Breaking the Sample Size Barrier in Model-Based Reinforcement Learning

Yuting Wei

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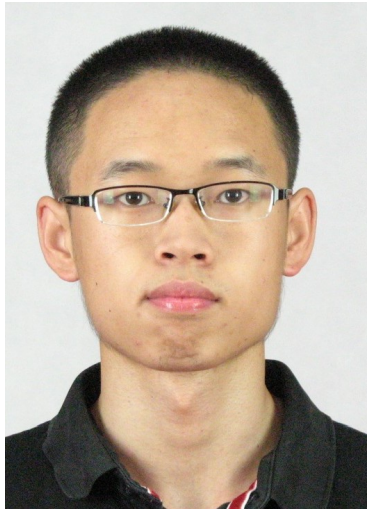
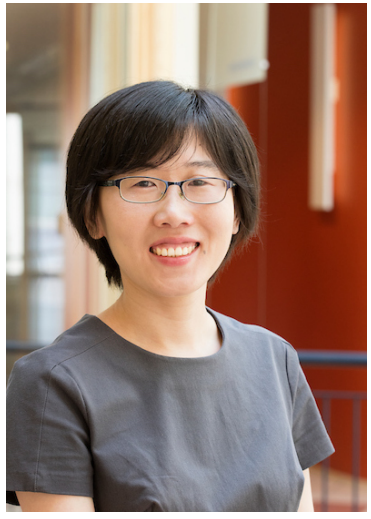
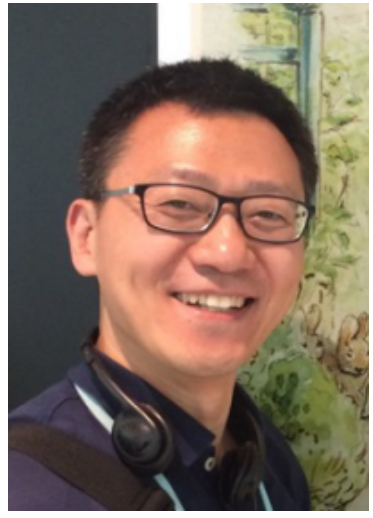
Nov, 2020



Gen Li
Tsinghua EE



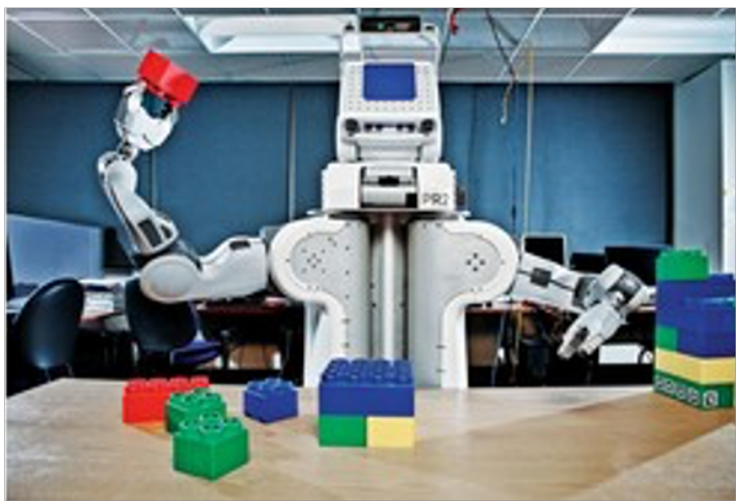
Yuejie Chi
CMU ECE



Yuantao Gu
Tsinghua EE

Yuxin Chen
Princeton EE

Reinforcement learning (RL)



RL challenges

- Unknown or changing environment
- Credit assignment problem
- Enormous state and action space

Provable efficiency

- Collecting samples might be expensive or impossible: sample efficiency
- Training deep RL algorithms might take long time: computational efficiency

This talk

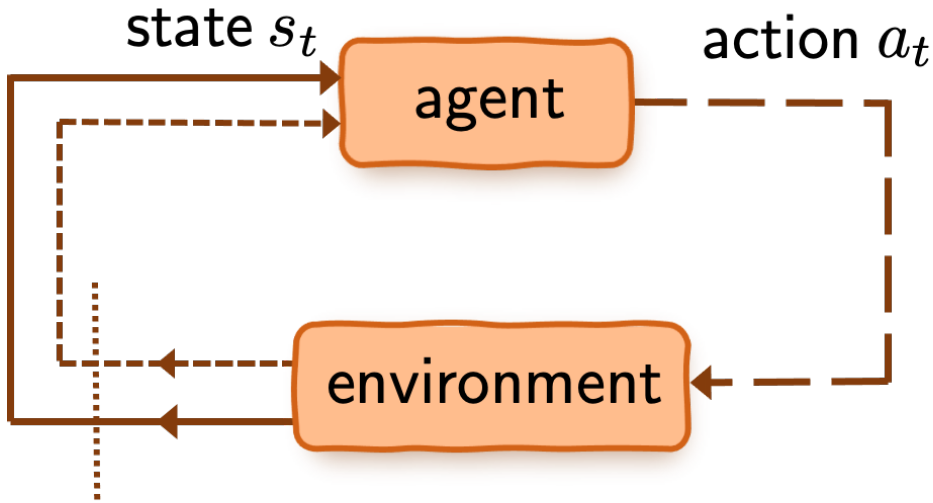
Question: can we design sample- and computation-efficient RL algorithms?

—— *inspired by numerous prior work*

[Kearns and Singh, 1999, Sidford et al., 2018a, Agarwal et al., 2019]...

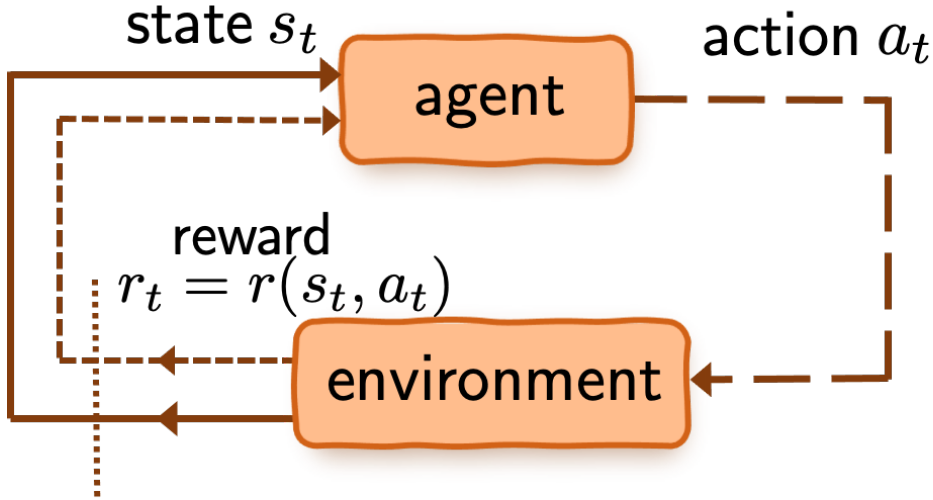
Background: Markov decision processes

Markov decision process (MDP)



- $\mathcal{S}$: state space
- $\mathcal{A}$: action space

Markov decision process (MDP)



- $\mathcal{S}$: state space
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- $r(s, a) \in [0, 1]$: immediate reward

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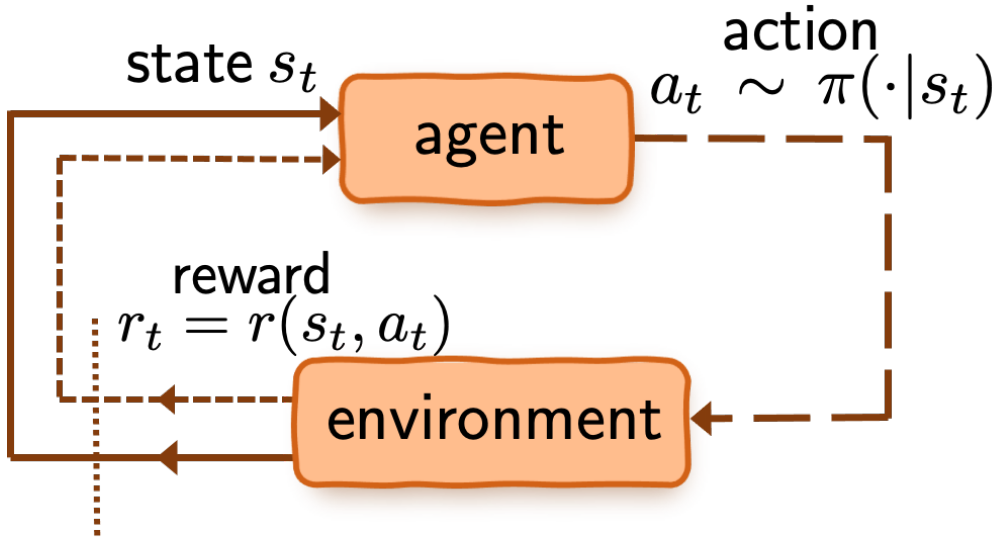

- $\mathcal{S}$: state space
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- $\pi(\cdot|s)$: policy (or action selection rule)

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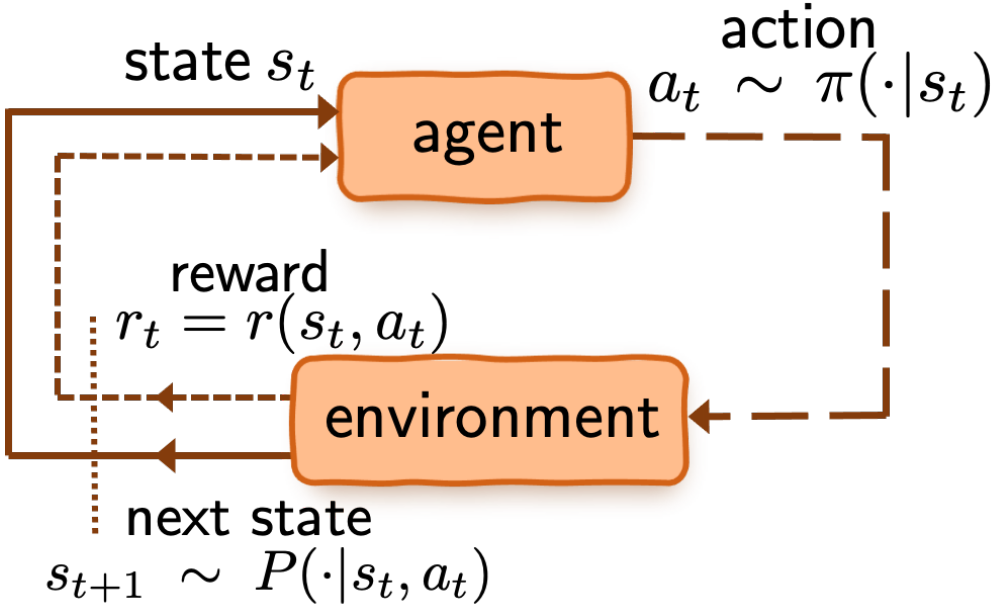
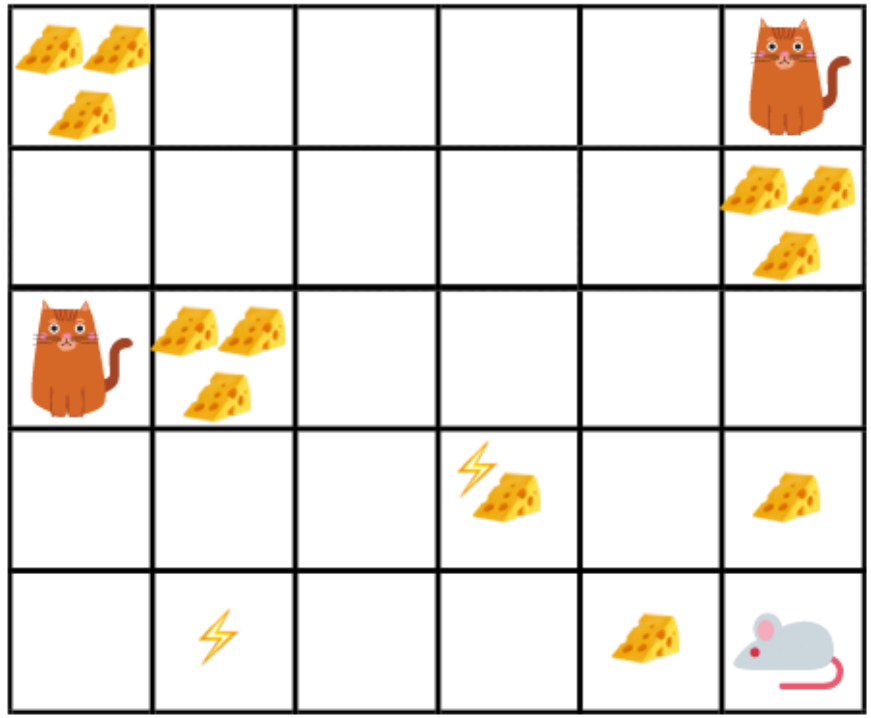
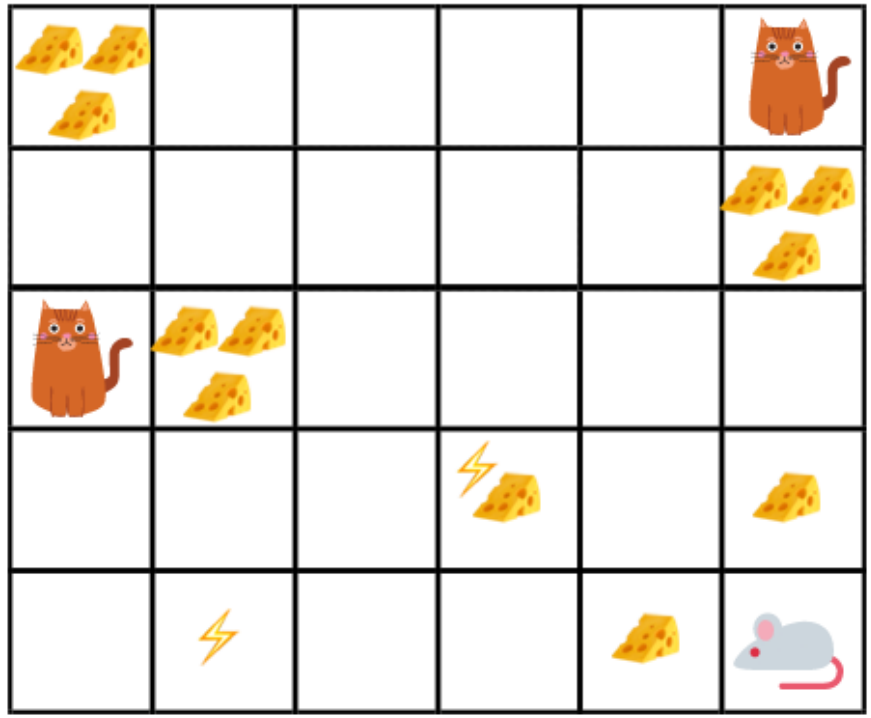


- $\mathcal{S}$: state space
- $\mathcal{A}$: action space
- $r(s, a) \in [0, 1]$: immediate reward
- $\pi(\cdot|s)$: policy (or action selection rule)
- $P(\cdot|s, a)$: unknown transition probabilities

Help the mouse!

