Interval-Valued Probability Measures

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Abstract: The purpose of this paper is propose a simple definition of an interval-valued probability measure and to examine some of the implications of this definition. This definition provides a method for characterizing certain types of imprecise probabilities. We develop some properties of this definition and propose a definition of integration with respect to it. We show that this definition is a generalization of a probability measure and encompasses probability measures, possibility measures and more generally clouds as defined by Neumaier in [5]. In the setting of interval-valued probability measures as defined below we are able to characterize the probability information contained in a cloud. We prove that continuously increasing clouds over the real line generate such "measures".

Keywords: Imprecise Probability, Possibility Theory, Probability Theory, Interval Analysis

1 Introduction

In this paper we establish a basis for linking various methods of uncertainty representation that have at their root, a common origin. They can all be constructed from inner and outer measures. Probability theory deals with sets in which the inner and outer measures converge to a single value, called measurable sets. For interval-valued probability measures as defined below, we relax this requirement and allow the inner and outer measures to remain apart. For example, this might come about if we wish to represent an unknown probability distribution when all we know about the distribution is the result of a finite sample of random experiments. Then we might form a bound on certain events such as with a Kolmogorov-Smirnoff statistic or using likelihood ratios. If one does have bounds on the probabilities of certain events this information can provide bounds on combinations of these events, and a natural desire is to determine the tightest possible bound for the information available.

In the first section of this paper we define, in a formal way, what we call an interval-valued probability measure. We note that the definition we propose is closely related to interval-probability as defined by Weichselberger [7] and particularly to F-probabilities. Weichselberger's definition begins with a set of probability measures and then defines an interval probability as a set function providing upper and lower bounds on the probabilities calculated from these measures. F-probabilities are simply the tightest bounds possible for the set of probability measures. The definition we propose is more fundamental in that it does not require a set of probability measures as a starting point and in fact does not require a definition of probability separate from itself. On the contrary the definition could be used in the other direction, as a basis for defining set of possible probability measures based on interval bounds on the probabilities of certain events.

Following the statement of the basic definition we derive various properties and implications. Following this formalization we propose a definition for integration with respect to such measures and show that they provide interval bounds on an uncertain integral.

In the second section of this paper we show how continuously increasing clouds over the real line, as defined by Neumaier [5], can be used to construct an interval-valued probability measure. This construction closely parallels the extension of a measure from a measure on a semi-ring as in Kolmogorov [2]. This construction shows how the interval-valued probability of a complex event can be approximated from simple events, just as the measure of a Borel set on the real line can be approximated by summing the lengths of a finite set of intervals.

In the last section we conclude the paper and discuss future directions.

2 Interval-valued Probability Measures

This section defines what we mean by an interval-valued probability measure, develops basic properties and proposes a definition for integration with respect to such a measure (although we do not develop this definition further in this paper). Interval-valued probability measures allow us to formally characterize uncertain probabilities when the uncertainty can be represented by interval bounds on certain events. This provides a structure from which bounds on the probability of more complex events can be constructed and utilized. Since many probabilities used in applications are estimated from finite samples, by selecting a single probability distribution constructed to be a "best" estimate for the data and application, the uncertainty in this selection may not be reflected in the results derived from utilizing the estimate. Interval-valued probability allows us to propagate the uncertainty through our calculations and thus into the solution.

2.1 Definition and Properties

We define an interval-valued probability measure and develop basic properties. This generalization of a probability measure includes both probability measures, and possibility/necessity measures. We think of this set function as a method for giving a partial representation for an unknown probability measure. Throughout, arithmetic operations involving set functions are in terms of interval arithmetic and $Int_{[0,1]} \equiv \{[a,b] \mid 0 \leq a \leq b \leq 1\}$.

Definition 1 *Given measurable space* $(S, \mathcal{A})$, $i_m : \mathcal{A} \to Int_{[0,1]}$ *satisfying the following properties:*
1) $i_m(\emptyset) = [0,0]$
2) $i_m(S) = [1,1]$
3) $\forall A \in \mathcal{A}$, $i_m(A) = [a^-(A), a^+(A)] \subseteq [0,1]$
4) for every countable collection of disjoint sets $\{A_{k\in K}\}, \{B_{j\in J}\} \subseteq \mathcal{A}$ *such that*$A = \cup_{k\in K} A_k$ *and* $(\cup_{k\in K} A_k) \cup (\cup_{i\in J} B_j) = S$

$$i_m(A) \subseteq \left[\max\left\{1 - \Sigma_{j\in J} a^+(B_j), \Sigma_{k\in K} a^-(A_k)\right\}, \min\left\{1 - \Sigma_{j\in J} a^-(B_j), \Sigma_{k\in K} a^+(A_k)\right\}\right]$$

is called an ***interval-valued probability measure*** *on* $(S, \mathcal{A})$

Property 1) and 2) simply state boundary conditions that every probability measure must satisfy. The third condition essentially states that the bound on any set in $\mathcal{A}$ is the tightest possible given the bounds on all other sets in $\mathcal{A}$. This is a basic consistency requirement. For example suppose $S = A \cup B \cup C \cup D$, all disjoint. Then given bounds on the probability of each set, e.g. $\Pr(A) \in [a^-(A), a^+(A)]$, we can state that

$$\begin{aligned} \Pr(A \cup B) \in & \left[\max\left\{1 - a^+(C) - a^+(D), a^-(A) + a^-(B)\right\},\right. \\ & \left.\min\left\{1 - a^-(C) - a^-(D), a^+(A) + a^+(B)\right\}\right] \end{aligned}$$

by considering how the maximum and minimum probabilities possible in each set bound the probability of the union and its complement.

The following three examples prove existence.

