



A Brief Review of Graph Theory

Lecture 7

Undirected Graphs

- An undirected graph $G = (V, E)$ consists of:
 - A set V of vertices (or nodes).
 - A set E of edges (or links) denoting *unordered* vertex pairs.

e.g.



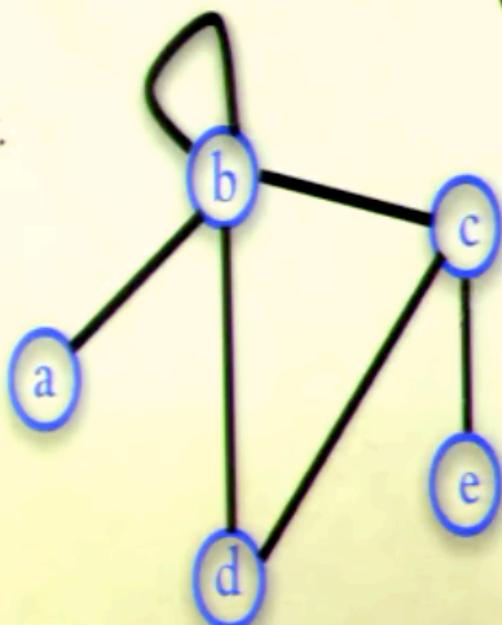
- We set $n = |V|$ to be the *cardinality* of the vertex set.
- We set $m = |E|$ to be the *cardinality* of the edge set.

Adjacency Matrix [Undirected Graphs]

- For an undirected graph, an adjacency matrix M has the properties that:
 - There is a **row** for each **vertex**.
 - There is **column** for each **vertex**.
 - The ij -th entry of the matrix is defined by:

$$M_{ij} = \begin{cases} 1 & (i, j) \in E \\ 0 & (i, j) \notin E \end{cases}$$

e.g.



$$M = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Adjacency Lists [Undirected Graphs]

- An undirected graph can also be stored using adjacency lists.
 - For each vertex i , we store a list of the neighbours of i .

The set of vertices that i shares an edge with.

Adjacency Lists [Directed Graphs]

- An directed graph can also be stored using adjacency lists.
 - For each vertex i , we store a list of the in-neighbours of i .
 - For each vertex i , we also store a list of the out-neighbours of i .

The set of vertices that i has arcs pointing to.

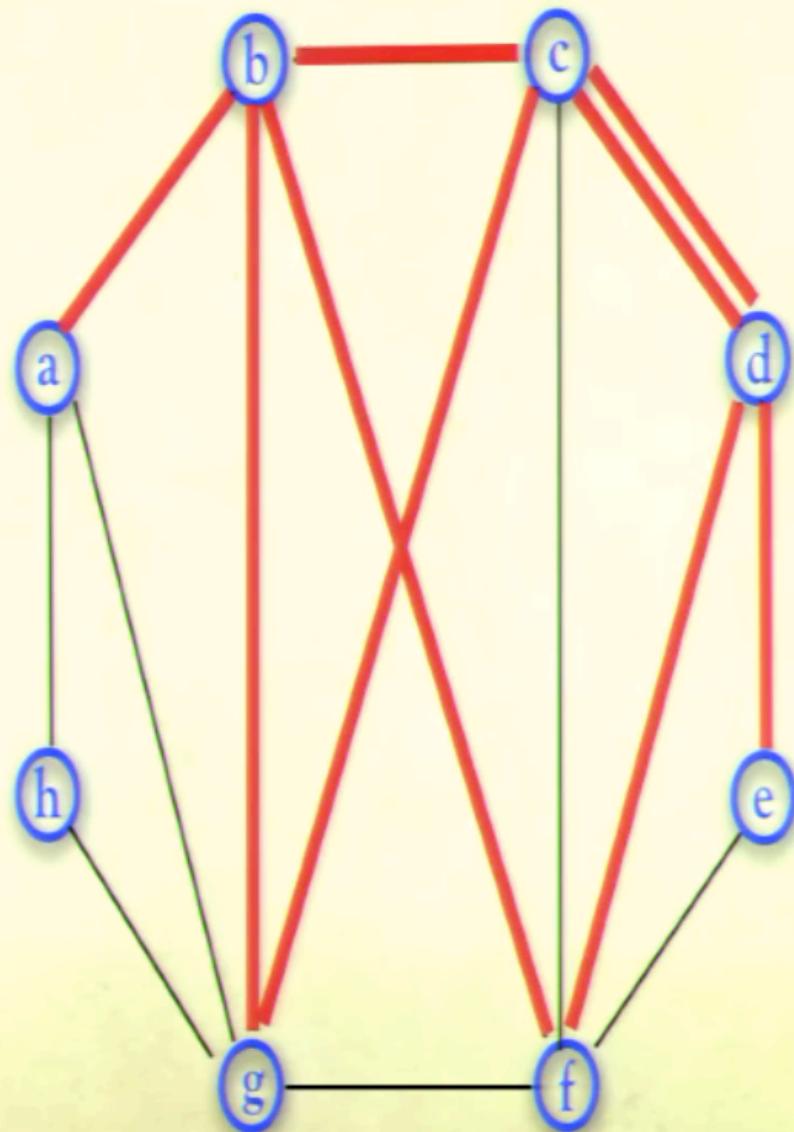


Graph Applications

- The number of practical applications for is extraordinarily large.
- Obviously there are useful in modelling networks.
 - Transportation Networks: e.g. Roads.
 - Social Networks: e.g. Facebook Graph.
 - Information Networks: e.g. the World Wide Web.
 - Financial Networks: e.g. Monetary Flows.

Walks

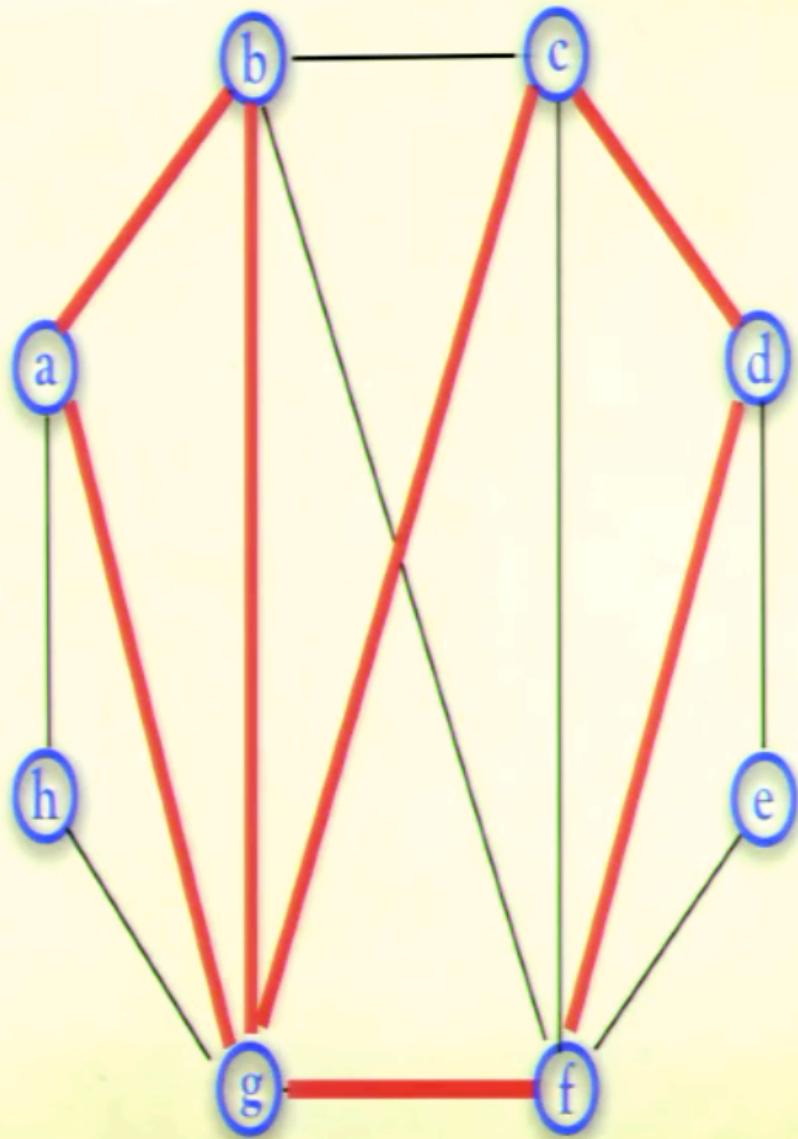
- A walk is list of vertices $\{v_0, v_1, v_2, \dots, v_l\}$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < l$.



