Lecture 8: Graph Algorithms: Breadth First Search

1 Graph Exploration

- Given a graph and two specified vertices r and v, a fundamental question is:
  - Does the graph contain a path from r to v?
- In fact, given a root vertex r, we will be able to answer this question simultaneously for all non-root vertices.
- That is, we can efficiently determine which vertices are reachable by **path traversal** starting at the root vertex.

2 Search Applications

- Graph Search has a **vast** number of applications.
- These can be simple applications:
  - How do we explore a maze?
  - How do I travel from Montreal to Toronto?
- Or, they can be much more substantive.
  - What is the best way to search the web?
  - How, and how far, will a disease spread through the population?

3 The Generic Search Algorithm

- The Generic Search Algorithm:

  The Search Algorithm
  Put r into a bag
    While the bag is non-empty
      Remove v from the bag
      If v is unmarked
        Mark v
        For each arc(v,w)
          Put w into the bag

- Formal pseudocode:

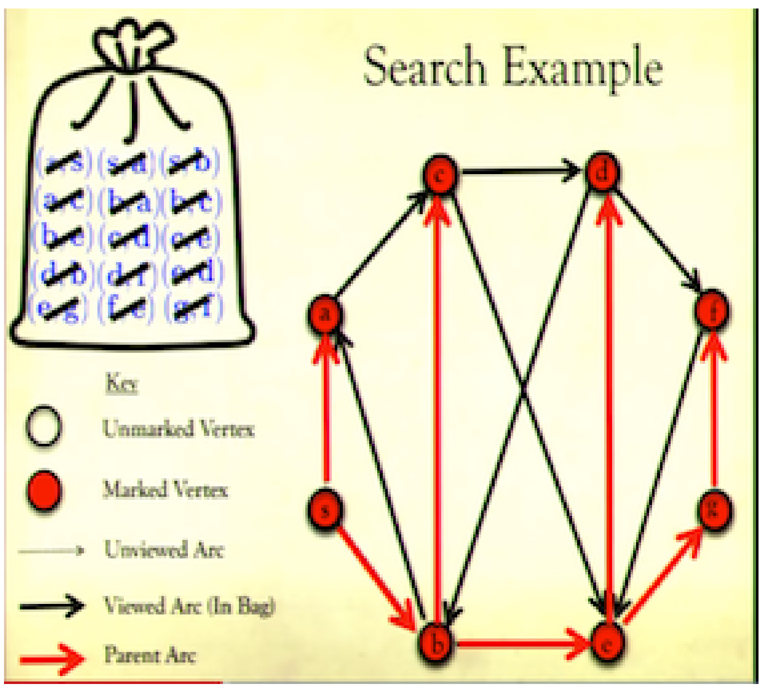
  Search(r)
  Set $B = \{r\}$
    While $B \neq 0$
      Let $v \in B$
      Set $B \leftarrow B \backslash \{v\}$
      If v is unmarked
        Mark v
        For each arc (v,w)
          Set $B \leftarrow B \cup \{w\}$

- We can view a vertex as having been discovered when it is **marked.**
- We will see that if the graph is connected then every vertex is discovered.
- But first observe that there are one **strange** thing about this algorithm.
  - We add a vertex to the bag when it is the end-point of an examined edge.
    - $\rightarrow$ Multiple copies of a vertex may be in the bag!
- Surprisingly this will not affect the performance of the algorithm.
  - Indeed, this will actually be useful to us if we modify the algorithm to account for this by "bagging" edges instead of vertices.

4 The Revised Generic Search Algorithm

- Updated algorithm:

  The Search Algorithm
  Put (*, r) into a bag
    While the bag is non-empty
      Remove (u, v) from the bag
      If v is unmarked
        Mark v
        Set $p(v) \leftarrow u$ *(Keep track of the predecessor)*
        For each arc(v,w)
          Put w into the bag



- Running Time:
  - We search each arc out of v only once, when v is first marked.
  - The arc is then added to the bag once and later removed from the bag once.
  - Runtime $= O(m)$

    - Linear time algorithm.

- Validity

  Theorem. Let G be a connected, undirected graph. Then the search algorithm finds every vertex in G.
  Proof.

    - We must show that each vertex v is **marked** by the search algorithm.
    - We prove this by induction on the smallest number k of edges in a path from the vertex to the root.

  Base Case: k = 0

    - Then v is the root vertex r.
    - But r is the first vertex marked in the algorithm, so the base case holds.

  Induction Hypothesis:

    - Assume any vertex v that has a path of (k - 1) (or fewer) edges to the root r is marked.

  Induction Step:

    - Assume there is a path P with k edges from v_k to r. Specifically let

$$P = \{v = v_k, v_{k-1}, ..., v_1, v_0 = r\}$$

    - Thus there is a path Q with (k - 1) edges from $u = v_{k-1}$ to r. That is.

$$Q = \{u = v_{k-1}, ..., v_1, v_0 = r\}$$

    - So, by the induction hypothesis, the vertex u is marked.
    - After we mark u, we place the edges incident to it in the bag,
      → The edge (u, v) is added to the bag.
    - Thus, when (u, v) is removed from the bag we will mark v. (If it is not already marked.)

