Lecture 1: Introduction: Good vs Bad Algorithms; Cryptography

1 The Central Paradigm of Computer Science

- The central **paradigm** in computer science is that an algorithm **A** is good if:

  - **A** runs in **polynomial time** in the input size n.
  - That is, **A** runs in time $T(n) = O(n^k)$ for some constant number k.
    - $\rightarrow T(n) = 100n + 55$
    - $\rightarrow T(n) = \frac{1}{2}n^2 + 999 \log n$
    - $\rightarrow T(n) = 6n^7 + 900000n^2 - \sqrt{n}$
  - An algorithm is **bad** if it runs in exponential time.
    - $\rightarrow T(n) = 2^n + 100n^5$
    - $\rightarrow T(n) = 1.000000001^n - n^3 - n$
  - An algorithm is **good** if it runs in **polynomial time** in the input size n.

e.g.

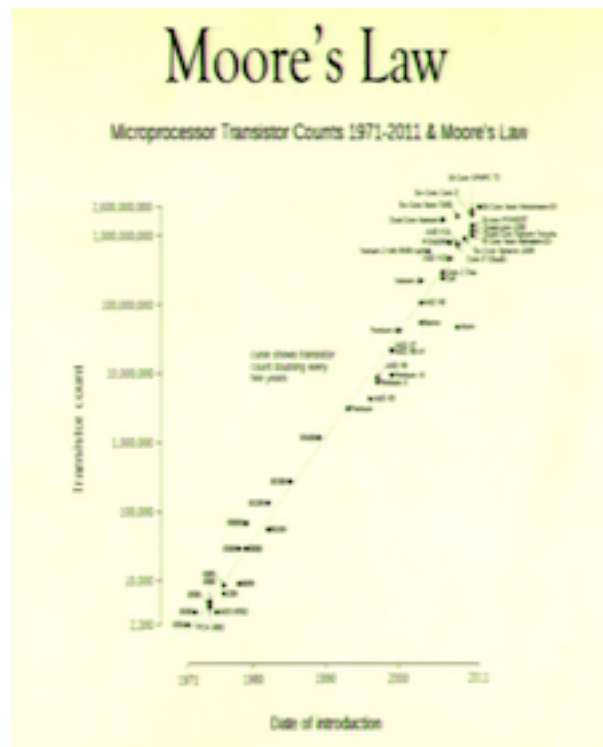|	Input Size n		
	10	100	1000
n	10	100	1000
Runtime of Algorithm n^2	100	10000	1000000
2^n	10^3	10^{30}	10^{300}

2 Good versus Bad (Algorithms)

- For example, consider the problem of sorting n numbers.

  - A Good Algorithm: **MergeSort** runs in time $O(n \cdot log n)$
  - A Bad Algorithm: **BruteForce Search** runs in time $O(n \cdot n!) \gg 2^n$

3 An Equivalent Characterization

- This central **paradigm** has an equivalent formulation

  - **A** runs in **polynomial time** in the input size n.
  - The input sizes that **A** can solve, in a fixed amount T of time, **scales multiplicatively** with increasing computational power.

Input Sizes solved in Time T		
	Power $= 1$	Power $= 2$
n	T	$2T$
Runtime of Algorithm n^2	$\sqrt{T}$	$\sqrt{2} \cdot \sqrt{T}$
2^n	$\log T$	$1 + \log T$

Moore's Law

Microprocessor Transistor Counts 1971-2011 & Moore's Law


- Moore's Law: Computational power *doubles* roughly every two years.
  - $\rightarrow$ Functional time algorithms will never be able to solve large problems.

- The practical implications are perhaps simpler to understand with this <u>latter</u> formulation.

- Thus, improvements in hardware will *never* overcome ***bad algorithm design***.

- Indeed, the current dramatic breakthroughs in computer science are based upon batter (faster and higher performance) algorithmic techniques.

4   Robustness

- This measure of quality or "goodness" is **robust**
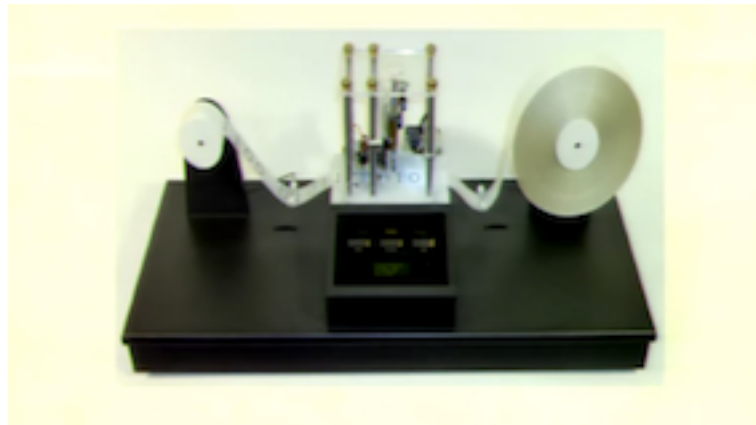
- All reasonable models of algorithms are polynomial time equivalent.

  - Otherwise one model could perform, say, an exponential number of operations in the time another model took to perform just one.

- The standard formal model is the **Turing Machine**.