Circuits

A circuit is a *closed walk*.

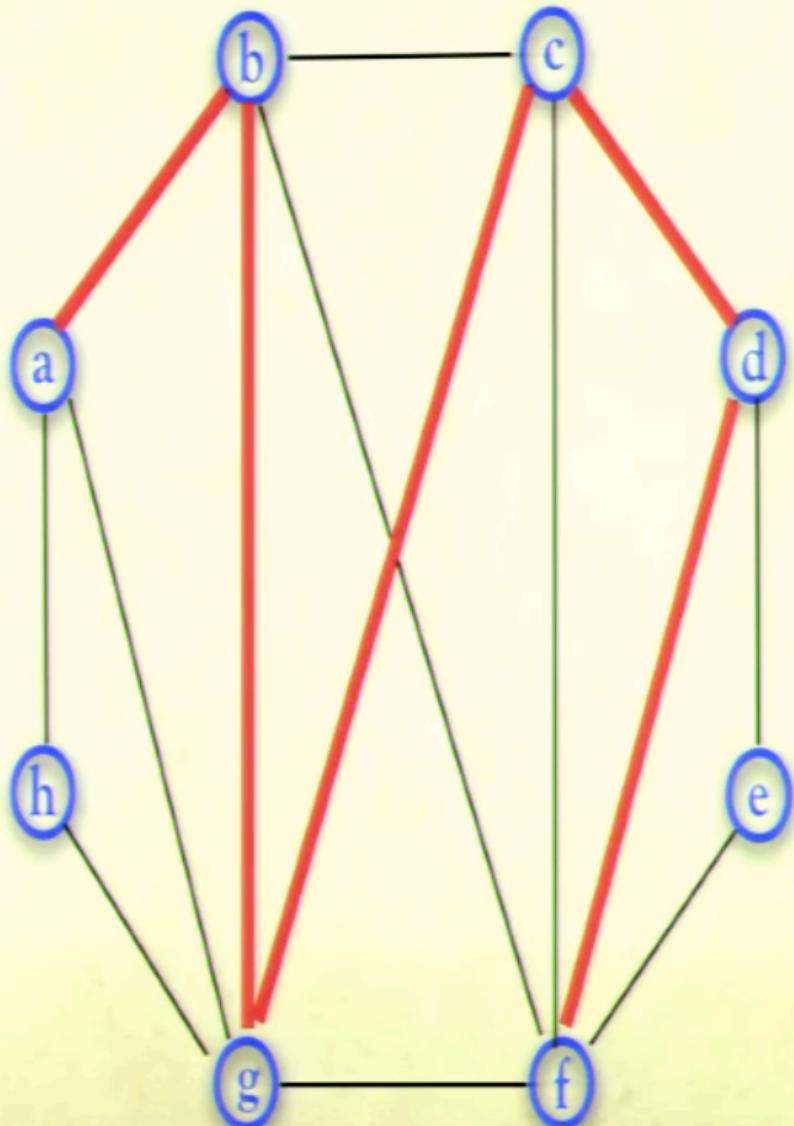
- A circuit is a walk $\{v_0, v_1, v_2, \dots, v_l\}$ where $v_0 = v_\ell$.



- An Eulerian Circuit is a circuit that uses every edge exactly once.

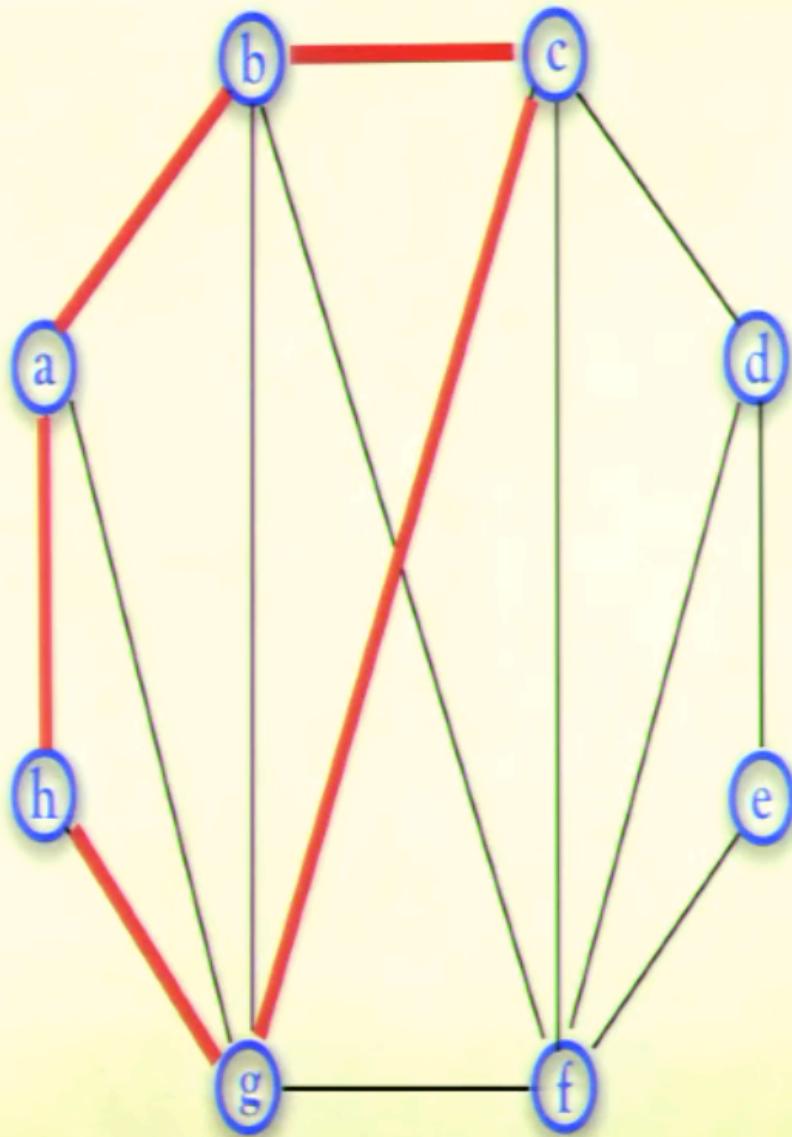
Paths

- A path is a walk where every vertex is distinct.



