Lecture 1

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1 The Central Paradigm of Computer Science

- The central ***paradigm*** in computer science is that an algorithm ***A*** is good if:
  - ***A*** runs in ***polynomial time*** in the input size n.
  - That is, ***A*** runs in time $T(\text{n}) = O(n^k)$ for some constant number k.
    - $T(\text{n}) = 100\text{n} + 55$
    - $T(\text{n}) = \frac{1}{2}n^2 + 999 \log n$
    - $T(\text{n}) = 6n^7 + 900000n^2 - \sqrt{n}$
  - An algorithm is ***bad*** if it runs in exponential time.
    - $T(\text{n}) = 2^n + 100n^5$
    - $T(\text{n}) = 1.000000001^n - n^3 - \text{n}$
  - An algorithm is ***good*** if it runs in ***polynomial time*** in the input size n.

e.g.

		Input Size n		
		10	100	1000
Runtime of Algorithm	n	10	100	1000
	n^2	100	10000	1000000
	2^n	10^3	10^{30}	10^{300}

2 Good versus Bad (Algorithms)

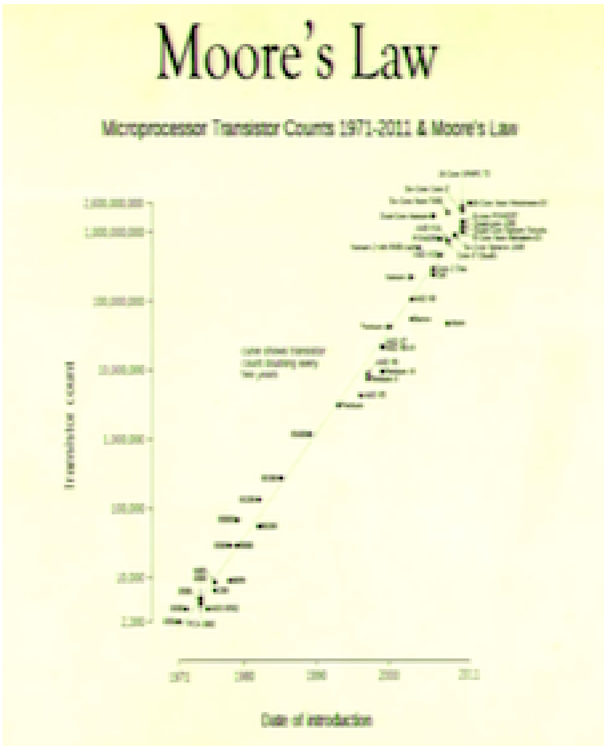
- For example, consider the problem of sorting n numbers.
  - A Good Algorithm: **MergeSort** runs in time $O(n \cdot logn)$
  - A Bad Algorithm: **BruteForce Search** runs in time $O(n \cdot n!) \gg 2^n$

3 An Equivalent Characterization

- This central ***paradigm*** has an equivalent formulation
  - ***A*** runs in ***polynomial time*** in the input size n.
  - The input sizes that ***A*** can solve, in a fixed amount T of time, ***scales multiplicatively*** with increasing computational power.

Input Sizes solved in Time T

		Power = 1	Power = 2
Runtime of Algorithm	n	T	$2T$
	n^2	$\sqrt{T}$	$\sqrt{2} \cdot \sqrt{T}$
	2^n	$\log T$	$1 + \log T$



