Multivariate Distributions **Notes 05**

Associated Reading: Wackerly 7, Chapter 5, Sections 1-4

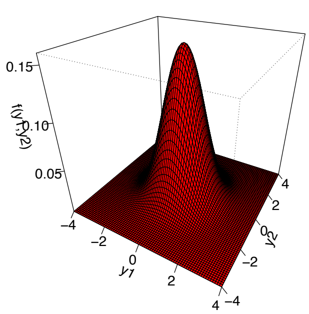
Up to now, we've concentrated on *univariate* probability distributions, i.e., distributions defined along any one axis (usually denoted y) in Cartesian space. So now we shift gears to *multivariate* probability distributions that are defined along n axes, which I and the book will denote as $y_1, y_2, y_3, \ldots$

First, a motivating example:

We conduct a survey of Penn students. Collect their weights, height ages. The data are sampled from a tri-variate population and could be modeled with tri-variate funtion.

The properties of multivariate probability distributions are, as you might guess, simple generalizations of the properties of univariate distributions:

	DISCRETE	**CONTINUOUS**
Name	Joint pmf	Joint pdf.
Symbol	$p(y_1, y_2, y_3 \cdots, y_k)$ $= P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k)$	$f(y_1, y_2 \cdots y_k)$. $P(a_1 \le Y_1 \le b_1, a_2 \le Y_2 \le b_2 \cdots a_k \le Y_k \le b_k)$ $= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} f(y_1 \cdots y_k)\, dy_k \cdots dy_1.$
Fundamental Properties (Theorems 5.1-5.3)	$0 \le p(y_1, y_2 \cdots y_k) \le 1$ $\sum_{(y_1 \cdots y_k)} p(y_1, y_2 \cdots, y_k) = 1$ all possible combinations	$f(y_1, y_2 \cdots y_k) \ge 0$ $\int_{y_1 \cdots y_k} f(y_1 \cdots y_k)\, dy_1 \cdots dy_k = 1$



CDF:

$F(y_1, y_2 \cdots y_k) = P(Y_1 \le y_1, \cdots Y_k \le y_k)$.

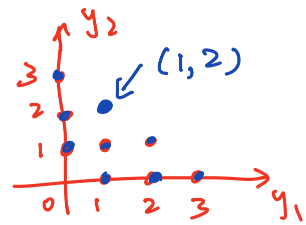
(comma $\Rightarrow$ $\cap$ or and).

→ **EXAMPLE.** Wackerly 7, Exercise 5.3

5.3 Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint probability function of Y_1 and Y_2.

Y_1 : # married executives selected $0 \le Y_1 \le 3$

Y_2 : # never married excutives selected. $0 \le Y_2 \le 3$



find joint pmf (Y_1, Y_2)

$0 \le Y_1 \le 3$

$0 \le Y_2 \le 3$

$1 \le Y_1 + Y_2 \le 3$

$$P(\underbrace{Y_1 = y_1}_{\text{out of 4}}, \underbrace{Y_2 = y_2}_{\text{out of 3}}) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}$$

$$\boxed{\begin{array}{l} 0 \le y_1 \le 3 \\ 0 \le y_2 \le 3 \\ 1 \le y_1 + y_2 \le 3 \end{array}}$$

⇒ **A Short Review of Double Integration**

Sample problem.

joint pdf.

$$f(y_1, y_2) = \begin{cases} c & \boxed{\begin{array}{l} 0 \le y_1 \le 2 - 2y_2 \\ 0 \le y_2 \le 1 \end{array}} \\ 0 & \text{otherwise.} \end{cases}$$

Question: How to find c?

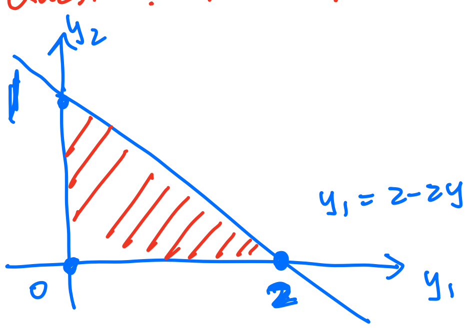


figure out the value of c?

$$\underbrace{\int_{\text{shaded area}} f(y_1, y_2)\, dy_1\, dy_2}_{} = 1.$$

$$c \times \underbrace{\text{area of shaded region}}_{1} = 1. \Rightarrow c = 1$$

Steps.

① draw the picture of support.

