

A Non-asymptotic Framework for Approximate Message Passing Algorithm



Yuting Wei

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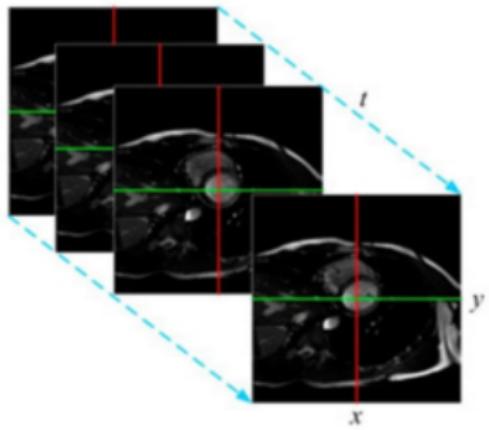
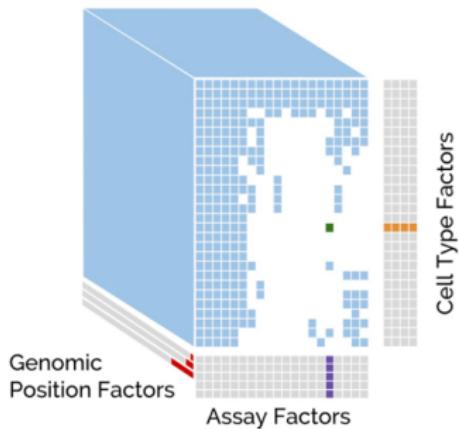
UC Davis, 2022



Gen Li, UPenn Statistics

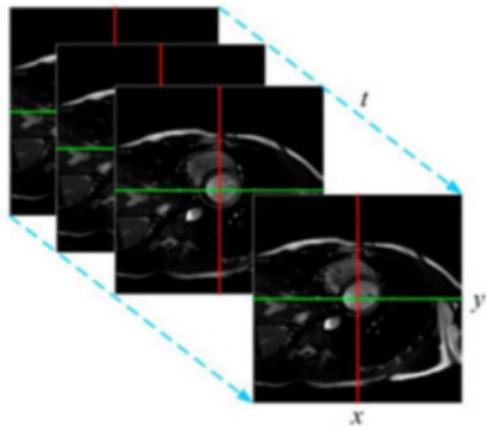
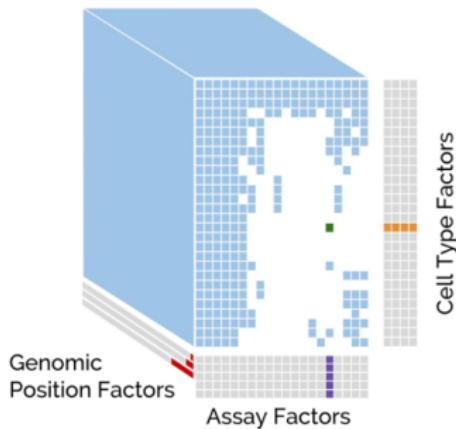
“A Non-Asymptotic Framework for Approximate Message Passing in Spiked Models,” Gen Li, Yuting Wei, *arxiv.2208.03313*

High-dimensional statistical tasks



Statistical tasks: linear regression, generalized linear models, low-rank matrix estimation, phase retrieval, tensor decomposition...

High-dimensional statistical tasks

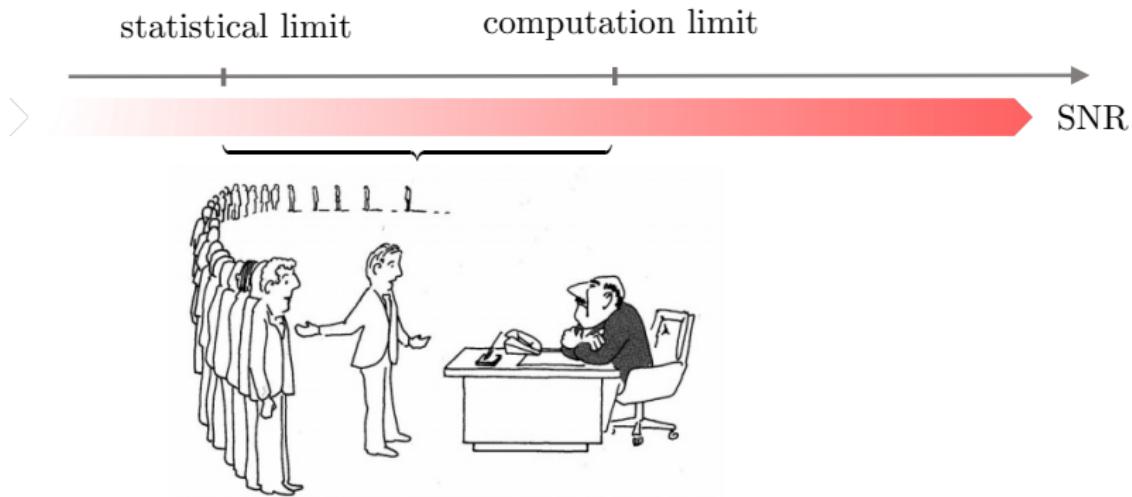


Statistical tasks: linear regression, generalized linear models, low-rank matrix estimation, phase retrieval, tensor decomposition...

When problem sizes are large, computation complexity is an issue!

Statistical accuracy vs. computation complexity

Problems with combinatorial nature (e.g. community detection, planted cliques, sparse principal component analysis, structured matrix models, sparse tensor models...)



"I can't find an efficient algorithm, but neither can all these people."