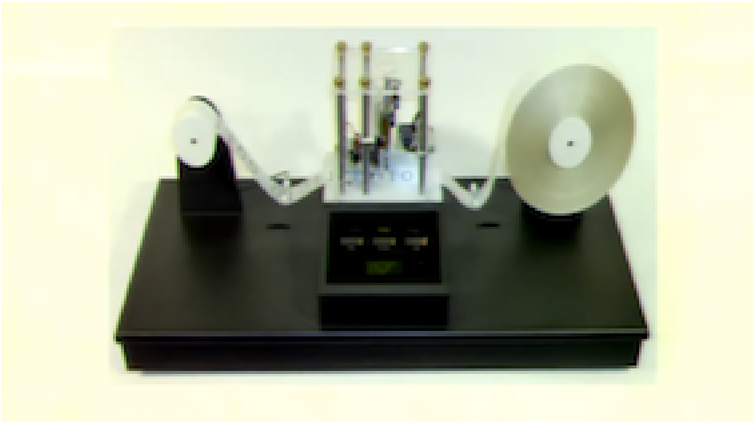
  - Moore's Law: Computational power *doubles* roughly every two years.

    → Functional time algorithms will never be able to solve large problems.

- The practical implications are perhaps simpler to understand with this latter formulation.
- Thus, improvements in hardware will *never* overcome ***bad algorithm design***.
- Indeed, the current dramatic breakthroughs in computer science are based upon batter (faster and higher performance) algorithmic techniques.

4 Robustness

- This measure of quality or "goodness" is ***robust***



- All reasonable models of algorithms are polynomial time equivalent.
  - Otherwise one model could perform, say, an exponential number of operations in the time another model took to perform just one.
- The standard formal model is the **Turing Machine**.

5 Cryptography (Just an example of the course)

- Alice wants to send Bob a message.
  But she is worried that Eve might intercept the message.
  She decides to encrypt the message M as $\hat{M} = f(M)$.
  Bob can then decrypt the message via $f^{-1}(\hat{M}) = f^{-1}(f(M)) = M$
  Eve cannot understand the message.
  Alice encrypts the message M with the (encryption) lock f.
  Bob decrypts the message $\hat{M}$ with the (decryption) key f^{-1}.
  Because Eve does not have the key she cannot decipher the message.
  Two major problems occur.
- Problem one:
  - Eve might be able to break the code.
    - → That is , given $\hat{M}$ she may be able to reconstruct f and then f^{-1}.
  - Standard techniques for code-breaking include:
    - → Frequency Analysis
      - · *e.g.* "e" is the most common letter and "the" is the most common word in English.

→ Cribs.

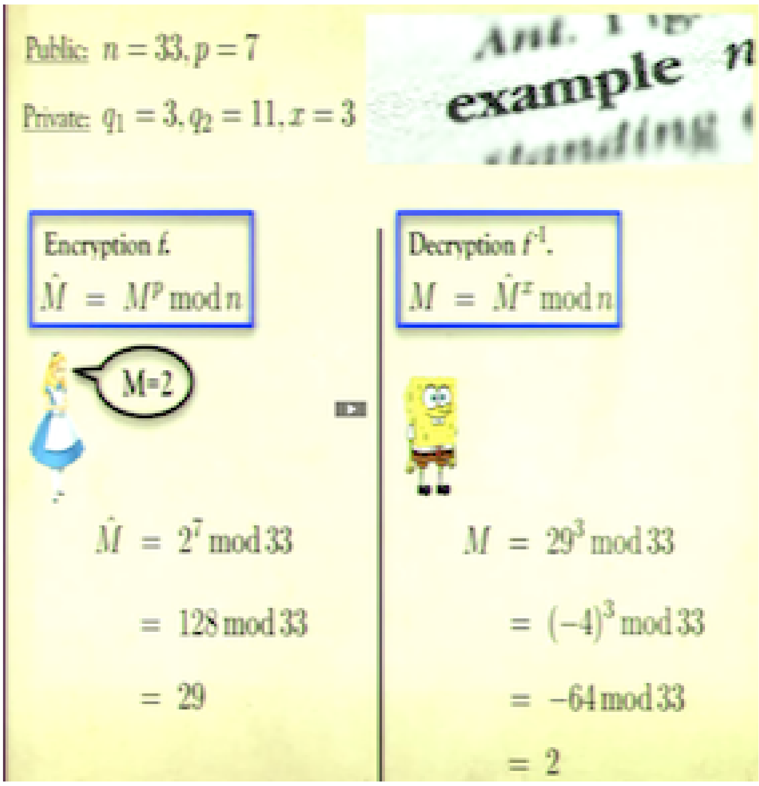
· *e.g.* German weather reports were exploited to help decode messages from the "unbreakable" **Enigma Machine**.

