

Lecture 2: Recursive Algorithms: MergeSort; Binary Search; The Master Theorem; The Recursive Tree Method

1 Reductions and Sub-Routines

- Solving a problem by **reducing** it (or a sub-problem of it) to another problem is the most fundamental technique in algorithm design.
- Specifically, algorithm A may use another algorithm B as a sub-routine.
- This has numerous advantages:
 - **Code Verification**: the correctness of A is independent of B .
 - **Code Reuse**: a great time-saver.
- A simple but very powerful special case of this paradigm is when the algorithm calls itself!
 - This method is called **recursion**.

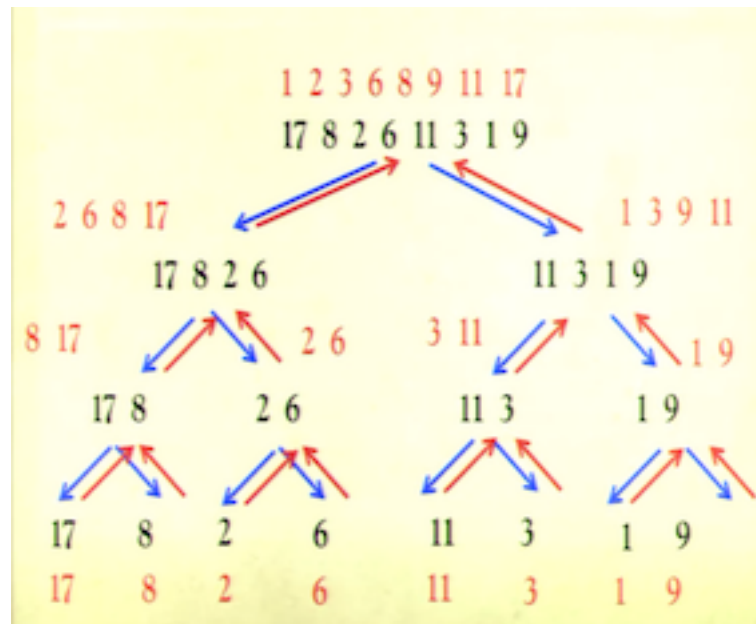
2 MergeSort

- We can sort n numbers into non-decreasing order using the following algorithm:

MergeSort(x_1, x_2, \dots, x_n)

If $n = 1$ then output x_1

Else output Merge{MergeSort($x_1, \dots, x_{\frac{n}{2}}$), MergeSort($x_{\frac{n}{2}+1}, \dots, x_n$)}



- Two Problems:

- Does the algorithm work?

Yes!

→ The algorithm calls itself on smaller instances

★ The division process terminates with a set of base cases of size 1.

→ MergeSort trivially works on the base cases.

→ So, given the validity of the Merge Step, the correctness of the algorithm follows by **strong induction**.

★ As long as base case is correct and merge step works, everything will be fine.

- If so, is it efficient (polynomial time)?

Yes! Look at the recursive formula.

→ To analyze this we represent the running time $T(n)$ via a **recurrence**:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

★ $2 \cdot T\left(\frac{n}{2}\right)$: Recurse on two problems with half the size.

★ $c \cdot n$: It takes linear time to merge two sorted lists.

Base Case: $T(1) = 1$

★ Or we can use $T(c) = O(1)$ for any constant c .

→ The Running Time of MergeSort

★ Theorem: MergeSort runs in time $O(n \cdot \log n)$

★ Proof:

1. By adding dummy numbers, we may assume n is a power of two: $n = 2^k$

2. We can unwind the recursive formula as follows:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Proof [cont.]

- But this unwinding operation can be repeated:

$$\begin{aligned}
 T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n \\
 &= 2 \cdot \left(2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}\right) + cn \\
 &= 2^2 \cdot T\left(\frac{n}{4}\right) + 2 \cdot cn \\
 &= 2^2 \cdot \left(2 \cdot T\left(\frac{n}{8}\right) + c \cdot \frac{n}{4}\right) + 2 \cdot cn \\
 &= 2^3 \cdot T\left(\frac{n}{8}\right) + 3 \cdot cn \\
 &= 2^3 \cdot \left(2 \cdot T\left(\frac{n}{16}\right) + c \cdot \frac{n}{8}\right) + 3 \cdot cn \\
 &= 2^4 \cdot T\left(\frac{n}{16}\right) + 4 \cdot cn \\
 &\vdots \\
 &= 2^k \cdot T(1) + k \cdot cn \quad \text{— Since } n = 2^k
 \end{aligned}$$

Proof [cont.]

- Thus we have:

$$T(n) = 2^k \cdot T(1) + k \cdot cn$$

$$= n \cdot T(1) + k \cdot cn$$

$$= n \cdot (1 + k \cdot c)$$

- But c is a constant so:

$$T(n) = O(n \cdot k)$$

- Furthermore, $k = \log n$, so we get that:

$$T(n) = O(n \cdot \log n)$$



3 Binary Search

- We can search for a key k in a sorted array of cardinality n using the binary search algorithm:

BinarySearch($a_1, a_2, \dots, a_n : k$)

