Probability

April 22, 2016

1 Probability Spaces

1.1 Classical Probability Spaces

Probability theory [5, 7, 16] is defined using the notions of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ. In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set, the space of events $\mathcal{E}$ as 2^{Ω}, the powerset of Ω, and the *probability measure* as a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i, the probability measures are additive $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example of a problem on a finite sample space (Two coin experiment) Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rclrcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

Note that the probability measure for disjoint events such as $\{HT\}$ and $\{TH\}$ do indeed add.

1.2 Quantum Probability Spaces

The mathematical framework above assumes that one has complete knowledge of the events and their relationships. But even in many classical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot be determined with certainty. In the quantum case, this partial knowledge is compounded by the fact that not all quantum events can be observed simultaneously. Indeed, in the quantum world, there are non-commuting events which cannot even happen simultaneously. To accommodate these more complex situations, we abandon the sample space Ω and define and reason directly about events. A quantum probability space consist of just two components: a set of events $\mathcal{E}$ and a probability measure $\mu : \mathcal{E} \to [0, 1]$. We give an example before giving the formal definition.

Consider the two-qubit Hilbert space with computational basis $|0\rangle$ and $|1\rangle$ and states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The set of events associated with this Hilbert space consists of all projections including the empty projection $\mathbb{0}$ and the unit projection $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$:

$$\{\mathbb{0}, |0\rangle\langle 0|, |1\rangle\langle 1|, |+\rangle\langle +|, |-\rangle\langle -|, \ldots, \mathbb{1}\}$$

Each event is interpreted as a possible post post-measurement state of a quantum system as follows: given some arbitrary current quantum state $|\psi\rangle$ to be measured, the event $|0\rangle\langle 0|$ states that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ states that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ states that the post-measurement state will be $|+\rangle$; the event $|-\rangle\langle -|$ states that the post-measurement state will be $|-\rangle$; the event $\mathbb{1}$ states that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the event $\mathbb{0}$ states that the post-measurement state will be the empty state.

Irrespective of the current state $|\psi\rangle$ and irrespective of the particular experiment, the probability of event $\mathbb{0}$ will always be 0 (it is an impossible event) and the probability of event $\mathbb{1}$ will always be 1 (it is a certain event). The probabilities attached to other events will depend on the particular state in question. If the state is $|0\rangle$, the probability of event $|0\rangle\langle 0|$ is 1; the probability of event $|1\rangle\langle 1|$ is 0; the probability of event $|+\rangle\langle +|$ is $\frac{1}{2}$; and the probability of event $|-\rangle\langle -|$ is $\frac{1}{2}$. If the state is $|+\rangle$, the probability of each event $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ will be $\frac{1}{2}$; the probability of event $|+\rangle\langle +|$ is 1; and the probability of event $|-\rangle\langle -|$ is 0.

We now formalize a *quantum probability space* as follows [3, 6, 15, 1, 11]. We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{E}$ as all *projections* on $\mathcal{H}$. Each quantum state $|\psi\rangle \in \mathcal{H} \setminus \{0\}$ induces a probability measure $\mu_\psi : \mathcal{E} \to [0, 1]$ on the space of events defined for any event $E \in \mathcal{E}$ as follows[1]:

$$\mu_\psi(E) = \langle \psi | E\psi \rangle \tag{1}$$

Similarly to the classical case, this probability measure must satisfy:

- $\mu(\mathbb{1}) = 1$, and
- for a collection of pairwise orthogonal E_i, we have $\mu(\sum_i E_i) = \sum_i \mu(E_i)$.