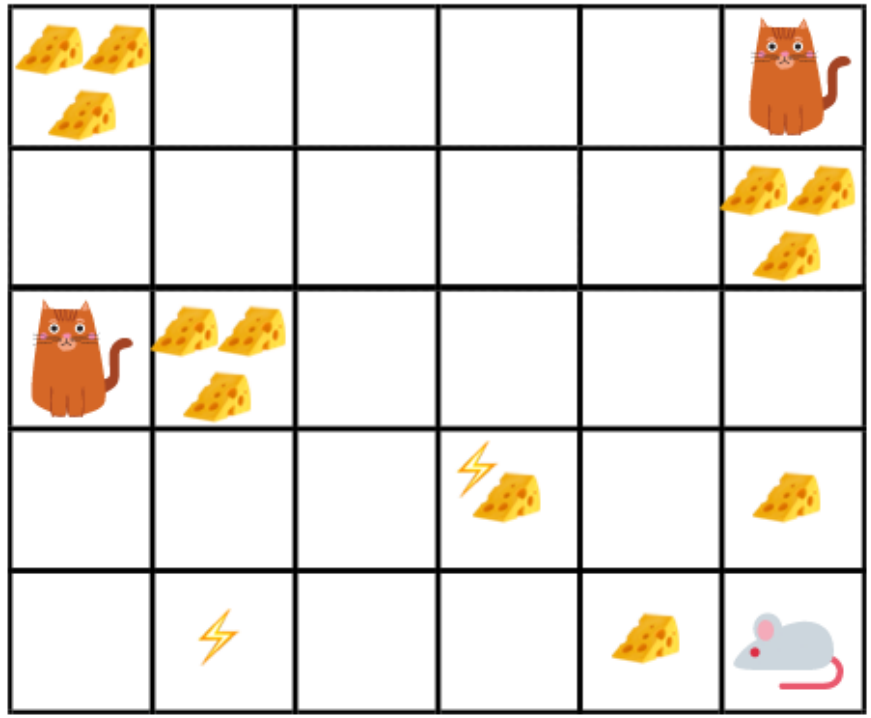


Help the mouse!



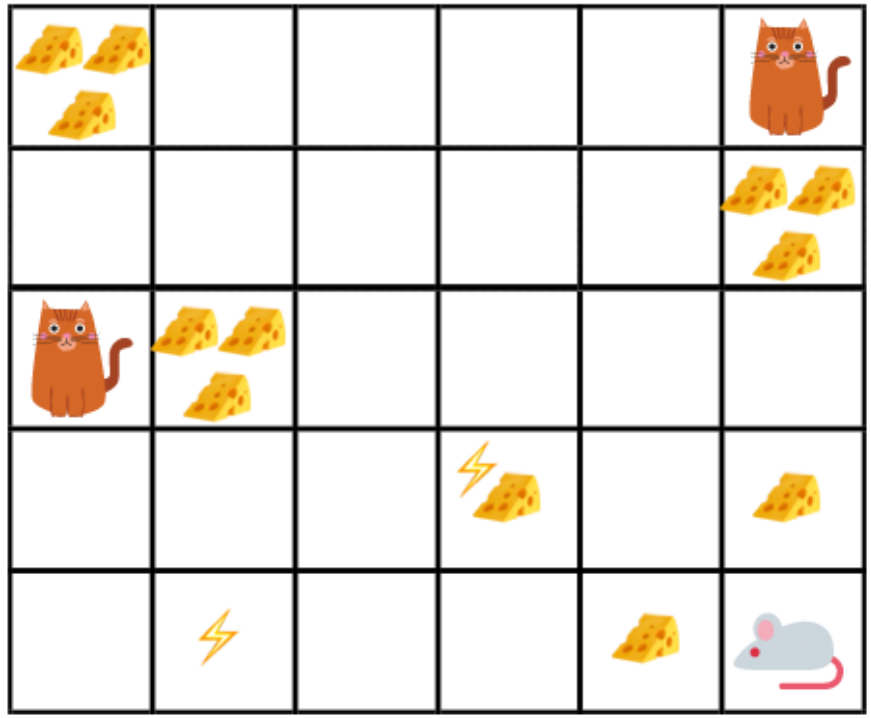
- state space $\mathcal{S}$: positions in the maze

Help the mouse!



- state space $\mathcal{S}$: positions in the maze
- action space $\mathcal{A}$: up, down, left, right

Help the mouse!



- state space $\mathcal{S}$: positions in the maze
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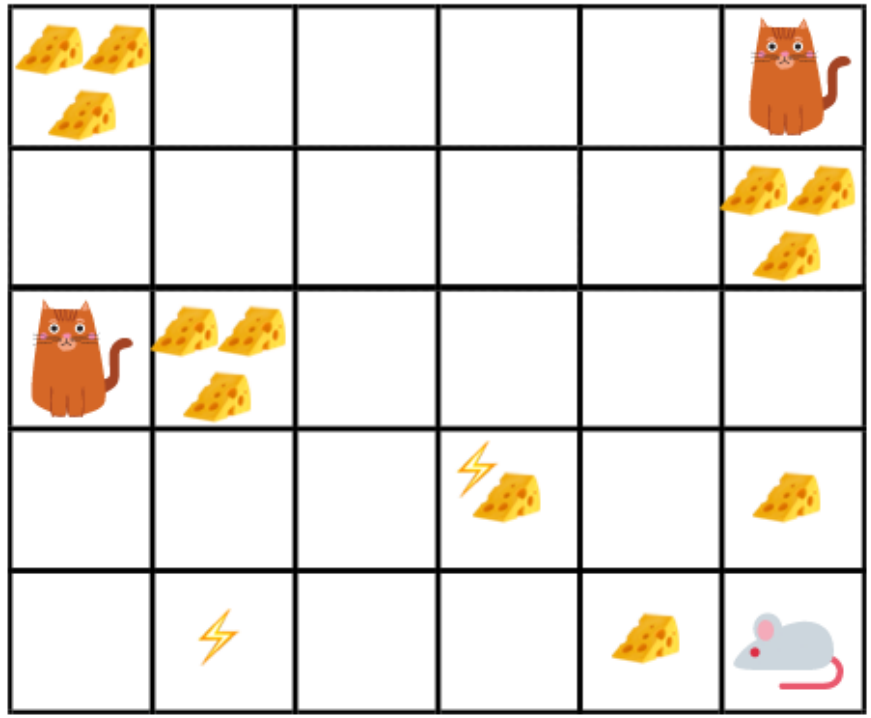
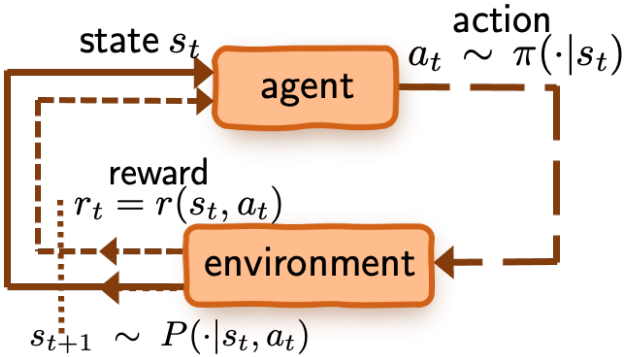
- immediate reward r: cheese, electricity shocks, cats

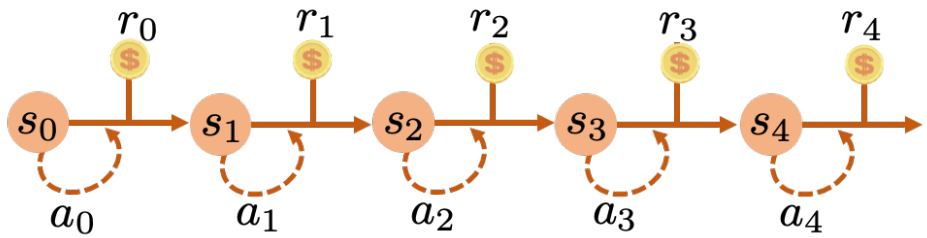
Help the mouse!



- state space $\mathcal{S}$: positions in the maze
- action space $\mathcal{A}$: up, down, left, right
- immediate reward r: cheese, electricity shocks, cats
- policy $\pi(\cdot|s)$: the way to find cheese

Value function

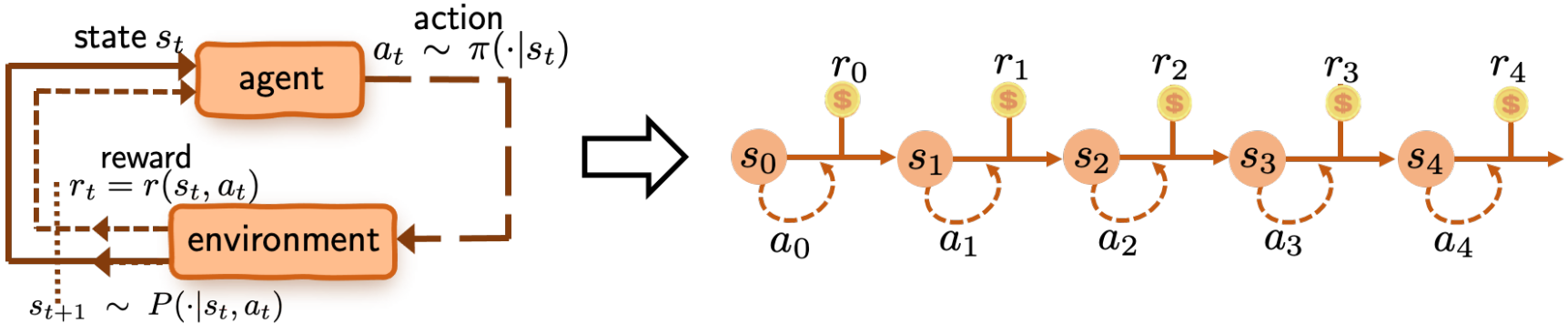




Value function of policy π: long-term discounted reward

$$\forall s \in \mathcal{S}: \quad V^{\pi}(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \,\middle|\, s_0 = s\right]$$

Value function



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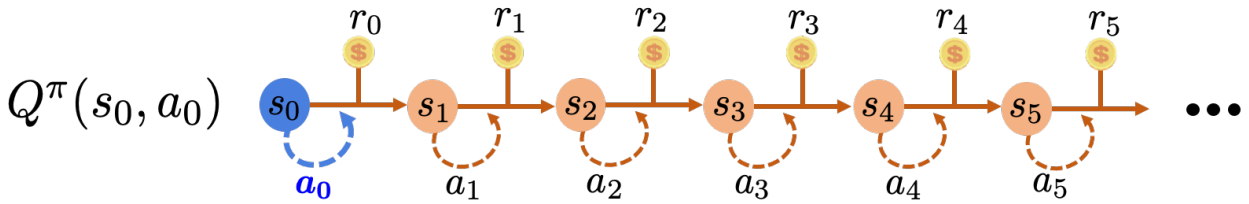
- $\gamma \in [0, 1)$: discount factor
- $(a_0, s_1, a_1, s_2, a_2, \cdots)$: generated under policy π

Action-value function (a.k.a. Q-function)



Q-function of policy π

$$\forall (s, a) \in \mathcal{S} \times \mathcal{A}: \quad Q^\pi(s, a) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \,\middle|\, s_0 = s, a_0 = a\right]$$

