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Game Theory, Social Choice, Decision Theory, and Optimization

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Set Functions, Games and Capacities in Decision Making

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(xx) $\vee, \wedge$ are lattice supremum and infimum. When applied to real numbers, the usual ordering on real numbers is meant, hence they reduce to maximum and minimum respectively when there is a finite number of arguments;

(xxi) Useful conventions: $\sum_{i\in\varnothing} x_i = 0$, $\prod_{i\in\varnothing} x_i = 1$, where the x_i's are real numbers. Considering quantities $x_1, x_2, \ldots$ defined on an interval $I \subseteq \mathbb{R}$, we set $\wedge_{i\in\varnothing} x_i = \bigvee I$, $\vee_{i\in\varnothing} x_i = \bigwedge I$, where $\bigvee I, \bigwedge I$ are respectively the supremum and infimum of I. Also, $0! = 1$.

1.2 General Technical Results

We begin by some useful combinatorial formulas.

Lemma 1.1 *Let X be any finite nonempty set.*

(i) For every set interval $[A, B]$, $A, B \subseteq X$

$$\sum_{C\in[A,B]} (-1)^{|C\setminus A|} = \sum_{C\in[A,B]} (-1)^{|B\setminus C|} = \begin{cases} 0, & \text{if } A \subset B \\ 1, & \text{if } A = B. \end{cases} \tag{1.1}$$

(ii) For every positive integer n

$$\sum_{\ell=0}^{k} (-1)^{\ell} \binom{n}{\ell} = (-1)^k \binom{n-1}{k} \quad (k < n). \tag{1.2}$$

For $k = n$, $\sum_{\ell=0}^{n} (-1)^{\ell} \binom{n}{\ell} = (1-1)^n = 0$.

(iii) For every set interval $[A, B]$ in X, any integer k such that $|A| \leqslant k < |B|$:

$$\sum_{\substack{C\in[A,B]\\ |C|\leqslant k}} (-1)^{|C\setminus A|} = (-1)^{k-|A|} \binom{|B\setminus A| - 1}{k - |A|}.$$

(iv) For all integers $n, k \geqslant 0$

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{k+j+1} = \frac{n!k!}{(n+k+1)!}.$$

(v) For all integers $n \geqslant 0, k > n$

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{k-j} = (-1)^n \frac{n!(k-n-1)!}{k!}.$$

intimately related. A table summarizing all known bases and transforms is given in Appendix A. Inclusion-exclusion coverings (Sect. 2.18) is related to the problem of decomposing a game into a sum of simpler games. Lastly, Sect. 2.19 considers games on infinite and finite universal sets X, whose domain is a subcollection of 2^X (called a set system).

2.1 Set Functions and Games

A *set function on* X is a mapping $\xi : 2^X \to \mathbb{R}$, assigning a real number to any subset of X. A set function can be

(i) *Additive* if $\xi(A \cup B) = \xi(A) + \xi(B)$ for every disjoint $A, B \in 2^X$;
(ii) *Monotone* if $\xi(A) \leqslant \xi(B)$ whenever $A \subseteq B$;
(iii) *Grounded* if $\xi(\varnothing) = 0$;
(iv) *Normalized* if $\xi(X) = 1$.

Note that an additive set function is uniquely determined by its value on elements of X, because $\xi(A) = \sum_{x \in A} \xi(\{x\})$.

Definition 2.1 A *game* $v : 2^X \to \mathbb{R}$ is a grounded set function.

As far as possible, throughout the book we distinguish by their notation the type of set functions (ξ for general set functions, v for games and μ for capacities, see Definition 2.5 below).

We denote the set of games on X by $\mathcal{G}(X)$. The set of set functions on X is simply $\mathbb{R}^{(2^X)}$.

A game v is *zero-normalized* if $v(\{x\}) = 0$ for every $x \in X$. We can already notice the following properties:

(i) If $\xi \geqslant 0$ (nonnegative) and additive, then ξ is monotone;
(ii) If ξ is additive, then $\xi(\varnothing) = \xi(\varnothing) + \xi(\varnothing)$, which entails $\xi(\varnothing) = 0$;
(iii) To any game v one can associate a zero-normalized game $v_0 = v - \beta$, with β an additive game defined by $\beta(\{x\}) = v(\{x\})$ for every $x \in X$.

To any set function ξ we associate its *conjugate* (a.k.a. *dual*) $\overline{\xi}$, which is a set function defined by

$$\overline{\xi}(A) = \xi(X) - \xi(A^c) \qquad (A \in 2^X). \tag{2.1}$$

Note that $\overline{\xi}(\varnothing) = \xi(X) - \xi(X) = 0$. The following properties are easy to show (try!).

Theorem 2.2 *Let ξ be a set function on X.*

(i) If $\xi(\varnothing) = 0$, then $\overline{\xi}(X) = \xi(X)$ and $\overline{\overline{\xi}} = \xi$;
(ii) If ξ is monotone, then so is $\overline{\xi}$.
(iii) If ξ is additive, then $\overline{\xi} = \xi$ (ξ is self-conjugate).

Remark 2.3 The term "game" may appear strange, although it is commonly used in decision theory and capacity theory. It comes from cooperative game theory (see, e.g., Owen [263], Peleg and Sudhölter [267], Peters [268]). A game v (in its full name, a *transferable utility game in characteristic function form*) represents the gain that can be achieved by cooperation of the players (more on this in Sect. 2.4). ◇

2.2 Measures

A *measure* is a nonnegative and additive set function. A normalized measure is called a *probability measure*. A *signed measure* is an additive set function, that is, it may take negative values. Measures are usually denoted by m, and $\mathcal{M}(X)$ denotes the set of measures on X.

Example 2.4 Let us give some easy examples of measures, apart from probability measures.

(i) The *counting measure* m_c just counts the elements in sets: $m_c(A) = |A|$ for all $A \in 2^X$.
(ii) Measure of length, volume, mass, etc., can be considered to be measures because they are additive and nonnegative. In $\mathbb{R}^n$, the *Lebesgue measure*[1] of a Cartesian product of real intervals $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is its volume $(b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.
(iii) Let $x_0 \in X$. The *Dirac measure centered at* x_0 is defined by

$$\delta_{x_0}(A) = \begin{cases} 1, & \text{if } x_0 \in A \\ 0, & \text{otherwise.} \end{cases}$$

◇

2.3 Capacities

Definition 2.5 A *capacity* $\mu : 2^X \to \mathbb{R}$ is a grounded monotone set function; i.e., $\mu(\varnothing) = 0$ and $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B$.

