Lecture 3: The Master Theorem

1 Quick Review

- Recall, a divide-and-conquer algorithm recursively breaks up a problem of size n in smaller sub-problems such that:
  - There are exactly a sub-problems.
  - Each sub-problem has size at most $\frac{1}{b} \cdot$ n
  - Once solved, the solutions to the sub-problems can be <u>combined</u> to produce a solution to the original problem in time $O(n^d)$
- So the run-time of a divide and conquer algorithm satisfies the recurrence:

$$T(\text{n}) = \text{a} \cdot T(\tfrac{n}{b}) + O(n^d)$$

- Examples: MergeSort and Binary Search

2 The Master Theorem

- **The Master Theorem**: If $T(\text{n}) = \text{a} \cdot T(\frac{n}{b}) + O(n^d)$ for constants a > 0, b > 1, and d ≥ 0 then:

$$T(n) = \begin{cases} O(n^d) & \text{if } a < b^d \quad \text{[Case I]} \\ O(n^d \cdot \log n) & \text{if } a = b^d \quad \text{[Case II]} \\ O(n^{\log_b a}) & \text{if } a > b^d \quad \text{[Case III]} \end{cases}$$

- Sanity Check: What does this give for MergeSort and Binary Search?

	a	b	d	Case	Runtime
`MergeSort`	2	2	1	II	$O(n \cdot \log n)$
`Binary Search`	1	2	0	II	$O(\log n)$

3 Proof of The Master Theorem

- Fact One:

Fact 1. $\displaystyle\sum_{k=0}^{\ell} \tau^k = \frac{1-\tau^{\ell+1}}{1-\tau}$ for any $\tau \neq 1$.

Proof.

- We have:

$$(1-\tau)\cdot\sum_{k=0}^{\ell}\tau^k = \sum_{k=0}^{\ell}\tau^k - \sum_{k=1}^{\ell+1}\tau^k$$

$$= \tau^0 - \tau^{\ell+1}$$

$$= 1 - \tau^{\ell+1}$$

- Dividing both sides by $(1-\tau)$ gives the result.

- Fact Two:

Fact 2. $x^{\log_b y} = y^{\log_b x}$ for any base b.

Proof.

$\boxed{\log_b z^p = p \cdot \log_b z}$

- Observe that, by the *power rule of logarithms*, we have:

$$\log_b x \cdot \log_b y = \log_b(y^{\log_b x})$$

- Similarly:

$$\log_b x \cdot \log_b y = \log_b(x^{\log_b y})$$

- Putting this together gives

$$\log_b(y^{\log_b x}) = \log_b(x^{\log_b y})$$

$$\Longrightarrow \quad x^{\log_b y} = y^{\log_b x}$$

- Proof of the Master Theorem

Proof.

- We may assume n is a power of b: $n = b^\ell$
- So we have:

$$\begin{aligned}
T(n) &= n^d + a \cdot \left(\frac{n}{b}\right)^d + a^2 \cdot \left(\frac{n}{b^2}\right)^d + \cdots + a^\ell \cdot \left(\frac{n}{b^\ell}\right)^d \\
&= n^d \cdot \left(1 + a \cdot \left(\frac{1}{b}\right)^d + a^2 \cdot \left(\frac{1}{b^2}\right)^d + \cdots + a^\ell \cdot \left(\frac{1}{b^\ell}\right)^d\right) \\
&= n^d \cdot \left(1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \cdots + \left(\frac{a}{b^d}\right)^\ell\right)
\end{aligned}$$

Proof [cont.] $T(n) = n^d \cdot \left(1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \cdots + \left(\frac{a}{b^d}\right)^\ell\right)$

Case I: $\frac{a}{b^d} < 1$

- Set $\tau = \frac{a}{b^d}$
- Then: $T(n) = n^d \cdot \sum_{k=0}^{\ell} \tau^k$
- Applying Fact 1, we know that:

$$\sum_{k=0}^{\ell} \tau^k = \frac{1-\tau^{\ell+1}}{1-\tau} \leq \frac{1}{1-\tau} = O(1)$$

As a, b, and d are constants so is $1-\tau$.