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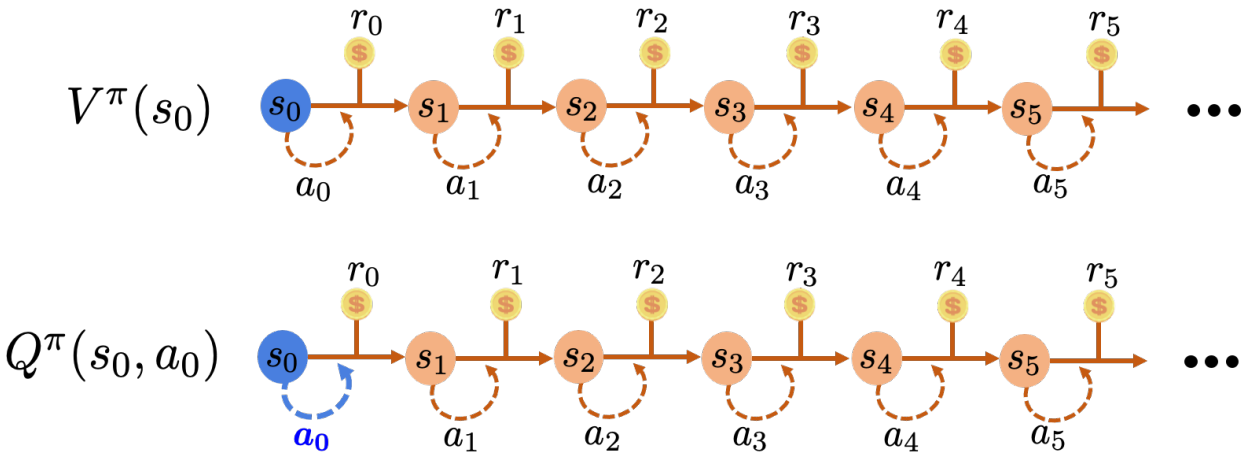
Action-value function (a.k.a. Q-function)



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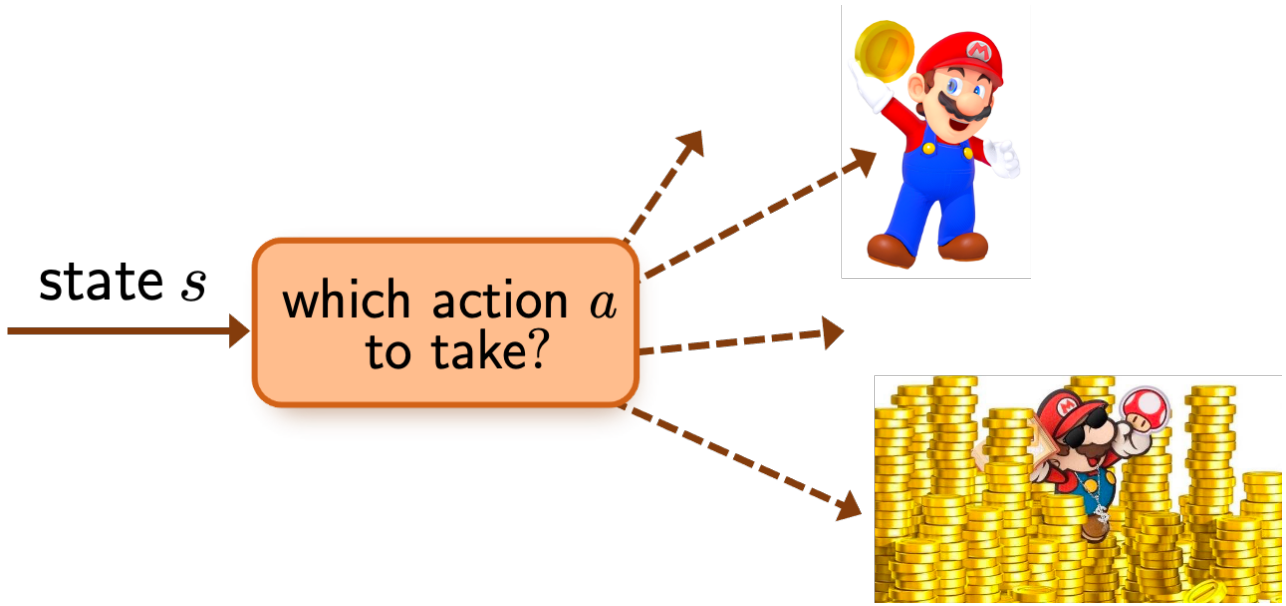
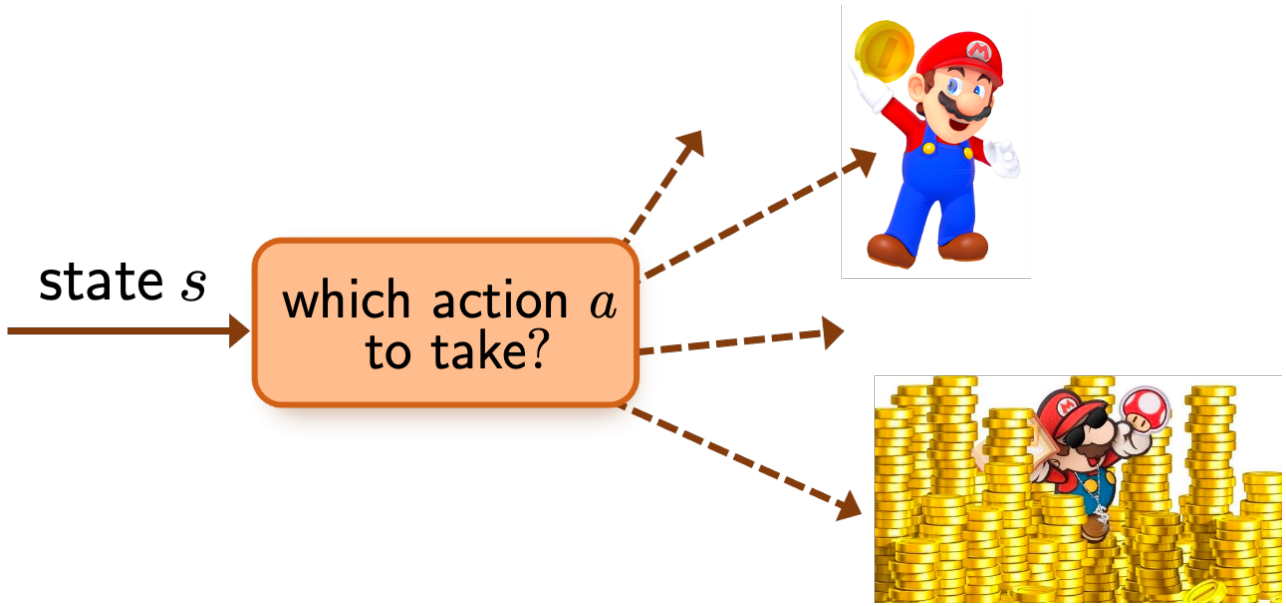
Optimal policy

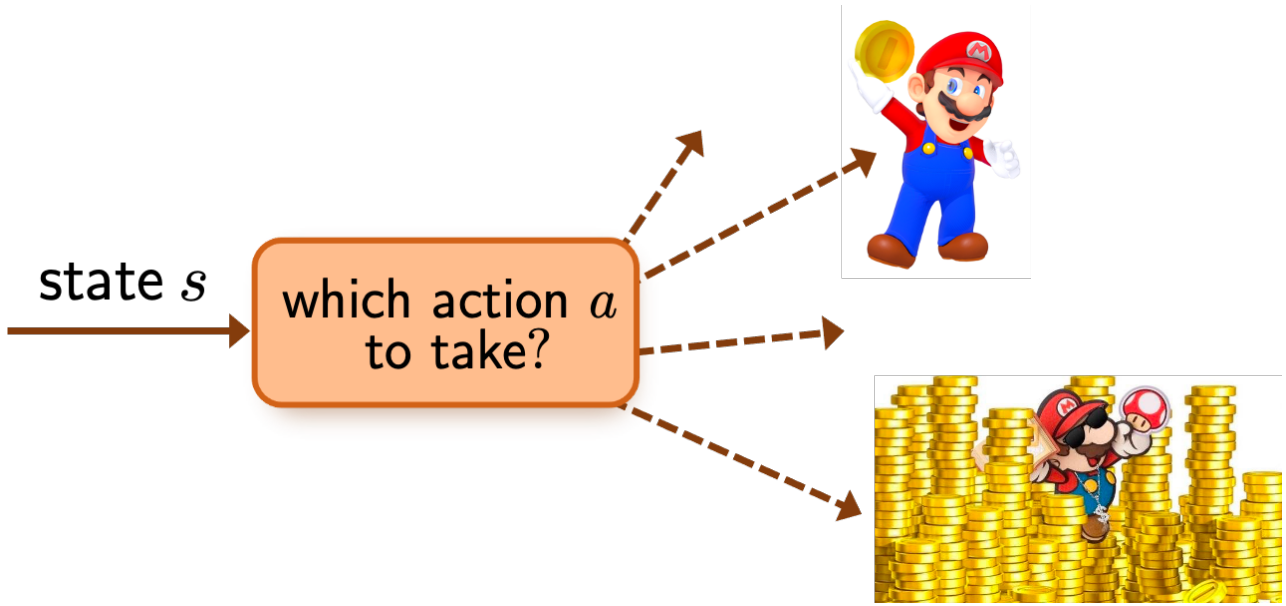


Optimal policy



- **optimal policy** $\pi^\star$: maximizing value function

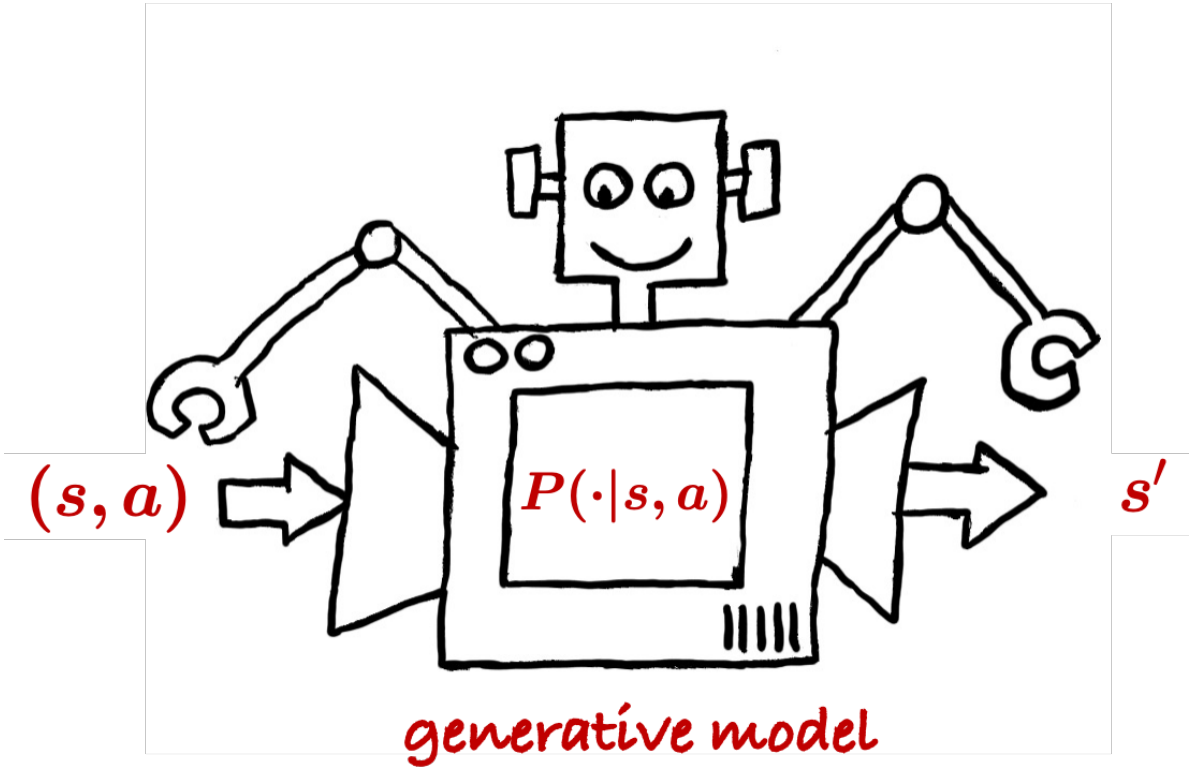
Optimal policy



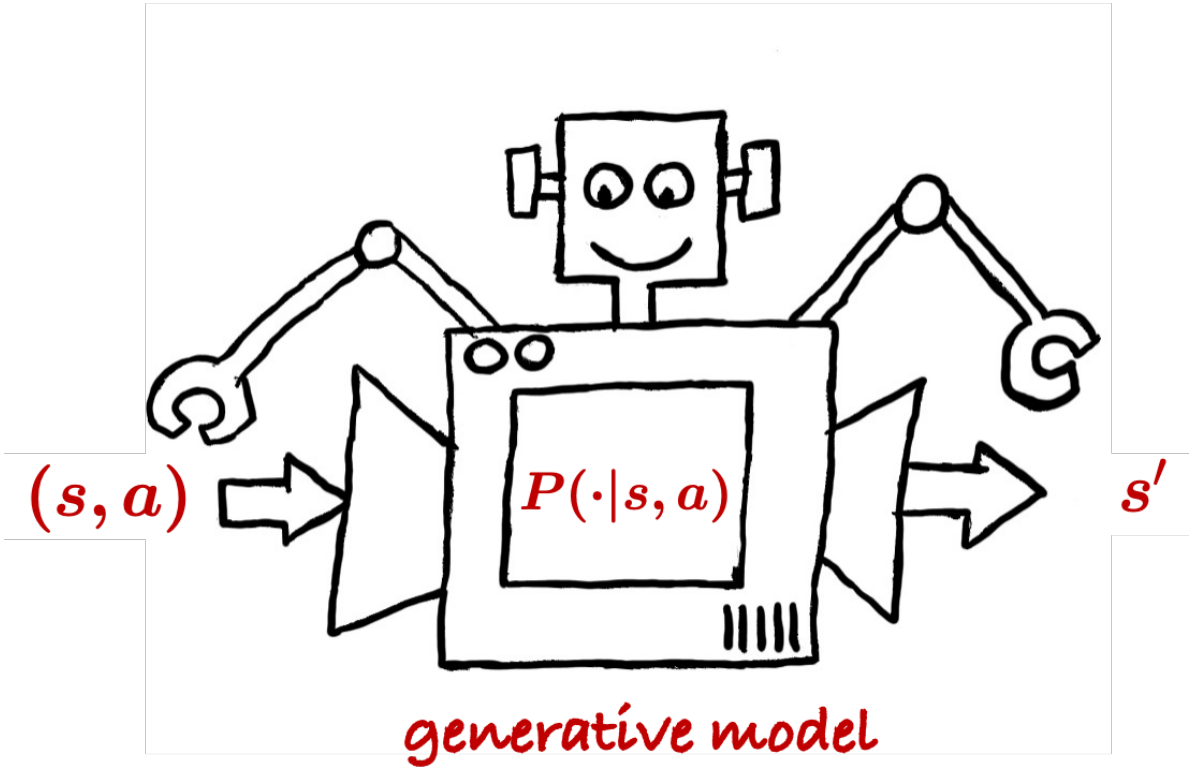
- **optimal policy** $\pi^\star$: maximizing value function
- optimal value / Q function: $V^\star := V^{\pi^\star}$; $Q^\star := Q^{\pi^\star}$

Practically, learn the optimal policy from data samples ...