Example 1 *Consider the set function $i_m : \mathcal{A} \to Int_{[0,1]}$ defined by*

$$i_m(A) = \begin{cases} [0,1] & \text{if } A \neq S \text{ and } A \neq \emptyset \\ [1,1] & \text{if } A = S \\ [0,0] & \text{if } A = \emptyset \end{cases}$$

which is equivalent to being totally ignorant with respect to an unknown probability distribution over $(S, \mathcal{A})$.
Then if $A \neq S$ or $\emptyset$, and if $(\cup_{k\in K} A_k)$ and $(\cup_{j\in J} B_j)$ are subsets of $\mathcal{A}$ satisfying condition 3) of the definition

$i_m(A) \equiv [0,1] \subseteq [\max\{1 - \Sigma_{j\in J}1, \Sigma_{k\in K}0\}, \min\{1 - \Sigma_{j\in J}0, \Sigma_{k\in K}1\}] = [0,1]$

If $S = \cup_{k\in K} A_k$ then since $S = \cup_{k\in K} A_k \cup \emptyset$

$$i_m(S) \equiv [1,1] \subseteq \left[\max\left\{1 - \emptyset^l, \Sigma_{k\in K}0\right\}, \min\{1 - \emptyset^u, \Sigma_{k\in K}1\}\right] = [1,1]$$

Thus i_m is an interval-valued probability measure.

Every probability measure is an interval-valued probability measure as we now show.

Example 2 *Let* $\Pr$ *be a probability measure over $(S, \mathcal{A})$. Define $i_m(A) = [\Pr(A), \Pr(A)]$ which is equivalent to having total knowledge about a probability distribution over S. If $(\cup_{k\in K} A_k) \cup (\cup_{j\in J} B_j) = S$ and $A = \cup_{k=1}^{\infty} A_k$, with all sets mutually disjoint then*
$\Pr(\cup_{k\in K} A_k) = \Sigma_{k\in K} a^-(A_k) = 1 - \Pr(\cup_{j\in J} B_j) = 1 - \Sigma_{j\in J} a^+(B_j)$ *and it is clear that*
$i_m(\cup_{k\in K} A_k) = [\max\{1 - \Sigma_{j\in J} a^+(B_j), \Sigma_{k\in K} a^-(A_k)\}, \min\{1 - \Sigma_{j\in J} a^-(B_j), \Sigma_{k\in K} a^+(A_k)\}]$.

A function $p : S \to [0,1]$ is called a regular possibility distribution function if $\sup\{p(x) \mid x \in S\} = 1$. Possibility distribution functions (see [6]) define a possibility measure, $Pos : S \to [0,1]$ where $Pos(A) = \sup\{p(x) \mid x \in A\}$ and it's dual necessity measure $Nec(A) = 1 - Pos(A^c)$ (we define $\sup\{p(x) \mid x \in \emptyset\} = 0$). We can also define a necessity distribution function $n : S \to [0,1]$ by setting $n(x) = 1 - p(x)$ and observe that $Nec(A) = \inf\{n(x) \mid x \in A^c\}$ (we define $\inf\{n(x) \mid x \in \emptyset\} = 1$).We now show that every regular possibility distribution function defines an interval-valued probability measure.

Example 3 *Let* $p : S \rightarrow [0,1]$ *be a regular possibility distribution function and let* Pos *be the associated possibility measure and* Nec *the dual necessity measure. Define* $i_m(A) = [Nec(A), Pos(A)]$ *where* A *is any subset of* S*. First note that* $0 \leq Nec(A) \leq Pos(A) \leq 1$ *which follows from the definition and regularity which requires that* $1 \leq Pos(A) + Pos(A^c)$*. This proves the definition is properly defined. Next observe that since* $i_m(S) = [Nec(S), Pos(S)] = [1,1]$ *and* $i_m(\emptyset) = [Nec(\emptyset), Pos(\emptyset)] = [0,0]$*, the boundary conditions are met. Finally, assume* $\cup_{k\in K} A_k = A$ *and* $(\cup_{k\in K} A_k) \cup (\cup_{i\in J} B_j) = S$ *all sets mutually disjoint. We need to show that condition 3) of the definition is met. This can be done from the following facts:*
Fact 1: $\forall \{A_{k\in K}\}$, $Pos(\cup_{k\in K} A_k) \leq \Sigma_{k\in K} Pos(A_k)$*. First assume* $Pos(\cup_{k\in K} A_k) = \alpha > 0$*. Then* $\sup(p(x) \mid x \in \cup_{k\in K} A_k) = \alpha$ *and this implies that either* $\sup(p(x) \mid x \in A_k) = \alpha$ *for at least one* $k \in I$ *or there exist an infinite subsequence of* A_{i_l} *such that* $\sup(p(x) \mid x \in A_{i_l}) \rightarrow \alpha$ *in which case* $\Sigma_{l=1}^{\infty} Pos(A_{i_l}) = \infty$*. In either case* $Pos(\cup_{k\in K} A_k) \leq \Sigma_{k\in K} Pos(A_k)$*. If* $Pos(\cup_{k\in K} A_k) = 0$ *then the inequality is clear since it is always true that* $Pos(A_k) \geq 0$*.*
Fact 2: $\forall \{A_{k\in K}\}$, $Nec(A_k) \leq Nec(\cup_{k\in K} A_k)$*. Note that* $(\cup_{k\in K} A_k)^c \subseteq (A_k)^c$ *implies that* $Pos((\cup_{k\in K} A_k)^c) \leq Pos((A_k)^c)$ *or* $Nec(A_k) = 1 - Pos((A_k)^c) \leq 1 - Pos((\cup_{k\in K} A_k)^c) = Nec(A)$*.*
Fact 3: If $Pos(\cup_{k\in K} A_k) = 1$ *then* $\forall j \in J$ $Nec(B_j) = 0$ *and for all but at most one* $k' \in K$, $Nec(A_{k'}) = 0$*. The first fact follows since* $Pos((\cup_{i\in J} B_j)^c) = Pos(\cup_{k\in K} A_k) = 1$*. Thus* $0 \leq Nec(B_j) \leq Nec(\cup_{i\in J} B_j) = 1 - Pos((\cup_{i\in J} B_j)^c) = 0$*. For the second assume first* $\exists k'$ *such that* $Pos(A_{k'}) = 1$*. Then* $A_{k'} \subseteq (A_{k\neq k'})^c$ *implies* $Nec(A_{k\neq k'}) = 0$*. On the other hand if* $\nexists$ *such a* k' *then for any* k' $Pos\left(\cup A_{k\in K|k\neq k'}\right) = 1$ *implies* $Nec(A_{k'}) = 0$*.*
We are now in a position to prove that property 3) holds.
Case 1: Assume $Pos(\cup_{k\in K} A_k) < 1$*. Then by regularity it must hold that* $Pos(\cup_{i\in J} B_j) = 1$ *and then fact 1 and 3, followed by 2 give*
$\min\{1 - \Sigma_{j\in J} Nec(B_j), \Sigma_{k\in K} Pos(A_k)\} \geq \min\{1 - Nec(B_k), Pos(A)\} \geq$
$\min\{1 - Nec(\cup_{i\in J} B_j), Pos(A)\} = Pos(A)$
and by fact 1 and 3
$\max\{1 - \Sigma_{j\in J} Pos(B_j), \Sigma_{k\in K} Nec(A_k)\} \leq$
$\max\{1 - Pos(\cup_{i\in J} B_j), 0\} = Nec(A)$
Case 2: Assume $Pos(\cup_{k\in K} A_k) = 1$ *(i.e.* $Pos(A) = 1$*). Then by fact 3*
$\min\{1 - \Sigma_{j\in J} Nec(B_j), \Sigma_{k\in K} Pos(A_k)\} \geq \min\{1, Pos(\cup_{k\in K} A_k)\} = 1$
and by fact 1 and 3 followed by 2
$\max\{1 - \Sigma_{j\in J} Pos(B_j), \Sigma_{k\in K} Nec(A_k)\} \leq \max\{1 - Pos(\cup_{i\in J} B_j), Nec(A_k)\} \leq$

