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A characterization of the core of convex games through Gateaux derivatives

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Abstract

We establish a calculus characterization of the core of supermodular games, which reduces the description of the core to the computation of suitable Gateaux derivatives of the Choquet integrals associated with the game. Our result generalizes a classic result of Shapley (Internat. J. Game Theory 1 (1971) 11) to infinite games. As an application, we show that this representation takes a stark form for supermodular measure games.

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1. Introduction

Even though the core of a transferable utility (TU) game is a fundamental solution concept, widely used in mathematical economics, fairly little is known about its structure in infinite games. In order to shed light on this issue, Epstein and Marinacci [7] and Marinacci and Montrucchio [13] have introduced a calculus and a subcalculus for set functions and used them to establish several characterizations of cores of infinite TU games. Besides providing a useful conceptual framework, this approach turned out to be fruitful in the study of infinite games having finite dimensional cores, as discussed at length in [13].

In this article we follow a different route and show that cores of infinite games can be also characterized by using the Gateaux derivatives of the associated Choquet

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integrals. Specifically, let $v : \Sigma \to \mathbb{R}$ be a bounded supermodular game defined on a σ-algebra Σ and let $B(\Sigma)$ be the set of all bounded Σ-measurable real valued functions. If we denote by $v(X) = \int X \, dv$ the Choquet integral of $X \in B(\Sigma)$ with respect to v, we can view v as a functional on $B(\Sigma)$. This makes it possible to talk of its Gateaux derivative $Dv(X)$ at X, which is a finitely additive measure on Σ, whose associated linear functional $\langle Dv(X), Y \rangle$ is the directional derivative in the direction $Y \in B(\Sigma)$.

Given a supermodular game $v : \Sigma \to \mathbb{R}$, our main result, Theorem 7, shows that under some standard topological conditions, we have

$$core(v) = \overline{co}^{w^*}(\{Dv(X) : X \text{ injective and in } B(\Sigma)\}), \tag{1}$$

namely, the core of v is the weak*-closed convex hull of the derivatives of v computed at all injective functions belonging to $B(\Sigma)$.

This result provides a calculus characterization of the core and reduces its description to the computation of suitable Gateaux derivatives. Moreover, it generalizes the classic result of Shapley [18] on the set of extreme points of cores of finite supermodular games. Section 3.1 will show how Shapley's result implicitly rests on the derivatives of injective functions.

To illustrate the usefulness of our calculus representation, in Section 4 we consider supermodular measure games and show that the general representation (1) takes a particularly stark form (see Theorem 12 below).

A secondary contribution of our work consists of a fairly detailed study of the Gateaux differentiability of Choquet integrals, which is a key issue for our representation. In particular, in Theorem 10 we derive in closed form the Gateaux derivatives of supermodular measure games.

To the best of our knowledge, the recent article by Carlier and Dana [2] is the only other work that has investigated similar issues. They consider the special case of positive scalar measure games and use different "rearrangements" techniques, which do not seem to extend easily beyond the scalar case. Nevertheless, their article contains several insightful results on both the structure of $core(f \circ P)$ and the Gateaux differentiability properties of the Choquet functional $v = f \circ P$.

The article is organized as follows. After some preliminaries in Section 2, in Section 3 we prove our representation for general supermodular games. Section 4 studies the representation for measure games. In the Related literature we discuss in detail the relations of our article with [2]. In Appendix A we gather several technical lemmas used throughout the article, while Appendix B contains the proofs of our main results.

2. Preliminaries

Throughout the article, Σ is a σ-algebra of sets of a space Ω. Subsets of Ω are understood to be in Σ even where not explicitly stated.

A set function $v : \Sigma \to \mathbb{R}$ is a *game* if $v(\emptyset) = 0$. The elements $A \in \Sigma$ are understood to be coalitions, while $v(A)$ is the worth of coalition A. A game v is

positive if $v(A) \geqslant 0$ for all A,

bounded if $\sup_{A \in \Sigma} |v(A)| < \infty$, *monotone* (or a capacity) if $v(A) \leqslant v(B)$ whenever $A \subseteq B$,

continuous if $\lim_{n\to\infty} v(A_n) = v(\Omega)$ whenever $A_n \uparrow \Omega$ and $\lim_{n\to\infty} v(A_n) = 0$ whenever $A_n \downarrow \emptyset$,

supermodular (or convex) if $v(A \cup B) + v(A \cap B) \geqslant v(A) + v(B)$ for all sets A and B,

additive (or a charge) if $v(A \cup B) = v(A) + v(B)$ for all pairwise disjoint sets A and B,

countably additive (or a measure) if $v(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} v(A_i)$ for all countable collections of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty}$.

The set of all charges (measures, resp.) that are bounded with respect to the variation norm $||\cdot||$ is denoted by $ba(\Sigma)$ ($ca(\Sigma)$, resp.). Generic elements of $ba(\Sigma)$ are denoted by m.

The set of all games that can be expressed as differences of two capacities is denoted by $bv(\Sigma)$. Aumann and Shapley [1] show that a game v belongs to $bv(\Sigma)$ if and only if the norm $||v|| \equiv \sup \sum_{i=1}^{n} |v(E_i) - v(E_{i-1})|$, where the sup is taken over all finite chains $\{E_i\}_{i=0}^{n}$, is finite.

The core of a game v is the set of all charges in $ba(\Sigma)$ that setwise dominate v, that is,

$$core(v) = \{m \in ba(\Sigma) : m(A) \geqslant v(A) \text{ for all } A \in \Sigma \text{ and } m(\Omega) = v(\Omega)\}.$$

The core is a (possibly empty) convex set and it is weak*-compact.

$B(\Sigma)$ denotes the set of all bounded Σ-measurable functions defined on Ω. The standard duality between $(ba(\Sigma), ||\cdot||)$ and $(B(\Sigma), ||\cdot||)$ will be denoted by $\langle X, m \rangle = \int X\, dm$, with $X \in B(\Sigma)$ and $m \in ba(\Sigma)$.

The following result (see [13]) plays an important role in understanding supermodular games; it is a generalization to real-valued supermodular games of well known properties of positive supermodular games, essentially due to Choquet [3].

Proposition 1. *Let $v : \Sigma \to \mathbb{R}$ be a bounded and supermodular game. Then $core(v) \neq \emptyset$ and*

(i) *Given any chain $\{E_i\}_{i\in I}$ in Σ, there is an extreme point m of $core(v)$ such that $m(E_i) = v(E_i)$ for all $i \in I$.*
(ii) *v is continuous if and only if $core(v) \subseteq ca(\Sigma)$.*
(iii) *v belongs to $bv(\Sigma)$.*

2.1. Choquet integrals and derivatives

Given a game v and a function $X \in B(\Sigma)$, the Choquet integral $v(X) \equiv \int X\, dv$ is defined as

$$v(X) = \int_0^{\infty} v(X \geqslant t)\, dt + \int_{-\infty}^{0} [v(X \geqslant t) - v(\Omega)]\, dt, \tag{2}$$

where on the right we have two Riemann integrals. The Choquet integral exists for all $X \in B(\Sigma)$ whenever $v \in bv(\Sigma)$ because $v(X \geqslant t)$ is of bounded variation in t and the Riemann integrals in (2) are well defined.

