Gleason's Theorem Is Not Constructively Provable
Author(s): Geoffrey Hellman
Source: *Journal of Philosophical Logic*, Vol. 22, No. 2 (Apr., 1993), pp. 193-203
Published by: Springer
Stable URL: http://www.jstor.org/stable/30226496
Accessed: 26-10-2016 06:30 UTC

GEOFFREY HELLMAN

GLEASON'S THEOREM IS NOT CONSTRUCTIVELY PROVABLE*

0. INTRODUCTION

Gleason's Theorem (Gleason 1957) characterizes all possible measures on the closed subspaces of a (separable) Hilbert space of dimension greater than two. Specifically:

GLEASON'S THEOREM. Let μ be a σ-additive measure on the closed subspaces of a (real or complex) Hilbert space $\mathscr{H}$ of dimension $\geqslant 3$; then there exists a positive self-adjoint operator W of trace class such that, for every closed subspace A, $\mu(A) = Tr(WP_A)$, where P_A is the projection operator of $\mathscr{H}$ onto A. (σ-additivity here means additivity with respect to countable families of mutually orthogonal subspaces.)

This deep theorem forms a cornerstone of the modern mathematical foundations of quantum mechanics. It is readily shown that the quantum mechanical pure and mixed states determine (normalized) probability measures on the lattice of the (closed) subspaces of the Hilbert space as the state space of the system. (The subspaces are generally taken to be in correspondence with the experimental or physical "propositions", "questions", or "eventualities" pertaining to the system.)[1] According to Gleason's Theorem, these are the *only* probability measures there are on the lattice of subspaces of a Hilbert space of dimension $\geqslant 3$. Moreover, Gleason's Theorem has deep physical significance. For instance, as a corollary, there can be no dispersion-free ($\{0, 1\}$-valued) measures on the subspaces of a Hilbert space of dimension three or greater. This rules out a major category of "hidden-variables theories" for quantum mechanics.

For the benefit of the reader unfamiliar with the concepts of quantum statics entering into Gleason's Theorem, we summarize them in this paragraph and the next. A *(quantum) probability measure* is a mapping μ: $\mathscr{L}(\mathscr{H}) \rightarrow [0, 1]$ such that $\mu(\mathscr{H}) = 1$ (normalization) and

Journal of Philosophical Logic **22**: 193–203, 1993.

$\mu(\vee_i A_i) = \Sigma_i \mu(A_i)$ for any sequence of pairwise orthogonal subspaces A_i (σ-additivity). (Here $\mathscr{L}(\mathscr{H})$ is the lattice of subspaces of the given Hilbert space $\mathscr{H}$, where by "subspace" we mean topologically closed (in the Hilbert space norm) linear submanifold of $\mathscr{H}$. $\vee$ denotes the lattice join, closed linear span.) The subspaces A of $\mathscr{H}$ are in one–one correspondence with the projection operators (projectors), P_A, projecting onto them, defined by the conditions $(\varphi, P\psi) = (P\varphi, \psi)$ (self-adjointness) and $P^2 = P$ (idempotence). $((\ ,\ )$ is the inner product of $\mathscr{H}$.) Such operators are also *positive*: $(\varphi, P\varphi) \geqslant 0$, all φ. Let ψ be a normalized vector and P_ψ the projector onto the one-dimensional subspace ("*ray*") spanned by ψ. Then P_ψ defines a quantum mechanical *pure state*. It induces a probability measure μ_{P_ψ} on $\mathscr{L}(\mathscr{H})$ by the rule: $\mu_{P_\psi}(A) \equiv \|P_A\psi\|^2 = (\psi, P_A\psi)$. (The required properties follow from linearity properties of the inner product and normalization of ψ.) (The rule can be motivated by the spectral theorem, cf. Jordan, 1969, section 24, pp. 78ff.)

