Rational trigonometry

Rational trigonometry is a proposed reformulation of metrical planar and solid geometries (which includes trigonometry) by Canadian mathematician Norman J. Wildberger, currently an associate professor of mathematics at the University of New South Wales. His ideas are set out in his 2005 book *Divine Proportions: Rational Trigonometry to Universal Geometry*.[1] According to New Scientist, part of his motivation for an alternative to traditional trigonometry was to avoid some problems that occur when infinite series are used in mathematics. Rational trigonometry avoids direct use of transcendental functions like sine and cosine by substituting their squared equivalents.[2] Wildberger draws inspiration from mathematicians predating Georg Cantor's infinite set-theory, like Gauss and Euclid, who he claims were far more wary of using infinite sets than modern mathematicians.[2][nb 1] To date, rational trigonometry is largely unmentioned in mainstream mathematical literature.

1 Approach

Rational trigonometry follows an approach built on the methods of linear algebra to the topics of elementary (high school level) geometry. Distance is replaced with its squared value (**quadrance**) and 'angle' is replaced with the squared value of the usual sine ratio (**spread**) associated to either angle between two lines. (Spread also corresponds to a scaled form of the inner product between the lines taken as vectors). The three main laws in trigonometry: Pythagoras's theorem, the sine law and the cosine law, given in rational (squared) form, are augmented by two further laws: the triple quad formula (relating the quadrances of three collinear points) and the triple spread formula (relating the spreads of three concurrent lines), giving the five main laws of the subject.

Rational trigonometry is otherwise broadly based on Cartesian analytic geometry, with *a point* defined as an ordered pair of rational numbers

$$(x, y)$$

and *a line*

$$ax + by + c = 0,$$

as a general linear equation with rational coefficients a, b and c .

By avoiding calculations that rely on square root operations giving only *approximate* distances between points, or standard trigonometric functions (and their inverses), giving only truncated polynomial *approximations* of angles (or their projections) geometry becomes entirely algebraic. There is no assumption, in other words, of the existence of real number solutions to problems, with results instead given over the field of rational numbers, their algebraic field extensions, or finite fields. Following this, it is claimed, makes many classical results of Euclidean geometry applicable in *rational* form (as quadratic analogs) over any field not of characteristic two.

The book *Divine Proportions* shows the application of calculus using rational trigonometric functions, including three-dimensional volume calculations. It also deals with rational trigonometry's application to situations involving irrationals, such as the proof that Platonic Solids all have rational 'spreads' between their faces.[nb 2]

2 Quadrance

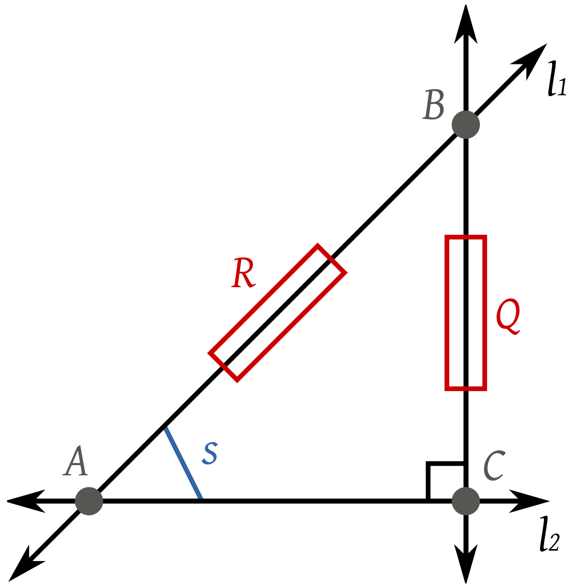
Quadrance and distance (as its square root) both measure separation of points in Euclidean space.[3] Following Pythagoras's theorem, the quadrance of two points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in a plane is therefore defined as the sum of squares of differences in the x and y coordinates:

$$Q(A_1, A_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Unlike vector addition of distances with line segments, adding the quadrances of two vectors to obtain their combined magnitude *always* entails finding a third leg of the associated triangle they form, including collinear segments as a special case of degenerate triangle where the same calculation made with distance-vectors simplifies to addition. In effect, the triangle inequality: $d_1 + d_2 - d_3 \geq 0$ is modified under rational trigonometry to a symmetric form: $(Q_1 + Q_2 - Q_3)^2 \geq 0$ equivalent to Pythagoras's theorem.

2.1 Quadrea

3 Spread



Suppose ℓ1 and ℓ2 intersect at the point A. Let C be the foot of the perpendicular from B to ℓ2. Then the spread is s = Q/R.

Spread gives one measure to the separation of two lines as a single dimensionless number in the range $[0, 1]$ (from *parallel* to *perpendicular*) for Euclidean geometry. It replaces the concept of angle but has several differences from angle, discussed in the section below. Spread can have several interpretations.

- *Trigonometric* (most elementary): it is the sine-ratio for the quadrances in a right triangle and therefore equivalent to the square of the sine of the angle.[3]
- *Vector:* as a rational function of the directions (practically, the slopes) of a pair of lines where they meet.
- *Cartesian:* as a rational function of the three coordinates used to ascribe the two vectors.
- *Linear algebra* (from the dot product) a normalized rational function: the square of the determinant of two vectors (or pair of intersecting lines) divided by the product of their quadrances.