- Therefore:

$$T(n) \leq n^d \cdot \frac{1}{1-\frac{a}{b^d}} = n^d \cdot \frac{b^d}{b^d - a} = O(n^d)$$

Proof [cont.] $T(n) = n^d \cdot \left(1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \cdots + \left(\frac{a}{b^d}\right)^\ell\right)$

<u>Case II:</u> $\frac{a}{b^d} = 1$

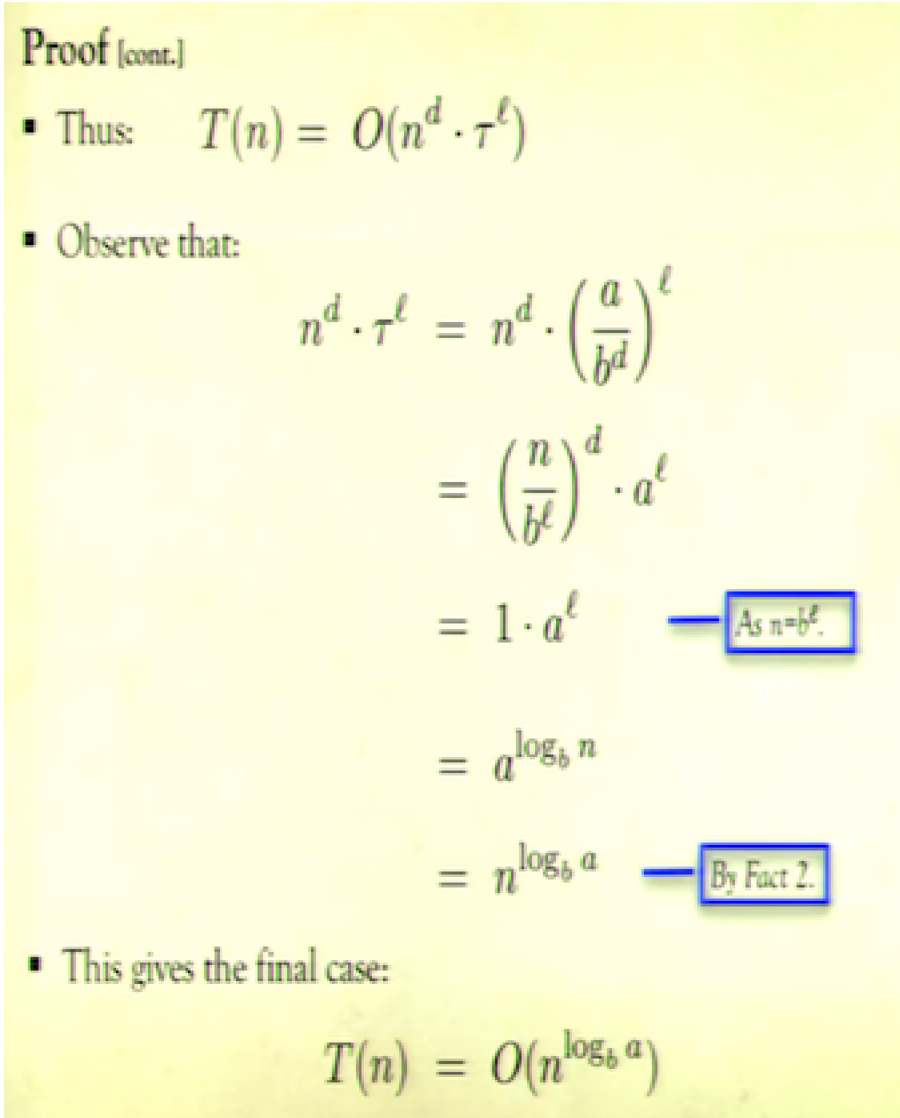
- Then: $T(n) = n^d \cdot (\ell + 1)$
- But $n = b^\ell$ so $\ell = \log_b n$.
- As b is a constant greater than one, this gives: $T(n) = O(n^d \cdot \log n)$

