510: **Fall 2019**

Lecture 4: Linear Maps & Matrix Representation Revisited

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References: Appendix A.4 of **[C&D]**; Chapter 3 of **[Ax]**

Last time we concluded with some examples which were leading up to representation of operations (transformations) on linear spaces with respect to a basis. Before we dive into this further, let us revisit the linear map concept from lecture 2.

1 Linear Maps

Recall the definition of a linear map: A **linear map** from V to W is a function $f : V \to W$ with the following properties:

additivity. $f(x + z) = f(x) + f(z)$ for all $x, z \in V$.
homogeneity. $f(ax) = af(x)$ for all $a \in \mathbb{F}$ and $x \in V$.

We denote by $\mathcal{L}(V, W)$ the set of all linear maps from V to W.

Suppose $(v_1, \ldots, v_n)$ is a basis of V and $\mathcal{A} : V \to W$ is linear. If $v \in V$, then we can write v in the form

$$v = a_1 v_1 + \cdots + a_n v_n$$

The linearity of $\mathcal{A}$ implies that

$$\mathcal{A}(v) = a_1 \mathcal{A}(v_1) + \cdots + a_n \mathcal{A}(v_n)$$

That is, the values of $\mathcal{A}(v_i)$, $i \in \{1, \ldots, n\}$ determine the values of $\mathcal{A}$ on arbitrary vectors in V.

Converse: Construction of Linear Map. Linear maps can be constructed that take on arbitrary values on a basis. Indeed, given a basis $(v_1, \ldots, v_n)$ of V and any choice of vectors $w_1, \ldots, w_n \in W$, we can construct a linear map $\mathcal{A} : V \to W$ such that $\mathcal{A}(v_j) = w_j$ for $j \in \{1, \ldots, n\}$. To do this, we define $\mathcal{A}$ as follows:

$$\mathcal{A}\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i w_i$$

where $a_i \in \mathbb{F}$, $i = 1, \ldots, n$ are arbitrary elements of $\mathbb{F}$.

Fact. The set of all linear maps $\mathcal{L}(V, W)$ is a vector space.

2 Null and Range Spaces

There are several important spaces associated with linear maps.

Null Space. For $\mathcal{A} \in \mathcal{L}(V, W)$, we define the **null space** of $\mathcal{A}$ to be the subset of V consisting of those vectors that map to zero under $\mathcal{A}$—i.e.,

$$\ker(\mathcal{A}) = \mathcal{N}(\mathcal{A}) = \{v \in V |\ \mathcal{A}(v) = 0\}$$

Examples.

1. Consider the map $\mathcal{A} \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$, where $P(\mathbb{R})$ is the set of polynomials over the field $\mathbb{R}$, defined by
$$(\mathcal{A}p)(x) = x^2 p(x), \;\; p(x) \in P(\mathbb{R})$$
Hence,
$$\ker(\mathcal{A}) = \{0\}$$
The reason for this is that the only polynomial $p(x)$ such that $p(x)x^2 = 0$ for all $x \in \mathbb{R}$ is the zero polynomial.
2. Recall that derivatives are linear maps. Indeed, consider $\mathcal{A} \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ defined by
$$\mathcal{A}(p) = Dp$$
The only functions whose derivative equals the zero function are the constant functions, so in this case the null space of $\mathcal{A}$ equals the set of constant functions.

Range Space. For $\mathcal{A} \in \mathcal{L}(V, W)$ the **range** of $\mathcal{A}$ is the subset of W consisting of those vectors that are of the form $\mathcal{A}(v)$ for some $v \in V$—i.e.
$$\text{range}(\mathcal{A}) = \mathcal{R}(\mathcal{A}) = \{\mathcal{A}(v) |\; v \in V\}$$

Examples.

