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Buffon's Needle Problem

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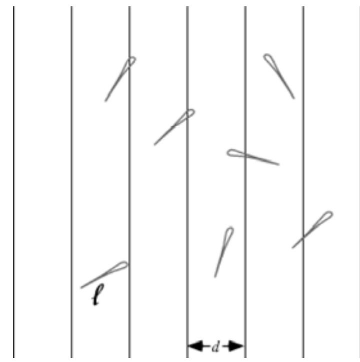


Buffon's needle problem asks to find the probability that a needle of length ℓ will land on a line, given a floor with equally spaced parallel lines a distance d apart. The problem was first posed by the French naturalist Buffon in 1733 (Buffon 1733, pp. 43-45), and reproduced with solution by Buffon in 1777 (Buffon 1777, pp. 100-104).

Define the size parameter x by

$$x \equiv \frac{\ell}{d}. \qquad (1)$$

For a short needle (i.e., one shorter than the distance between two lines, so that $x = \ell/d < 1$), the probability $P(x)$ that the needle falls on a line is

$$\begin{aligned} P(x) &= \int_0^{2\pi} \frac{\ell\,|\cos\theta|}{d}\,\frac{d\theta}{2\pi} && (2)\\ &= \frac{2\ell}{\pi d}\int_0^{\pi/2} \cos\theta\,d\theta && (3)\\ &= \frac{2\ell}{\pi d} && (4)\\ &= \frac{2x}{\pi}. && (5) \end{aligned}$$

For $x = \ell/d = 1$, this therefore becomes

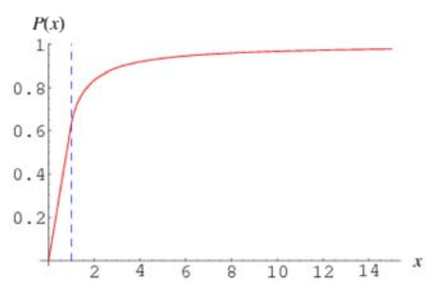
$$P(x=1) = \frac{2}{\pi} = 0.636619\ldots \qquad (6)$$

(OEIS A060294).

For a long needle (i.e., one longer than the distance between two lines so that $x = \ell/d > 1$), the probability that it intersects at least one line is the slightly more complicated expression

$$P(x) = \frac{2}{\pi}\left(x - \sqrt{x^2-1} + \sec^{-1} x\right), \qquad (7)$$

where (Uspensky 1937, pp. 252 and 258; Kunkel).



Writing

$$P(x) = \begin{cases} \frac{2x}{\pi} & \text{for } x \leq 1 \\ \frac{2}{\pi}\left(x - \sqrt{x^2-1} + \sec^{-1} x\right) & \text{for } x > 1 \end{cases} \qquad (8)$$

then gives the plot illustrated above. The above can be derived by noting that

(9)

$$P(x) = \int_0^{\pi/2} \int_{\ell \sin\phi/2} f_s\, f_\phi\, d\, s\, d\, \phi,$$

where

$$f_s = \begin{cases} \frac{2}{d} & \text{for } 0 \le x \le \frac{1}{2} d \\ 0 & \text{for } x > \frac{1}{2} d \end{cases} \quad (10)$$

$$f_\phi = \frac{2}{\pi} \quad (11)$$

are the probability functions for the distance s of the needle's midpoint s from the nearest line and the angle ϕ formed by the needle and the lines, intersection takes place when $0 \le s \le (\ell \sin\phi)/2$, and ϕ can be restricted to $[0, \pi/2]$ by symmetry.

Let N be the number of line crossings by n tosses of a short needle with size parameter x. Then N has a binomial distribution with parameters n and $2x/\pi$. A point estimator for $\theta = 1/\pi$ is given by

$$\hat{\theta} = \frac{N}{2\, x\, n}, \quad (12)$$

which is both a uniformly minimum variance unbiased estimator and a maximum likelihood estimator (Perlman and Wishura 1975) with variance

$$\text{var}(\hat{\theta}) = \frac{\theta}{2\, n} \left(\frac{1}{x} - 2\,\theta\right), \quad (13)$$

which, in the case $x = 1$, gives

$$\text{var}(\hat{\theta}) = \frac{\theta^2 (1 - 2\,\theta)}{2\,\theta\, n}. \quad (14)$$

The estimator $\hat{\pi} = 1/\hat{\theta}$ for π is known as Buffon's estimator and is an asymptotically unbiased estimator given by

$$\hat{\pi} = \frac{2\, x\, n}{N}, \quad (15)$$

where $x = \ell/d$, n is the number of throws, and N is the number of line crossings. It has asymptotic variance

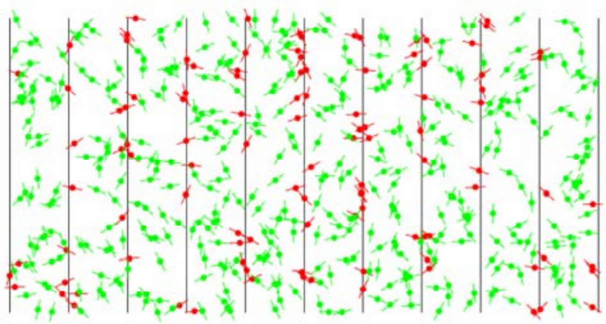
$$\text{avar}(\hat{\pi}) = \frac{\pi^2}{2\, n} \left(\frac{\pi}{x} - 2\right), \quad (16)$$

which, for the case $x = 1$, becomes

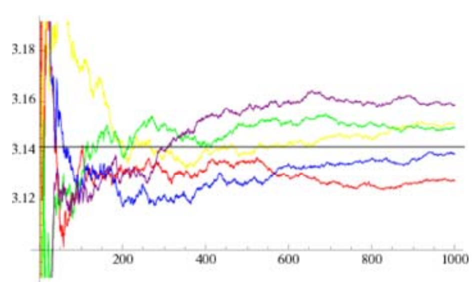
$$\text{avar}(\hat{\pi}) = \frac{\pi^2 \left(\frac{1}{2}\pi - 1\right)}{n} \quad (17)$$

$$\approx \frac{5.6335339}{n} \quad (18)$$

(OEIS A114598; Mantel 1953; Solomon 1978, p. 7).



The above figure shows the result of 500 tosses of a needle of length parameter $x = 1/3$, where needles crossing a line are shown in red and those missing are shown in green. 107 of the tosses cross a line, giving $\hat{\pi} = 3.116 \pm 0.073$.



Several attempts have been made to experimentally determine π by needle-tossing. π calculated from five independent series of tosses of a (short) needle are illustrated above for one million tosses in each trial $x = 1/3$. For a discussion of the relevant statistics and a critical analysis of one of the more accurate (and least believable) needle-tossings, see Badger (1994). Uspensky (1937, pp. 112-113) discusses experiments conducted with 2520, 3204, and 5000 trials.

The problem can be extended to a "needle" in the shape of a convex polygon with generalized diameter less than d. The probability that the boundary of the polygon will intersect one of the lines is given by

$$P = \frac{p}{\pi\, d}, \quad (19)$$

where p is the perimeter of the polygon (Uspensky 1937, p. 253; Solomon 1978, p. 18).

A further generalization obtained by throwing a needle on a board ruled with two sets of perpendicular lines is called the Buffon-Laplace needle problem.

SEE ALSO:
Buffon-Laplace Needle Problem, Clean Tile Problem

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