3.1 Calculating spread

- **Trigonometric**

Suppose two lines, ℓ_1 and ℓ_2, intersect at the point A as shown at right. Choose a point $B \neq A$ on ℓ_1 and let C be the foot of the perpendicular from B to ℓ_2. Then the spread s is

$$s(\ell_1, \ell_2) = \frac{Q(B,C)}{Q(A,B)} = \frac{Q}{R}.$$ [3]

- **Vector/slope** (two-variable)

Like angle, spread depends only on the relative slopes of two lines (constant terms being eliminated) and is invariant under translation (i.e. it is preserved when lines are moved keeping parallel with themselves). So given two lines whose equations are

$$a_1x + b_1y = \text{constant and } a_2x + b_2y = \text{constant}$$

we may rewrite them as two lines which meet at the origin $(0, 0)$ with equations

$$a_1x + b_1y = 0 \text{ and } a_2x + b_2y = 0$$

In this position the point $(-b_1, a_1)$ satisfies the first equation and $(-b_2, a_2)$ satisfies the second and the three points $(0, 0)$, $(-b_1, a_1)$ and $(-b_2, a_2)$ forming the spread will give three quadrances:

$$Q_1 = (b_1^2 + a_1^2),$$
$$Q_2 = (b_2^2 + a_2^2),$$
$$Q_3 = (b_1 - b_2)^2 + (a_1 - a_2)^2$$

The *cross law* – see below – in terms of spread is:

$$1 - s = \frac{(Q_1 + Q_2 - Q_3)^2}{4Q_1Q_2}.$$

which becomes:

$$1 - s = \frac{(a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1 - b_2)^2 - (a_1 - a_2)^2)^2}{4(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

This simplifies, in the numerator, to: $(2a_1a_2 + 2b_1b_2)^2$, giving:

$$1 - s = \frac{(a_1a_2 + b_1b_2)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

Then, using the important identity due to Fibonacci: $(a_2b_1 - a_1b_2)^2 + (a_1a_2 + b_1b_2)^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2)$, the standard expression for spread in terms of slopes (or directions) of two lines becomes:

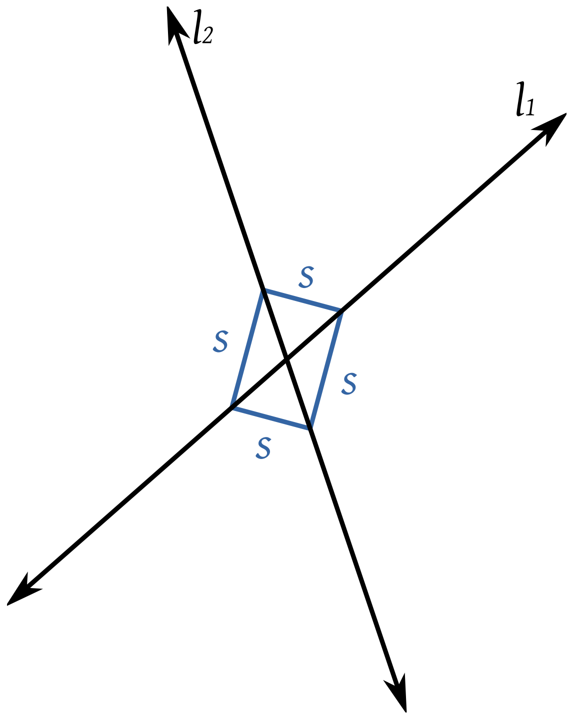
$$s = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

- **Cartesian** (three-variable)

This replaces $(-b_1, a_1)$ with $(x_1, y_1), (-b_2, a_2)$ with (x_2, y_2) and the origin $(0, 0)$ (as the point of intersection of two lines) with (x_3, y_3) in the previous result:

$$s = \frac{((y_1 - y_3)(x_2 - x_3) - (y_2 - y_3)(x_1 - x_3))^2}{((y_1 - y_3)^2 + (x_1 - x_3)^2)((y_2 - y_3)^2 + (x_2 - x_3)^2)}.$$

3.2 Spread compared to angle



The spread of two lines can be measured in four equivalent positions.

Unlike angle, which can define a relationship between *rays* emanating from a point, by a circular measure parametrization, and where a pair of lines can be considered four pairs of rays, forming four angles, 'spread' is a fundamental concept in rational trigonometry, describing *two lines* by a single measure of a rational function (see above).[3] Being equivalent to the *square* of a sine of the corresponding angle θ (and to the haversine of the chord-based double-angle Δ=2θ), the spread of both an angle and its supplementary angle are equal.

Spread is not proportional, however, to the separation between lines as angle would be; with spreads of 0, 1/4, 1/2, 3/4, and 1 corresponding to unevenly spaced angles 0, 30, 45, 60 and 90 degrees.