$\max\{1 - Pos(\cup_{i\in J} B_j), Nec(\cup_{k\in K} A_k)\} = Nec(A)$.
This establishes the set inclusion required in property 3) of the definition.

In section 3 we will show that clouds [5] provide a convenient representation for non-trivial interval-valued probability measures. But first we establish some of the properties of interval-valued probability measures.

Proposition 1 *The following relationships hold*
1) $i_m(A^c) = [1 - a^+(A), 1 - a^-(A)] = 1 - i_m(A)$.
2) $A \subset B \Rightarrow i_m(A) \le_I i_m(B)$ *where* $[a,b] \le_I [c,d] \Rightarrow a \le c$ *and* $b \le d$.
3) $\forall \{A_k\}$ *disjoint* $i_m(\cup_{k=1}^{\infty} A_k) \subseteq \Sigma_{k=1}^{\infty} i_m(A_k)$.
4) $i_m(A \cap B) = 1 - i_m(A^c \cup B^c)$, $i_m(A - B) = i_m(A \cap B^c)$.
5) $\forall \cup_{k=1}^{\infty} A_k$ *disjoint,* $\Sigma_{k=1}^{\infty} a^-(A_k) \le 1$.
6) If $\cup_{k=1}^{\infty} A_k = S$, *then* $\Sigma_{k=1}^{\infty} a^+(A_k) \ge 1$.

Proof:
1) $i_m(A^c) = [a^-(A^c), a^+(A^c)] \subseteq [\max\{1 - a^+(A), a^-(A^c)\}, \min\{1 - a^-(A), a^+(A^c)\}]$
$\Rightarrow 1 - a^+(A) \le a^-(A^c)$ and $1 - a^-(A) \ge a^+(A^c)$ but by a similar argument
$1 - a^+(A^c) \le a^-(A)$ and $1 - a^-(A^c) \ge a^+(A)$ which taken together
$\Rightarrow i_m(A^c) = [1 - a^+(A), 1 - a^-(A)] = 1 - i_m(A)$

2) $A \subset B \Rightarrow S = B^c \cup A \cup (B - A) \Rightarrow$
$i_m(A \cup (B - A)) = [a^-(B), a^+(B)] \subseteq$
$[\max\{1 - a^+(B^c), a^-(A) + a^-(B - A)\}, \min\{1 - a^-(B^c), a^+(A) + a^+(B - A)\}]$
$= [\max\{a^-(B), a^-(A) + a^-(B - A)\}, \min\{a^+(B), a^+(A) + a^+(B - A)\}]$
(from property 1)
$\Rightarrow a^-(A) + a^-(B - A) \le a^-(B) \Rightarrow a^-(A) \le a^-(B)$
similarly since $B^c \subset A^c$
$\Rightarrow a^-(B^c) + a^-(A^c - B^c) \le a^-(A^c)$ and $a^+(A^c) \le a^+(B^c) + a^+(A^c - B^c)$
$\Rightarrow 1 - a^+(B) + a^-(A^c - B^c) \le 1 - a^+(A) \Rightarrow a^+(A) + a^-(A^c - B^c) \le a^+(B) \Rightarrow a^+(A) \le a^+(B)$

3) This is clear from property 3) of the definition.

4) $i_m(A^c \cup B^c) = i_m((A \cap B)^c) = 1 - i_m(A \cap B)$ (property 1)
$\Rightarrow i_m(A \cap B) = 1 - i_m(A^c \cup B^c)$ since $A^c \cup B^c = (A \cap B)^c$ then
$i_m(A - B) = i_m(A \cap B^c) = 1 - i_m(A^c \cup B)$ since $A - B = A \cap B^c$
and the identity $A \triangle B = (A \cup B) - (A \cap B)$ gives the final equation.

5) Since
$i_m\left(\cup_{k=1}^{\infty}A_k\right) \subseteq \left[\max\left\{1-\Sigma_{j=1}^{\infty}a^+\left(B_j\right), \Sigma_{k=1}^{\infty}a^-\left(A_k\right)\right\}, \min\left\{1-\Sigma_{j=1}^{\infty}a^-\left(B_j\right), \Sigma_{k=1}^{\infty}a^+\left(A_k\right)\right\}\right] \subseteq$
$[0,1] \Rightarrow$
$\max\left\{1-\Sigma_{j=1}^{\infty}a^+\left(B_j\right), \Sigma_{k=1}^{\infty}a^-\left(A_k\right)\right\} \leq 1.$

6) This follows since $\cup_{k=1}^{\infty}A_k \cup \emptyset = S$ requires that $\min\left\{1-0, \Sigma_{k=1}^{\infty}a^+\left(A_k\right)\right\} =$
$1.\square$

Note that this definition of an interval-valued probability measure does not lead to a contribution. For example, if $\cup_{k\in K}A_k = S$, i.e. $\cup_{k\in K}A_k \cup \emptyset = S$ then
$i_m\left(\cup_{k=1}^{\infty}A_k\right) \subseteq \left[\max\left\{1-\emptyset^u, \Sigma_{k=1}^{\infty}a^-\left(A_k\right)\right\}, \min\left\{1-\emptyset^l, \Sigma_{k=1}^{\infty}a^+\left(A_k\right)\right\}\right] = [1,1]$
and $i_m\left(\emptyset\right) \subseteq \left[\max\left\{1-\Sigma_{k=1}^{\infty}a^+\left(A_k\right), 0\right\}, \min\left\{1-\Sigma_{k=1}^{\infty}a^-\left(A_k\right), 0\right\}\right] = [0,0]$ by 5) of the proposition. This supports the consistency of statement 4) and statements 1) and 2) of the definition.