<u>Case III:</u> $\frac{a}{b^d} > 1$

- Again set $\tau = \frac{a}{b^d}$
- Then: $T(n) = n^d \cdot \sum_{k=0}^{\ell} \tau^k$
- Applying Fact 1 gives:

As a, b, and d are constants so is τ-1.

$$\sum_{k=0}^{\ell} \tau^k = \frac{\tau^{\ell+1} - 1}{\tau - 1} \leq \frac{\tau^{\ell+1}}{\tau - 1} = O(\tau^{\ell+1}) = O(\tau^\ell)$$



- Specifically, what matters is **not** the statement of the Master Theorem but the **ideas** underlying its proof.
  - First, if we understand the proof then we can easily reconstruct the theorem.
  - Second, if we understand the proof then we can easily apply the method to a much broader class of problems. For example:
    - → The sub-problems have different sizes.

e.g. The Deterministic Selection Algorithm.

$$T(n) \leq T(\frac{7n}{10}) + T(\frac{n}{5}) + O(n)$$

$\rightarrow$ The combination function is not of the form $f(\mathrm{n}) = n^d$.

e.g. Euclid's Greatest Common Divisor Algorithm.

$$T(n) = T(\frac{n}{2}) + O(\log n)$$

$\rightarrow$ The parameters a, b, and d are <u>not</u> constants.

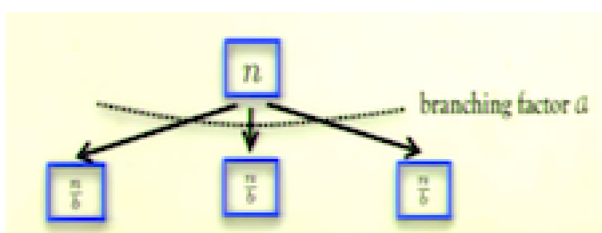
e.g. $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + O(n^{\frac{1}{\log \log n}})$

4 The Recursion Tree Method

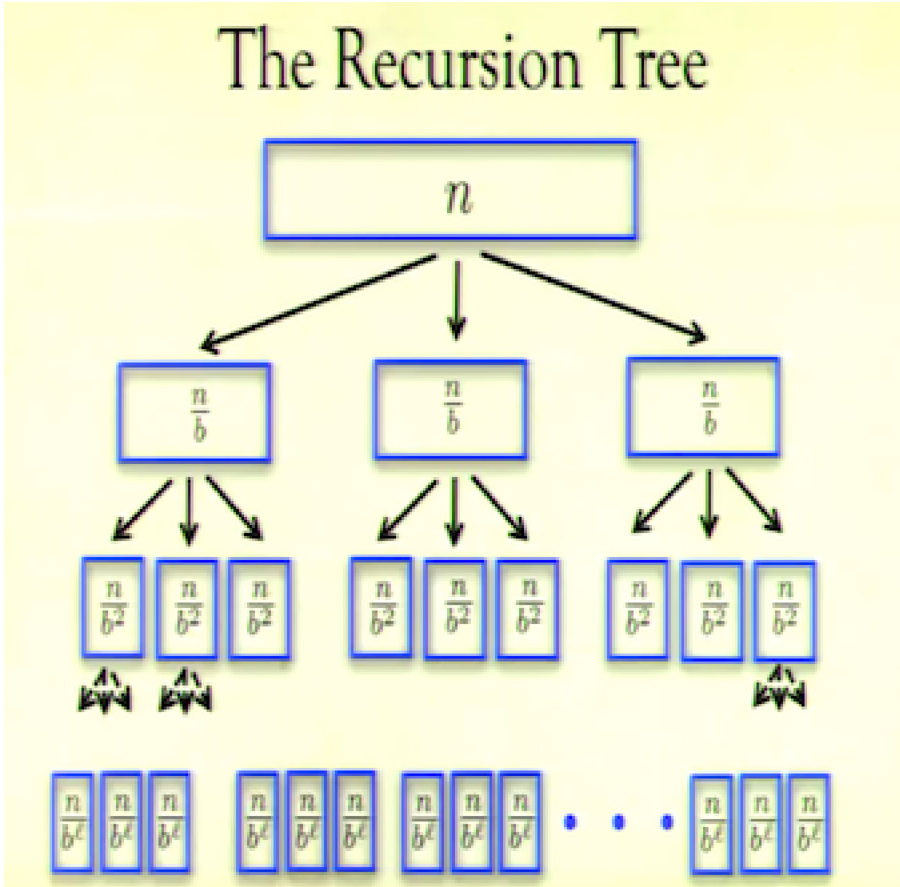
- The Master Theorem is a special case the the **recursion tree method**.
- Specifically, we model the *divide and conquer* recursive formula by a tree:

$$T(\mathrm{n}) = \mathrm{a} \cdot T(\tfrac{n}{b}) + O(n^d)$$

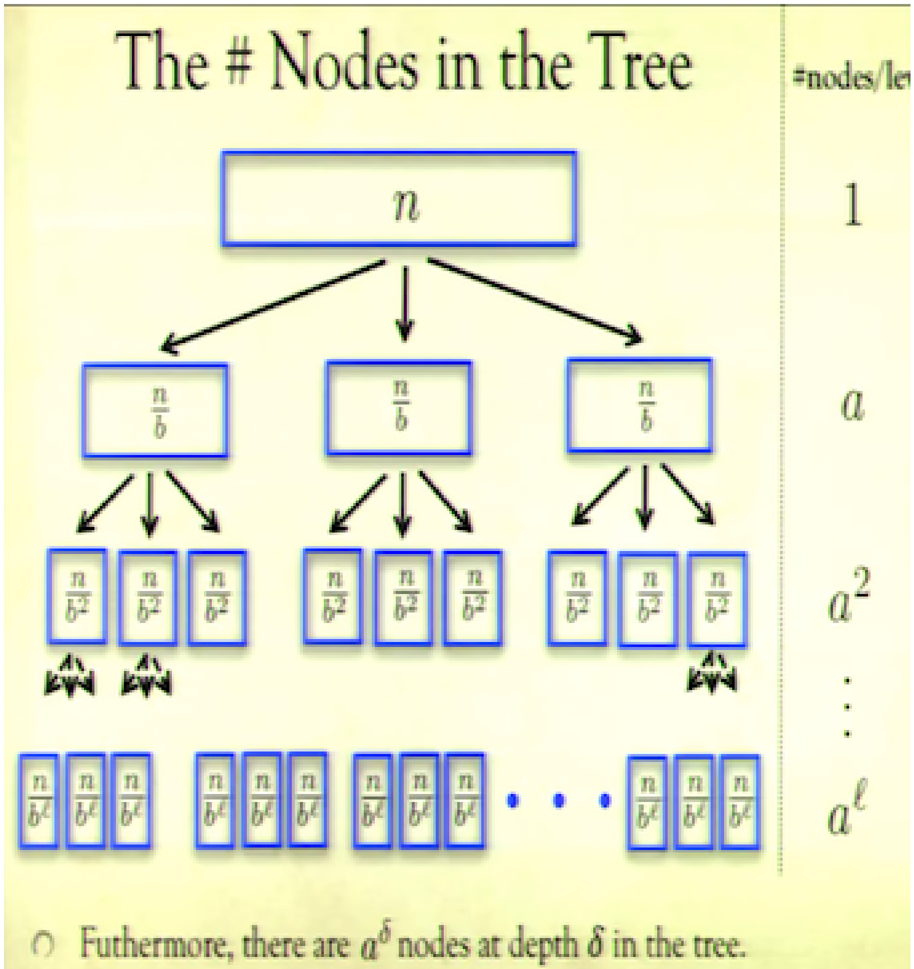
  - The **root** node of the trees has a label n.
  - The root has a children each with label $\frac{n}{b}$.



