

Problem Set #3

MACS 30250, Dr. Evans

Due Wednesday, May. 20 at 1:30pm

1. **Stochastic i.i.d. cake eating problem (5 points).** Assume a cake eating problem similar to the one in Section 5 of the [dynamic programming introduction](#) from last term in which cake size in the current period is W , cake size next period is W' , current period consumption is the difference in the cake size $c = W - W'$, and the utility of consumption is a monotonically increasing concave function $u(c)$. Also assume that the individuals preferences fluctuate each period according to some i.i.d. shock ε . The Bellman equation can be written in the following way to incorporate the uncertainty,

$$V(W, \varepsilon) = \max_{W' \in [0, W]} e^\varepsilon u(W - W') + \beta E_{\varepsilon'} [V(W', \varepsilon')] \quad (1)$$
$$\text{where } u(W - W') \equiv \frac{(W - W')^{1-\gamma} - 1}{1 - \gamma}$$

where E is the unconditional expectations operator over all values in the support of ε . Assume that the preference shock ε can take on 5 possible values $\boldsymbol{\varepsilon}$ that have the following unconditional probability $\boldsymbol{\pi}$ each period.

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} -1.40 \\ -0.55 \\ 0.00 \\ 0.55 \\ 1.40 \end{bmatrix} \quad \boldsymbol{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.4 \\ 0.2 \\ 0.1 \end{bmatrix}$$

- (a) Assume that $\beta = 0.9$ and $\gamma = 2.2$. Use value function iteration with interpolation over the expected value of the value function next period $E_{\varepsilon'} V(W', \varepsilon')$ to solve for the equilibrium solution value function $V(W, \varepsilon)$ and policy function $W' = \psi(W, \varepsilon)$ that solve the Bellman equation (1). For your grid, use 30 equally spaced points of cake size W between $W_{min} = 0.1$ and $W_{max} = 10.0$, and use the five points in the ε dimension listed above. [HINT: First integrate out the shock next period ε' just using the five-element stochastic vector $\boldsymbol{\pi}$. Then use a one-dimensional interpolant in the expected W' dimension.]
- (b) Display a plot of your equilibrium value function as five line plots $V(W, \varepsilon_j)$, one for each value of ε_j for $j = 1, 2, \dots, 5$ with the value function $V(W, \varepsilon)$ on the y -axis and with cake size W on the x -axis. Make sure to include a legend to show which line belongs to which preference shock.
- (c) Display a plot of your equilibrium policy function as five line plots $W' = \psi(W, \varepsilon_j)$, one for each value of ε_j for $j = 1, 2, \dots, 5$ with the policy function $\psi(W, \varepsilon)$ on the y -axis and with cake size W on the x -axis. Make sure to include a legend to show which line belongs to which preference shock.

2. **Persistent AR(1) stochastic cake eating problem (5 points).** Assume a cake eating problem similar to the one in Section 6 of the [dynamic programming introduction](#) from last term in which cake size in the current period is W , cake size next period is W' , current period consumption is the difference in the cake size $c = W - W'$, and the utility of consumption is a monotonically increasing concave function $u(c)$. Also assume that the individuals preferences fluctuate each period according to some persistent AR(1) shock ε . The Bellman equation can be written in the following way to incorporate the uncertainty,

$$V(W, \varepsilon) = \max_{W' \in [0, W]} e^{\varepsilon} u(W - W') + \beta E_{\varepsilon'|\varepsilon} [V(W', \varepsilon')] \quad (2)$$

$$\text{where } u(W - W') \equiv \frac{(W - W')^{1-\gamma} - 1}{1-\gamma}$$

where E is the conditional expectations operator over all values in the support of ε . Assume that the preference shock ε can take on 5 possible values $\boldsymbol{\varepsilon}$ that have the following Markov transition probability matrix $\boldsymbol{\pi}$ each period.

$$\boldsymbol{\varepsilon} = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4 \quad \varepsilon_5]^T = [-1.40 \quad -0.55 \quad 0.00 \quad 0.55 \quad 1.40]^T$$

$$\boldsymbol{\pi} = \pi_{j,k} = Pr(\varepsilon'_k | \varepsilon_j) = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} & \pi_{1,5} \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \pi_{2,4} & \pi_{2,5} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} & \pi_{3,5} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} & \pi_{4,5} \\ \pi_{5,1} & \pi_{5,2} & \pi_{5,3} & \pi_{5,4} & \pi_{5,5} \end{bmatrix} = \begin{bmatrix} 0.40 & 0.28 & 0.18 & 0.10 & 0.04 \\ 0.20 & 0.40 & 0.20 & 0.13 & 0.07 \\ 0.10 & 0.20 & 0.40 & 0.20 & 0.10 \\ 0.07 & 0.13 & 0.20 & 0.40 & 0.20 \\ 0.04 & 0.10 & 0.18 & 0.28 & 0.40 \end{bmatrix}$$

where $\pi_{j,k}$ represents the probability of having preference shock ε'_k next period given current preference shock ε_j . So the rows of Markov transition matrix $\boldsymbol{\pi}$ represent the different values of ε in the current period, and the columns of $\boldsymbol{\pi}$ represent the different values of ε next period. The values in each row of $\boldsymbol{\pi}$ sum to one.

- Use value function iteration with interpolation over the expected value of the value function next period $E_{\varepsilon'|\varepsilon} V(W', \varepsilon')$ to solve for the equilibrium solution value function $V(W, \varepsilon)$ and policy function $W' = \psi(W, \varepsilon)$ that solve the Bellman equation (2). For your grid, use 30 equally spaced points of cake size W between $W_{min} = 0.1$ and $W_{max} = 10.0$, and use the five points in the ε dimension listed above. [HINT: First integrate out the shock next period ε' just using the 5×5 Markov transition matrix $\boldsymbol{\pi}$. In this case you will just use a different row of $\boldsymbol{\pi}$ corresponding to your current value of ε . Then use a one-dimensional interpolant in the expected W' dimension.]
- Display a plot of your equilibrium value function as five line plots $V(W, \varepsilon_j)$, one for each value of ε_j for $j = 1, 2, \dots, 5$ with the value function $V(W, \varepsilon)$ on the y -axis and with cake size W on the x -axis. Make sure to include a legend to show which line belongs to which preference shock.

- (c) Display a plot of your equilibrium policy function as five line plots $W' = \psi(W, \varepsilon_j)$, one for each value of ε_j for $j = 1, 2, \dots, 5$ with the policy function $\psi(W, \varepsilon)$ on the y -axis and with cake size W on the x -axis. Make sure to include a legend to show which line belongs to which preference shock.