Next since $\max\left\{1-\Sigma_{j=1}^{\infty}a^+\left(B_j\right), \Sigma_{k=1}^{\infty}a^-\left(A_k\right)\right\} \leq \min\left\{1-\Sigma_{j=1}^{\infty}a^-\left(B_j\right), \Sigma_{k=1}^{\infty}a^+\left(A_k\right)\right\}$ and $\Sigma_{k=1}^{\infty}a^-\left(A_k\right) \leq \Sigma_{k=1}^{\infty}a^+\left(A_k\right)$ and $1-\Sigma_{j=1}^{\infty}a^+\left(B_j\right) \leq 1-\Sigma_{j=1}^{\infty}a^-\left(B_j\right)$ by definition of i_m, it must hold that $1-\Sigma_{j=1}^{\infty}a^+\left(B_j\right) \leq \Sigma_{k=1}^{\infty}a^+\left(A_k\right)$ and $\Sigma_{k=1}^{\infty}a^-\left(A_k\right) \leq 1-\Sigma_{j=1}^{\infty}a^-\left(B_j\right)$. But these follow from 5) of the proposition, since $\Sigma_{k=1}^{\infty}a^+\left(A_k\right)+\Sigma_{j=1}^{\infty}a^+\left(B_j\right) \geq 1$ and $\Sigma_{k=1}^{\infty}a^-\left(A_k\right)+\Sigma_{j=1}^{\infty}a^-\left(B_j\right) \leq 1$. Thus statement 4) of the definition is internally consistent.

Now by definition of i_m, $\Sigma_{k=1}^{\infty}a^-\left(A_k\right) \geq 0$ implies $\max\left\{1-\Sigma_{j=1}^{\infty}a^+\left(B_j\right), \Sigma_{k=1}^{\infty}a^-\left(A_k\right)\right\} \geq$ 0 and by 5) of the proposition $\Sigma_{j=1}^{\infty}a^-\left(B_j\right) \leq 1$ gives $\min\left\{1-\Sigma_{j=1}^{\infty}a^-\left(B_j\right), \Sigma_{k=1}^{\infty}a^+\left(A_k\right)\right\} \leq$ 1 thus $i_m\left(\cup_{k=1}^{\infty}A_k\right) \subseteq [0,1]$ which shows that statement 3) of the definition is consistent, i.e. $i_m : \mathcal{A} \rightarrow Int_{[0,1]}$.

Finally, note that
$[1,1] = i_m\left(A \cup A^c\right) = \left[\max\left\{1-\emptyset^u, a^-\left(A\right)+a^-\left(A^c\right)\right\}, \min\left\{1-\emptyset^l, a^+\left(A\right)+a^+\left(A^c\right)\right\}\right] =$
$\left[\max\left\{1, a^-\left(A\right)+1-a^+\left(A\right)\right\}, \min\left\{1, a^+\left(A\right)+1-a^-\left(A\right)\right\}\right] = [1,1] \Rightarrow$
$a^+\left(A\right)+1-a^-\left(A\right) \geq 1 \Rightarrow a^+\left(A\right) \geq a^-\left(A\right)$ and $a^-\left(A\right)+1-a^+\left(A\right) \leq 1 \Rightarrow$
$a^+\left(A\right) \geq a^-\left(A\right)$
so definitions 1), 2) and 4) in combination are consistent with statement 3) of the definition.$\square$

2.2 Integration

In this section we briefly explore how the definition of an interval-valued probability measure might be applied to define an interval-valued integral. This suggests how the definition might be applied. This definition will be explored in more detail in future work.Given an interval-valued probability measure, i_m, defined on measurable space $(S, \mathcal{A})$ and an integrable function $f : S \to R$ we define integration in the sense of Lebesgue as follows:

First if f is an $\mathcal{A}-$measurable function such that $f(x) = \left\{ \begin{array}{ll} y & x \in A \\ 0 & x \notin A \end{array} \right.$ with $A \in \mathcal{A}$, we define in the natural way

$$\int_A f(x)\, di_m = y i_m(A)$$

Now suppose f only takes values $\{y_k \mid k \in K\}$ on disjoint measurable sets $\{A_k \mid k \in K\}$ that is, $f(x) = \left\{ \begin{array}{ll} y_k & x \in A_k \\ 0 & x \notin A \end{array} \right|$ where $A = \cup_{k \in K} A_k$. The maximum probability in A is $a^+(A)$ Thus, the maximum possible value for the integral is

$$a^+\left(\int_A f(x)\, di_m\right) = \sup\left\{\Sigma_{k\in K} y_k a_k \mid \Sigma_{k \in K} a_k = a^+(A), a^-(A_k) \le a_k \le a^+(A_k)\right\}$$

Similarly, the minimum possible value is

$$a^-\left(\int_A f(x)\, di_m\right) = \inf\left\{\Sigma_{k\in K} y_k a_k \mid \Sigma_{k \in K} a_k = a^-(A), a^-(A_k) \le a_k \le a^+(A_k)\right\}$$

Then we define

$$\int_A f(x)\, di_m = \left[a^-\left(\int_A f(x)\, di_m\right), a^+\left(\int_A f(x)\, di_m\right)\right]$$

Note that $\{\Sigma_{k\in K} y_k a_k \mid a^-(A_k) \le a_k \le a^+(A_k)\} = \left[a^-\left(\int_A f(x)\, di_m\right), a^+\left(\int_A f(x)\, di_m\right)\right]$ so the definition intuitively makes sense in that it is equal to the set of all possible values of the unknown probability measure being modeled.

In general, assume f is an integrable function and $\{f_k\}$ is a sequence of simple functions converging uniformly to f. Then we define

$$\int_A f(x)\, di_m = \lim_{k\to\infty} \int_A f_k(x)\, di_m$$

Where $\lim_{k\to\infty} \int_A f_k(x)\, di_m = \left[\lim_{k\to\infty} a^-\left(\int_A f_k(x)\, di_m\right), \lim_{k\to\infty} a^+\left(\int_A f_k(x)\, di_m\right)\right]$ provided the limits exist.

