# 1 Important Background

[ *Fundamental Theorem of Algebra* ] Every non-zero, single-variable, degree $n$ polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

Application in matrix and eigenvalues: Every $n \times n$ matrix characteristic polynomial $C(\lambda)$ can be factor into linear factors as $C(\lambda) = (\lambda -_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$. Therefor, every $n \times n$ matrix has exactly $n$ eigenvalues, counted with multiplicity.

# 2 Problem Statement

Every square complex matrix $A : \mathbb{C}^n \to \mathbb{C}^n$ can be reduced into Jordan Normal Form.

i.e. Every $A$ can be expressed as $A = CA_J C^{-1}$, where $A_J$ composed by Jordan Blocks, $C$ changing basis.

# 3 Proof Structure

Method used here is to simply the matrix by choosing suitable new basis. When there are eigenvalues' multiplicity $> 1$ and eigenvectors not enough to make a basis, pull root vectors of repeated eigenvalues, basis of root subspaces to form a linearly independent set as basis.

The proof is consisted by three steps.

STEP 1. Every square complex matrix can be reduced into matrix $A$ in upper triangular form.

(skip in this proof)

STEP 2. Upper triangular $A$ can be expressed as

$$A = \begin{bmatrix} A|V^{\lambda_1} & & & 0 \\ & A|V^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & A|V^{\lambda_k} \end{bmatrix}$$

, where $A|V^{\lambda_i}$ is restricted $A$ on root subspace $V^{\lambda_i}$ of each distinct eigenvalues $\lambda_i$.

$$A|V^{\lambda_i} = \begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$$

Further, $\mathbb{C}^n = V^{\lambda_1} \bigoplus V^{\lambda_2} \bigoplus ... \bigoplus V^{\lambda_k}$.

STEP 3. Every upper triangular $A|V^{\lambda_i}$ can be reduced into Jordan Block.

# 4 STEP 2: Understand Root Subspace $V^{\lambda_i}$

- Find $A|V^{\lambda_i}$.

  By definition of root vector, $v$ is a root vector if for some $m$, $(A-\lambda Id)^m v = 0$. Root subspace $V^\lambda(A) = ker(A-\lambda Id)^m$, is the space of all root vectors of $\lambda$.

  Also, $ker(A-\lambda Id) \subset ker(A-\lambda Id)^2 \subset ... \subset ker(A-\lambda Id)^m = ker(A-\lambda Id)^{m+1}$.

  Suppose $e_1, e_2$ basis of $ker(A-\lambda Id)$, then $e_1, e_2$ eigenvectors of $A$. $Ae_1 = \lambda e_1$, $Ae_2 = 0e_1 + \lambda e_2$.

  Further suppose $e_1, e_2, e_3, e_4$ basis of $ker(A-\lambda Id)^2$. Then $(A-\lambda Id)(A-\lambda Id)e_3 = 0$. $(A-\lambda Id)e_3 \in ker(A-\lambda Id)$, is linearly combination of $e_1, e_2$. $A(e_3) - \lambda e_3 = ae_1 + be_2$, then $A(e_3) = ae_1 + be_2 + \lambda e_3$.

  Similarly, we can get basis of $V^{\lambda_i}$ such that $A(e_i) = l.c.\{e_1, e_2, ..e_{i-1}\} + \lambda e_i$.

  So that, $A|V^{\lambda_i}$ **is upper triangular matrix with $\lambda$ on all diagonal entries** .

$$A|V^{\lambda_i} = \begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$$

  Assume $dimV^\lambda(A) = k$ , then subordinate basis $\{e_1, .., e_k\}$, $A|V^{\lambda_i}$ is $k \times k$ matrix.

  Further, characteristic polynomial of $A|V^{\lambda_i}$ will be $det(A|V^{\lambda_i} - tId|V^{\lambda_i}) = (\lambda - t)^k$.

Lemma 1. **For $\lambda_i \neq \lambda_j$, $(A-\lambda_j Id)|V^{\lambda_i}(A)$ is invertible.**

*Proof:* Diagonal entries: $\lambda_i - \lambda_j \neq 0$. Thus, $det = (\lambda_i - \lambda_j)^k \neq 0$. Invertible.