The Choquet integral is positively homogeneous and Lipschitz continuous (see Lemma A.1). It is also monotone when v is a capacity, while it is superadditive (and so concave) when v is supermodular. Finally, it is additive on any pair of comonotonic functions, that is, on any pair $X, Y \in B(\Sigma)$ such that $[X(\omega) - X(\omega')][Y(\omega) - Y(\omega')] \geqslant 0$ for any $\omega, \omega' \in \Sigma$ (see [17]).

By Proposition 1(i),

$$v(X) = \min_{m \in core(v)} \langle X, m \rangle \quad \text{for all } X \in B(\Sigma), \tag{3}$$

whenever v is supermodular. Consequently, in the supermodular case the Choquet integral can be viewed as a support function.[1]

Given $v : B(\Sigma) \to \mathbb{R}$ and $X \in B(\Sigma)$, if there exists an element $Dv(X) \in ba(\Sigma)$ such that

$$\langle Y, Dv(X) \rangle = \lim_{t \downarrow 0} \frac{v(X + tY) - v(X)}{t} \tag{4}$$

for all $Y \in B(\Sigma)$, then we say that v is Gateaux differentiable at X, and $Dv(X)$ is the Gateaux derivative of v at X.

Finally, $\partial v(X)$ denotes the standard superdifferential:

$$\partial v(X) = \{m \in ba(\Sigma) : \langle Y - X, m \rangle \geqslant v(Y) - v(X) \text{ for all } Y \in B(\Sigma)\}.$$

3. General results

3.1. Shapley's Theorem

Shapley [18] has characterized the extreme points of $core(v)$ when Ω is finite and Σ is its power set. Recall that a maximal chain $\mathscr{C}$ of a finite set $\Omega = \{\omega_1, \ldots, \omega_N\}$ is a collection of sets

$$\{\omega_{\sigma(1)}\}, \{\omega_{\sigma(1)}, \omega_{\sigma(2)}\}, \ldots, \{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(N)}\}, \tag{5}$$

where σ is a permutation over $\{1, \ldots, N\}$.

Theorem 2 (Shapley). *Let v be a supermodular game defined on the power set of a finite set Ω. Then, a charge m is an extreme point of $core(v)$ if and only if there is a maximal chain $\mathscr{C}$ such that $v(A) = m(A)$ for all $A \in \mathscr{C}$.*

We now provide a perspective on Shapley's result that permits our generalization. Denote by $A_{\sigma(1)} \subseteq \cdots \subseteq A_{\sigma(N)}$ the maximal chain (5); its associated extreme point m^σ is given by $m^\sigma(\omega_{\sigma(i)}) = v(A_{\sigma(i)}) - v(A_{\sigma(i-1)})$ for each $i = 1, \ldots, N$, where $A_{\sigma(0)} \equiv \emptyset$. The vector m^σ is usually interpreted as a marginal worth vector (see [9]).

[1] The converse is false, as support functions are in general not comonotonic additive.

When Ω is finite, each maximal chain can be viewed as the collection of the upper sets of a suitable injective function on Ω. Up to comonotonicity, such injective function is unique. In fact, two injective functions X_1 and X_2 share the same collection of upper sets if and only if they are comonotonic. Therefore, modulo comonotonicity, there is a one-to-one natural correspondence between injective functions and maximal chains.

Using this correspondence, we can replace maximal chains with injective functions $X : \Omega \to \mathbb{R}$ and then look at the differentiability properties at injective functions of the Choquet functional $v : B(\Sigma) \to \mathbb{R}$ associated with the game v. It turns out that for any game v, not even necessarily supermodular, its associated Choquet functional is differentiable at all X injective and that its derivative is a marginal worth vector, that is, $Dv(X)(\omega_{\sigma(i)}) = v(A_{\sigma(i)}) - v(A_{\sigma(i-1)})$.

Summing up these observations, we have the following equivalent form of Shapley's Theorem.

Proposition 3. *Let v be a supermodular game defined on the power set of a finite set Ω. Then, a charge m is an extreme point of core(v) if and only if it belongs to the set*

$$\{Dv(X) : X \text{ is an injective function on } \Omega\}.$$

It is this perspective on Shapley's result that permits our generalization to infinite games given by Theorem 7.

3.2. Core representation

We begin with a lemma that establishes some properties of $\partial v(X)$ in the supermodular case.

Lemma 4. *Let $v : \Sigma \to \mathbb{R}$ be a bounded and supermodular game. Then, for all $X \in B(\Sigma)$, $\partial v(X) \subseteq core(v)$ is non-empty and*

$$\partial v(X) = \{m \in core(v) : \langle X, m \rangle = v(X)\}. \tag{6}$$

Moreover, given any $X_1, X_2 \in B(\Sigma)$, the three following conditions are equivalent:

(i) $\partial v(X_1) \cap \partial v(X_2) \neq \emptyset$,
(ii) $v(X_1 + X_2) = v(X_1) + v(X_2)$,
(iii) $\partial v(X_1) \cap \partial v(X_2) = \partial v(X_1 + X_2)$.

If v is supermodular, by a classic result in Convex Analysis the set $\partial v(X)$ is a singleton if and only if the Choquet integral $v : B(\Sigma) \to \mathbb{R}$, which is Lipschitz continuous, is Gateaux differentiable at X (see [15]). The following result is thus an immediate consequence of the second part of Lemma 4. It says that the derivative is invariant under comonotonicity.

Corollary 5. *Let $v : \Sigma \to \mathbb{R}$ be a bounded and supermodular game. If the Choquet integral $v : B(\Sigma) \to \mathbb{R}$ is Gateaux differentiable at X_1 and at X_2, then $Dv(X_2) = Dv(X_1)$ whenever X_1 and X_2 are comonotonic.*

We can now turn to our first result, in which we prove that the Choquet functional is Gateaux differentiable at all injective functions in $B(\Sigma)$. This is a significant class of functions in $B(\Sigma)$, as proved by Lemma A.2, which shows that the class of all injective functions is dense in $B(\Sigma)$.

To prove this result, assume that v is continuous and that (Ω, Σ) is a (standard) Borel space, that is, (Ω, Σ) is isomorphic to a pair (Ω', Σ'), where Ω' is a Borel subset of some Polish space and Σ' is its Borel σ-algebra (see [19]).

Theorem 6. *Let (Ω, Σ) be a Borel space and let $v : \Sigma \to \mathbb{R}$ be a bounded, continuous, and supermodular game. If $X \in B(\Sigma)$ is injective, then $\partial v(X)$ is a singleton consisting of the Gateaux derivative $Dv(X)$.*

The next example shows that continuity is needed in Theorem 6.