Instead, (recalling the supplementary property) two equal, co-terminal spreads determine a third spread, whose value will be a solution to the triple spread formula for a triangle (or three concurrent lines) with spreads of s, s and r :

$$(2s + r)^2 = 2(2s^2 + r^2) + 4s^2 r$$

$$4s^2 + 4sr + r^2 = 4s^2 + 2r^2 + 4s^2 r$$

giving the quadratic polynomial (in s):

$$r^2 + 4s^2 r - 4sr = 0$$

$$r^2 - 4s(1 - s)r = 0$$

and solutions

$$r = 0$$

$$r = 4s(1 - s)$$

This is equivalent to the trigonometric identity :

$$\sin^2(2\theta) = 4\sin^2\theta(1 - \sin^2\theta)$$

of the angles θ, θ and $180 - 2\theta$ of a triangle, using

$$S_2(s) = S_2(\sin^2\theta) = \sin^2(2\theta) = r(s)$$

to denote a *second* spread polynomial in s .

Tripling spreads likewise involves a triangle (or three concurrent lines) with one spread of r (the previous solution), one spread of s and obtaining a *third* spread polynomial, t in s . This turns out to be:

$$S_3(s) = s(3 - 4s)^2 = t(s)$$

Further multiples of any basic spread of lines can be generated by continuing this process using the triple spread formula.

Every *multiple* of a spread which is rational will thus be rational, but the converse does not apply. For example, by the half-angle formula, two lines meeting at a 15° (or 165°) angle have spread of:

$$\mathrm{hav}(30^\circ) = \sin^2(30^\circ/2) = (1 - \cos 30^\circ)/2 = (1 - \sqrt{3}/2)/2 = (2 - \sqrt{3})/4$$

and thus exists by algebraic extension of the rational numbers.

3.3 Turn and coturn

See also: Turn (geometry)

3.4 Twist

See also: Twist (mathematics)

4 Spread polynomials

As seen for double and triple spreads, the *n*th multiple of any spread, s gives a polynomial in that spread, denoted $S_n(s)$, as one solution to the triple spread formula.

In the conventional language of circular functions, these *n*th-degree *spread polynomials*, for *n* = 0, 1, 2, ..., can be characterized by the identity:

$$\sin^2(n\theta) = S_n(\sin^2\theta).$$

4.1 Identities

4.1.1 Explicit formulas

$$S_n(s) = s\sum_{k=0}^{n-1} \frac{n}{n-k}\binom{2n-1-k}{k}(-4s)^{n-1-k}.$$

(Michael Hirschhorn, Shuxiang Goh)[1]

$$S_n(s) = \frac{1}{2} - \frac{1}{4}\left(1-2s+2\sqrt{s^2-s}\right)^n - \frac{1}{4}\left(1-2s-2\sqrt{s^2-s}\right)^n.$$

$$S_n(s) = -\frac{1}{4}\left(\left(\sqrt{1-s}+i\sqrt{s}\right)^{2n}-1\right)^2\left(\sqrt{1-s}-i\sqrt{s}\right)^{2n}.$$

From the definition it immediately follows that

$$S_n(s) = \sin^2\left(n \arcsin\left(\sqrt{s}\right)\right).$$

4.1.2 Recursion formula

$$S_{n+1}(s) = 2(1-2s)S_n(s) - S_{n-1}(s) + 2s.$$

[1]

4.1.3 Relation to Chebyshev polynomials

The spread polynomials are related to the Chebyshev polynomials of the first kind, T_n by the identity

$$1 - 2S_n(s) = T_n(1-2s).$$

This implies

$$S_n(s) = \frac{1-T_n(1-2s)}{2} = 1 - T_n^2\left(\sqrt{1-s}\right).$$ [1]

The second equality above follows from the identity

$$2T_n^2(x) - 1 = T_{2n}(x)$$

on Chebyshev polynomials.

4.1.4 Composition

The spread polynomials satisfy the composition identity

$$S_n(S_m(s)) = S_{nm}(s).$$ [1]

4.1.5 Coefficients in finite fields

When the coefficients are taken to be members of the finite field F_p, then the sequence $\{S_n\}_{n=0,1,2,\ldots}$ of spread polynomials is periodic with period $(p^2-1)/2$. In other words, if $k = (p^2-1)/2$, then $S_{n+k} = S_n$, for all n.

4.1.6 Orthogonality

When the coefficients are taken to be real, then for $n \neq m$, we have

$$\int_0^1 \left(S_n(s)-\tfrac{1}{2}\right)\left(S_m(s)-\tfrac{1}{2}\right)\frac{ds}{\sqrt{s(1-s)}} = 0.$$

[1]

For $n = m$, the integral is $\pi/8$ unless $n = m = 0$, in which case it is $\pi/4$.