— see survey [Bandeira, Perry, Wein \(2018\)](#)

Approximate message passing (AMP) algorithm

- AMP is a low-complexity, iterative algorithm

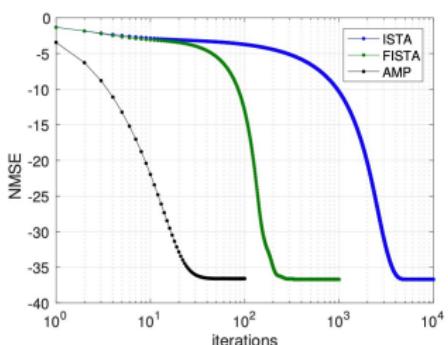
[Donoho, Maleki, Montanari (2009, 2010a, 2011b), Bayati & Montanari (2011)]

Approximate message passing (AMP) algorithm

- AMP is a low-complexity, iterative algorithm
[Donoho, Maleki, Montanari (2009, 2010a, 2011b), Bayati & Montanari (2011)]
- Theoretically optimal vs. computationally feasible estimators
[Reeves, Pfister (2019), Barbier et al. (2017), Lelarge & Miolane (2019), Montanari & Ramji (2019), Celentano & Montanari (2019)]

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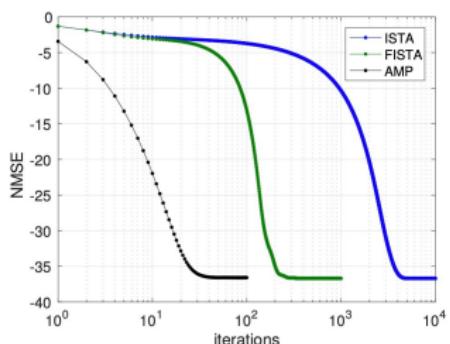
AMP in computing LASSO

Advantages of AMP:

- fast convergence
- asymptotically exact characterization
- easily combine with prior info on signal structure

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AMP in computing LASSO

— *tutorial, Feng, Venkataraman, Rush, Samworth (2022)*

Prior theory of AMP

Exact asymptotics: for constant # iterations t (e.g. $t = 20$), empirical distribution of the coordinates of AMP iterate \mathbf{x}_t is approximately Gaussian ($n \rightarrow \infty$), with variance given by low-dimensional recursion:

$$\text{state evolution: } \tau_{t+1} = F(\tau_t)$$

τ_t captures the variance at iteration t

[Bayati & Montanari (2011), Javanmard & Montanari (2013), Schniter & Rangan (2014)]

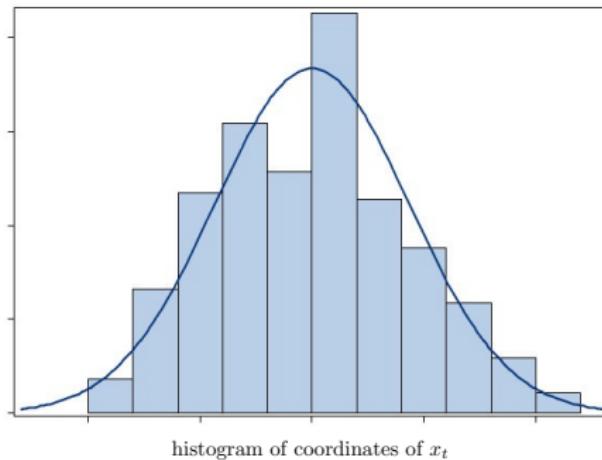
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Another benefit: AMP as a tool to analyze statistical procedures

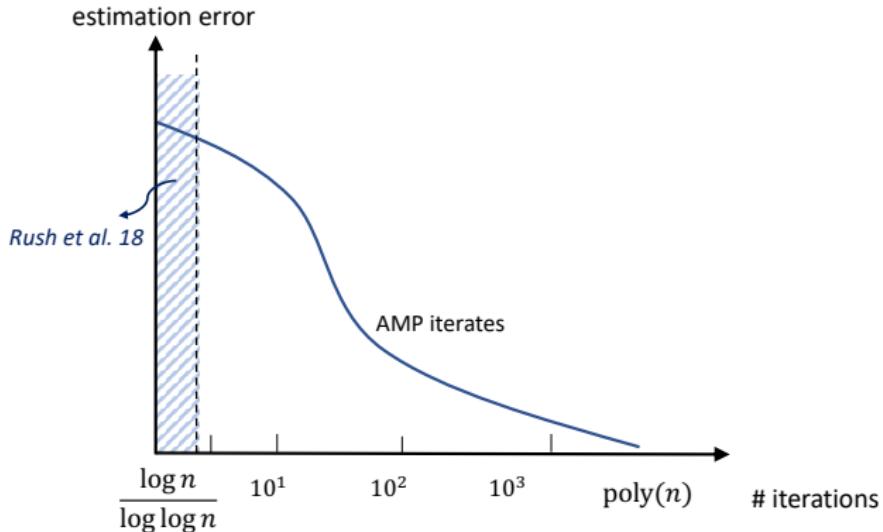
[Donoho, Maleki, Montanari (2009), Donoho & Montanari (2016), Sur, Chen, Candès. (2017), Bu et al. (2020), Fan & Wu (2021), Li & Wei (2021)...]

Non-asymptotic analyses are quite limited so...

- compare to other optimization methods
- compare to other analysis techniques



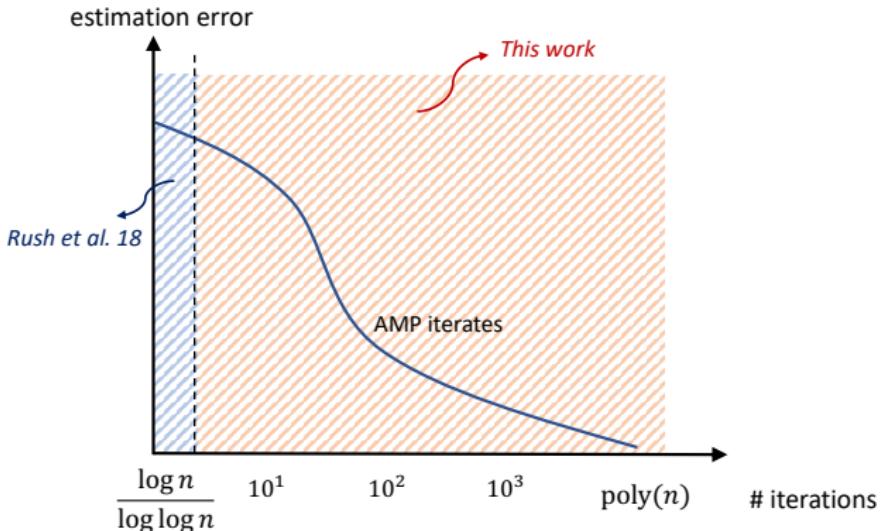
Non-asymptotic analysis?



Non-asymptotic result: Rush & Venkataraman (2018)

#iterations = $o(\log n / \log \log n)$ (*based on state-evolution analysis*)

Non-asymptotic analysis?



Question: Is it possible to develop non-asymptotic analysis of AMP beyond $o(\log n / \log \log n)$ iterations?