- Validity in Directed Graphs

    - For directed graphs, a similar argument proves that each vertex that has a directed path to it from the root r is **marked.**

5 Search Tree

- Observe that each non-root vertex has exactly one predecessor.
  Theorem. Let G be a connected, undirected graph. Then the predecessor edges form a tree rooted at r.
  Proof.
  - By induction on the number of k of marked vertices.

  Base Case: k = 1
  - Then v is the first vertex marked.
    → So the base case holds.

  Induction Hypothesis:
  - Assume the predecessor edges for the first (k -1) marked vertices form a tree rooted at r.

  Induction Step:
  - Let v be the kth vertex to be marked.
  - Assume v was marked when we removed the edge (u, v).
    → $u = p(v)$
  - But (u, v) was added to the bag when we marked vertex u.
    → Vertex u is in the set S of the first (k - 1) vertices to be marked.
  - By the induction hypothesis, the predecessor edges for S form a tree T rooted at r.
  - $T \cup (p(v), v)$ is a tree rooted at r on the first k marked vertices.

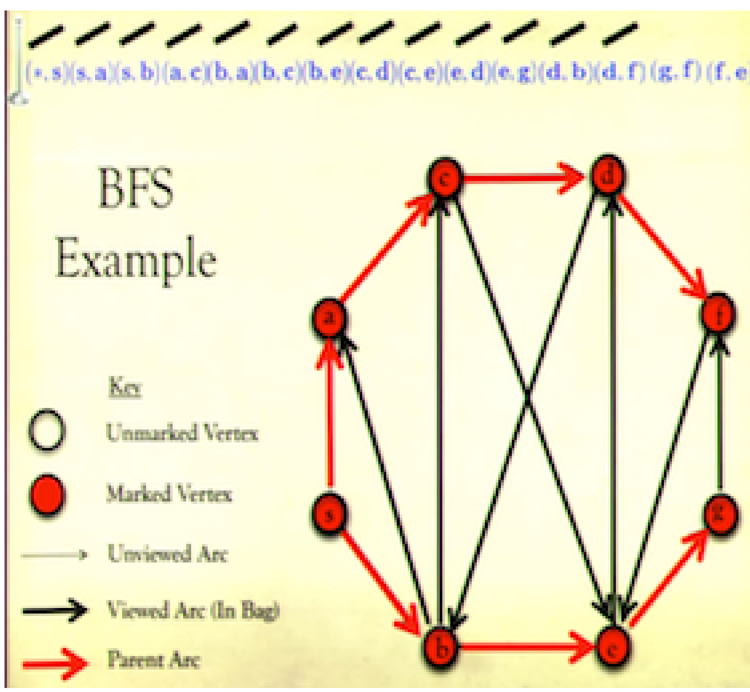
6 Inplementational Aspects

- There is some flexibility in how to implement the search algorithm?
  - If the bag contains many arcs, which one should we remove?
- In fact, what in computer science is a "bag"?
- The "bag" is just short-hand for a data structure.

- Moreover, the choice of data structure used has important consequences.
- Three choices of data structure for the bag give fundamental algorithms:
  - Queue: Breadth First Search
  - Stack: Depth First Search
  - Priority Queue: Minimum Spanning Tree Algorithm

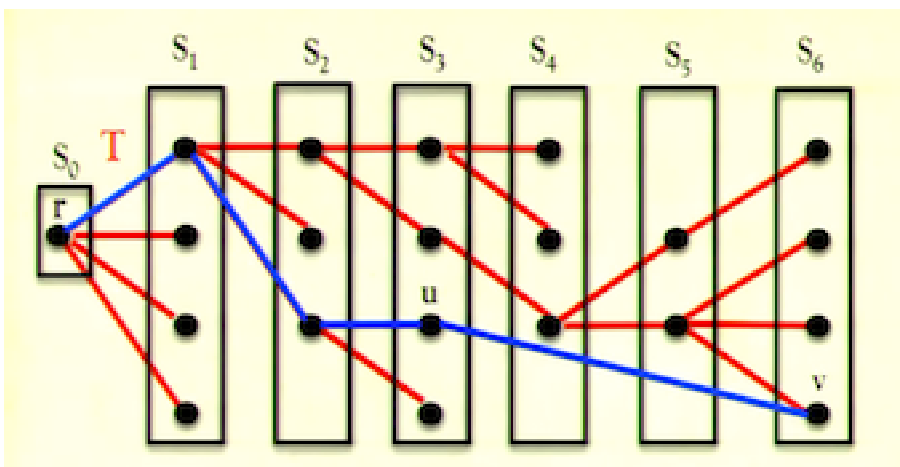
7 Breadth First Search

- Using a **Queue** (FIFO) data structure produces:
  Breadth First Search Algorithm
  Add (*, r) to a Queue
  While the Queue is non-empty
  Remove the first arc (u, v) from the Queue
  If v is unmarked
  Mark v
  Set $p(v) \leftarrow u$
  For each arc(v,w)
  Add (v, w) into the back of the Queue