We will examine the properties of this integral to a future paper.

3 Continuously Increasing Clouds on R as Generator Functions

In this section we define clouds and show that continuously increasing clouds on the real line define an interval-valued probability measure on the Borel sets. This gives the most general setting for intervval-valued probability measures.

The concept of a cloud was introduced by Neumaier in [5] as follows:

Definition 2 *A cloud over set S is a mapping c such that:*
1) $\forall s \in S,\ c(s) = [n(s), p(s)]$ *with* $0 \leq n(s) \leq p(s) \leq 1$
2) $(0,1) \subseteq \cup_{s \in S} c(s) \subseteq [0,1]$
In addition, random variable X taking values in S is said to belong to cloud c (written $X \in c$) iff
3) $\forall \alpha \in [0,1],\ \Pr(n(X) \geq \alpha) \leq 1 - \alpha \leq \Pr(p(X) > \alpha)$

In [1] we showed that possibility distributions could be constructed which satisfy the following consistency definition.

Definition 3 *Let $p : S \to [0,1]$ be a regular possibility distribution function with associated possibility measure Pos and necessity measure Nec. Then p is said to be* **consistent** *with random variable X if $\forall$ measurable sets A, $Nec(A) \leq \Pr(X \in A) \leq Pos(A)$.*

For our purposes we will combine these two notions and use the following equivalent representation for a cloud.

Proposition 2 *Let $\bar{p}$, $\underline{p}$ be a pair of regular possibility distribution functions over set S such that $\forall s \in S$ $\bar{p}(s) + \underline{p}(s) \geq 1$. Then the mapping $c(s) = [\underline{n}(s), \bar{p}(s)]$ where $\underline{n}(s) = 1 - \underline{p}(s)$ (i.e. the dual necessity distribution function) is a cloud. In addition, if X is a random variable taking values in S and the possibility measures associated with $\bar{p}$, $\underline{p}$ are consistent with X then X belongs to cloud c. Conversely, every cloud defines such a pair of possibility distribution functions and their associated possibility measures are consistent with every random variable belonging to c.*

Proof:
$\Rightarrow$1) $\bar{p}$, $\underline{p}$: $S \to [0,1]$ and $\bar{p}(s) + \underline{p}(s) \geq 1$ imply property 1) of definition 2

2) Since all regular possibility distributions satisfy $\sup \{p(s) \mid s \in S\} = 1$ property 2) of definition 2 holds.
Therefore c is a cloud. Now assume consistency. Then
$\alpha \geq Pos \{s \mid p(s) \leq \alpha\} \geq \Pr \{s \mid p(s) \leq \alpha\} = 1 - \Pr \{s \mid p(s) > \alpha\}$ gives the right-hand side of the required inequalities and
$1-\alpha \geq Pos \{s \mid p(s) \leq 1-\alpha\} \geq \Pr \{s \mid p(s) \leq 1-\alpha\} = \Pr \{s \mid 1-p(s) \geq \alpha\} = \Pr (\underline{n}(X) \geq \alpha)$ gives the left-hand side.
The opposite identity was proven in section 5 of [5].$\square$.

Assume the cloud $c : R \to Int_{[0,1]}$ (with $c(x) = [n(x), p(x)]$) satisfies the property that each function n,$p : R \to [0,1]$ is continuously increasing with $\lim_{x\to\infty} n(x), p(x) = 1$ and $\lim_{x\to-\infty} n(x), p(x) = 0$. Assume random variable $X \in c$ and for any Borel set A, let $\Pr(A) = \Pr(X \in A)$.

We begin by developing probability bounds for members of the family of sets

$$\mathcal{I} = \{(a,b], (-\infty,a], (a,\infty), (-\infty,\infty), \emptyset \mid a < b\}$$

For $I = (-\infty, b]$ the consistency of n, p with X gives

$$\Pr(I) \in [\inf \{n(x) \mid x \in I^c\}, \sup \{p(x) \mid x \in I\}] = [n(b), p(b)]$$

which gives, for $I = (a, \infty)$ (since $I^c = (-\infty, a]$)

$$\Pr(I) \in [1 - p(a), 1 - n(a)]$$

and for $I = (a,b]$, since $I = (-\infty, b] - (-\infty, a]$ and considering minimum and maximum probabilities in each set,

$$\Pr(I) \in [\max \{n(b) - p(a), 0\}, p(b) - n(a)].$$

Therefore, if we extend the definition of p, n by defining $p(-\infty) = n(-\infty) = 0$ and $p(\infty) = n(\infty) = 1$, we can make the following general definition.

Definition 4 *For any $I \in \mathcal{I}$, if $I \neq \emptyset$, define*

$$i_m(I) = \left[a^-(I), a^+(I)\right] = [\max \{n(b) - p(a), 0\}, p(b) - n(a)]$$

where a and b are the left and right endpoints of I otherwise set

$$i_m(\emptyset) = [0,0]$$

Remark 1 *Note that with this definition* $i_m((-\infty,\infty)) = [\max\{n(\infty) - p(-\infty), 0\}, p(\infty) - n(-\infty)] = [1,1]$ *which matches our intuition and thus it is easy to see that* $\Pr(I) \in i_m(I)\ \forall I \in \mathcal{I}$.

We can extend this to include finite unions of elements of $\mathcal{I}$. For example if $E = I_1 \cup I_2 = (a,b] \cup (c,d]$ with $b < c$, then we consider the probabilities $\Pr((a,b]) + \Pr((c,d])$ and $1 - (\Pr((-\infty,a]) + \Pr((b,c]) + \Pr((d,\infty)))$ (the probability of the sets that make up E versus one less the probability of the intervals that make up the complement),and consider the minimum and maximum probability for each case as a function of the minimum and maximum of each set. The minimum for the first sum is $\max(0, n(d) - p(c)) + \max(0, n(b) - p(a))$ and the maximum is $p(d) - n(c) + p(b) - n(a)$. The minimum for the second is $1 - (p(\infty) - n(d) + p(c) - n(b) + p(a) - n(-\infty)) = n(d) - p(c) + n(b) - p(a)$ and the maximum is $1 - (\max(0, n(\infty) - p(d)) + \max(0, n(c) - p(b)) + \max(0, n(a) - p(-\infty))) = p(d) - \max(0, n(c) - p(b)) - n(a)$.
This gives
$p(E) \geq \max\left\{\begin{array}{l} n(d) - p(c) + n(b) - p(a) \\ \max(0, n(d) - p(c)) + \max(0, n(b) - p(a)) \end{array}\right.$ and
$p(E) \leq \min\left\{\begin{array}{l} p(d) - \max(0, n(c) - p(b)) - n(a) \\ p(d) - n(c) + p(b) - n(a) \end{array}\right.$
so $P(E) \in [\max(0, n(d) - p(c)) + \max(0, n(b) - p(a)), p(d) - \max(0, n(c) - p(b)) - n(a)]$
where the final line is arrived at by noting that
$\forall x, y\ n(x) - p(y) \leq \max(0, n(x) - p(y))$.

