Lecture 5: Recursive Algorithms: The Median; Randomized Selection; Deterministic Selection

1 The Median Problem

- How do we find the **median** of a set $S = \{x_1, x_2, \cdot \cdot \cdot, x_n\}$?
  - We could sort the list and then output the $[\frac{n}{2}]$th number.
  - Using **Mergesort**, or otherwise, this will take time $O(\mathrm{n} \cdot \log n)$.
- Is there a **faster** way to find the median?
  - We only need the median number. Sorting all numbers seems overkill. We cannot do better than $O(\mathrm{n})$ since we need to read n numbers in linear time.

2 The Selection Problem

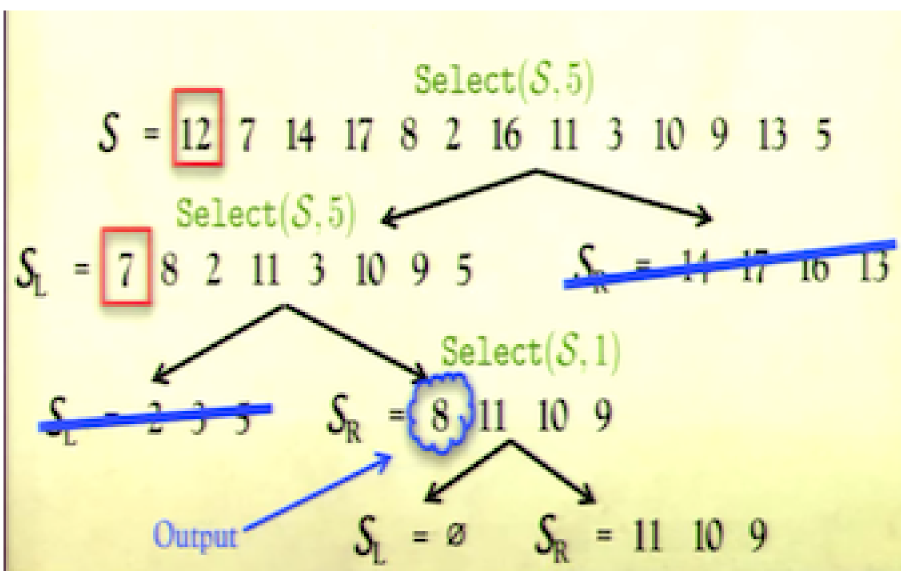
- How do we find the k^{th} smallest number in $S = \{x_1, x_2, \cdot \cdot \cdot, x_n\}$?
  - Again, we could **sort** the list and then output the k^{th} number.
  - Using **Mergesort**, this takes time $O(\mathrm{n} \cdot \log n)$.
- We can do this much faster using recursion...

The Selection Algorithm

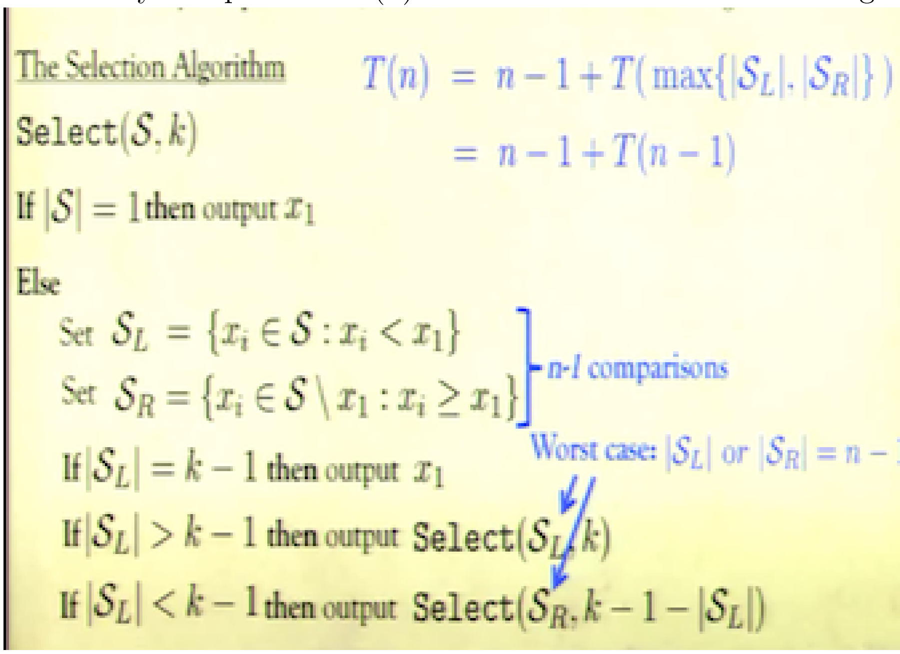
Select(S, k)
If $|S| = 1$ **then** output x_1
Else
  Set $S_L = \{x_i \in S : x_i < x_1\}$
  Set $S_R = \{x_i \in S \setminus x_1 : x_i \geq x_1\}$
  If $|S_L|$ = k - 1 **then** output x_1
  If $|S_L|$ > k - 1 **then** output **Select**(S_L, k)
  If $|S_L|$ < k - 1 **then** output **Select**(S_R, k - 1 - $|S_L|$)