The natural generalization of this to *mixtures* (worked out by von Neumann) uses the trace formalism. A positive operator W is *of trace class* if $\Sigma_i(\varphi_i, W\varphi_i)$ is finite, where the φ_i are an orthonormal basis. This sum can be shown independent of the orthonormal basis and is called the *trace* of W, $\mathrm{Tr}(W)$. Tr is itself a linear functional on bounded operators (which include projectors). A positive, self-adjoint operator W such that $\mathrm{Tr}(W) = 1$ is called a *density matrix*. (If we think of an orthonormal basis as corresponding to the mutually exclusive and jointly exhaustive possible outcomes of experiments to test an observable, with different bases corresponding to different experimental arrangements for different observables, we see the plausibility of using density matrices to represent statistical states.) Projectors onto rays are examples, and for $W = P_\psi$ as above we have $\mathrm{Tr}(WP_A) = (\psi, WP_A\psi) = (W\psi, P_A\psi) = (\psi, P_A\psi) = \mu_{P_\psi}(P_A)$, as above. The remaining examples are the *mixtures*, generated by taking convex linear combinations of projectors onto rays (which need not be mutually orthogonal, but which, by a theorem of von Neumann, can be chosen to be so). (See Jordan, 1969, Theorem 22.1, p. 73.) Where $W = \Sigma_i w_i P_{\varphi_i}$, with $w_i > 0$, $\Sigma_i w_i = 1$, and the φ_i orthonormal, we have $\mathrm{Tr}(WP_A) = \Sigma_i w_i(\varphi_i, P_A\phi_i)$, and it is clear that this defines a probability measure on $\mathscr{L}(\mathscr{H})$, as it is a convex sum of such measures. Gleason's remarkable theorem then guarantees that, provided the dimension of $\mathscr{H}$ is $\geqslant 3$, the density matrices give rise to all mathematically possible quantum probability measures.

Known proofs of Gleason's Theorem employ non-constructive steps, relying essentially on the classical Bolzano – Weierstrass theorem ("Every bounded set of reals contains an accumulation point"). (Gleason 1957; Cooke, Keane, and Moran 1985 [hereinafter "CKM"]. The latter, while elementary in character, makes heavy reliance on non-constructive corollaries of Bolzano – Weierstrass (e.g. sequential compactness).) A recent examination of the mathematical foundations of quantum mechanics from a constructive point of view concluded by calling attention to two open problems, the first of which was, "Find a constructive version of Gleason's Theorem." (Bridges 1981, p. 272.) The Propositions of this note, below, show that no constructive proof of Gleason's Theorem – indeed of the restriction of that theorem to the unit sphere S of the real three-dimensional Hilbert space $\mathbb{R}^3$ – is possible.[2]

1. PRELIMINARIES

Let S be the unit sphere of $\mathbb{R}^3$, as a three-dimensional Hilbert space. For points s, s' of S (as vectors), define $s \perp s'$ iff the angle $\varphi(s, s')$ between s and s' (from the origin) $= \pi/2$. Define a *frame* to be a triple (p, q, r) of pairwise orthogonal elements of S, i.e., such that $p \perp q$, $q \perp r$, and $p \perp r$. A *frame function* is then defined as a function $f\colon S \to \mathbb{R}$ such that

$$f(p) + f(q) + f(r)$$

has the same value for every frame (p, q, r). This value, denoted w_f, is called the *weight* of f. For f a frame function, if f is bounded, we denote its supremum, $\sup(f)$, by M_f, and its infimum, $\inf(f)$, by m_f. (Where no confusion arises, the subscript may be dropped.)

As a key example of frame functions, note that if A is a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^3$, then the quadratic form $Q(s) \equiv (s, A(s))$ restricted to S is a frame function with weight $= \mathrm{Tr}(A)$, the trace of A. (This is easy to see in case A is *symmetric* ($(u, Av) = (Au, v)$ for arbitrary vectors u, v): then, expressed in an orthonormal eigenbasis

p, q, r of A, let $s = xp + yq + zr$; then, after expansion and elimination of vanishing terms, $(s, As) = ax^2 + by^2 + cz^2$, where a, b, c are the eigenvalues of A ($Ap = ap$, $Aq = bq$, and $Ar = cr$). By definition, $w_Q = a + b + c = \mathrm{Tr}(A)$. In the general case in which A is assumed merely linear, one obtains the same result by considering the symmetric operator $\frac{1}{2}(A + A^t)$, where t is transposition.) Furthermore, clearly the normalized probability measures on (the subspaces of) $\mathbb{R}^3$ are bounded frame functions (more precisely, are generated by them, through the association of unit vectors with one-dimensional subspaces).

We now can state the restricted version of Gleason's Theorem on which we shall concentrate:

GLEASON (restricted). Let f be a bounded frame function on S, and define $\gamma \equiv \gamma_f \equiv w_f - M_f - m_f$. There exists a frame (p, q, r) such that, for any $s \in S$, if the coordinates of s with respect to (p, q, r) are (x, y, z), then

$$f(s) = Mx^2 + \gamma y^2 + mz^2. \qquad \text{(Cf. CKM, §3.)}$$

This implies that all bounded frame functions are given by a (in fact, symmetric) linear operator as in the preceding example. (A can be constructed as the corresponding weighted sum, $MP_p + \gamma P_q + mP_r$, of projectors onto the respective rays of the given frame, (p, q, r).)

2. THE NON-CONSTRUCTIVITY OF GLEASON

It is crucial to observe that a constructive proof of Gleason (restricted) would provide a constructive proof that any (constructive) bounded frame function attains its extremal values, which in turn would provide a constructive method of calculating not only those values but at least some points at which they are attained. (The display of Gleason (restricted) tells us that f attains its supremum at p and its infimum at r, and the theorem – read constructively – would tell us how to find the frame (p, q, r). (Strictly speaking, we should subscript all the logical and mathematical symbols in the statements of "Gleason's Theorem" to indicate that it is those statements, *constructively understood*, that are at issue. From a radical constructivist standpoint, the classical statements are couched in a language inaccessible (in crucial places, especially infinitistic quantification) to constructive mathematics.)) It is this implication that can be readily shown to lead to unacceptable conclusions from the constructive point of view. The case is similar to the well-known general observation that the classical theorem, that a uniformly continuous real-valued function on a compact metric space attains its extrema, is essentially non-constructive. (See, e.g., Dummett 1977, p. 115.)

The following facts are constructively unproblematic:

(1) If f is a (constructive) bounded frame function, so is αf for any (constructive) real number α.

(2) For any real α and bounded frame function f, if $\alpha > 0$, αf attains its supremum at the same points as f, and αf attains its infimum at the same points as f, which we may write,

$$\alpha > 0 \rightarrow (\alpha f)^{-1}[\sup(\alpha f)] = f^{-1}[\sup(f)] \;\&$$
$$(\alpha f)^{-1}[\inf(\alpha f)] = f^{-1}[\inf(f)].$$

(Again, all objects and operations are understood to be constructive or computable in the constructivist's favorite sense. Thus, this formula is to be read as saying that one can pass constructively from a proof that $\alpha > 0$ to a proof that the points (if any) at which αf can be shown to attain its supremum are just the points at which f can be shown to attain its supremum, etc.)

(3) Under the same conditions as (2), if $\alpha < 0$, αf attains its supremum at the points where f attains its infimum, and αf attains its infimum at the points where f attains its supremum, i.e.,

$$\alpha < 0 \rightarrow (\alpha f)^{-1}[\sup(\alpha f)] = f^{-1}[\inf(f)] \;\&$$
$$(\alpha f)^{-1}[\inf(\alpha f)] = f^{-1}[\sup(f)].$$

As already remarked, we also have:

(4) Under the hypothesis that Gleason (restricted) is constructively provable, if f is a (constructive) bounded frame function, there exist (i.e., can be fount) at least one point p such that f attains its supremum at p and a point $r \perp p$ at which f attains its infimum. (Note also

that $f(-s) = f(s)$. Thus, Gleason (restricted) really gives us pairs of points, $(p, -p)$, $(r, -r)$, satisfying the respective conditions.)

(5) Fix a vector $p \in S$ (think of p as the North pole); the function

$$f(s) = \cos^2 \varphi(p, s)$$

is a constructive bounded frame function (of weight 1) such that,

(i) f attains its supremum, $M_f = 1$, uniquely at p and $-p$;

(ii) for any r, if f attains its infimum at r, $f(r) = m_f = 0$, then $r \perp p$, i.e., r lies on the equator, $E \equiv \{s \in S\colon s \perp p\}$..

We now can proceed to the main result:

PROPOSITION I. *Gleason (restricted) cannot be constructively proved.*

We prove this by the method of counterexamples (cf., e.g., Bishop 1967; Bridges 1979; Dummett 1977), that is, we show that constructive solubility of any undecided number theoretic problem of a general form (e.g., whether Goldbach's conjecture has a least counterexample $\equiv 0 \bmod 4$) is reducible to construct provability of Gleason (restricted). To this end, introduce a real number generator $\beta(n)$ as follows: let $A(k)$ and $B(k)$ be decidable properties of natural numbers such that it is known neither whether $\exists n A(n)$ nor whether the least n, if it exists, such that $A(n)$ satisfies B or not. (For example, $A(k)$ can be "a sequence of four consecutive 5's begins at the k'th place in the decimal expansion of π", and $B(k)$ can be "k is even".) Now define

$$\beta(n) = \begin{cases} 0, \text{ if } \forall m \leqslant n \neg A(m) \\ 2^{-k} \text{ if } A(k) \ \& \ \forall m < k(\neg A(m)) \ \& \ B(k) \\ -2^{-k} \text{ if } A(k) \ \& \ \forall m < k(\neg A(m)) \ \& \ \neg B(k). \end{cases}$$

Here all logical symbols are to be understood constructively. Note that this does define a constructively acceptable real number generator, as the Cauchy convergence condition, $\forall k \exists m \forall n > m(|\beta(m) - \beta(n)| < 2^{-k})$, is clearly constructively satisfied.

Now let β designate the real number defined by $\beta(n)$ and let f be the bounded frame function of (5), $f(s) = \cos^2 \varphi(p, s)$, and assume that Gleason (restriced) were constructively provable. βf is a constructive bunded frame function (by (1)). By hypothesis, βf attains its supremum computably at a point p' (cf. (4)). Classically, we know that, if $\beta \neq 0$, either p' is one of the poles, p or $-p$ ($p' \perp E$), or else it lies on the equator ($p' \in E$). Constructively, we cannot say this, but, since the poles and E are well separated, we can assert (in fact, for arbitrary p') that either $p' \perp E$ fails or else $p' \in E$ fails (in the intuitionistic sense of implying absurdity). To see this, note that E is a *located* subset of S, that is, where d is the usual geodesic metric on S, $d(s, E) \equiv \inf \{d(s, e)\colon e \in E\}$ is computable, since we can compute to arbitrary accuracy the shortest distance from s to E along a line of longitude (for any computable s).[3] Thus by the triangle inequality we have

$$\vdash_c d(p, p') + d(p', E) \geqslant d(p, E),$$

and similarly with $-p$ in place of p. (Cf. Bishop 1967, pp. 82–3. Here, '$\vdash_c$' means "it is constructively provable that".) Also, of course, $d(p, E) > 0$ and $d(-p, E) > 0$. Hence,

$$\vdash_c d(p, p') + d(p', E) > 0 \quad \text{and}$$

$$\vdash_c d(-p, p') + d(p', E) > 0.$$

Then, by an elementary constructive property of reals, we have,

$$\vdash_c d(p, p') > 0 \vee d(p', E) > 0 \quad \text{and}$$

$$\vdash_c d(-p, p') > 0 \vee d(p', E) > 0,$$

where '$\vee$' is constructive "or". (The property of reals is just $\alpha_1 + \cdots + \alpha_k > 0 \rightarrow \alpha_j > 0$, some j, $1 \leqslant j \leqslant k$. See Bridges 1979, p. 16, 6.5.) Since only finitely many alternatives are involved (one each for p and $-p$), we may distribute, inferring

$$\vdash_c [d(p, p') > 0 \;\&\; d(-p, p') > 0] \vee d(p', E) > 0$$

where the right disjunct implies, $\forall e \in E\colon d(p', e) > 0$. We have established,

