

COLOURING THE SPHERE

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ABSTRACT. Let G be the graph with the points of the unit sphere in \mathbb{R}^3 as its vertices, by defining two unit vectors to be adjacent if they are orthogonal as vectors. We show that the chromatic number of this graph is four, thereby answering a question of Peter Frankl. We also prove that the subgraph of G induced by the unit vectors with rational coordinates is 3-colourable.

1. INTRODUCTION

Let G be the graph with the points of the unit sphere in three dimensions as its vertices, and two vertices are adjacent if and only if they are orthogonal as unit vectors.

1.1. Lemma. *The chromatic number of G is four.*

Proof. It is clear that G has chromatic number at least three. Suppose that we have a 3-colouring of G using the colors red, white and black. If x is a vertex in G we may view it as the north pole. If x is coloured red then the points on the equator with respect to x must be coloured black and white and, furthermore, points of both colours must occur (since any great circle on the sphere contains orthogonal pairs of vectors). It follows in turn that the South pole with respect to x must also be coloured red. Thus we have shown that in a 3-colouring of G , any antipodal pair of points must have the same colour. This means that any 3-colouring of G determines a 3-colouring of the real projective plane, with the property that the points on any line are coloured with at most two colours.

But it has been shown [2] that the 3-colourings of the projective plane over a field K are in one-to-one correspondence with the non-Archimedean valuations of K . A non-Archimedean valuation on a field K is a real-valued function ν on K such that:

- (a) $\nu(x) \geq 0$ for all x in K , and $\nu(x) = 0$ if and only if $x = 0$.
- (b) $\nu(xy) = \nu(x) + \nu(y)$ for all x and y in K .
- (c) $\nu(x + y) \leq \max\{\nu(x), \nu(y)\}$.

The colouring corresponding to a non-Archimedean valuation ν is described as follows. A point with homogeneous coordinates (x, y, z) in the projective plane

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over K is coloured

$$\begin{aligned} \text{red if: } & \nu(x) > \nu(y), \quad \nu(x) > \nu(z), \\ \text{white if: } & \nu(x) \leq \nu(y), \quad \nu(y) > \nu(z), \\ \text{red if: } & \nu(x) \leq \nu(z), \quad \nu(y) \leq \nu(z). \end{aligned}$$

If L is a subfield of K then ν is also a valuation on L . Conversely, if a valuation on L is given then it can be extended to a valuation on K . By way of (a pertinent) example, we describe a non-Archimedean valuation on the rationals \mathbb{Q} . Choose a prime p (any prime). Every rational number b can now be written in the form ap^n , where the numerator and denominator of a are not divisible by p . Define $\nu(b)$ to be 2^{-n} ; this is the required valuation. This is known as the p -adic valuation on \mathbb{Q} . Every non-Archimedean valuation on \mathbb{Q} is p -adic for some prime p . (For more information on valuations see, e.g., [1, 3].)

Since the real projective plane can be 3-coloured, it follows that the sphere in \mathbb{R}^3 can be 3-coloured in such a way that antipodal pairs of points have the same colour, and all great circles are 2-coloured. But we need a colouring with the property that orthogonal vectors have different colours.

We will call a vector x in the unit sphere a \mathbb{Q} -point if the line it spans contains a point with coordinates in \mathbb{Q} . Any 3-colouring of the real projective plane determines a 3-colouring of the rational projective plane, and hence a 3-colouring of the \mathbb{Q} -points on the unit sphere. As noted above, such a colouring comes from one of the p -adic valuations, and the question now is whether any of these give rise to a 3-colouring of the \mathbb{Q} -points on the sphere satisfying the orthogonality condition. But if p is odd then the points $(1, 1, 0)$ and $(-1, 1, 0)$ in the projective plane will be coloured white. These two points determine orthogonal points on the sphere. If p is equal to 2 then we see that the points $(-7, 0, 1)$ and $(1, 0, 7)$ are both coloured black, and again determine orthogonal points on the sphere. Hence no 3-colouring of the sphere works. (We could also have observed that the points $(-1, 3, 1)$ and $(-5, 2, 1)$ will both be coloured black, for any prime p , but the argument used actually shows that the subgraph of G induced by the points in $\mathbb{Q}(\sqrt{2})$ is not 3-colourable.)