This talk: sampling from a generative model

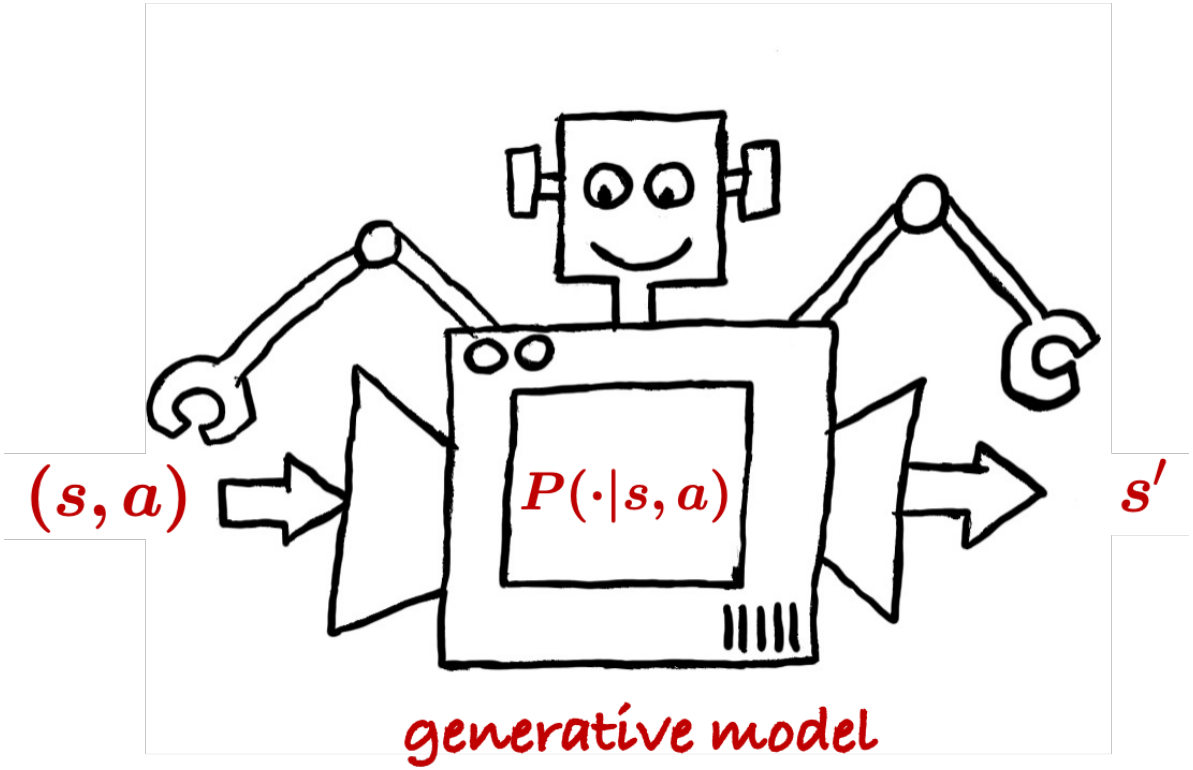


This talk: sampling from a generative model



For each state-action pair (s, a), collect N samples $\{(s, a, s'_{(i)})\}_{1 \le i \le N}$

This talk: sampling from a generative model



For each state-action pair (s, a), collect N samples $\{(s, a, s'_{(i)})\}_{1 \le i \le N}$

How many samples are sufficient to learn an ε-optimal policy?

An incomplete list of prior art

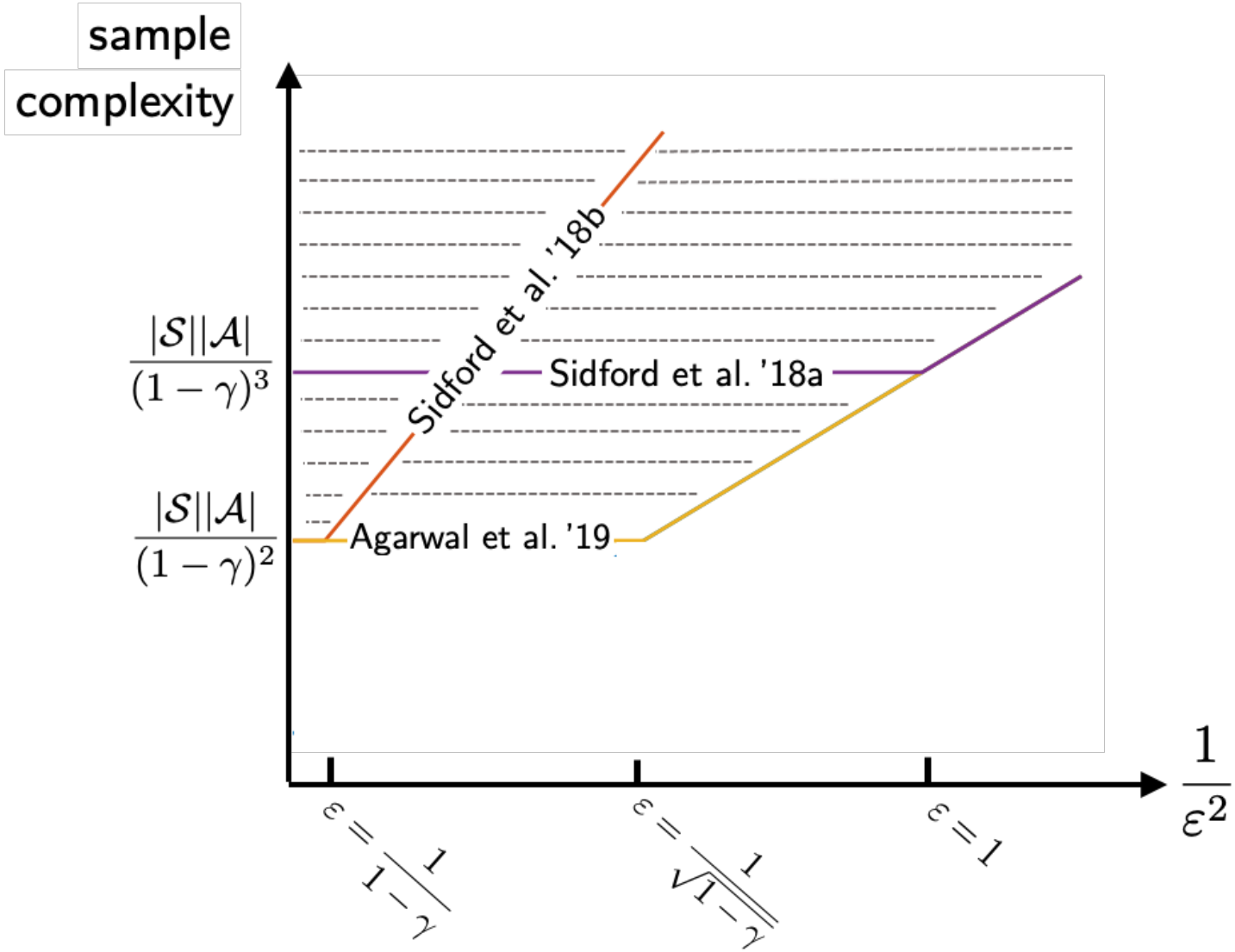
- [Kearns and Singh, 1999]
- [Kakade, 2003]
- [Kearns et al., 2002]
- [Azar et al., 2012]
- [Azar et al., 2013]
- [Sidford et al., 2018a]
- [Sidford et al., 2018b]
- [Wang, 2019]
- [Agarwal et al., 2019]
- [Wainwright, 2019a]
- [Wainwright, 2019b]
- [Pananjady and Wainwright, 2019]
- [Yang and Wang, 2019]
- [Khamaru et al., 2020]
- [Mou et al., 2020]
- ...

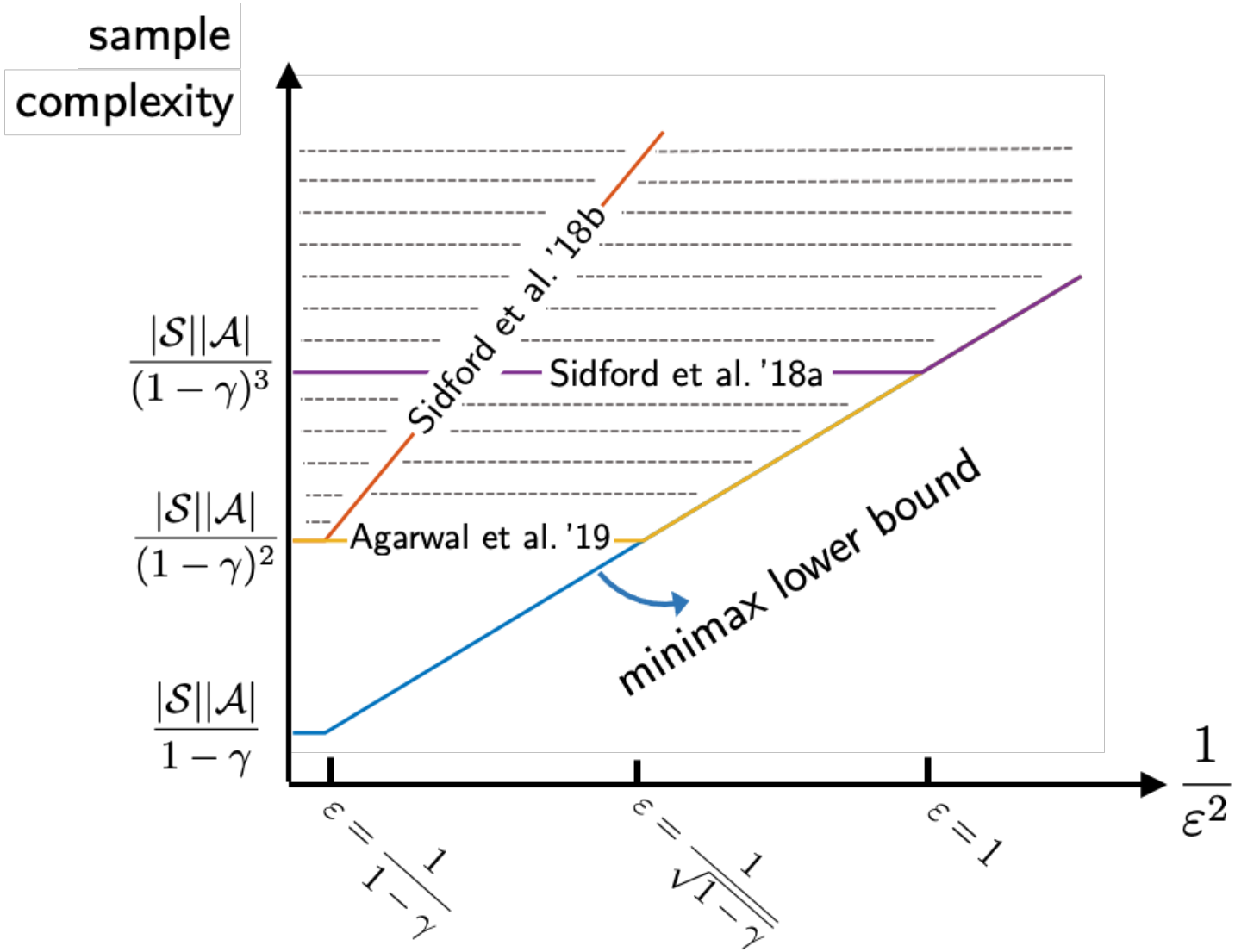
An even shorter list of prior art

algorithm	sample size range	sample complexity	ε-range
Empirical QVI [Azar et al., 2013]	$\lceil \frac{\lvert\mathcal{S}\rvert^2\lvert\mathcal{A}\rvert}{(1-\gamma)^2}, \infty)$	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^3\varepsilon^2}$	$(0, \frac{1}{\sqrt{(1-\gamma)\lvert\mathcal{S}\rvert}}]$
Sublinear randomized VI [Sidford et al., 2018b]	$\lceil \frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^2}, \infty)$	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^4\varepsilon^2}$	$(0, \frac{1}{1-\gamma}]$
Variance-reduced QVI [Sidford et al., 2018a]	$\lceil \frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^3}, \infty)$	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^3\varepsilon^2}$	$(0, 1]$
Randomized primal-dual [Wang, 2019]	$\lceil \frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^2}, \infty)$	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^4\varepsilon^2}$	$(0, \frac{1}{1-\gamma}]$
Empirical MDP + planning [Agarwal et al., 2019]	$\lceil \frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^2}, \infty)$	$\frac{\lvert\mathcal{S}\rvert\lvert\mathcal{A}\rvert}{(1-\gamma)^3\varepsilon^2}$	$(0, \frac{1}{\sqrt{1-\gamma}}]$