Cycles

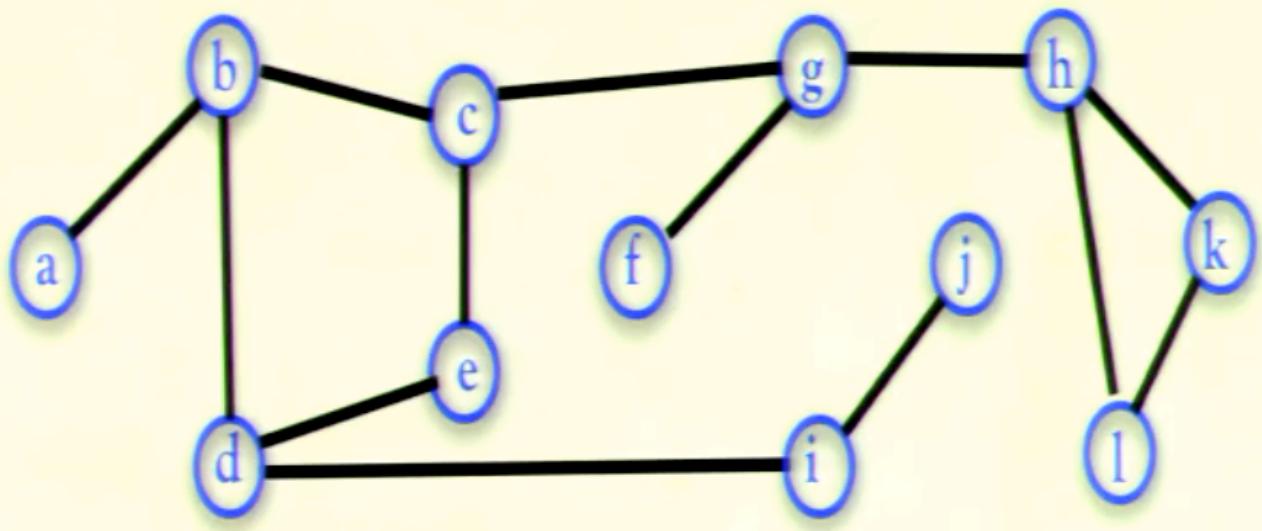
- A cycle is a walk $\{v_0, v_1, v_2, \dots, v_l\}$ where every vertex is distinct except for the end-vertices $v_0 = v_\ell$.



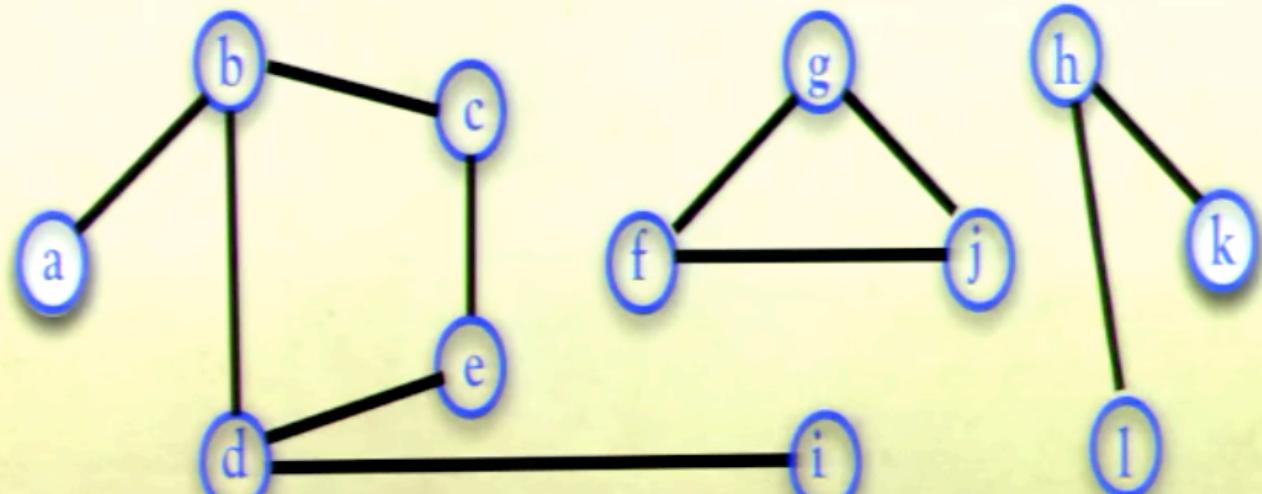
- An Hamiltonian Cycle is a cycle that uses every vertex *exactly once*.

Connected Graphs

- A graph is **connected** if for every pair of vertices $u, v \in V$ it is possible to walk from u to v .

