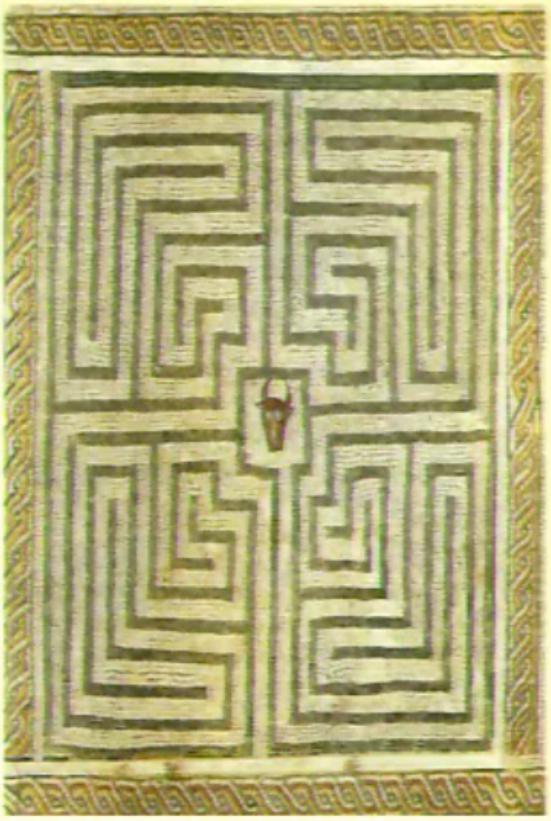
The Generic Search Algorithm



- Recall our **generic graph search algorithm:**

The Search Algorithm

Put $(*, r)$ into a bag

While the bag is non-empty

Remove (u,v) from the bag

If v is unmarked

Mark v

Set $p(v) \leftarrow u$

For each arc (v,w)

Put (v,w) into the bag

Data Structures

- Recall, three choices of **data structure** give fundamental algorithms:

Queue.

Breadth First Search.

Stack.

Depth First Search.

Priority Queue.

Minimum Spanning Tree Algorithm.

Depth First Search

- Using a **Stack** (LIFO) data structure produces:

Depth First Search Algorithm

Add $(*, r)$ to a Stack

While the Stack is non-empty

Remove the first arc (u,v) from the Stack

If v is unmarked

Mark v

Set $p(v) \leftarrow u$

For each arc (v,w)

Add (v,w) into the back of the Stack

DFS Example

Key

Unmarked Vertex

Marked Vertex

Unviewed Arc

Parent Arc

Viewed Arc (In Bag)

(a, c)
~~(c, e)~~
~~(c, d)~~
~~(d, f)~~
~~(d, b)~~
~~(f, e)~~
~~(g, f)~~
~~(e, g)~~
~~(e, d)~~
~~(b, e)~~
~~(b, c)~~
~~(b, a)~~
~~(r, b)~~
(r, a)
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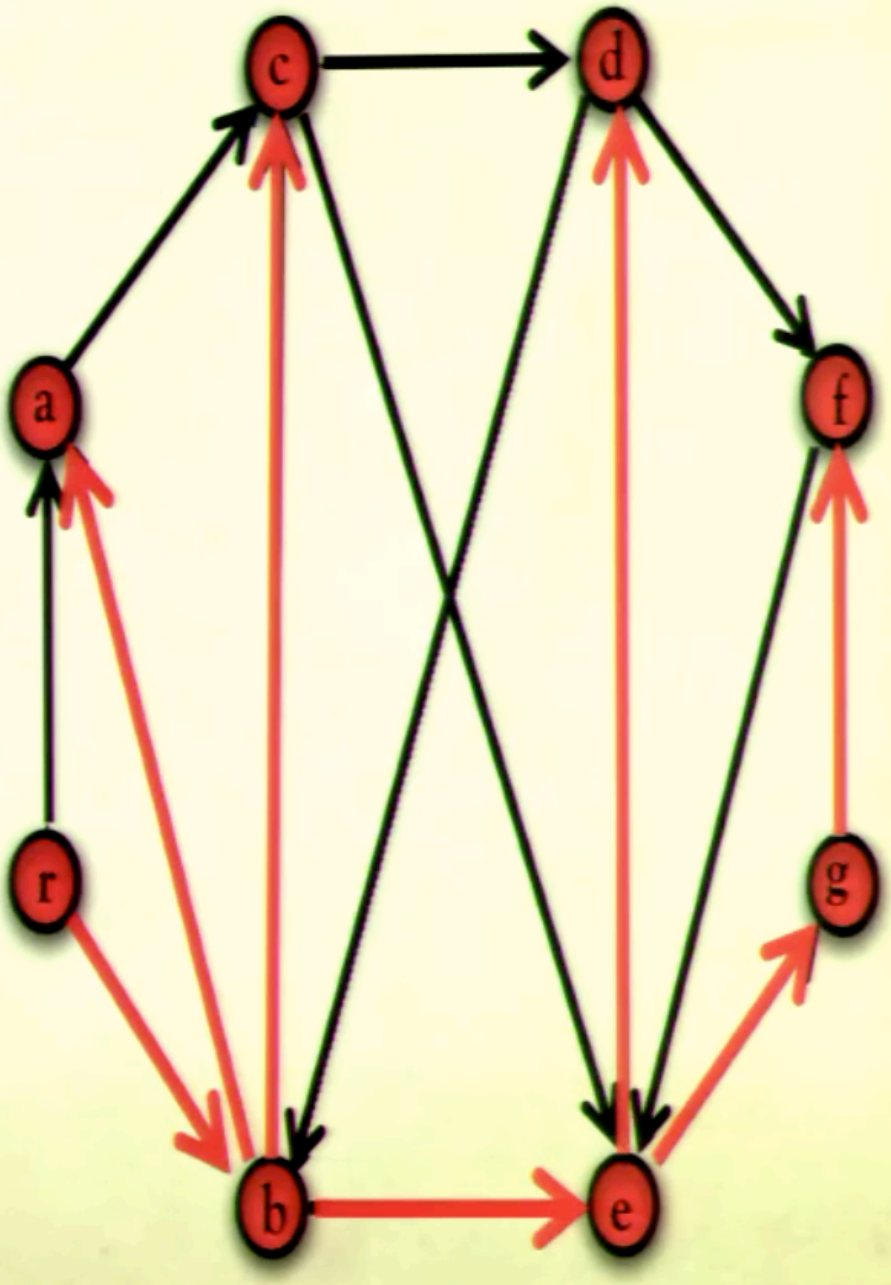

Depth First Search Trees

- Recall, we proved that the search algorithm produces a **search tree**.

Theorem. *Let G be a connected, undirected graph. Then the predecessor edges form a tree rooted at r.*

- However, the structure of the DFS tree is quite different from that of a BFS tree.
- So let's investigate this structure...

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Theorem. *Let G be a connected, undirected graph. Then the predecessor edges form a tree rooted at r.*

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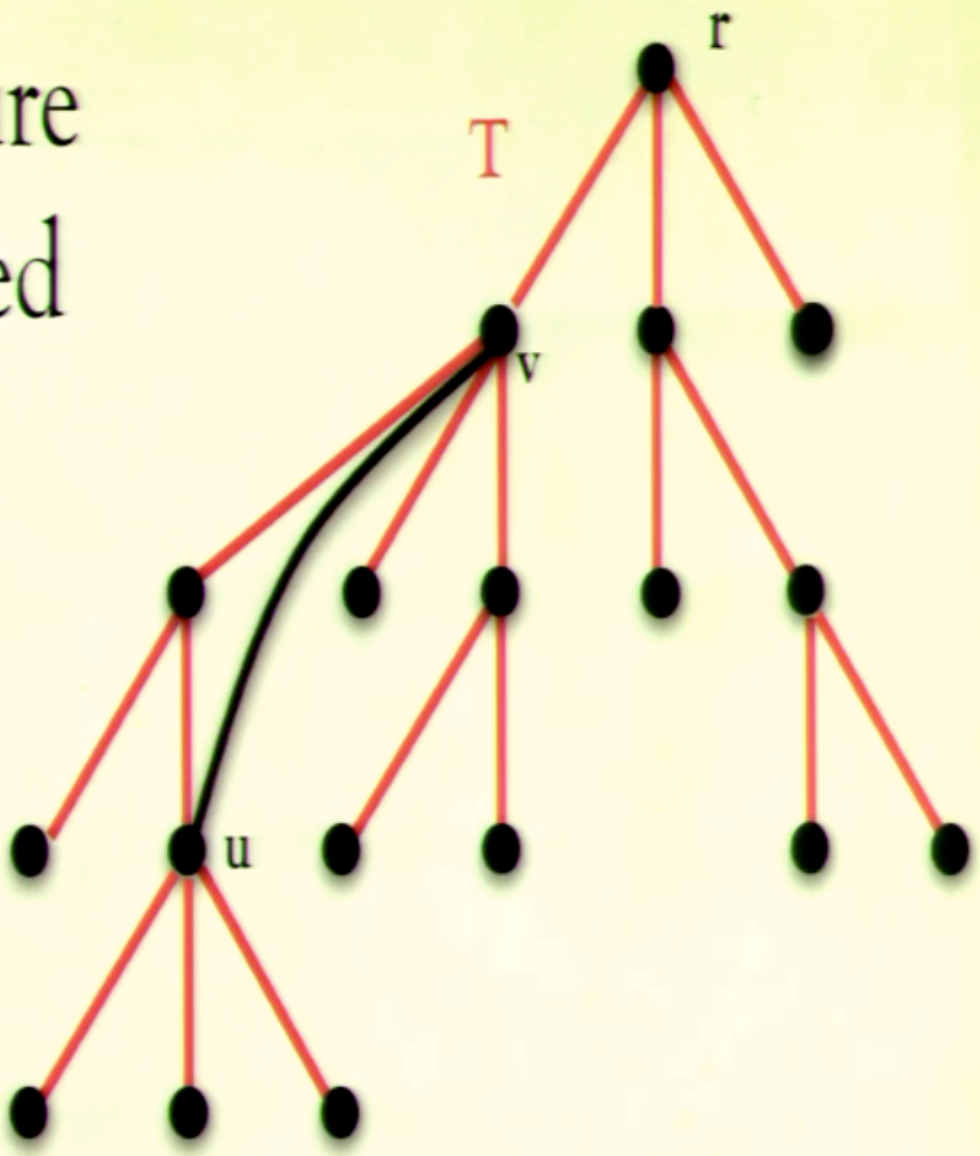
- So let's investigate this structure...

Edge Structure in Undirected Graphs

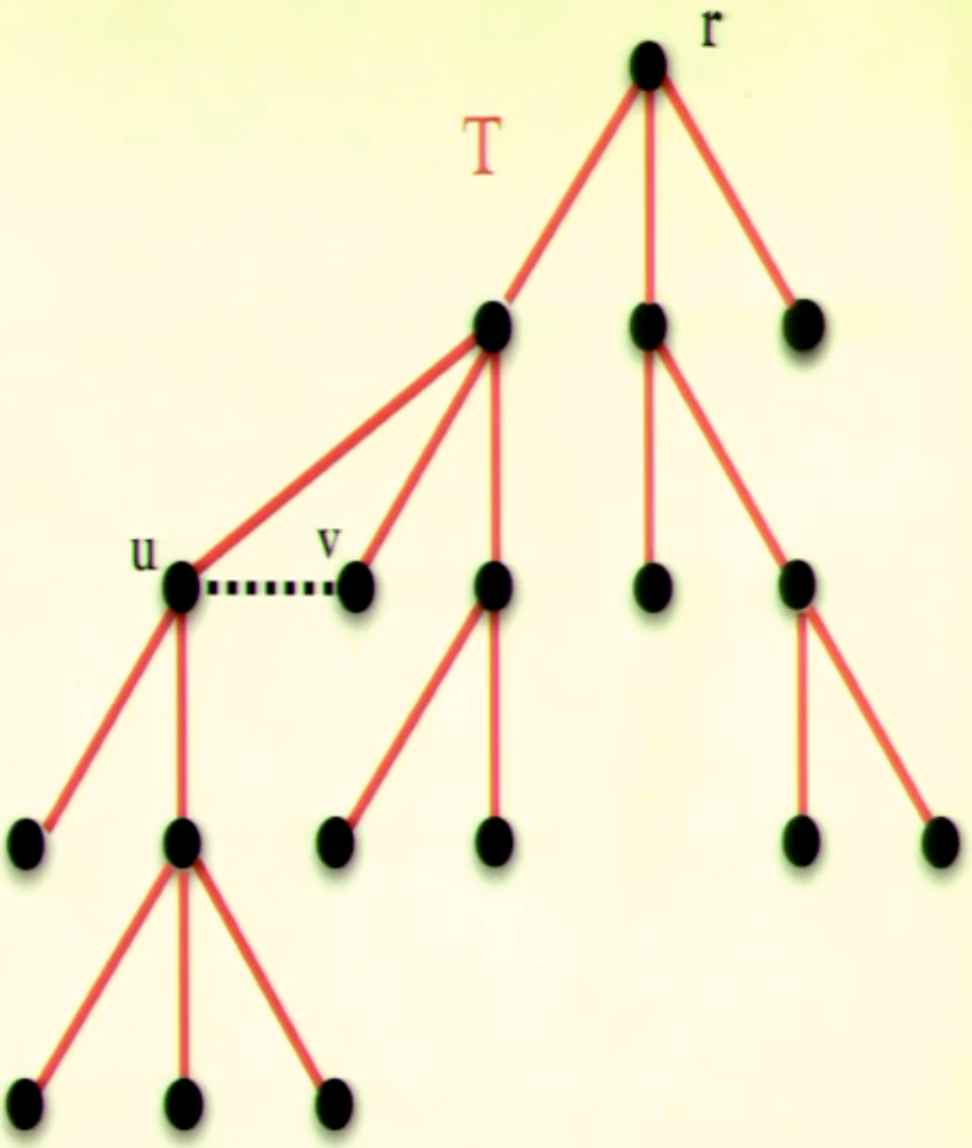


- Depth First Search partitions the edges of an undirected graph into two types:

  Tree Edges. *Predecessor edges in the DFS tree T.*

  Back Edges. *Edges where one endpoint is an ancestor of the other endpoint in T.*

Cross Edges



- When we perform a DFS, there no crosses of the following type:

Cross Edges. *Edges where neither endpoint is an ancestor of the other in T.*

Depth First Search (Recursive Description)

- The Depth First Search algorithm can be defined **recursively**:

```
RecursiveDFS(r)
Mark r
For each edge (r, v)
    If v is unmarked
        Set p(v) ← r
        RecursiveDFS(v)
```

- Using this formulation the aforementioned structure of the graph edges is easy to show...

```
RecursiveDFS(u)
Mark u
For each edge (u, v)
    If v is unmarked
        Set p(v) ← u
        RecursiveDFS(v)
```

Ancestral Edges [cont.]

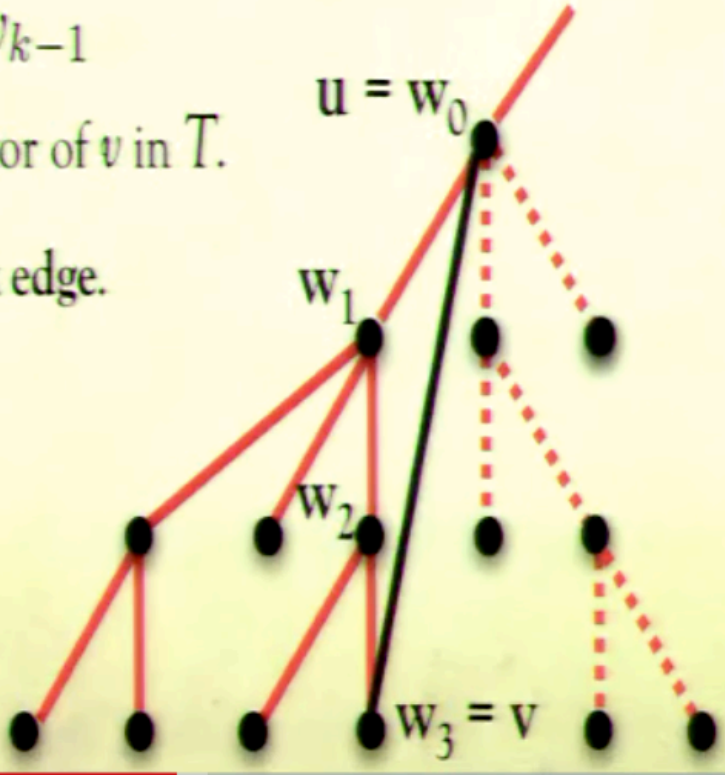
Proof [cont.]

Case 2: v is marked when `RecursiveDFS`(u) examines (u,v).

- But v was marked after u, so it was marked during `RecursiveDFS`(u)
- Thus, we have a series of vertices $\{u = w_0, w_1, \ldots, w_{\ell-1}, w_\ell = v\}$ where $p(w_k) = w_{k-1}$

  $\Longrightarrow$ u is an ancestor of v in T.

  $\Longrightarrow$ (u,v) is a **back edge**.



□

On DFS Edges: Tree, Back and Cross

Theorem. Let T be a DFS tree in an undirected graph G. Then, for every edge (u,v), either u is an ancestor of v in T or v is an ancestor of u.

Corollary. Let T be a DFS tree in an undirected graph G. Then, every non-tree edge is a back edge.

Previsit and Postvisit

- We can add a **clock** to record when we visit a vertex for the first (*previsit*) and last time (*postvisit*):

clock $\leftarrow 0$

`RecursiveDFS`(r)

Mark r

pre$(r) \leftarrow$ clock

clock $\leftarrow$ clock $+ 1$

For each edge (r, v)

If v is unmarked

Set $p(v) \leftarrow r$

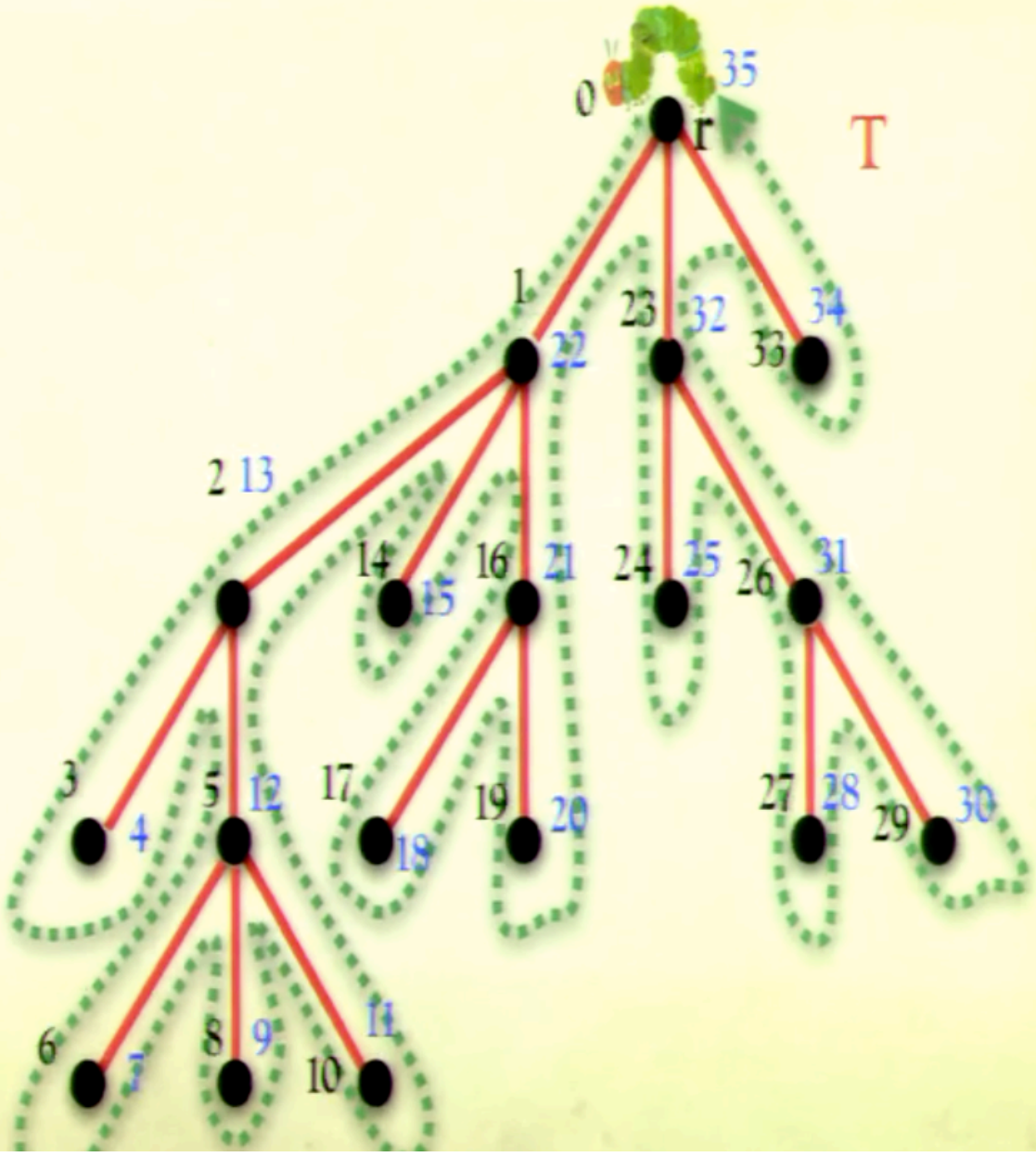
`RecursiveDFS`(v)

post$(r) \leftarrow$ clock

clock $\leftarrow$ clock $+ 1$

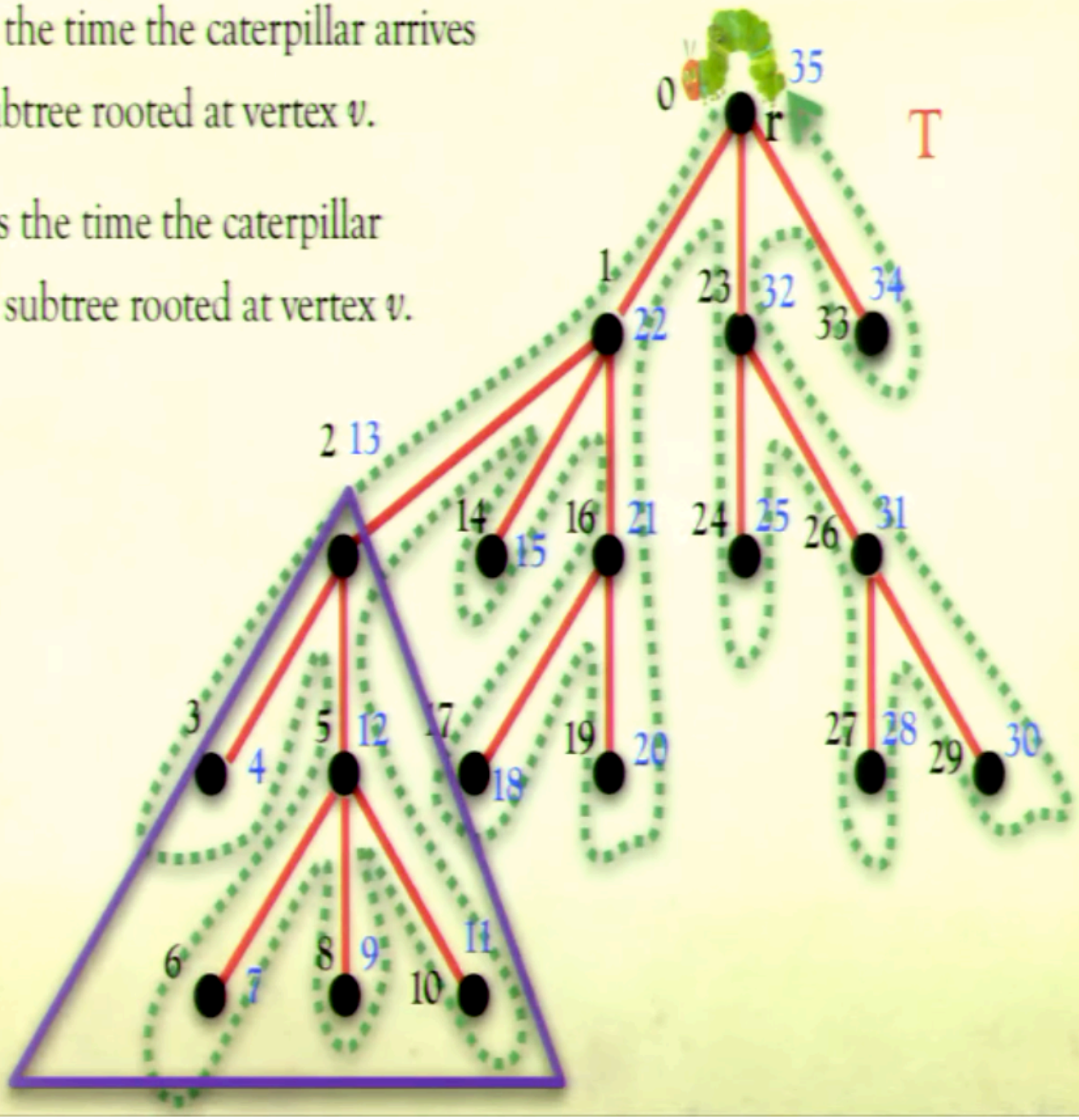
- *The previsit and postvisit times will highlight more properties of DFS...*

Previsit and Postvisit Times



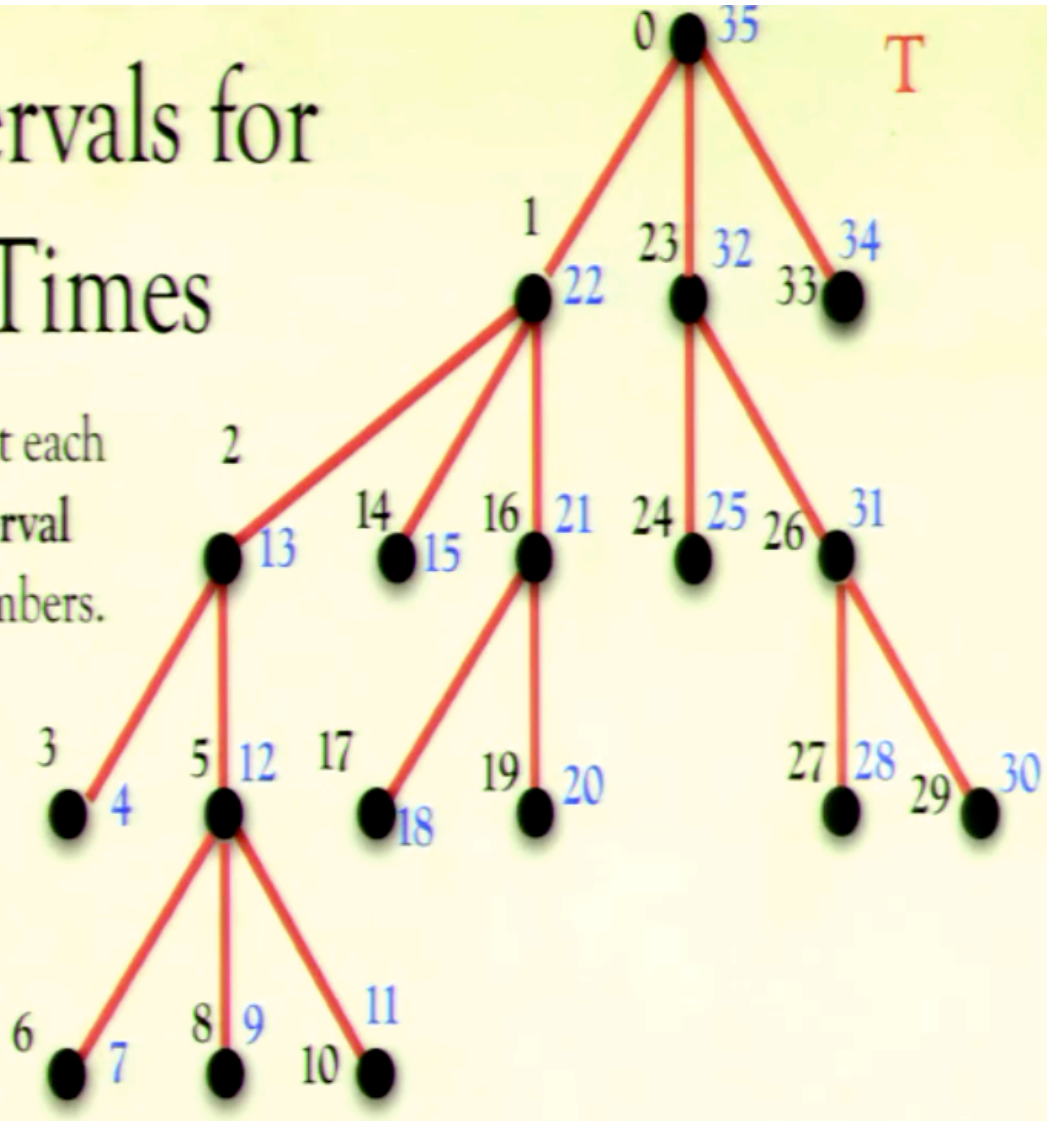
Previsit and Postvisit Times

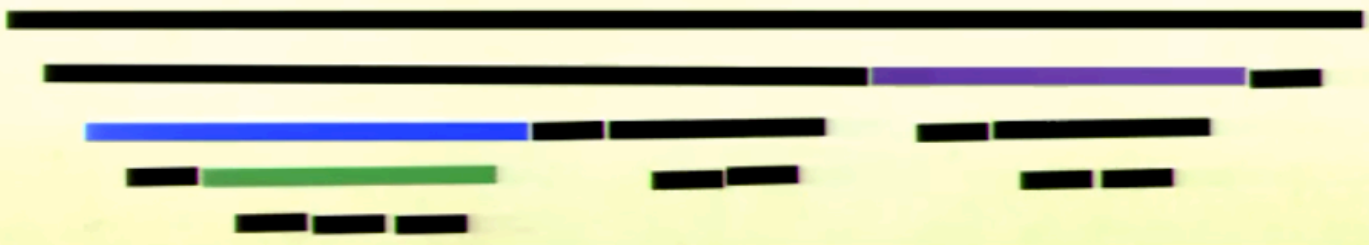
- Pre(v) is the time the caterpillar arrives at the subtree rooted at vertex v.
- Post(v) is the time the caterpillar exits the subtree rooted at vertex v.

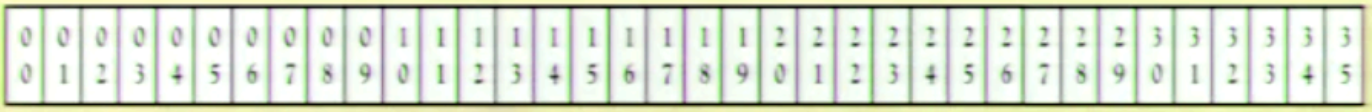


The Intervals for Visit Times

- We can represent each vertex by an **interval** over the real numbers.



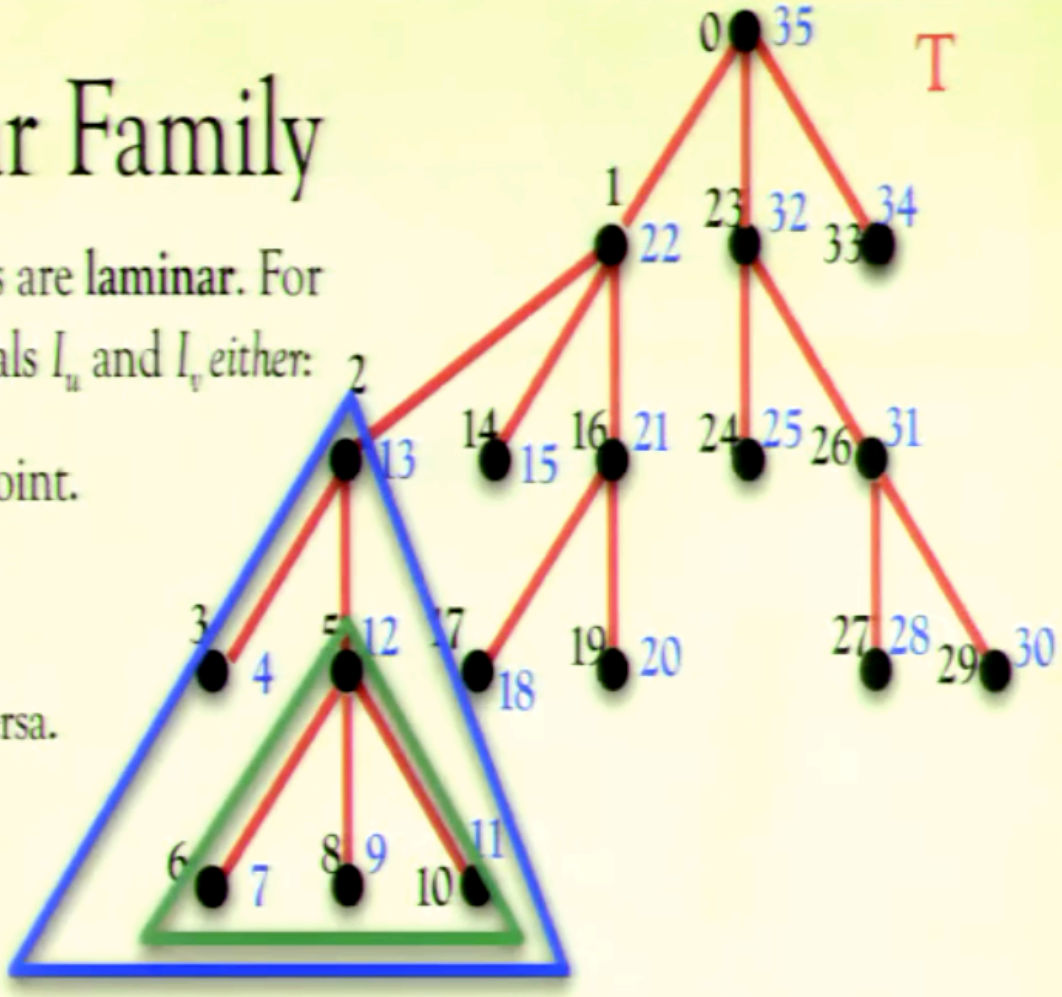




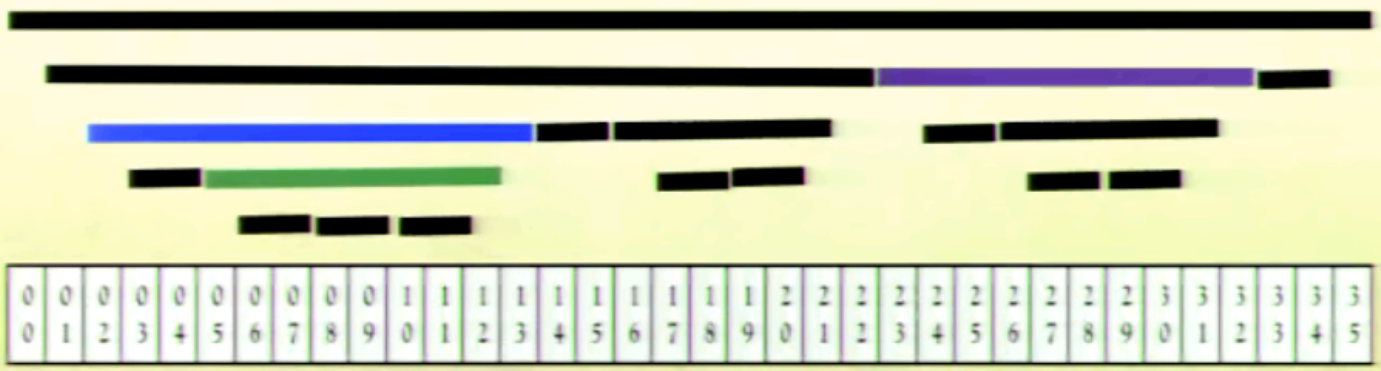
A Laminar Family

- The set of intervals are **laminar**. For each pair of intervals I_u and I_v *either*:
  - I_u and I_v are disjoint.

    or
  - $I_u \subseteq I_v$ or vice versa.

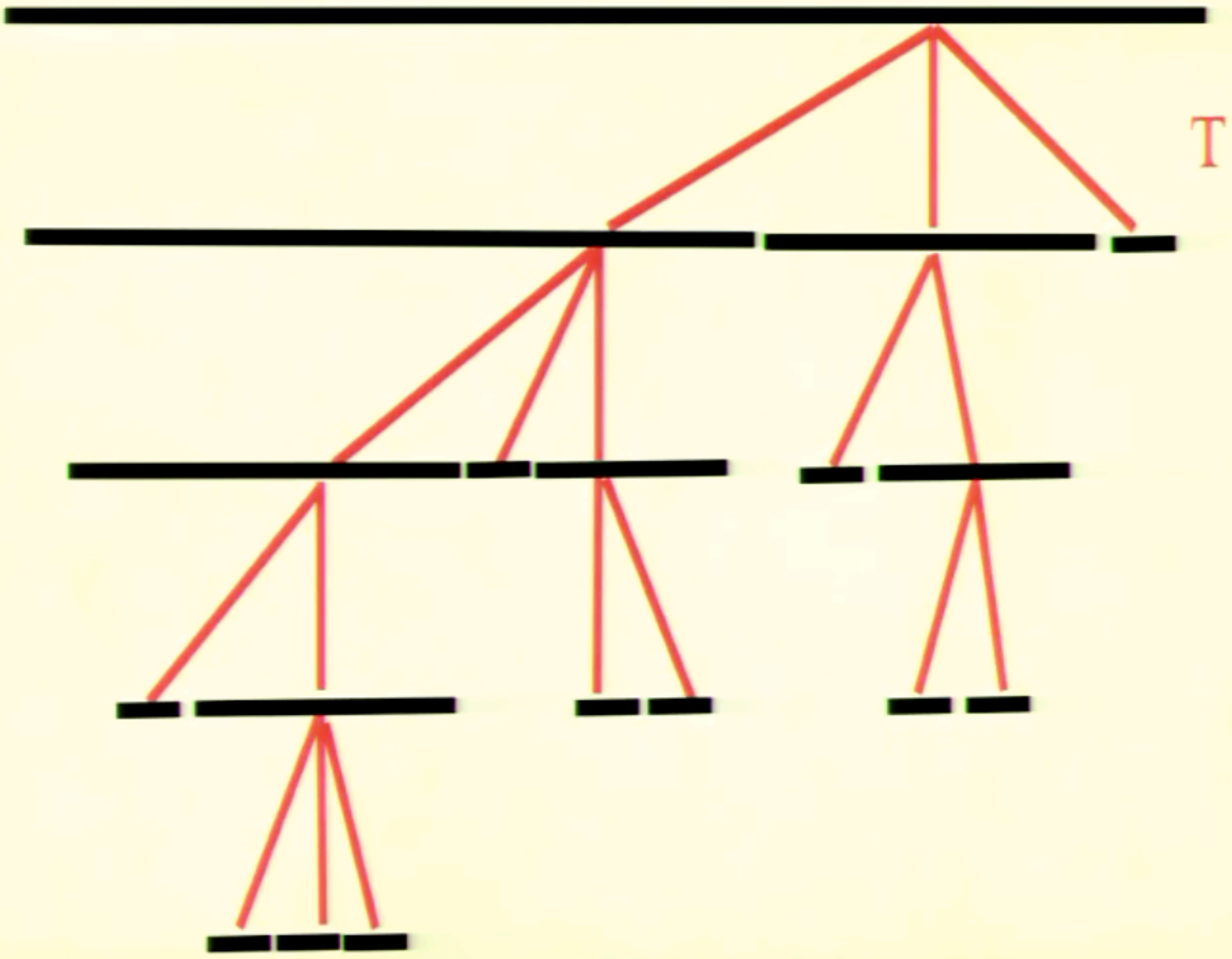


- This follows from the definition of the *worm crawl*, as the intervals correspond to subtrees.



The Tree of Intervals

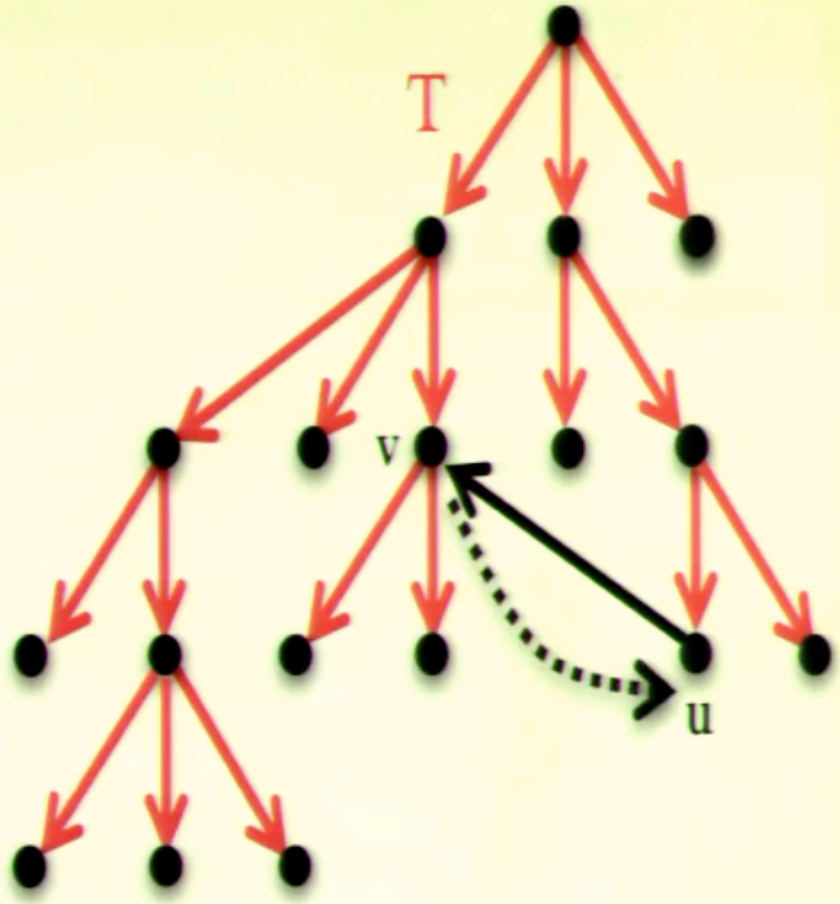
- Add an edge between an interval and the smallest interval that contains it.



- This is **exactly** the DFS tree!

00	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35

Arc Structure in Directed Graphs



- Depth First Search partitions the edges of an directed graph into four types:

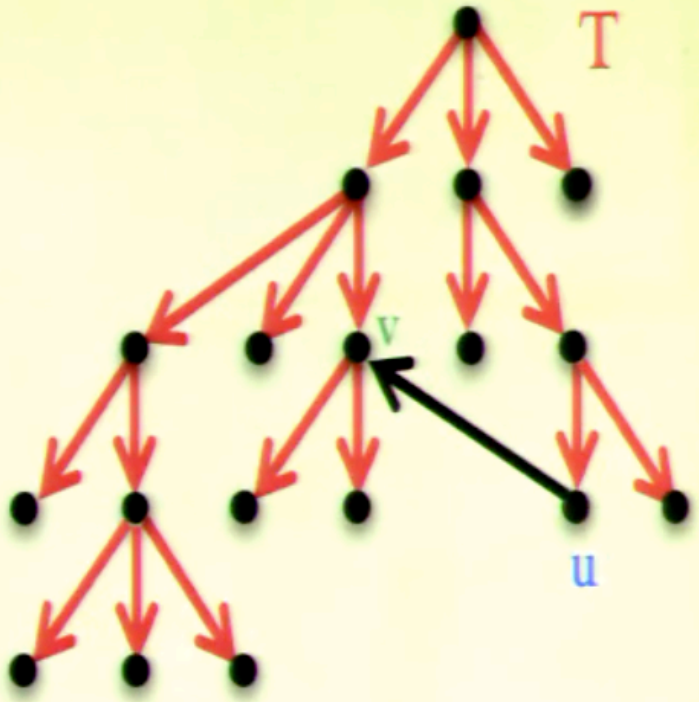
Tree Arcs. *Predecessor edges in the DFS tree T.*

Forward Arcs. *Arcs (u,v) where u is an ancestor of v.*

Backward Arcs. *Arcs (u,v) where u is descendant of v.*

Cross Arcs. *Non-Ancestral arcs (u,v) where u is marked after v.*

The Intervals for Directed Graphs



- The intervals for an arc (u,v) have the following properties:

Tree Arcs. *Predecessor edges in the DFS tree T.*

Forward Arcs. *Arcs (u,v) where u is an ancestor of v.*

Backward Arcs. *Arcs (u,v) where u is descendant of v.*

Cross Arcs. *Non-Ancestral arcs (u,v) where u is marked after v.*

Post-Visit Times

- The *post-visit* times for an arc (u, v) satisfy the following:

Tree Arcs.  post(v) < post(u)

Forward Arcs.  post(v) < post(u)

Backward Arcs.  post(u) < post(v)

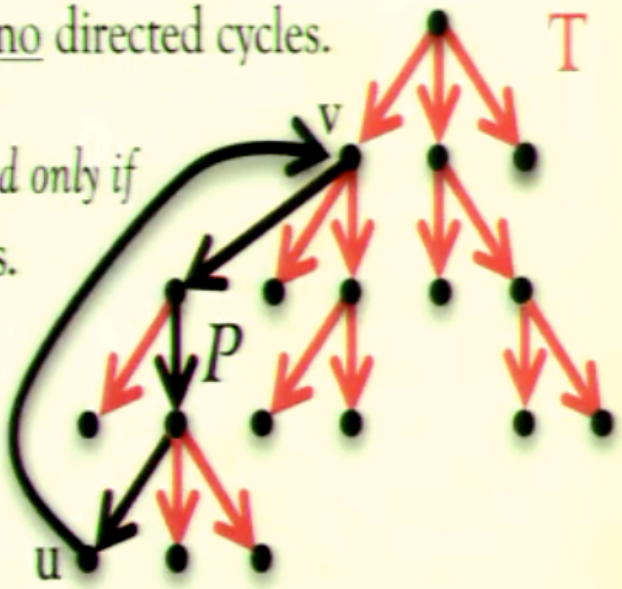
Cross Arcs.  post(v) < post(u)

- Observe that post(u) < post(v) only for backward arcs.

Directed Acyclic Graphs

- A directed graph is **acyclic** if it contains no directed cycles.

Theorem. A directed graph G is acyclic *if and only if* depth first search produces no backward arcs.



Proof.

$(\Rightarrow)$

- Let DFS give a backward arc (u, v).
- By definition, u is a descendant of v in the DFS tree T.
- So there is a path $P = \{v = v_0, v_1, \ldots, v_k = u\}$ in T.

$\Longrightarrow P \cup (u, v)$ is a directed cycle in G.

Theorem. A directed graph is acyclic *if and only if* depth first search produces no backward arcs.

Proof [cont.]

$(\Leftarrow)$

- Assume DFS gives no *backward arcs.*
- Suppose there is a directed cycle $C = \{v_0, v_1, \ldots, v_k, v_0\}$ in T.
- As there are no *backward arcs* we have that:

$$post(v_0) > post(v_1) > post(v_2) \cdots > post(v_k) > post(v_0)$$

- Thus we obtain the contradiction that $post(v_0) > post(v_0)$.
- So G is **acyclic.** □

Corollary. There is a linear time algorithm to test whether or not a directed graph is acyclic

Proof.

- Run DFS and then check if any arc is a **backward arc.** □

Topological Orderings

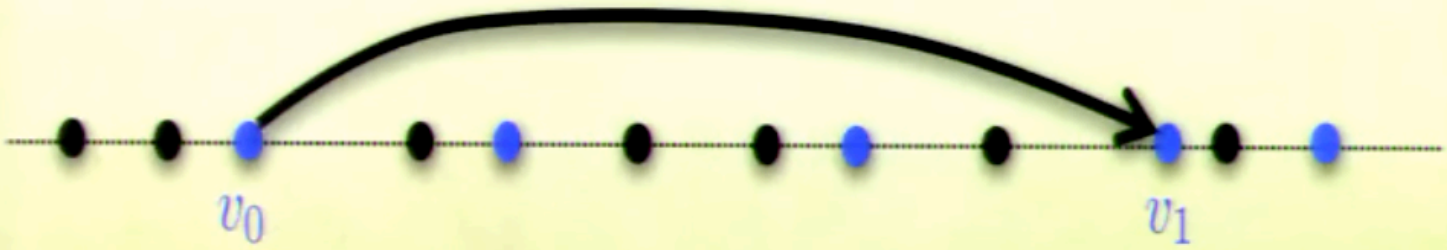
- A directed graph has a **topological ordering** (*linear ordering*) if the vertices can be horizontally ordered such that every arc is from right to left.

Theorem. A directed graph G has a topological ordering *if and only if* depth first search produces no backward arcs.

Proof.

$(\Rightarrow)$

- If DFS produces a **backward arc** then G contains a cycle C.
- Let the cycle be $C = \{v_0, v_1, \ldots, v_k, v_0\}$ where *wlog* v_0 is the **leftmost** vertex of the cycle in the order.



$\Longrightarrow$ The arc (v_0, v_1) goes from *left to right*.

Proof [cont.]

$(\Leftarrow)$

- Assume DFS gives no *backward arcs*.
- Then for every arc (u, v) we have :

$$post(u) > post(v)$$

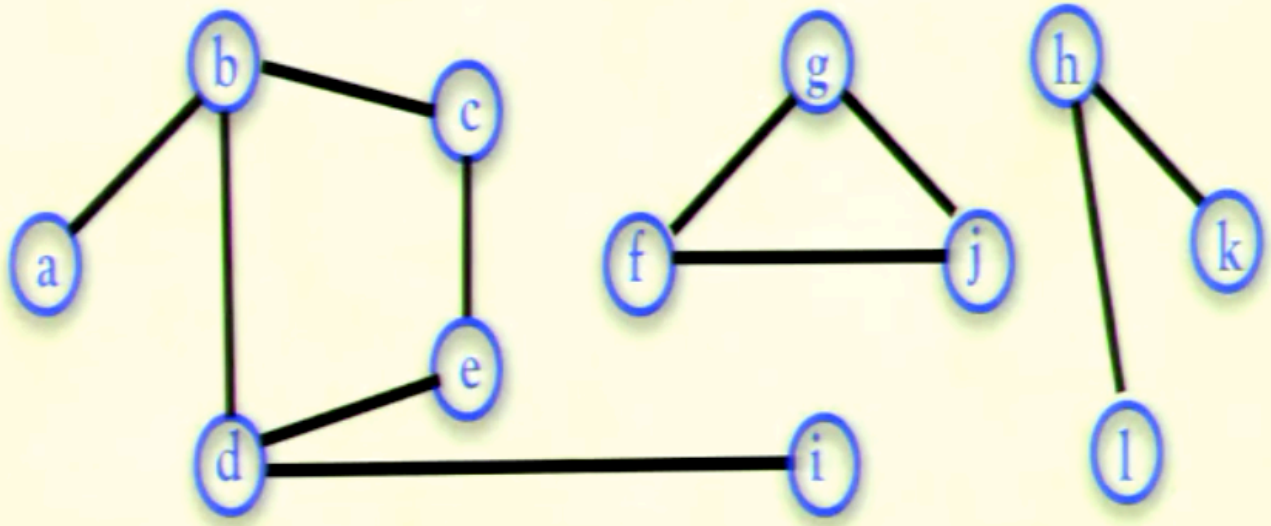
- Thus, if we place each vertex v at coordinate $post(v)$ on the x-axis then every arc goes from *right to left*.
- Thus G has a **topological ordering**. □

Corollary. There is a linear time algorithm to test whether or not a directed graph has a topological ordering. □

Corollary. A directed graph is acyclic *if and only if* it has a topological ordering. □

Disconnected Graphs

- So far, for undirected graphs, we have focussed on **connected** graphs.
- For **disconnected** undirected graphs, a DFS search will find the entire component containing the root vertex.



- Thus, after we have searched one component, we must run DFS from a **new** root node to find the next component.

Strongly Connected Components

- In a directed graph, a subgraph $S \subseteq V$ is **strongly connected** if for every pair $u, v \in S$ there is a directed path from u to v and vice versa.

- Surprisingly, DFS can be used to decompose a directed graph into its strongly connected components extremely quickly...

Theorem. There is a linear time algorithm to find the strongly connected components of a directed graph.

Proof.

- *See the Dasgupta or Erickson texts.*

□