② if f is constant on its support. then

$$\int f = \text{constant} \times \text{area of support}.$$

⇒ constant.

③ if f is not constant.

⇒ double integration.

we need to reduce the prob. to one way integration.

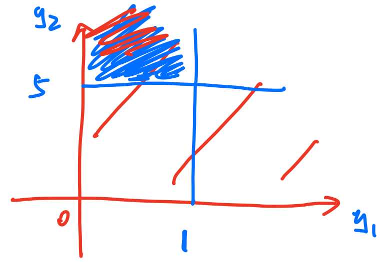
→ **EXAMPLE.** Wackerly 7, Exercise 5.7

5.7 Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a What is $P(Y_1 < 1, Y_2 > 5)$?

b What is $P(Y_1 + Y_2 < 3)$?

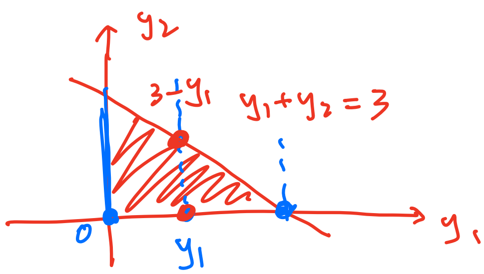


a) $P(Y_1 < 1, Y_2 > 5)$

$$= \int_0^1 \int_5^{+\infty} e^{-(y_1+y_2)}\, dy_2\, dy_1 = \int_0^1 e^{-y_1} \cdot \int_5^{\infty} e^{-y_2}\, dy_2\, dy_1$$

$$= \int_0^1 e^{-y_1} \cdot \left(-e^{-y_2} \Big|_5^{\infty} \right) dy_1$$

$$= e^{-5} \cdot \int_0^1 e^{-y_1}\, dy_1 = \boxed{e^{-5}(1 - e^{-1})}$$



b) $P(Y_1 + Y_2 < 3) = \int_{\text{triangle}} e^{-(y_1+y_2)}\, dy_1\, dy_2.$

$$= \int_0^3 \int_0^{3-y_1} e^{-(y_1+y_2)}\, dy_2\, dy_1$$

⚠ range.

$$= \int_0^3 e^{-y_1} \cdot \boxed{\int_0^{3-y_1} e^{-y_2}\, dy_2}\, dy_1$$

Check offline.

$\Rightarrow = 1 - 4e^{-3}$

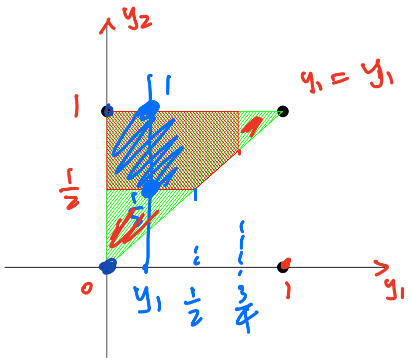
→ **EXAMPLE.** Wackerly 7, Exercise 5.9

5.9 Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} k(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of k that makes this a probability density function.

b Find $P(Y_1 \le 3/4, Y_2 \ge 1/2)$.



a). $\int f(y_1, y_2)\, dy_1\, dy_2 = 1.$

gree shaded area

$$1 = \int_0^1 \int_{y_1}^1 f(y_1, y_2)\, dy_2\, dy_1$$

$$= \int_0^1 \int_{y_1}^1 k(1-y_2)\, dy_2\, dy_1$$

$$= k\int_0^1 \left((1-y_1) - \frac{y_2^2}{2}\Big|_{y_1}^1 \right) dy_1$$

$$= k \cdot \int_0^1 (1-y_1) - \frac{1}{2} + \frac{y_1^2}{2}\, dy_1$$

$$\Rightarrow \quad k = 6$$

$$1 = \int_0^1 \int_0^{y_2} \boxed{k(1-y_2)}\, dy_1\, dy_2$$

$$1 = \int_0^1 k\,(1-y_2)\cdot y_2\, dy_2$$

Beta($\alpha = 2$, $\beta = 2$)

$$k = \frac{1}{B(2,2)} = \frac{1}{\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}} = 6$$

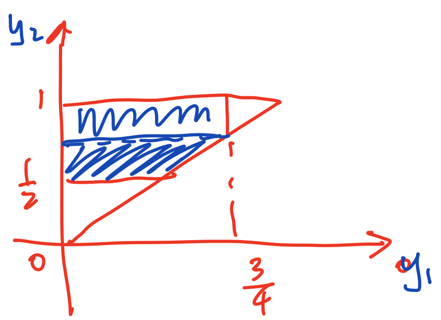
b). $P(Y_1 \le \frac{3}{4}, Y_2 \ge \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 6(1-y_2)\, dy_2\, dy_1$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \int_{y_1}^1 6(1-y_2)\, dy_2\, dy_2$$

$$= \frac{31}{64}$$

Check this.