AMP for signal recovery in spiked models

Spiked Wigner model

$$M = \lambda v^* + W$$

Diagram illustrating the Spiked Wigner model:

The matrix M is represented as the sum of three components:

- A scalar multiple of the vector v^* , represented by a vertical column of red and light red blocks.
- A transpose of the vector v^* , represented by a horizontal row of red and light red blocks.
- A Wigner matrix W , represented by a grid of blue and light blue blocks.

The equation $M = \lambda v^* v^{*\top} + W$ summarizes this decomposition.

Johnstone (2001),

Spiked Wigner model

$$M = \lambda v^{\star\top} + W$$

The diagram illustrates the Spiked Wigner model. It shows a matrix M as the sum of three components: a scalar multiple of a vector v^\star , its transpose $v^{\star\top}$, and a random matrix W .
- λ is a scalar value.
- v^\star is a vertical vector with red and light red segments.
- $v^{\star\top}$ is a horizontal vector with red and light red segments.
- W is a 5x5 matrix with a central blue block and smaller blue blocks at the corners.

- $W_{ij} = W_{ji} \sim \mathcal{N}(0, \frac{1}{n})$ and $W_{ii} \sim \mathcal{N}(0, \frac{2}{n})$

Johnstone (2001),

Spiked Wigner model

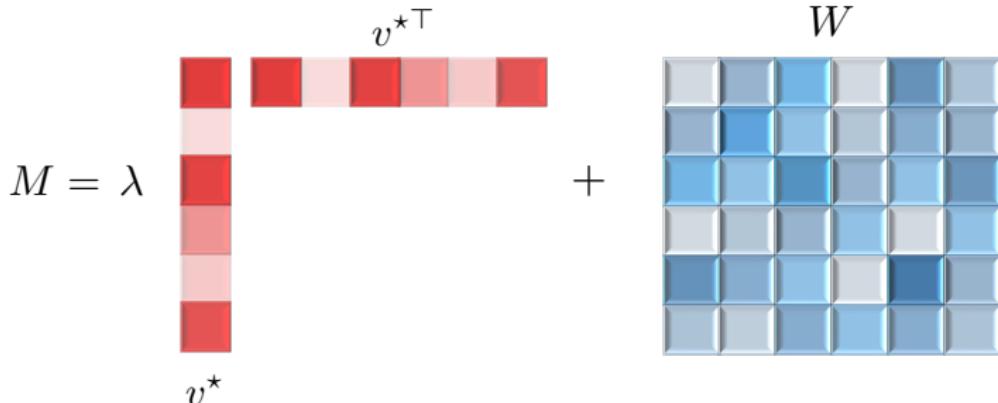
$$M = \lambda v^{\star\top} + W$$

The diagram illustrates the Spiked Wigner model. On the left, a vertical vector v^{\star} is shown as a stack of colored blocks: red at the top and bottom, and light pink in the middle. Above it, its transpose $v^{\star\top}$ is shown as a horizontal row of colored blocks: red, light pink, light pink, red, light pink, red. To the right of the sum symbol (+) is a large square matrix W , which is mostly blue with a sparse pattern of white and darker blue blocks.

- $W_{ij} = W_{ji} \sim \mathcal{N}(0, \frac{1}{n})$ and $W_{ii} \sim \mathcal{N}(0, \frac{2}{n})$
- $\lambda = \text{SNR}$ (signal-to-noise ratio) with $\|v^{\star}\|_2 = 1$
- **Goal:** estimate v^{\star} from M

Johnstone (2001),

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- $W_{ij} = W_{ji} \sim \mathcal{N}(0, \frac{1}{n})$ and $W_{ii} \sim \mathcal{N}(0, \frac{2}{n})$
- $\lambda = \text{SNR}$ (signal-to-noise ratio) with $\|v^*\|_2 = 1$
- **Goal:** estimate v^* from M
- **Phase transition at $\lambda > 1$:** the top eigenvalue separates from bulk, eigenvector correlates non-trivially with v^*

Johnstone (2001), Johnstone & Lu (2004), Péché (2006), Baik & Silverstein (2006), Capitaine, Donati-Martin & Féral (2009), Féral & Péché (2007)...

Spiked Wigner model with structures

$$M = \lambda v^{\star\top} + W$$

The diagram illustrates the Spiked Wigner model with structures. It shows the decomposition of a matrix M into three components separated by plus signs ($+$).
1. On the left, a vertical vector v^{\star} is shown, consisting of several colored segments: light pink at the top, followed by red, then dark red, and finally black at the bottom.
2. In the middle, the transpose of v^{\star} is shown as a horizontal row vector $v^{\star\top}$, which has the same color segments as v^{\star} .
3. On the right, a square matrix W is shown, composed of a grid of colored squares. The colors form a pattern where the center square is blue, surrounded by white, then blue, then white, and so on, creating a checkerboard-like appearance with a central spike.

Spiked Wigner model with structures

$$M = \lambda v^{\star\top} + W$$

The diagram illustrates the Spiked Wigner model. It shows a matrix M represented by a vertical vector v^{\star} (consisting of red and light red segments) multiplied by its transpose $v^{\star\top}$ (consisting of light red and red segments), plus a matrix W represented by a grid of blue and white squares.

Applications: spin-glass problems, community detection, image alignment, angular synchronization

Spiked Wigner model with structures

$$M = \lambda v^* \mathbf{v}^{*\top} + W$$

The diagram illustrates the Spiked Wigner model. On the left, a vertical vector v^* is shown as a stack of colored blocks, transitioning from light red at the top to dark red at the bottom. Above it, the expression $M = \lambda$ is written. To the right of the plus sign (+) is a horizontal vector $v^{*\top}$, also composed of colored blocks, transitioning from light red to dark red. To the right of the plus sign is a large square matrix W , which is mostly light blue with a distinct checkerboard pattern of darker blue and white squares.

Applications: spin-glass problems, community detection, image alignment, angular synchronization

- \mathbb{Z}_2 synchronization: $\sqrt{n}v_i^* \stackrel{\text{i.i.d.}}{\sim} \text{Unif}\{+1, -1\}$

Singer (2011), Panchenko (2013), Deshpande, Abbe & Montanari (2016), Perry, Wein, Bandeira, Moitra (2018), Javanmard, Montanari & Ricci-Tersenghi (2016)...