Note that the constant function 0 is a capacity. Also, a capacity is a monotone game, and takes only nonnegative values. A capacity is *normalized* if in addition $\mu(X) = 1$. Note that an additive normalized capacity is a probability measure. The set of

[1] Henri-Léon Lebesgue (Beauvais, 1875 – Paris, 1941), French mathematician, famous for his major contributions to measure and integration theory.

Theorem 2.17 *Let v be a $\{0,1\}$-valued game (i.e., whose range is $\{0,1\}$). Then*

$$\mathbf{mc}(v)(A) = 1 \text{ if and only if } A \in \uparrow \mathcal{B}_0$$

where $\mathcal{B}_0$ is the set of minimal subsets of $\mathcal{B} = \{B : v(B) = 1\}$.

We recall that $\uparrow \mathcal{B}_0$ is the upset generated by $\mathcal{B}_0$ (Sect. 1.3.2).

2.7 Properties

We give the main properties of capacities and games.

Definition 2.18 Let v be a game on X. We say that v is

(i) *superadditive* if for any $A, B \in 2^X$, $A \cap B = \varnothing$,

$$v(A \cup B) \geqslant v(A) + v(B).$$

The game is said to be *subadditive* if the reverse inequality holds;

(ii) *supermodular* if for any $A, B \in 2^X$,

$$v(A \cup B) + v(A \cap B) \geqslant v(A) + v(B).$$

The game is said to be *submodular* if the reverse inequality holds. A game that is both supermodular and submodular is said to be *modular*. Supermodular games are often improperly called *convex* games, while submodular games are called *concave* (see Remark 2.24);

(iii) *k-monotone* (for a fixed integer $k \geqslant 2$) if for any family of k sets $A_1, \ldots, A_k \in 2^X$,

$$v\Big(\bigcup_{i=1}^{k} A_i\Big) \geqslant \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcap_{i \in I} A_i\Big).$$

v is *totally monotone* (or *∞-monotone*) if it is k-monotone for any $k \geqslant 2$;

(iv) *k-alternating* (for a fixed integer $k \geqslant 2$) if for any family of k sets $A_1, \ldots, A_k \in 2^X$,

$$v\Big(\bigcap_{i=1}^{k} A_i\Big) \leqslant \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcup_{i \in I} A_i\Big).$$

v is *totally alternating* (or *∞-alternating*) if it is k-alternating for any $k \geqslant 2$;

been considered (see the seminal work of Aumann and Shapley [11]), and in the finite case, it is not uncommon to consider games with *restricted cooperation*, that is, defined on a proper subset of 2^X. Indeed, in many real situations, it is not reasonable to assume that any coalition or group can form, and coalitions that can actually form are called *feasible*. If X is a set of political parties, leftist and rightist parties will never form a feasible coalition. Also, if some hierarchy exists among players, feasible coalitions should correspond to sets including all subordinates, or all superiors, depending on the interpretation of what a coalition represents. A last example concerns games induced by a communication graph. A feasible coalition is then a group of players who can communicate, in other terms, it corresponds to a connected component of the graph.

In this section, we briefly address the infinite case (a complete treatment of set functions on infinite sets would take a whole monograph, including in particular classical measure theory (Halmos [188]), and nonclassical measure theory, as it can be found in Denneberg [80], König [215], Pap [264, 265], Wang and Klir [343]), and focus on the finite case. We will present several possible algebraic structures for the subcollections of 2^X.

We use the general term *set system* to denote the subcollection of 2^X where set functions are defined. Its precise definition is as follows.

Definition 2.101 A *set system* $\mathcal{F}$ on X is a subcollection of 2^X containing $\varnothing$ and such that $\bigcup_{A\in\mathcal{F}} A = X$.

$\mathcal{F}$ endowed with set inclusion is therefore a poset, and $\varnothing$ is its least element (see Sect. 1.3.2 for all definitions concerning posets and lattices). We recall that $A \subset\cdot\, B$ means that $A \subset B$ and there is no C such that $A \subset C \subset B$. Elements of $\mathcal{F}$ are *feasible sets*. Definitions of set functions, games and capacities remain unchanged, only the domain changes. In particular, a *game v on* $(X, \mathcal{F})$ is a mapping $v : \mathcal{F} \subseteq 2^X \to \mathbb{R}$ satisfying $v(\varnothing) = 0$. We denote by $\mathcal{G}(X, \mathcal{F})$ the set of games on $\mathcal{F}$.[16]

2.19.1 Case Where X Is Arbitrary

(see Halmos [188, Chaps. 1 and 2])

Definition 2.102

(i) A nonempty subcollection $\mathcal{F}$ of 2^X is an *algebra on* X if it is closed under finite union and complementation:

$$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}; \qquad A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F};$$

[16]This notation implies that our previous notation $\mathcal{G}(X)$ is a shorthand for $\mathcal{G}(X, 2^X)$. The omission of the set system means that we consider the Boolean lattice 2^X. We keep this convention throughout the book.

(ii) A nonempty subcollection $\mathcal{R}$ of 2^X is a *ring on X* if it is closed under finite union and set difference:

$$A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R} \text{ and } A \setminus B \in \mathcal{R}.$$

Observe that:

(i) An algebra $\mathcal{F}$ is closed under finite $\cap$, and $\varnothing, X \in \mathcal{F}$. Hence an algebra is a set system;
(ii) For a ring $\mathcal{R}$, $\varnothing \in \mathcal{R}$ but X is not necessarily an element of $\mathcal{R}$;
(iii) Every algebra is a ring; Every ring containing X is an algebra.

Definition 2.103 An algebra $\mathcal{F}$ is a *σ-algebra* if it is closed under countable unions:

$$\{A_n\} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Observe that a σ-algebra is closed under countable intersection. A similar definition exists for σ-rings.

The set of finite subsets of X with their complement is an algebra, while the set of countable subsets of X with their complement is a σ-algebra.

We introduce some additional properties of set functions.

Definition 2.104 Let $\mathcal{X}$ be a nonempty subcollection of 2^X and ξ be a set function on $(X, \mathcal{X})$.