⚠ decomposition motived by the shape of area of interest



$\Longrightarrow$ integration wrt y_1 first

& then wrt y_2 (exercise).

So we have seen that multivariate probability distributions are just like univariate ones, except that working with them is more mathematically complex. This leads to the natural question: *are there new concepts in probability that can only be illustrated using multivariate distributions*? In other words, why is Chapter 5 so long if all we are doing is generalizing the theorems and definitions of Chapters 3 and 4? Indeed, there are many new concepts that will be discussed over the remaining parts of the chapter:

- marginal probability distributions;
- conditional probability distributions;
- independence of random variables;
- assessing covariance of random variables; and
- conditional expectations.

Note 5 (first three items); Note 6. (last two items)

$f(y_1 y_2 \cdots y_k) \Rightarrow f(y_1)$

$P(Y_2 | Y_1 = y_1)$

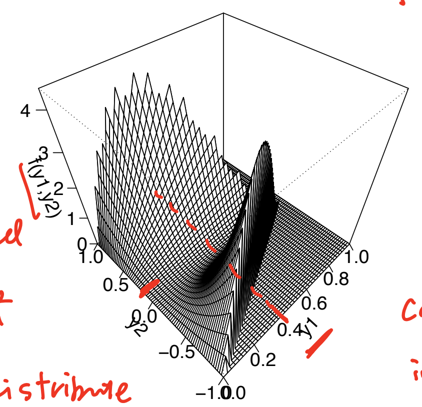
We'll deal with the first three in this notes set, and with the others in the next notes set.

Illustration of Marginal and Conditional Distributions

(Y_1, Y_2)

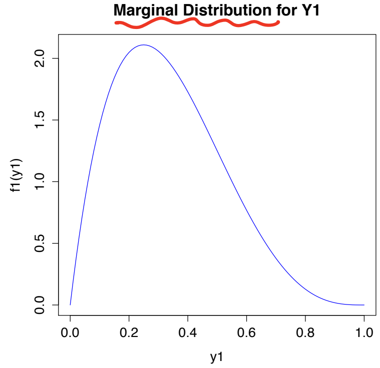
$f(y_1, y_2)$ $\quad y_1 \in [0,1]$, $\quad y_2 \in [0,1]$

Suppose we have multivariate r.v.s $Y_1\ Y_2 \cdots Y_n$ but we are interested in only a subset of them / how they distribute

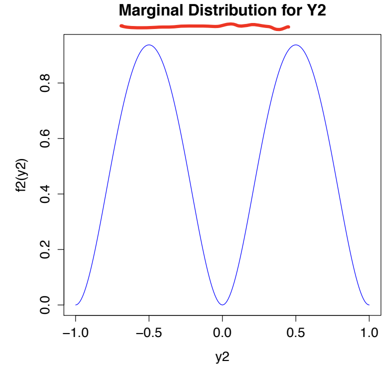


Care about distribution of y_1: integrate over y_2 to get marginal dist of y_1

Marginal Distribution for Y1



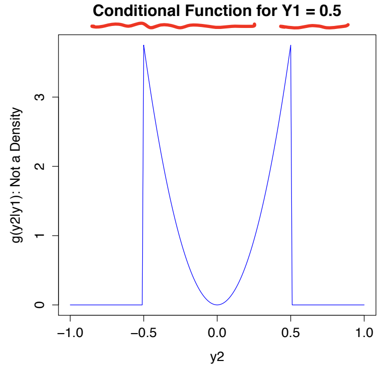
Marginal Distribution for Y2



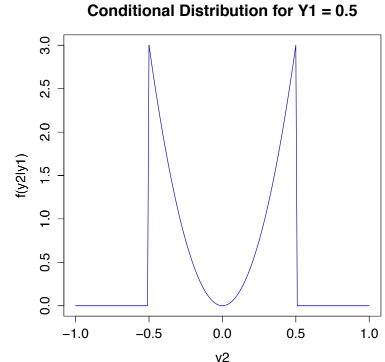
marginal dist of Y_1 → $\boxed{f(y_1)} = \int_{\text{domain } y_2 : D_2} f(y_1, y_2)\, dy_2$

$f(y_2) = \int_{D_1} f(y_1, y_2)\, dy_1$

Conditional Function for Y1 = 0.5



Conditional Distribution for Y1 = 0.5



Conditional function: $f(y_1 = 0.5, y_2)$

↓

integration of y_2 is not necessarily equal to 1. normalize it → to a proper pdf.