Spiked Wigner model with structures

$$M = \lambda v^{\star T} + W$$

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Applications: spin-glass problems, community detection, image alignment, angular synchronization

- \mathbb{Z}_2 synchronization: $\sqrt{n}v_i^{\star} \stackrel{\text{i.i.d.}}{\sim} \text{Unif}\{+1, -1\}$
- sparse Wigner model: $\|v^{\star}\|_0 = k$

Singer (2011), Panchenko (2013), Deshpande, Abbe & Montanari (2016), Perry, Wein, Bandeira, Moitra (2018), Javanmard, Montanari & Ricci-Tersenghi (2016)...

Spiked Wigner model with structures

$$M = \lambda v^* \mathbf{v}^{*\top} + W$$

The diagram illustrates the Spiked Wigner model. On the left, a vertical vector v^* is shown with red segments of increasing length from top to bottom. To its right is a plus sign. Above the plus sign is the transpose symbol $\mathbf{v}^{*\top}$. To the right of the plus sign is a matrix W , which is a 10x10 grid of blue and white squares. The first column of W has a darker blue shade than the other columns.

Applications: spin-glass problems, community detection, image alignment, angular synchronization

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- sparse Wigner model: $\|v^*\|_0 = k$
- non-negative Wigner model: $v_i^* \geq 0$

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Applications: spin-glass problems, community detection, image alignment, angular synchronization

- \mathbb{Z}_2 synchronization: $\sqrt{n}v_i^* \stackrel{\text{i.i.d.}}{\sim} \text{Unif}\{+1, -1\}$
- sparse Wigner model: $\|v^*\|_0 = k$
- non-negative Wigner model: $v_i^* \geq 0$
- cone-constrained spiked models: $v^* \in \mathcal{K}$ (e.g. monotone, convex)

Singer (2011), Panchenko (2013), Deshpande, Abbe & Montanari (2016), Perry, Wein, Bandeira, Moitra (2018), Javanmard, Montanari & Ricci-Tersenghi (2016)...

An incomplete list of prior art

\mathbb{Z}_2 synchronization:

- Panchenko'13
- Baik, Arous, Péché'05
- Javanmard et al.'16
- Montanari & Sen'16
- Lelarge & Miolane'19
- Deshpande, Abbe, Montanari'17
- Celentano, Fan, Mei'21

general convex cones:

- Deshpande, Montanari, Richard'14
- Lesieur, Krzakala, Zdeborová'17
- Bandeira, Kunisky, Wein'19

sparse PCA (Wigner / Wishart)

- Johnstone & Lu'09
- d'Aspremont et al.'04
- Vu & Lei'12
- Berthet & Rigollet'13
- Ma'13
- Lesieur, Krzakala, Zdeborová'15
- Deshpande & Montanari'14
- Wang, Berthet, Samworth'16
- Ding, Kunisky, Wein, Bandeira'19

positive Wigner models

- Montanari & Richard'16

Idealistic estimators

Maximum likelihood estimator $\coloneqq \arg \min_{\substack{v \in \mathcal{S}^{n-1} \\ v \text{ with structures}}} \|M - \lambda vv^\top\|_F^2$

Bayes optimal estimator $\coloneqq \mathbb{E}[vv^\top \mid M]$

AMP for spiked models

AMP for spiked models:

$$x_{t+1} = M\eta_t(x_t) - \langle \eta'_t(x_t) \rangle \cdot \eta_{t-1}(x_{t-1}), \text{ for } t \geq 1$$

where $\langle x \rangle := \frac{1}{n} \sum_{i=1}^n x_i$.

- η_t : denoising function selected *a priori* (tailored to structure of v^*)
 - ▶ **\mathbb{Z}_2 synchronization:** $\eta_t(x) = \rho_t \tanh(x)$
 - ▶ **sparse estimation:** $\eta_t(x) = \rho_t \cdot \text{sign}(x)(|x| - \tau_t)_+$
 - ▶ **general cone:** $\eta_t(x) = \rho_t \cdot \text{Proj}_{\mathcal{K}}(x)$

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 - ▶ **general cone:** $\eta_t(x) = \rho_t \cdot \text{Proj}_{\mathcal{K}}(x)$
- effectiveness of AMP — *Onsager correction term*
 $\langle \eta'_t(x_t) \rangle \cdot \eta_{t-1}(x_{t-1})$

Prior results: exact asymptotics

Theorem (Montanari & Venkataraman (2019))

Suppose the empirical distribution $\{v_i^*\}_{i=1}^n \rightarrow \mu_V$ on \mathbb{R} , with $\mathbb{E}[V^2] = 1$. For constant # iterations t (independent of n), it satisfies,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (x_{t,i} - v_i^*)^2 = \mathbb{E} \left[(\alpha_t V + \beta_t G - V)^2 \right], \quad \text{almost surely}$$

where $V \sim \mu_V$ and $G \sim \mathcal{N}(0, 1)$ are independent.

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- State Evolution (SE) via the recursion

$$(\alpha_{t+1}, \beta_{t+1}) = F(\alpha_t, \beta_t) = \begin{cases} \alpha_{t+1} = \lambda \mathbb{E}[V \cdot \eta_t(\alpha_t V + \beta_t G)] \\ \beta_{t+1}^2 = \mathbb{E}[\eta_t^2(\alpha_t V + \beta_t G)] \end{cases}$$

- **Challenges for non-asymptotic guarantees:** deal with statistical dependence between iterations

AMP for spiked models:

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This work: a new decomposition of AMP

Theorem (Li & Wei (2022))

Initialize AMP with x_1 independent of W . For every $1 \leq t \leq n$, AMP yields the decomposition

$$x_{t+1} = \alpha_{t+1} v^* + \sum_{k=1}^t \beta_t^k \phi_k + \xi_t, \quad (*)$$

for $\phi_k \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{n} I_n)$.