(will be used in proof of Proposition 2)

- Proposition 1

  **Let $k = dimV^\lambda(A)$, then multiplicity of $\lambda$ in characteristic polynomial of $A$ is $k$.**

  *Proof:* Expend $e_1, ..., e_k$ to basis of $\mathbb{C}^n$, then $A$ can be written as:

$$\begin{array}{c} \quad e_1 \quad .... \quad e_k, e_{k+1}... \\ A = \begin{bmatrix} \lambda_i & & * & \\ & \ddots & & * \\ 0 & & \lambda_i & \\ & 0 & & B \\ & & & \end{bmatrix} \end{array}$$

  The characteristic polynomial of A: $det(A - tId) = (\lambda - t)^k \times det(B - tId)$.

  If not a root of $det(B - tId)$, $\lambda$ only exists in $(\lambda - t)^k$, then multiplicity is $k$.

  Assume $\lambda$ is a root $det((B - tId)$, then $\lambda$ is an eigenvalue of $B$. There is $w \neq 0$, $Bw = \lambda w$.

  Then extend $w$ into $\mathbb{C}^n$ with 0 on $e_1, .., e_k$, we can get $w'$. $Aw' = \lambda w' + v$, where $v \in V^\lambda(A)$.

Then $(A - \lambda Id)w' = v \in V^\lambda(A)$, which means$w'$ also in $V^\lambda(A)$.

But $w' = (0, ..0, w_1, .., w_{n-k})$, can not be expressed in $e_1, ...e_k$ described above, not a root vector of $\lambda$. Contradiction.

Thus, there is no $\lambda$ in $det(B - tId)$, the multiplicity of $\lambda$ is $k$.

Then, for each distinct eigenvalues $\lambda_1, \lambda_2, .., \lambda_k$ of $A$, there are root subspaces $V^{\lambda_1}, V^{\lambda_2}, ..., V^{\lambda_k}$ of each eigenvalues. $\lambda_i$ has multiplicity $m_i = dimV^{\lambda_i}$.

According to *Fundamental Theorem of Algebra*, $m_1 + m_2 + ... + m_k = n$. Then $dimV^{\lambda_1} + dimV^{\lambda_2} + ... + dimV^{\lambda_k} = n$.

To prove $\mathbb{C}^n = V^{\lambda_1} \bigoplus V^{\lambda_2} \bigoplus ... \bigoplus V^{\lambda_k}$, we still need to prove that set of all vectors in bases of $V^{\lambda_1}, V^{\lambda_2}, ..., V^{\lambda_k}$ is linearly independent.

Lemma 2. Let $m = dimV^\lambda(A)$, $v \in V^\lambda(A)$. Then $v \in ker(A - \lambda Id)^m$.

*Proof:* By definition of root vector, $(A - \lambda Id)^k v = 0$, for some $k$. $V^\lambda(A) = ker(A - \lambda Id)^k$.

Assume $k > m$.

Then $ker(A - \lambda Id) \subset ... \subset ker(A - \lambda Id)^m \subset ... \subset ker(A - \lambda Id)^k = V^\lambda(A)$.

Each time degree of $(A - \lambda Id$ increase by 1, dimension of set increase at least by 1.

Then $dimV^\lambda(A) > m$. Contradiction. Only $k \leqslant m$ possible.

Thus, $(A - \lambda Id)^m v = (A - \lambda Id)^{m-k} v \times (A - \lambda Id)^k v = 0$. $v \in ker(A - \lambda Id)^m$.

- Proposition 2.

Let $e_1 \in V^{\lambda_1}, ..., e_k \in V^{\lambda_k}$, $e_1, ...e_k \neq 0$. Then $\{e_1, ..., e_k\}$ linearly independent.

*Proof:* To prove linearly independence, suppose $\mu_1 e_1 + \mu_2 e_2 + ... + \mu_k e_k = 0$.

When $k = 2$:

To kill $e_1$, let $dimV^{\lambda_1} = m_1$. Then by Lemma 2, $(A - \lambda_1 Id)^{m_1} e_1 = 0$.

Multiply by $(A - \lambda_1 Id)^{m_1}$, get $\mu_1 0 + \mu_2 (A - \lambda_1 Id)^{m_1} e_2 = 0$.