Conditional dist of Y_2 given y_1. → $\boxed{f(y_2 | y_1)} = \dfrac{f(y_1, y_2)}{f(y_1)} = \dfrac{f(y_1, y_2)}{\int f(y_1, y_2)\, dy_2}$

marginal of y_1

easy check: $\int f(y_2 | y_1)\, dy_2$

$=1$

Here we'll define the marginal probability mass function and marginal probability density function for discrete and continuous r.v.'s (assuming bivariate distributions):

pmf ... pdf.

discrete. joint pmf

Y_1 $\quad p_1(y_1) = \sum_{y_2 \in D_2} p(y_1, y_2)$

Y_2 $\quad p_2(y_2) = \sum_{y_1 \in D_1} p(y_1, y_2)$

continuous. joint pdf

$f_1(y_1) = \int_{D_2} f(y_1, y_2)\, dy_2$

$f_2(y_2) = \int_{D_1} f(y_1, y_2)\, dy_1$

→ **EXAMPLE.** Wackerly 7, Exercise 5.19

5.19 In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

y_2 \ y_1	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

a Find the marginal probability distribution of Y_1.

b According to results in Chapter 4, Y_1 has a binomial distribution with $n = 2$ and $p = 1/3$. Is there any conflict between this result and the answer you provided in part (a)?

a). to compute $p_1(y_1)$.

$p_1(y_1) = \sum_{y_2} p(y_1, y_2)$

y_1	$p_1(y_1)$
0	4/9
1	4/9
2	1/9

⇒ marginal dist of Y_1.

b) $Y_1 \sim Bin(n=2, p=\frac{1}{3})$.

$P(Y_1=0) = \binom{2}{0}(\frac{1}{3})^0(1-\frac{1}{3})^2 = \frac{4}{9}$

$P(Y_1=1) = \binom{2}{1}(\frac{1}{3})\cdot(1-\frac{1}{3}) = \frac{4}{9}$

$P(Y_1=2)$ ...

pmf for Bin.

⇒ conclude marginal dist in a) ⇒ $Bin(2, \frac{1}{3})$

→ **EXAMPLE.** Wackerly 7, Exercise 5.31(a-b)

5.31 In Exercise 5.13, the joint density function of Y_1 and Y_2 is given by

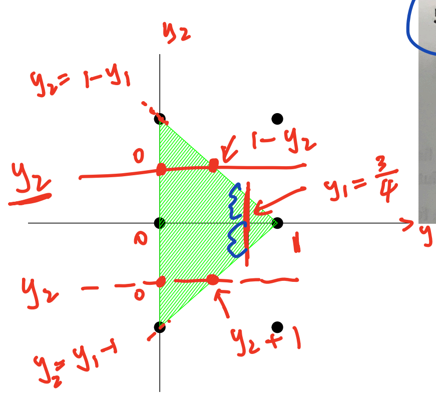
$$f(y_1, y_2) = \begin{cases} 30 y_1 y_2^2, & y_1 - 1 \le y_2 \le 1 - y_1,\ 0 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Show that the marginal density of Y_1 is a beta density with $\alpha = 2$ and $\beta = 4$.

b Derive the marginal density of Y_2.

c Derive the conditional density of Y_2 given $Y_1 = y_1$.

d Find $P(Y_2 > 0 | Y_1 = .75)$.



a) $\boxed{f_1(y_1)} = \int_{D_2} f(y_1, y_2)\, dy_2$

$= \int_{y_1 - 1}^{1 - y_1} 30 y_1 y_2^2\, dy_2$

$= 30 y_1 \int_{y_1-1}^{1-y_1} y_2^2\, dy_2 = \boxed{20 \cdot y_1 (1 - y_1)^3}$ $\quad y_1 \in [0,1]$

Check.