Note:

1. If $|S_L|$ = k - 1, then x_1 is the k^{th} smallest number.
2. On the other hand, suppose smallest S_L contains at least (k - 1) elements, which means it gets k elements, in other words, k^{th} smallest element is actually in S_L. The k^{th} smallest element of S must also be the k^{th} smallest element of S_L.
3. S_L union x_1 has the most (k - 1) elements, which means the k^{th} smallest element must be in the set S_R. Since everything in S_R is at least bigger than in x_1, so that it is also bigger than in S_L, to find the k^{th} smallest element, we need (k - 1 - $|S_L|$).



- How many comparisons T(n) does this recursive selection algorithm make?



- So in the **worst case** we have is:

$$
\begin{aligned}
T(\text{n}) &= (\text{n - 1}) + T(\text{n - 1}) \\
&= (\text{n - 1}) + (\text{n - 2}) + T(\text{n - 2}) \\
&. \\
&. \\
&. \\
&= (\text{n - 1}) + (\text{n - 2}) + (\text{n - 3}) + \cdots + 2 + 1 \\
&= \tfrac{1}{2}\text{n}(\text{n - 1}) \\
&= \Omega(n^2)
\end{aligned}
$$

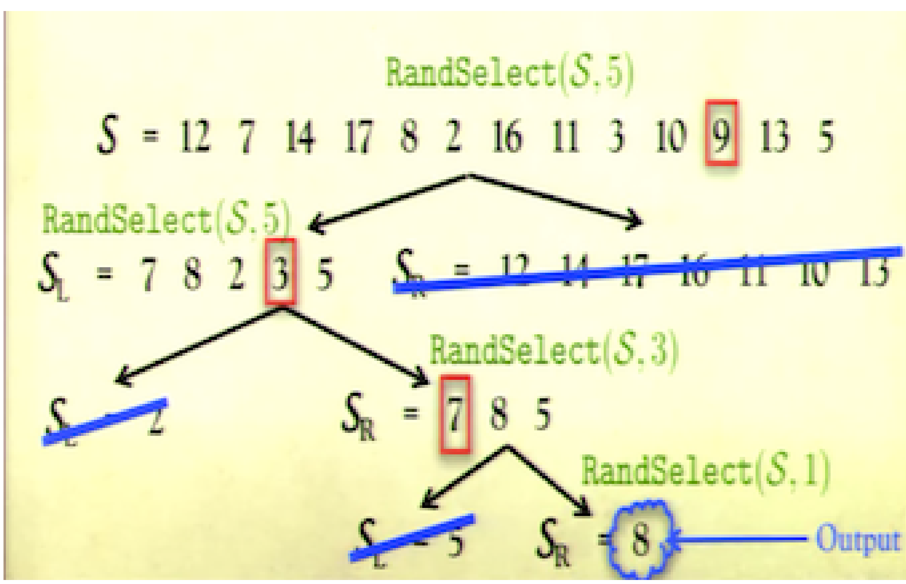
- This is terrible - sorting would have been faster!
- How to fix it? Use Balanced Pivots!
    - The problem is the algorithm repeatedly **pivots** on the first number in the current list.
        → But if we are unlucky this pivot could be very **unbalanced**. That is:
        $\max\{|S_L|, |S_R|\} \approx$ n
        $\min\{|S_L|, |S_R|\} \approx 0$