1. Consider the map $\mathcal{A} \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$, defined by
$$(\mathcal{A}p)(x) = x^2 p(x), \;\; p(x) \in P(\mathbb{R})$$
Then the range of $\mathcal{A}$ is the set of polynomials of the form
$$a_2 x^2 + \cdots + a_m x^m$$
where $a_2, \ldots, a_m \in \mathbb{R}$.
2. Consider $\mathcal{A} \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ defined by
$$\mathcal{A}(p) = Dp$$
Then
$$\text{range}(\mathcal{A}) = P(\mathbb{R})$$
since every polynomial $q \in P(\mathbb{R})$ is differentiable.

2.1 Some results on Null and Range Spaces

Proposition 1. If $\mathcal{A} \in \mathcal{L}(V, W)$, then $\mathcal{N}(\mathcal{A})$ is a subspace of V.

Proof. **Q**: What do we need to check? **A**: (1) $0 \in V$, (2) $u, v \in V \implies u + v \in V$, (3) $u \in V, \; a \in \mathbb{F} \implies au \in V$. OR (1) check $\mathcal{N}(\mathcal{A}) \subset V$ and (2) $\forall\; u, v \in V, \; \forall a \in \mathbb{F}, \; av_1 + av_2 \in V$.

So, let's try the former:

(1). Suppose $\mathcal{A} \in \mathcal{L}(V, W)$. By additivity,
$$\mathcal{A}(0) = \mathcal{A}(0 + 0) = \mathcal{A}(0) + \mathcal{A}(0) \implies \mathcal{A}(0) = 0 \implies 0 \in \mathcal{N}(\mathcal{A})$$

(2). Suppose $u, v \in \mathcal{N}(\mathcal{A})$, then
$$\mathcal{A}(u + v) = \mathcal{A}(u) + \mathcal{A}(v) = 0 + 0 = 0 \implies u + v \in \mathcal{N}(\mathcal{A})$$

(3). Suppose $u \in \mathcal{N}(\mathcal{A})$, $a \in \mathbb{F}$, then

$$\mathcal{A}(au) = a\mathcal{A}(u) = a \cdot 0 = 0 \implies au \in \mathcal{N}(\mathcal{A})$$

$\square$

Definition 1. A linear map $\mathcal{A} : V \to W$ is **injective** if for any $u, v \in V$

$$\mathcal{A}(u) = \mathcal{A}(v) \quad \implies \quad u = v$$

Because of the next proposition, only Example 1—i.e. $\mathcal{A}(p) = x^2 p(x)$—above is injective.

Proposition 2. Suppose $\mathcal{A} \in \mathcal{L}(V, W)$.

$$\mathcal{A} \text{ injective} \iff \mathcal{N}(\mathcal{A}) = \{0\}$$

DIY Exercise.

Proof. ($\implies$) Suppose that $\mathcal{A}$ is injective.
WTS. $\mathcal{N}(\mathcal{A}) = \{0\}$.

$$\text{Prop 1} \quad \implies \quad \{0\} \subset \mathcal{N}(\mathcal{A})$$

It is sufficient to show that $\mathcal{N}(\mathcal{A}) \subset \{0\}$:
Suppose $v \in \mathcal{N}(\mathcal{A})$.

$$[\mathcal{A}(v) = 0 = \mathcal{A}(0)] \text{ and } \mathcal{A} \text{ injective} \quad \implies \quad v = 0$$

which shows the claim.

($\impliedby$). Suppose that $\mathcal{N}(\mathcal{A}) = \{0\}$.
WTS. $\mathcal{A}$ is injective.
Suppose $u, v \in V$ and $\mathcal{A}(u) = \mathcal{A}(v)$. Then,

$$0 = \mathcal{A}(u) - \mathcal{A}(v) = \mathcal{A}(u - v) \quad \implies \quad u - v \in \mathcal{N}(\mathcal{A}) = \{0\} \quad \implies \quad u - v = 0 \quad \implies \quad u = v$$

which shows the claim. $\square$

Proposition 3. For any $\mathcal{A} \in \mathcal{L}(V, W)$, $\mathcal{R}(\mathcal{A})$ is a subspace of W.

DIY Exercise. proof.

Definition 2. A linear map $\mathcal{A} : V \to W$ is **surjective** if its range is all of the co-domain W.