$\overset{?}{\sim}$ Beta $(\alpha = 2, \beta = 4)$.

easy check by comparing pdf.

b) $f_2(y_2) = \int_{D_1} f(y_1, y_2)\, dy_1$

ranges of y_1 for integration are different

① $y_2 \in [0, 1]$ $\quad f_2(y_2) = \int_0^{1-y_2} 30 y_1 y_2^2\, dy_1$

$= \boxed{15 y_2^2 (1 - y_2)^2}$

② $y_2 \in [-1, 0)$ $\quad f_2(y_2) = \int_0^{1+y_2} 30 y_1 y_2^2\, dy_1 = \boxed{15 y_2^2 (y_2 + 1)^2}$.

The conditional pmf's and pdf's for discrete and continuous r.v.'s are given by . . .

discrete r.v.

$\rightarrow p(y_1 \mid y_2) = \dfrac{p(y_1, y_2)}{p_2(y_2)}$

$\rightarrow p(y_2 \mid y_1) = \dfrac{p(y_1, y_2)}{p_1(y_1)}$

continuous r.v.

$f(y_1 \mid y_2) = \dfrac{f(y_1, y_2)}{f_2(y_2)}$

$f(y_2 \mid y_1) = \dfrac{f(y_1, y_2)}{f_1(y_1)}$

. . . while the conditional cdf is given by . . .

$F(y_1 \mid y_2) = P(Y_1 \le y_1 \mid Y_2 = y_2)$

$F(y_2 \mid y_1) = P(Y_2 \le y_2 \mid Y_1 = y_1)$

→ **EXAMPLE.** Wackerly 7, Exercise 5.31(c-d)

5.31 In Exercise 5.13, the joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 30 y_1 y_2^2, & y_1 - 1 \le y_2 \le 1 - y_1,\ 0 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Show that the marginal density of Y_1 is a beta density with $\alpha = 2$ and $\beta = 4$.
b Derive the marginal density of Y_2.
c Derive the conditional density of Y_2 given $Y_1 = y_1$.
d Find $P(Y_2 > 0 \mid Y_1 = .75)$.

conditional c.d.f

c). $f(y_2 \mid y_1) = \boxed{\dfrac{f(y_1, y_2)}{f_1(y_1)}} = \dfrac{30 y_1 y_2^2}{20 y_1 (1-y_1)^3} = \boxed{\dfrac{3 y_2^2}{2(1-y_1)^3}}$ $\begin{cases} 0 \le y_1 < 1 \\ y_1 - 1 \le y_2 \le 1 - y_1 \end{cases}$

⚠ otherwise $f(y_2 \mid y_1) = 0$

d) $\boxed{P(Y_2 > 0 \mid Y_1 = 0.75)} = \displaystyle\int_{y_2 > 0} f(y_2 \mid y_1)\, dy_2.$

$= \displaystyle\int_0^{1-y_1} \frac{3 y_2^2}{2(1-y_1)^3}\, dy_2, \quad (y_1 = 0.75)$

$= \displaystyle\int_0^{\frac{1}{4}} \frac{3 y_2^2}{2 \cdot (1/4)^2}\, dy_2 = \frac{1}{2}.$

The r.v.'s Y_1 and Y_2 are *independent* if and only if:

$\boxed{Y_1 \perp Y_2 \text{ or } Y_1 \perp\!\!\!\perp Y_2}$

	DISCRETE	**CONTINUOUS**
PMF/PDF	$p(y_1, y_2) = p_1(y_1) \cdot p_2(y_2)$	$f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$
CDF	$F(y_1, y_2) = F_1(y_1) \cdot F_2(y_2)$	

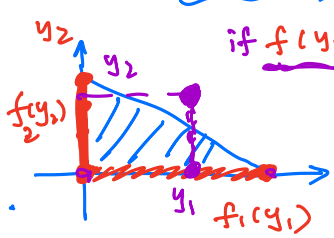
An easier way (sometimes) to answer the question "Are Y_1 and Y_2 independent random variables" is given by Theorem 5.5:

$\boxed{\text{Ask:}}$ ① is the region where $f(y_1, y_2) > 0$ ($p(y_1, y_2)$) "rectangular"?

suppost. (Could be infinite / semi-infinite).

if "No" ⇒ not independent.

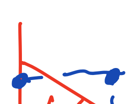
if "yes" ⇒ $\boxed{\text{Ask:}}$ Can I write $f(y_1, y_2)$ ($p(y_1, y_2)$) as $f(y_1) \cdot f(y_2)$.



if $f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$
>0 >0 $\boxed{>0}$

if "yes" ⇒ independent.

if "no" ⇒ not independent.