(i) ξ is *σ-additive* if it satisfies

$$\xi\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \xi(A_n)$$

for any family $\{A_n\}$ of pairwise disjoint sets in $\mathcal{X}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{X}$;
(ii) ξ is *continuous from below at a set* $A \in \mathcal{X}$ if for every countable family $\{A_n\}$ of sets in $\mathcal{X}$ such that $A_1 \subseteq A_2 \subseteq \cdots$ and $\lim_{n\to\infty} A_n = A$, it holds

$$\lim_{n\to\infty} \xi(A_n) = \xi(A).$$

ξ is continuous from below if this holds for every $A \in \mathcal{X}$;
(iii) ξ is *continuous from above at a set* $A \in \mathcal{X}$ if for every countable family $\{A_n\}$ of sets in $\mathcal{X}$ such that $A_1 \supseteq A_2 \supseteq \cdots$, $\xi(A_m) < \infty$ for some m, and $\bigcap_{n=1}^{\infty} A_n = A$, it holds

$$\lim_{n\to\infty} \xi(A_n) = \xi(A).$$

ξ is continuous from above if this holds for every $A \in \mathcal{X}$;
(iv) ξ is *continuous* if it is continuous from below and from above.

A *measure*[17] m is a nonnegative σ-additive set function on a ring, such that $m(\varnothing) = 0$. Observe that by the latter property, every measure is finitely additive. A measure m is *finite* if $m(X) < \infty$. A probability measure is a normalized measure. A *charge* is a finitely additive nonnegative set function vanishing at the empty set.

The continuity properties and σ-additivity are intimately related.

Theorem 2.105 *Let ξ be a finite, nonnegative, and finitely additive set function on a ring $\mathcal{R}$.*

(i) If ξ is either continuous from below at every $A \in \mathcal{X}$ or continuous from above at $\varnothing$, then μ is σ-additive, i.e., it is a (finite) measure;
(ii) If μ is a measure on $\mathcal{R}$, then it is continuous from below and continuous from above.

Remark 2.106

(i) In probability theory, algebras and σ-algebra are often called *fields* and σ-*fields*. σ-additivity is also called *countable additivity*, and continuity from above (respectively, below) is sometimes called *outer* (respectively, *inner*) *continuity*.
(ii) σ-additivity and σ-algebras are related to the famous *Problem of Measure* (see Aliprantis and Border [7, pp. 372–373] for a more detailed discussion): Given a set X, is there any probability measure defined on its power set so that the probability of each singleton is 0? The motivation for this question is that most often in applied sciences, to each point of the real line we assign measure zero. Returning to the Problem of Measure, if X is countable, then σ-additivity entails that no such probability measure exists, therefore sets of higher cardinality must be chosen. The *Continuum Hypothesis* asserts that the smallest uncountable cardinality is the cardinality of the interval $[0, 1]$. However, Banach and Kuratowski have shown that under this hypothesis, still no probability measure can have measure zero on singletons. It follows that in order to make probability measures satisfy this requirement, there are two choices: either σ-additivity is abandoned, or measurability of every set (that is, $\mathcal{F} = 2^X$) is abandoned. The latter choice is the most common one, and leads to σ-algebras.

Null Sets

The notion of null sets is well known in classical measure theory, where it indicates a set that cannot be "seen" by a (signed) measure, in the sense that its measure, as

[17] This is the classical definition. It generalizes the definition given in Sect. 2.2 for finite sets.

well as the measure of all its subsets, is zero. For more general set functions, this notion can be extended as follows.

Definition 2.107 (Murofushi and Sugeno [252]) Let v be a game on $(X, \mathcal{F})$, where $\mathcal{F}$ is a set system. A set $N \in \mathcal{F}$ is called a *null set* w.r.t. v if

$$v(A \cup M) = v(A) \qquad (\forall M \subseteq N \text{ s.t. } A \cup M \in \mathcal{F}), (\forall A \in \mathcal{F}).$$

We give the main properties of null sets.

Theorem 2.108 *Let v be a game on $(X, \mathcal{F})$. The following holds.*

(i) *The empty set is a null set;*
(ii) *If N is a null set, then $v(N) = 0$;*
(iii) *If N is a null set, then every $M \subseteq N$, $M \in \mathcal{F}$ is a null set;*
(iv) *If $\mathcal{F}$ is closed under finite unions, the finite union of null sets is a null set;*
(v) *If $\mathcal{F}$ is closed under countable unions and if v is continuous from below, the countable union of null sets is a null set;*
(vi) *Assume $\mathcal{F}$ is an algebra. Then N is a null set if and only if $v(A \setminus M) = v(A)$ (equivalently, $v(A \Delta M) = v(A)$), for all $M \subseteq N$, $M \in \mathcal{F}$, and for all $A \in \mathcal{F}$;*
(vii) *If v is monotone, N is a null set if and only if $v(A \cup N) = v(A)$ for all $A \in \mathcal{F}$;*
(viii) *If v is additive, N is a null set if and only if $v(M) = 0$ for every $M \subseteq N$, $M \in \mathcal{F}$;*
(ix) *If v is additive and nonnegative, N is null if and only if $v(N) = 0$.*

The proof of these statements is immediate from the definitions, and is left to the readers. Statement (viii) shows that our definition of null sets is an extension of the classical one.

Supermodular and Convex Games

The definition of supermodularity [see Definition 2.18(ii)] is left unchanged on algebras, because they are closed under finite union and intersection. When X is infinite, the equivalence between convexity and supermodularity [see Corollary 2.23(ii)] is lost in general, even if $\mathcal{F} = 2^X$ and X is countable. This is shown by the following example (Fragnelli et al. [145]).

Example 2.109 Consider $X = \mathbb{N}$ and the game v defined by:

$$v(S) = \begin{cases} 1, & \text{if } |S| = +\infty \\ 0, & \text{otherwise.} \end{cases}$$

It satisfies $v(S \cup i) - v(S) \leqslant v(T \cup i) - v(T)$ for every $S \subseteq T \subseteq \mathbb{N} \setminus \{i\}$, and therefore v is convex. However, consider S and T being respectively the set of odd and even numbers. Then $v(S) = v(T) = v(S \cup T) = 1$, and $v(S \cap T) = 0$. Therefore v is not supermodular. ◇

In the whole chapter, we consider a finite set N, with $|N| = n$.[1] The chapter makes an extensive use of Sects. 1.3.3–1.3.6 on polyhedra and linear programming.

3.1 Definition and Interpretations of the Core

Definition 3.1 Let us consider a game $v \in \mathcal{G}(N, \mathcal{F})$, where $\mathcal{F}$ is any set system on N (Definition 2.101). The *core* of v is defined by

$$\mathbf{core}(v) = \{x \in \mathbb{R}^N : x(S) \geqslant v(S), \forall S \in \mathcal{F}, \quad x(N) = v(N)\}, \tag{3.1}$$

where $x(S)$ is a shorthand for $\sum_{i \in S} x_i$. By convention, $x(\varnothing) = 0$.