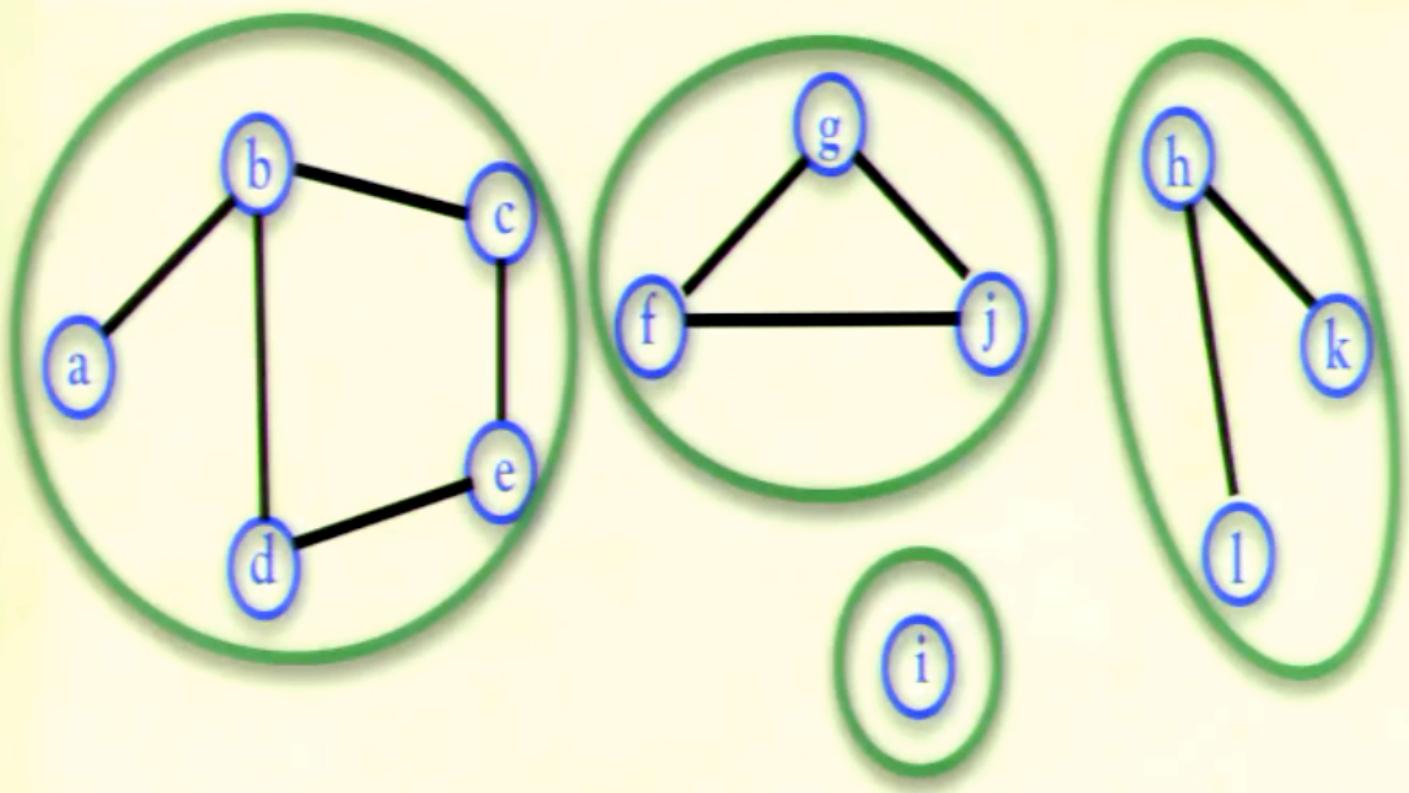


- A graph is **disconnected** if there exists a pair of vertices $u, v \in V$ for which there is no possible walk from u to v .



Graph Components

- A connected subgraphs are called the components of the graph.

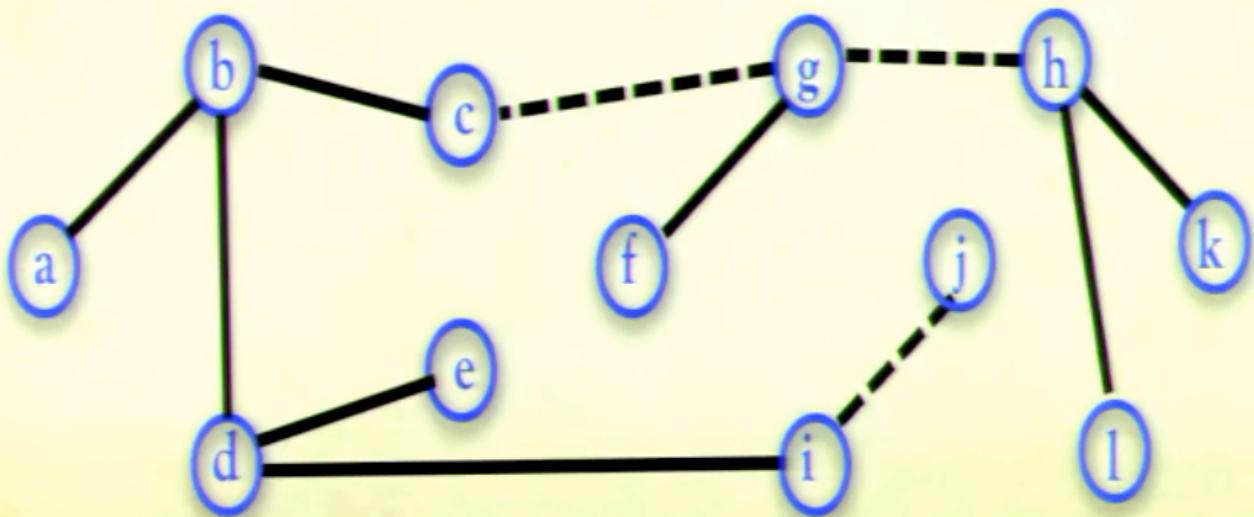


- Thus a *connected graph* has exactly one component.

Trees



- A tree is a connected component with no cycles.
- A forest is a graph whose components are all trees.

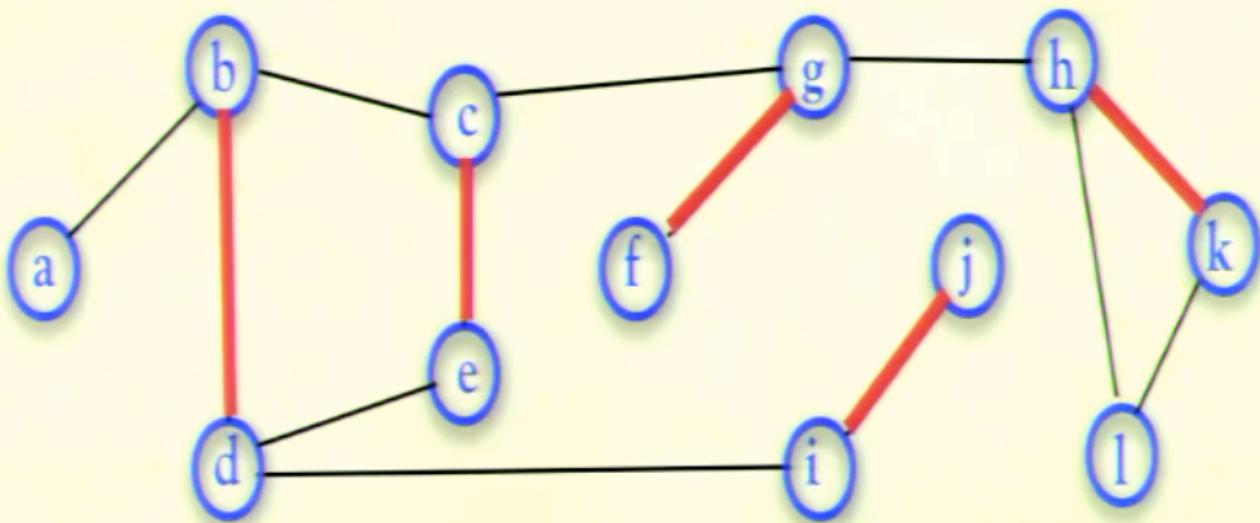


- A tree is **spanning** if it contains every vertex in the graph.