- What we would like is to select a pivot that is **balanced**. That is:
  $\max\{|S_L|, |S_R|\} \approx \frac{n}{2}$
  $\min\{|S_L|, |S_R|\} \approx \frac{n}{2}$

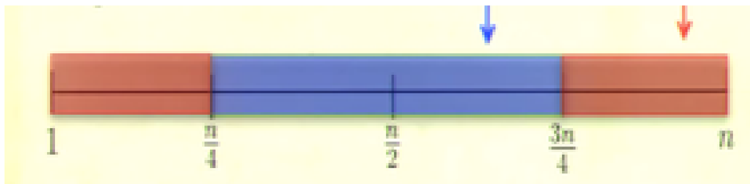
- Randomization
  - The current algorithm is **deterministic** in the choice of the pivot.
  - To fix the problem we consider a **randomized** implementation.
    - → Do not pivot deterministically on x_1.
    - → Instead choose the pivot at <u>random</u> from $S = \{x_1, x_2, \cdots, x_n\}$

<u>Randomized Selection</u>

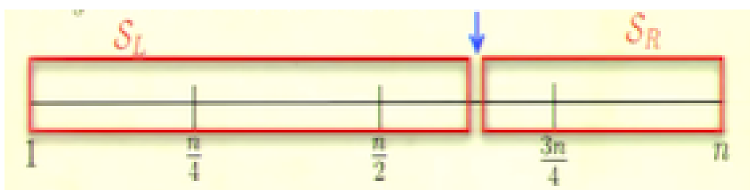
RandSelect(S, k)
If $|S| = 1$ **then** output x_1
Else
  Pick a random pivot $x_\tau \in \{x_1, x_2, \cdots, x_n\}$
  Set $S_L = \{x_i \in S : x_i < x_\tau\}$
  Set $S_R = \{x_i \in S \setminus x_\tau : x_i \geq x_\tau\}$
  If $|S_L| =$ k - 1 **then** output x_τ
  If $|S_L| >$ k - 1 **then** output **RandSelect**(S_L, k)
  If $|S_L| <$ k - 1 **then** output **RandSelect**(S_R, k - 1 - $|S_L|$)



- Good vs Bad Pivots
  - Imagine all the numbers in the **sorted** order.



  - With probability $\frac{1}{2}$ the pivot x_τ lies between the 1st and 3rd quartiles.
    - → We say that such a pivot is good.
    - → We say that such a pivot is bad.
  - **Key Observation**: If the pivot is good then $\max\{|S_L|, |S_R|\} \leq \frac{3}{4} \cdot$ n



- Expected Runtime
  - In the worst case, the *randomized algorithm* will pick the <u>worst</u> pivot! The probability of its happening is very small.
  - So, for randomized algorithms, we are always interested in the **expected runtime** $\bar{T}(n) = E(T(n))$, not the worst case run time.
  - Using our observation, we then have that:

$$\bar{T}(n) \leq \frac{1}{2} \cdot \bar{T}(\frac{3n}{4}) + \frac{1}{2} \cdot \bar{T}(n) + O(n)$$

Note:

1. The first term: the probability of $\frac{1}{2}$ comes from I make a good pivot. If I make a good pivot, I know both of my subsets will have at most the size of $\frac{3n}{4}$.

2. The second term: If I get a bad pivot, the size of the next problem might be (n - 1), which is $\bar{T}(n)$ in the worst case.

$$\bar{T}(n) \leq \frac{1}{2} \cdot \bar{T}(\frac{3n}{4}) + \frac{1}{2} \cdot \bar{T}(n) + O(n)$$

