On coloring the rational quantum sphere

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Abstract

We discuss types of colorings of the rational quantum sphere similar to the one suggested recently by Meyer [1], in particular the consequences for the Kochen-Specker theorem and for the correlation functions of entangled subsystems.

1 Introduction

Recently, Godsil and Zaks [2] published a constructive coloring of the rational unit sphere with the property that for any orthogonal tripod formed by rays extending from the origin of the points of the sphere, exactly one ray is red, white and black. They also showed that any consistent coloring of the real sphere requires an additional color.

Based on this very interesting result, Meyer [1] suggested that the physical impact of the Kochen-Specker theorem [3] is "nullified," since for all practical purposes it is impossible to operationalize the difference between any dense set of rays and the continuum of Hilbert space rays.

We shall argue here that Meyer's result is itself "nullified" for a formal and for a physical reason: (i) the non-closedness of the resulting set of propositions under quantum logical operations; in particular under the `nor`-operation; (ii) the continuity of Bell-type two-particle correlation function rules out the possibility that "unperformed experiments have results" [4].

2 Mathematical aspects

2.1 Colorings

In what follows we shall consider "rational rays." A "rational ray" is the linear span of a non-zero vector of $\mathbb{Q}^n \subset \mathbb{R}^n$.

Let p be a prime number. A coloring of the rational rays of $\mathbb{R}^n$, $n \geq 1$, using $p^{n-1} + p^{n-2} + \ldots + 1$ colors can be constructed in a straightforward manner. We refer to [5, 6, 7] for the theoretical background of the following construction.

Each rational ray is the linear span of a vector $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$, where $x_1, x_2, \ldots, x_n$ are coprime. Such a vector is unique up to a factor ± 1.

Next, let $\mathbb{Z}_p$ be the field of residue classes modulo p. The vector space $\mathbb{Z}_p^n$ has $p^n - 1$ non-zero vectors; each ray through the origin of $\mathbb{Z}_p^n$ has $p-1$ non-zero vectors. So there are exactly $(p^n - 1)/(p-1) = p^{n-1} + p^{n-2} + \ldots + 1$ distinct rays through the origin which can be colored with $p^{n-1} + p^{n-2} + \ldots + 1$ distinct colors.

Finally, assign to the ray $Sp(x_1, x_2, \ldots, x_n)$ ("Sp" denotes linear span) the color of the ray of $\mathbb{Z}_p^n$ which is obtained by taking the modulus of the coprime integers $x_1, x_2, \ldots, x_n$ modulo p. Observe that $x_1, x_2, \ldots, x_n$ cannot vanish simultaneously modulo p and that $\pm(x_1, x_2, \ldots, x_n)$ yield the same color. Obviously, all $p^{n-1} + p^{n-2} + \ldots + 1$ colors are actually used.

In what follows, we consider the case $p = 2$, $n = 3$. Here all rational rays $Sp(x, y, z)$ (with $x, y, z \in \mathbb{Z}$ coprime) are colored according to the property which ones of the components x, y, z are even (E) and odd (O). There are exactly 7 of such triples OEE, EOE, EEO, OOE, EOO, OEO, OOO which are associated with one of seven different colors #1, #2, #3, #4, #5, #6, #7. Only the EEE triple is excluded. Those seven colors can be identified with the seven points of the projective plane over $\mathbb{Z}_2$; cf. Fig. 1.

Next, we restrict our attention to those rays which meet the rational unit sphere $S^2 \cap \mathbb{Q}^3$. The following statements on a triple $(x, y, z) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$ (not necessarily coprime) are equivalent:

(i) The ray $Sp(x, y, z)$ intersects the unit sphere at two rational points; i.e., it contains the rational points $\pm\,(x, y, z)\,/\sqrt{x^2 + y^2 + z^2} \in S^2 \cap \mathbb{Q}^3$.

(ii) The Pythagorean property holds, i.e., $x^2 + y^2 + z^2 = n^2,\ n \in \mathbb{N}$.