4.1.7 Generating functions

The ordinary generating function is

$$\sum_{n=1}^{\infty} S_n(s)x^n = \frac{sx(1+x)}{(1-x)^3+4sx(1-x)}.$$

(Michael Hirschhorn)[1]

The exponential generating function is

$$\sum_{n=1}^{\infty} \frac{S_n(s)}{n!}x^n = \frac{1}{2}e^x\left[1-e^{-2sx}\cos\left(2x\sqrt{s(1-s)}\right)\right].$$

4.1.8 Differential equation

$S_n(s)$ satisfies the second order linear non-homogeneous differential equation

$$s(1-s)\,y'' + \left(\tfrac{1}{2}-s\right)y' + n^2\left(y-\tfrac{1}{2}\right) = 0.$$

5 Spread periodicity theorem

For every integer s and every prime p, there is a natural number m such that $Sn(s)$ is divisible by p precisely when m divides n. This number m is a divisor of either $p-1$ or $p+1$. The proof of this number theoretical property was first given in a paper by Shuxiang Goh and N. J. Wildberger.[4] It involves considering the projective analogue to quadrance in the finite projective line $\mathbf{P}^1(Fp)$.

5.1 Table of spread polynomials, with factorizations

The first several spread polynomials are as follows:

$$S_0(s) = 0$$

$$S_1(s) = s$$

$$\begin{aligned} S_2(s) &= 4s - 4s^2 \\ &= 4s(1-s) \end{aligned}$$

$$\begin{aligned} S_3(s) &= 9s - 24s^2 + 16s^3 \\ &= s(3-4s)^2 \end{aligned}$$

$$\begin{aligned} S_4(s) &= 16s - 80s^2 + 128s^3 - 64s^4 \\ &= 16s(1-s)(1-2s)^2 \end{aligned}$$

$$\begin{aligned} S_5(s) &= 25s - 200s^2 + 560s^3 - 640s^4 + 256s^5 \\ &= s(5 - 20s + 16s^2)^2 \end{aligned}$$

$$\begin{aligned} S_6(s) &= 36s - 420s^2 + 1792s^3 - 3456s^4 + 3072s^5 - 1024s^6 \\ &= 4s(1-s)(1-4s)^2(3-4s)^2 \end{aligned}$$

$$\begin{aligned} S_7(s) &= 49s - 784s^2 + 4704s^3 - 13440s^4 + 19712s^5 - 14336s^6 + 4 \\ &= s(7 - 56s + 112s^2 - 64s^3)^2 \end{aligned}$$

$$\begin{aligned} S_8(s) &= 64s - 1344s^2 + 10752s^3 - 42240s^4 + 90112s^5 - 106496s^6 \\ &\quad + 65536s^7 - 16384s^8 \\ &= 64s(s-1)(1-2s)^2(1-8s+8s^2)^2 \end{aligned}$$

$$\begin{aligned} S_9(s) &= 81s - 2160s^2 + 22176s^3 - 114048s^4 + 329472s^5 - 559104s \\ &\quad + 552960s^7 - 294912s^8 + 65536s^9 \\ &= s(-3+4s)^2(-3+36s-96s^2+64s^3)^2 \end{aligned}$$

$$\begin{aligned} S_{10}(s) &= 100s - 3300s^2 + 42240s^3 - 274560s^4 + 1025024s^5 \\ &\quad - 2329600s^6 + 3276800s^7 - 2785280s^8 + 1310720s^9 - 2 \\ &= 4s(1-s)(5-20s+16s^2)^2(1-12s+16s^2)^2 \end{aligned}$$

$$\begin{aligned} S_{11}(s) &= 121s - 4840s^2 + 75504s^3 - 604032s^4 + 2818816s^5 \\ &\quad - 8200192s^6 + 15319040s^7 - 18382848s^8 + 13697024s^9 \\ &= s(11 - 220s + 1232s^2 - 2816s^3 + 2816s^4 - 1024s^5)^2 \end{aligned}$$

6 Laws of rational trigonometry

Wildberger states that there are five basic laws in rational trigonometry. He also states that these laws can be verified using high-school level mathematics. Some are equivalent to standard trigonometrical formulae with the variables expressed as quadrance and spread.[3]

In the following five formulas, we have a triangle made of three points $A_1, A_2, A_3,$. The spreads of the angles at those points are $s_1, s_2, s_3,$, and $Q_1, Q_2, Q_3,$ are the quadrances of the triangle sides opposite A_1, A_2, and A_3, respectively. As in classical trigonometry, if we know three of the six elements $s_1, s_2, s_3, , Q_1, Q_2, Q_3$, and these three are not the three *s*, then we can compute the other three.

6.1 Triple quad formula

The three points A_1, A_2, A_3 are collinear if and only if:

$$(Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2)$$

where Q_1, Q_2 and Q_3 represent the quadrances between A_1, A_2 and A_3 respectively. It can either be proved by analytic geometry (the preferred means within rational trigonometry) or derived from Heron's formula, using the condition for collinearity that the triangle formed by the three points has zero area.