$$\vdash_c \neg(p' \perp E) \vee \neg(p' \in E). \tag{*}$$

(Note that this much holds for arbitrary p', and is not conditional on any assumption about β.)[4] But we also have

$$\vdash_c \beta > 0 \rightarrow p' \perp E \qquad (**)$$

(by (2) and (5)(i)); furthermore,

$$\vdash_c \beta < 0 \rightarrow p' \in E \qquad (***)$$

(by (3) and (5)(ii), i.e. $\beta f(p') = M_{\beta f} = m_f = 0 \rightarrow p' \perp p$). Then, combining (*), (**), and (***) and the definition of β, we have (constructively) either

$$\vdash_c \neg \exists k(A(k) \ \& \ B(k) \ \& \ \forall m < k \neg A(m))$$

or

$$\vdash_c \neg \exists k(A(k) \ \& \neg B(k) \ \& \ \forall m < k \neg A(m)).$$

But we are in no position to assert either of these conclusions. This is thus a counterexample to the constructive provability of Gleason (restricted), as claimed.

Gleason's original theorems for $\mathbb{R}^3$ (1957, Theorem 2.8, Theorem 2.3) characterized *non-negative* frame functions rather than the whole class of bounded frame functions. Since this is slightly more restricted than what we have called "Gleason (restricted)", one may ask whether one may still hope for a constructive proof of Gleason (restricted), restricted to non-negative frame functions. (Call this "Gleason (restricted, non-negative)".) A straightforward adaptation of the above argument provides a negative answer.

PROPOSITION II. *The statement "Gleason (restricted, non-negative)" cannot be constructively proved.*

Let f be a non-negative frame function on S, and let β be as in the proof of Proposition I. Define

$$g(s) = [\beta f + M_f](s),$$

where M_f is $\sup(f)$. Then g is a non-negative frame function (of weight $\beta w_f + 3M_f$). Now carry out the same argument as in Proposition I, applied to g instead of βf.

Finally, we may consider the still more restricted class of *normalized* (i.e. of weight 1) non-negative frame functions, of special relevance to probability. Taking f to be *non-trivial* (somewhere > 0) and g as in Proposition II, we have $w_g > 0$, and then $h(s) \equiv g(s)/w_g$ is normalized. The argument proving Proposition I applies to h, and so, as a corollary, we have,

PROPOSITION III. *The statement "Gleason (restricted, non-negative, normalized)" cannot be constructively proved.*

3. CONCLUDING REMARKS

Let us remark briefly on the method of proof by counterexample. The "counterexample" produced is not of the usual sort, that is, it does not imply the (ordinary) negation of the proposition in question. Rather it is a counterexample, given our current state of knowledge, to the claim to have a (constructive) method of proving the proposition. The method gains its force from the consideration that, at any stage of mathematical inquiry, we shall almost certainly be able to supply unsolved problems with corresponding conditions serving in the roles of 'A' and 'B' of the argument. Due in part to the open-ended (and perhaps vague) character of the notion of "constructive method", there is no formalization of constructive mathematics whose adequacy is generally recognized (among constructivists) comparable to, say, the system of Zermelo–Fraenkel set theory as a formalization of (ordinary) classical mathematics. Thus, in order to show a problem to be constructively insoluble, we have little choice but to reduce given problems, whose insolubility is taken for granted, to the problem in question. The method of counterexamples is just a special case of this, in which what is taken for granted is the insolubility of "find a uniform constructive method of answering all number-theoretic questions of *this* particular form". (Whether in fact the constructivist can even formulate these reflections on the method entirely within constructivist language is a question of some poignancy. See Hellman (1989). This need not concern us here, however, as we can perfectly well understand the reflections classically.)

To conclude, let us remark on the significance of limitative results such as these. Whether constructivist mathematics (of any given variety) can do justice to scientific applications is, from our perspective, a question of central importance in the foundations and philosophy of mathematics. Resistence to Brouwer's program of intuitionism has, since its inception, stemmed in part from doubts as to the power of constructivist mathematics on this score. The work of Bishop and others, however, can be said to have breathed new life into constructivist mathematics: it shows that a great deal of applicable mathematics can indeed be constructivized. A great deal, however, is not all, and, if our assessment is sound, it is in any case not enough.

