lower limiting profile of this category and is not at least as good as its upper limiting profile (the relation "at least as good as" being $\mathcal{S}_\lambda$).

Definition (Optimistic version of ELECTRE TRI)

For each $a \in A$,

- Increase k from 1 until the first value k such that $\pi^k \; \mathcal{P}_\lambda \; a$.
- Assign alternative a to C^{k-1}.

The Optimistic version of ELECTRE TRI ETRI-B-pd assigns an alternative a to the category C^k such that the upper limiting profile of this category is better than a and the lower limiting profile of this category is not better than a (the relation "better than" being $\mathcal{P}_\lambda$).

Remark

It is proved that, if $a \in A$ is assigned to the category C^k by the pessimistic version and to the category C^l by the Optimistic version, then $k \leq l$.

Majority Rule sorting procedure (MR-Sort)

- MR-Sort is a simplified version of the ELECTRE TRI sorting model directly inspired by the work of Bouyssou and Marchant (2007) who provide an axiomatic characterization of non-compensatory sorting methods.
- The general principle of MR-Sort (without veto) is to assign alternatives by comparing their performances to those of profiles delimiting the categories.
- An alternative is assigned to a category "above" a profile if and only if it is at least as good as the profile on a (weighted) majority of criteria.

MR-Sort's Principles

The condition for an alternative $a \in A$ to be assigned to a category C^k is expressed as follows:

$$\sum_{i: g_i(a) \geq g_i(\pi^k)} w_i \geq \lambda \quad \text{and} \quad \sum_{i: g_i(a) \geq g_i(\pi^{k+1})} w_i < \lambda$$

The MR-Sort assignment rule involves $r \times n + 1$ parameters, i.e., n weights, $(r-1) \times n$ profiles evaluations and 1 majority threshold.

Learning parameters of MR-Sort

- The problem of learning the parameters of a MR-Sort model on the basis of assignment examples can be formulated as a mixed integer linear program (MILP) but only instances of modest size can be solved in reasonable computing times.
  - A problem involving 1000 alternatives, 10 criteria and 5 categories requires 21000 binary variables.
  - For a similar program, it is mentioned that problems with less than 400 binary variables can be solved within 90 minutes.
- Learning only the weights and the majority threshold of an MR-Sort model on the basis of assignment examples can be done using an ordinary linear program (without binary or integer variables).
- On the contrary, learning profiles evaluations is not possible by linear programming without binary variables.

Exercise

	Crit 1	Crit 2	Crit 3	Crit 4
a	5	7	2	5
b	8	4	6	2
c	4	8	7	5
d	6	4	5	7
e	2	6	2	8
f	3	5	6	4
	$w_1 = 0.4$	$w_2 = 0.3$	$w_3 = 0.1$	$w_4 = 0.2$

By using the MR-Sort approach, assign these six alternatives to one of the three categories $C^1 \equiv \mathrm{Bad}$ (the worst category), $C^2 \equiv \mathrm{Medium}$ and $C^3 \equiv \mathrm{Good}$ (the best category), where their limiting profiles are the following:

	Crit 1	Crit 2	Crit 3	Crit 4
π^4	10	10	10	10
π^3	6	6	5	5
π^2	5	5	3	4
π^1	1	1	1	1

We set $\lambda = 0.6$.