Matchings

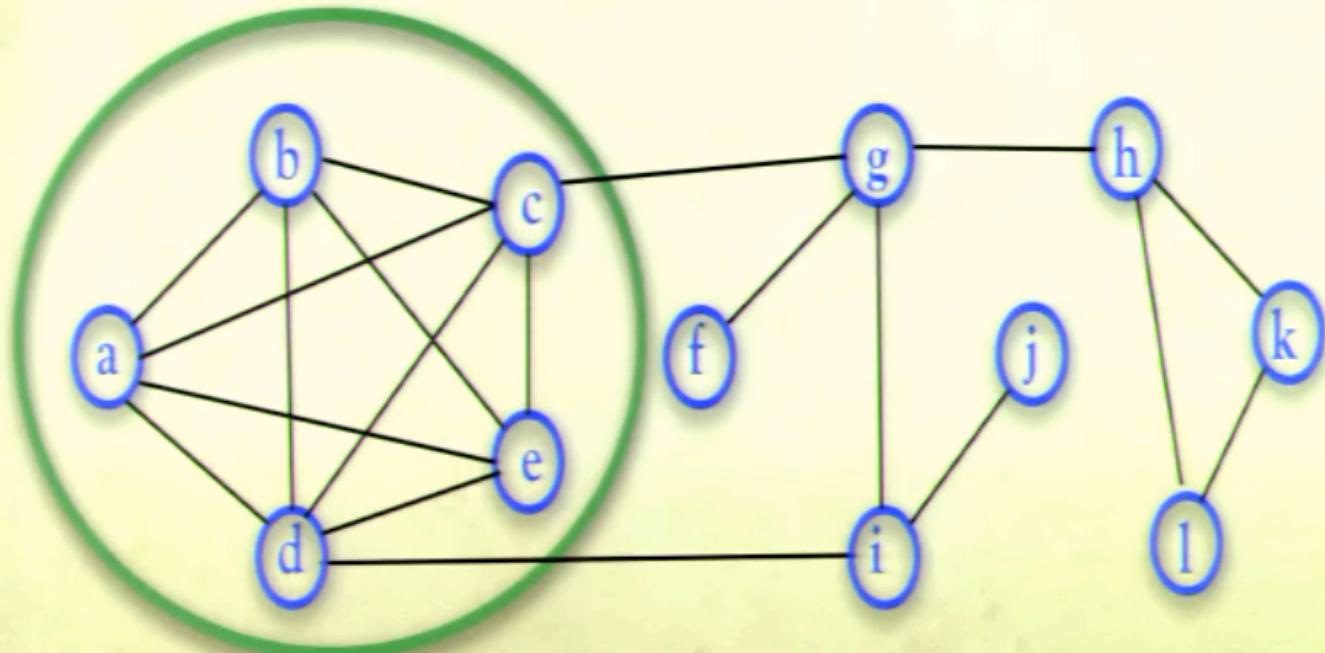
- A matching is a set of vertex-disjoint edges.



- Hence, each vertex is incident to *at most* one edge in the matching.
- A matching is **perfect** if every vertex is incident to an edge in the matching.



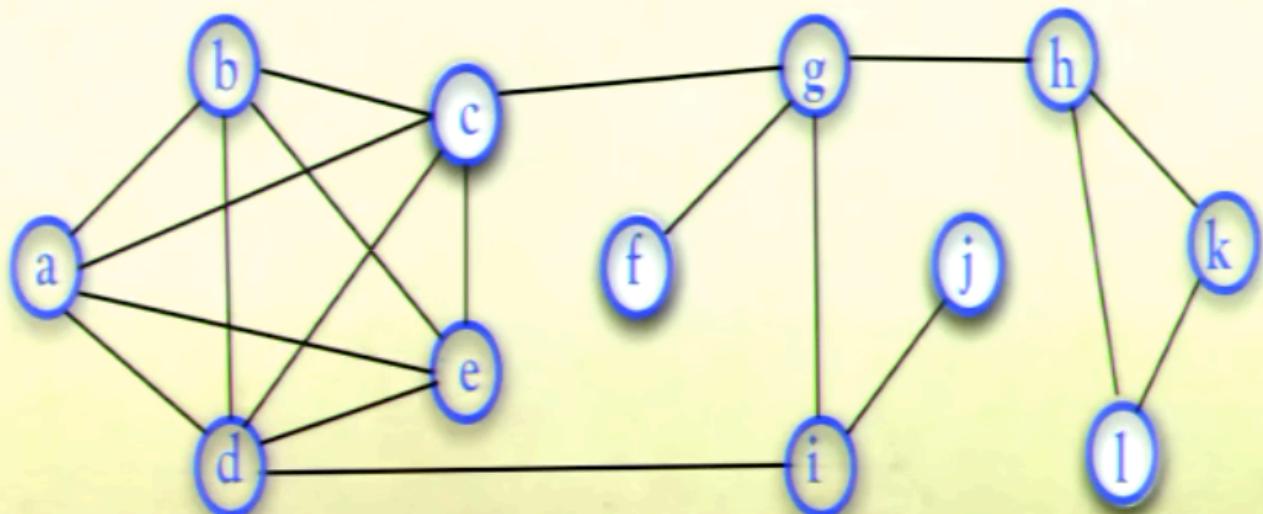
- A clique is a set of pairwise adjacent vertices.





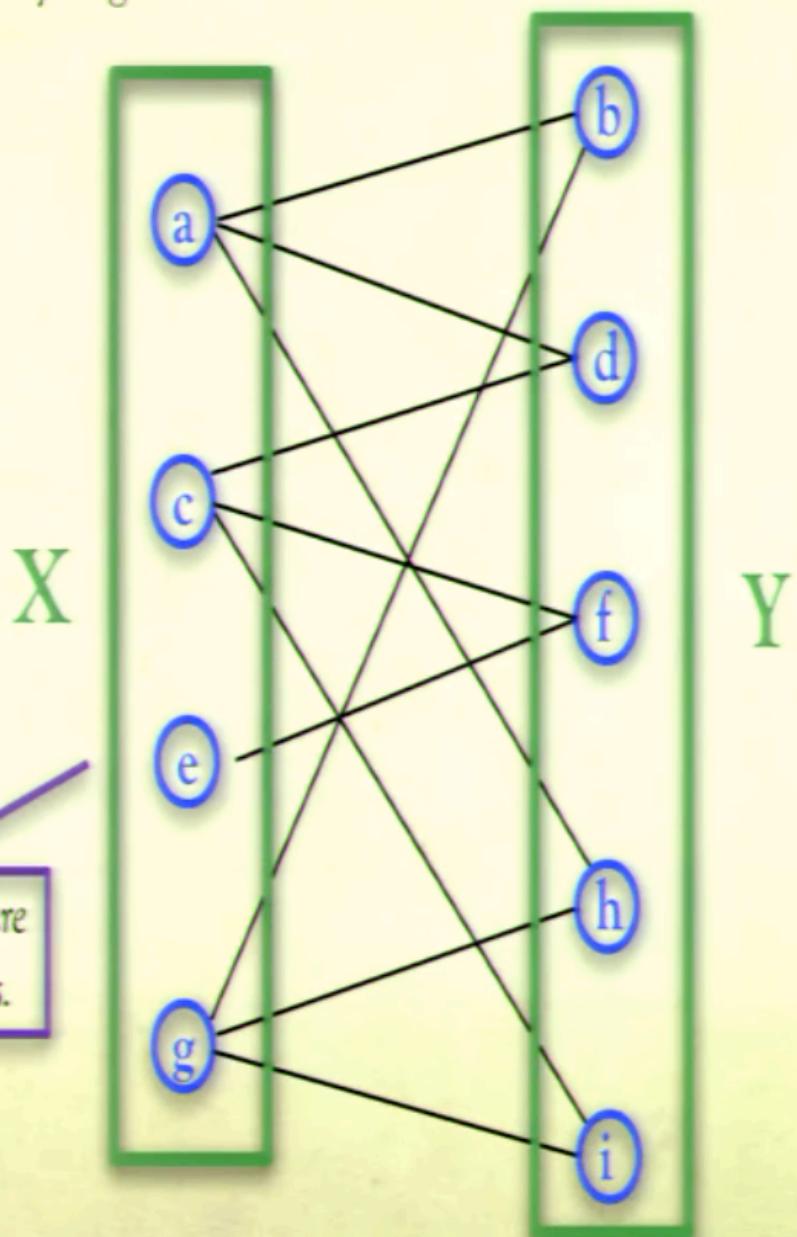
Independent Sets

- O An independent set (stable set) is a set of pairwise non-adjacent vertices.