important parameters $\Longrightarrow$

- # states $|\mathcal{S}|$, # actions $|\mathcal{A}|$
- the discounted complexity $\frac{1}{1-\gamma}$
- approximation error $\varepsilon \in (0, \frac{1}{1-\gamma}]$


sample
complexity
$\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^3}$
$\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^2}$
Sidford et al. '18b
Sidford et al. '18a
Agarwal et al. '19
$\frac{1}{\varepsilon^2}$
$\varepsilon = \frac{1}{1-\gamma}$
$\varepsilon = \frac{1}{\sqrt{1-\gamma}}$
$\varepsilon = 1$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minimax lower bound
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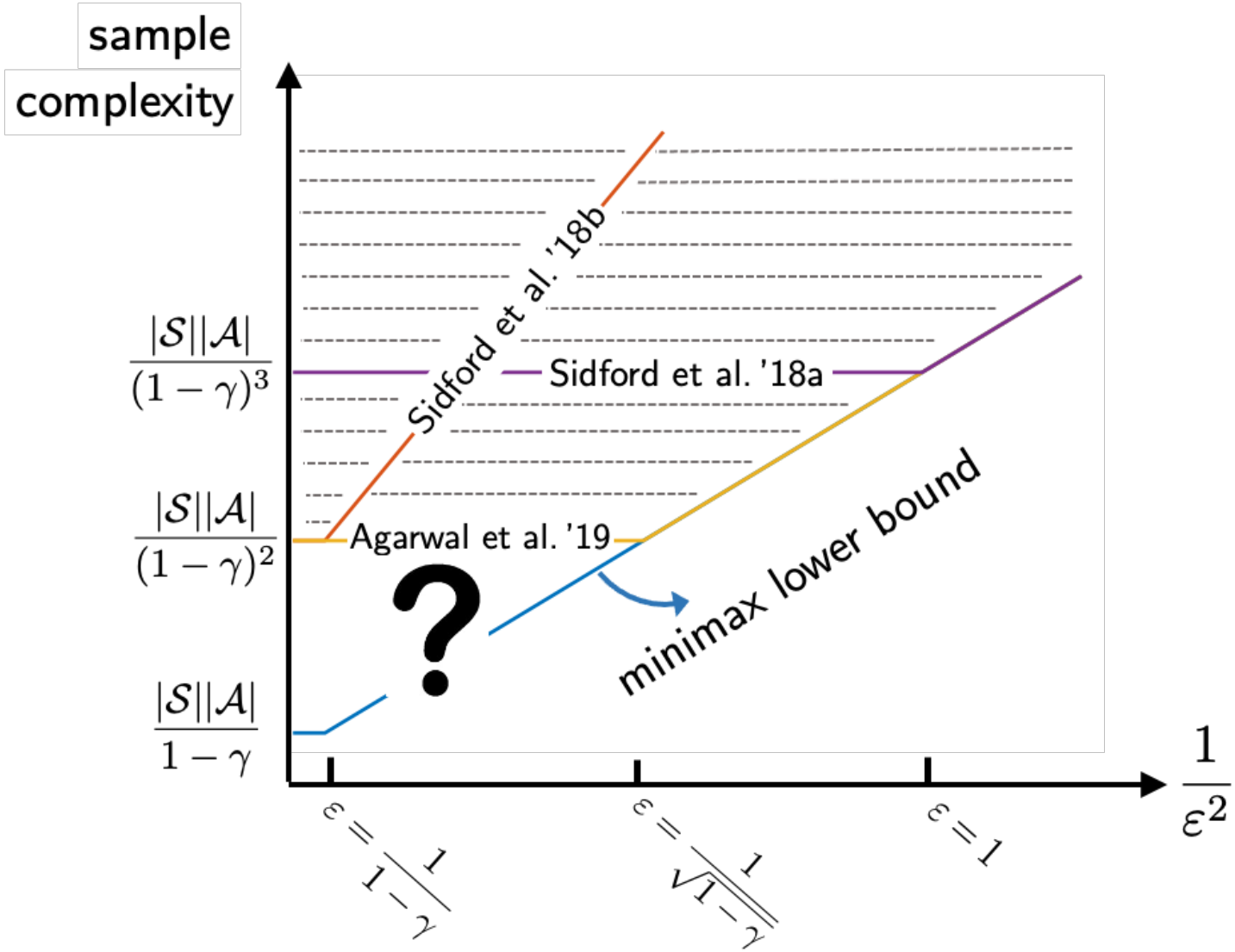
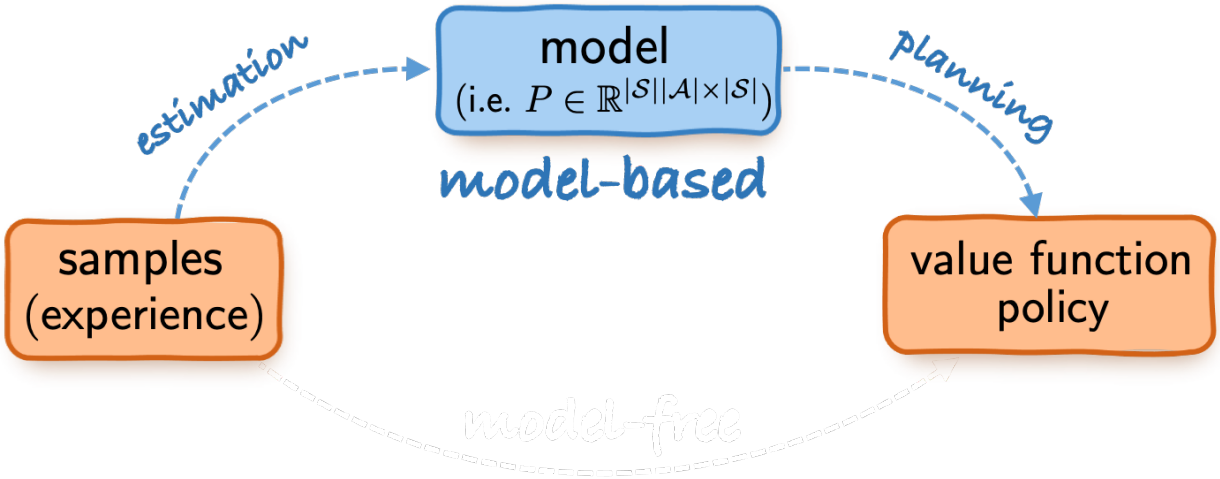


All prior theory requires sample size $> \underbrace{\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^2}}_{\text{sample size barrier}}$

This talk: break the sample complexity barrier

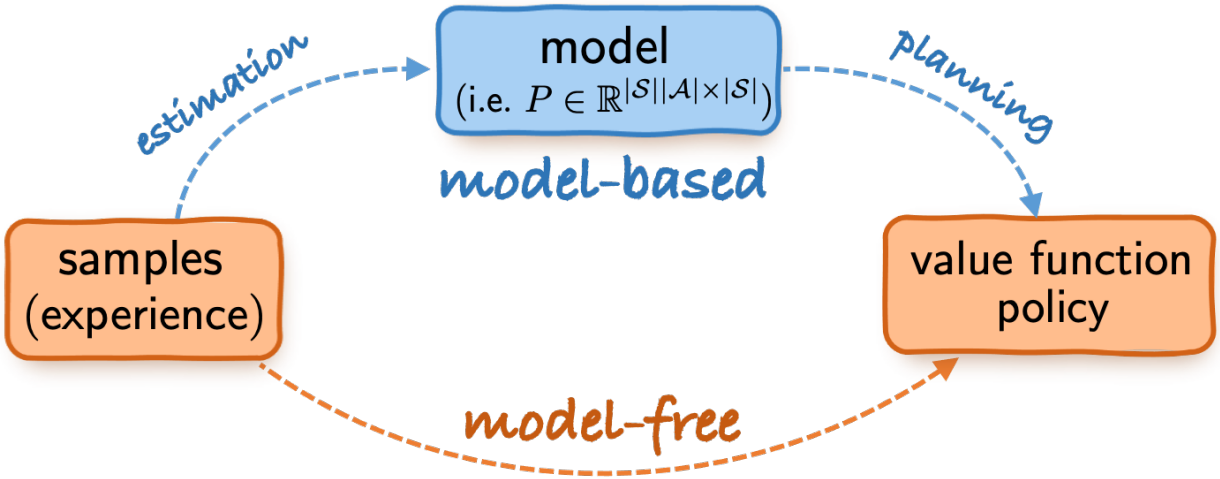
Two approaches



Model-based approach ("plug-in")

1. build empirical estimate $\widehat{P}$ for P
2. planning based on empirical $\widehat{P}$

Two approaches



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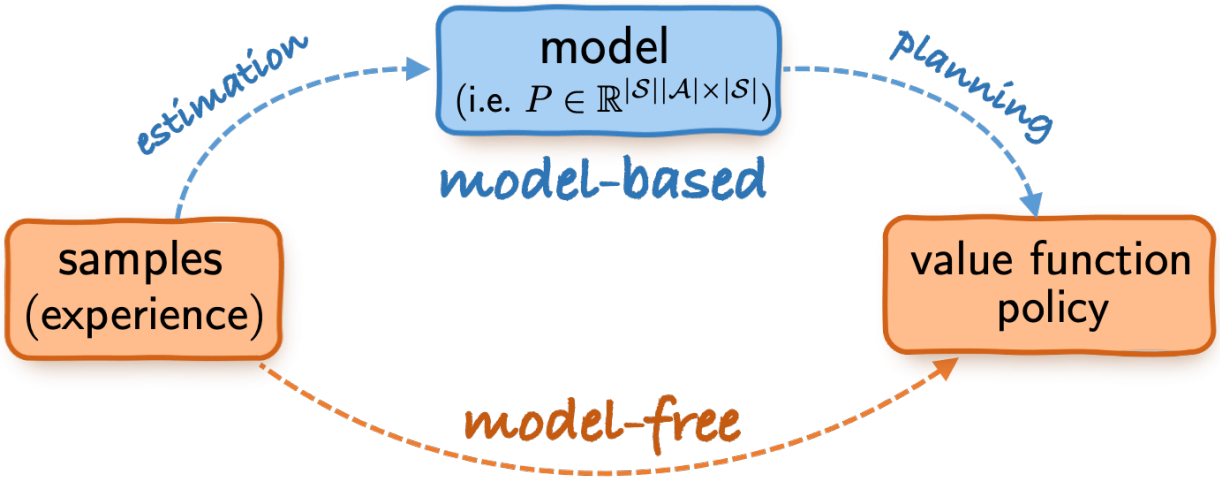
Model-free approach

