

演習

以下を証明せよ.

12. $A \vee A \supset A$

13. $A \supset A \vee A$

14. $A \vee B \supset B \vee A$

12.

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \\ A \vee A & A & A \end{array} \\ \hline A \quad 2,3 \\ \hline A \vee A \supset A \quad 1 \end{array}$$

13.

$$\frac{\frac{1}{A}}{A \vee A} \\ \frac{A \vee A}{A \supset A \vee A}^1$$

14.

$$\begin{array}{ccc} & 2 & 3 \\ & A & B \\ & \hline & B \vee A & B \vee A \\ 1 & & \\ A \vee B & & \\ \hline & B \vee A & 2,3 \\ & \hline & A \vee B \supset B \vee A & 1 \end{array}$$

ルール適用に関する注意点

例:演習14.

正解

$$\begin{array}{c}
 \begin{array}{ccc}
 1 & 2 & 3 \\
 A \vee B & \frac{A}{B \vee A} & \frac{B}{B \vee A} \\
 \hline
 & B \vee A & \\
 \hline
 & B \vee A & \\
 \hline
 & A \vee B \supset B \vee A & 1
 \end{array}
 \end{array}$$

誤解

$$\begin{array}{c}
 \begin{array}{cc}
 1 & 1 \\
 \frac{A \vee B}{A} & \frac{A \vee B}{B} \\
 \hline
 B \vee A & B \vee A \\
 \hline
 B \vee A \\
 \hline
 A \vee B \supset B \vee A & 1
 \end{array}
 \end{array}$$

一見正しいように見えるのだが

$$\frac{\frac{A \vee B}{A}}{B \vee A}$$

の部分が単体では辻褄が合わない
(対応するルールが存在しない)

演習

以下を証明せよ.

$$15. (A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$$

$$16. (A \wedge C) \vee (B \wedge C) \supset (A \vee B) \wedge C$$

$$17. (A \wedge B) \vee C \supset (A \vee C) \wedge (B \vee C)$$

$$18. (A \vee C) \wedge (B \vee C) \supset (A \wedge B) \vee C$$

15.

$$\begin{array}{c}
 \begin{array}{c} 1 \\ \hline (A \vee B) \wedge C \\ \hline A \vee B \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} 1 \\ (A \vee B) \wedge C \\ \hline A \quad C \end{array} \\
 \hline
 \begin{array}{c} 2 \\ A \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} 1 \\ (A \vee B) \wedge C \\ \hline B \quad C \end{array} \\
 \hline
 \begin{array}{c} 3 \\ B \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} A \wedge C \\ \hline (A \wedge C) \vee (B \wedge C) \end{array}
 \quad
 \begin{array}{c} B \wedge C \\ \hline (A \wedge C) \vee (B \wedge C) \end{array}
 \end{array}$$

$$\begin{array}{c}
 (A \wedge C) \vee (B \wedge C) \quad 2,3 \\
 \hline
 \end{array}$$

$$\begin{array}{c}
 \hline
 (A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C) \quad 1
 \end{array}$$

15を例に, 証明のやり方を示す.
ただし, この方法が唯一でもなければ最善というわけでもない.

$$(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$$

証明したい式

$F \supset G$ の形なので \supset の左(F)を仮定する.

$$\begin{array}{c} 1 \\ (A \vee B) \wedge C \end{array}$$

$$\frac{(A \wedge C) \vee (B \wedge C)}{(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)} 1$$

証明したい式

$F \supset G$ の形なので \supset の左(F)を仮定する.

$$\frac{1 \quad (A \vee B) \wedge C}{A \vee B}$$

$$\frac{1 \quad (A \vee B) \wedge C}{C}$$

$$\frac{(A \wedge C) \vee (B \wedge C)}{(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)} 1$$

仮定した式から情報を取り出す.
 $A \vee B$, C が各々単独で使える.
 同じ仮定を2回用いる.