Bipartite Graphs

- O In a **bipartite graph** the vertex set can be partitioned as $V = X \cup Y$ such that every edge has one end-vertex in X and one end-vertex in Y .



Note that X and Y are both independent sets.

The Handshaking Lemma

- Let $\Gamma(v) = \{u : (u, v) \in E\}$ be the set of neighbours of v .
- The degree, $\deg(v)$, of a vertex v is the cardinality of $\Gamma(v)$.

The Handshaking Lemma. In an undirected graph, there are an even number of vertices with odd degree.

Proof.

- We have:

$$2 \cdot |E| = \sum_{v \in V} \deg(v)$$

Double count the number of pairs (v, e)
where e is an edge incident to v .

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- We have:

$$\begin{aligned}2 \cdot |E| &= \sum_{v \in V} \deg(v) \\&= \sum_{v \in \mathcal{O}} \deg(v) + \sum_{v \in \mathcal{E}} \deg(v)\end{aligned}$$

\mathcal{O} is set of vertices with odd degree, and
 \mathcal{E} is set of vertices with even degree.

The Handshaking Lemma

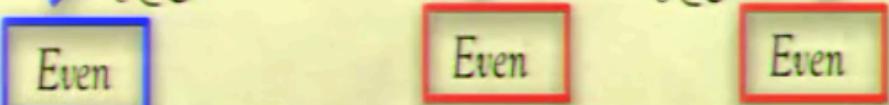
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The Handshaking Lemma. In an undirected graph, there are an even number of vertices with odd degree.

Proof.

- We have:

$$\begin{aligned}2 \cdot |E| &= \sum_{v \in V} \deg(v) \\&= \sum_{v \in O} \deg(v) + \sum_{v \in E} \deg(v) \\ \Rightarrow \sum_{v \in O} \deg(v) &= 2 \cdot |E| - \sum_{v \in E} \deg(v)\end{aligned}$$





Euler's Theorem

- The first result in Graph Theory is the following.

Theorem. [Euler 1736, Hierholzer 1873] An undirected graph contains an Euler Circuit if and only if every vertex has even degree.

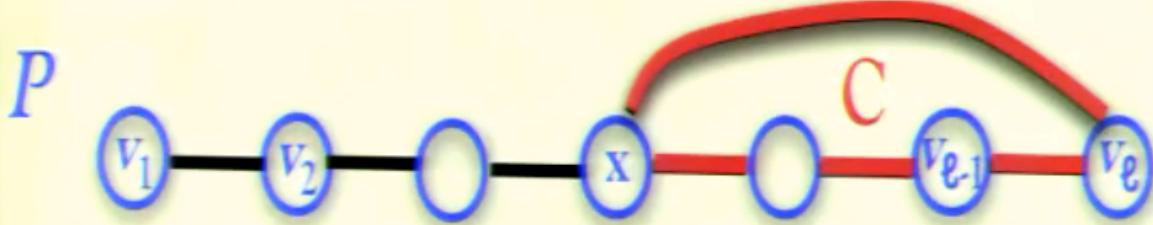
Leaves

- A vertex with degree one in a tree is called a leaf.

Lemma. A tree T with $n \geq 2$ vertices has at least one leaf vertex.

Proof.

- A tree is connected \implies There are no vertices with degree 0 as $n \geq 2$.
- For a contradiction, assume that every vertex has degree at least 2.
- Take the longest path $P = \{v_1, v_2, \dots, v_{\ell-1}, v_\ell\}$ in T .



- But $\deg(v_\ell) \geq 2$ so v_ℓ has a neighbour $x \neq v_{\ell-1}$
- We must have $x = v_j$ for some $1 \leq j \leq \ell - 2$
otherwise $\{v_1, v_2, \dots, v_\ell, x\}$ is a longer path than P .
- But then $C = \{x = v_j, v_{j+1}, \dots, v_\ell, x\}$ is a cycle, contradiction. □

The Number of Edges in a Tree

Theorem. A tree with n vertices has $n-1$ edges.

Proof.

- Let's prove this by induction.

Base Case:

- A tree on one vertex has zero edges.

Induction Hypothesis

- Assume that any tree on $n-1$ vertices has $n-2$ edges.

Induction Step

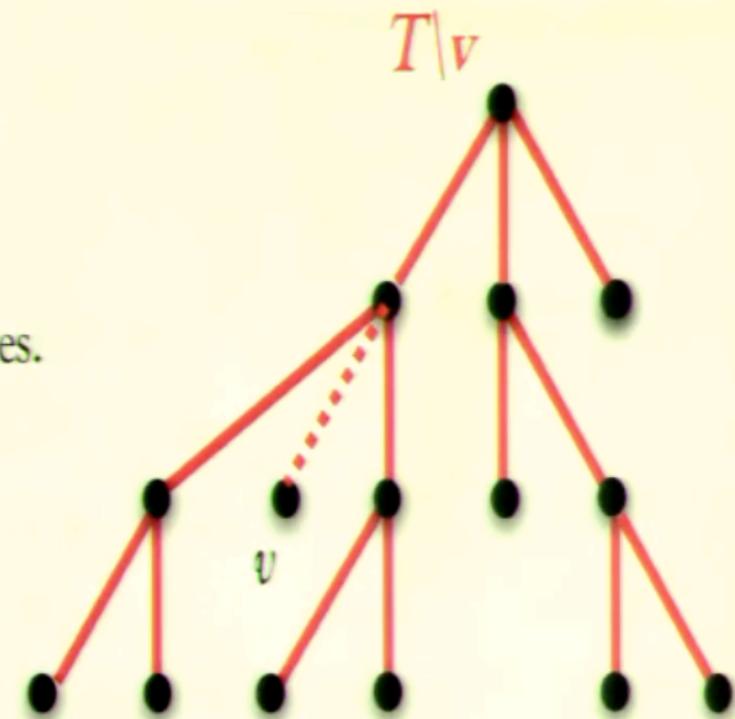
- Take a tree T with $n \geq 2$ vertices.