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here $(\alpha_{t+1}, \beta_t, \xi_t)$ obeys

$$\alpha_{t+1} = \lambda v^{*\top} \eta_t(x_t),$$

$$\beta_t^k = \langle \eta_t(x_t), z_k \rangle \quad \text{for an explicit-defined basis } \{z_k\}$$

$$\|\xi_t\|_2 = \left\langle \sum_{k=1}^{t-1} \mu^k \phi_k, \delta_t \right\rangle - \langle \delta'_t \rangle \sum_{k=1}^{t-1} \mu^k \beta_{t-1}^k + \Delta_t + O\left(\sqrt{\frac{t \log n}{n}} \|\beta_t\|_2\right) \quad \text{w.h.p.}$$

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- x_t behaves like $\alpha_t v^* + \sum_{k=1}^{t-1} \beta_{t-1}^k \phi_k$ if $\|\xi_{t-1}\|_2$ is small

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- if η_t are nice (*smooth & with finite jumps*), we can track how $\|\xi_t\|_2$ depends on λ, t, n

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- if η_t are nice (*smooth & with finite jumps*), we can track how $\|\xi_t\|_2$ depends on λ, t, n
- decomposition $(*)$ can be extended for spectral initialization

Applications in two examples:
 \mathbb{Z}_2 synchronization & sparse Wigner model

Example 1: \mathbb{Z}_2 Synchronization

- Setting: $M = \lambda v^* v^{*\top} + W$ where $\sqrt{n}v_i^* \sim \text{Unif}(\{\pm 1\})$
- Goal: recover v^* given M

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Deshpande, Abbe, Montanari (2016) characterizes the theoretical limit of this problem

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\| v^* v^{*\top} - \underbrace{\mathbb{E}[vv^\top | M]}_{\text{Bayes estimator}} \|_F^2 \right] = \begin{cases} 1 & \lambda \leq 1; \\ 1 - q^*(\lambda)^2 & \lambda > 1. \end{cases}$$

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- ✓ initialize by spectral methods Montanari & Venkataramanan (2019)

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A hybrid procedure proposed in Celentano, Fan, Mei (2021)

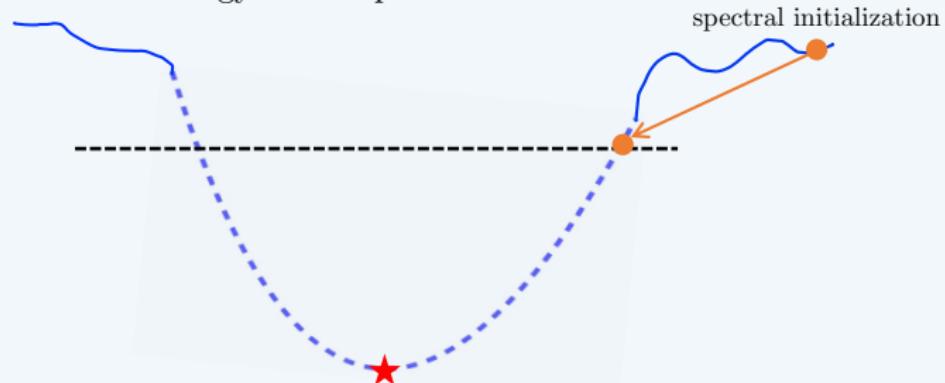
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A hybrid procedure proposed in Celentano, Fan, Mei (2021)

TAP free energy landscape



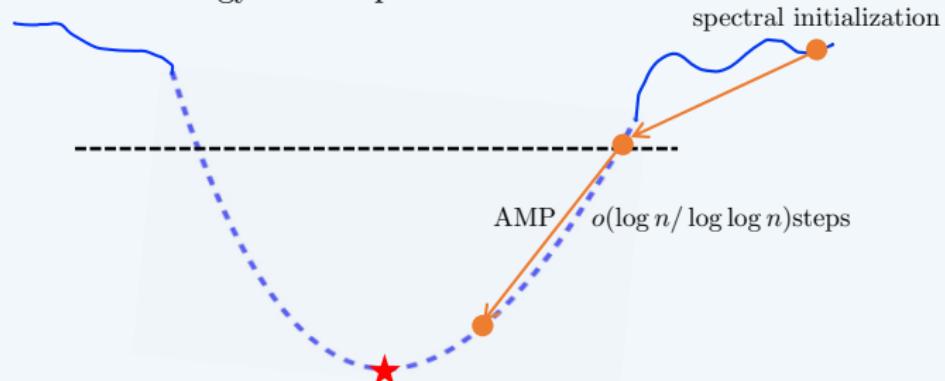
Example 1: \mathbb{Z}_2 Synchronization

- Setting: $M = \lambda v^* v^{*\top} + W$ where $\sqrt{n}v_i^* \sim \text{Unif}(\{\pm 1\})$
- Goal: recover v^* given M

— AMP succeeds in a fixed t , large n limit

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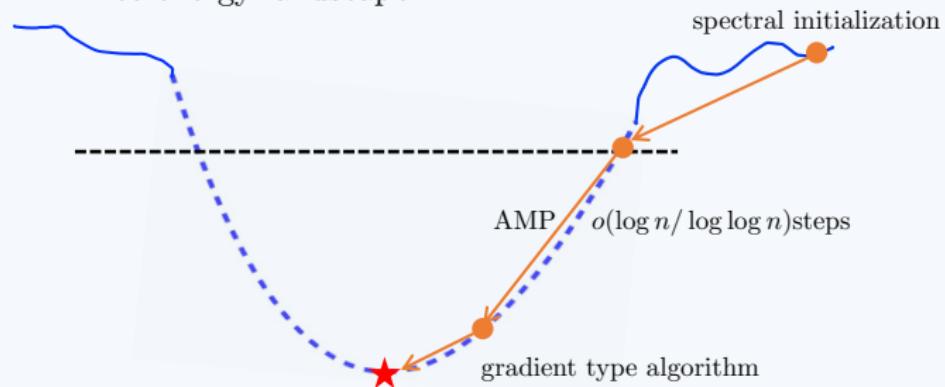
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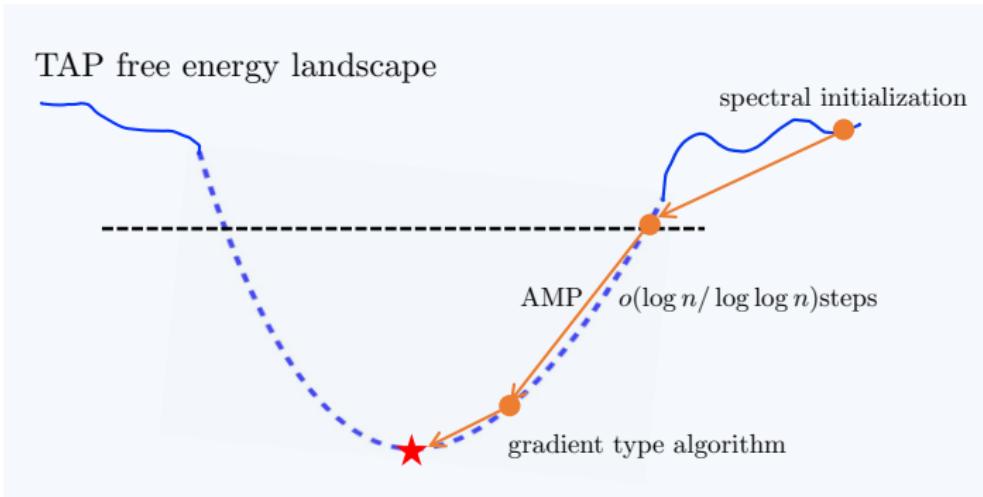


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Open question: spectral-initialized AMP converges for $\lambda > 1$?