$$\frac{1 \quad (A \vee B) \wedge C}{A \vee B}$$

$$\frac{2 \quad A \quad \frac{1 \quad (A \vee B) \wedge C}{C}}{C}$$

$$\frac{3 \quad B \quad \frac{1 \quad (A \vee B) \wedge C}{C}}{C}$$

$$\frac{(A \wedge C) \vee (B \wedge C)}{(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)} \quad 1$$

$A \vee B$ から, A の場合, B の場合と場合わけを行う. (\vee -除去)
 $(A \vee B) \wedge C$ から C を導いた木を2回用いる.

$$\frac{1 \quad (A \vee B) \wedge C}{A \vee B}$$

$$\frac{2 \quad A \quad \frac{1 \quad (A \vee B) \wedge C}{C}}{A \wedge C}$$

$$\frac{3 \quad B \quad \frac{1 \quad (A \vee B) \wedge C}{C}}{B \wedge C}$$

$$\frac{(A \wedge C) \vee (B \wedge C)}{(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)} 1$$

\wedge -導入により $A \wedge C$, $B \wedge C$ を得る.

$$\frac{1 \quad (A \vee B) \wedge C}{A \vee B}$$

$$\frac{2 \quad A \quad \frac{1 \quad (A \vee B) \wedge C}{C}}{A \wedge C} \quad \frac{A \wedge C}{(A \wedge C) \vee (B \wedge C)}$$

$$\frac{3 \quad B \quad \frac{1 \quad (A \vee B) \wedge C}{C}}{B \wedge C} \quad \frac{B \wedge C}{(A \wedge C) \vee (B \wedge C)}$$

$$\frac{(A \wedge C) \vee (B \wedge C)}{(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)} \quad 1$$

\vee -導入により $(A \wedge C) \vee (B \wedge C)$ を得る。
 これは、先ほどの場合わけで、 A の場合でも B の場合でも
 $(A \wedge C) \vee (B \wedge C)$ を得ることを示している。

$$\begin{array}{c}
 \begin{array}{c} 1 \\ \hline (A \vee B) \wedge C \\ \hline A \vee B \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} 2 \quad 1 \\ A \quad \frac{(A \vee B) \wedge C}{C} \\ \hline A \wedge C \end{array}
 \quad
 \begin{array}{c} 3 \quad 1 \\ B \quad \frac{(A \vee B) \wedge C}{C} \\ \hline B \wedge C \end{array} \\
 \hline
 (A \wedge C) \vee (B \wedge C) \quad (A \wedge C) \vee (B \wedge C)
 \end{array}$$

$$\begin{array}{c}
 (A \wedge C) \vee (B \wedge C) \\
 \hline
 (A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C) \quad 1
 \end{array}$$

2,3

∨-除去を適用して証明を完成させる.

16.

	2	3		4	5
	$A \wedge C$	$B \wedge C$		$A \wedge C$	$B \wedge C$
	<hr style="width: 80%; margin: 0 auto;"/>	<hr style="width: 80%; margin: 0 auto;"/>		<hr style="width: 80%; margin: 0 auto;"/>	<hr style="width: 80%; margin: 0 auto;"/>
	A	B		C	C
	<hr style="width: 80%; margin: 0 auto;"/>	<hr style="width: 80%; margin: 0 auto;"/>		<hr style="width: 80%; margin: 0 auto;"/>	<hr style="width: 80%; margin: 0 auto;"/>
1	$A \vee B$	$A \vee B$	1	C	C
$(A \wedge C) \vee (B \wedge C)$			$(A \wedge C) \vee (B \wedge C)$		
<hr style="width: 100%; margin: 0 auto;"/>			<hr style="width: 100%; margin: 0 auto;"/>		
		2,3			4,5
	$A \vee B$			C	
	<hr style="width: 100%; margin: 0 auto;"/>				
	$(A \vee B) \wedge C$				
	<hr style="width: 80%; margin: 0 auto;"/>				
	1				
$(A \wedge C) \vee (B \wedge C) \supset (A \vee B) \wedge C$					