Example. Set $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$. Consider the filter game $v : \Sigma \to \{0, 1\}$ defined as follows: $v(A) = 1$ if and only if $A \subseteq \mathbb{N}$ is cofinite (see [12]). This two-valued game v is supermodular and discontinuous at Ω. It is easy to verify that $\int X\,dv = \liminf_{n\to\infty} X(n)$ for each $X \in B(\Sigma)$. This functional is nowhere Gateaux differentiable (see [15, Example 1.21]).

The derivatives $Dv(X)$ of Theorem 6 are extreme points of $core(v)$. In fact, by (3) and (6), we have $\langle X, Dv(X)\rangle < \langle X, m\rangle$ whenever $core(v) \ni m \neq Dv(X)$. Thus, all the Gateaux derivatives $Dv(X)$ are exposed, and so extreme, points of $core(v)$ (cf. [15]).

This suggests the possibility of representing $core(v)$ as a weak*-closed convex hull of these derivatives. This is achieved in Theorem 7, our main result. Let $\sim$ be the comonotonic relation on $B(\Sigma)$, that is, $X_1 \sim X_2$ if X_1 and X_2 are comonotonic. Then, $\sim$ is an equivalence relation when restricted to the collection $BI(\Sigma)$. As usual, $BI(\Sigma)/\sim$ denotes the set of equivalence classes determined by $\sim$, while, with a slight abuse of notation, $X \in BI(\Sigma)/\sim$ means that X is a representative of one of the equivalence classes determined by $\sim$.

Theorem 7. *Let (Ω, Σ) be a Borel space and $v : \Sigma \to \mathbb{R}$ be a bounded, continuous and supermodular game. Then*

$$core(v) = \overline{co}^{w^*}\{Dv(X) : X \in BI(\Sigma)/\sim\} \tag{7}$$

and, for all $Y \in B(\Sigma)$,

$$v(Y) = \inf\{\langle Dv(X), Y\rangle : X \in BI(\Sigma)/\sim\}, \tag{8}$$

where the inf *is a* min *if and only if there is an injective $X \in B(\Sigma)$ such that $v(X + Y) = v(X) + v(Y)$.*

Theorem 7 is the announced calculus characterization of the core. In the next section we provide an illustration of its usefulness by studying the important class of measure games.

El Kaabouchi [6] provides a generalization of Shapley's Theorem for supermodular Choquet capacities defined on a compact metric space. In his richer setting, he shows that the core of a supermodular Choquet capacity v is the weak*-closed convex hull of all the measures $m \in core(v)$ such that $\int X\, dm = \int X\, dv$ for some injective Borel function. Relative to his article, a contribution of our work is the observation that such a set is, for a general function $X \in B(\Sigma)$, the superdifferential of the Choquet integral at X. This observation then permits our differential approach.

We close this section by extending to our setting Ichiishi's Theorem (see [9]), which in a finite setting completes Shapley's Theorem by showing that a game is supermodular whenever each marginal worth vector of the game belongs to its core.

Theorem 8. *Let (Ω, Σ) be a Borel space and let $v \in bv(\Sigma)$. Suppose v is Gateaux differentiable at all $X \in BI(\Sigma)$. Then, v is supermodular provided $Dv(X) \in core(v)$ for each $X \in BI(\Sigma)$.*

When Ω is finite, we get back to Ichiishi's Theorem. In fact, in this case the Choquet functional is differentiable over $BI(\Sigma)$. By Theorem 6, for infinite games supermodularity is a sufficient condition under which Choquet functionals are differentiable over $BI(\Sigma)$. A limitation of Theorem 8 is that it is not easy to establish more general conditions under which this is the case.

4. Measure games

A game $v : \Sigma \to \mathbb{R}$ is a (*non-atomic*) *measure game* if there exists a vector measure $P = (P_1, \ldots, P_n) : \Sigma \to \mathbb{R}^n_+$, where each P_i is a non-atomic probability measure on Σ, and a function $f : R(P) \to \mathbb{R}$ defined over the range $R(P)$ of P and with $f(0) = 0$, such that

$$v(E) = f(P(E)) \quad \text{for all } E \in \Sigma.$$

When $n = 1$, $v = f \circ P$ is called a *scalar measure game*. By the Lyapunov Theorem, $R(P)$ is a compact and convex subset of $\mathbb{R}^n$.

Measure games play an important role in mathematical economics. Standard examples include exchange economies with transferable utilities and models of production technology (see [1,8]). However, even for a measure game, little is known about its core, except in the special case when its elements are linear combinations of the underlying vector measure P, that is, when $core(f \circ P) \subseteq span\{P_1, \ldots, P_n\}$. As far as we know, outside this special "linear" case (see [13] and the references therein) the only attempt to characterize the core of a measure game can be found in [2], in which Carlier and Dana consider scalar measure games.

The purpose of this section is to show that for supermodular measure games Theorem 7 delivers a particularly sharp description of the core. This is achieved through a study of the differentiability properties of the associated Choquet integrals $\int X\, d(f \circ P)$. In order to do this, we first characterize supermodular measure games in terms of the underlying function $f : R(P) \to \mathbb{R}$. Consider the following class of functions:

Definition 1. A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ defined on a convex set A is ultramodular if

$$f(x+h) - f(x) \leqslant f(y+h) - f(y)$$

for all $x, y \in A$ with $x \leqslant y$ and for all $h \geqslant 0$ such that $x + h$ and $y + h$ belong to A.

In other words, a function is ultramodular if its second difference

$$f(x+h+k) - f(x+h) - f(x+k) + f(x)$$

is non-negative for $h, k \geqslant 0$. Ultramodular functions are supermodular when A is a lattice. The converse in general does not hold and this is why we call them "ultramodular."[2]

Proposition 9. *A measure game $f \circ P : \Sigma \to \mathbb{R}$ is supermodular whenever $f : R(P) \to R$ is ultramodular. The converse holds when $R(P) = [0, 1]^n$.*

Bounded ultramodular functions have several interesting characterizations, detailed in [14]. The simplest one is the following: If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice differentiable, then f is ultramodular if and only if $\partial^2 f / \partial x_i \partial x_j \geqslant 0$ for all i, j.

When $n = 1$, ultramodularity is equivalent to convexity, provided f is continuous. Proposition 9 then reduces to the well known fact that a scalar measure game $f \circ P$ is supermodular if and only if f is convex, provided f is continuous. Without continuity, convexity and ultramodularity are no longer equivalent, and so the standard result fails, whereas Proposition 9 still holds.

When $n > 1$, ultramodularity and convexity are independent notions. There are convex functions that are not ultramodular (e.g., $f(x) = ||x||$) and, vice versa, ultramodular functions that are not convex (e.g., $f(x) = \prod_{i=1}^n x_i$). We refer to [14] for a study of this class of functions.

4.1. Gateaux differentiability

We can now study the Gateaux differentiability of the Choquet functional $\int X\, d(f \circ P)$, which is by Theorem 7 a key step in characterizing *core*$(f \circ P)$.

We need some notation. For a given $X \in B(\Sigma)$, $G_X^i(q)$ denotes the distribution function $G_X^i(q) = P_i(X \geqslant q)$ for $i = 1, 2, \ldots, n$, while $G_X : \mathbb{R} \to R(P)$ is the vector

[2] Choquet [3, p. 172] defines a similar class of functions, even though he also requires the first difference to be non-negative, and so the function itself to be non-decreasing. Proposition 9 as well is mainly due to [3, pp. 193–194].

mapping defined by

$$G_X(q) = (G_X^1(q), \ldots, G_X^n(q)).$$

The symbol $\bar{P}$ denotes the measure $\bar{P} = P_1 + \cdots + P_n$. For a vector $x \in \mathbb{R}^n$, $|x|_1 = \sum_{i=1}^n |x_i|$ is the l^1-norm of $\mathbb{R}^n$.

By Theorem 6, under general conditions the Choquet functional is Gateaux differentiable at all injective functions in $B(\Sigma)$. We now show that a stronger result can be proved for measure games, thanks to their special structure.

As usual, a function $f : R(P) \to \mathbb{R}$ is differentiable on $R(P)$ whenever it can be extended to a differentiable function on some open set containing $R(P)$.

Theorem 10. *Let $v = f \circ P$ be a measure game over a Borel space (Ω, Σ), with f ultramodular. Given $X \in B(\Sigma)$, suppose one of the following holds*:

(i) *X is injective and f is differentiable and Lipschitz on $R(P)$*;
(ii) *X has a continuous distribution G_X and f is continuously differentiable.*

Then, the game v is Gateaux differentiable at X and its differential is

$$\langle Dv(X), Y \rangle = \sum_{i=1}^n \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, Y\, dP_i. \tag{9}$$

There is a trade-off between conditions (i) and (ii): condition (i) requires more on X (injectivity rather than just a continuous distribution function), whereas condition (ii) requires more on f (continuous differentiability rather than "plain" differentiability).

Remarks. (1) By Lemma A.2, the set of all functions $X \in B(\Sigma)$ having a continuous G_X is dense in $B(\Sigma)$. Such a set is actually a G_δ subset of $B(\Sigma)$, and so in this case the domain of differentiability is given by a broad class of functions. To see this property of the domain, let $\alpha_n \downarrow 0$ and set $V_n = \{X \in B(\Sigma) : \bar{P}(X = q) < \alpha_n \text{ for all } q \in \mathbb{R}\}$. Each set V_n is open, and so the set $\bigcap_n V_n$ is the desired G_δ dense subset. Since $B(\Sigma)$ is not a weak Asplund space (see [15]), it is noteworthy that the domain of Gateaux differentiability of a concave Choquet functional contains a dense G_δ subset of $B(\Sigma)$.

(2) In the special case $n = 1$, the differentiability assumptions in Theorem 10 can be weakened. For a scalar convex and continuous f, the game $f \circ P$ is Gateaux differentiable at X, with

$$\langle Dv(X), Y \rangle = \int f'_+(G_X \circ X)\, Y\, dP, \tag{10}$$

provided either X is injective or X has a continuous distribution function G_X and f is Lipschitz.[3] This scalar version of Theorem 10 sharpens an earlier result of Carlier and Dana [2], who proved (10) for a strictly increasing and differentiable f.

[3] Here, f'_+ is the right derivative.

4.2. Core representation

We can now state the version of Theorem 7 for measure games. One of its features is the use of densities, made possible by the existence of an underlying vector probability P. In this regard, we have the following simple result (cf. [13]).

Proposition 11. *Let $v = f \circ P$ be a measure game, not necessarily supermodular, and suppose f is lower semicontinuous at 0 and at $P(\Omega)$. Then, $core(v) \subseteq ca(\Sigma)$ and each $m \in core(v)$ is absolutely continuous with respect to $\bar{P}$.*

In view of Proposition 11, we can consider $core(v) \subseteq L_1(\Omega, \Sigma, \bar{P})$ by identifying $m \in core(v)$ with its density $dm/d\bar{P} \in L_1(\Omega, \Sigma, \bar{P})$. In particular, $core(v)$ can be viewed as a $\sigma(L_1, L_\infty)$-compact subset of $L_1(\Omega, \Sigma, \bar{P})$.

To state the result, we need two pieces of notation:

(i) $\overline{co}$ denotes the closed convex hull in the norm topology of $L_1(\Omega, \Sigma, \bar{P})$;
(ii) $\int_E (\partial f/\partial x_i)(G_X \circ X)\, dP_i$ denotes the measure naturally associated with the linear functional $Y \to \int (\partial f/\partial x_i)(G_X \circ X)\, Y\, dP_i$.

Theorem 12. *Let $v = f \circ P$ be a measure game over a Borel space (Ω, Σ), and suppose f is differentiable, Lipschitz and ultramodular. Then,*

$$core(v) = \overline{co}\left\{ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_X \circ X) \frac{dP_i}{d\bar{P}} \colon X \in BI(\Sigma)/\sim \right\}$$
$$= \overline{co}^{w^*}\left\{ \sum\nolimits_{i=1}^{n} \int_E \frac{\partial f}{\partial x_i}(G_X \circ X)\, dP_i : X \in BI(\Sigma)/\sim \right\}, \qquad (11)$$

where $dP_i/d\bar{P}$ is the Radon–Nikodym derivative of P_i with respect to $\bar{P}$.

The above representation of $core(v)$ can be further sharpened thanks to the $\sigma(L_1, L_\infty)$ -compactness of $core(v)$, which makes it possible to apply the classic Vitali Convergence Theorem (see, e.g., [5, p. 325]). We thus get the following topology-free representation, based on almost sure convergence. In the statement, $\bar{P} - \lim f_n$ denotes a $\bar{P}$-a.e. limit of the sequence $\{f_n\}_n$.

Corollary 13. *Let $v = f \circ P$ be a measure game and suppose the hypotheses of Theorem 12 hold. If we set*

$$\mathscr{D} \equiv \left\{ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_X \circ X) \frac{dP_i}{d\bar{P}} \colon X \in BI(\Sigma)/\sim \right\},$$

then,

$$core(v) = \{\bar{P} - \lim f_n : f_n \in co(\mathscr{D})\}.$$

All the above representation formulas hold under weaker assumptions in the scalar case, along the lines of Remark (2) of the previous subsection. In this case the differentiability assumptions are no longer needed, and for a scalar game $v = f \circ P$ representation (11) becomes:

$$core(v) = \overline{co}\{f'_+(G_X \circ X) : X \in BI(\Sigma)/\sim\} \tag{12}$$

provided f is convex and continuous. Note that this is true since in (A.3) of Lemma A.3 we can set $f(y) = \int_0^y \rho(t)\,dt$, with $\rho = f'_+$. Using some non-trivial properties of ultramodular functions, in [14] we show that in Theorems 10 and 12 it is possible to dispose of the differentiability assumptions even in the non-scalar case. For brevity, we refer the reader to [14] for details.