Remark 2 *Note the two extreme cases for* $E = (a,b] \cup (c,d]$. *For* $p(x) = n(x)\ \forall x$, *what Neumaier calls a thin cloud, then, as expected,*

$$\Pr(E) = p(d) - p(c) + p(b) - p(a) = \Pr((a,b]) + \Pr((c,d])$$

i.e. it is a probability measure. Moreover, for $n(x) = 0\ \forall x$,

$$P(E) \in [0, p(d)].$$

Thus what Neumaier calls a fuzzy cloud yields the same result as for example 3 since for this E, $Nec(E) = 0$ ($\exists x \in [b,c) \subseteq E^c$ *and* $n(x) = 0$) *i.e. the cloud defines a possibility measure.*

Let

$$\mathcal{E} = \left\{ \cup_{k=1}^{K} I_k \mid I_k \in \mathcal{I} \right\}$$

i.e. $\mathcal{E}$ is the algebra of sets generated by I. Note that every element of E has a unique representation as a union of the minimum number of elements of $\mathcal{I}$ (or, stated differently, as a union of disconnected elements of $\mathcal{I}$). Note also that $R \in \mathcal{E}$ and $\mathcal{E}$ is closed under complements.

Assume $E = \cup_{k=1}^{K} I_k$ and $E^c = \cup_{j=1}^{J} M_j$ are the unique representations of E and E^c in $\mathcal{E}$ in terms of elements of $\mathcal{I}$. Then, considering minimum and maximum possible probabilities of each interval it is clear that

$$\Pr(E) \in \left[\max\left(\Sigma_{k=1}^{K} a^-(I_k), 1 - \Sigma_{j=1}^{J} a^+(M_j)\right), \min\left(\Sigma_{k=1}^{K} a^+(I_k), 1 - \Sigma_{j=1}^{J} a^-(M_j)\right)\right].$$

This can be made more concise using the following result.

Proposition 3 *If $E = \cup_{k=1}^{K} I_k$ and $E^c = \cup_{j=1}^{J} M_j$ are the unique representations of E and $E^c \in \mathcal{E}$ then $\Sigma_{k=1}^{K} a^-(I_k) \geq 1 - \Sigma_{j=1}^{J} a^+(M_j)$ and $\Sigma_{k=1}^{K} a^+(I_k) \geq 1 - \Sigma_{j=1}^{J} a^-(M_j)$.*

Proof:
We need only prove $\Sigma_{k=1}^{K} a^-(I_k) \geq 1 - \Sigma_{j=1}^{J} a^+(M_j)$ since we can exchange the roles of E and E^c giving $\Sigma_{j=1}^{J} a^-(M_j) \geq 1 - \Sigma_{k=1}^{K} a^+(I_k)$ proving the second inequality.
Note $\Sigma_{k=1}^{K} a^-(I_k) + \Sigma_{j=1}^{J} a^+(M_j)$ is of the form $\Sigma_{k=1}^{K} \max(0, n(b_k) - p(a_k)) + \Sigma_{j=1}^{J} p(a_{j+1}) - n(b_j) \geq \Sigma_{k=1}^{K} (n(b_k) - p(a_k)) + \Sigma_{j=1}^{J} p(a_{j+1}) - n(b_j)$. Since the union of the disjoint intervals yields all of the real line we have either $p(\infty)$ or $n(\infty)$ less either $p(-\infty)$ or $n(-\infty)$ which is one regardless.$\square$

Next i_m is extended to $\mathcal{E}$.

Proposition 4 *For any $E \in \mathcal{E}$ let $E = \cup_{k=1}^{K} I_k$ and $E^c = \cup_{j=1}^{J} M_j$ be the unique representations of E and E^c in terms of elements of $\mathcal{I}$ respectively. If*

$$i_m(E) = \left[\Sigma_{k=1}^{K} a^-(I_k), 1 - \Sigma_{j=1}^{J} a^-(M_j)\right]$$

then $i_m : \mathcal{E} \to Int_{[0,1]}$, is an extension of $\mathcal{I}$ to $\mathcal{E}$ and is well defined.

Proof:
First assume $E = (a, b] \in \mathcal{I}$, then $E^c = (-\infty, a] \cup (b, \infty)$ so by the definition
$i_m(E) =$
$[\max\{n(b) - p(a), 0\}, 1 - (\max\{n(a) - p(-\infty), 0\} + \max\{n(\infty) - p(b), 0\})]$

$= 1 - (n(a) + (1 - p(a))) = p(a) - n(a)$ which matches the definition for i_m on $\mathcal{I}$. The other cases for $E \in \mathcal{I}$ are similar. Thus it is an extension.
Well defined is easy, since the representation of any element in $\mathcal{E}$ in terms of the minimum number of elements of $\mathcal{I}$ is unique. In addition it is clear that $0 \leq \Sigma_{k=1}^{K} a^{-}(I_k)$ and $1 - \Sigma_{j=1}^{J} a^{-}(M_j) \leq 1$. So we only need to show that $\Sigma_{k=1}^{K} a^{-}(I_k) \leq 1 - \Sigma_{j=1}^{J} a^{-}(M_j)$. i.e. that $\Sigma_{k=1}^{K} a^{-}(I_k) + \Sigma_{j=1}^{J} a^{-}(M_j) \leq 1$. If we relabel the endpoints of all these intervals as $-\infty = c_1 < c_2 ... < c_N = \infty$ then
$\Sigma_{k=1}^{K} a^{-}(I_k) + \Sigma_{j=1}^{J} a^{-}(M_j) = \Sigma_{n=1}^{N-1} \max\{n(c_{n+1}) - p(c_n), 0\} \leq$
$\Sigma_{n=1}^{N-1} \max\{p(c_{n+1}) - p(c_n), 0\} = \Sigma_{n=1}^{N-1} p(c_{n+1}) - p(c_n) = 1$. Thus $\Sigma_{k=1}^{K} a^{-}(I_k) + \Sigma_{j=1}^{J} a^{-}(M_j) \leq 1.\square$