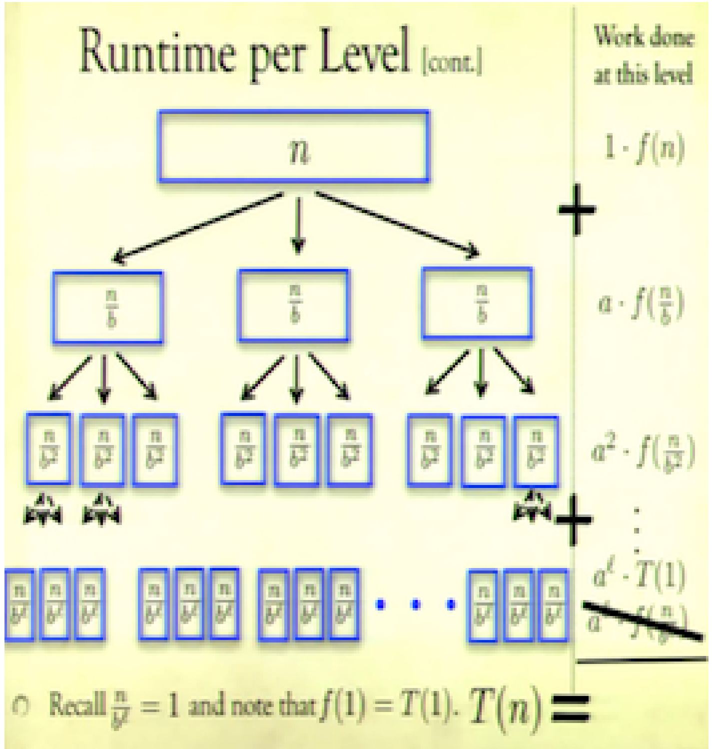
- This pattern then repeats at the children, then grandchildren, etc.