5. Related literature

Delbaen [4] provides a characterization of the exposed points of cores of continuous supermodular games by means of chains. He does not use either the Choquet extension of games or any calculus formalism. He also suggests that the core could be described as the closed convex hull of its exposed points via the classic Amir–Lindenstrauss Theorem.

Carlier and Dana [2] investigate the core of a distortion $v = f \circ P$, where the function $f : [0,1] \to [0,1]$ is assumed to be strictly convex, increasing and differentiable. A function $s : \Omega \to [0,1]$ is measure preserving (m.p.) with respect to P if $\lambda(B) = P(s^{-1}(B))$ for all Borel sets $B \subseteq [0,1]$. Using "rearrangements" techniques (see, e.g., [16]), Carlier and Dana [2] proved that

$$core(f \circ P) = \overline{co}\{f'(s) : s \text{ is a m.p. function}\}.$$

Unlike the representation in Proposition 12, this representation does not have a closed form. The next result (whose proof is omitted) is key to understand the relations between their result and ours.

Proposition 14. *Let P be a non-atomic probability measure defined on a σ-algebra Σ of subsets of a space Ω. Then, a function $s : \Omega \to [0,1]$ is measure preserving if and only if there exists $X \in B(\Sigma)$ with a continuous distribution function and such that $G_X \circ X = s$.*

Hence, Proposition 12 improves their result since, under weaker hypotheses, it establishes a more economical representation of the core of a supermodular scalar measure game, which only uses the collection of injective functions rather than the collection of all functions having continuous distribution functions. To see that the latter collection is larger than the former, consider $\Omega = [0,1]$ equipped with the Lebesgue measure λ, and the tent map $T : [0,1] \to [0,1]$ defined by $T(x) = 1 - |2x - 1|$ for all $x \in [0,1]$. The tent map and all its iterates T^n are m.p. functions with respect to λ, and so by Proposition 14 they have a continuous distribution function G_{T^n}. But, they are not injective; therefore, while they are included in the Carlier and Dana representation, they are absent in ours.

The cost of our improvement is that, unlike [2], we have to assume that (Ω, Σ) is a Borel space.

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Appendix A. Lemmas

Lemma A.1. *Let $v : \Sigma \to \mathbb{R}$ be a game in $bv(\Sigma)$. The Choquet integral $v : B(\Sigma) \to \mathbb{R}$ is Lipschitz continuous, with*

$$|v(X) - v(Y)| \leqslant ||v||||X - Y|| \tag{A.1}$$

for all $X, Y \in B(\Sigma)$.

Proof. Let us first prove the result when v itself is a capacity. Suppose $v(X) \geqslant v(Y)$. As $X \leqslant Y + ||X - Y||$, by monotonicity $v(X) \leqslant v(Y) + ||X - Y||v(\Omega)$, which implies

$$|v(X) - v(Y)| \leqslant v(\Omega)||X - Y|| \tag{A.2}$$

when v is monotonic. Pick now any $v \in bv(\Sigma)$. We know that v is representable as $v = v^+ - v^-$ where v^+ and v^- are the positive and negative semivariations of v respectively. v^+ and v^- are monotonic and $||v|| = v^+(\Omega) + v^-(\Omega)$. Thanks to (A.2), it follows straightforwardly

$$|v(X) - v(Y)| \leqslant [v^+(\Omega) + v^-(\Omega)]||X - Y||$$

which is (A.1). $\square$

Lemma A.2. *The set $BI(\Sigma)$ is dense in $B(\Sigma)$ whenever it is non-empty (e.g., when Ω is a Borel space).*

Proof. Since the simple functions are dense in $B(\Sigma)$, it will suffice to prove that, given any simple function X, there are injective functions ε close to X, for all $\varepsilon > 0$. W.l.o.g., set $||X_0|| = 1$, where X_0 is the existing injective function. Let us prove that, for $\lambda > 0$ small enough, the elements $X + \lambda X_0$ are injective. Let $R(X)$ be the (finite) range of X. Define

$$\sigma = \min\{|x_i - x_j| : x_i, x_j \in R(X),\ \ x_i \neq x_j\}.$$

The function $Y = X + \lambda X_0$ is injective when $0 < \lambda < \sigma/2$. For, take any two elements $\omega_1 \neq \omega_2$. If $X(\omega_1) = X(\omega_2)$, then $Y(\omega_1) \neq Y(\omega_2)$ for all $\lambda > 0$. Suppose, in contrast, that $X(\omega_1) \neq X(\omega_2)$, say $X(\omega_1) > X(\omega_2)$. Then,

$$\begin{aligned} Y(\omega_1) - Y(\omega_2) &= [X(\omega_1) - X(\omega_2)] + \lambda[X_0(\omega_1) - X_0(\omega_2)] \\ &\geqslant \sigma - 2\lambda > 0, \end{aligned}$$

as desired. □

To prove Theorem 10, we need a lemma which provides a general condition, under which the Gateaux derivative of the Choquet functional $\int X\, d(f \circ P)$ admits a closed form at an injective function $X \in BI(\Sigma)$. Note that the mapping G_X can be viewed as a curve in $R(P)$ with endpoints $\underline{0}$ and $\underline{1}$. We denote by $C_X \subseteq R(P)$ its range, that is,

$$C_X = \{x \in \mathbb{R}^n : x = G_X(q) \text{ for some } q \in \mathbb{R}\}.$$

Lemma A.3. *Let $v = f \circ P$ be a supermodular measure game, with f lower semicontinuous at 0 and $P(\Omega)$. If $X \in BI(\Sigma)$ and if there exists a locally integrable function $\rho_X : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$f(y) = \int_0^{|y|_1} \rho_X(t)\, dt \tag{A.3}$$

for all $y \in C_X$, then the Gateaux derivative is given by

$$\langle Dv(X), Y \rangle = \int_\Omega \rho_X(|G_X|_1 \circ X)\, Y\, d\bar{P}, \tag{A.4}$$

where $|G_X|_1 = \sum_{i=1}^n G_X^i$.

Proof. The curve $q \to G_X(q)$ is continuous since X is injective and each P_i is non-atomic. Consider the arc-length parametrization $\gamma : [0, n] \to C_X$, with $|\gamma(t)|_1 = t$ for each $t \in [0, n]$; that is, γ is the inverse of the map $x \to |x|_1$ restricted to C_X.