The core of a game v is therefore a set of real vectors x having the property that the additive game generated by x is greater than v. Since it is defined by a set of linear inequalities plus one linear equality, it is a convex closed polyhedron of dimension at most $n - 1$, which may be empty.

The next two sections study in depth the properties of this polyhedron. Beforehand, we make some remarks on the interpretations of the core. To this end, we recall the two main interpretations of games and capacities given in Sect. 2.4.1.

In the first interpretation, N is a set of players, agents, etc., and $v(S)$ is the "worth" of coalition $S \subseteq N$. This pertains to cooperative game theory, social choice and group decision making, however the notion of core is best suited to cooperative game theory, and we therefore stick to this framework here. For a better understanding, we develop a little bit more its presentation (see Driessen [96], Owen [263], Peleg and Sudhölter [267] and Peters [268] for monographs on the topic, and Examples 2.6 and 2.8 for illustrations of this situation).

In most cases of interest, the function v represents the maximum benefit (or minimum cost, in which case inequalities in (3.1) have to be reversed) a coalition can achieve by cooperation of its members (or by using in common a resource). If all players in N cooperate, the quantity $v(N)$ represents the achieved benefit (or paid cost) in total.[2] Let us assume that the coalition N eventually forms. Then each player in N would like to be rewarded for his cooperation, for having contributed to the realization of the total benefit $v(N)$. This amounts to defining an *allocation*

[1]We apologize for the change of notation from X to N, since X is the universal set in Chaps. 2 and 4. We chose X for these chapters of general interest, as being "neutral," compared to the more specific Ω (obviously related to uncertainty), E (standing for the set of edges, which is common in combinatorial optimization), N (standard in game theory and for pseudo-Boolean functions), etc. We have chosen N in this chapter because it is more closely related to game theory. Also, throughout the chapter, vectors in $\mathbb{R}^n$ are used, more conveniently denoted by x, y, z, which could have caused some confusion with elements of X.

[2]Generally, people think benefits are positive amounts, however $v(N)$ could be negative and is considered then to be a loss. The following discussion works as well when $v(N)$ is a loss.

Chapter 4
Integrals

It is well known that in the case of classical (additive) measures, the Lebesgue integral is the usual definition of an integral with respect to a measure, and it allows the computation of the expected value of random variables. The question which is addressed in this chapter is: How to define the integral of a function with respect to a nonadditive measure, i.e., a capacity or a game? As we will see, the answer is not unique, and there exist many definitions in the literature. Nevertheless, two concepts of integrals emerge: the one proposed by Choquet in 1953, and the one proposed by Sugeno in 1974. Both are based on the decumulative distribution function of the integrand w.r.t. the capacity, the Choquet integral being the area below the decumulative function, and the Sugeno integral being the value at the intersection with the diagonal. Most of the other concepts of integral are also based on the decumulative function, like the Shilkret integral, but other approaches are possible. For example, the concave integral proposed by Lehrer is defined as the lower envelope of a class of concave and positively homogeneous functionals.

Integrals being naturally defined for functions on an arbitrary (infinite) universe, we suppose in this chapter that X is an arbitrary nonempty set, in contradiction with the general philosophy of the book, which is to work on finite sets. As far as heavy topological and measure-theoretic notions are not needed, we give definitions and establish results in the general (infinite) case, before specializing to the discrete case. More detailed expositions in a fully measure-theoretic framework can be found in Denneberg [80], Marinacci and Montrucchio [235], Wang and Klir [343], see also Murofushi and Sugeno [250, 252, 255].

The chapter mainly studies in parallel the Choquet integral and the Sugeno integral. Their definition are first given for nonnegative functions (Sect. 4.2) and then extended to real-valued functions (Sect. 4.3), which lead to two kinds of integrals, the symmetric and the asymmetric one. In Sect. 4.4, the case of simple functions is studied, which leads naturally to the discrete case (Sect. 4.5). The properties of both integrals are studied in depth in Sect. 4.6, followed by

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results on characterization (Sect. 4.8). Other minor topics are studied (expression w.r.t. various transforms, particular cases, integrands defined on the real line, etc.) before introducing other integrals (Sect. 4.11): the Shilkret integral, the concave integral, the decomposition integral and various pseudo-integrals. The chapter ends with an extension of the Choquet integral to nonmeasurable functions (Sect. 4.12).

Throughout the chapter, all capacities and games are finite; i.e., $\mu(X) < \infty$.

4.1 Simple Functions

Let X be arbitrary. A function $f : X \to \mathbb{R}$ is *simple* if its range $\mathbf{ran}f$ is a finite set. We give several ways of decomposing a simple nonnegative function f using characteristic functions. We assume $\mathbf{ran}f = \{a_1, \dots, a_n\}$, supposing $0 \leqslant a_1 < a_2 < \cdots < a_n$. One can easily check that

$$
\begin{aligned}
f &= \sum_{i=1}^{n} a_i 1_{\{x \in X : f(x) = a_i\}} \\
&= \sum_{i=1}^{n} (a_i - a_{i-1}) 1_{\{x \in X : f(x) \geqslant a_i\}} \qquad (4.1)
\end{aligned}
$$

letting $a_0 = 0$. These decompositions are respectively called the *vertical* and the *horizontal* decompositions. These names should be clear from Fig. 4.1 illustrating them.

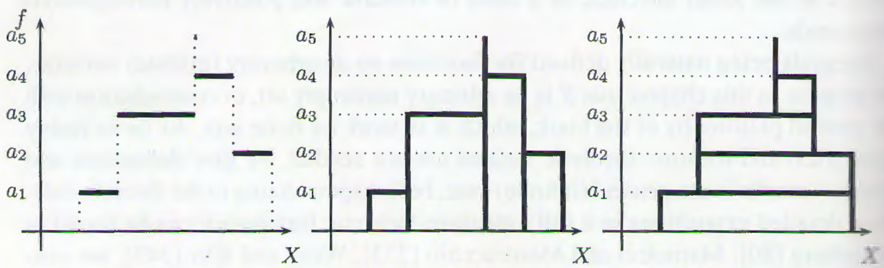

Fig. 4.1 Decompositions of a simple function f on X (function on *left*): vertical (*middle*), and horizontal decomposition (*right*)

4.2 The Choquet and Sugeno Integrals for Nonnegative Functions

We consider an arbitray set X, together with an algebra $\mathcal{F}$ (Definition 2.102). A function $f : X \to \mathbb{R}$ is $\mathcal{F}$*-measurable* if the sets $\{x : f(x) > t\}$ and $\{x : f(x) \geqslant t\}$ belong to $\mathcal{F}$ for all $t \in \mathbb{R}$.