8 Breadth First Search Trees

- Because BFS uses a queue it is easy to verify that:
  - The edges are added to the queue in order of their distance from r/
  - The vertices are marked in order of their distance from r.
- Specifically:
  Theorem. For any vertex v, the path from v to r given by the search tree T of predecessor edges is a shortest path.
  Exercise: Verify this formally.
- Structure:
  - Let S_l be the set or "layer" of vertices at distance l from r in T.
  - Then a vertex $v \in S_l$ is also at distance l from r in the whole graph G.
    - → Thus implies for every non-tree edge (u, v), u and v are either in the same layer or in adjacent layers.
    - → If not, we can find a shorter path to the root from u or v.



9 BFS and Bipartite Graphs

- Recall in a **bipartite graph** the vertex set can be partitioned as $V = X \cup Y$ such that every edge has one end-vertex in X and one end-vertex in Y.

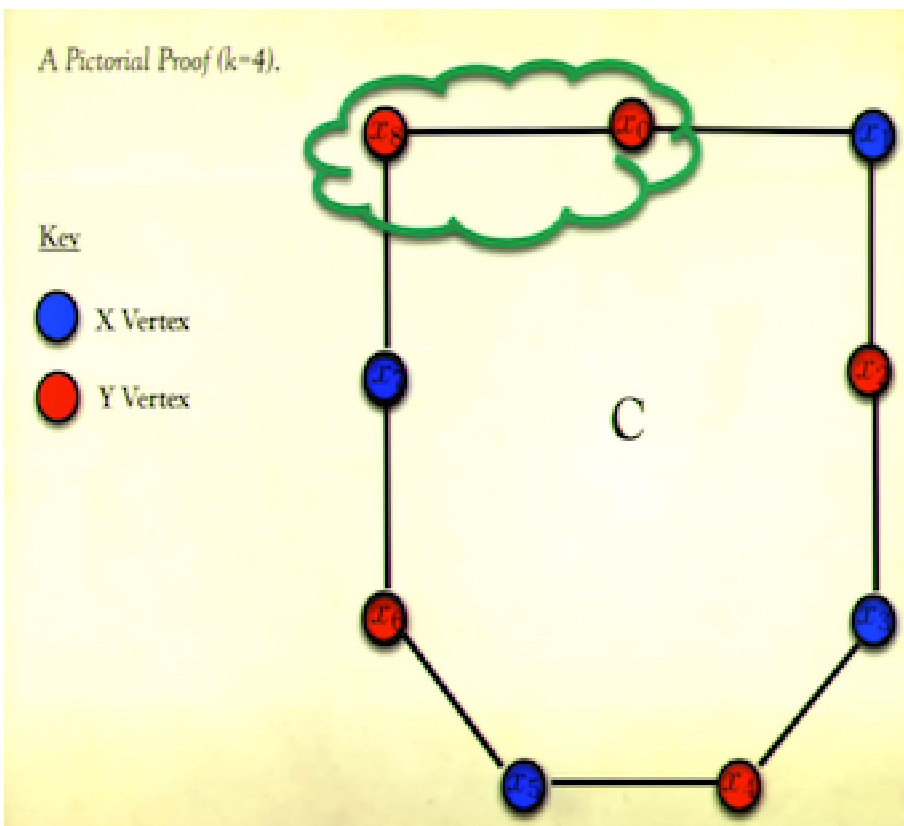
Theorem. A graph G is bipartite if and only if it contains no odd length cycles.
Proof.
($\Rightarrow$)

- Assume G contains as a subgraph an odd length cycle C.
- let $C = \{v_0, v_1, v_2, ..., v_{2k}\}$
- Wlog we may assume vertex $v_0 \in Y$. Therefore:

$$v_0 \in Y \Rightarrow v_1 \in Y \Rightarrow \cdots \Rightarrow v_{2k} \in Y$$

- But $(v_0, v_{2k}) \in E$. Thus

$$v_0 \in Y \Rightarrow v_{2k} \in Y$$



(⇐)

- Assume G contains no odd length cycles.
- Select an arbitrary root vertex r and run the BFS algorithm.
- Recall that for every non-tree edge (u, v), u and v are either in the same layer or in adjacent layers.
- Now set:

$$X = \bigcup_{l\ odd} S_l \quad Y = \bigcup_{l\ even} S_l$$

- Suppose, for every non-tree edge (u, v), that u and v are in adjacent layers.
- Now, for every tree edge (u, v) we also have u and v in adjacent layers.
- So assume, there is non-tree edge (u, v) with u and v in the same layer.
- Let z be the closest common ancestor of u and v in the search tree T.
- Let P be the path in T from u to z; let Q be the path in T from v to z.
- Since u and v in the same layer we have: $|P| = |Q|$
- But then the cycle $C = P \cup Q \cup (u, v)$ has an odd number of edges as: