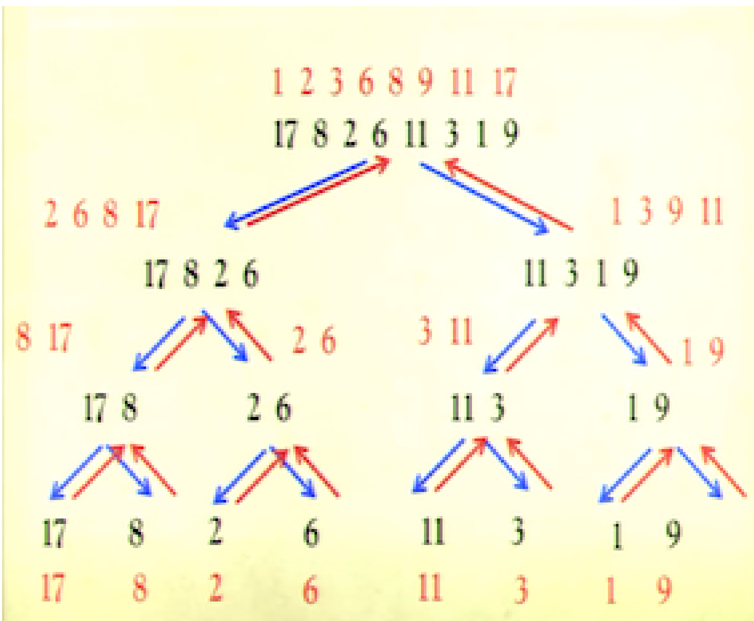
Lecture 2: Recursive Algorithms

1 Reductions and Sub-Routines

- Solving a problem by **reducing** it (or a sub-problem of it) to another problem is the most fundamental technique in algorithm design.
- Specifically, algorithm A may use another algorithm B as a sub-routine.
- This has numerous advantages:
  - **Code Verification**: the correctness of A is independent of B.
  - **Code Reuse**: a great time-saver.
- A simple but very powerful special case of this paradigm is when the algorithm calls itself!
  - This method is called **recursion**.

2 MergeSort

- We can sort n numbers into non-decreasing order using the following algorithm:
  MergeSort(x_1, x_2, ... , x_n)
  If n = 1 then output x_1
  Else output Merge{MergeSort(x_1, ... , $x_{\frac{n}{2}}$), MergeSort($x_{\frac{n}{2}+1}$, ... , x_n)}



- Two Problems:
  - Does the algorithm work?
    Yes!
    - → The algorithm calls itself on smaller instances
      - · The division process terminates with a set of base cases of size 1.
    - → MergeSort trivially works on the base cases.
    - → So, given the validity of the Merge Step, the correctness of the algorithm follows by **strong induction**.
      - · As long as base case is correct and merge step works, everything will be fine.
  - If so, is it efficient (polynomial time)?
    Yes! Look at the recursive formula.
    - → To analyze this we represent the running time T (n) via a **recurrence**:

$$\text{Recursive Formula: } T(\mathrm{n}) = 2 \cdot T(\tfrac{n}{2}) + \mathrm{c} \cdot \mathrm{n}$$

      - · $2 \cdot T(\frac{n}{2})$: Recuse on two problems with half the size.
      - · $\mathrm{c} \cdot \mathrm{n}$: It takes linear time to merge two sorted lists.

$$\text{Base Case: } T(1) = 1$$

      - · Or we can use $T(\mathrm{c}) = O(1)$ for any constant c.
    - → The Running Time of MergeSort
      - · Theorem: MergeSort runs in time $O(\mathrm{n} \cdot \log n)$
      - · Proof:
        1. By adding dummy numbers, we may assume n is a power of two: $\mathrm{n} = 2^k$
        2. We can unwind the recursive formula as follows:

$$T(\mathrm{n}) = 2 \cdot T(\tfrac{n}{2}) + \mathrm{c} \cdot \mathrm{n}$$

Proof [cont.]

- But this **unwinding operation** can be repeated:

$$\begin{aligned}
T(n) &= 2\cdot T\left(\frac{n}{2}\right)+c\cdot n\\
&= 2\cdot\left(2\cdot T\left(\frac{n}{4}\right)+c\cdot\frac{n}{2}\right)+cn\\
&= 2^2\cdot T\left(\frac{n}{4}\right)+2\cdot cn\\
&= 2^2\cdot\left(2\cdot T\left(\frac{n}{8}\right)+c\cdot\frac{n}{4}\right)+2\cdot cn\\
&= 2^3\cdot T\left(\frac{n}{8}\right)+3\cdot cn\\
&= 2^3\cdot\left(2\cdot T\left(\frac{n}{16}\right)+c\cdot\frac{n}{8}\right)+3\cdot cn\\
&= 2^4\cdot T\left(\frac{n}{16}\right)+4\cdot cn\\
&\ \vdots\\
&= 2^k\cdot T(1)+k\cdot cn \quad \text{— } \boxed{\text{Since } n=2^k}
\end{aligned}$$

Proof [cont.]

- Thus we have:

$$\begin{aligned}
T(n) &= 2^k\cdot T(1)+k\cdot cn\\
&= n\cdot T(1)+k\cdot cn\\
&= n\cdot(1+k\cdot c)
\end{aligned}$$

- But c is a constant so:

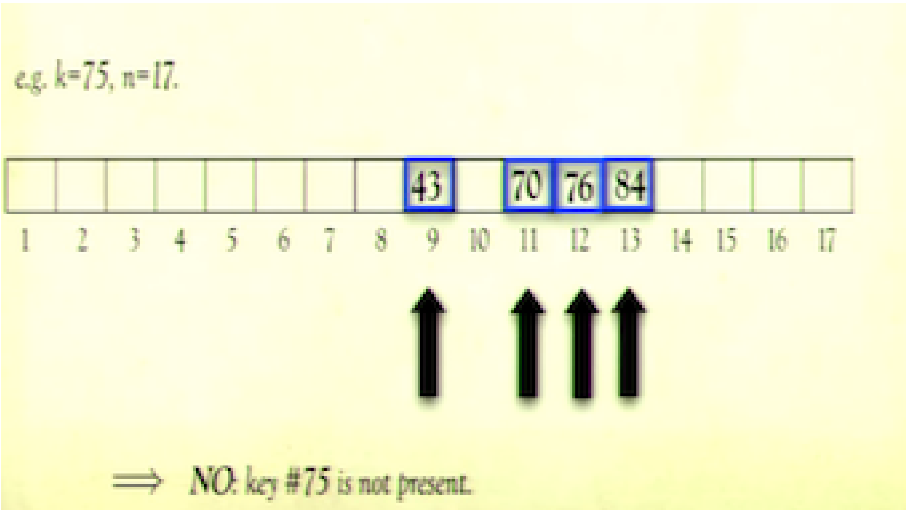
$$T(n) = O(n\cdot k)$$

- Furthermore, $k=\log n$, so we get that:

$$T(n) = O(n\cdot\log n) \qquad \square$$

3 Binary Search

- We can search for a key k in a sorted array of cardinality n using the binary search algorithm:
  BinarySearch(a_1, a_2, ... , a_n : k)
  While n > 0 do:
  If $a_{\frac{n}{2}}$ = k output YES
  Else if $a_{\frac{n}{2}}$ > k output BinarySearch(a_1, a_2, ... , $a_{\frac{n}{2}-1}$: k)
  Else if $a_{\frac{n}{2}}$ < k output BinarySearch($a_{\frac{n}{2}+1}$, ... , a_n: k)
  Output NO



- Does this work?
  - The validity of the binary search follows simply by strong induction.(The base case is trivially true.)
- Running Time?
  - Recurrence:

$$\text{Recursive Formula: } T(\text{n}) = T(\tfrac{n}{2}) + \text{c}$$
$$\text{Base Case: } T(1) = 1$$

  - Theorem: Binary Search runs in time $O(\log n)$
    1. By adding dummy numbers, we may assume n is a power of two: n = 2^k
    2. We can unwind the recursive formula as follows:

$$T(n) = T\left(\frac{n}{2}\right) + c$$

$$= \left(T\left(\frac{n}{4}\right) + c\right) + c$$

$$= T\left(\frac{n}{4}\right) + 2 \cdot c$$

- Again this unwinding operation can be **repeated:**

$$\begin{aligned} T(n) &= T\left(\frac{n}{4}\right) + 2 \cdot c \\ &= \left(T\left(\frac{n}{8}\right) + c\right) + 2 \cdot c \\ &= T\left(\frac{n}{8}\right) + 3 \cdot c \\ &\vdots \\ &= T\left(\frac{n}{2^k}\right) + k \cdot c \end{aligned}$$

- Hence:

$$\begin{aligned} T(n) &= T(1) + k \cdot c \quad \text{— } \boxed{\text{Since } n = 2^k} \\ &= 1 + \log n \cdot c \end{aligned}$$

- This gives the claimed running time:

$$T(n) = O(\log n)$$

4 Divide and Conquer Algorithms

- A **divide and conquer** algorithm recursively breaks up a problem of size n in smaller sub-problems such that:
  - → There are exactly a sub-problems.
  - → Each sub-problem has size at most $\frac{1}{b} \cdot$ n
  - → Once solved, the solutions to the sub-problems can be combined to produce a solution to the original problem in time $O(n^d)$
- So the run-time of a divide and conquer algorithm satisfies the recurrence:

$$T(\mathrm{n}) = \mathrm{a} \cdot T(\tfrac{n}{b}) + O(n^d)$$

- MergeSort and Binary Search are indeed **divide and conquer** algorithms.

	Recursion Formula	a	b	d
MergeSort	$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n^1)$	2	2	1
Binary Search	$T(n) = 1 \cdot T\left(\frac{n}{2}\right) + O(n^0)$	1	2	0

5 Non-Military Applications of Divide and Conquer

- Divide and Conquer has many other non-military, practical applications:
  - Big Data
  - Distributed Algorithms
  - Clustering and Classification
  - MapReduce

6 Dummy Entries

- MergeSort actually has the recurrence:

$$\hat{T}(n) = \hat{T}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + \hat{T}\left(\left\lceil\frac{n}{2}\right\rceil\right) + c \cdot n$$

- Recall we got around this by adding dummy entries:
  - We found $\hat{n}$ the smallest power of 2 greater than n.
  - For this case, MergeSort then does have recurrence:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

  - But we also have:

$$\hat{T}(n) \leq T(\bar{n}) = O(\bar{n} \cdot \log \bar{n}) = O(n \cdot \log n)$$

- Here is another way to solve the recurrence:

$$\hat{T}(n) = \hat{T}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \hat{T}\left(\left\lceil \frac{n}{2} \right\rceil\right) + c \cdot n$$

  - As we only want to upper bound the running time, we can use:

$$\hat{T}(n) \leq \hat{T}\left(\frac{n}{2} + 1\right) + c \cdot n$$

  - This +1 does not seem to fit with our methodology, but we can fix this by applying a **domain transformation**.

- Domain Tranformation
  - For the domain transformation, simply set: $T(\mathrm{n}) = \hat{T}(\mathrm{n} + 2)$
  - Thus we have: $T(\mathrm{n}) = T(\frac{n}{2}) + \hat{c} \cdot \mathrm{n}$

$$
\begin{aligned}
T(n) &= \hat{T}(n+2) \\
&\leq \hat{T}\left(\frac{n+2}{2}+1\right)+c\cdot(n+2) \\
&\leq \hat{T}\left(\frac{n+2}{2}+1\right)+\hat{c}\cdot n \\
&= \hat{T}\left(\frac{n}{2}+2\right)+\hat{c}\cdot n \\
&= T\left(\frac{n}{2}\right)+\hat{c}\cdot n
\end{aligned}
$$

- Of course, we can solve this recurrence as: $T(\mathrm{n}) = O(\mathrm{n} \cdot \log n)$
- Therefore, $\hat{T}(\mathrm{n}) = T(\mathrm{n}\ \text{-}\ 2) = O(\mathrm{n} \cdot \log n)$
- As well as ceilings and floors, domain transformations can be used to $\underline{simplify}$ many other recurrences; e.g. removing lower order terms.