Proposition 5 *The set function* $i_m : \mathcal{E} \to Int_{[0,1]}$ *satisfies the following properties:*
(1) If $E \subseteq F$ *then* $i_m(E) \leq_I i_m(F)$
(2) If $E = \cup_{k=1}^{K} E_k$ *with the* E_k *mutually disjoint then* $i_m(E) \subseteq \left[\Sigma_{k=1}^{K} a^{-}(E_k), \Sigma_{k=1}^{K} E_i^{u}\right]$
(3) $i_m(E) = 1 - i_m(E^c)$

Proof
(1) Assume $E \subseteq F$ are two sets in $\mathcal{E}$ with $E = \cup_{k=1}^{K} I_k$ and $F = \cup_{j=1}^{J} M_j$ being the unique representations as elements of $\mathcal{I}$. Then $\forall k \exists j$ such that $I_k \subseteq M_j$. Assume the left and right endpoints are a_k, b_k, c_j, d_j respectively Then $c_j \leq a_k \leq b_k \leq d_j$., the monotonicity of n and p and since $\forall x,\ n(x) \leq p(x)$ combine to give $\max\{n(b_k) - p(a_k), 0\} \leq \max\{n(d_j) - p(c_j), 0\}$ which gives $\Sigma_{k=1}^{K} a^{-}(I_k) \leq \Sigma_{j=1}^{J} a^{-}(M_j)$ thus $a^{-}(E) \leq a^{-}(F)$. The inequality in terms of upper bounds is derived by the same argument applied to the representations for E^c and F^c which combined proves $i_m(E) \leq_I i_m(F)$
(2) First note that if $E = (a, c]$, $E_1 = (a, b]$ and $E_2 = (b, c]$ so that $E = E_1 \cup E_2$ then since
$n(c) - p(a) \geq n(b) - p(a)$ and $n(c) - p(a) \geq n(c) - p(b)$ and
$n(c) - p(a) = n(c) - p(b) + p(b) - p(a) \geq n(c) - p(b) + n(b) - p(a)$ then
$\max\{n(c) - p(a), 0\} \geq \max\{n(b) - p(a), 0\} + \max\{n(c) - p(b), 0\}$ so $a^{-}(E) \geq a^{-}(E_1) + a^{-}(E_2)$ and
$p(c) - n(a) = p(c) - n(b) + n(b) - n(a) \leq p(c) - n(b) + p(b) - n(a)$ gives $a^{+}(E) \leq a^{+}(E_1) + a^{+}(E_2)$
so that $i_m(E) \subseteq [a^{-}(E_1) + a^{-}(E_2), a^{+}(E_1) + a^{+}(E_2)]$
Next let $E = E_1 \cup E_2$ where $E_1 = (a, b]$ and $E_2 = (c, d]$ with $b < c$ then since $a^{-}(E) = \max\{n(b) - p(a), 0\} + \max\{n(d) - p(c), 0\} = a^{-}(E_1) + a^{-}(E_2)$

and
$a^+(E) = p(d) - \max(0, n(c) - p(b)) - n(a) \leq p(d) - (n(c) - p(b)) - n(a) = a^+(E_1) + a^+(E_2)$ giving
$i_m(E) \subseteq [a^-(E_1) + a^-(E_2), a^+(E_1) + a^+(E_2)]$ the proposition statement is just an extension of these two facts.
(3) $[1,1] - \left[\Sigma_{k=1}^{K} a^-(I_k), 1 - \Sigma_{j=1}^{J} a^-(M_j)\right] = \left[\Sigma_{j=1}^{J} a^-(M_j), 1 - \Sigma_{k=1}^{K} a^-(I_k)\right] = i_m(E^c)$.□

The family of sets,$\mathcal{E}$,is a ring of sets generating the Borel sets $\mathcal{B}$. For an arbitrary Borel set S, then it is clear that

$$\Pr(S) \in \left[\sup\left\{a^-(E) \mid E \subseteq S, E \in \mathcal{E}\right\}, \inf\left\{a^+(F) \mid S \subseteq F, F \in \mathcal{E}\right\}\right]$$

This prompts the following:

Proposition 6 *Let* $i_m :.\mathcal{B} \rightarrow [0,1]$ *be defined by*

$$i_m(A) = \left[\sup\left\{a^-(E) \mid E \subseteq A, E \in \mathcal{E}\right\}, \inf\left\{a^+(F) \mid A \subseteq F, F \in \mathcal{E}\right\}\right] \quad (1)$$

The i_m *is an extension from* $\mathcal{E}$ *to* $\mathcal{B}$ *and is well-defined.*

Proof:
Property (1) of proposition 5 insures it is an extension since, for example, if $E \subseteq F$ are two elements of $\mathcal{E}$ then $a^+(E) \leq a^+(F)$ so $\inf\{a^+(F) \mid E \subseteq F, F \in \mathcal{E}\} = a^+(E)$ similarly it ensures that $\sup\{a^-(F) \mid F \subseteq E, F \in \mathcal{E}\} = a^-(E)$.
Next we show that i_m is well defined. In proposition 4 we showed that $\forall E \in \mathcal{E}$, $i_m(E) \subseteq [0,1]$. So $0 \leq \sup\{a^-(E) \mid E \subseteq S\}$ and $\inf\{a^+(E) \mid S \subseteq E\} \leq 1$. But $\sup\{a^-(E) \mid E \subseteq S\} \leq \inf\{a^+(F) \mid S \subseteq F\}$.follows by (1) of proposition 5.□

Proposition 7 $\forall A \in \mathcal{B}$. $i_m(A^c) = [1,1] - i_m(A)$

Proof:
Since for all $E \in \mathcal{E}$ property (3) of proposition 5 holds, $\mathcal{E}$ is closed under complements and $E \subseteq A \Longleftrightarrow A^c \subseteq E^c$, we have
$a^-(A) = \sup\{a^-(E) \mid E \subseteq A, E \in \mathcal{E}\} = \inf\{1 - a^-(E) \mid E \subseteq A, E \in \mathcal{E}\} = \inf\{a^+(E) \mid A^c \subseteq E, E \in \mathcal{E}\} = 1 - (A^c)^u$.□

Theorem 8 *The function* $i_m : \mathcal{B} \rightarrow Int_{[o,1]}$ *is an interval-valued probability measure provided we define* $\inf \emptyset = 1$ *and* $\sup \emptyset = 0$.