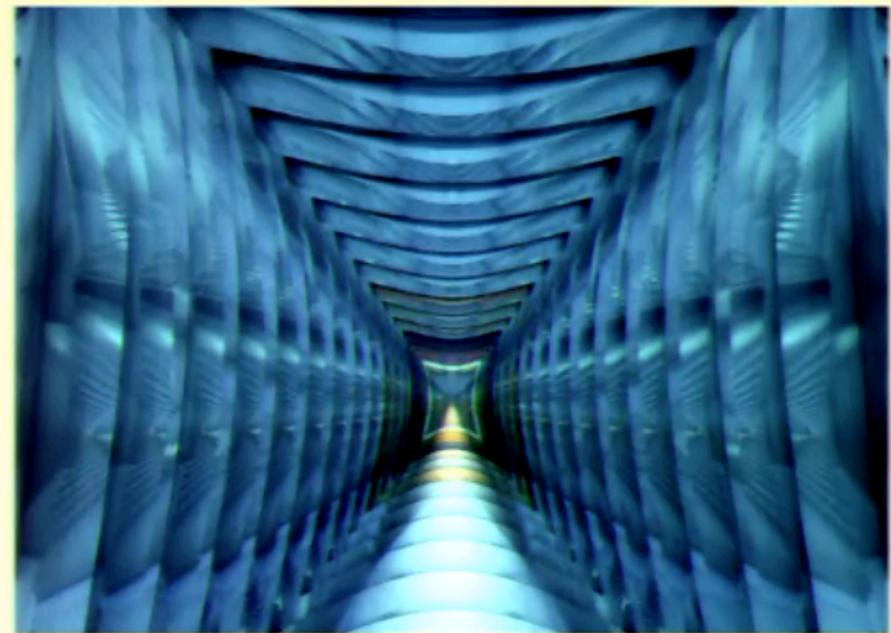
- By the previous lemma, this tree contains a leaf vertex v .

$$\implies T \setminus \{v\} \text{ is a tree on } n-1 \text{ vertices.}$$

$$\implies \text{By the induction hypothesis, } T \setminus \{v\} \text{ is a tree with } n-2 \text{ edges.}$$

$$\implies T \text{ is a tree with } n-1 \text{ edges.}$$





Hall's Theorem

- How do we know if a bipartite graph $G = (X \cup Y, E)$ contains a perfect matching?
- This is actually easy to test using Hall's Condition.

Hall's Condition: $\forall B \subseteq X, \quad |\Gamma(B)| \geq |B|$

Hall's Theorem. A bipartite graph, with $|X|=|Y|$, contains a perfect matching if and only if Hall's Condition is satisfied.*

Hall's Theorem

Hall's Theorem. A bipartite graph, with $|X| = |Y|$, contains a perfect matching if and only if $\forall B \subseteq X, |\Gamma(B)| \geq |B|$

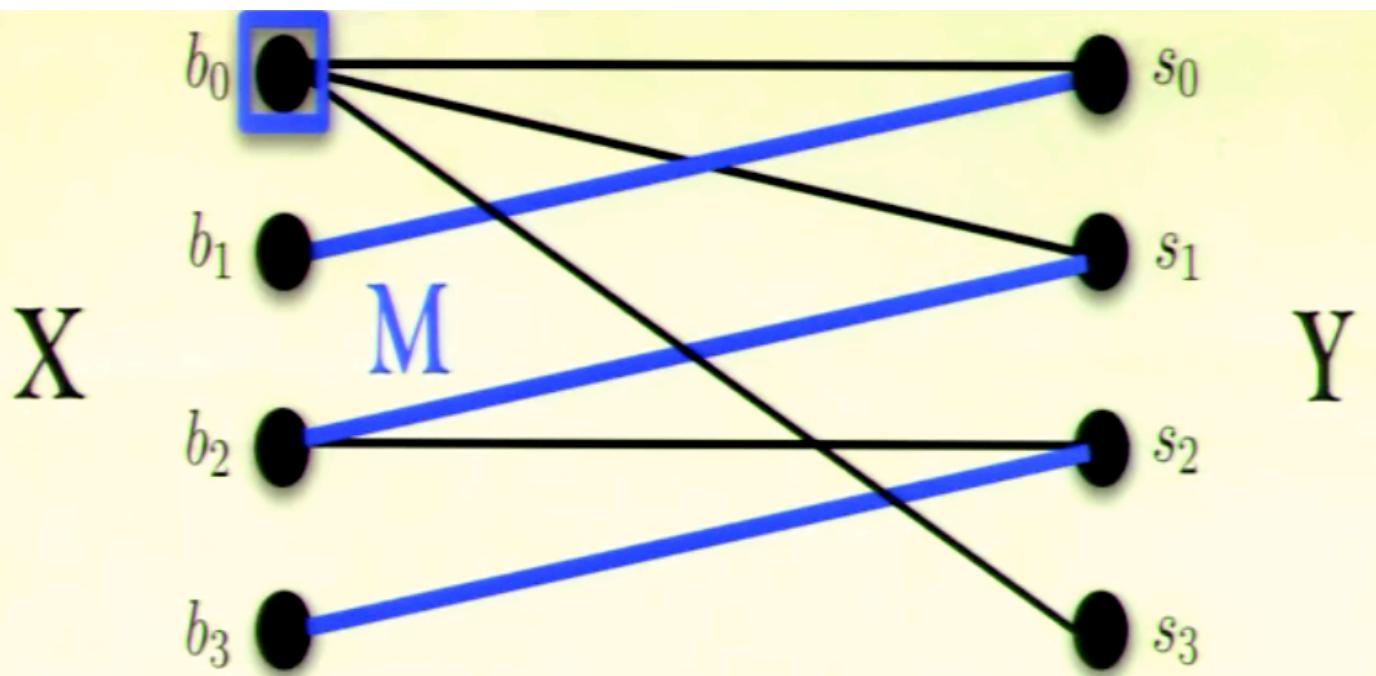
Proof.

(\Rightarrow)

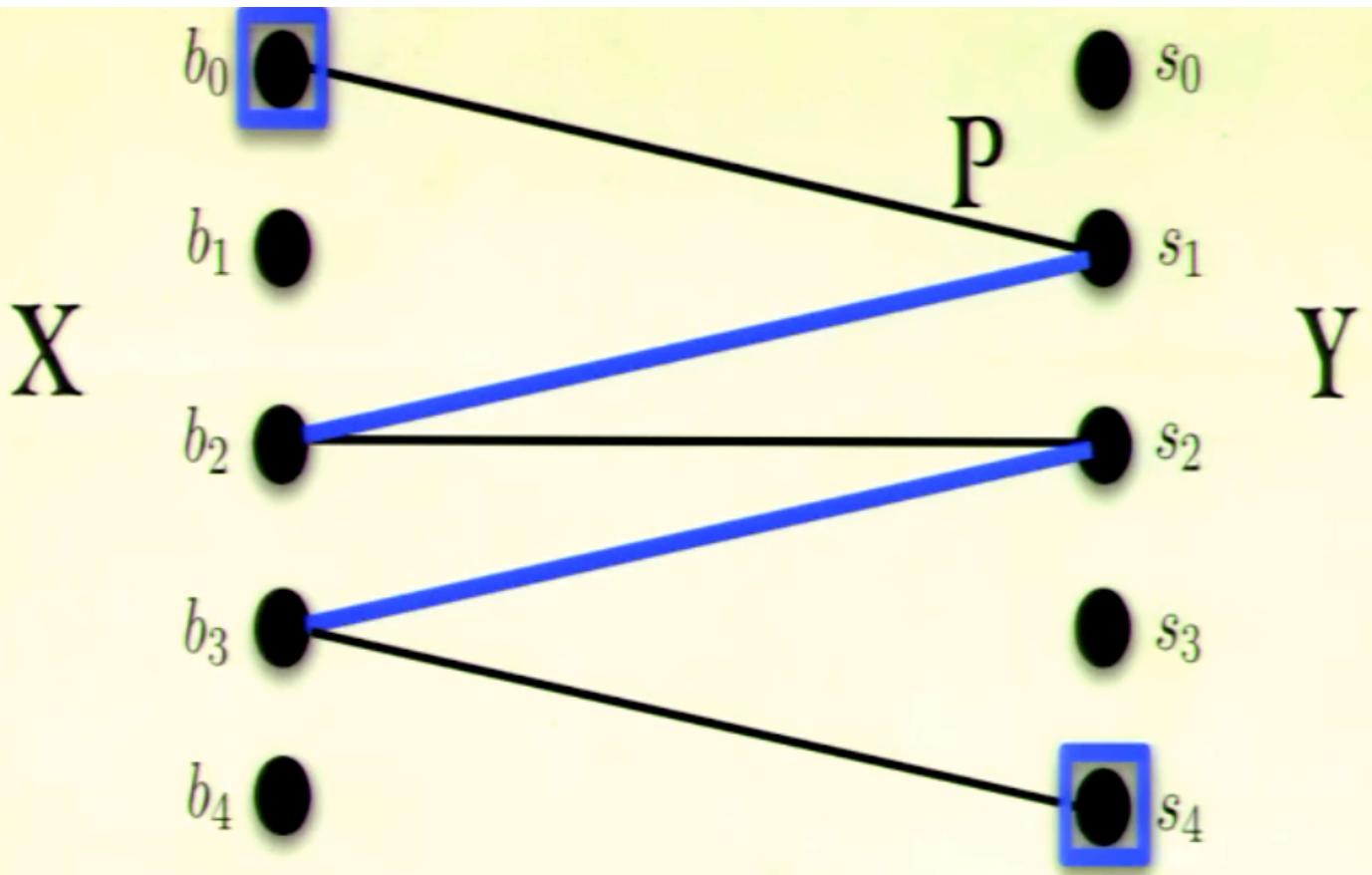
- If there is an set $B \subseteq X$ with $|\Gamma(B)| < |B|$ then the graph cannot have a perfect matching.