Recall Example 2—i.e., $\mathcal{A}(p) = Dp$. This map is surjective since $\mathcal{R}(\mathcal{A}) = P(\mathbb{R})$.

Combining the above propositions and definitions leads us to a key result in linear algebra which is worth internalizing.

Theorem 1. If V is finite dimensional and $\mathcal{A} \in \mathcal{L}(V, W)$, then $\mathcal{R}(\mathcal{A})$ is a finite-dimensional subspace of W and

$$\dim(V) = \dim(\mathcal{N}(\mathcal{A})) + \dim(\mathcal{R}(\mathcal{A}))$$

Lemma 1. Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof. Suppose V is a finite-dimensional vector space of dimension n. Let $\mathcal{A} \in \mathcal{L}(V, W)$ and let $(u_1, \ldots, u_m)$ be a basis of $\mathcal{N}(\mathcal{A})$ so that $\dim(\mathcal{N}(\mathcal{A})) = m$.

We can extend $(u_1, \ldots, u_m)$ to a basis of V—that is, we can find $n - m$ vectors $u_{m+1}, \ldots, u_n$ such that

$$(u_1, \ldots, u_m, u_{m+1}, \ldots, u_n)$$

is a basis for V.

<u>WTS.</u> $\mathcal{R}(\mathcal{A})$ is finite-dimensional with dimension $n - m$.

Claim. $(\mathcal{A}(u_{m+1}), \ldots, \mathcal{A}(u_n))$ is a basis for $\mathcal{R}(\mathcal{A})$.

Need to show: (1) the set spans $\mathcal{R}(\mathcal{A})$ and (2) it is linearly independent.

Let $v \in V$. Since $\mathrm{span}(u_1, \ldots, u_m, u_{m+1}, \ldots, u_n) = V$,

$$v = \sum_{i=1}^{n} a_i u_i, \;\; a_i \in \mathbb{F}, \;\; \forall i$$

so that

$$\mathcal{A}(v) = \sum_{i=1}^{n} a_i \mathcal{A}(u_i) = \sum_{i=m+1}^{n} a_i \mathcal{A}(u_i) \tag{1}$$

since $u_1, \ldots, u_m \in \mathcal{N}(\mathcal{A})$. Now,

$$(1) \implies \mathrm{span}(\mathcal{A}(u_j))_{j=m+1}^{n} = \mathcal{R}(\mathcal{A}).$$

Let's show (2). Suppose that $c_{m+1}, \ldots, c_n \in \mathbb{F}$ and

$$c_{m+1}\mathcal{A}(u_{m+1}) + \cdots + c_n \mathcal{A}(u_n) = 0 \implies \mathcal{A}(c_{m+1}u_{m+1} + \cdots + c_n u_n) = 0 \implies \sum_{j=m+1}^{n} c_j u_j \in \mathcal{N}(\mathcal{A})$$

Now, since $\mathrm{span}(u_1, \ldots, u_m) = \mathcal{N}(\mathcal{A})$ and $(u_j)_{j=1}^{n}$ are linearly independent,

$$\sum_{j=m+1}^{n} c_j u_j = \sum_{i=1}^{m} d_i u_i, \;\; d_i \in \mathbb{F} \implies c_j, d_i = 0 \; \forall i, j$$

This shows the claim and also that $\dim(\mathcal{R}(\mathcal{A})) = n - m$.

$\square$

There are two corollaries to this theorem which are important and can be used as sanity checks.

Corollary 1. If V and W are finite-dimensional vector spaces such that $\dim(V) > \dim(W)$, then <u>no</u> linear map from V to W is injective.

Proof.

$$\dim(V) > \dim(W) \implies \dim(\mathcal{N}(\mathcal{A})) = \dim(V) - \dim(\mathcal{R}(\mathcal{A})) \geq \dim(V) - \dim(W) > 0$$

$\square$

Corollary 2. If V and W are finite-dimensional vector spaces such that $\dim(V) < \dim(W)$, then <u>no</u> linear map from V to W is surjective.