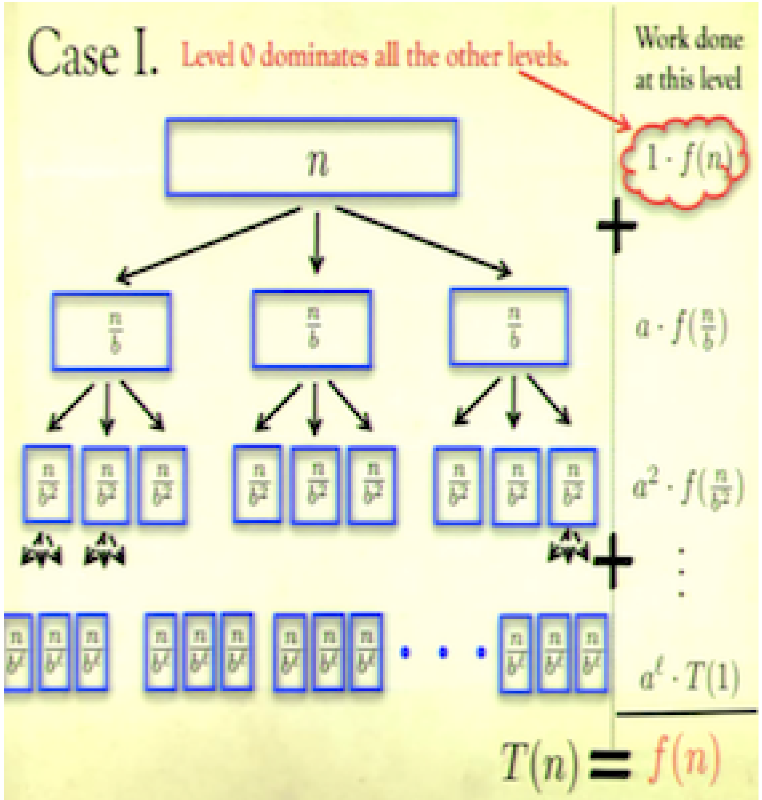


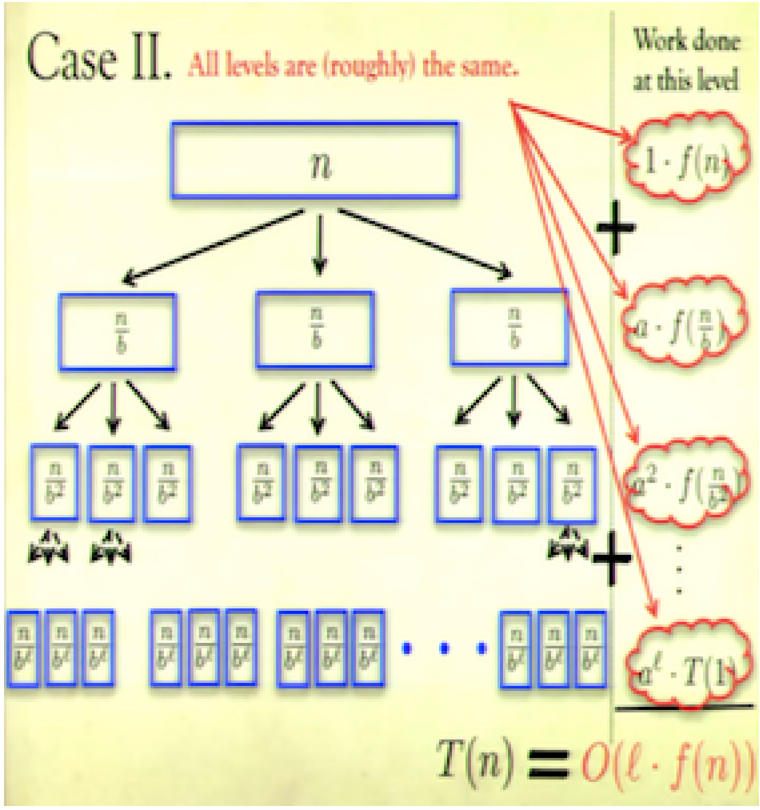
- This process stops at the **leaves** (base cases) which have label $\frac{n}{b^l}$ = 1. (As n = b^l)

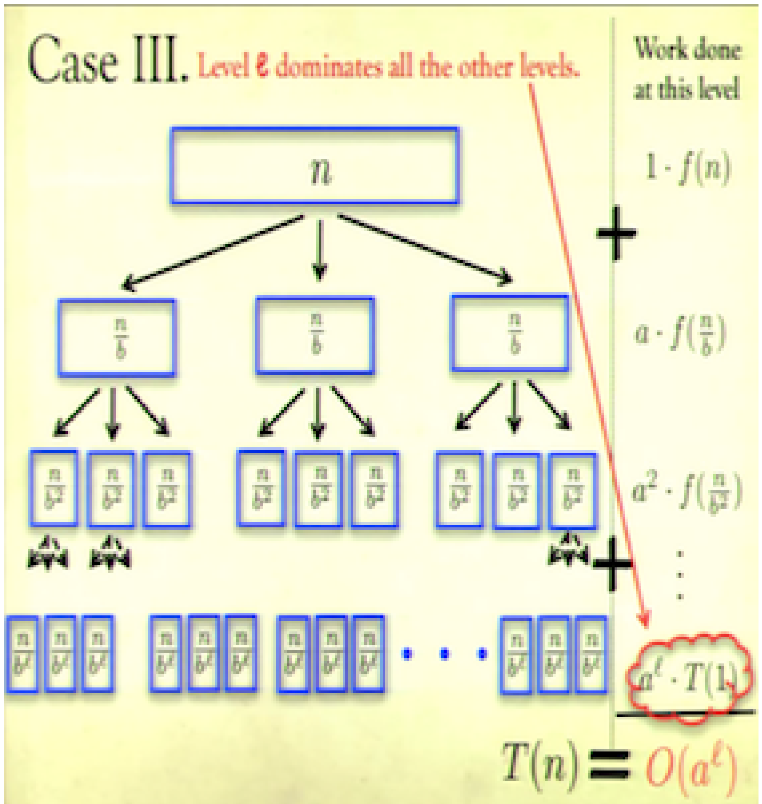


- How much time do we spend at each level?