Proof (click at right to show/hide)
The line AB has the general form:

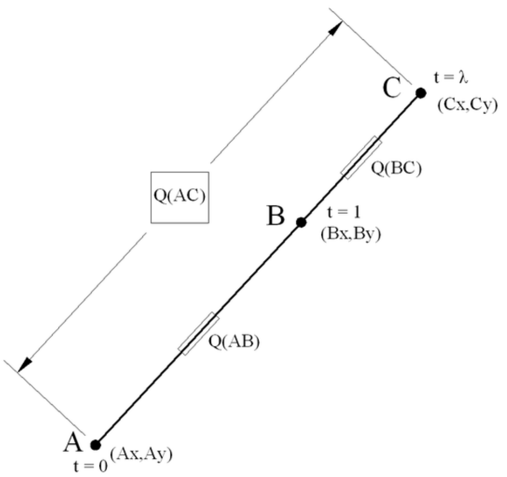


Illustration of nomenclature used in the proof.

$$ax + by + c = 0$$

where the (non-unique) parameters *a*, *b* and *c*, can be expressed in terms of the coordinates of points *A* and *B* as:

$$a = A_y - B_y$$

$$b = B_x - A_x$$

$$c = A_x B_y - A_y B_x$$

so that, everywhere on the line:

$$(A_y - B_y)x + (B_x - A_x)y + (A_x B_y - A_y B_x) = 0.$$

But the line can also be specified by two simultaneous equations in a parameter *t*, where *t* = 0 at point *A* and *t* = 1 at point *B*:

$$x = (B_x - A_x)t + A_x \text{ and } y = (B_y - A_y)t + A_y$$

or, in terms of the original parameters:

$$x = bt + A_x \text{ and } y = -at + A_y.$$

If the point *C* is collinear with points *A* and *B*, there exists some value of *t* (for distinct points, not equal to 0 or 1), call it *λ*, for which these two equations are simultaneously satisfied at the coordinates of the point *C*, such that:

$$C_x = b\lambda\ + A_x \text{ and } C_y = -a\lambda\ + A_y.$$

Now, the quadrances of the three line segments are given by the squared differences of their coordinates, which can be expressed in terms of λ:

$$\begin{aligned}
Q(AB) &\equiv (B_x - A_x)^2 + (B_y - A_y)^2 \\
&= b^2 + (-a)^2 \\
&= a^2 + b^2
\end{aligned}$$

$$\begin{aligned}
Q(BC) &\equiv (C_x - B_x)^2 + (C_y - B_y)^2 \\
&= ((b\lambda\ + A_x) - B_x)^2 + ((-a\lambda\ + A_y) - B_y)^2 \\
&= (b\lambda\ + (A_x - B_x))^2 + (-a\lambda\ + (A_y - B_y))^2 \\
&= (b\lambda\ + (-b))^2 + (-a\lambda\ + a)^2 \\
&= b^2(\lambda\ - 1)^2 + a^2(-\lambda\ + 1)^2 \\
&= b^2(\lambda\ - 1)^2 + a^2(\lambda\ - 1)^2 \\
&= (a^2 + b^2)(\lambda\ - 1)^2
\end{aligned}$$

$$\begin{aligned}
Q(AC) &\equiv (C_x - A_x)^2 + (C_y - A_y)^2 \\
&= ((b\lambda\ + A_x) - A_x)^2 + ((-a\lambda\ + A_y) - A_y)^2 \\
&= (b\lambda\ + A_x - A_x)^2 + (-a\lambda\ + A_y - A_y)^2 \\
&= (b\lambda)^2 + (-a\lambda)^2 \\
&= b^2\lambda^2 + (-a)^2\lambda^2 \\
&= b^2\lambda^2 + a^2\lambda^2 \\
&= (a^2 + b^2)\lambda^2
\end{aligned}$$

where use was made of the fact that $(-\lambda\ + 1)^2 = (\lambda\ - 1)^2$.

Substituting these quadrances into the equation to be proved:

$$(Q(AB)+Q(BC)+Q(AC))^2 = 2(Q(AB)^2+Q(BC)^2+Q(AC)^2)$$

$$((a^2+b^2)+(a^2+b^2)(\lambda-1)^2+(a^2+b^2)\lambda^2)^2 = 2((a^2+b^2)^2+((a^2+b^2)(\lambda -$$

$$(a^2+b^2)^2(1+(\lambda-1)^2+\lambda^2)^2 = 2(a^2+b^2)^2(1+((\lambda-1)^2)^2+(\lambda^2)^2)$$

Now, if A and B represent distinct points, such that $(a^2 + b^2)$ is not zero, we may divide both sides by $Q(AB)^2 = (a^2 + b^2)^2$:

$$(1+\lambda^2-2\lambda+1+\lambda^2)^2 = 2(1+(\lambda^2-2\lambda+1)^2+\lambda^4)$$

$$(2\lambda^2-2\lambda+2)^2 = 2(1+\lambda^4-2\lambda^3+\lambda^2-2\lambda^3+4\lambda^2-2\lambda+\lambda^2-2\lambda+1+\lambda^4)$$

$$4(\lambda^2-\lambda+1)^2 = 2(2\lambda^4-4\lambda^3+6\lambda^2-4\lambda+2)$$

$$4(\lambda^4-\lambda^3+\lambda^2-\lambda^3+\lambda^2-\lambda+\lambda^2-\lambda+1) = 4(\lambda^4-2\lambda^3+3\lambda^2-2\lambda+1)$$

$$\lambda^4-2\lambda^3+3\lambda^2-2\lambda+1 = \lambda^4-2\lambda^3+3\lambda^2-2\lambda+1$$

6.2 Pythagoras's theorem

The lines A_1A_3 (of quadrance Q_1) and A_2A_3 (of quadrance Q_2) are perpendicular (their spread is 1) if and only if:

$$Q_1 + Q_2 = Q_3.$$

where Q_3 is the quadrance between A_1 and A_2.