5   Cryptography (Just an example of the course)

- Alice wants to send Bob a message.
  But she is worried that Eve might intercept the message.
  She decides to encrypt the message M as $\hat{M} = f(M)$.
  Bob can then decrypt the message via $f^{-1}(\hat{M}) = f^{-1}(f(M)) = M$
  Eve cannot understand the message.

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Alice encrypts the message M with the (encryption) lock f.
Bob decrypts the message $\hat{M}$ with the (decryption) key f^{-1}.
Because Eve does not have the key she cannot decipher the message.
Two major problems occur.

- Problem one:

    ○ Eve might be able to break the code.

    → That is , given $\hat{M}$ she may be able to reconstruct f and then f^{-1}.

    ○ Standard techniques for code-breaking include:

    → Frequency Analysis

    ⋆ *e.g.* "e" is the most common letter and "the" is the most common word in English.

    → Cribs.

    ⋆ *e.g.* German weather reports were exploited to help decode messages from the "unbreakable" **Enigma Machine**.

- Problem Two:

    ○ Alice and Bob need to agree on what the encryption code (lock) f is.

    ○ But to do this they need to exchange a message discussing the code.

- Public-Key Cryptography

    ○ In face, Bob gives absolutely everyone a copy of the (encryption) lock.

    ○ Everyone can send the message and Bob has the key.

    ○ The lock is made public. This is called public-key cryptography.

    ○ But this idea sounds completely crazy.

    → Doesn't this solution to Problem Two make Problem One inevitable?

    → That is, if Eve has the lock f then won't she use it to decode $f(M)$?

    ○ No, not if f is hard to invert for anybody except Bob himself.

    ○ But do such functions f that are hard to invert **exist**?

    ○ Yes!

- RSA Encryption

1. Bob chooses two large prime numbers q_1, q_2 and a large number p that is co-prime to $(q_1 - 1) \cdot (q_2 - 1)$

2. Bob's public key is (p, n) where $n = q_1 \cdot q_2$
$$\text{Encryption: } \hat{M} = M^p \bmod n$$

3. Bob's prime key is (q_1, q_2, x) where x is the inverse of p modular $(q_1 - 1) \cdot (q_2 - 1)$
$$\text{Decryption: } M = \hat{M}^x \bmod n$$

   ○ $p = 2^{16} + 1$
   ○ $q_1, q_2 = 2048$ bits

4. An Example

   (a) Prime numbers q_1, q_2 and number p co-prime to $(q_1 - 1) \cdot (q_2 - 1)$
   $p = 7$, $q_1 = 3$, $q_2 = 11$ valid as $\gcd(7,20) = 1$

   (b) Public key (p,n) where $n = q_1 \cdot q_2$
   $n = 33$

   (c) Private Key (q_1, q_2, x) where x is the inverse of p mod $(q_1 - 1) \cdot (q_2 - 1)$
   $x = 33$ as $3 \cdot 7 = 1 \bmod 20$
   Public: $n = 33$, $p = 7$
   Private $q_1 = 3$, $q_2 = 11$, $x = 3$

Public: $n = 33, p = 7$

Private: $q_1 = 3, q_2 = 11, x = 3$

Ant. ... example n ...tanding

Encryption f.
$$\hat{M} = M^P \bmod n$$

Decryption f^{-1}.
$$M = \hat{M}^x \bmod n$$

M=2

$$\hat{M} = 2^7 \bmod 33$$
$$= 128 \bmod 33$$
$$= 29$$

$$M = 29^3 \bmod 33$$
$$= (-4)^3 \bmod 33$$
$$= -64 \bmod 33$$
$$= 2$$

- Is RSA Encryption Safe?

  ○ Public-key cryptography lies at the heart of the modern economy.
    → Financial Services
    → Online Shopping
    → Secure Messaging
  ○ So it is extremely important that the method is safe.
  ○ We claim it is because:
    → Bob has a good algorithm for decryption.
    → Eve only has a bad algorithm for decryption.

- Bob has a Good Decryption Algorithm

  ○ Initially, Bob can do the following in polynomial time.
    → Choose the primes q_1, q_2
    → Choose a number p that is co-prime with $(q_1 - 1) \cdot (q_2 - 1)$

$\rightarrow$ Find x the inverse (Using Euclid's Algorithm) of p and $(q_1 - 1) \cdot (q_2 - 1)$

○ Using fast exponentiation, encoding and decoding is polynomial time.

Encryption: $\hat{M} = M^p \bmod n$

- Eve has a Bad Decryption Algorithm

Decryption: $M = \hat{M}^x \bmod n$

○ To decrypt Eve needs to find x the inverse of p and $(q_1 - 1) \cdot (q_2 - 1)$

$\rightarrow$ She knows p, but does not know q_1, q_2

$\rightarrow$ Instead, she only knows $n = q_1 \cdot q_2$

○ So to find q_1, q_2, she needs to find the prime factorization of n.

○ But it is believed that finding the prime factors of a b-bit number is hard.

$\rightarrow$ Instead, the obvious algorithm attempts to divide n by each integer in the range $\{2,3, ... ,2^b\} \leftarrow$ 4096 bits

○ In fact, if Eve could find x without q_1, q_2, then she could use x to find q_1, q_2. That is, she could factor n.

$\Rightarrow$ Eve only has exponential time algorithm to decrypt.