(\Leftarrow)

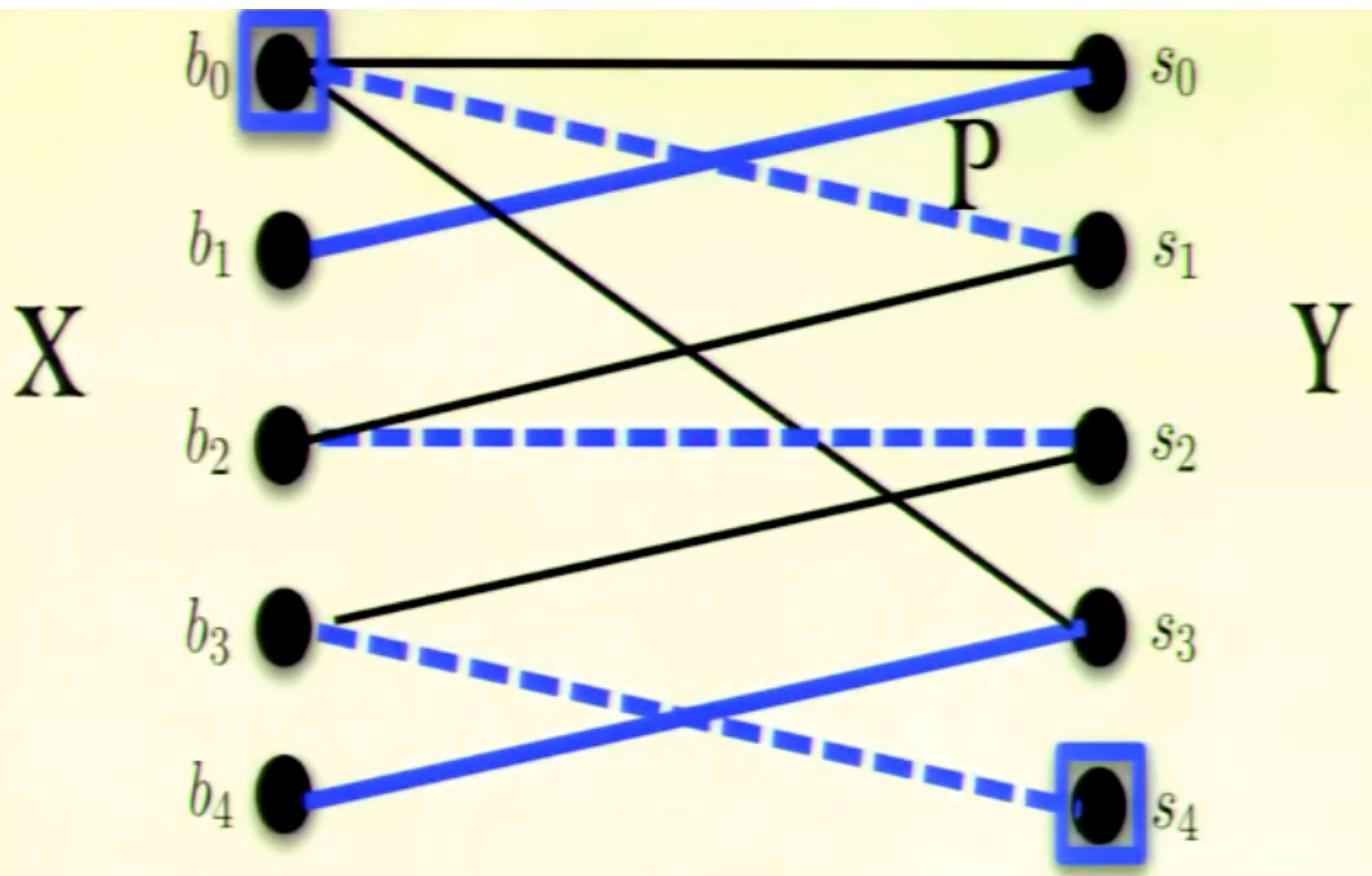
- Suppose Hall's Condition is satisfied: $\forall B \subseteq X, |\Gamma(B)| \geq |B|$
- Take a maximum cardinality matching M in the graph.
- If M is perfect we are done.
- So we may assume M is not perfect and there is an unmatched vertex b_0 in X .



- By Hall's condition: $|\Gamma(\{b_0, b_1, b_2\})| \geq |\{b_0, b_1, b_2\}| = 3$
- So $\{b_0, b_1, b_2\}$ have a neighbour $s_2 \neq \{s_0, s_1\}$
- Let s_2 be matched to b_3 in M .
- By Hall's condition: $|\Gamma(\{b_0, b_1, b_2, b_3\})| \geq |\{b_0, b_1, b_2, b_3\}| = 4$
- So $\{b_0, b_1, b_2, b_3\}$ have a neighbour $s_3 \neq \{s_0, s_1, s_2\}$
- Let s_3 be matched to b_4 in M , etc...



- The graph contains a *finite number of nodes* so this process must terminate.
- The process can only terminate if we reach an *unmatched node* $s_k \in S$
- Using the edges we have found we can trace back a **path** P from s_k to b_0 that *alternates* between using non-matching edges and matching edges.
- The path P is called an **alternating path**.



- Swapping the matching and non-matching edges gives *one extra matching edge*.
- Note this is still a valid matching. Firstly, the internal nodes of P are still incident to exactly one matching edge.
- Secondly, its end-nodes, s_k and b_0 , were previously unmatched so are now incident to exactly one edge in the new matching.
- This contradicts the fact that M was a *maximum cardinality matching*. □