This is equivalent to the Pythagorean theorem (and its converse).

There are many classical proofs of Pythagoras's theorem; this one is framed in the terms of rational trigonometry.

The *spread* of an angle is the square of its sine. Given the triangle *ABC* with a spread of 1 between sides *AB* and *AC*,

$$Q(AB) + Q(AC) = Q(BC)$$

where *Q* is the "quadrance", i.e. the square of the distance.

Proof

Construct a line *AD* dividing the spread of 1, with the point *D* on line *BC*, and making a spread of 1 with *DB* and *DC*. The triangles *ABC*, *DBA* and *DAC* are similar (have the same spreads but not the same quadrances).

This leads to two equations in ratios, based on the spreads of the sides of the triangle:

$$s_C = \frac{Q(AB)}{Q(BC)} = \frac{Q(BD)}{Q(AB)} = \frac{Q(AD)}{Q(AC)}.$$

$$s_B = \frac{Q(AC)}{Q(BC)} = \frac{Q(DC)}{Q(AC)} = \frac{Q(AD)}{Q(AB)}.$$

Now in general, the two spreads resulting from dividing a spread into two parts, as line *AD* does for spread *CAB*, do not add up to the original spread since spread is a non-linear function. So we first prove that dividing a spread

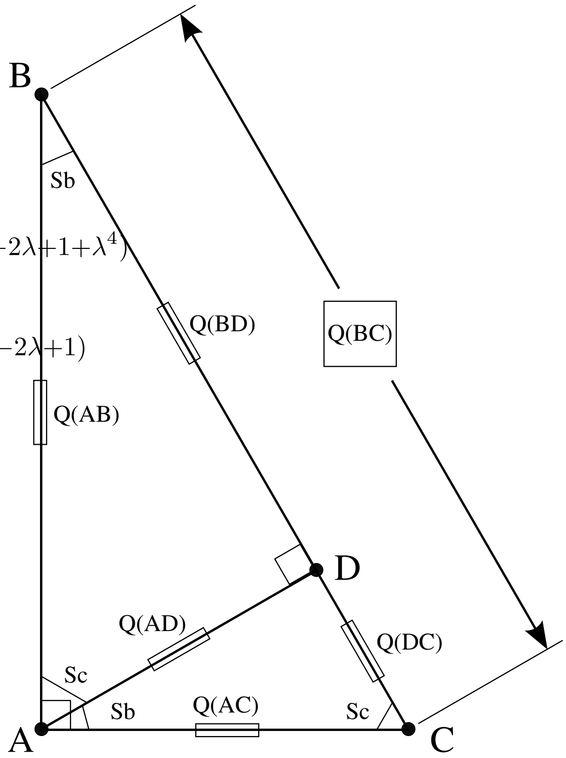


Illustration of nomenclature used in the proof.

of 1, results in two spreads that do add up to the original spread of 1.

For convenience, but with no loss of generality, we orient the lines intersecting with a spread of 1 to the coordinate axes, and label the dividing line with coordinates (x_1, y_1) and (x_2, y_2) . Then the two spreads are given by:

$$s_1 = \frac{(x_2-x_2)^2+(y_2-y_1)^2}{(x_2-x_1)^2+(y_2-y_1)^2} = \frac{(y_2-y_1)^2}{(x_2-x_1)^2+(y_2-y_1)^2},$$

$$s_2 = \frac{(x_2-x_1)^2+(y_2-y_2)^2}{(x_2-x_1)^2+(y_2-y_1)^2} = \frac{(x_2-x_1)^2}{(x_2-x_1)^2+(y_2-y_1)^2}.$$

Hence:

$$s_1 + s_2 = \frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)^2+(y_2-y_1)^2} = 1.$$

So that:

$$s_C + s_B = 1.$$

Using the first two ratios from the first set of equations, this can be rewritten:

$$\frac{Q(AB)}{Q(BC)} + \frac{Q(AC)}{Q(BC)} = 1.$$

Multiplying both sides by $Q(BC)$:

$$Q(AB) + Q(AC) = Q(BC).$$

Q.E.D.

6.3 Spread law

For any triangle $\overline{A_1A_2A_3}$ with nonzero quadrances:

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$ [1]

This is the law of sines, just squared.

6.4 Cross law

For any triangle $\overline{A_1A_2A_3}$,

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$ [1]

This is analogous to the law of cosines. It is called 'cross law' because $(1 - s_3)$, the square of the cosine of the angle, is called the 'cross'.

