U, and hence is bounded above on a translate V of U which includes the origin. But then τ^{-1} is bounded on the set $V \cap (-V)$, which is a neighborhood of the origin. Since τ^{-1} is rationally homogeneous, it follows that τ^{-1} is continuous at the origin, and hence by additivity at each point of R^2. Since, however, τ^{-1} maps R^2 biuniquely onto R, it cannot be continuous, and the contradiction completes the proof.

References

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NOTE ON CONVEX FUNCTIONS

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E. M. Wright in an interesting note [1] on a convex inequality states that it is unknown whether functions exist which satisfy condition (i) below and not condition (ii). It is not difficult to construct such a function, making use of a Hamel basis. (This makes use of the axiom of choice. See [2] or [3].) The conditions are as follows:

$$\text{(i)} \qquad f\left(\frac{a+b}{2}\right) \leqq \frac{f(a)+f(b)}{2} \qquad \text{for real } a \text{ and } b.$$

(ii) If $a \leqq b$ and $\delta > 0$ then $f(a+\delta) - f(a) \leqq f(b+\delta) - f(b)$.

Let H be a Hamel basis for the real numbers over the rationals. Suppose without loss of generality that 1 and π belong to H. Then each real number x has the unique representation $x = \sum_{h \in H} r_{x,h} \cdot h$, where the $r_{x,h}$ are rational numbers, only a finite number of which are not zero. Let

$$f(x) = \sum_{h \in H} r_{x,h}^2 \qquad \text{for each real } x.$$

It is easy to check that (i) holds. To see that (ii) does not hold, let $a = 1$, $b = \pi$, and $\delta = 1$. Then $a < b$, $\delta > 0$, and $f(a+\delta) - f(a) = 4 - 1 > 2 - 1 = f(b+\delta) - f(b)$.

The function f may be modified so that (i) is still satisfied; and so that (ii) is satisfied for any preassigned set of values of $\delta > 0$ of power less than the continuum, but not for all $\delta > 0$.

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