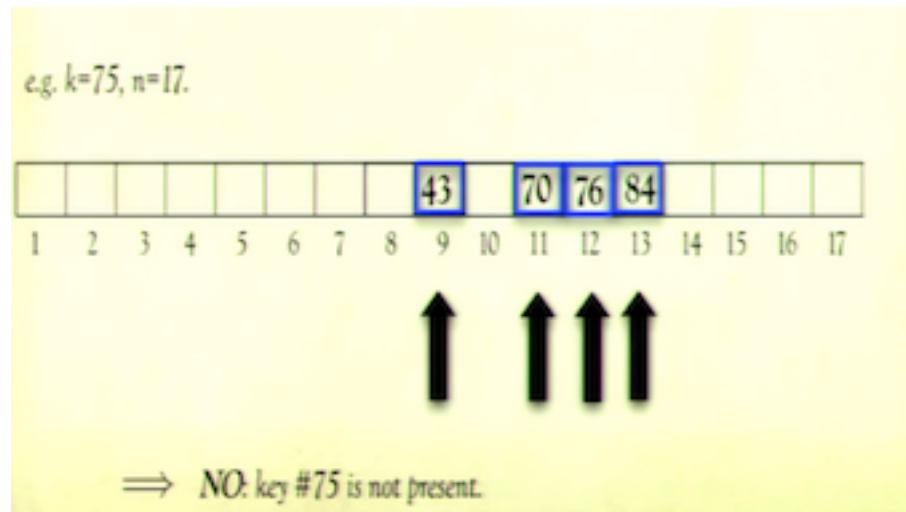
While $n > 0$ do:

If $a_{\frac{n}{2}} = k$ output YES

Else if $a_{\frac{n}{2}} > k$ output BinarySearch($a_1, a_2, \dots, a_{\frac{n}{2}-1} : k$)

Else if $a_{\frac{n}{2}} < k$ output BinarySearch($a_{\frac{n}{2}+1}, \dots, a_n : k$)

Output NO



- Does this work?
 - The validity of the binary search follows simply by strong induction. (The base case is trivially true.)
- Running Time?
 - Recurrence:

Recursive Formula: $T(n) = T(\frac{n}{2}) + c$

Base Case: $T(1) = 1$
 - Theorem: Binary Search runs in time $O(\log n)$
 1. By adding dummy numbers, we may assume n is a power of two: $n = 2^k$
 2. We can unwind the recursive formula as follows:

$$\begin{aligned}T(n) &= T\left(\frac{n}{2}\right) + c \\&= \left(T\left(\frac{n}{4}\right) + c\right) + c \\&= T\left(\frac{n}{4}\right) + 2 \cdot c\end{aligned}$$

- Again this unwinding operation can be repeated:

$$\begin{aligned}T(n) &= T\left(\frac{n}{4}\right) + 2 \cdot c \\&= \left(T\left(\frac{n}{8}\right) + c\right) + 2 \cdot c \\&= T\left(\frac{n}{8}\right) + 3 \cdot c \\&\vdots \\&= T\left(\frac{n}{2^k}\right) + k \cdot c\end{aligned}$$

- Hence:

$$\begin{aligned}T(n) &= T(1) + k \cdot c \quad \text{— Since } n = 2^k \\&= 1 + \log n \cdot c\end{aligned}$$

- This gives the claimed running time:

$$T(n) = O(\log n)$$

4 Divide and Conquer Algorithms

- A **divide and conquer** algorithm recursively breaks up a problem of size n in smaller sub-problems such that:
 - There are exactly a sub-problems.
 - Each sub-problem has size at most $\frac{1}{b} \cdot n$
 - Once solved, the solutions to the sub-problems can be combined to produce a solution to the original problem in time $O(n^d)$
- So the run-time of a divide and conquer algorithm satisfies the recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$$

- MergeSort and Binary Search are indeed **divide and conquer** algorithms.

	Recursion Formula	a	b	d
MergeSort	$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n^1)$	2	2	1
Binary Search	$T(n) = 1 \cdot T\left(\frac{n}{2}\right) + O(n^0)$	1	2	0

5 Non-Military Applications of Divide and Conquer

- Divide and Conquer has many other non-military, practical applications:
 - Big Data
 - Distributed Algorithms
 - Clustering and Classification
 - MapReduce

6 Dummy Entries

- MergeSort actually has the recurrence:

$$\hat{T}(n) = \hat{T}(\lceil \frac{n}{2} \rceil) + \hat{T}(\lfloor \frac{n}{2} \rfloor) + c \cdot n$$

- Recall we got around this by adding dummy entries:
 - We found \bar{n} the smallest power of 2 greater than n .
 - For this case, MergeSort then does have recurrence:

$$T(n) = 2 \cdot T(\frac{n}{2}) + c \cdot n$$

- But we also have:

$$\hat{T}(n) \leq T(\bar{n}) = O(\bar{n} \cdot \log \bar{n}) = O(n \cdot \log n)$$

- Here is another way to solve the recurrence:

$$\hat{T}(n) = \hat{T}(\lceil \frac{n}{2} \rceil) + \hat{T}(\lfloor \frac{n}{2} \rfloor) + c \cdot n$$

- As we only want to upper bound the running time, we can use:

$$\hat{T}(n) \leq \hat{T}(\frac{n}{2} + 1) + c \cdot n$$

Note: This +1 does not seem to fit with our methodology, but we can fix this by applying a **domain transformation**.

- Domain Transformation
 - For the domain transformation, simply set: $T(n) = \hat{T}(n + 2)$
 - Thus we have: $T(n) = T(\frac{n}{2}) + \hat{c} \cdot n$

$$\begin{aligned} T(n) &= \hat{T}(n+2) \\ &\leq \hat{T}\left(\frac{n+2}{2} + 1\right) + c \cdot (n+2) \\ &\leq \hat{T}\left(\frac{n+2}{2} + 1\right) + \hat{c} \cdot n \\ &= \hat{T}\left(\frac{n}{2} + 2\right) + \hat{c} \cdot n \\ &= T\left(\frac{n}{2}\right) + \hat{c} \cdot n \end{aligned}$$

- Of course, we can solve this recurrence as: $T(n) = O(n \cdot \log n)$
- Therefore, $\hat{T}(n) = T(n - 2) = O(n \cdot \log n)$
- As well as ceilings and floors, domain transformations can be used to simplify many other recurrences; e.g. removing lower order terms.