To complete the proof, we give a four colouring of G . Begin by colouring the two points on the x -axis red, the points on the y -axis white and those on the z -axis black. There are exactly three great circles which pass through two of these three pairs of points. We colour these using the two colours used on the four points that they pass through. These circles divide the sphere into eight open octants. Colour the four octants in the half space $z > 0$ red, white, black and blue in some order. Give the remaining four octants the colour of their antipodal octant. The reader is invited to check that this works. \square

In contrast to the above result, we have the following:

1.2. Lemma. *The chromatic number of the subgraph of G induced by the unit vectors with all coordinates rational is three.*

Proof. To show this, we use the 3-colouring of the rational projective plane corresponding to the 2-adic valuation on \mathbb{Q} . This plane can be coordinatized by the

integers. If we do this then (x, y, z) is coloured:

red if: x is odd, y and z are even,
white if: y is odd and z is even,
black if: z is odd.

Note that we assume that x , y and z have no common factor, and so one of these three integers must be odd.

The point (x, y, z) in the rational projective plane determines a rational point on the sphere if and only if $x^2 + y^2 + z^2$ is a square. A necessary condition for this to occur is that precisely one of x , y and z be odd. (This follows from the observation that any square is congruent to 0 or 1, modulo 4, and from the fact that at least one of x , y and z is odd.) It follows that if (x, y, z) is coloured white then x is also even, and if it is coloured black then x and y must be even. Thus, if (x, y, z) and (x', y', z') are two points in the rational projective plane with the same colour and if these points correspond to rational points on the sphere, the inner product $xx' + yy' + zz'$ must be odd. Hence it is not zero, and so the subgraph of G induced by the rational points of the sphere is 3-chromatic. \square

Each colour class in the above 3-colouring is dense in the sphere. To prove this, let α be chosen so that $\sin \alpha = \frac{3}{5}$ and $\cos \alpha = \frac{4}{5}$. Then α is not a rational multiple of π , and therefore $\sin n\alpha$ and $\cos n\alpha$ are non-zero for all integers n . Let F be the matrix which represents rotation about the z -axis through an angle α . It follows that the image \mathcal{I} , under the powers of F , of the point $(1, 0, 0)$ is a dense subset of the equator. Suppose the point

$$u := \left(\frac{a}{d}, \frac{b}{d}, 0 \right)$$

is on the unit sphere and that a and d are odd and b is even. Then u has the same colour as $(1, 0, 0)$, and so does Fu . This proves that \mathcal{I} is monochromatic. Now let G represent rotation through an angle of α about the y -axis. Then the same arguments show that the image of \mathcal{I} under the powers of G is a dense monochromatic set of the rational points on the unit sphere. Since the map which sends (x, y, z) to (y, z, x) permutes the three colour classes cyclicly, it follows that all three colour classes are dense. (This also shows that the rational points on the unit sphere in \mathbb{R}^3 are dense. The easiest way to verify this is to note that, under stereographic projection, the rational points on the sphere, excepting the North pole, are in one-to-one correspondence with the rational points in the affine plane. The key to this is that stereographic projection is a birational mapping. See e.g., Chapter XI, in particular Exercise 23, of [4].)

Despite the fact that the colour classes of this colouring are dense, there are many monochromatic circles. To see this, choose two unit vectors u and v with the same colour. Then, if $u \neq -v$, the unit vectors x such that $x \cdot u = v \cdot u$ form a circle. It is easy to verify that if

$$u = \frac{1}{d}(a, b, c), \quad v = \frac{1}{d'}(a', b', c')$$

then $(aa' + bb' + cc')$ is odd, and hence that the given circle is monochromatic. (If v has a different colour from u then this circle will contain only rational points with a different colour from u , since in this case $(aa' + bb' + cc')$ is even.)

Since $\chi(G) = 4$ it follows from a well known result of Erdős and Szekeres that G contains finite 4-chromatic subgraphs, but we have not yet found one. Note that if we have a set of n points in the sphere inducing a finite 4-chromatic subgraph H of G , the points in this set cannot all have rational coordinates. (Because the rational points can be 3-coloured and H cannot be.)

NOTE

This article first appeared as research report CORR 88-12 from the Department of Combinatorics and Optimization at the University of Waterloo (dated March 1988). David Roberson transcribed this into latex. Apart from a few small corrections (and this note), this version is as close to the original as a reasonable amount of effort could make it.

The report was never published because the main result (Lemma 1.1) was already known. However the proof of Lemma 1.1 is new and Lemma 1.2 is also new. Since this report was widely cited but not widely available, David felt that it would be worth posting a version on the arxiv. The authors thank him for his effort.

REFERENCES

- [1] J. W. S. Cassels. *Local fields*, volume 3 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1986.
- [2] A. W. Hales and E. G. Straus. Projective colorings. *Pacific J. Math.*, 99(1):31–43, 1982.
- [3] Serge Lang. *Algebra*. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.
- [4] J. G. Semple and G. T. Kneebone. *Algebraic Projective Geometry*. Oxford, at the Clarendon Press, 1952.