All hw should be uploaded to canvas as a pdf. All homework should be legible and oriented correctly so make sure your scans are correctly done.

Problem 1. Consider a linear map $\mathcal{L} : U \to \mathbb{F}$. Let $\mathcal{N}(\mathcal{L}) = \{u \in U |\ \mathcal{L}(u) = 0\}$. Prove that if $u \in U$ is not in $\mathcal{N}(\mathcal{L})$, then $U = \mathcal{N}(\mathcal{L}) \oplus \{au|\ a \in \mathbb{F}\}$.
Hint: use Proposition 2 from Lec 3.

Problem 2. Prove that if $\{v_1, \dots, v_n\}$ is linearly independent in V, then so is the set $\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}$.

Problem 3. Let V be a subspace of $\mathbb{R}^n$ and let $v_1, \dots, v_\ell$ and $w_1, \dots, w_m$ in V. Prove that if $v_1, \dots, v_\ell$ are linearly independent and $w_1, \dots, w_m$ span V, then $\ell \leq m$.

Problem 4. Let $\mathcal{E} = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and let

$$\mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

be anotehr basis for $\mathbb{R}^2$. Consider the linear map $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ which is the counterclockwise rotation through $\pi/4$. Its matrix representation is a rotation matrix of the form

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

1. Write out A explicitly?
2. Given that vectors in the domain on which $\mathcal{A}$ operates are represented with respect to basis $\mathcal{E}$—i.e. $x \in \mathbb{R}^2$ has coordinates wrt basis $\mathcal{E}$ and we denote it as $(x)_\mathcal{E}$—and suppose that vectors in the co-domain are represented with respect to $\mathcal{V}$—i.e. $y \in \mathbb{R}^2$ has coordinates wrt basis $\mathcal{V}$ and we denote it as $(y)_\mathcal{V}$. Find the matrix representation of $\mathcal{A}$ in these bases. Hint: take what we did in class for writing a vector in one basis in another basis and apply that to $\mathcal{A}(e_i)$ for $i = 1, 2$.

Problem 5. Let

$$S = \{2 + x + x^2 + 3x^3, 4 + 2x + 4x^2 + 6x^3, 6 + 3x + 8x^2 + 7x^3, 2 + x + 5x^3, 4 + x + 9x^3\}$$

and $V = \text{span}(S)$ be the subspace of order three polynomials with real coefficients (denoted by $\mathbb{R}_3[x]$) generated by S.

1. Find a basis for $\mathbb{R}_3[x]$. What is the dimension of $\mathbb{R}_3[x]$?
2. Find the dimension of V by reducing (if necessary) S to a basis for V.

Problem 6. Consider the linear map $\mathcal{A} : U \to V$.

1. Prove that $\mathcal{R}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$ are subspaces of V and U, respectively.
2. If $b \in \mathcal{R}(\mathcal{A})$, then prove that

$$[\mathcal{A}(u) = b \text{ has a unique solution}] \iff [\mathcal{N}(\mathcal{A}) = \{0\}]$$

3. Suppose $b \in \mathcal{R}(\mathcal{A})$, and let u_0 be such that $\mathcal{A}(u_0) = b$. Prove that

$$[\mathcal{A}(u) = b] \iff [u - u_0 \in \mathcal{N}(\mathcal{A})]$$

Problem 7. Let $A : (U, \mathbb{F}) \to (V, \mathbb{F})$ with $\dim U = n$ and $\dim V = m$ be a linear map with $\mathrm{rank}(A) = k$. Show that there exist bases $(u_i)_{i=1}^n$ and $(v_j)_{j=1}^m$ of U, V respectively such that with respect to these bases A is represented by the block diagonal matrix

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

What are the dimensions of the different blocks?

Problem 8. Suppose that V and W are finite dimensional and that U is a subspace of V. Prove that there exists $A \in \mathcal{L}(V, W)$ such that $\ker(A) = U$ if and only if $\dim U \geq \dim V - \dim W$.