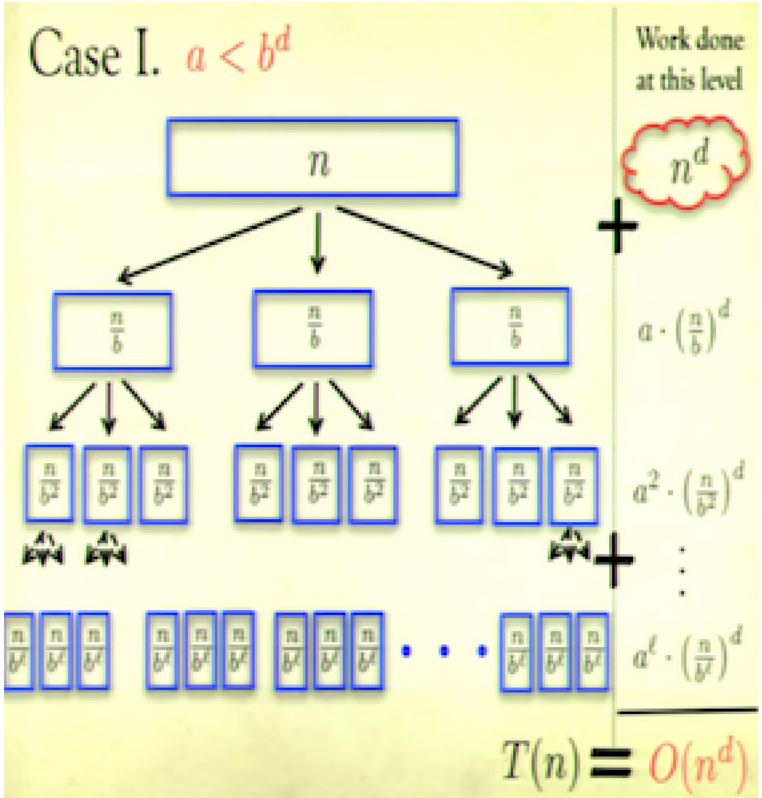

Runtime per Level [cont.]
Work done at this level
n
$\frac{n}{b}$
$\frac{n}{b^2}$
$1 \cdot f(n)$
$a \cdot f(\frac{n}{b})$
$a^2 \cdot f(\frac{n}{b^2})$
$a^\ell \cdot T(1)$
Recall $\frac{n}{b^\ell} = 1$ and note that $f(1) = T(1)$. $T(n) =$


Case I.
Level 0 dominates all the other levels.
Work done at this level
n
$1 \cdot f(n)$
$a \cdot f(\frac{n}{b})$
$a^2 \cdot f(\frac{n}{b^2})$
$a^\ell \cdot T(1)$
$T(n) = f(n)$