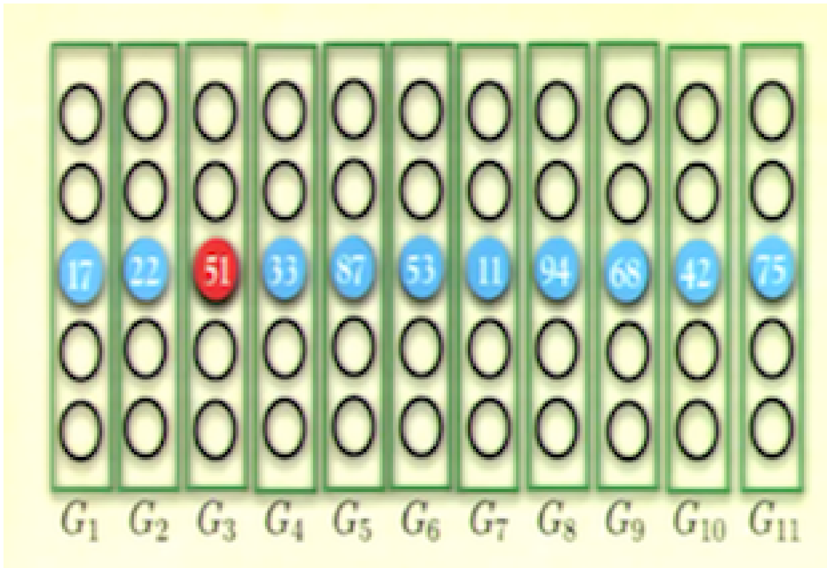
$$\Rightarrow \frac{1}{2} \cdot \bar{T}(n) \leq \frac{1}{2} \cdot \bar{T}(\frac{3n}{4}) + O(n)$$

$$\Rightarrow \quad \bar{T}(n) \leq \bar{T}(\frac{3n}{4}) + O(n)$$

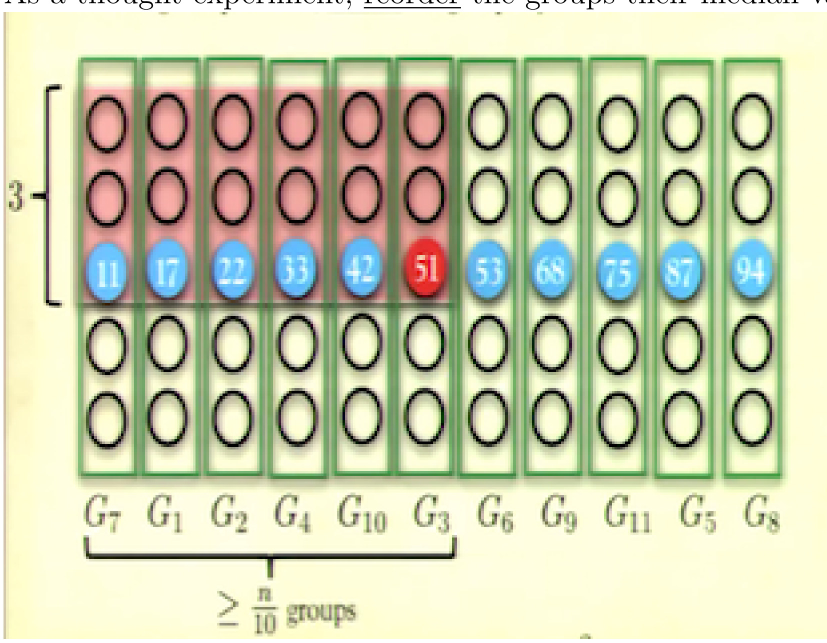
- Apply the Master Theorem
  - → a = 1, b = $\frac{4}{3}$, and d = 1
  - → This is Case 1 of the Master Theorem.
  - → Runtime = $O(n^d) = O(n)$

3 Deterministic Selection

- So we have a **linear time** randomized algorithm for the selection problem.
- Is there a linear time deterministic algorithm?
- To do this, what we need is a deterministic method to find a **good pivot**.
- The idea is to find "the median of the medians."
- Divide $S = \{x_1, x_2, \cdots, x_n\}$ into groups of cardinality five: $G_1 = \{x_1, \cdots, x_5\}, G_2 = \{x_6, \cdots, x_{10}\}, \cdots, G_{\frac{n}{5}} = \{x_{n-4}, \cdots, x_n\}$
- Now sort each group and let z_i be the median of the froup G_i



- Let m be the **median** of $Z = \{z_1, z_2, \cdots, z_{\frac{n}{5}}\}$
- As a thought experiment, reorder the groups their median values:

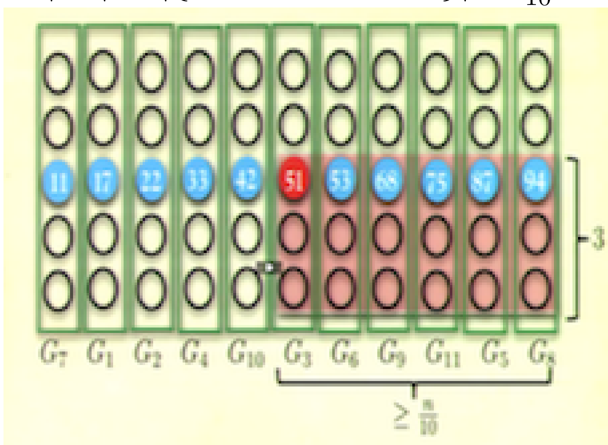


- The **median of the medians** is greater than at least $\frac{3}{10} \cdot (n-1)$ numbers in S

$\Rightarrow |S_R| = |\{x_i \in S \setminus m : x_i \geq m\}| \leq \frac{7}{10} \cdot n$

- There are at least $\frac{3}{10} \cdot (n-1)$ numbers in S that are at least as big as m.

  $\Rightarrow |S_L| = |\{x_i \in S : x_i < m\}| \leq \frac{7}{10} \cdot n$



- Thus, using m as a **pivot** we have:

  $\Rightarrow max\{|S_L|, |S_R|\} \leq \frac{7}{10} \cdot n$

- Thus the median of the medians is a **good** pivot.

- But how actually do we find the median of the medians?

  - Using the same **deterministic recursive algorithm**!

Deterministic Selection

DetSelect(S, k)
If $|S| = 1$ **then** output x_1
Else
  Partition S into $[\frac{n}{5}]$ groups of 5.
  For $j = \{1, 2, \cdots, \frac{n}{5}\}$
    Let z_j be the median of group G_j
  Let $Z = \{z_1, z_2, \cdots, z_{[\frac{n}{5}]}\}$
  Set m $\leftarrow$ **DetSelect**(Z, $[\frac{n}{10}]$)
  Set $S_L = \{x_i \in S : x_i <$ m$\}$
  Set $S_R = \{x_i \in S \setminus \{m\} : x_i \geq$ m$\}$
  If $|S_L| =$ k - 1 **then** output m

If $|S_L|$ > k - 1 **then** output **DetSelect**(S_L, k)
If $|S_L|$ < k - 1 **then** output **DetSelect**(S_R, k - 1 - $|S_L|$)

- Thus, using m as a **pivot** we have:

  $\Rightarrow max\{|S_L|, |S_R|\} \leq \frac{7}{10} \cdot n$

- The recursive formula for the running time is then:

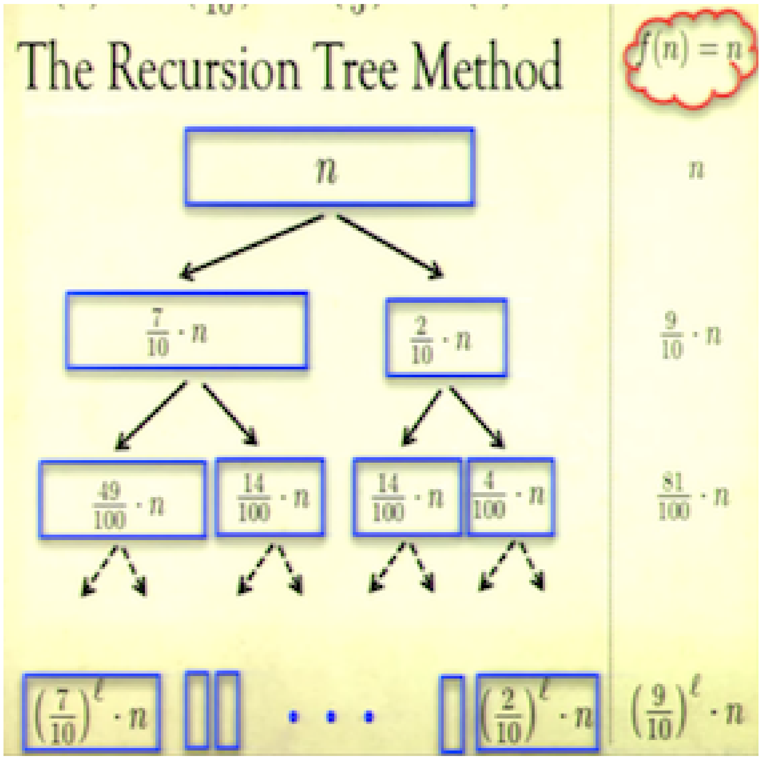
  $\Rightarrow T(n) \leq T(\frac{7n}{10}) + T(\frac{n}{5}) + O(n)$