— learning w/o constructing a model explicitly

Two approaches



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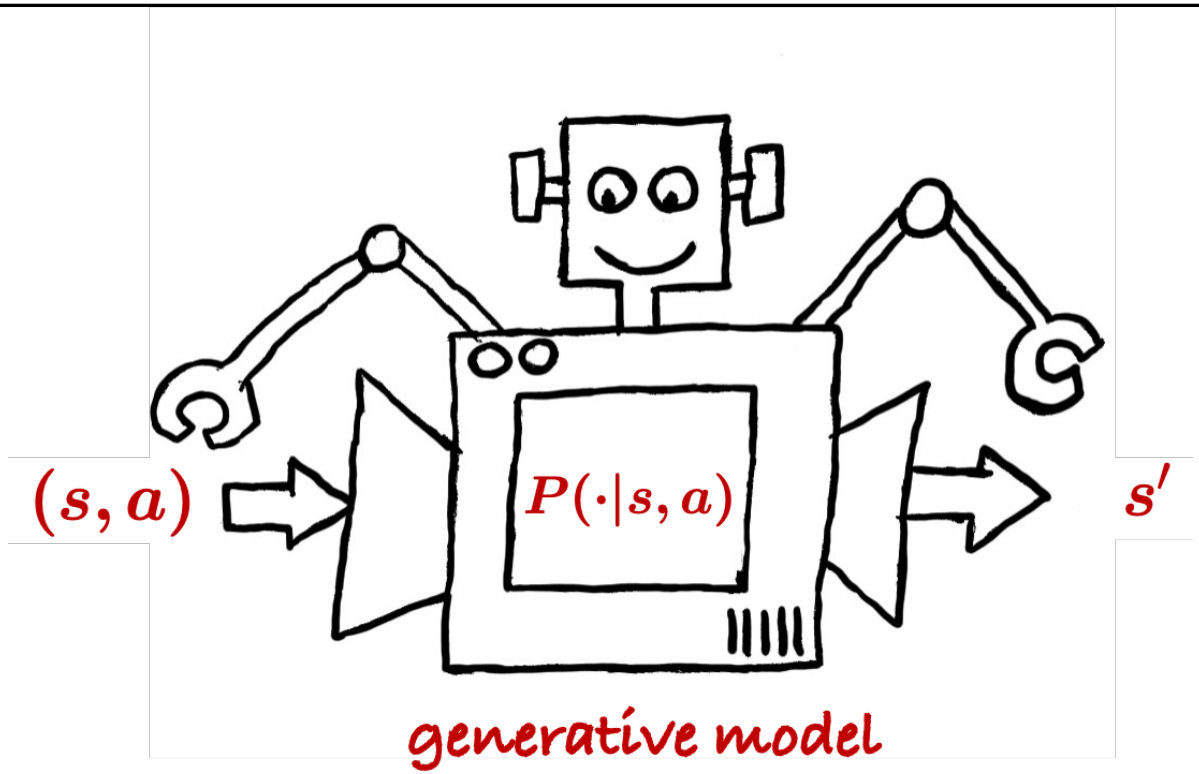
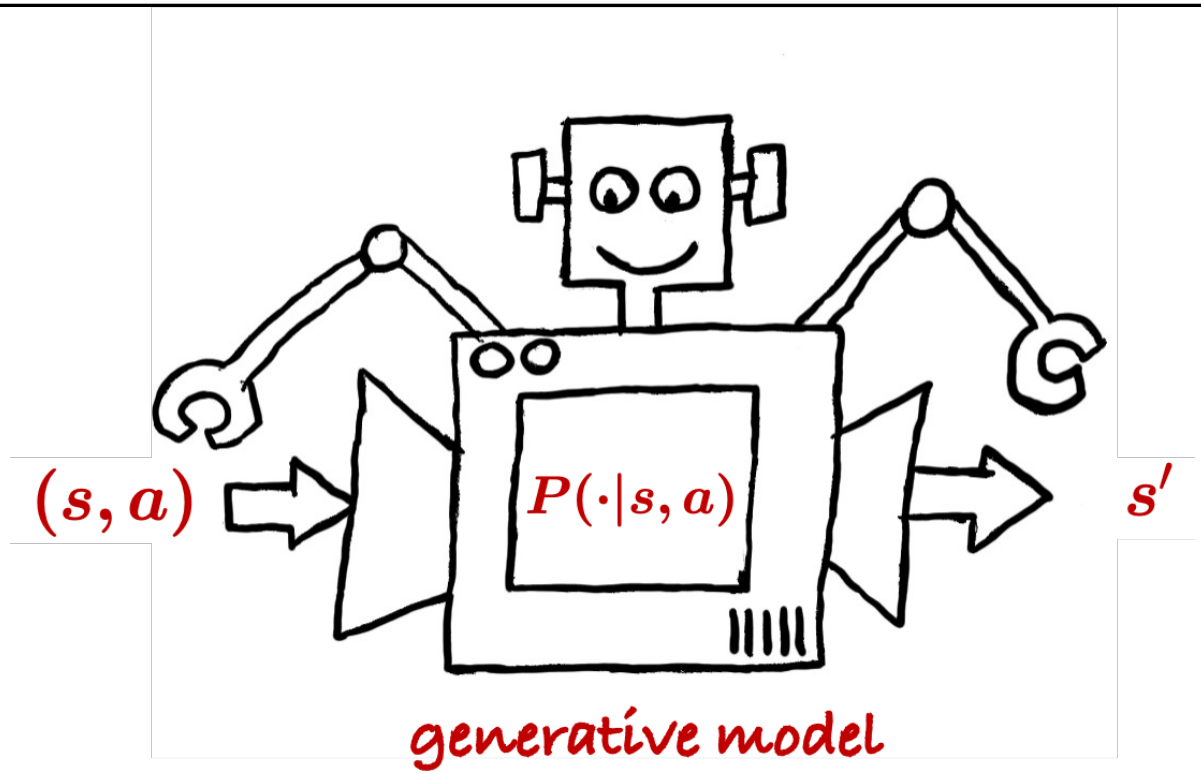
Model estimation



Sampling: for each (s, a), collect N ind. samples $\{(s, a, s'_{(i)})\}_{1 \le i \le N}$

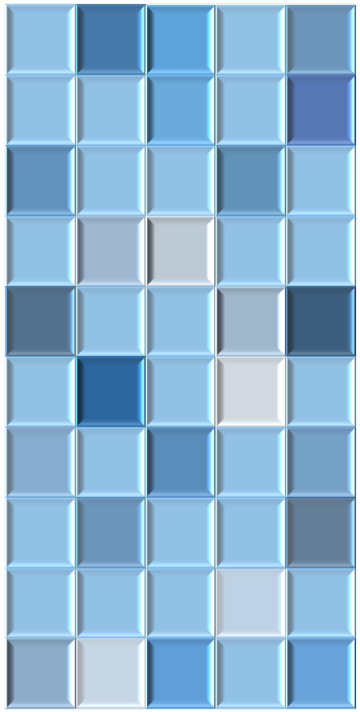
Model estimation



Sampling: for each (s, a), collect N ind. samples $\{(s, a, s'_{(i)})\}_{1 \le i \le N}$

Empirical estimates: estimate $\widehat{P}(s'|s, a)$ by $\underbrace{\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}\{s'_{(i)} = s'\}}_{\text{empirical frequency}}$

Our method: plug-in estimator + perturbation



truth: $P \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}| \times |\mathcal{S}|}$ $r \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$

original MDP $(\mathcal{S}, \mathcal{A}, P, r, \gamma)$ $\Longrightarrow$ $\pi^\star = \arg\max_\pi V^\pi$

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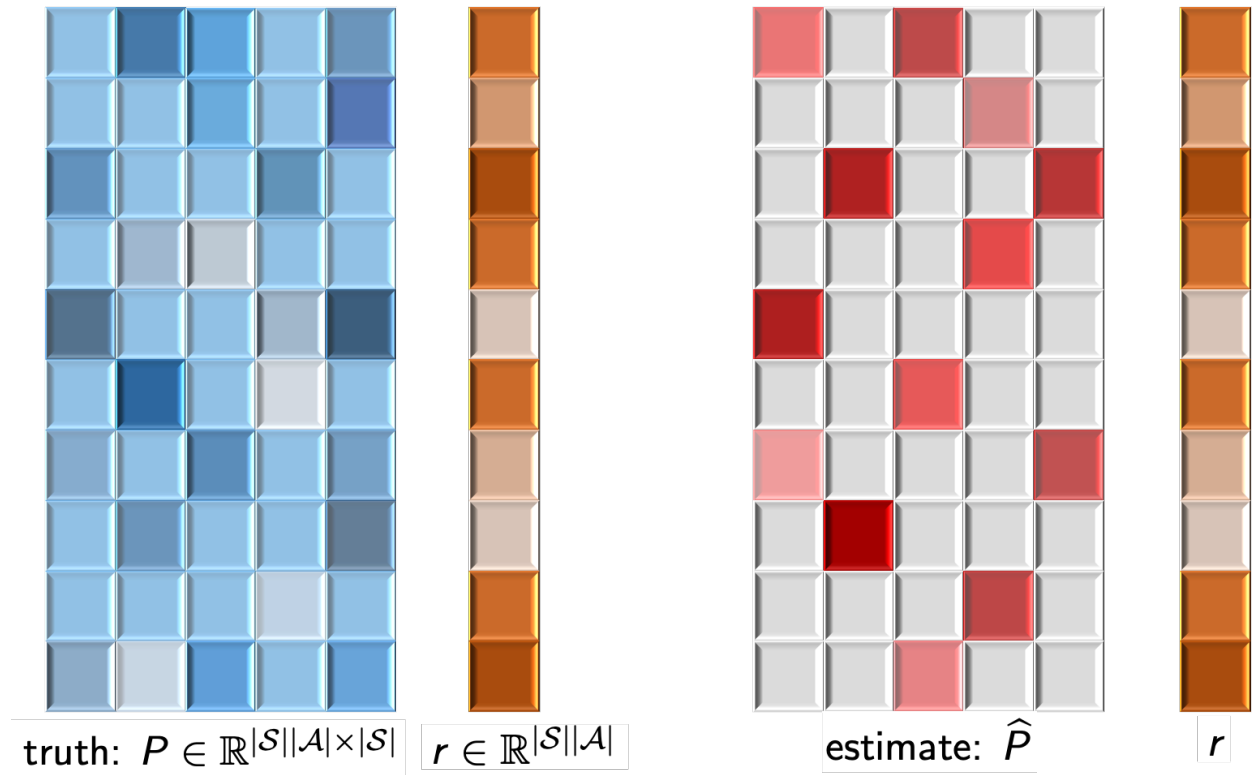
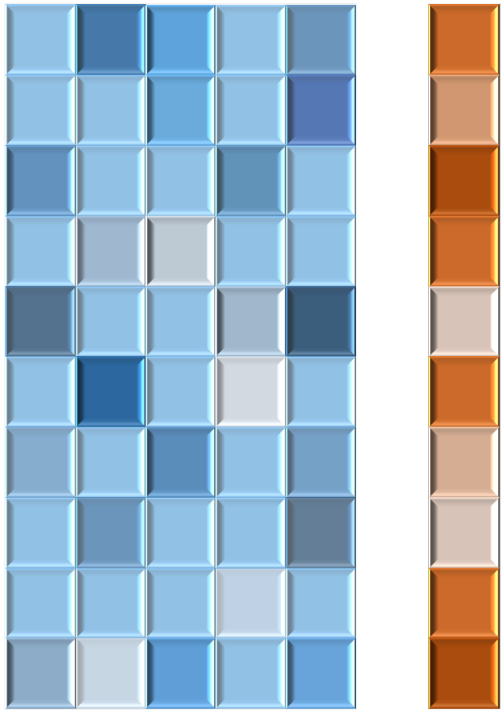


original MDP $(\mathcal{S}, \mathcal{A}, P, r, \gamma)$ $\Longrightarrow$ $\pi^\star = \arg\max_\pi V^\pi$

empirical MDP $(\mathcal{S}, \mathcal{A}, \widehat{P}, r, \gamma)$ $\Longrightarrow$ $\widehat{\pi}^\star = \arg\max_\pi \widehat{V}^\pi$

Our method: plug-in estimator + perturbation



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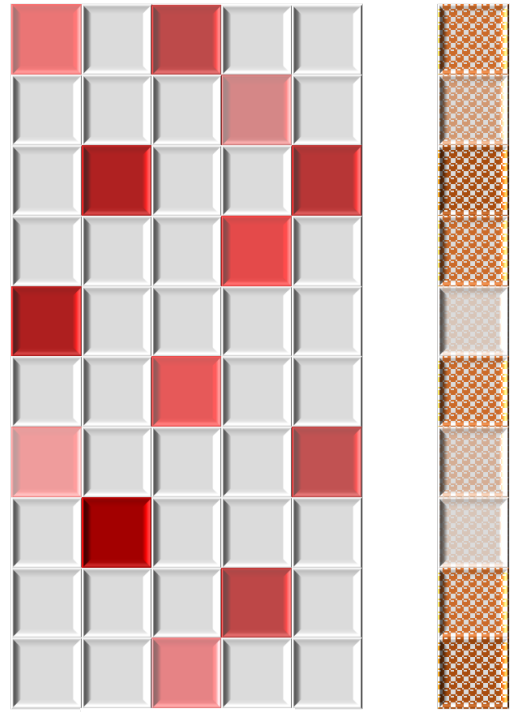

estimate: $\widehat{P}$ $\widetilde{r}$

$$\text{original MDP } (\mathcal{S}, \mathcal{A}, P, r, \gamma) \quad \Longrightarrow \quad \pi^\star = \arg\max_\pi V^\pi$$

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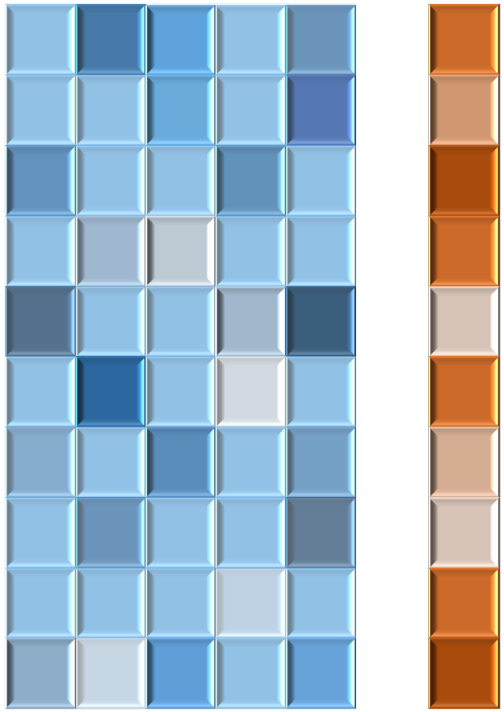
Our method: plug-in estimator + perturbation