\mathbb{Z}_2 Synchronization: our results

Theorem (Li & Wei (2022))

Spectrally initialized AMP satisfies

$$x_{t+1} = \alpha_{t+1} v^* + \sum_{k=1}^t \beta_t^k \phi_k + \xi_t,$$

with

$$\alpha_{t+1} = \mathbb{E} \left[\lambda v^{*\top} \eta_t \left(\alpha_t v^* + \frac{1}{\sqrt{n}} G \right) \right] + O \left(\sqrt{\frac{t \log n}{(\lambda - 1)^3 n}} \right),$$
$$\|\beta_t\|_2 = 1, \quad \|\xi_t\|_2 \lesssim O \left(\sqrt{\frac{t \log n}{(\lambda - 1)^3 n}} + \sqrt{\frac{\log^7 n}{(\lambda - 1)^9 n}} \right)$$

w.h.p. provided that $t \lesssim \frac{(\lambda - 1)^{10}}{\log^7 n} n$.

- denoising functions:

$$\eta_t(x) := \tanh(\pi_t x) / \|\tanh(\pi_t x)\|_2, \quad \text{where } \pi_t^2 = n(\|x_t\|_2^2 - 1)$$

- record (asymptotic) State Evolution:

$$\tau_{t+1} := \lambda^2 \int \tanh(\tau_t + \sqrt{\tau_t}x) \varphi(dx)$$

then

$$\alpha_t^2 - \tau_{t+1} = O\left(\sqrt{\frac{t \log n}{(\lambda - 1)^8 n}} + \sqrt{\frac{\log^7 n}{(\lambda - 1)^{14} n}}\right)$$

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$$\alpha_t^2 - \tau^* = (\lambda^2 - 1)(1 - (\lambda - 1))^t + O\left(\sqrt{\frac{t \log n}{(\lambda - 1)^8 n}} + \sqrt{\frac{\log^7 n}{(\lambda - 1)^{14} n}}\right)$$

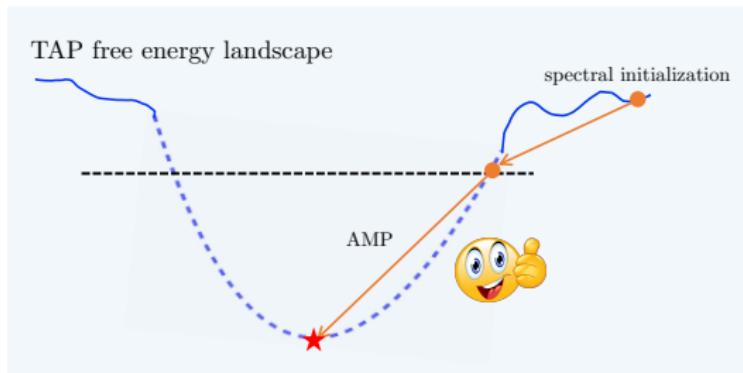
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- answer the open question ([Celentano, Fan & Mei \(2021\)](#)) positively:
spectral-initialized AMP is enough!



\mathbb{Z}_2 Synchronization: simulations

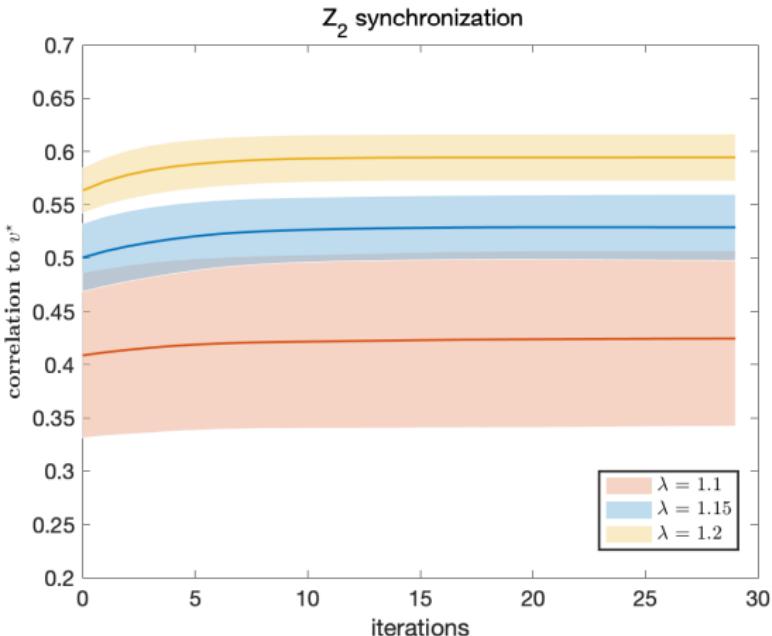


Figure: Convergence of spectrally initialized AMP for different signal strengths with $n = 10000$. Repeat 40 times.

Example 2: sparse PCA

- Setting: $M = \lambda v^* v^{*\top} + W$ where $\|v^*\|_0 = k$
- Goal: recover v^* given M

$$\lambda \approx \sqrt{\frac{k \log n}{n}}$$

statistical limit

$$\lambda \approx \sqrt{\frac{k^2}{n}}$$

computation limit

reduction to planted cliques:
Berthet & Rigollet (2013)

SNR



Zou et al. (2006)
Amini and Wainwright (2008)
Ma (2013)
Deshpande and Montanari (2014b)
Hopkins et al. (2017)

"I can't find an efficient algorithm, but neither can all these people."

