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Fuzzy Sets in Foundations of Quantum Mechanics

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80.1 Introduction

Three times in my scientific life I fell in love. For the first time, in the middle of the Seventies of the XX Century the object of my love was (and still is, since real love never ends) the theory of quantum logics. Quantum logics are mathematical structures that are encountered in the very foundations of quantum mechanics. Formally, they are order-theoretic structures more general than Boolean algebras: orthomodular partially ordered sets or lattices, that are believed to represent properties of quantum objects in the same way as Boolean algebras represent properties of objects that conform to laws of classical physics.

The object of my second scientific love was fuzzy set theory. From the very beginning of my acquaintance with this theory the very idea of a set the boundaries of which are not sharp but vanish gradually was for me so visible and beautiful that I could not resist. Specially, that very soon it occurred that the objects of my first and second scientific love intertwine or, more precisely, the second one embraces the first. But this needs more detailed explanation.

80.2 Mączyński's Functions and Giles Weakly Disjoint Sets

Professor Maciej Mączyński, the supervisor of my PhD Thesis, proved in 1973 [1] that any quantum logic possessing so called ordering set of probability measures (only such quantum logics are interesting from the physical point of view!) can be isomorphically represented as a family of $[0,1]$-valued functions L such that:

a) 0 (the null function) belongs to L,
b) if f belongs to L, then $1-f$ also belongs to L,
c) for any (finite or countable) sequence f_i of functions that belong to L such that $f_i + f_j \leq 1$ (such functions were called in [1] *pairwise orthogonal*), pointwise algebraic sum of these functions belongs to L,

and, conversely, any family of functions fulfilling conditions a) - c) is a quantum logic in the traditional, order-theoretic sense.

Obviously, I was well acquainted with Mączyński's Functional Representation Theorem, so as soon as I learned the rudiments of fuzzy set theory I noticed that Mączyński's functions could be treated as membership functions of fuzzy sets and that out of his three conditions that characterize any quantum logic in its functional

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form, the first two conditions could be in a straightforward way expressed in the language of fuzzy set theory. Some time later I noticed that also a part of the third of Mączyński's conditions is easily expressible in the language of fuzzy sets: pairwise orthogonality of functions defined by the condition $f_i + f_j \leq 1$ is in fact equivalent to the condition: $f_i \cap f_j = 0$, where $\cap$ denotes intersection of fuzzy sets defined by $f \cap g = max(f + g - 1, 0)$ This intersection was called *bold intersection* by Giles [2] who was the first to study it in the domain of fuzzy sets. Nowadays it is usually called *Łukasiewicz intersection*. Since Giles called in [2] *weakly disjoint* two fuzzy sets f and g such that $f \cap g = 0$, Mączyński's conditions can be expressed as follows:

a') the empty set belongs to L,
b') if f belongs to L, then its fuzzy complement also belongs to L,
c') for any (finite or countable) sequence of pairwise weakly disjoint sets that belong to L, their pointwise algebraic sum also belongs to L.

As we can see, after such reformulation only the second part of the third condition is not expressed by standard fuzzy set operations. Therefore, my clear aim was to replace algebraic sum that appeared in the condition c') by a "genuine" fuzzy set operation. In the meantime, the results stated above were announced at the Second International Fuzzy Systems Association Congress held in Tokyo in July 1987 [3].

80.3 Finally, Only Łukasiewicz Operations

Expressing quantum logic entirely in terms of "genuine" fuzzy set operations was not an easy task and it took me a couple of years before I made it. Finally, it occurred that it is De Morgan triple consisting of the standard fuzzy complement and Łukasiewicz operations (union being defined via De Morgan law: $f \& g = (f' \cap g')' = min(f + g, 1))$ that solves the problem. However, it occurred that in order to replace pointwise algebraic sum appearing in the condition c') by Łukasiewicz union, it was necessary to add one more, fortunately very natural, condition. The result, announced for the first time in [4], was as follows:

Any quantum logic with an ordering set of probability measures can be isomorphically represented as a family L of fuzzy sets such that:

a") the empty set belongs to L,
b") if f belongs to L, then its fuzzy complement also belongs to L,
c") for any (finite or countable) sequence of pairwise weakly disjoint sets that belong to L, their Łukasiewicz union also belongs to L.
d") the empty set is the only set in L that is weakly disjoint with itself,

and, conversely, any family of fuzzy sets satisfying a") - d") is a quantum logic in the traditional, order-theoretic sense. Let us note that the condition c") is obviously fulfilled in any family of crisp sets when fuzzy operations are degenerated to traditional set-theoretic operations.