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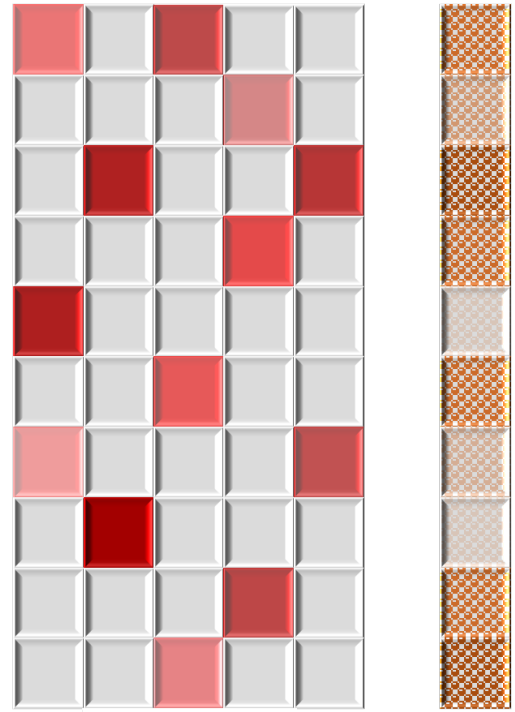

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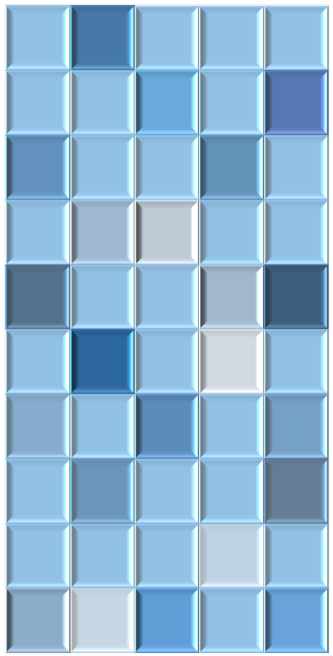
planning (policy iteration, Q-value iteration, $\cdots$)

Challenges in the sample-starved regime



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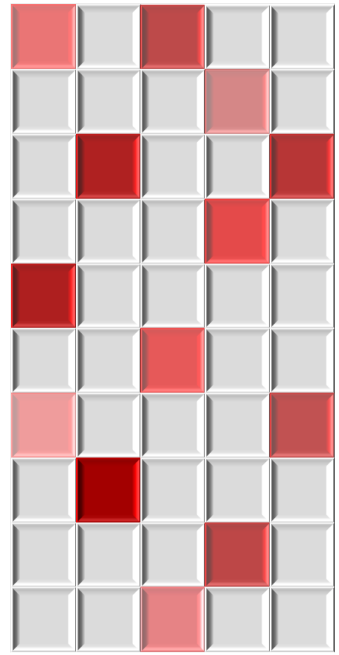

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- can't recover P faithfully if sample size $\ll |\mathcal{S}|^2|\mathcal{A}|$!

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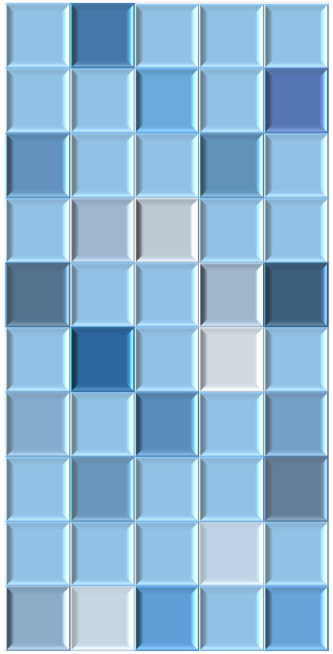

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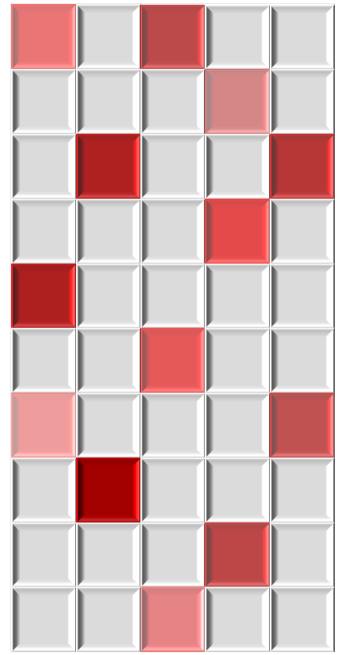

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Can we trust our policy estimate when reliable model estimation is infeasible?

Main result

Theorem (Li, Wei, Chi, Gu, Chen '20)

For every $0 < \varepsilon \leq \frac{1}{1-\gamma}$, *policy* $\widehat{\pi}^{\star}_{\mathrm{p}}$ *of perturbed empirical MDP achieves*

$$\|V^{\widehat{\pi}^{\star}_{\mathrm{p}}} - V^{\star}\|_{\infty} \leq \varepsilon \quad \textit{and} \quad \|Q^{\widehat{\pi}^{\star}_{\mathrm{p}}} - Q^{\star}\|_{\infty} \leq \gamma\varepsilon$$

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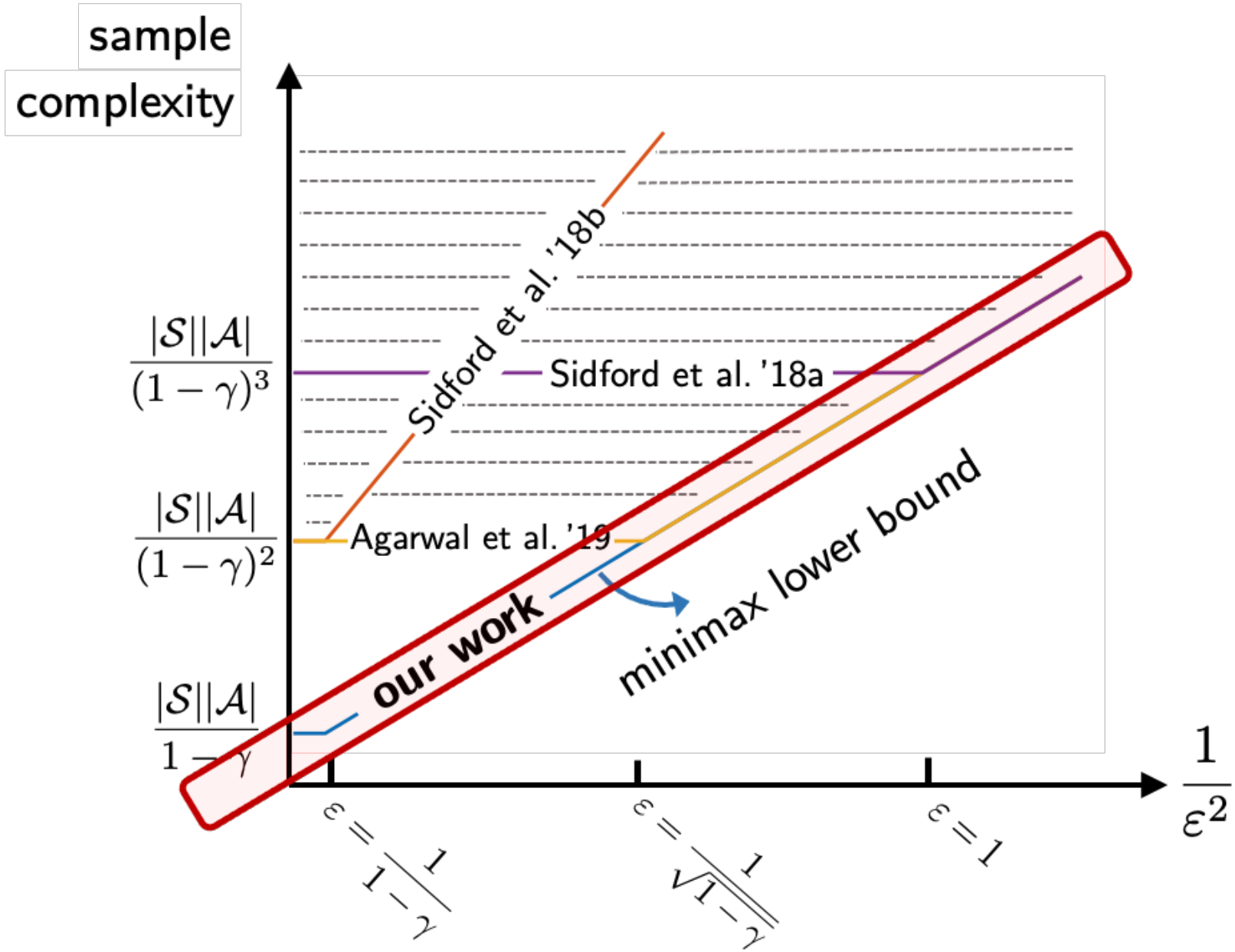
- $\widehat{\pi}^{\star}_{\mathrm{p}}$: obtained by empirical QVI or PI within $\widetilde{O}\big(\frac{1}{1-\gamma}\big)$ iterations
- minimax lower bound: $\widetilde{\Omega}\big(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^3\varepsilon^2}\big)$ [Azar et al., 2013]


sample
complexity
$\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^3}$
$\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^2}$
$\frac{|\mathcal{S}||\mathcal{A}|}{1-\gamma}$
Sidford et al. '18b
Sidford et al. '18a
Agarwal et al. '19
our work
minimax lower bound
$\varepsilon = \frac{1}{1-\gamma}$
$\varepsilon = \frac{1}{\sqrt{1-\gamma}}$
$\varepsilon = 1$
$\frac{1}{\varepsilon^2}$

A sketch of the main proof ingredients

Notation and Bellman equation

- V^π: true value function under policy π
  - Bellman equation: $V = (I - \gamma P_\pi)^{-1} r$ [Sutton and Barto, 2018]

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- $V^\star := V^{\pi^\star}$: optimal values under true models
- $\widehat{V}^\star := \widehat{V}^{\widehat{\pi}^\star}$: optimal values under empirical models

Proof ideas (cont.)

Elementary decomposition:

$$V^\star - V^{\widehat{\pi}^\star} = (V^\star - \widehat{V}^{\pi^\star}) + (\widehat{V}^{\pi^\star} - \widehat{V}^{\widehat{\pi}^\star}) + (\widehat{V}^{\widehat{\pi}^\star} - V^{\widehat{\pi}^\star})$$

Proof ideas (cont.)

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Proof ideas (cont.)

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- **Step 2:** control $\widehat{V}^{\widehat{\pi}^\star} - V^{\widehat{\pi}^\star}$
  (**decouple statistical dependence**)

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Bernstein's inequality: $\big|\big(\widehat{P}_\pi - P_\pi\big) V^\pi\big| \le \sqrt{\frac{\mathrm{Var}[V^\pi]}{N}} + \frac{\|V^\pi\|_\infty}{N}$

Byproduct: policy evaluation

Theorem (Li, Wei, Chi, Gu, Chen'20)

Fix any policy π. For every $0 < \varepsilon \leq \frac{1}{1-\gamma}$, plug-in estimator $\widehat{V}^{\pi}$ obeys

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- tackle sample size barrier: prior work requires sample size $> \frac{|\mathcal{S}|}{(1-\gamma)^2}$ [Agarwal et al., 2019, Pananjady and Wainwright, 2019, Khamaru et al., 2020]

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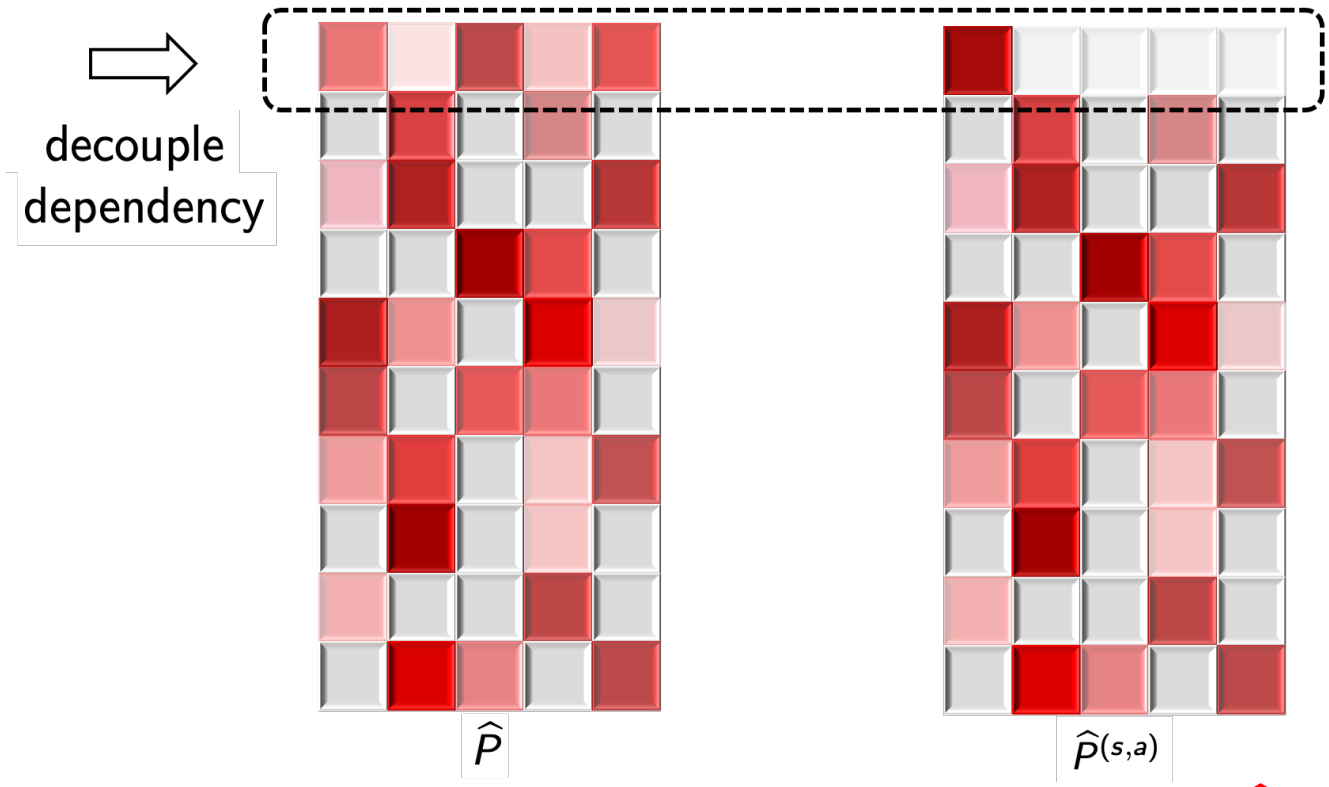
key idea 2: a leave-one-out argument to decouple stat. dependency btw $\widehat{\pi}$ and samples