6.5 Triple spread formula

For any triangle $\overline{A_1A_2A_3}$,

$$(s_1+s_2+s_3)^2 = 2(s_1^2+s_2^2+s_3^2)+4s_1s_2s_3.$$ [1]

This relation can be derived from the formula for the sine of a compound angle: in a triangle (whose three angles sum to 180°) we have,

$$\sin(a) = \sin(b + c) = \sin(b)\cos(c) + \sin(c)\cos(b)$$

Equivalently, it describes the relationship between the spreads of three concurrent lines, as spread (like angle) is unaffected when the sides of a triangle are moved parallel to themselves to meet in a common point.

Knowing two spreads allows the third spread to be calculated by solving the associated quadratic formula but, because two solutions are possible, further *triangle spread rules* must be used to select the appropriate one. (The relative complexity of this process contrasts with the much simpler method of obtaining a supplementary angle of two others.)

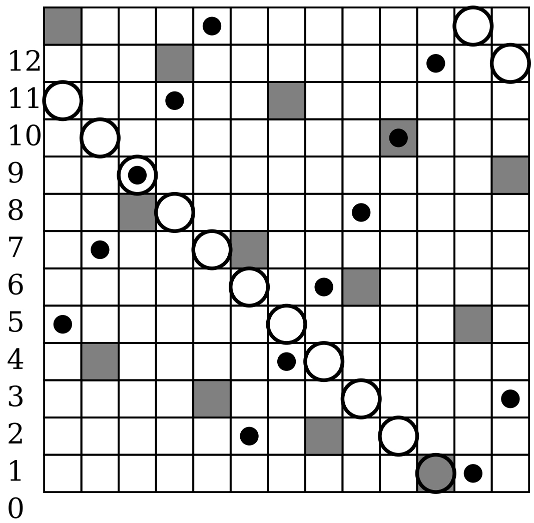
7 Trigonometry over arbitrary fields

As the laws of rational trigonometry give algebraic (and not transcendental) relations, they apply in generality to algebraic number fields beyond the rational numbers. Specifically, any finite field which does not have characteristic 2 reproduces a form of these laws, and thus a finite field geometry.[5] The 'plane' formed by a finite field F_p is the cartesian product $F_p \times F_p$ of all ordered pairs of field elements, with opposite edges identified forming the surface of a discrete torus. Individual elements correspond to standard 'points' whereas 'lines' are sets of no more than p points related by incidence (an initial point) plus direction or slope given in lowest terms (say all points '2 over and 1 up') that 'wrap' the plane before repeating.

7.1 Example: (verify the spread law in F_{13})

The figure (right) shows a *triangle* of three such lines in the finite field setting $F_{13} \times F_{13}$:

Each line has it own symbol and the intersections of lines (*vertices*) is marked by *two* symbols present at points:



A Triangle through the points (2, 8), (9, 9), and (10, 0) of the finite field-plane $F_{13} \times F_{13}$.

(2, 8), (9, 9) and (10, 0).

Using *Pythagoras's theorem* with arithmetic modulo 13, we find these sides have quadrances of:

$$(9 - 2)^2 + (9 - 8)^2 = 50 \equiv 11 \bmod 13$$

$$(9 - 10)^2 + (9 - 0)^2 = 82 \equiv 4 \bmod 13$$

$(10-2)^2 + (0-8)^2 = 128 \equiv 11 \bmod 13$

Rearranging the *Cross law* (see above) gives separate expressions for each spread, in terms of the three quadrances:

$1-(4+11-11)^2/(4.4.11) = 1-3/7 \equiv 8 \bmod 13$

$1-(11+11-4)^2/(4.11.11) = 1-12/3 \equiv 10 \bmod 13$

$1-(4+11-11)^2/(4.4.11) = 1-3/7 \equiv 8 \bmod 13$

In turn we note these ratios are all equal – as per the *Spread law* (at least in mod 13):

8/11 : 10/4 : 8/11

Since first and last ratios match (making the triangle *isosceles*) we just cross multiply, and take differences, to show equality with the middle ratio also:

$(11)(10) - (8)(4) \equiv 78$ (0 mod 13)

Otherwise, the standard Euclidean plane is taken to consist of just rational points, $\mathbb{Q} \times \mathbb{Q}$, omitting any non-algebraic numbers as solutions. Properties like incidence of objects, representing the solutions or 'content' of geometric theorems, therefore follow a number theoretic approach that differs and is more restrictive than one allowing real numbers. For instance, *not all* lines passing through a circle's centre are considered to meet the circle at its circumference. To be incident such lines must be of the form: $ax + by = 0, a^2 + b^2 = c^2 (a, b, c \in \mathbb{Q})$ and necessarily meet the circle in a *rational* point.

8 Computation – complexity and efficiency

Rational trigonometry makes nearly all problems solvable with only addition, subtraction, multiplication or division, as trigonometric functions (of angle) are purposively avoided in favour of trigonometric ratios in quadratic form.[3] At most, therefore, results required as distance (or angle) can be approximated from an exact-valued rational equivalent of quadrance (or spread) after these simpler operations have been carried out. To make use of this advantage however, each problem must either be given, or set up, in terms of prior quadrances and spreads, which entails additional work.[6]

The laws of rational trigonometry, being algebraic and 'exact-valued', introduce subtleties into the solutions of problems, such as the non-additivity of quadrances of collinear points (in the case of the triple quad formula) or the spreads of concurrent lines (in the case of the triple spread formula) absent from the classical subject, where linearity is incorporated into distance and circular measure of angles, albeit 'transcendental' techniques, necessitating approximation in results. Additional complexity is also introduced by the need to have 'rules' to handle the dual solutions these quadratic relations generate.