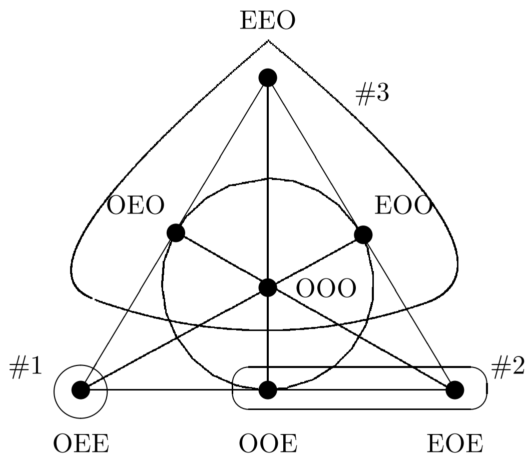

Figure 1: The projective plane over $\mathbb{Z}_2$. and the reduced coloring scheme discussed.

This equivalence can be demonstrated as follows. All points on the rational unit sphere can be written as $\mathbf{r} = \left(\frac{a}{a'}, \frac{b}{b'}, \frac{c}{c'}\right)$ with $a, b, c \in \mathbb{Z}$, $a', b', c' \in \mathbb{Z} \setminus \{0\}$, and $\left(\frac{a}{a'}\right)^2 + \left(\frac{b}{b'}\right)^2 + \left(\frac{c}{c'}\right)^2 = 1$. Multiplication of $\mathbf{r}$ with ${a'}^2{b'}^2{c'}^2$ results in a vector of $\mathbb{Z}^3$ satisfying (ii). Conversely, from $x^2 + y^2 + z^2 = n^2$, $n \in \mathbb{N}$, we obtain the rational unit vector $\left(\frac{x}{n}, \frac{y}{n}, \frac{z}{n}\right) \in S^2 \cap \mathbb{Q}^3$.

Notice that this Pythagorean property is rather restrictive. Not all rational rays intersect the rational unit sphere. For a proof, consider $Sp(1,1,0)$ which intersects the unit sphere at $\pm(1/\sqrt{2})(1,1,0) \notin S^2 \cap \mathbb{Q}^3$. Although both the set of rational rays as well as $S^2 \cap \mathbb{Q}^3$ are dense, there are "many" rational rays which do not have the Pythagorean property.

If x, y, z are chosen coprime then a necessary condition for $x^2 + y^2 + z^2$ being a non-zero square is that precisely one of x, y, and z is odd. This is a direct consequence of the observation that any square is congruent to 0 or 1, modulo 4, and from the fact that at least one of x, y, and z is odd. Hence our coloring of the rational rays induces the following coloring of the rational unit sphere with those three colors that are represented by the standard basis of $\mathbb{Z}_2^3$:

color #1 if x is odd, y and z are even,

color #2 if y is odd, z and x are even,

color #3 if z is odd, x and y are even.

All three colors occur, since the vectors of the standard basis of $\mathbb{R}^3$ are colored differently.

Suppose that two points of $S^2 \cap \mathbb{Q}^3$ are on rays $Sp(x, y, z)$ and $Sp(x', y', z')$, each with coprime entries. The inner product $xx' + yy' + zz'$ is even if and only if the inner product of the corresponding basis vectors of $\mathbb{Z}_2^3$ is zero or, in other words, the points are colored differently. In particular, three points of $S^2 \cap \mathbb{Q}^3$ with mutually orthogonal position vectors are colored differently.

From our considerations above, three colors are sufficient to obtain a coloring of the rational unit sphere $S^2 \cap \mathbb{Q}^3$ such that points with orthogonal position vectors are colored differently, but clearly this cannot be accomplished with two colors. So the "chromatic number" for the rational unit sphere is three. This result is due to Godsil and Zaks [2]; they also showed that the chromatic number of the real unit sphere is four. However, they obtained their result in a slightly different way. Following [6] all rational rays are associated with three colors by making the following identification:

$$\begin{aligned} &\#1, \\ \#2 = &\#4, \\ \#3 = \#5 = \#6 = &\#7. \end{aligned}$$

This 3-coloring has the property that coplanar rays are always colored by using only two colors; cf. Fig. 1. According to our approach this intermediate 3-coloring is not necessary, since rays in colors #4, #5, #6, #7 do not meet the rational unit sphere.