Proof.

$$\dim(V) < \dim(W) \implies \dim(\mathcal{R}(\mathcal{A})) = \dim(V) - \dim(\mathcal{N}(\mathcal{A})) \leq \dim(V) < \dim(W)$$

$\square$

Implications for Systems of Linear Equations. Consider

$$a_{ji} \in \mathbb{F},\ j \in \{1, \ldots, m\},\ i \in \{1, \ldots, n\}$$

Define $\mathcal{A} : \mathbb{F}^n \to \mathbb{F}^m$ by

$$\mathcal{A}(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} a_{1i} x_i, \ldots, \sum_{i=1}^{n} a_{mi} x_i \right)$$

Consider $\mathcal{A}(x) = 0$ so that

$$\sum_{i=1}^{n} a_{ji} x_i = 0,\ j \in \{1, \ldots, m\}$$

Suppose we know the a_{ji}'s and want to find x's that satisfy these equations. There are clearly m equations and n unknowns. Trivially,

$$x_1 = \cdots = x_n = 0$$

is a solution. Are there others? i.e.,

$$\dim(\mathcal{N}(\mathcal{A})) > \dim(\{0\}) = 0$$

This happens exactly when $\mathcal{A}$ is not injective which is equivalent to $n > m$.

Take away: a homogeneous system of linear equations in which there are more variables than equations must have nonzero solutions.

Examining $\mathcal{A}(x) = c$ for some $c \in \mathbb{F}^m$, we can draw an analogous conclusion via similar reasoning.

Take away: an inhomogeneous system of linear equations in which there are more equations than variables has no solution for some choice of the constant terms.

3 More on the Matrix Representation Theorem

Key Idea: Any linear map between finite dimensional linear spaces can be represented as a matrix multiplication.

$$\begin{tikzcd}
x \in U \arrow[r, "\mathcal{A}"] \arrow[d, "Q"'] & y = \mathcal{A}(x) \in V \arrow[d, "P"] \\
Qx \arrow[r, "A"] & AQx = Py
\end{tikzcd}$$

Let $\mathcal{A} : U \to V$ be a linear map from (U, F) to (V, F) where $\dim(U) = n$ and $\dim(V) = m$. Let $\mathcal{U} = \{u_j\}_{j=1}^{n}$ be a basis for U and let $\mathcal{V} = \{v_j\}_{j=1}^{m}$ be a basis for V.

We will get a matrix representation of A wrt to $\mathcal{U}$ and $\mathcal{V}$ in a few steps:

(step 1) represent x in terms of a basis for U;

(step 2) use linearity to get the map applied to each basis element for U

(step 3) use the uniqueness of a representation wrt a basis to get a representation in terms of basis vectors for V for each map of the basis vectors of U

(step 4) compile these representations to get a matrix

Now in detail...

(step 1) We showed last time that given a basis, the coordinate representation of each x is unique—that is, for any $x \in U$, $\exists!\ \xi = (\xi_1, \dots, \xi_n) \in F^n$ such that

$$x = \sum_{j=1}^{n} \xi_j u_j$$

where $\xi \in F^n$ is the <u>component vector</u>.

(step 2) By linearity,

$$\mathcal{A}(x) = \mathcal{A}\left(\sum_{j=1}^{n} \xi_j u_j\right) = \sum_{j=1}^{n} \xi_j \mathcal{A}(u_j)$$

(step 3) Now each $\mathcal{A}(u_j) \in V$, thus each $\mathcal{A}(u_j)$ has a *unique* representation in terms of the $\{v_j\}_{j=1}^{m}$ such that

$$\mathcal{A}(u_j) = \sum_{i=1}^{m} a_{ij} v_i \quad \forall\ j \in \{1, \dots, n\}$$

i.e.

