CentraleSupélec

MASTER BDMA
DECISION MODELING

EXAM DECEMBER 2022

2 hours 15 min

All the documents are allowed. The access to the Internet is prohibited (except going to my web page to download the subject of the exam to read on your computer and maybe some other documents)

Exercise 1: Voting rules

1. Let us consider the following preferences of $n = 17$ voters, given on a set $X = \{a, b, c, d\}$ of $m = 4$ candidates:

$$\begin{array}{ll} \text{5 voters :} & c \succ b \succ a \succ d \\ \text{4 voters :} & b \succ a \succ c \succ d \\ \text{4 voters :} & d \succ b \succ c \succ a \\ \text{2 voters :} & a \succ c \succ b \succ d \\ \text{2 voters :} & d \succ b \succ a \succ c \end{array}$$

Which candidate is elected by using the following voting rules or principles:

(a) plurality voting ?

(b) plurality runoff voting (plurality with two rounds) ?

(c) Condorcet principle ?

(d) Borda principle ?

(e) Copeland principle ?

The Copeland principle associates to each candidate x a score calculated as follows:

$$S_{Cop}(x) = \sum_{\substack{y \in X \\ y \neq x}} Cop(x, y)$$

where X is the set of candidates and $Cop(x, y) = \begin{cases} +1 & \text{if a majority of voters prefers } x \text{ to } y \quad (\geq 50\%) \\ -1 & \text{if a majority of voters prefers } y \text{ to } x \quad (\geq 50\%) \\ 0 & \text{if the both two previous situations arise simultaneously} \end{cases}$

The candidate with the highest Copeland score is elected.

(f) Kramer-Simpson principle?

The Kramer-Simpson principle associates to each candidate x a score

$$KS(x) = \min_{\substack{y \in X \\ y \neq x}} n(x, y)$$

where X is the set of candidates and $n(x, y)$ being the number of voters who prefer x to y.

The candidate with the highest Kramer-Simpson score is elected.

2. In general, does the Copeland principle elect the Condorcet winner, if this latter exists? Justify your answer.

3. In general, does the Kramer-Simpson principle elect the Condorcet winner, if this latter exists? Justify your answer.

4. Let us consider the following preferences of 4 voters given on a set $X = \{a, b, c, d, e\}$ of 5 candidates:

$$\begin{array}{ll} \text{1 voter :} & a \succ b \succ c \succ d \succ e \\ \text{1 voter :} & b \succ c \succ d \succ a \succ e \\ \text{1 voter :} & c \succ d \succ a \succ b \succ e \\ \text{1 voter :} & d \succ a \succ b \succ c \succ e \end{array}$$

   (a) The Borda principle is chosen to elect the winner of this election.
      i. Who is elected?
      ii. By adding 3 new voters to the previous 4 one (and we have now a total of 7 voters), is it possible to provide the preferences of these new voters such that the candidate e is elected (the winner is not necessary unique)? Justify your answer.
      iii. By adding 4 new voters to the previous one (and we have now a total of 8 voters), is it possible to provide the preferences of these new voters such that the candidate e is elected (the winner is not necessary unique)? Justify your answer.

   (b) Is it possible to add new voters to the previous 4 one (by keeping the same 4 types of preferences) such that a is the unique Condorcet winner ? Justify your answer.

Exercise 2: Ranking or sorting?

We consider the following six students a, b, c, d, e, f evaluated on three subjects $N = \{1; 2; 3\}$. The scores on each criterion are given in the interval $[0, 100]$. The strict preference, given by the Decision Maker (DM), is denoted by $\succ$, while his indifference preference is denoted by $\sim$. The performance matrix of the students evaluations is the following:

	1: Mathematics	2: Statistics	3: Language
a	85	90	75
b	80	70	70
c	80	65	70
d	85	90	60
e	50	65	75
f	50	70	60

The two parts below are independent and can be solved separately.

Part 1: Ranking

1. We assume that the Decision-Maker (DM) provides the following preference information: $a \succ b$; $c \succ d$ and $e \succ f$. Are these preferences representable by an additive model ? Justify your answer.

2. We assume that the Decision-Maker (DM) provides the following preference information: $a \succ b$; $c \succ d$ and $e \sim f$. Are these preferences representable by an additive model ? Justify your answer.

3. Now, we assume that the marginal utility function u_i associated to the criterion i, is exactly the obtained marks, i.e., $u_i(x_i) = x_i$, where x_i is a value on the criterion i.

   (a) Determine the ranking $\succsim_1$ of the 6 students by using a weighted sum, where the weights (ECTS) associated to the 3 subjects is the vector $(w_1 = 6; w_2 = 3; w_3 = 2)$, w_i being the weight associated to the criterion i.

   (b) Is it possible to represent the following preferences by a weighted sum model ?

   - $d \succ c$ and $f \succ e$
   - Language is strictly more important than Mathematics
   - There is no weight (of a criterion) equals to zero.

   Justify your answer.

Part 2: Sorting

We aim at developing a multi-criteria method assigning the 8 students to some ordered categories. The envisaged method is based on the elaboration of an outranking relation, as it is done, for instance, in MR-Sort method. However, unlike MR-Sort where each alternative is compared to reference profiles representing the boundaries of the categories, the outranking relation here is defined on the given set of the alternatives.

- In the sequel, we denote by $A = \{a, b, c, \ldots\}$ the set of alternatives to assign and N the set of n criteria.

- The outranking relation $\succsim$ means "at least as good as", with $\succ$ its asymmetric part and $\sim$ is symmetric part. The binary relation $\succsim$ is defined by

$$a \succsim b \Leftrightarrow \sum_{i|g_i(a)\geq g_i(b)} w_i \geq \lambda$$

  where $w_i \geq 0$ represents the weight associated to the criterion i ($\sum_{i\in N} w_i = 1$), $g_i(a)$ represents the value of the alternative a on the criterion i and $\lambda \in [0.5; 1]$ is the majority threshold.

- Two alternatives a and b are said "incomparable" if $[\mathrm{not}(a \succsim b)$ and $\mathrm{not}(b \succsim a)]$.

- The p ordered categories we consider are denoted $C_1, C_2, \ldots, C_p$ (C_1 and C_p being respectively the worst and the best category).

  ⋆ $C(a)$ represents the category where the alternative a is assigned.

  ⋆ $C(a) \geq C(b)$ means that a is assigned to a category greater than the category where b is assigned.

  ⋆ $C(a) > C(b)$ means that a is assigned to a category strictly greater than the category where b is assigned.

Let us consider the following assignment principles:

$$\forall a, b \in A, \quad C(a) > C(b) \Rightarrow a \succsim b \tag{1}$$

$$\forall a, b \in A, \quad a \succsim b \Rightarrow C(a) \geq C(b) \tag{2}$$

To assign the 6 students, we consider the following parameters and preferences:

- We have 4 categories $C_1; C_2; C_3$ and C_4;

- The majority threshold is $\lambda = 0.7$;
- The weight vector is $(w_1 = 0.3; w_2 = 0.3; w_3 = 0.4)$
- The students a belongs to the category C_4;
- The student f belongs to the category C_1.

Determine the assignments of the other students by using these preferences and the adopted assignment principle (Equation (2)).