Probability

April 18, 2016

1 Probability Spaces

1.1 Classical Probability Spaces

Textbook probability theory [1, 2, 4] is defined using the notions of a *sample space* Ω, a space of *events* $\mathcal{A}$, and a *probability measure* μ. In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events $\mathcal{A}$ as, 2^{Ω}, the powerset of Ω. A *probability measure* is a function $\mu : \mathcal{A} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i, we have $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two coin experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rclrcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

1.2 Quantum Probability Spaces

The mathematical framework above assumes that one has complete knowledge of the events and their relationships. But even in many classical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot be determined with certainty. In the quantum case, this partial knowledge is compounded by the fact that not all quantum events can be observed simultaneously. Indeed, in the quantum world, there are non-commuting events which cannot even happen simultaneously. To accommodate these more complex situations, we completely abandon the sample space Ω and define and reason directly about events. Thus a quantum probability space will consist of just two components: a set of events $\mathcal{A}$ and a probability measure $\mu : \mathcal{A} \to [0, 1]$. We give an example before giving the formal definition.

Consider the two-qubit Hilbert space with computational basis $|0\rangle$ and $|1\rangle$ and states:

$$|{+}\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |{-}\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} .$$

A possible space of events is the set $\{\mathbb{0}, |0\rangle\langle 0|, |1\rangle\langle 1|, \mathbb{1}\}$ consisting of the projection operators $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ as well as the empty projection $\mathbb{0}$ and the unit projection $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$. Each event is interpreted as a possible post-measurement state of a quantum system as follows: given some arbitrary current quantum $|\psi\rangle$ to be measured, the event $|0\rangle\langle 0|$ states that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ states that the post-measurement state will be $|1\rangle$; the event $\mathbb{1}$ states that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the event $\mathbb{0}$ states that the post-measurement state will be the empty state. Clearly irrespective of the current state $|\psi\rangle$ and irrespective of the particular experiment, the probability of event $\mathbb{0}$ will always be 0 (it is an impossible event) and the probability of event $\mathbb{1}$ will always be 1 (it is a certain event). The probabilities attached to other events will depend on the particular state in question. If the state is $|0\rangle$, the probability of event $|0\rangle\langle 0|$ is 1 and the probability of event $|1\rangle\langle 1|$ is 0. If the state is $|+\rangle$, the probability of both events $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ will be $\frac{1}{2}$. The same Hilbert space setup admits other possible spaces of events. For example, another space of events could be $\{\mathbb{0}, |+\rangle\langle +|, |-\rangle\langle -|, \mathbb{1}\}$. Using that space of events, the state $|0\rangle$ maps both events $|+\rangle\langle +|$ and $|-\rangle\langle -|$ to the probability $\frac{1}{2}$.

We now formalize a *quantum probability space* as follows [3, 5]. We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{A}$ as all linear combinations of *projections* on $\mathcal{H}$. Specifically, let $P_1, P_2, \ldots, P_k$ be mutually orthogonal projections on $\mathcal{H}$ with sum $\mathbb{1}$ and define the event space $\mathcal{A}$ to be the linear span of these operators:

$$\mathcal{A} = \left\{ \sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{C} \right\}.$$

Each quantum state $|\psi\rangle$ induces a probability measure $\mu_\psi : \mathcal{A} \to [0, 1]$ on the space of events defined as follows:

$$\mu_\psi(A) = \langle \psi | A\psi \rangle$$

Similarlly to the classical case, this probability measure satisfies:

- $\mu(\mathbb{1}) = 1$, and
- for all $A \in \mathcal{A}$, we have $\mu(A^* A) \geq 0$.

1.3 Plan

Several assumptions are woven in the definition of a quantum probability space:

- the Hilbert space $\mathcal{H}$;
- the real interval $[0, 1]$;
- the fact that each state induces a probability measure, i.e., the Born rule;
- the fact that every probability measure is induced by a state, i.e., Gleason's theorem

In the remainder of the paper, we examine each of these assumptions and consider variations motivated by computation in a world with limited resources. In particular, we will consider a variant of the Hilbert space over finite fields $\mathbb{F}^d_{p^2}$. Instead of $[0, 1]$, we will consider set-valued probability measures, in particular $\{0\}$, impossible, and $(0, \infty)$, possible. Surprisingly, some combinations of space and probability will result in no probability measure or an unique probability measure. In these cases, there is no need to discuss whether there is a Born rule, because we do not have enough probability to correspond to every state.

If there may be more than one probability measure, we will discuss whether there is a Born rule to generate a probability measure from a state. When the space is $\mathbb{C}^d$, we will try to induced a Born rule from the conventional Born rule; when the space is $\mathbb{F}^d_{p^2}$, there is no natural way to induce a probability measure from a state, so we will set some conditions a Born $\tilde{\pi}$ should have:

- Given a pure state $|\Psi\rangle \in \mathbb{F}^{d\,*}_{p^2}$, a Born-rule $\tilde{\pi}$ should give a probability $\tilde{\pi}_\Psi$;

- $\langle\Psi|\Phi\rangle = 0 \Leftrightarrow \tilde{\pi}_\Psi(|\Phi\rangle) = \tilde{0}$, where $\tilde{0}$ is 0 while considering $[0,1]$ and $\tilde{0}$ is impossible while considering $\{\text{impossible}, \text{possible}\}$.
- $\tilde{\pi}_\Psi(|\Phi\rangle) = \tilde{\pi}_{\mathbf{U}|\Psi\rangle}(\mathbf{U}|\Phi\rangle)$,where $|\Psi\rangle, |\Phi\rangle \in \mathbb{F}_{p^2}^{d\,*}$ and $\mathbf{U}$ is any unitary map, i.e., $\mathbf{U}^\dagger\mathbf{U} = \mathbb{1}$.

Notice that when the space is $\mathbb{C}^d$, every Born rule we consider will satisfy these three conditions.

Finally, if there is a Born rule, we will see whether every probability measure is induced by a state, and establish Gleason's theorem.

References

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