$$\mathcal{A}(u_1) = \sum_{i=1}^{m} a_{i,1} v_i, \quad \mathcal{A}(u_2) = \sum_{i=1}^{m} a_{i,2} v_i, \quad \dots$$

remark: $(a_{ij})_i$ is the j–th column of A

Thus $\{a_{i,j}\}_{i=1}^{m}$ is the representation of $\mathcal{A}(u_j)$ in terms of $\{v_1, v_2, \dots, v_m\}$. In fact, each $\mathcal{A}(u_j)$ forms a column of the matrix representation

$$\mathcal{A}(u_j) \longleftrightarrow \begin{bmatrix} a_{1,j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Therefore,

$$\mathcal{A}(x) = \sum_{j=1}^{n} \xi_j \mathcal{A}(u_j) = \sum_{j=1}^{n} \xi_j \sum_{i=1}^{m} a_{ij} v_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \xi_j\right) v_i = \sum_{i=1}^{m} \eta_i v_i$$

Thus, the representation of $\mathcal{A}(x)$ with respect to $\{v_1, v_2, \dots, v_m\}$ is

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix} \in F^m$$

where

$$\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \quad \text{for each } i \in \{1, \dots, m\}$$

so that

$$\eta = A\xi, \;\; a \in F^{m\times n}$$

(step 4) recall that the unique representation of x with respect to $\{u_1, u_2, \dots, u_n\}$ is

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \in F^n$$

And, we just argued that η was the unique representation of

$$(\mathcal{A}(u_1), \dots, \mathcal{A}(u_n))$$

wrt $\mathcal{V}$. Hence, A is the matrix representation of the linear operator $\mathcal{A}$ from U to V.

In most applications we replace vectors and linear maps by their representative, viz. component vectors and matrices, e.g., we write $\mathcal{N}(A)$ instead of $\mathcal{N}(\mathcal{A})$.

Theorem 2 (Matrix Representation)**.** Let (U, F) have basis $\{u_j\}_{j=1}^n$ and let (V, F) have basis $\{v_i\}_{i=1}^m$. Let $\mathcal{A} : U \to V$ be a linear map. Then, w.r.t. these bases, the linear map A is represented by the $m \times n$ matrix

$$A = (a_{ij})_{i=1,j=1}^{m,n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \dots & \ddots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in F^{m\times n}$$

where the j–th column of A is the component vector of $\mathcal{A}(u_j)$ w.r.t. the basis $\{v_i\}_{i=1}^m$.

Example. (Observable Canonical Form). Let $\mathcal{A} : (\mathbb{R}^n, \mathbb{R}) \to (\mathbb{R}^n, \mathbb{R})$ be a linear map and suppose that

$$\mathcal{A}^n = -\alpha_1\mathcal{A}^{n-1} - \alpha_2\mathcal{A}^{n-2} - \cdots - \alpha_{n-1}\mathcal{A} - \alpha_n I, \quad \alpha_i \in \mathbb{R}$$

where I is the identity map from $\mathbb{R}^n$ to itself. Let $b \in \mathbb{R}^n$. Suppose

$$(b, \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b))$$

is a basis for $\mathbb{R}^n$. Show that, w.r.t. this basis, the vector b and the linear map $\mathcal{A}$ are represented by

$$\bar{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{n-2} \\ \vdots & & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{bmatrix}$$

Proof.

$$\bar{b} = 1 \cdot b + 0 \cdot \mathcal{A}(b) + \cdots + 0 \cdot \mathcal{A}^{n-1}(b)) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(1st column)}$$

$$\mathcal{A}(b) = 0 \cdot b + 1 \cdot \mathcal{A}(b) + 0 \cdot \mathcal{A}^2(b) + \cdots + 0 \cdot \mathcal{A}^{n-1}(b) \text{ (2nd column)}$$

$$\mathcal{A}(\mathcal{A}^{n-1}(b)) = \mathcal{A}^n(b) = -\alpha_n b - \alpha_{n-1}\mathcal{A}(b) + \cdots - \alpha_1 \mathcal{A}^{n-1}(b) \text{ (n-th column)}$$

hence

$$A = \begin{bmatrix} 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{bmatrix}$$

as expected. □

Remark. We will see later that this is the observable canonical form.

Fact. The composition of two linear operators is just matrix multiplication.

4 Change of Basis for Matrix Representation

We can apply this result to institute a change of basis.