2.2 Reduced two-coloring

As a corollary, the rational unit sphere can be colored by two colors such that, for any arbitrary orthogonal tripod spanned by rays through its origin, one vector is colored by color #1 and the other two rays are colored by color #2. This can be easily verified by identifying colors #2 & #3 from the above scheme. (Two equivalent two-coloring schemes result from a reduced chromatic three-coloring scheme by requiring that color #1 is associated with x or y being odd, respectively.)

2.3 Denseness of single colors

It can also be shown that each color class in the above coloring schemes is dense in the sphere. To prove this, Godsil and Zaks consider α such that $\sin\alpha = \frac{3}{5}$ and thus $\cos\alpha = \frac{4}{5}$. α is not a rational multiple of π; hence $\sin(n\alpha)$ and $\cos(n\alpha)$ are non-zero for all integers n. Let F be the rotation matrix about the z-axis through an angle α; i.e.,

$$F = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the image I, under the powers of F, of the point $(1,0,0)$ is a dense subset of the equator.

Now suppose that the point $u = \left(\frac{a}{c}, \frac{b}{c},\right)$ is on the rational unit sphere and that a, c are odd and thus b is even. In the coloring scheme introduced above, u has the same color as $(1,0,0)$ (identify $a = c = 1$ and $b = 0$); and so does Fu. This proves that I (the image of all powers of F of the points u) is dense. We shall come back to the physical consequences of this property later.

In the reduced two-color setting, if the two "poles" $\pm(0,0,1)$ acquire color #1, then the entire equator acquires color #2. Thus, for example, for the two tripods spanned by $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(3,4,0),(-4,3,0),(0,0,1)\}$, the first two legs have color #2, while $(0,0,1)$ has color #1.

2.4 Non-closedness

The above coloring scheme of the rays through the origin meeting the rational sphere is not closed under certain geometrical operations such as taking two orthogonal ray of the subspace spanned by two non-collinear rays (the cross product of the associated vectors). This can be easily demonstrated by considering the two vectors

$$\left(\frac{3}{5},\frac{4}{5},0\right),\ \left(0,\frac{4}{5},\frac{3}{5}\right) \in S^2 \cap \mathbb{Q}^3.$$

The cross product thereof is

$$\left(\frac{12}{25},\frac{-9}{25},\frac{12}{25}\right) \notin S^2 \cap \mathbb{Q}^3.$$

Indeed, if instead of $S^2 \cap \mathbb{Q}^3$ we would start with three non-orthogonal, non-collinear rational rays and generate new ones by the cross product, we would end up with *all* rational rays [8].

From a quantum logic point of view, this non-closedness under elementary operations such as the `nor`-operation might be considered a serious deficiency which rules out the above model as an alternative candidate for Hilbert space quantum mechanics. Indeed, it is just the relative (with respect to other sets such as the rational rays) "thinness" which guarantees colorability, but which on the other hand does not allow closedness under the quantum logical operations.

It may, however, be argued that the non-closedness is among counterfactual, non-commuting properties which have no direct operational meaning. But then it would be entirely senseless to consider any but six points of the rational sphere corresponding to intersections with a single tripod (that one being measured), which would make any coloring trivial.

Of course, this leaves open the question as to whether or not there exist dense sets of chromatic number three which are closed under quantum logical operations, in particular under the `nor`-operation.