We begin by proving that the map $s = \gamma^{-1} \circ G_X \circ X : \Omega \to [0, n]$ is measure preserving, namely,

$$\lambda(A) = \bar{P}(s^{-1}(A)) \tag{A.5}$$

for all Borel set $A \subseteq [0, n]$, where λ is the Lebesgue measure on $\mathbb{R}$. For, take an interval $[t_1, t_2] \subseteq [0, n]$, with $t_1 \leqslant t_2$. Let $x_1, x_2 \in C_X$ the unique points on the curve such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. Hence, $|x_1|_1 = t_1$ and $|x_2|_1 = t_2$. Since G_X is monotone, we have

$$\gamma([t_1, t_2]) = \{x \in C_X : x_1 \leqslant x \leqslant x_2\}.$$

Clearly,

$$G_X^{-1}(\gamma([t_1, t_2])) = [q_1, q_2]$$

where q_1 is the minimal element for which $G_X(q_1) = x_2$ and q_2 is the maximal element such that $G_X(q_2) = x_1$ (such elements exist because G_X is continuous). Now

$$\begin{aligned}\bar{P}(s^{-1}([t_1, t_2])) &= \bar{P}(\{q_1 \leqslant X \leqslant q_2\}) = \sum_{i=1}^{n} P_i\{q_1 \leqslant X \leqslant q_2\} \\ &= \sum_{i=1}^{n} [G_X^i(q_1) - G_X^i(q_2)] = |G_X(q_1) - G_X(q_2)|_1 \\ &= |x_2 - x_1|_1 = |t_2 - t_1| = \lambda([t_1, t_2]),\end{aligned}$$

which proves that s is measure preserving.

Since the linear functional $Y \to \int_\Omega \rho(|G_X \circ X|_1) Y \, d\bar{P}$ is continuous over $B(\Sigma)$, it can be viewed as a signed measure $\overline{m}$ over $B(\Sigma)$. Thanks to (A.3), fixed any scalar $q^* \in \mathbb{R}$, we have

$$f(G_X(q^*)) = \int_0^{|G_X(q^*)|_1} \rho(t) \, dt.$$

In view of (A.5), we can change variable by means of $t = \gamma \circ G_X \circ X = |G_X \circ X|_1$. This implies:

$$f(G_X(q^*)) = \int_{X \geqslant q'} \rho(|G_X \circ X|_1) \, d\overline{P},$$

where q' is the minimal $q \in \mathbb{R}$ such that $G_X(q') = G_X(q^*)$. As $\overline{P}\{q' \leqslant X < q^*\} = 0$, we have

$$f(G_X(q^*)) = \int_{X \geqslant q^*} \rho(|G_X \circ X|_1) \, d\overline{P} = \overline{m}\{X \geqslant q^*\}.$$

As q^* is arbitrary, we conclude that for each $q \in \mathbb{R}$ it holds:

$$\overline{m}\{X \geqslant q\} = f(G_X(q)) = f(P(X \geqslant q)) = v(X \geqslant q). \tag{A.6}$$

By Theorem 6, the Gateaux derivative $Dv(X)$ exists at X. On the other hand, the same uniqueness argument used to prove Theorem 6 shows that the equality $\overline{m}\{X \geqslant q\} = v(X \geqslant q)$ of (A.6) implies $\overline{m} = Dv(X)$, as desired. □

Appendix B. Proofs

Proposition 1. Point (i) is proved in [13]. As to (ii), in view of (i), the "if" part is trivial. Actually, if $E_n \uparrow \Omega$, there exists some $m \in core(v)$, such that $m(E_n) = v(E_n)$. Consequently, $v(E_n) \to v(\Omega)$, analogously for $E_n \downarrow \emptyset$. As to the "only if" part, let $E_n \uparrow \Omega$ and assume that v is continuous. Let $m \in core(v)$. Then $\liminf_n m(E_n) \geqslant \lim_n v(E_n) = m(\Omega)$ and $\liminf_n m(E_n^c) \geqslant \lim_n v(E_n^c) = 0$. In turn, the latter inequality implies that $\limsup_n m(E_n) = m(\Omega) - \liminf_n m(E_n^c) \leqslant m(\Omega)$, so that $\liminf_n m(E_n) \geqslant m(\Omega) \geqslant \limsup_n m(E_n)$. We conclude that $m \in ca(\Sigma)$.

We conclude with (iii). Since $core(v)$ is weak*-compact, it is norm bounded and so there is $M > 0$ such that $||m|| \leqslant M$ for all $m \in core(v)$. Hence, for each finite chain

$\{E_i\}_{i=0}^n$, we have $\sum_{i=1}^n |v(E_i) - v(E_{i-1})| \leqslant \sum_{i=1}^n |m(E_i) - m(E_{i-1})| \leqslant M$. This implies that $||v|| < +\infty$. $\square$

Lemma 4. A routine separation argument proves the first part. In particular, as to the non-emptiness of $\partial v(X)$, by Proposition 1 there exists $m \in core(v)$ such that $m(X \geqslant t) = v(X \geqslant t)$ for all $t \in \mathbb{R}$. Let us prove the equivalence among the three statements.

(i) $\Rightarrow$ (ii). Let $m \in \partial v(X_1) \cap \partial v(X_2)$. Hence, $v(X_1) = \langle X_1, m \rangle$ and $v(X_2) = \langle X_2, m \rangle$, and so

$$v(X_1) + v(X_2) = \langle X_1 + X_2, m \rangle \geqslant v(X_1 + X_2).$$

As v is superadditive, we conclude that $v(X_1) + v(X_2) = v(X_1 + X_2)$.

(ii) $\Rightarrow$ (iii). We first prove that $\partial v(X_1) \cap \partial v(X_2) \subseteq \partial v(X_1 + X_2)$. Suppose that $\partial v(X_1) \cap \partial v(X_2) \neq \emptyset$, the inclusion being trivially true otherwise. Let $m \in \partial v(X_1) \cap \partial v(X_2)$, so that

$$v(X_1) = \langle X_1, m \rangle \quad \text{and} \quad v(X_2) = \langle X_2, m \rangle.$$

Hence, $v(X_1 + X_2) = v(X_1) + v(X_2) = \langle X_1 + X_2, m \rangle$, which implies $m \in \partial v(X_1 + X_2)$. Conversely, let us prove that $\partial v(X_1 + X_2) \subseteq \partial v(X_1) \cap \partial v(X_2)$. Pick any $m \in \partial v(X_1 + X_2)$ and suppose, *per contra*, that $m \notin \partial v(X_1) \cap \partial v(X_2)$, say $m \notin \partial v(X_1)$. We have

$$v(X_1 + X_2) = \langle X_1 + X_2, m \rangle = \langle X_1, m \rangle + \langle X_2, m \rangle > v(X_1) + v(X_2),$$

a contradiction.