Setup:

- Let $\mathcal{A} : U \to V$, where U, V are vector spaces over the same field F, be a linear map with $\dim U = m$ and $\dim V = n$.
- Let U with elements x have bases $\{u_j\}_{j=1}^m$ and $\{\bar{u}_j\}_{j=1}^m$ generating component vectors ξ and $\bar{\xi}$, resp., in F^m.
- Let V with elements y have bases $\{v_j\}_{j=1}^n$ and $\{\bar{v}_j\}_{j=1}^n$ generating component vectors η and $\bar{\eta}$, resp., in F^n.
- Let A be the matrix representation of $\mathcal{A} : U \to V$ w.r.t. the bases $\{u_j\}_{j=1}^m$ and $\{v_j\}_{j=1}^n$.
- Let $\bar{A}$ be the matrix representation of $\mathcal{A} : U \to V$ w.r.t. the bases $\{\bar{u}_j\}_{j=1}^m$ and $\{\bar{v}_j\}_{j=1}^n$.

Note. A and $\bar{A}$ are said to be equivalent, and $\bar{A} = QAP$ is said to be a similarity transform.

Moreover, if $U = V$ and $\{\bar{v}_j\}_{j=1}^n = \{\bar{u}_j\}_{j=1}^n$, then

$$\bar{A} = P^{-1}AP$$

Then, by the above proposition, $x \in U$ has generating component vectors ξ and $\bar{\xi}$ for bases $\{u_j\}_{j=1}^m$ and $\{\bar{u}_j\}_{j=1}^m$, resp. Moreover,

$$\xi = P\bar{\xi}$$

with non-singular $P \in F^{m\times m}$ given by

$$P = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^{-1} \begin{bmatrix} \bar{u}_1 & \cdots & \bar{u}_m \end{bmatrix}$$

Similarly, $y \in V$ has generating component vectors η and $\bar{\eta}$ for bases $\{v_i\}_{i=1}^n$ and $\{\bar{v}_i\}_{i=1}^n$, resp., and

$$\bar{\eta} = Q\eta$$

with non-singular $Q \in F^{n\times n}$ given by

$$Q = \begin{bmatrix} \bar{v}_1 & \cdots & \bar{v}_n \end{bmatrix}^{-1} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

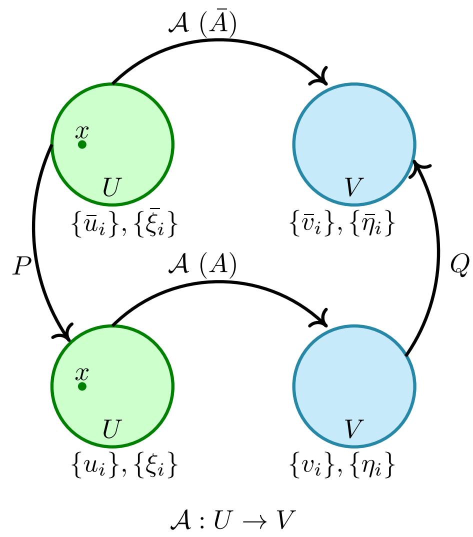

Figure 1: Change of Basis

By the matrix representation theorem (Theorem 2), we know that $\eta = A\xi$. Hence,

$$\bar{\eta} = Q\eta = QA\xi = QAP\bar{\xi} = \bar{A}\bar{\xi}$$

where $\bar{A} = QAP$.

So, if A is the matrix representation of $\mathcal{A}$ with respect to $\{u_i\}, \{v_j\}$, then $\bar{A} = QAP$ is the matrix representation of $\mathcal{A}$ with respect to $\{\bar{u}_i\}, \{\bar{v}_j\}$.

DIY Exercise. Let $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Consider

$$B = \{b_1, b_2, b_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$C = \{c_1, c_2, c_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Clearly, B and C are bases for $\mathbb{R}^3$. Suppose $\mathcal{A}$ maps vectors in B in the following way:

$$\mathcal{A}(b_1) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \mathcal{A}(b_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{A}(b_3) = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

Write down the matrix representation of $\mathcal{A}$ with respect to B and then C.