Sparse PCA: our results

Theorem (Li & Wei (2022))

Suppose $0 < \lambda \lesssim 1$. Given an informative initialization (with non-vanishing correlation with v^*), AMP satisfies

$$x_{t+1} = \alpha_{t+1} v^* + \sum_{k=1}^t \beta_t^k \phi_k + \xi_t,$$

with

$$\alpha_{t+1} = \mathbb{E} \left[\lambda v^{*\top} \eta_t \left(\alpha_t v^* + \frac{1}{\sqrt{n}} G \right) \right] + \sqrt{\frac{k + t \log^3 n}{n}},$$

$$\|\beta_t\|_2 = 1, \quad \|\xi_t\|_2 \lesssim \sqrt{\frac{k + t \log^3 n}{n}} \quad \text{w.h.p.}$$

provided that $\frac{t \log^3 n}{n \lambda^2} \lesssim 1$ and $\frac{k \log n}{n \lambda^2} \lesssim 1$.

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denoising functions:

$$\eta_t(x) = \gamma_t \text{sign}(x)(|x| - \tau_t)_+ \quad \text{where } \gamma_t^{-1} := \|(|x_t| - \tau_t)_+\|_2, \tau_t \asymp \sqrt{\frac{\log n}{n}}$$

Several remarks

- record (asymptotic) State Evolution:

$$\alpha_{t+1}^* := \frac{\lambda v^{*\top} \int S\mathbf{T}_{\tau_t} \left(\alpha_t^* v^* + \frac{x}{\sqrt{n}} \right) \varphi_n(dx)}{\sqrt{\int \left\| S\mathbf{T}_{\tau_t} \left(\alpha_t^* v^* + \frac{x}{\sqrt{n}} \right) \right\|_2^2 \varphi_n(dx)}}$$

then

$$|\alpha_{t+1} - \alpha_{t+1}^*| \lesssim \sqrt{\frac{k \log n + t \log^3 n}{n}}$$

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- two sufficient initialization schemes:

► AMP with **diagonal maximization**: $\lambda \|v^*\|_\infty \gtrsim \sqrt{\frac{k \log n}{n}}$

► AMP with **sample-split initialization**: $\lambda \gtrsim \sqrt{\frac{k^2}{n}}$ and $\|v^*\|_\infty \lesssim \frac{\log n}{k}$

Sparse PCA: simulations

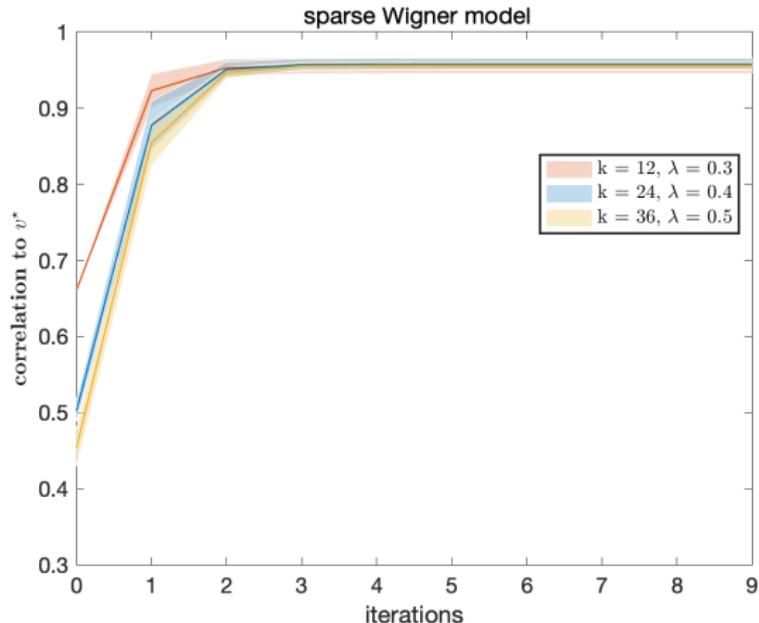


Figure: Convergence of AMP with diagonal maximization for different signal strengths with $n = 10000$. Repeat 40 times.

Questions?

A glimpse of our main proof idea...

— *decomposition:* $x_{t+1} = \alpha_{t+1} v^* + \sum_{k=1}^t \beta_t^k \phi_k + \xi_t$

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- write $U_{t-1} := [z_k]_{1 \leq k \leq t-1} \in \mathbb{R}^{n \times (t-1)}$ and denote

$$z_t := \frac{(I - U_{t-1} U_{t-1}^\top) \eta_t(x_t)}{\|(I - U_{t-1} U_{t-1}^\top) \eta_t(x_t)\|_2} \quad \text{Gram-Schmidt orthogonalization,}$$

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- write $\eta_t(x_t) = \sum_{k=1}^t \beta_t^k z_k$, for $\beta_t^k := \langle \eta_t(x_t), z_k \rangle$

Main proof idea: a new decomposition

- AMP updates:

$$x_{t+1} = M\eta_t(x_t) - \langle \eta'_t(x_t) \rangle \cdot \eta_{t-1}(x_{t-1}), \text{ where } M = \lambda v^* v^{*\top} + W$$

- Goal: $x_{t+1} = \alpha_{t+1} v^* + \sum_{k=1}^t \beta_t^k \phi_k + \xi_t$

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$$M\eta_t(x_t)$$

$$= v^* \underbrace{\lambda v^{*\top} \eta_t(x_t)}_{\alpha_{t+1}} + \left\{ W_t + \sum_{k=1}^{t-1} \underbrace{\left[W_k z_k z_k^\top + z_k z_k^\top W_k - z_k z_k^\top W_k z_k z_k^\top \right]}_{W_k - W_{k+1}} \right\} \cdot \underbrace{\sum_{k=1}^t \beta_t^k z_k}_{\eta_t(x_t)}$$

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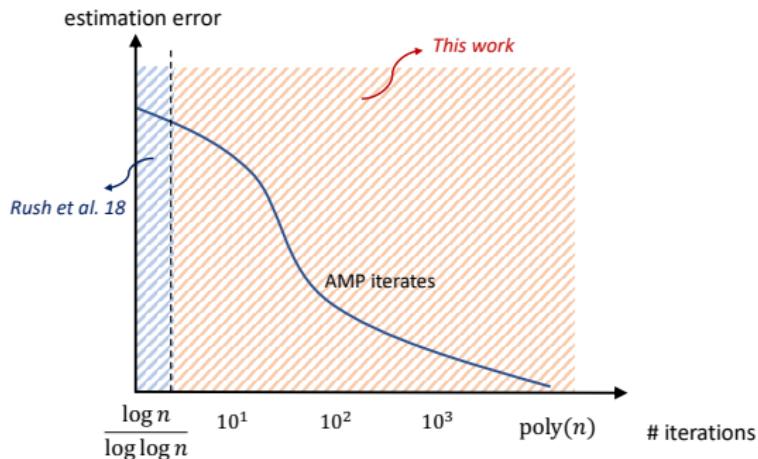
$$= \underbrace{v^* \lambda v^{*\top} \eta_t(x_t)}_{\alpha_{t+1}} + \left\{ \underbrace{W_t + \sum_{k=1}^{t-1} \left[W_k z_k z_k^\top + z_k z_k^\top W_k - z_k z_k^\top W_k z_k z_k^\top \right]}_{W_k - W_{k+1}} \right\} \cdot \underbrace{\sum_{k=1}^t \beta_t^k z_k}_{\eta_t(x_t)}$$