Proof:
(1) Consider elementary sets $[-b-1,-b]$. Then $\lim_{b\to\infty}[p(-b)-n(-b-1)] = 0 \Rightarrow \inf\{a^+(E) \mid \emptyset \subseteq E\} = 0$. Also, since there are no elementary sets with $E \subseteq \emptyset$, $\sup\{a^-(E) \mid E \subseteq \emptyset\} = \sup\emptyset = 0$. Combined this gives $i_m(\emptyset) = [0,0]$
(2) Consider elementary sets $[-b,b]$. Then $\lim_{b\to\infty}\max\{n(b)-p(-b),0\} = 1 \Rightarrow \sup\{a^-(E) \mid E \subseteq R\} = 1$. Also, since there are no elementary sets with $R \subseteq E$, $\inf\{a^+(E) \mid R \subseteq E\} = \inf\emptyset = 1$. Combined give $i_m(R) = [1,1]$
(3) Assume $S = \cup_{k\in K}A_k$ and $(\cup_{k\in K}A_k) \cup (\cup_{i\in J}B_j) = R$ with all set A_k, B_j being mutually disjoint and all sets being Borel sets. We need to show that
$i_m(A) \subseteq [\max\{1-\Sigma_{j\in J}a^+(B_j), \Sigma_{k\in K}a^-(A_k)\}, \min\{1-\Sigma_{j\in J}a^-(B_j), \Sigma_{k\in K}a^+(A_k)\}]$
First we show that $a^-(A) \geq \Sigma_{k\in K}a^-(A_k)$ and $a^+(A) \leq \Sigma_{k\in K}a^+(A_k)$.
For each A_k, we can choose an $E_k \in \mathcal{E}$ such that $E_k \subseteq A_k$ and $a^-(E_k)$ is arbitrarily close to $a^-(A_k)$ so $\forall\epsilon > 0 \exists\{E_k \mid i = 1, n\}$ such that $0 \leq \Sigma_{k\in K}a^-(A_k) - \Sigma_{k=1}^K a^-(E_k) < \epsilon$. Also $\cup_{k=1}^K E_k \subseteq A$ giving (using proposition 5) $\Sigma_{k=1}^K a^-(E_k) \leq a^-\left(\cup_{k=1}^K E_k\right) \leq a^-(A)$. This proves $a^-(A) \geq \Sigma_{k\in K}a^-(A_k)$. A similar argument gives the inequality $a^+(A) \leq \Sigma_{k\in K}a^+(A_k)$.
$1-\Sigma_{j\in J}a^+(B_j) \leq a^-(A)$
To obtain the inequality $1-\Sigma_{j\in J}a^+(B_j) \leq a^-(A)$ we need only observe that this is equivalent to $1-a^-(A) \leq \Sigma_{j\in J}a^+(B_j)$ which follows from the first inequality and proposition 7 since $a^+(A^c) = 1-a^-(A)$. The inequality $a^+(A) \leq 1-\Sigma_{j\in J}a^-(B_j)$ is proven in a similar manner.□

Finally we have the following result that shows that the interval-valued probability measure, i_m, is the best possible for the cloud that generated it.

Theorem 9 *Let c, i_m be a continuously increasing cloud and it's generated interval-valued probability measure. If X is a random variable, then X belonging to cloud c if and only if $\Pr(X) \in i_m(A)$ for all measurable A.*

Proof:
$\Rightarrow$ If X belongs to c then $\Pr(X) \in i_m(I)$ for all $I \in \mathcal{I}$ by construction and the consistency of n and p with X. Then for $E = \cup_{k=1}^K I_k$ and $E^c = \cup_{j=1}^J M_j$ with the I_k, M_j disjoint elements of $\mathcal{I}$, by the summability of a probability measure it must hold that $\Pr(E) = \Sigma_{k=1}^K \Pr(I_k) = 1-\Sigma_{j=1}^J \Pr(M_j)$ which gives $\Pr(E) \geq \Sigma_{k=1}^K a^-(I_k)$ and $\Pr(E) \leq 1-\Sigma_{j=1}^J a^-(M_j)$ thus $\Pr(E) \in i_m(E)$ consequently $\Pr(X) \in i_m(E)$ for all $E \in \mathcal{E}$. Finally, if $A \subseteq E$ with $E \in \mathcal{E}$ then $\Pr(A) \leq \Pr(E)$ thus $\Pr(A) \leq \sup\{a^-(E) \mid S \subseteq E, E \in \mathcal{E}\} = a^+(A)$ similarly $\Pr(A) \geq a^-(A)$ giving $\Pr(X) \in i_m(A)$,
$\Leftarrow$ Let $\Pr(X) \in i_m(A)$ for all measurable A. We need to show that n, p are

consistent with X which by proposition 2 means X belongs to c. Let S be an arbitrary subset of R and let $\sup\{x \in S\} = a$. Since p is continuously increasing we know that $Pos(S) = \sup\{p(x) \mid x \in S\} = p(a)$ then since $\Pr((-\infty, a]) \in i_m((-\infty, a]) = [0, p(a)]$ we know that $\Pr(S) \leq \Pr(A) \leq p(a) = Pos(S)$ showing that X is consistent with p. A similar argument shows the consistency with n and thus X belongs to c.$\square$

4 Conclusion

The definition of an interval-valued probability measure provides a formal setting in which the concepts of probability, possibility and clouds can be combined. We have shown how such measures can be constructed from the simple setting of a cloud which lays the groundwork for development of approximation techniques. Future research will focus on the applications of such measures to problems in optimization in which uncertainty can not be fully captured by probability alone. We intend to extend the results in [3] which provided an approximation technique for probability measures via inner and outer measures to approximation techniques for functions of cloudy random variables.

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