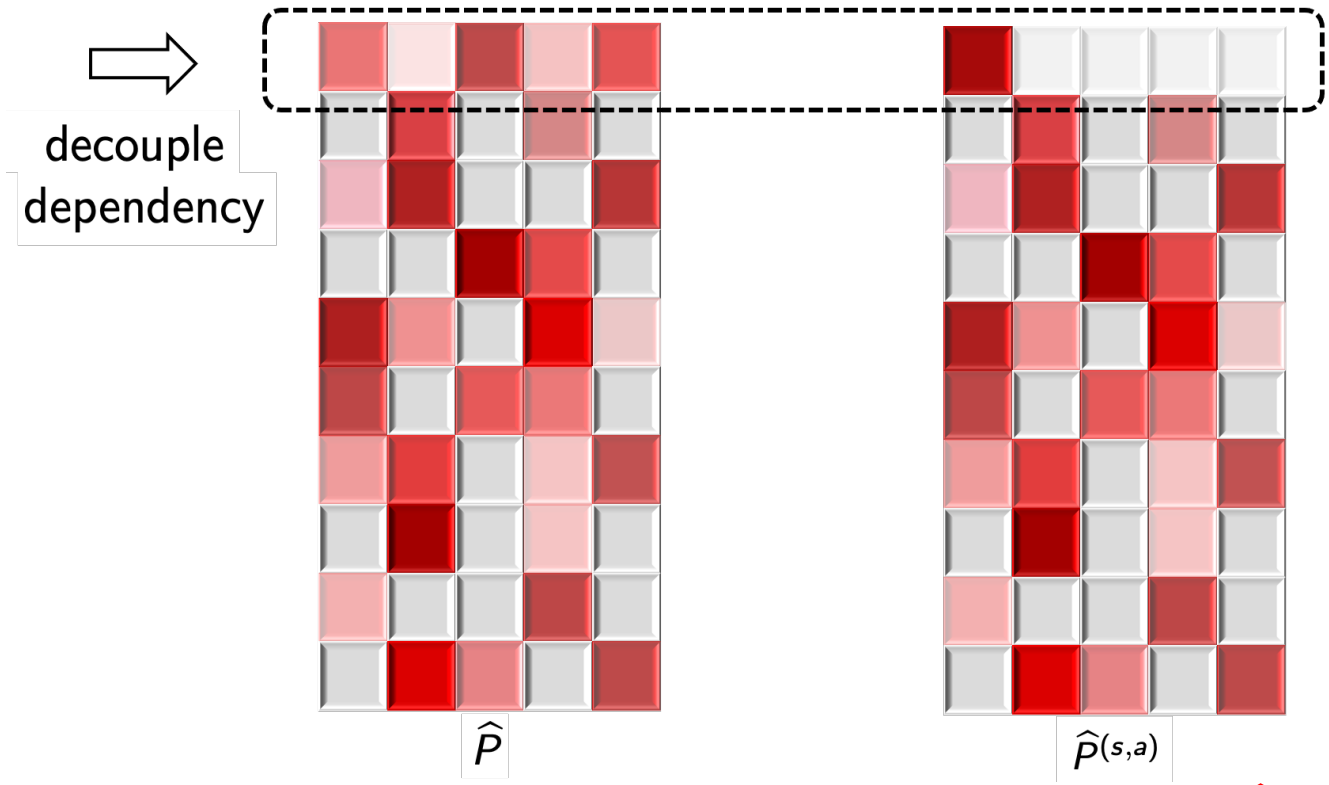
— inspired by [Agarwal et al., 2019] but quite different . . .

Key idea 2: leave-one-out argument



- state-action absorbing MDP for each (s, a): $(\mathcal{S}, \mathcal{A}, \widehat{P}^{(s,a)}, r, \gamma)$

Key idea 2: leave-one-out argument



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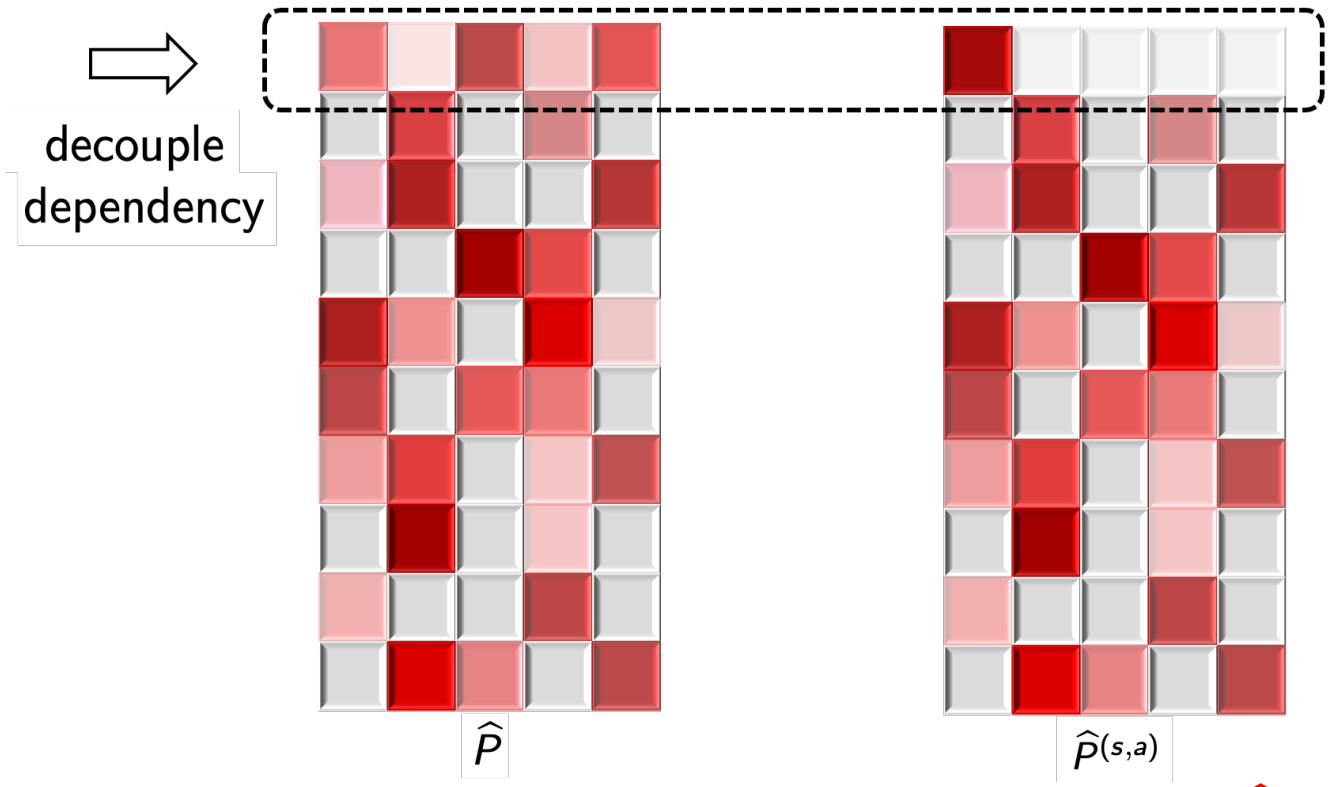

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Caveat: require $\widehat{\pi}^\star$ to stand out from other policies

Key idea 3: tie-breaking via perturbation

- How to ensure the optimal policy stand out from other policies?

$$\forall s \in \mathcal{S}, \quad \widehat{Q}^{\star}(s, \widehat{\pi}^{\star}(s)) - \max_{a : a \neq \widehat{\pi}^{\star}(s)} \widehat{Q}^{\star}(s, a) \geq \omega$$

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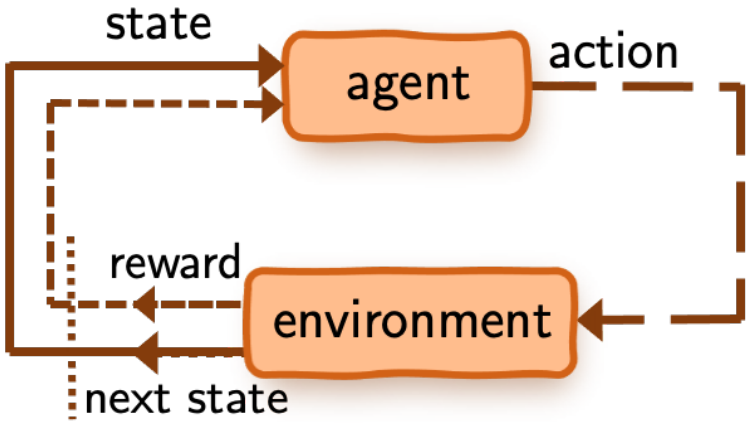
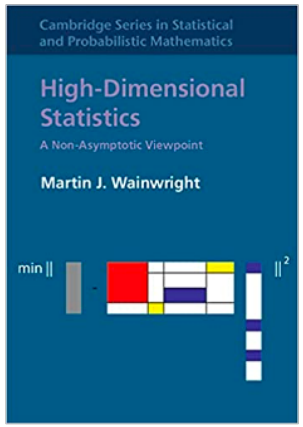
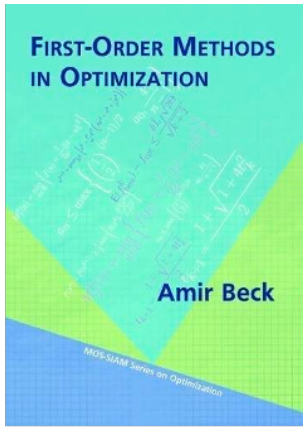
- **Solution:** slightly perturb rewards $r \implies \widehat{\pi}^{\star}_{\mathrm{p}}$
  - ensures the uniqueness of $\widehat{\pi}^{\star}_{\mathrm{p}}$
  - $V^{\widehat{\pi}^{\star}_{\mathrm{p}}} \approx V^{\widehat{\pi}^{\star}}$

Concluding remarks

Understanding RL requires modern statistics and optimization







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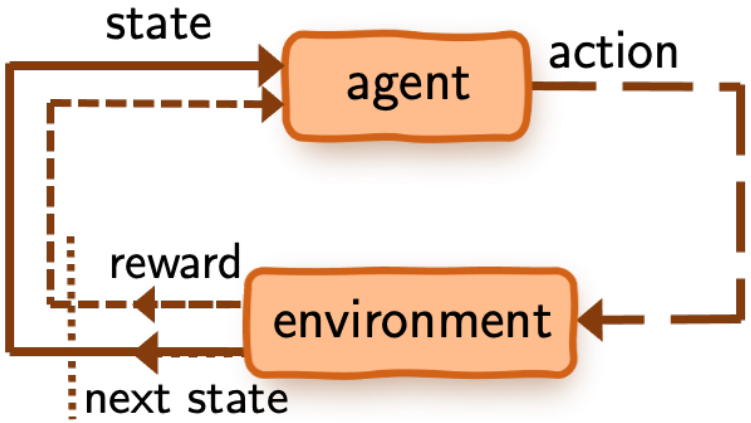

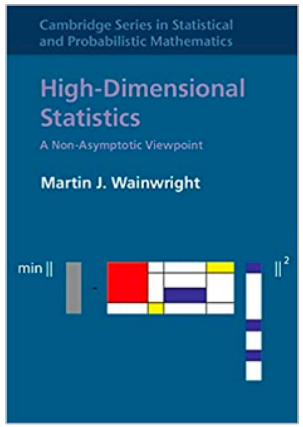

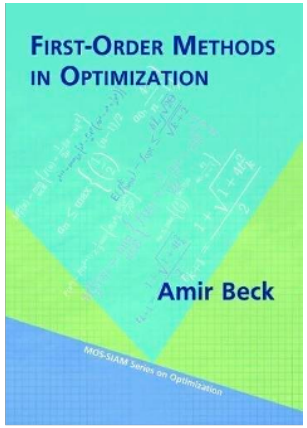

Future directions

- beyond the tabular setting

  [Feng et al., 2020, Jin et al., 2019, Duan and Wang, 2020]

- finite-horizon episodic MDPs

  [Dann and Brunskill, 2015, Jiang and Agarwal, 2018, Wang et al., 2020]

Paper:

"Breaking the sample size barrier in model-based reinforcement learning with a generative model," G. Li, Y. Wei, Y. Chi, Y. Gu, Y. Chen, arxiv:2005.12900, 2020