Since $\lambda_1 \neq \lambda_2$, diagonal entries of $(A - \lambda_1 Id)|V^{\lambda_2} = \lambda_2 - \lambda_1 \neq 0$.

Then $(A - \lambda_1 Id)^m$ can not send $e_2$ gradually to 0. $(A - \lambda_1 Id)^{m_1} e_2 \neq 0$.

$\mu_2 (A - \lambda_1 Id)^{m_1} e_2 = 0$, thus $\mu_2$ must be 0. $e_1 \neq 0$, then $\mu_1$ also must be 0.

Therefore, $\{e_1, e_2\}$ linearly independent.

When k = 3:

Similarly, $\mu_1 0 + \mu_2 (A - \lambda_1 Id)^{m_1} e_2 + \mu_3 (A - \lambda_1 Id)^{m_1} e_3 = 0$

To kill $e_2$, let $dimV^{\lambda_2} = m_2$. Then by Lemma 2, $(A - \lambda_2 Id)^{m_2} e_2 = 0$.

Multiply, get $\mu_2 (A - \lambda_1 Id)^{m_1} (A - \lambda_2 Id)^{m_2} e_2 + \mu_3 (A - \lambda_1 Id)^{m_1} (A - \lambda_2 Id)^{m_2} e_3 = 0$.

Similarly, by Lemma 1, $(A - \lambda_1 Id)^{m_1} (A - \lambda_2 Id)^{m_2} | V^{\lambda_3}$ $det \neq 0$, invertible.

$(A - \lambda_1 Id)^{m_1} (A - \lambda_2 Id)^{m_2} e_3 \neq 0$, then $\mu_3$ must be 0.

Then $\mu_2 (A - \lambda_1 Id)^{m_1} (A - \lambda_2 Id)^{m_2} e_2 = 0$, $\mu_2$ must 0. Similarly, $\mu_1 = 0$.

Therefore, $\{e_1, e_2, e_3\}$ linearly independent.

For $k$ root spaces, repeated this process applying $(A - \lambda_i Id)^{m_i}$ to kill terms $e_1, .., e_{k-1}$.

Then we can get $\mu_k, \mu_{k-1}, .., \mu_1 = 0$, $\{e_1, ..., e_k\}$ linearly independent.

$V^{\lambda_2}, ..., V^{\lambda_k}$ each has adapted basis. Let basis of $V^{\lambda_j}$ be $E_j = \{e_1^{\lambda_j}, e_2^{\lambda_j}, .., e_{m_k}^{\lambda_j}\}$

Then we can let $\sum_{i=1}^{m_1} \mu_i^{\lambda_1} e_i^{\lambda_1} + \sum_{i=1}^{m_2} \mu_i^{\lambda_2} e_i^{\lambda_2} + .... + \sum_{i=1}^{m_k} \mu_i^{\lambda_k} e_i^{\lambda_k} = 0$

By Proposition 2, each $\sum_{i=1}^{m_j} \mu_i^{\lambda_j} e_i^{\lambda_j}$ is 0. Then every $\mu$ is 0. Set of all vectors $E_1, E_2, ..., E_k$ is linearly independent.

Thus consider $E = \cup E_j$ for $j = 1, .., k$. By Proposition 1, $dimE = n$. $E$ is a basis of $\mathbb{C}^n$.

$\mathbb{C}^n = V^{\lambda_1} \bigoplus V^{\lambda_2} \bigoplus ... \bigoplus V^{\lambda_k}$

$$A = \begin{bmatrix} A|V^{\lambda_1} & & & 0 \\ & A|V^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & A|V^{\lambda_k} \end{bmatrix}$$

## 5 STEP 3: Jordan Block

- Proposition 3.

  Let $N : V \to V$ nilpotent, then $\exists v_1, ..., v_q$, $height(v_i) = m_i$ such that $v_1.Nv_1, ...N^(m_1 - 1)v_1$

  *Proof:*

## 6 Special Case: Diagonalization of Symmetirc Matrix

## 7 Note

0. Jordan
1. just one version of story telling
2. open the door to modern linear algebra to me
3. further edit on this
4. Example with notation.