9 Notability and criticism

Rational trigonometry is mentioned in only a modest number of mathematical publications, besides Wildberger's own articles and book. *Divine Proportions* was dismissed by reviewer Paul J. Campbell, writing in Mathematics Magazine: “the author claims that this new theory will take 'less than half the usual time to learn'; but I doubt it. and it would still have to be interfaced with the traditional concepts and notation.” Reviewer, William Barker, Isaac Henry Wing Professor of Mathematics at Bowdoin College, also writing for the MAA, was more approving: "*Divine Proportions* is unquestionably a valuable addition to the mathematics literature. It carefully develops a thought provoking, clever, and useful alternate approach to trigonometry and Euclidean geometry. It would not be surprising if some of its methods ultimately seep into the standard development of these subjects. However, unless there is an unexpected shift in the accepted views of the foundations of mathematics, there is not a strong case for rational trigonometry to replace the classical theory” [7] New Scientist's Amanda Gefter described the approach of Wildberger as an example of finitism.[2] James Franklin in the Mathematical Intelligencer argued that the book deserved careful consideration.[8]

An analysis by Michael Gilsdorf of the same example trigonometric problems used by the author in an earlier paper, found the claim that rational trigonometry takes fewer steps to solve most problems compared to classical methods could be untrue, if free selection of classical methods is available for optimal solution of a given problem; like using the cross product formula for the area of a triangle from the coordinates of its vertices, or applying Stewart's theorem directly to (and in the special case of) the median of a triangle. Concerning pedagogy, and whether the quadratic measures introduced by rational trigonometry offered real benefits over traditional teaching and learning of the subject, the analysis made further observations that classical trigonometry was not based on the use of Taylor series to approximate angles, but rather on measurements of chord (twice the sine of an angle), so with a proper understanding students could reap advantages from continued use of linear measurement without the claimed logical inconsistencies when circular parametrization of angles is subsequently introduced.[9]

10 See also

- Finitism
- Ultrafinitism
- Universal hyperbolic geometry
- History of trigonometry

11 Notes

[1] For Wildberger's views on the history of infinity, see the Gefter New Scientist article, but also see Wildberger's History of Mathematics and Math Foundations lectures, University of New South Wales, circa 2009–2014 in more than 120 videos and lectures, available online @youtube

[2] See *Divine Proportions* for numerous examples of calculus done with rational trigonometric functions, as well as problems involving the application of rational trigonometry to situations containing irrationals.

12 References

- A comparison of classical and rational trigonometry
- Rational Trigonometry Applied to Robotics, by João Pequito Almeida
- The Impossibility of Trisecting and Angle with Straightedge and Compass: An Approach Using Rational Trigonometry, by David G. Poole
- How to multiply and divide triangles, by Maurice Craig

[1] Wildberger, Norman John (2005). *Divine Proportions: Rational Trigonometry to Universal Geometry* (1 ed.). Australia: Wild Egg Pty Ltd. ISBN 0-9757492-0-X. Retrieved 2015-12-01.

[2] "Infinity's end: Time to ditch the never-ending story?" by Amanda Gefter, New Scientist, 15 August 2013

[3] Wildberger, Norman J. (2007). "A Rational Approach to Trigonometry". *Math Horizons* (Washington, DC: Mathematical Association of America). November 2007: 16–20. ISSN 1072-4117.

[4] Shuxiang Goh, N. J. Wildberger (November 5, 2009). "Spread polynomials, rotations and the butterfly effect". arXiv:0911.1025.

[5] Le Anh Vinh, Dang Phuong Dung (July 17, 2008). "Explicit tough Ramsey graphs". arXiv:0807.2692., page 1. Another version of this article is at Le Anh Vinh, Dang Phuong Dung (2008), "Explicit tough Ramsey Graphs", *Proceedings of International Conference on Relations, Orders and Graphs: Interaction with Computer Science 2008*, Nouha Editions, 139–146.

[6] Olga Kosheleva (2008), "Rational trigonometry: computational viewpoint", Geombinatorics, Vol. 1, No. 1, pp. 18–25.

[7] http://www.maa.org/publications/maa-reviews/divine-proportions-rational-trigonometry-to-universal-geometry

[8] J. Franklin, Review of *Divine Proportions*, *Mathematical Intelligencer* 28 (3) (2006), 73-4.

[9] http://web.maths.unsw.edu.au/~{ }norman/papers/TrigComparison.pdf

13 External links

- Wildberger's rational trigonometry site, including downloadable papers and sections of his book
- Spread polynomials, rotations and the butterfly effect
- Euler Math Toolbox implementation of Rational Trigonometry
- Youtube channel

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14.1 Text

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