(iii) $\Rightarrow$ (i). Since $\partial v(X_1 + X_2) \neq \emptyset$, we have $\partial v(X_1) \cap \partial v(X_2) \neq \emptyset$. $\square$

Theorem 6. It suffices to prove that $\partial v(X)$ is a singleton. Note that the continuity of v at Ω implies that $core(v) \subseteq ca(\Sigma)$ and, consequently, $\partial v(X) \subseteq ca(\Sigma)$ (see Proposition 1). Take any element $m \in \partial v(X)$ and, w.l.o.g., assume that $X \geqslant 0$. It satisfies

$$\int_0^\infty v(X \geqslant t)\, dt = \int_0^\infty m(X \geqslant t)\, dt. \tag{B.1}$$

The function $m(X \geqslant t) - v(X \geqslant t)$ is of bounded variation and so it is continuous for all $t \in \mathbb{R} \backslash C$, where C contains at most countably many elements. As $m(X \geqslant t) - v(X \geqslant t) \geqslant 0$, in view of (B.1), we infer that $m(X \geqslant t) = v(X \geqslant t)$ for all $t \in \mathbb{R} \backslash C$. Therefore, given two supergradients $m_1, m_2 \in \partial v(X)$, we have

$$m_1(X \geqslant t) = m_2(X \geqslant t) \quad \text{for all } t \in [a, b] \backslash C,$$

where $a < \inf_{\omega \in \Omega} X(\omega)$ and $b > \sup_{\omega \in \Omega} X(\omega)$ and C is as above. Consider the chain

$$\mathscr{C} = \{X \geqslant t\}_{t \in [a,b] \backslash C}.$$

Our objective is that of choosing a countable subchain of $\mathscr{C}$. As $\lambda\{[a, b] \backslash C\} = b - a$, the set $[a, b] \backslash C$ is dense in $[a, b]$. Therefore, we can construct a sequence $q_n \in [a, b] \backslash C$ which is dense in $[a, b] \backslash C$. Consider now the family

$$\mathscr{C}^* = \{q_m > X \geqslant q_n\}_{n,m}$$

with $q_m > q_n$. Clearly, m_1 and m_2 agree over the sets $\{q_m > X \geqslant q_n\}$. Since $\mathscr{C}^*$ is a π system, m_1 and m_2 agree over $\sigma(\mathscr{C}^*)$ as well. On the other hand, it is obvious that $\mathscr{C}^*$ is a separating system, provided X is injective. Actually, for any pair ω, $\omega' \in \Omega$, with $\omega \neq \omega'$, there exists some q_n such that $X(\omega) < q_n < X(\omega')$. Therefore, $\omega \in \{q_n > X \geqslant q_s\}$ for some q_s, while $\omega' \notin \{q_n > X \geqslant q_s\}$. If $X(\omega) > X(\omega')$ the argument is analogous. Hence, by Mackey's Theorem [11], $\sigma(\mathscr{C}^*)$ is the Borel σ -algebra Σ. We deduce $m_1 = m_2$ and this completes the proof. $\square$

Theorem 7. By Lemma A.2, the collection of injective functions $BI(\Sigma)$ is dense in $B(\Sigma)$. In view of Theorem 6, the concave and Lipschitz functional $v(\cdot)$ is differentiable on $BI(\Sigma)$. Hence, $BI(\Sigma)$ is a "D -system" and, by Jofre and Thibault [10, Corollary 3.5], the superdifferentials admit a "D-representation". Hence, we have

$$\partial v(X) = \overline{co}^{w^*}\{w^* - \lim Dv(X_n) : ||X_n - X|| \to 0, \quad X_n \in BI(\Sigma)\} \tag{B.2}$$

for all $X \in B(\Sigma)$.

By (6) of Lemma 4, $core(v) = \partial v(0)$ and, for all $\alpha > 0$ and $\beta \in \mathbb{R}$, it holds

$$\partial v(\alpha X + \beta) = \partial v(X)$$

whenever $X \in B(\Sigma)$. Hence, since $\alpha X \xrightarrow{||\cdot||} 0$ as $\alpha \downarrow 0$, by (B.2) we have $\{Dv(X) : X \in BI(\Sigma)\} \subseteq \partial v(0)$, and so $\overline{co}^{w^*}(\{Dv(X) : X \in BI(\Sigma)\}) \subseteq \partial v(0)$. Again by (B.2), also the converse inclusion holds, and we conclude that $core(v) = \overline{co}^{w^*}(\{Dv(X) : X \in BI(\Sigma)\})$. The restriction to $BI(\Sigma)/\sim$ follows from Corollary 5.

The last part of the theorem is a direct consequence of Lemma 4. For, suppose there exists some $X \in BI(\Sigma)$ such that $v(Y + X) = v(Y) + v(X)$. By Lemma 4, $\partial v(Y) \cap \partial v(X) \neq \emptyset$, i.e., $Dv(X) \in \partial v(Y)$. Hence, $v(Y) = \langle Y, Dv(X) \rangle$ and the infimum is thus attained. Conversely, suppose the infimum is a minimum, i.e., $v(Y) = \langle Y, Dv(X) \rangle$ for some $X \in BI(\Sigma)$. This implies $Dv(X) \in \partial v(Y)$ and, in turn, $\partial v(Y) \cap \partial v(X) \neq \emptyset$. By Lemma 4 we conclude that $v(Y + X) = v(Y) + v(X)$. $\square$

Theorem 8. Assume v is differentiable at X. The function $1_{\{X \geqslant t\}}$ is comonotone to X. Hence, $v(X + \varepsilon 1_{\{X \geqslant t\}}) = v(X) + \varepsilon v(X \geqslant t)$ for each $\varepsilon \geqslant 0$. As v is differentiable at X, then $Dv(X)(X \geqslant t) = v(X \geqslant t)$, which proves the first claim.

Let us prove the second statement. Consider the Lipschitz continuous concave functional $v_e(X) : B(\Sigma) \to \mathbb{R}$ given by

$$v_e(X) = \min_{m \in core(v)} \langle X, m \rangle.$$

By the definition of Choquet integral, we know that $\langle m, X \rangle \geqslant v(X)$ for all $m \in core(v)$ and all $X \in B(\Sigma)$. Consequently, $v(X) \leqslant v_e(X)$ on $B(\Sigma)$. On the other hand, by the definition of marginal worth charge, we have $\langle m_X, X \rangle = v(X)$ for all $X \in BI(\Sigma)$. As $m_X \in core(v)$, we have $v(X) = v_e(X)$ on $BI(\Sigma)$. Since $BI(\Sigma)$ is dense in $B(\Sigma)$, it follows $v(X) = v_e(X)$ for all $X \in B(\Sigma)$. Hence, the Choquet functional v is concave and v is supermodular. For instance, if $A, B \in \Sigma$, we have

$$\begin{aligned} v(A \cup B) + v(A \cap B) &= v(1_{A \cup B}) + v(1_{A \cap B}) = v(1_{A \cup B} + 1_{A \cap B}) \\ &= v(1_A + 1_B) \geqslant v(1_A) + v(1_B) = v(A) + v(B). \qquad \square \end{aligned}$$

Proposition 9. We only prove the converse, as the other direction is due to [3]. Since $R(P) = [0,1]^n$, there is a collection $\{E_i\}_{i=1}^n$ such that, for each i, $P_i(E_i) = 1$ and $P_i(E_j) = 0$ for all $i \neq j$. For each i, set $E_i^* = E_i \cap (\bigcup_{j \neq i} E_j)^c$. The collection $\{E_i^*\}_{i=1}^n$ is pairwise disjoint and is such that $P_i(E_i^*) = 1$ and $P_i(E_j^*) = 0$ for all $i \neq j$. Given $y \in R(P)$, since each component y_i belongs to $[0,1]$, by non-atomicity for each i there exists $E_i^y \subseteq E_i^*$ such that $P_i(E_i^y) = y_i$. Set $E^y = \bigcup_{i=1}^n E_i^y$. The sets $\{E_i^y\}_{i=1}^n$ are pairwise disjoint, and so $P_i(E^y) = \sum_{j=1}^n P_i(E_j^y) = y_i$. Hence, $P(E^y) = y$.