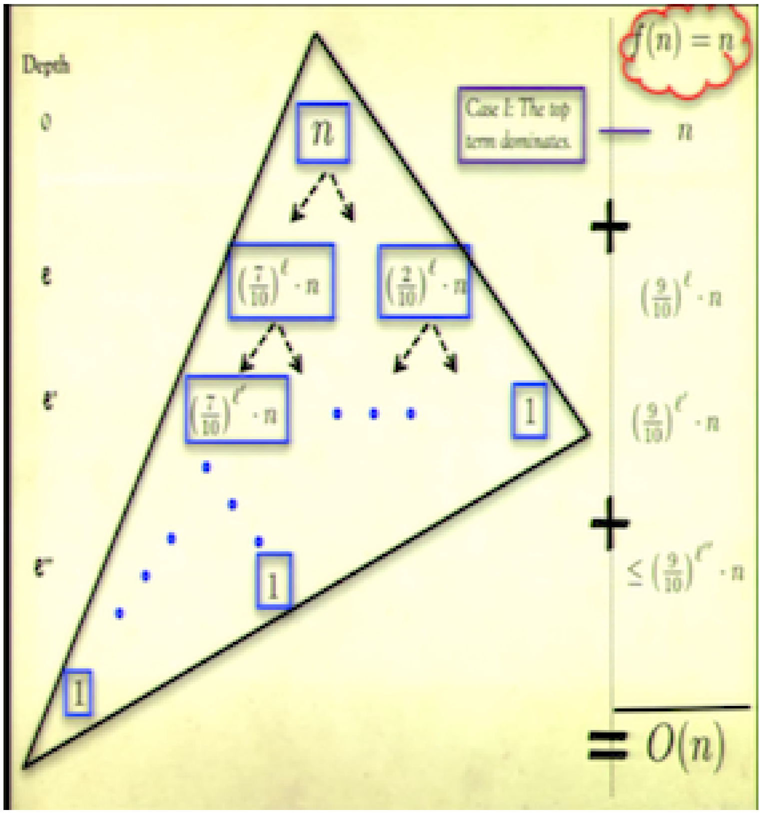
  Note:

  1. $T(\frac{n}{5})$ term comes from finding the median of the medians.
  2. $T(\frac{7n}{10})$ comes from that pivoting on the median of the medians gives a significantly smaller sub-problem.
  3. $O(n)$ comes from breaking in groups of size 5, finding the median of each group, and pivoting on the median of medians.

- But this does **not** fit with the Master Theorem!
  - The problem is not broken into the same sized sub-problems. One is $\frac{7n}{10}$, the other is $\frac{n}{5}$.

- This does not matter as we understand the proof of the Master Theorem,
  $\Rightarrow$ Apply the Recursion Tree Method!


The Recursion Tree Method
$f(n) = n$
n
$\frac{7}{10} \cdot n$
$\frac{2}{10} \cdot n$
$\frac{9}{10} \cdot n$
$\frac{49}{100} \cdot n$
$\frac{14}{100} \cdot n$
$\frac{14}{100} \cdot n$
$\frac{4}{100} \cdot n$
$\frac{81}{100} \cdot n$
$\left(\frac{7}{10}\right)^{\ell} \cdot n$
$\left(\frac{2}{10}\right)^{\ell} \cdot n$
$\left(\frac{9}{10}\right)^{\ell} \cdot n$



- Runtime: $T(n) = O(n)$
- Thus, we have a deterministic **linear time** algorithm to solve the selection problem (and, specifically, to find the median).
  - The reason why finding the selection problem is useful is that we try to find the median, but when we break it into two sub-problems, we are not finding the medians in the sub-problems since things would be shifted, and we might be finding the n over 10th problem.
  - This works a lot in induction. When you do induction proof, here we are using a stronger algorithm as our sub-routines, using kth selection problem, rather than median problem, which assumes that you have a stronger induction hypothesis. Often we use induction, it is hard to prove an easy result, since we can prove a general result, using induction.