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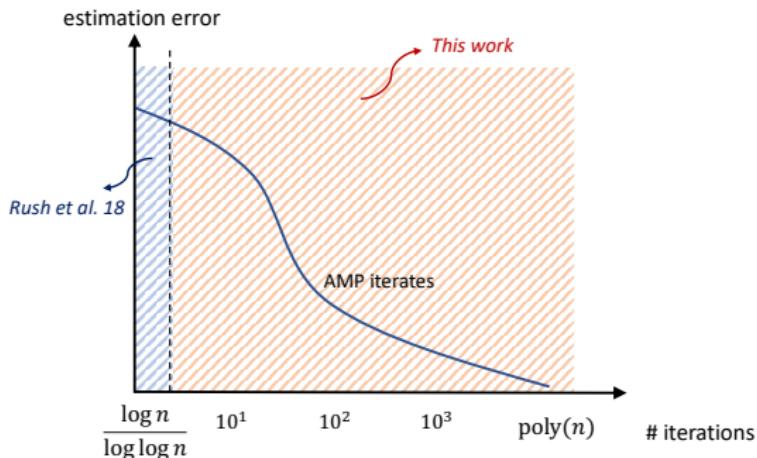
$$\xi_t = \sum_{k=1}^{t-1} z_k \left[\langle W_k z_k, \eta_t(x_t) \rangle - \langle \eta'_t(x_t) \rangle \beta_{t-1}^k - \beta_t^k z_k^\top W_k z_k \right] - \sum_{k=1}^t \beta_t^k \zeta_k$$

Concluding remarks



- A new non-asymptotic framework of AMP for spiked models that allows for # iterations $O(\frac{n}{\text{poly}(\log n)})$

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- A new non-asymptotic framework of AMP for spiked models that allows for # iterations $O(\frac{n}{\text{poly}(\log n)})$
- Apply our theory to two specific examples: \mathbb{Z}_2 synchronization & sparse PCA

Concluding remarks: future extensions

- Other settings: regression (*upcoming*), GLMs, phase retrieval, etc?



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- Other settings: regression (*upcoming*), GLMs, phase retrieval, etc?
- Can we understand AMP for general cone structures (*non-separable denoising functions*)?
- Universality beyond Gaussian design (e.g. uniform Bernoulli, correlated columns, DFT)
- Connections between AMP and other optimization procedures



Thanks for your attention! Questions?

Paper:

"A Non-Asymptotic Framework for Approximate Message Passing in Spiked Models," G. Li, Y. Wei, *arxiv.2208.03313*

"Non-Asymptotic Analysis for Approximate Message Passing with Applications to Sparse and Robust Regression," G. Li, Y. Wei, *upcoming*

Non-asymptotic guarantees

- best known results: Rush & Venkataraman (2016)
 $\#\text{iterations} = o(\log n / \log \log n)$ (*based on state-evolution analysis*)

$$\begin{aligned} & \mathbb{P}(\text{residual at time } t \geq \epsilon) \\ &= \mathbb{P}\left(\sum_{i=0}^{t-1} r_i^t \geq \epsilon\right) \leq \sum_{i=0}^{t-1} \mathbb{P}\left(r_i^t \leq \frac{\epsilon}{t}\right) \leq tC_{t-1} \exp\left(-\frac{c_{t-1}}{t^2} n \epsilon^2\right) \end{aligned}$$

requires $\frac{n}{(t!)^2} \rightarrow \infty \rightarrow t = o(\log n / \log \log n)$

- main challenges: deal with statistical dependence between iterations

Conditioning technique

AMP updates $x_{t+1} = Wm_t - \gamma_t m_{t-1}$
where $m^t = \eta_t(x_t), \quad \gamma_t = \langle \eta'_t(x_t) \rangle$

- $m_{-1} = 0, x_0 = 0$ and $x_1 = W\eta_t(0)$
- σ -algebra \mathcal{F}_t generated by $\{x_0, x_1, \dots, x_t\}$, conditioning on \mathcal{F} is equivalent to conditioning on event

$$\mathcal{E}_t := \left\{ x_1 + \gamma_0 m_{-1} = Wm_0, x_2 + \gamma_1 m_1 = Wm_1, \dots, x_t + \gamma_{t-1} m_{t-1} = Wm_{t-1} \right\}$$

- W conditioning on linear observations

$$W|_{\mathcal{F}_t} \stackrel{\text{d}}{=} \mathbb{E}[W|\mathcal{F}_t] + P_t^\perp W^{\text{new}} P_t^\perp$$

$$W|_{\mathcal{F}_t} m^t \stackrel{\text{d}}{=} \underbrace{W^{\text{new}} P_t^\perp m^t}_{\text{Gaussian term}} + \underbrace{W^{\text{new}} (I - P_t^\perp) m^t + \mathbb{E}[W|\mathcal{F}_t] m^t}_{\text{non-Gaussian term}}$$

Bolthausen (2006), Donoho (2006), Bayati & Montanari (2011), Rush & Venkataramanan (2016), Berthier, Montanari & Nguyen (2020)

Auxiliary details

Define $\zeta_k := \left(\frac{\sqrt{2}}{2} - 1\right) z_k z_k^\top W_k z_k + \sum_{i=1}^{k-1} g_i^k z_i$

$$W_k z_k + \zeta_k = \phi_k \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \frac{1}{n} I_n)$$

- conditioning on $\{z_i\}_{i < k}$, W_k is a Wigner matrix in subspace U_{k-1}^\perp
- W_k is independent of $\{W_i z_i\}_{i < k}$ and x_k and z_k only depend on $\{W_i z_i\}_{i < k}$
- $W_k z_k$ has zero variance along the directions of $\{z_i\}_{i < k}$ and $\frac{2}{n}$ variance along the direction of z_k