Now, let $R(P) \ni x \leqslant y$. Since each x_i belongs to $[0, y_i]$, again by non-atomicity for each i there exists $E_i^x \subseteq E_i^y$ such that $P_i(E_i^x) = x_i$. If we set $E^x = \bigcup_{i=1}^n E_i^x$, by proceeding as before it is easy to see that $P(E^x) = x$. Let $h \geqslant 0$ be such that $y + h \in R(P)$. Again, by proceeding as before, there exists $E^{y+h} \supseteq E^y$ such that $P(E^{y+h}) = y + h$.

Set $A = E^y$ and $B = E^x \cup (E^{y+h} \backslash E^y)$. Then, $A \cup B = E^{y+h}$ and $A \cap B = E^x$. By the supermodularity of $f \circ P$,

$$\begin{aligned} f(y) + f(x+h) &= f(P(E^y)) + f(P(E^x \cup (E^{y+h} \backslash E^y))) \\ &= f(P(A)) + f(P(B)) \leqslant f(P(A \cup B)) + f(P(A \cap B)) \\ &= f(P(E^{y+h})) + f(P(E^x)) = f(y+h) + f(x), \end{aligned}$$

which shows that f is ultramodular. $\square$

Theorem 10. *Case* (i). We invoke Lemma A.3 and the notation adopted in that Lemma. In particular, the arc-length parametrization $\gamma : [0,n] \to C_X$ is an isometry. Since f is Lipschitz, the function $f \circ \gamma$ is Lipschitz over $[0,n]$. Hence, $f \circ \gamma$ is absolutely continuous, and so, γ being differentiable a.e., we have $f(\gamma(t)) = \int_0^t d/du[f(\gamma(u))]\,du$ for all $t \in [0,n]$. If we set $\rho(t) = d/dt[f(\gamma(t))]$, we have $f(x) = \int_0^{|x|_1} \rho(u)\,du$ for all $x \in C_X$. On the other hand, if $\nabla f(x)$ denotes the gradient of f, by the chain rule we have:

$$d/dt[f(\gamma(t))] = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle.$$

Hence, $\rho(t) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle$. Plugging it into (A.4), we get that the derivative is

$$\langle Dv(X), Y \rangle = \int \langle \nabla f(G_X \circ X), \gamma'(|G_X|_1 \circ X) \rangle \, Y \, d\overline{P}. \tag{B.3}$$

Eq. (B.3) holds for all f which are differentiable and ultramodular. Fix now X and set $f_i(x) = x_i$, with $i = 1, \ldots, n$. The corresponding game is $v = f_i \circ P = P_i$. Eq. (B.3) becomes

$$\int Y\,dP_i = \int \gamma_i'(|G_X|_1 \circ X)\,Y\,d\overline{P},$$

where $\gamma_i'(t)$ is the ith component of the vector $\gamma'(t)$. If we set $Y = 1_E$, where E is any element of Σ, we have

$$P_i(E) = \int_E \gamma_i'(|G_X|_1 \circ X)\,d\overline{P}. \tag{B.4}$$

As P_i is absolutely continuous with respect to $\overline{P}$, by (B.4), $\gamma_i'(|G_X| \circ X)$ is the Radon–Nikodym derivative $dP_i/d\overline{P}$. Consequently, getting back to (B.3), we can write

$$\begin{aligned}\langle Dv(X), Y\rangle &= \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\gamma_i'(|G_X|_1 \circ X)\,Y\,d\overline{P} \\ &= \sum\nolimits_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\,\frac{dP_i}{d\overline{P}}\,Y\,d\overline{P} = \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\,Y\,dP_i,\end{aligned}$$

which is the desired result.

Case (ii): Let X be such that the distribution G_X is continuous. In view of the D-representation (B.2) of superdifferentials, we have

$$\partial v(X) = \overline{co}^{w^*}\{w^* - \lim Dv(X_n)\},$$

where X_n are injective and $||X_n - X|| \to 0$. By case (i),

$$\langle Dv(X_n), Y\rangle = \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_{X_n} \circ X_n)\,Y\,dP_i.$$

As G_X is continuous, then $G_{X_n} \to G_X$ uniformly. Consequently, $G_{X_n} \circ X_n \to G_X \circ X$, and so $\partial f/\partial x_i(G_{X_n} \circ X_n) \to \partial f/\partial x_i(G_X \circ X)$. Since $|\partial f/\partial x_i(G_{X_n} \circ X_n)| \leqslant M$, by the Lebesgue dominated convergence theorem, we have

$$\langle Dv(X_n), Y\rangle \to \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\,Y\,dP_i.$$

Therefore, all the weak*-limits are identical and, consequently, $\partial v(X)$ is a singleton. This prove the differentiability of v at X and formula (9) as well. $\square$

Theorem 12. We know that $core(v) \subseteq L^1(\Sigma, \overline{P}) \subseteq ca(\Sigma)$. The weak* topology of $ca(\Sigma)$ agrees over $L^1(\Sigma, \overline{P})$ with the weak topology $\sigma(L^1, L^\infty)$. Eq. (9) of Theorem 10 can be written as

$$\langle Dv(X), Y \rangle = \sum\nolimits_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, Y\, dP_i$$
$$= \int \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_X \circ X) \frac{dP_i}{d\overline{P}} \right) Y\, d\overline{P}.$$

Hence, by Theorem 7, we have

$$core(v) = \overline{co}^{\sigma(L^1, L^\infty)} \left\{ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_X \circ X) \frac{dP_i}{d\overline{P}} : X \in BI(\Sigma) \Big/ \sim \right\}.$$

On the other hand, it is well known that the closed convex hull in the norm and in weak topology coincide, and so the first claim follows. The second formula is just the one of Theorem 7. $\square$

Corollary 13. Let $g \in \overline{co}^s\{\mathscr{D}\}$. By definition, there exists a sequence $\{g_n\}_n \subseteq co(\mathscr{D})$ such that $g_n \to g$ in L^1. This implies that $g_n \to g$ in measure, which in turn implies the existence of a subsequence g_{n_k} such that, $\overline{P} - a.s.$, $\lim_k g_{n_k} = g$. Conversely, let $g = \overline{P} - \lim_{n \to \infty} g_n$ for a sequence $\{g_n\}_n \subseteq co(\mathscr{D})$. This implies that $g_n \to g$ in measure. Moreover, as $\{g_n\}_n \subseteq core(v)$, by the $\sigma(L^1, L^\infty)$-compactness of $core(v)$ the sequence $\{g_n\}_n$ is uniformly integrable (see [5, Corollary IV.8.11]). Hence, by the Vitali Converge Theorem (see [5, Theorem IV.10.9]), we have $g_n \to g$ in L_1. $\square$

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