16.
(別解)

証明は一通りではないことに注意

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} 2 \\ \hline A \wedge C \\ \hline A \\ \hline A \vee B \end{array} & \begin{array}{c} 2 \\ \hline A \wedge C \\ \hline C \end{array} & \begin{array}{c} 3 \\ \hline B \wedge C \\ \hline B \\ \hline A \vee B \end{array} \\
 \begin{array}{c} 1 \\ \hline (A \wedge C) \vee (B \wedge C) \end{array} & \begin{array}{c} \hline (A \vee B) \wedge C \end{array} & \begin{array}{c} 3 \\ \hline B \wedge C \\ \hline C \\ \hline (A \vee B) \wedge C \end{array} \\
 \hline & & 2,3 \\
 & \begin{array}{c} (A \vee B) \wedge C \\ \hline \end{array} & 1 \\
 & \begin{array}{c} (A \wedge C) \vee (B \wedge C) \supset (A \vee B) \wedge C \end{array}
 \end{array}
 \end{array}$$

17.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & 2 & & 2 & \\
 & A \wedge B & & A \wedge B & \\
 & \hline
 & A & & B & \\
 & \hline
 & A \vee C & & B \vee C & \\
 & \hline
 1 & (A \wedge B) \vee C & & (A \vee C) \wedge (B \vee C) & \\
 \hline
 & & & & \begin{array}{ccccc}
 & 3 & & 3 & \\
 & C & & C & \\
 & \hline
 & A \vee C & & B \vee C & \\
 & \hline
 & (A \vee C) \wedge (B \vee C) & & & \\
 & & & & 2,3
 \end{array}
 \end{array}
 \\
 \hline
 \begin{array}{c}
 (A \vee C) \wedge (B \vee C) \\
 \hline
 1 \\
 (A \wedge B) \vee C \supset (A \vee C) \wedge (B \vee C)
 \end{array}
 \end{array}$$

17.
(別解)

$$\begin{array}{c}
 \begin{array}{ccc}
 & 2 & \\
 & \frac{A \wedge B}{A} & \\
 1 & & 3 \\
 (A \wedge B) \vee C & \frac{A}{A \vee C} & \frac{C}{A \vee C} \\
 \hline
 & & 2,3 \\
 & A \vee C & \\
 \hline
 \end{array}
 \qquad
 \begin{array}{ccc}
 & 4 & \\
 & \frac{A \wedge B}{B} & \\
 1 & & 5 \\
 (A \wedge B) \vee C & \frac{B}{B \vee C} & \frac{C}{B \vee C} \\
 \hline
 & & 4,5 \\
 & B \vee C & \\
 \hline
 \end{array}
 \\
 \hline
 \begin{array}{c}
 (A \vee C) \wedge (B \vee C) \\
 \hline
 1 \\
 (A \wedge B) \vee C \supset (A \vee C) \wedge (B \vee C)
 \end{array}
 \end{array}$$

18.

$$\begin{array}{c}
 \begin{array}{c}
 1 \\
 \hline
 (A \vee C) \wedge (B \vee C) \\
 \hline
 A \vee C
 \end{array}
 \qquad
 \begin{array}{c}
 1 \\
 \hline
 (A \vee C) \wedge (B \vee C) \\
 \hline
 B \vee C
 \end{array}
 \qquad
 \begin{array}{c}
 2 \quad 4 \\
 A \quad B \\
 \hline
 A \wedge B
 \end{array}
 \qquad
 \begin{array}{c}
 5 \\
 C \\
 \hline
 C
 \end{array}
 \qquad
 \begin{array}{c}
 3 \\
 C \\
 \hline
 C
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \hline
 (A \wedge B) \vee C
 \end{array}$$

$$\begin{array}{c}
 \hline
 (A \vee C) \wedge (B \vee C) \supset (A \wedge B) \vee C \quad 1
 \end{array}$$