$p(y_1, y_2) = 0$

$\longrightarrow p(y_1, y_2) = p(y_1) \cdot p(y_2)$

$p(y_1) \cdot p(y_2) \neq 0$

→ **EXAMPLE.** Wackerly 7, Exercise 5.49

5.49 In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week and Y_2, the proportion of the capacity sold during the week, given by

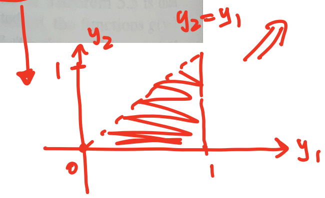
$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that Y_1 and Y_2 are dependent.

region is not "rectangular"

⇓

$Y_1 \not\perp Y_2$.



→ **EXAMPLE.** Wackerly 7, Exercise 5.51(a)

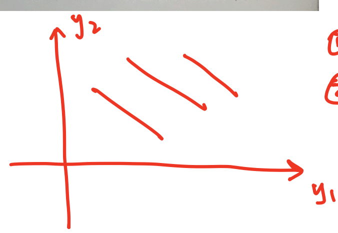
5.51 In Exercise 5.7, we considered Y_1 and Y_2 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Are Y_1 and Y_2 independent?

b Does the result from part (a) explain the results you obtained in Exercise 5.25 (d)–(f)? Why?

a).



① rectangular ✓.

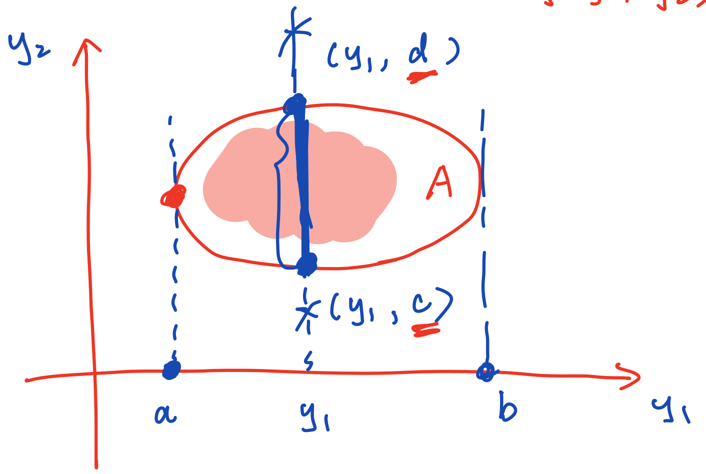
② decompose $f(y_1, y_2) = e^{-(y_1+y_2)}$

$= e^{-y_1} \cdot e^{-y_2}$

$= g(y_1) \cdot h(y_2)$

$\Rightarrow \boxed{Y_1 \perp Y_2.}$

$f(y_1, y_2)$



$\int_A f(y_1, y_2)\, dy_1 dy_2$.

Steps:

① Find the left most & right most values that y_1 can take.

② draw a vertical line. for any value $y_1 \in (a, b)$

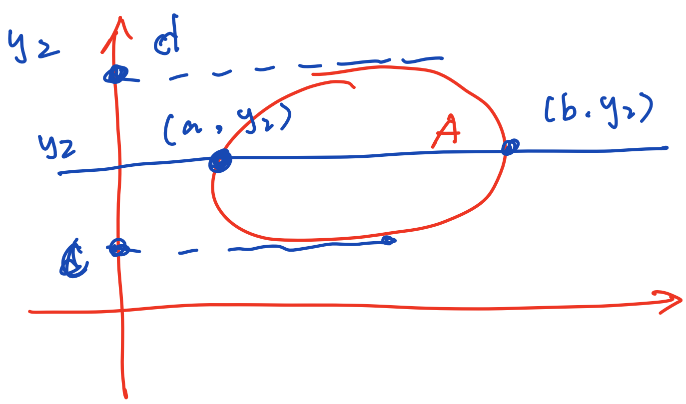
d. c change with y_1.

$\Rightarrow$ functions of y_1.

③ $\int_A f(y_1, y_2)\, dy_1 dy_2$

$$= \int_a^b \int_{c(y_1)}^{d(y_1)} f(y_1, y_2)\, dy_2\, dy_1$$

also do.



$\int_A f(y_1, y_2)\, dy_1 dy_2$

$$= \int_c^d \int_{a(y_2)}^{b(y_2)} f(y_1, y_2)\, dy_1\, dy_2.$$