We denote by $B(\mathcal{F})$ the set of bounded $\mathcal{F}$-measurable functions, and by $B^+(\mathcal{F})$ the set of bounded $\mathcal{F}$-measurable nonnegative functions.

Lemma 4.1 *The set $B(\mathcal{F})$ endowed with the usual order on functions is a lattice; i.e., $f, g \in B(\mathcal{F})$ imply that $f \vee g$ and $f \wedge g$ belong to $B(\mathcal{F})$.*

(The proof is left to the readers.) Evidently, the same holds for $B^+(\mathcal{F})$.

For any function $f \in B(\mathcal{F})$ and a capacity μ, we introduce the *decumulative distribution function* or *survival function* $G_{\mu,f} : \mathbb{R} \to \mathbb{R}$, which is defined by

$$G_{\mu,f}(t) = \mu(\{x \in X : f(x) \geqslant t\}) \quad (t \in \mathbb{R}). \tag{4.2}$$

We notice that $G_{\mu,f}$ is well-defined because f is $\mathcal{F}$-measurable. Some authors replace "$\geqslant$" by "$>$," but as we will see by Lemma 4.5, this is unimportant.

For convenience, we often use the shorthands $\mu(f \geqslant t)$ and $\mu(f > t)$ for $\mu(\{x \in X : f(x) \geqslant t\})$ and $\mu(\{x \in X : f(x) > t\})$.

We establish basic properties of the decumulative distribution function. Before that, we introduce the notions of essential supremum and infimum.

Definition 4.2 For any $f \in B(\mathcal{F})$ and any capacity μ on $(X, \mathcal{F})$, the *essential supremum* and *essential infimum* of f w.r.t. μ are defined by

$$\text{ess sup}_\mu f = \inf\{t : \{x \in X : f(x) > t\} \text{ is null w.r.t. } \mu\}$$

$$\text{ess inf}_\mu f = \sup\{t : \{x \in X : f(x) < t\} \text{ is null w.r.t. } \mu\}$$

respectively (see Definition 2.107 for the definition of a null set).

Lemma 4.3 *Let $f \in B^+(\mathcal{F})$ and μ be a capacity on $(X, \mathcal{F})$. Then $G_{\mu,f} : \mathbb{R} \to \mathbb{R}$*

(i) *is a nonnegative nonincreasing function, with $G_{\mu,f}(0) = \mu(X)$;*
(ii) *$G_{\mu,f}(t) = \mu(X)$ on the interval $[0, \text{ess inf}_\mu f]$;*
(iii) *has a compact support, namely $[0, \text{ess sup}_\mu f]$.*

Proof (i) Obvious by monotonicity of μ and the fact that $t > t'$ implies $\{x : f(x) \geqslant t\} \subseteq \{x : f(x) \geqslant t'\}$.

(ii) By definition, $N = \{f < \text{ess inf}_\mu f\}$ is a null set, hence $G_{\mu,f}(\text{ess inf}_\mu f) = \mu(X \setminus N) = \mu(X)$ by Theorem 2.108(vi).

(iii) Since f is bounded, so is its essential supremum. Now, by definition $\{x : f(x) > \text{ess sup}_\mu f\}$ is a null set, therefore $G_{\mu,f}(t) = 0$ if $t > \text{ess sup}_\mu f$. □

Figure 4.2 illustrates these definitions and properties.

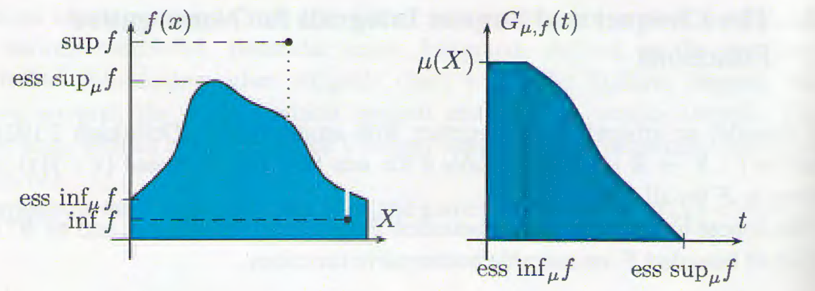

Fig. 4.2 A bounded nonnegative measurable function (*left*) and its decumulative distribution (*right*), supposing that singletons are null sets

Definition 4.4 Let $f \in B^+(\mathcal{F})$ and μ be a capacity on $(X, \mathcal{F})$. The *Choquet integral* of f w.r.t. μ is defined by

$$\int f \,\mathrm{d}\mu = \int_0^\infty G_{\mu,f}(t)\,\mathrm{d}t, \tag{4.3}$$

where the right hand-side integral is the Riemann integral.

Let us check if the Choquet integral is well-defined. As shown in Lemma 4.3, the decumulative function is a decreasing function bounded by $\mu(X) < \infty$, with compact support. Hence it is Riemann-integrable, so the Choquet integral is well-defined.

We prove now that it is equivalent to put a strict inequality in the definition of the decumulative function.

Lemma 4.5 *Let $f \in B^+(\mathcal{F})$ and μ be a capacity. Then*

$$\int_0^\infty \mu(f \geqslant t)\,\mathrm{d}t = \int_0^\infty \mu(f > t)\,\mathrm{d}t.$$

Proof (We follow Marinacci and Montrucchio [235].) Set for simplicity $G'_{\mu,f}(t) = \mu(\{x : f(x) > t\})$ for each $t \in \mathbb{R}$. We have for each $t \in \mathbb{R}$ and each $n \in \mathbb{N}$

$$\left\{x : f(x) \geqslant t + \frac{1}{n}\right\} \subseteq \{x : f(x) > t\} \subseteq \{x : f(x) \geqslant t\}$$

which yields

$$G_{\mu,f}\Big(t + \frac{1}{n}\Big) \leqslant G'_{\mu,f}(t) \leqslant G_{\mu,f}(t).$$

If $G_{\mu,f}$ is continuous at t, we have

$$G_{\mu,f}(t) = \lim_{n\to\infty} G_{\mu,f}\Big(t + \frac{1}{n}\Big) \leqslant G'_{\mu,f}(t) \leqslant G_{\mu,f}(t)$$

hence equality holds throughout. Otherwise, as $G_{\mu,f}$ is a nonincreasing function, it is discontinuous on an at most countable set $T \subseteq \mathbb{R}$. Hence both functions are equal for all $t \notin T$, which in turn implies that $\int_0^\infty G'_{\mu,f}(t)\,\mathrm{d}t = \int_0^\infty G_{\mu,f}(t)\,\mathrm{d}t$ by standard results on Riemann integration. □

We turn now to the Sugeno integral.