NOTES

*This material is based upon work supported by the National Science Foundation (USA) under Grant No. DIR-8922435. I am grateful to Philip Ehrlich and to Stewart Shapiro for useful comments on an earlier draft.

[1]As Mackey (1963) puts it in his Axiom VII: "The partially ordered set of all questions in quantum mechanics [basic two-valued propositions of the form "the value of observable T lies in (would, on suitable measurement, be found to lie in) Borel set b"] is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite dimensional Hilbert space." Cf. also von Neumann (1955) and Stein (1972). The program of grounding this in more basic assumptions is a principal aim of the algebraic approach to foundations of quantum mechanics known as "quantum logic". (Cf., e.g., Jauch 1968; Varadarajan 1968; Piron 1976.) In any case, the (complete, orthocomplemented, orthomodular) lattice $\mathscr{L}(\mathscr{H})$ of subspaces of a Hilbert space $\mathscr{H}$ is generally taken to be the appropriate domain of definition of probability measures in quantum mechanics.

[2]As will emerge, what are shown constructively unprovable are direct cnstructive readings of theorems of Gleason (1957), stated precisely above and below. We do not investigate other statements which might conceivably be proposed as "constructive versions of Gleason's theorem".

[3]A point s is computable just in case its coordinates are.

[4]Note that if (*) were conditional on $\beta \neq 0$, so would be our counterexample. Intuitionistically, this would make no diffeerence, since – lacking Markov's principle – we would not have $\beta \neq 0 \rightarrow \beta > 0 \vee \beta < 0$. It is worth pointing out, however, that the present argument does not thus rely on non-acceptance of Markov's principle.

REFERENCES

Bishop, E. (1967) *Foundations of Constructive Analysis* (New York: McGraw-Hill).
Bridges, D. S. (1979) *Constructive Functional Analysis* (London: Pitman).
Bridges, D. S. (1981) "Towards a Constructive Foundation for Quantum Mechanics," in F. Richman, ed., *Constructive Mathematics* (Berlin: Springer), pp. 260–273.

Cooke, R., Keane, M., and Moran, W. (1985) "An elementary Proof of Gleason's Theorem," *Mathematical Proceedings of the Cambridge Philosophical Society* (Cambridge, England: Cambridge University Press); reprinted in Hughes, R.I.G., *The Structure and Interpretation of Quantum Mechanics* (Cambridge, Mass.: Harvard University Press, 1989), pp. 321–337.

Dummett, M. (1977) *Elements of Inuitionism* (Oxford: Oxford University Press).

Gleason, A.M. (1957) "Measures on the Closed Subspaces of a Hilbert Space," *Journal of Mathematics and Mechanics* **6**(6), 885–893.

Hellman, G. P. (1989) "Never say 'Never'! On the Communication Problem between Intuitionism and Classicism," *Philosophical Topics* **17**(2), 47–67.

Jauch, J. M. (1968) *Foundations of Quantum Mechanics* (Reading, Mass.: Addison-Wesley).

Jordan, T. F. (1969) *Linear Operators for Quantum Mechanics* (New York: Wiley).

Mackey, G. W. (1963) *The Mathematical Foundations of Quantum Mechanics* (Reading, Mass.: Benjamin).

Piron, C. (1976) *Foundations of Quantum Physics* (Reading, Mass.: Benjamin).

Stein, H. (1972) "On the Conceptual Structure of Quantum Mechanics," in Colodny, R. A., ed., *Paradigms and Paradoxes: The Philosophical Challenge of the Quantum Domain* (Pittsburgh: University of Pittsburgh Press), pp. 367–438.

Varadarajan, V. S. (1968) *Geometry of Quantum Thoery, Volume I* (Princeton: Van Nostrand).

von Neumann, J. (1955) [1932] *Mathematical Foundations of Quantum Mechanics*, translated from the German edition by R. T. Beyer (Princeton: Princeton University Press).

Department of Philosophy,
University of Minnesota,
Minneapolis, MN 55455, U.S.A.