- Problem Two:
  - Alice and Bob need to agree on what the encryption code (lock) f is.
  - But to do this they need to exchange a message discussing the code.
- Public-Key Cryptography
  - In face, Bob gives absolutely everyone a copy of the (encryption) lock.
  - Everyone can send the message and Bob has the key.
  - The lock is made public. This is called public-key cryptography.
  - But this idea sounds completely crazy.
    - → Doesn't this solution to Problem Two make Problem One inevitable?
    - → That is, if Eve has the lock f then won't she use it to decode $f(M)$?
  - No, not if f is hard to invert for anybody except Bob himself.
  - But do such functions f that are hard to invert **exist**?
  - Yes!
- RSA Encryption
  1. Bob chooses two large prime numbers q_1, q_2 and a large number p that is co-prime to $(q_1$ - 1) · $(q_2$ - 1)
  2. Bob's public key is (p, n) where $n = q_1 \cdot q_2$

     Encryption: $\hat{M} = M^p$ mod n
  3. Bob's prime key is (q_1, q_2, x) where x is the inverse of p modular $(q_1$ - 1) · $(q_2$ - 1)

     Decryption: $M = \hat{M}^x$ mod n

     - p = 2^{16} + 1
     - q_1, q_2 = 2048 bits
  4. An Example
     (a) Prime numbers q_1, q_2 and number p co-prime to $(q_1$ - 1) · $(q_2$ - 1)
     p = 7, q_1 = 3, q_2 = 11 valid as gcd(7,20) = 1
     (b) Public key (p,n) where n = $q_1 \cdot q_2$
     n = 33
     (c) Private Key (q_1, q_2, x) where x is the inverse of p mod $(q_1$ - 1) · $(q_2$ - 1)
     x = 33 as 3 · 7 = 1 mod 20
     Public: n = 33, p = 7
     Private q_1 = 3, q_2 = 11, x = 3



- Is RSA Encryption Safe?
  - Public-key cryptography lies at the heart of the modern economy.
    - → Financial Services
    - → Online Shopping
    - → Secure Messaging
  - So it is extremely important that the method is safe.
  - We claim it is because:
    - → Bob has a good algorithm for decryption.
    - → Eve only has a bad algorithm for decryption.

- Bob has a Good Decryption Algorithm
  - Initially, Bob can do the following in polynomial time.
    - → Choose the primes q_1, q_2
    - → Choose a number p that is co-prime with $(q_1 - 1) \cdot (q_2 - 1)$
    - → Find x the inverse (Using Euclid's Algorithm) of p and $(q_1 - 1) \cdot (q_2 - 1)$
  - Using fast exponentiation, encoding and decoding is polynomial time.

$$\text{Encryption: } \hat{M} = M^p \text{ mod n}$$

- Eve has a Bad Decryption Algorithm

$$\text{Decryption: } M = \hat{M}^x \text{ mod n}$$

  - To decrypt Eve needs to find x the inverse of p and $(q_1 - 1) \cdot (q_2 - 1)$
    - → She knows p, but does not know q_1, q_2
    - → Instead, she only knows n = $q_1 \cdot q_2$
  - So to find q_1, q_2, she needs to find the prime factorization of n.

- But it is believed that finding the prime factors of a b-bit number is hard.
  - $\rightarrow$ Instead, the obvious algorithm attempts to divide n by each integer in the range $\{2,3, \ldots ,2^b\} \leftarrow$ 4096 bits
- In fact, if Eve could find x without q_1, q_2, then she could use x to find q_1, q_2. That is, she could factor n.
  $\Rightarrow$ Eve only has exponential time algorithm to decrypt.