Definition 4.6 Let $f \in B^+(\mathcal{F})$ be a function and μ be a capacity on $(X, \mathcal{F})$. The *Sugeno integral* of f w.r.t. μ is defined by

$$\fint f\,\mathrm{d}\mu = \bigvee_{t\geqslant 0}(G_{\mu,f}(t) \wedge t) = \bigwedge_{t\geqslant 0}(G_{\mu,f}(t) \vee t). \tag{4.4}$$

In words, the Sugeno integral is the abscissa of the intersection point between the diagonal and the decumulative function, while the Choquet integral is the area below the decumulative function (Fig. 4.3).

One can easily check that the second equality holds in (4.4).

Remark 4.7 As for the Choquet integral, $G_{\mu,f}(t)$ can be replaced by $G'_{\mu,f}(t) = \mu\{x : f(x) > t\}$ without change. Indeed, we have proved that $G_{\mu,f}$ and $G'_{\mu,f}$ only differ at discontinuity points, and for those points, it can be checked that the two definitions lead to the same result. ◇

We immediately give a useful alternative formula for the Sugeno integral.

Lemma 4.8 *Let $f \in B^+(\mathcal{F})$ be a function and μ be a capacity on $(X, \mathcal{F})$. The Sugeno integral can be written as follows:*

$$\fint f\,\mathrm{d}\mu = \bigvee_{A\in\mathcal{F}}\Big(\bigwedge_{x\in A} f(x) \wedge \mu(A)\Big). \tag{4.5}$$

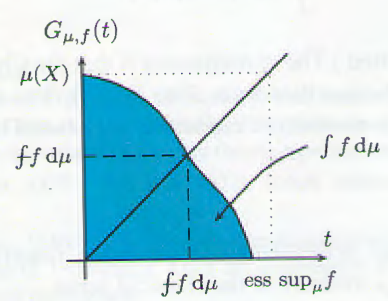

Fig. 4.3 The Choquet and Sugeno integrals

Proof For any $t \geqslant 0$, because $\bigwedge_{x:f(x)\geqslant t} f(x) \geqslant t$ and $\{f \geqslant t\} \in \mathcal{F}$ we have

$$t \wedge \mu(f \geqslant t) \leqslant \bigvee_{A\in\mathcal{F}} \Big(\bigwedge_{x\in A} f(x) \wedge \mu(A)\Big),$$

which yields

$$⨍ f \,\mathrm{d}\mu = \bigvee_{t\geqslant 0} \Big(t \wedge \mu(f \geqslant t)\Big) \leqslant \bigvee_{A\in\mathcal{F}} \Big(\bigwedge_{x\in A} f(x) \wedge \mu(A)\Big). \tag{4.6}$$

Now, for any given $A \in \mathcal{F}$, taking $t' = \bigwedge_{x\in A} f(x)$, we get $A \subseteq \{f \geqslant t'\}$. Applying monotonicity of μ, we obtain

$$\bigwedge_{x\in A} f(x) \wedge \mu(A) \leqslant t' \wedge \mu(f \geqslant t') \leqslant \bigvee_{t\geqslant 0} \Big(t \wedge \mu(f \geqslant t)\Big) = ⨍ f \,\mathrm{d}\mu$$

for any $A \in \mathcal{F}$. Consequently,

$$\bigvee_{A\in\mathcal{F}} \Big(\bigwedge_{x\in A} f(x) \wedge \mu(A)\Big) \leqslant ⨍ f \,\mathrm{d}\mu,$$

which, with (4.6), permits to conclude. □

This result is already in the original work of Sugeno [319] (see also Wang and Klir [343, Theorem 9.1]).

A fundamental fact is the following.

Lemma 4.9 *Let $A \in \mathcal{F}$ (i.e., 1_A is measurable). Then for every capacity μ*

$$\int 1_A \,\mathrm{d}\mu = \mu(A). \tag{4.7}$$

(Proof is obvious and omitted.) The consequence is that the Choquet integral can be viewed as an extension of capacities from $\mathcal{F}$ to $B^+(\mathcal{F})$. The same statement holds for the Sugeno integral for *normalized* capacities only (see Theorem 4.43(iii) for a more general statement).

Remark 4.10

(i) The Choquet integral generalizes the Lebesgue integral, and the latter is recovered when μ is a measure in the classical sense.

(ii) As the name indicates, the Choquet integral was introduced by Gustave Choquet[1] [53], although this reference does not mention explicitly any notion of integral. As many great ideas, the Choquet integral was rediscovered many times. The first appearance seems to be due to Vitali[2] [332], whose integral for inner and outer Lebesgue measures is exactly the Choquet integral for these measures. We mention also Šipoš [334], who introduced the symmetric version of the Choquet integral (Sect. 4.3.1) as the limit of finite sums computed over finite subsets of $\mathbb{R}$ containing 0. Also, the expression of the Choquet integral in the discrete case can be found in the 1967 paper of Dempster on upper and lower probabilities [77, Eq. (2.10)], as well as in the works of Lovász [226] (known under the name of Lovász extension; see Sect. 2.16.4). Up to the knowledge of the author, the first appearance of the name "Choquet integral" is due to Schmeidler [286] in 1986, followed independently by Murofushi and Sugeno in 1989 [250]. As mentioned by Chateauneuf and Cohen [48, Footnote 10], Schmeidler in fact rediscovered the Choquet integral, and became aware that is was previously introduced by Choquet through private discussions with Jean-François Mertens, who drew his attention to the 1971 paper by Dellacherie [76], showing that the Choquet integral is comonotonically additive and monotone (a fact, by the way, duly acknowledged by Schmeidler himself in [286]).

(iii) The Sugeno integral was introduced by Michio Sugeno[3] in 1972 [318–320] under the name of *fuzzy integral*.[4] As for the Choquet integral, this functional was in fact known as early as 1944, under the name of Ky Fan[5] distance [137]. This distance is defined as

$$\|f - g\|_0 = \bigvee \{x : x > 0, G_{\nu,|f-g|}(x)/x < 1\}$$

with ν a σ-additive probability. Hence, the Sugeno integral of f corresponds to the Ky Fan distance of f to the null function; i.e., $\|f - 0\|_0$.