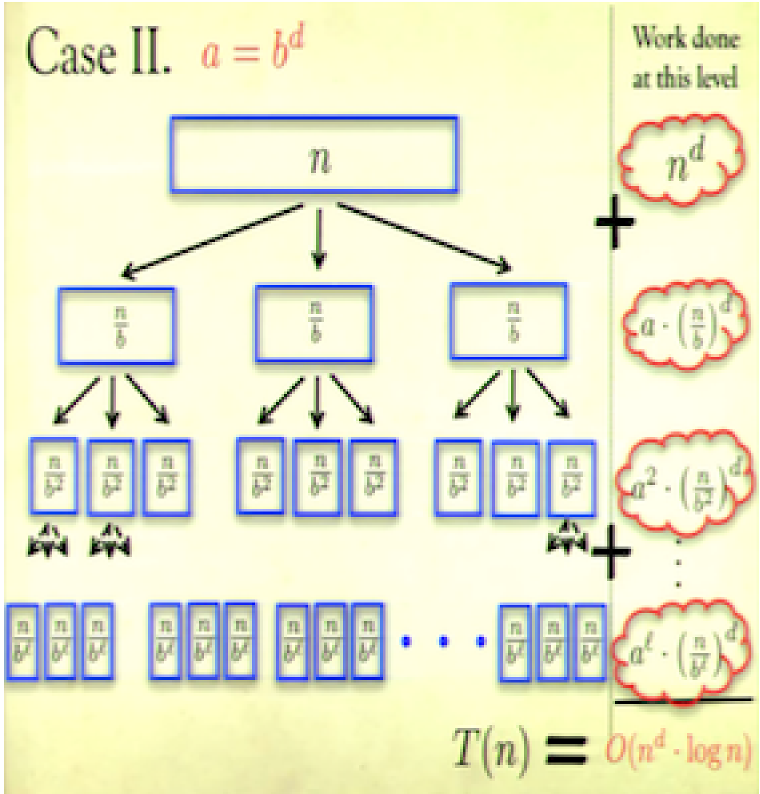

Case II. All levels are (roughly) the same.
Work done at this level
n
$\frac{n}{b}$
$\frac{n}{b^2}$
$\frac{n}{b^\ell}$
$1 \cdot f(n)$
+
$a \cdot f(\frac{n}{b})$
$a^2 \cdot f(\frac{n}{b^2})$
+
$a^\ell \cdot T(1)$
$T(n) = O(\ell \cdot f(n))$



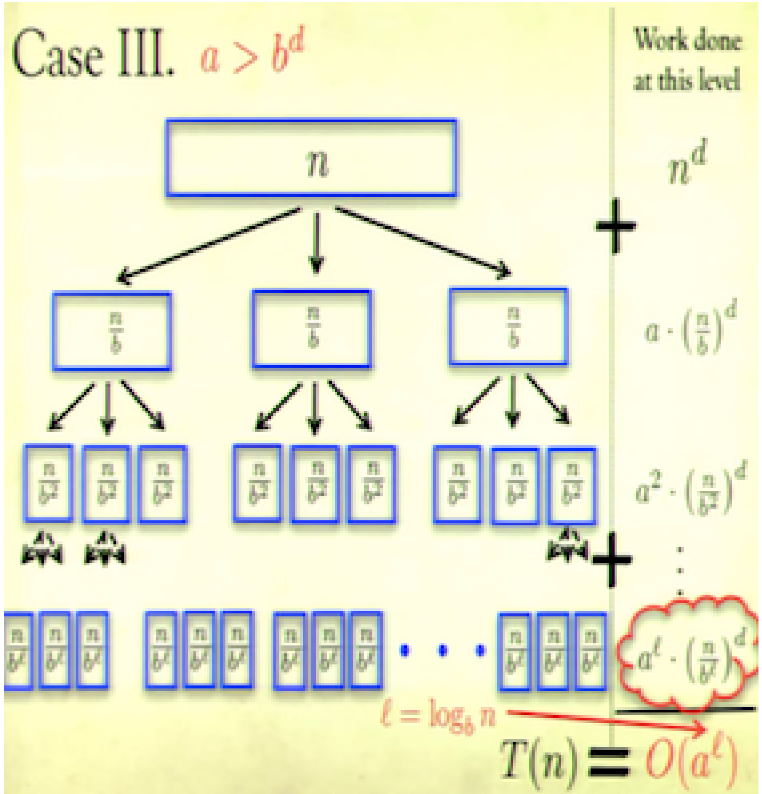
- This gives us the proof of the Master Theorem:



Case I. $a < b^d$
Work done at this level
n
n^d
$\frac{n}{b}$
$a \cdot \left(\frac{n}{b}\right)^d$
$\frac{n}{b^2}$
$a^2 \cdot \left(\frac{n}{b^2}\right)^d$
$\frac{n}{b^\ell}$
$a^\ell \cdot \left(\frac{n}{b^\ell}\right)^d$
$T(n) = O(n^d)$



Case II. $a = b^d$
Work done at this level
n
$\frac{n}{b}$
$\frac{n}{b^2}$
$\frac{n}{b^\ell}$
n^d
$a \cdot \left(\frac{n}{b}\right)^d$
$a^2 \cdot \left(\frac{n}{b^2}\right)^d$
$a^\ell \cdot \left(\frac{n}{b^\ell}\right)^d$
$T(n) = O(n^d \cdot \log n)$



Case III. $a > b^d$
Work done at this level
n
n^d
$\frac{n}{b}$
$a \cdot \left(\frac{n}{b}\right)^d$
$\frac{n}{b^2}$
$a^2 \cdot \left(\frac{n}{b^2}\right)^d$
$\frac{n}{b^\ell}$
$a^\ell \cdot \left(\frac{n}{b^\ell}\right)^d$
$\ell = \log_b n$
$T(n) = O(a^\ell)$