

Probability

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1 Probability Spaces

1.1 Classical Probability Spaces

Textbook probability theory [1, 2, 4] is defined using the notions of a *sample space* Ω , a space of *events* \mathcal{F} , and a *probability measure* μ . In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events \mathcal{F} as, 2^Ω , the powerset of Ω . A *probability measure* is a function $\mu : \mathcal{F} \rightarrow [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i , we have $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two coin experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is “heads” is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$\mu(\emptyset)$	$= 0$	$\mu(\{HT, TH\})$	$= 2/3$
$\mu(\{HH\})$	$= 1/3$	$\mu(\{HT, TT\})$	$= 0$
$\mu(\{HT\})$	$= 0$	$\mu(\{TH, TT\})$	$= 2/3$
$\mu(\{TH\})$	$= 2/3$	$\mu(\{HH, HT, TH\})$	$= 1$
$\mu(\{TT\})$	$= 0$	$\mu(\{HH, HT, TT\})$	$= 1/3$
$\mu(\{HH, HT\})$	$= 1/3$	$\mu(\{HH, TH, TT\})$	$= 1$
$\mu(\{HH, TH\})$	$= 1$	$\mu(\{HT, TH, TT\})$	$= 2/3$
$\mu(\{HH, TT\})$	$= 1/3$	$\mu(\{HH, HT, TH, TT\})$	$= 1$

1.2 Quantum Probability Spaces

A classical model decides the occurrence or non-occurrence of all events simultaneously which is inconsistent with quantum mechanics. Indeed, in the quantum world, there are (non-commuting) events which cannot happen simultaneously. To accommodate this situation, we completely abandon the sample space Ω and define and reason directly about events. Thus a quantum probability space will consist of just two components: a set of events \mathcal{A} and a probability measure $\phi : \mathcal{A} \rightarrow [0, 1]$. These components are defined as follows [3, 5].

We first assume an ambient Hilbert space \mathcal{H} and define the set of events \mathcal{A} as *projections* on \mathcal{H} . Similarly to the classical case, a probability measure is a function $\phi : \mathcal{A} \rightarrow [0, 1]$ satisfying:

- $\phi(\mathbb{1}) = 1$, and
- for all $A \in \mathcal{A}$, we have $\phi(A^*A) \geq 0$.

Yu-Tsung says: If we follow [3, 5], then we also need

- ϕ can be extended to a linear functional $\phi : \text{alg}(\mathcal{A}) \rightarrow \mathbb{C}$, where $\text{alg}(\mathcal{A})$ is the minimal $*$ -algebra generated by \mathcal{A} .

Also,

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should be

- for all $A \in \text{alg}(\mathcal{A})$, we have $\phi(A^*A) \geq 0$.

because we have $\phi(A^*A) = \phi(A^2) = \phi(A)$ if A is a projection.

As an example, let P_1, P_2, \dots, P_k be mutually orthogonal projections on \mathcal{H} with sum $\mathbb{1}$ and define the event space \mathcal{A} to be the linear span of these operators:

$$\mathcal{A} = \left\{ \sum_{j=1}^k \lambda_j P_j \mid \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}.$$

Yu-Tsung says: $\left\{ \sum_{j=1}^k \lambda_j P_j \mid \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}$ is the minimal $*$ -algebra generated by P_1, P_2, \dots, P_k , but it contains all possible observables P_1, P_2, \dots, P_k can generate (and something more) not just projections. For example, $2\mathbb{1} \in \left\{ \sum_{j=1}^k \lambda_j P_j \mid \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}$.

Each state $|\psi\rangle$ of the Hilbert space induces a probability measure $\phi_\psi : \mathcal{A} \rightarrow [0, 1]$ defined as follows:

$$\phi_\psi(A) = \langle \psi | A \psi \rangle$$

Yu-Tsung says: So ϕ_ψ maps the projections generated by P_1, P_2, \dots, P_k to $[0, 1]$, and maps $\left\{ \sum_{j=1}^k \lambda_j P_j \mid \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}$ to \mathbb{C} ...

Concrete example: consider the two qubit Hilbert space with computational bases $|0\rangle$ and $|1\rangle$. First, consider the two states

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

and consider the following families of projections:

- Family I: $|0\rangle\langle 0|, |1\rangle\langle 1|$
- Family II: $|+\rangle\langle +|, |-\rangle\langle -|$

In family I, all operators can be expressed as $\{\lambda_1|0\rangle\langle 0| + \lambda_2|1\rangle\langle 1| \mid \lambda_1, \lambda_2 \in \mathbb{C}\}$. In order to identify projectors among them, we need to solve the following two equations.

$$\begin{aligned} \lambda_1|0\rangle\langle 0| + \lambda_2|1\rangle\langle 1| &= (\lambda_1|0\rangle\langle 0| + \lambda_2|1\rangle\langle 1|)^* = \lambda_1^*|0\rangle\langle 0| + \lambda_2^*|1\rangle\langle 1| \\ \lambda_1|0\rangle\langle 0| + \lambda_2|1\rangle\langle 1| &= (\lambda_1|0\rangle\langle 0| + \lambda_2|1\rangle\langle 1|)^2 = \lambda_1^2|0\rangle\langle 0| + \lambda_2^2|1\rangle\langle 1| \end{aligned}$$

Therefore, we actually have $\lambda_1, \lambda_2 \in \{0, 1\}$, and there are only four projections: 0 , $|0\rangle\langle 0|$, $|1\rangle\langle 1|$, and $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$. Then, $\phi_{|+\rangle}$ and $\phi_{|-\rangle}$ of each projections are

$$\begin{array}{ll} \phi_{|+\rangle}(0) &= 0 \\ \phi_{|+\rangle}(|0\rangle\langle 0|) &= 1/2 \\ \phi_{|+\rangle}(|1\rangle\langle 1|) &= 1/2 \\ \phi_{|+\rangle}(\mathbb{1}) &= 1 \end{array} \quad \begin{array}{ll} \phi_{|-\rangle}(0) &= 0 \\ \phi_{|-\rangle}(|0\rangle\langle 0|) &= 1/2 \\ \phi_{|-\rangle}(|1\rangle\langle 1|) &= 1/2 \\ \phi_{|-\rangle}(\mathbb{1}) &= 1 \end{array}$$

Similarly, $\phi_{|+\rangle}$ and $\phi_{|-\rangle}$ gives two probability for family II as well.

$$\begin{array}{ll} \phi_{|+\rangle}(0) & = 0 \\ \phi_{|+\rangle}(|+\rangle\langle +|) & = 1 \\ \phi_{|+\rangle}(|-\rangle\langle -|) & = 0 \\ \phi_{|+\rangle}(\mathbb{1}) & = 1 \end{array} \quad \begin{array}{ll} \phi_{|-\rangle}(0) & = 0 \\ \phi_{|-\rangle}(|+\rangle\langle +|) & = 0 \\ \phi_{|-\rangle}(|-\rangle\langle -|) & = 1 \\ \phi_{|-\rangle}(\mathbb{1}) & = 1 \end{array}$$

1.3 Plan

Several assumptions are woven in the definition of a quantum probability space:

- the Hilbert space \mathcal{H} ;
- the real interval $[0, 1]$;
- the fact that each state induces a probability measure, i.e., the Born rule;
- the fact that every probability measure is induced by a state, i.e., Gleason's theorem

In the remainder of the paper, we examine each of these assumptions and consider variations motivated by computation in a world with limited resources. In particular, we will consider a variant of the Hilbert space over finite fields; we will consider set-valued probability measures; we will consider ways other than the Born rule in which a state can induce a probability measure, and we will consider probability measures that may come from information beyond the quantum states.

References

- [1] William G. Faris. Appendix: Probability in quantum mechanics. In *The infamous boundary : seven decades of controversy in quantum physics*. Boston : Birkhauser, 1995.
- [2] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. A foundation for computer science. Addison-Wesley, 1994.
- [3] Hans Maassen. Quantum probability and quantum information theory. In *Quantum information, computation and cryptography*, pages 65–108. Springer, 2010.
- [4] V.K. Rohatgi and A.K.M.E. Saleh. *An Introduction to Probability and Statistics*. Wiley Series in Probability and Statistics. Wiley, 2011.
- [5] JM Swart. Introduction to quantum probability. *Lecture Notes*, 2013.