---

[1] Gustave Choquet (Solesmes, 1915 – Lyon, 2006) is a French mathematician. He was professor at Université Pierre et Marie Curie in Paris and at École Polytechnique, and his main contributions include functional analysis, potential and capacity theory, topology and measure theory.

[2] Giuseppe Vitali (Ravenna, 1875 – Bologna, 1932), Italian mathematician. His contributions concern measure theory.

[3] Michio Sugeno (Yokohama, 1940–), Japanese computer scientist and mathematician. He has been professor at Tokyo Institute of Technology. Apart his contribution to measure theory, he mainly works in the field of artificial intelligence.

[4] As mentioned on p.28, Sugeno used instead of "capacity" the term "fuzzy measure," which he introduced, in the idea of representing human subjectivity.

[5] Ky Fan (Hangzhou, 1914 – Santa Barbara CA, 2010) Chinese mathematician. He received his D.Sc. degree in Paris under the supervision of M. Fréchet, and then did all his career in the United States, mainly at UCSB. He worked in convex analysis and topology. The "Fan inequality" is famous and generalizes Cauchy-Schwarz inequality.

(iv) An important observation is that the Sugeno integral can live on very poor structures for the range of the integrands and capacities: the richness of the real line is not needed, contrarily to the Choquet integral, and the definition of the Sugeno integral works on any totally ordered set L, like $\mathbb{N}$ or any finite subset of it, provided L is considered to be a set of "positive" values (see Sect. 4.3.2 for the general case where negative values are allowed). In particular, L can be taken as a *qualitative scale*; i.e., a finite totally ordered set of qualitative degrees of evaluation, like {*very bad, bad, medium, good, very good*}. The only requirement is that the range of μ and f should be included in L.

(v) The Choquet and Sugeno integrals are defined for (finite) capacities, but their definitions still work for games, provided they are of bounded variation norm (Sect. 2.19.1) in the case of the Choquet integral. However, note that in this case, the decumulative function is no longer nonincreasing in general, which causes the second equality in (4.4) not to hold any more! Therefore, it is better to consider that the Sugeno integral is not well defined in the case of nonmonotonic games. Alternatively, one may decide to define the Sugeno integral by, e.g., the expression with the supremum.

Lastly, note that the Choquet integral is not defined for set functions ξ such that $\xi(\varnothing) \neq 0$. Indeed, the area under the decumulative function may become infinite in this case.

(vi) We may define these integrals on a restricted domain $A \subseteq X$: in this case, we write

$$\int_A f \, \mathrm{d}\mu = \int_0^{\infty} \mu(\{f \geqslant t\} \cap A) \, \mathrm{d}t, \quad ⨍_A f \, \mathrm{d}\mu = \bigvee_{t \geqslant 0} (\mu(\{f \geqslant t\} \cap A) \wedge t). \tag{4.8}$$

◇

4.3 The Case of Real-Valued Functions

We suppose now that f is a bounded measurable real-valued function. We decompose f into its positive and negative parts f^+, f^-:

$$f = f^+ - f^-, \text{ with } f^+ = 0 \vee f, \quad f^- = (-f)^+. \tag{4.9}$$

Note that both f^+, f^- are nonnegative functions in $B^+(\mathcal{F})$ (bounded and measurable).

Suppose on the contrary that $\pi \neq \mathrm{Id}$ does not order f in decreasing order. Without loss of generality, consider that $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n$. It is a standard result from combinatorics that one can go from the identity permutation to π by elementary switches exchanging only 2 neighbor elements; i.e., we have the sequence

$$\sigma = \mathrm{Id} \to \cdots \pi' \to \pi'' \to \cdots \to \pi$$

with in each step $\pi'(j) = \pi''(j)$ except for $j = i, i+1$ for some $1 \leqslant i < n$, where $\pi'(i) = \pi''(i+1)$ and $\pi'(i+1) = \pi''(i)$. Consider two consecutive π', π'' in the sequence differing on $i, i+1$; we have by (4.37),

$$\begin{aligned}
\int f \, \mathrm{d}\phi^{\pi',v} &= \sum_{j=1}^{n} f_{\pi'(j)}\big(v(A^{\downarrow}_{\pi'}(j)) - v(A^{\downarrow}_{\pi'}(j-1))\big) = \sum_{j=1}^{n} (f_{\pi'(j)} - f_{\pi'(j+1)}) v(A^{\downarrow}_{\pi'}(j)) \\
&= \sum_{j=1}^{i-2} (f_{\pi'(j)} - f_{\pi'(j+1)}) v(A^{\downarrow}_{\pi'}(j)) + (f_{\pi'(i-1)} - f_{\pi'(i)}) v(A^{\downarrow}_{\pi'}(i-1)) \\
&\quad + (f_{\pi'(i)} - f_{\pi'(i+1)}) v(A^{\downarrow}_{\pi'}(i)) + (f_{\pi'(i+1)} - f_{\pi'(i+2)}) v(A^{\downarrow}_{\pi'}(i+1)) \\
&\quad + \sum_{j=i+2}^{n} (f_{\pi'(j)} - f_{\pi'(j+1)}) v(A^{\downarrow}_{\pi'}(j)) \\
&\leqslant \sum_{j=1}^{i-2} (f_{\pi''(j)} - f_{\pi''(j+1)}) v(A^{\downarrow}_{\pi''}(j)) \\
&\quad + (f_{\pi''(i-1)} - f_{\pi''(i+1)}) v(A^{\downarrow}_{\pi''}(i-1)) \\
&\quad + (f_{\pi''(i+1)} - f_{\pi''(i)})\big(v(A^{\downarrow}_{\pi''}(i+1)) + v(A^{\downarrow}_{\pi''}(i-1)) - v(A^{\downarrow}_{\pi''}(i))\big) \\
&\quad + (f_{\pi''(i)} - f_{\pi''(i+2)}) v(A^{\downarrow}_{\pi''}(i+1)) + \sum_{j=i+2}^{n} (f_{\pi''(j)} - f_{\pi''(j+1)}) v(A^{\downarrow}_{\pi''}(j)) \\
&= \sum_{j=1}^{n} f_{\pi''(j)}\big(v(A^{\downarrow}_{\pi''}(j)) - v(A^{\downarrow}_{\pi''}(j-1))\big) = \int f \, \mathrm{d}\phi^{\pi'',v},
\end{aligned}$$

where in the inequality we have used supermodularity of v, and the fact that $f_{\pi'(i)} - f_{\pi'(i+1)} \geqslant 0$, because $\pi'(i) < \pi'(i+1)$ (by construction, i and $i+1$ have not been switched before). It follows that $\int f \, \mathrm{d}v \leqslant \int f \, \mathrm{d}\phi^{\pi,v}$. □

From the above lemma, the following fundamental result is immediate (see Definition 3.1 for a definition of **core**(v)).