3 Physical aspects

3.1 Elements of physical reality

The coloring schemes discussed above have a physical interpretation as follows. Any linear subspace $Sp\,\mathbf{r}$ of a vector $\mathbf{r}$ can be identified with the associated projection operator $E_{\mathbf{r}}$ and with the quantum mechanical proposition "the physical system is in a pure state $E_{\mathbf{r}}$" [9]. The coloring of the associated point on the unit sphere (if it exists) is equivalent with a valuation or two-valued probability measure

$$Pr : E_{\mathbf{r}} \mapsto \{0, 1\}$$

where $0 \sim \#2$ and $1 \sim \#1$. That is, the two colors #1, #2 can be identified with the classical truth values and with the EPR- type "elements of physical reality" [10]:

"It is true that the physical system is in a pure state $E_{\mathbf{r}}$."

"It is false that the physical system is in a pure state $E_{\mathbf{r}}$."

Since, as has been argued before, the rational unit sphere has chromatic number three, two colors suffice for a reduced coloring generated under the assumption that the colors of two rays in any orthogonal tripod are identical. This effectively generates consistent valuations or "elements of physical reality" associated with the dense subset of physical properties corresponding to the rational unit sphere [1]. For an extension of these arguments, see Kent [11] and Clifton and Kent [12].

3.2 Continuity of physical quantities

We may indeed take for granted that for all practical (operational) physical purposes, a dense subset cannot be distinguished from the usual Hilbert space based upon real or complex number fields. This is, at least to our knowledge, one of the very rare instances where it *does* make a difference whether we consider a real rational quantity or merely its rational double. The two cases lie on different sides of a demarcation line of classicality and (maybe) an "understanding" of quantum mechanics.

To put it differently: given any nonzero measurement uncertainty ε and any non-colorable Kochen-Specker graph $\Gamma(0)$ [3, 13], there exists another Graph (in fact, a denumerable infinity thereof) $\Gamma(\delta)$ which lies inside the range of measurement uncertainty $\delta \leq \varepsilon$ [and thus cannot be discriminated from the non-colorable $\Gamma(0)$] which *can* be colored. Such a graph, however, might not be connected in the sense that the associated subspaces can be cyclically rotated into itself by local transformation along single axes. The set $\Gamma(\delta)$ might thus correspond to a collection of tripods such that none of the axes coincides with any other axis and some of those non-identical single axes are located within δ apart from each other.

If this is indeed the case, then the reason why a Kochen-Specker type contradiction does not occur is the impossibility to "close" the argument; to complete the cycle: the necessary elements of physical reality are simply not available in the rational sphere model. (For the same reason, a triangle of equal length does not exist in $\mathbb{Q}^2$ [14].)

3.3 Non-local correlations

However, in addition to the non-closedness under elementary quantum logical operations we would like to point out another, rather serious problem. Due to the density of points colored with any single color, any such coloring is non-continuous in the sense that between any two points of equal color there is a point (indeed, an infinity thereof) of different color.

Let us then assume that any such value-definiteness might apply also to non-local setups and consider Bell-type correlation functions for spin-$\frac{1}{2}$ state measurements. For singlet states along two directions which are an angle θ apart, the quantum probabilities to find identical two particle states $++$ or $--$ is $P^{=} = P^{++} + P^{--} = \sin^2(\theta/2)$, whereas for the non-identical states $+-$ and $-+$ it is $P^{\neq} = P^{+-} + P^{-+} = \cos^2(\theta/2)$. The corresponding classical quantities are $P^{=} = \theta/\pi$ and $P^{\neq} = 1 - \theta/\pi$ [15, 16, 17]. In the quantum case, a statistical argument [4, 16, 17] demonstrates that "elements of physical reality" do not exist, whereas in the classical case they can be defined.

Thus if there exists any coloring of the associated fourdimensional model, any such coloring should be in accordance with the physical findings which support quantum mechanics. In particular, any probability theory derived from the valuations of dense subsets of Hilbert spaces should be able to reproduce the well-known quantum correlations which violate value-definiteness as well as locality. This, however, is neither the case for the usual classical value-definiteness, nor with the discontinuity originating in the denseness of any single color.

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