Yu-Tsung says: Claim: If $\sum_i |\psi_i\rangle\langle\psi_i|$ is a projection, then they are orthogonal.
Proof: If $\sum_i |\psi_i\rangle\langle\psi_i|$ is a projection, then

$$\begin{aligned} |\psi_j\rangle &= \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\psi_j\rangle \\ &= \sum_i |\psi_i\rangle\langle\psi_i|\psi_j\rangle \end{aligned}$$

Therefore, we have

$$0 = \sum_{i \neq j} |\psi_i\rangle\langle\psi_i|\psi_j\rangle$$

1.3 Plan

In the remainder of the paper, we consider variations of quantum probability spaces motivated by computation of numerical quantities in a world with limited resources:

[1] Recently, people extend the domain of μ_ψ to all operators $\mathcal{A}$ on $\mathcal{H}$ and consider $\mu_\psi : \mathcal{A} \to \mathbb{C}$ [11, 17]. When an operator $A \in \mathcal{A}$ is Hermitian, $\mu_\psi(A)$ is the expectation value of A. We does not take this approach because we want to focus only on probability.

- Instead of the Hilbert space $\mathcal{H}$ (constructed over the uncountable and uncomputable complex numbers $\mathbb{C}$), we will consider variants constructed over finite fields [10, 9, 8].
- Instead of real-valued probability measures producing results in the uncountable and uncomputable interval $[0, 1]$, we will consider set-valued probablitity measures [2, 14].

We will then ask if it is possible to construct variants of quantum probability spaces under these conditions. The main question is related to the definition of probability measures: is it possible to still define a probability measure as a function that depends on a single state? Specifically,

- given a state $|\psi\rangle$, is there a probability measure mapping events to probabilities that only depends on $|\psi\rangle$? In the conventional quantum probability space, the answer is yes by the Born rule [4, 12] and the map is given by: $E \mapsto \langle\psi|E\psi\rangle$.
- given a probability measure μ mapping each event E to a probability, is there is a *unique* state ψ such that $\mu(E) = \langle\psi|E\psi\rangle$. In the conventional case, the answer is yes by Gleason's theorem [6, 13, 15].

Amr says: The rest is NO NO NO. We are inventing some other conditions on probability measures that are not part of the framework !!!

If there may be more than one probability measure, we will discuss whether we will keep using the Born rule (1) or there is another formula $\tilde{\mu}$ such that $\tilde{\mu}_\psi$ is a probability measure for all $|\psi\rangle \in \mathcal{H}\backslash\{0\}$.

Then, Gleason's theorem states given a probability measure μ there is a mixed state $|\psi\rangle$ such that $\mu = \mu_\psi$, i.e., the Born rule is surjective[2]. For any other formula $\tilde{\mu}$, we can ask whether $\tilde{\mu}$ is surjective as well.

Obveriously, we don't want arbirarily assign a state $|\psi\rangle \in \mathcal{H}\backslash\{0\}$ with a proabability measure $\tilde{\mu}_\psi$, for example, assigning every state to the same probability measure. We want $\tilde{\mu}_\psi$ satisfying the following properties with some physical meaning:

- $\langle\psi|\phi\rangle = 0 \Leftrightarrow \tilde{\mu}_\psi\left(|\phi\rangle\langle\phi|\right) = \tilde{0}$, where $\tilde{0}$ is 0 for $[0, 1]$ and $\tilde{0}$ is impossible for $\mathscr{L}_2 = \{\text{impossible}, \text{possible}\}$.
- $\tilde{\mu}_\psi\left(|\phi\rangle\langle\phi|\right) = \tilde{\mu}_{\mathbf{U}|\psi\rangle}\left(\mathbf{U}|\phi\rangle\langle\phi|\mathbf{U}^\dagger\right)$,where $|\psi\rangle, |\phi\rangle$ are states and $\mathbf{U}$ is any unitary map, i.e., $\mathbf{U}^\dagger\mathbf{U} = \mathbb{1}$.

And the results can be summarized in the following table:

State space $\mathcal{H}$	Probability values	Is there a $\tilde{\mu}$ satisfying the given conditions?	Is the $\tilde{\mu}$ surjective?
$\mathbb{C}^d$ for $d \geq 3$	$[0, 1]$	Yes	Yes
$\mathbb{C}^d$	$\mathscr{L}_2$	Yes	No
$\mathbb{F}_{p^2}^{d}{}^*$ for $d \geq 3$ except $d = p = 3$	$[0, 1]$	No	
$\mathbb{F}_{p^2}^{d}{}^*$	$\mathscr{L}_2$	Yes	No

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