Theorem 4.39 (The Choquet integral as a lower expected value) *Suppose $|X| = n$ and $\mathcal{F} = 2^X$. Then for any function f on X, the game v is supermodular if and* *only if*

$$\int f\,\mathrm{d}v = \min_{\phi \in \mathbf{core}(v)} \int f\,\mathrm{d}\phi, \tag{4.51}$$

*where $\phi \in$ **core**(v) is identified with an additive measure.*

Proof Suppose that v is supermodular. By Theorem 3.15, we know that any core element ϕ is a convex combination of all marginal vectors: $\phi = \sum_\pi \lambda_\pi \phi^{\pi,v}$ with $\lambda_\pi \geqslant 0$ and $\sum_\pi \lambda_\pi = 1$. Using Lemma 4.38, we have by linearity of the integral [Theorem 4.24(ix)]

$$\int f\,\mathrm{d}v = \sum_\pi \Big(\lambda_\pi \int f\,\mathrm{d}v\Big) \leqslant \sum_\pi \Big(\lambda_\pi \int f\,\mathrm{d}\phi^{\pi,v}\Big) = \int f\,\mathrm{d}\Big(\sum_\pi \lambda_\pi \phi^{\pi,v}\Big) = \int f\,\mathrm{d}\phi$$

for any core element ϕ. Since by Lemma 4.38, equality is satisfied for at least one $\phi^{\pi,v}$, (4.51) holds.

Conversely, suppose that (4.51) holds for any f. Take any function f such that $f_{\sigma(1)} \geqslant \cdots \geqslant f_{\sigma(n)}$. Letting $A_\sigma^\downarrow(i) = \{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}$, there exists a core element ϕ such that

$$\int f\,\mathrm{d}v = \sum_{i=1}^n (f_{\sigma(i)} - f_{\sigma(i+1)}) v(A_\sigma^\downarrow(i)) = \sum_{i=1}^n (f_{\sigma(i)} - f_{\sigma(i+1)}) \phi(A_\sigma^\downarrow(i)) = \int f\,\mathrm{d}\phi. \tag{4.52}$$

Since $\phi \in$ **core**(v), we have $\phi(A_\sigma^\downarrow(i)) \geqslant v(A_\sigma^\downarrow(i))$, hence nonnegativity of $f_{\sigma(i)} - f_{\sigma(i+1)}$ and (4.52) force $\phi(A_\sigma^\downarrow(i)) = v(A_\sigma^\downarrow(i))$; i.e., ϕ is the marginal vector $\phi^{\sigma,v}$ [see (3.8) and (3.9)]. This being true for any f on X, it follows that for any permutation σ on X, the marginal vector $\phi^{\sigma,v}$ belongs to the core, a condition that is equivalent to supermodularity of v (see Theorem 3.15). □

Remark 4.40

- Again, as explained in Remark 4.36, results similar to Lemma 4.38 and Theorem 4.39 hold for submodular games, with inequalities inverted, min changed to max and the core changed to the anticore (that is, the set of efficient vectors ϕ satisfying $\phi(S) \leqslant v(S)$ for all $S \in 2^X$; see Sect. 3.1): the Choquet integral for submodular games is an upper expected value on the anticore.
- Dempster [77, Sect. 2] has shown that (4.51) holds for belief measures, a particular case of supermodular capacities.
- A result similar to Theorem 4.39 holds for the Sugeno integral; see Sect. 7.7.4.

◇

Recalling from Sect. 1.3.7 the notion of support function of a convex set, Theorem 4.39 merely says that for supermodular games, the Choquet integral is the support function of the core, because the right-hand of (4.51) can be rewritten as $\min_{x\in \mathbf{core}(v)}\langle f, x\rangle$, considering f, x as vectors in $\mathbb{R}^n$. A simple application of Theorem 1.12 leads to the following corollary.

Corollary 4.41 (The core as the superdifferential of the Choquet integral) *(Danilov and Koshevoy [66]) Suppose $|X| = n$ and $\mathcal{F} = 2^X$. Then for any supermodular[8] game v on X,*

$$\mathbf{core}(v) = \partial\Big(\int \cdot \,\mathrm{d}v\Big)(\mathbf{0}).$$

Remark 4.42 By Theorem 1.12 again, the support function is positively homogeneous and concave (or equivalently, superadditive). Hence, Lemma 4.38 and Theorem 4.39 constitute another proof of the equivalence of (i) and (ii) in Theorem 4.35. ◇

4.6.2 The Sugeno Integral

Theorem 4.43 *Let f be a function in $B^+(\mathcal{F})$, and μ a capacity on $(X, \mathcal{F})$. The following properties hold.*

(i) Positive $\wedge$-homogeneity:

$$-\!\!\!\!\!\!\int (\alpha 1_X \wedge f)\,\mathrm{d}\mu = \alpha \wedge -\!\!\!\!\!\!\int f\,\mathrm{d}\mu \qquad (\alpha \geqslant 0)$$

(ii) Positive $\vee$-homogeneity if $\operatorname{ess\,sup}_\mu f \leqslant \mu(X)$:

$$-\!\!\!\!\!\!\int (\alpha 1_X \vee f)\,\mathrm{d}\mu = \alpha \vee -\!\!\!\!\!\!\int f\,\mathrm{d}\mu \qquad (\alpha \in [0, \operatorname{ess\,sup}_\mu f]).$$

(iii) Hat function: for every $\alpha \geqslant 0$ and for every $A \in \mathcal{F}$,

$$-\!\!\!\!\!\!\int \alpha 1_A\,\mathrm{d}\mu = \alpha \wedge \mu(A)$$

(iv) Scale inversion: if $\operatorname{ess\,sup}_\mu f \leqslant \mu(X)$,

$$-\!\!\!\!\!\!\int (\mu(X)1_X - f)\,\mathrm{d}\mu = \mu(X) - -\!\!\!\!\!\!\int f\,\mathrm{d}\overline{\mu},$$

where $\overline{\mu}$ is the conjugate capacity;

---

[8] In Danilov and Koshevoy [66], the condition of supermodularity was overlooked.