80.4 Philosophical Consequences

Elements of quantum logics represent properties of studied physical systems (I shall keep the word "quantum" by an abuse of language since these structures represent also properties of classical physical systems). On the other hand in the developed fuzzy set representation of quantum logics these elements are represented by fuzzy subsets of a set of all pure states (*phase space*) of a studied physical system. If a physical system obeys deterministic laws of classical physics, then, whatever is a state of this system, it either possesses or does not possess each of its properties. This implies that a subset of a phase space representing a property is a crisp set and it occurs that a "quantum logic" of such a system is a Boolean algebra.

Quantum mechanics is not a deterministic theory. According to the nowadays most popular interpretation its laws allow only to calculate *probabilities* of results of future experiments, in particular experiments designed to check whether a quantum system possesses or not any of its properties. In the developed fuzzy set representation these probabilities are reinterpreted as degrees to which a quantum system that is in a specific pure state possesses all its properties even before they are measured.

It should be mentioned that according to the orthodox Copenhagen interpretation of quantum mechanics one is not even allowed to say that a quantum system possesses or not any of its properties before a suitable experiment is carried out, which gave rise to the famous statement: *Unperformed experiments have no results* [5]. According to the propounded "fuzzy interpretation" of quantum mechanics one should replace this statement by a statement: *Unperformed experiments have all their possible results, each of them to the degree defined by suitable quantum-mechanical calculations*.

It is nothing strange, of course, that according to the propounded interpretation a quantum system can possess a property to the degree, say, a, and simultaneously does not possess it to the degree $1-a$. For example a photon that is in a state of linear polarization oriented under the angle α to the direction of a polarization filter possesses the property of *being able to pass through the filter* to the degree $cos^2\alpha$ and simultaneously it possesses the property of *not being able to pass through the filter* to the degree $1-cos^2\alpha = sin^2\alpha$. This results in the fact that when an experiment is repeated many times, the fraction $cos^2\alpha$ of identically prepared photons passes through the filter and the fraction $sin^2\alpha$ does not.

80.5 Possible Mathematical Consequences

Another argument that the statement *unperformed experiments have no results* is true, seems to come from mathematics. Already in 1964 J. S. Bell [6] proved that an attempt to endow quantum objects with well-defined sharp properties *before they are measured* in some cases leads to numerical results not compatible with quantum-mechanical calculations that were many times confirmed by experiments. These results took form of the famous Bell inequalities which are in some cases violated by quantum objects. Obviously, Bell's considerations were based on the classical paradigm according to which a property of an object can be only either entirely possessed or entirely not possessed by the object, and on utilizing the classical Kolmogorovian probability calculus based on the Boolean algebra of crisp random events. Therefore, basing these considerations on fuzzy set concepts, in particular on the concept that an object can possess a property "partially", should allow to annihilate the apparent discrepancy between the "common sense" considerations and experimental results.

Moreover, it is possible to show that Bell-type inequalities do not have to hold in some versions of fuzzy probability calculus [7], which again indicates that the mathematical formalism of quantum mechanics should be rather based on fuzzy mathematics than traditional crisp mathematics.

Another problem is the problem of the possibility of constructing phase-space representation of quantum mechanics considered already by E. Wigner in 1932 [8] with the aim of making quantum mechanics "more similar" to classical statistical mechanics. Although Wigner succeeded in constructing such representation, his "pseudo-probability distribution" is cursed with one unacceptable feature: it unavoidably becomes negative in some regions of the phase space. In my opinion this might be an artifact caused by unjustified use of traditional crisp mathematics in the area where fuzzy mathematics is a proper tool.

80.6 Prospects for the Future

According to the results reported in the previous Sections of this note, fuzzy set ideas, like the idea that a physical object can possess its properties only partially, seem to be better suited for description of quantum objects than traditional, crisp ones. I do hope that changing the language of description of quantum objects from the traditional language based on crisp sets to the language based on fuzzy sets would allow to avoid numerous paradoxes that plague quantum mechanics and will make this theory more comprehensible.

References

1. Mączyński, M.J.: The Orthogonality Postulate in Quantum Mechanics. International Journal of Theoretical Physics 8, 353–360 (1973)
2. Giles, R.: Łukasiewicz Logic and Fuzzy Set Theory. International Journal of Man-Machine Studies 67, 313–327 (1976)
3. Pykacz, J.: Quantum Logics as Families of Fuzzy Subsets of the Set of Physical States. Preprints of the Second IFSA Congress, Tokyo, July 20-25, vol. 2, pp. 437–440 (1987)
4. Pykacz, J.: Fuzzy Quantum Logics and Infinite-valued Łukasiewicz Logic. International Journal of Theoretical Physics 33, 1403–1416 (1994)
5. Peres, A.: Unperformed Experiments Have no Results. American Journal of Physics 46, 745–747 (1978)
6. Bell, J.S.: On the Einstein Podolsky Rosen Paradox. Physics 1, 195–200 (1964)

7. Pykacz, J., D'Hooghe, B.: Bell-type Inequalities in Fuzzy Probability Calculus. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 9, 263–275 (2001)
8. Wigner, E.: On the Quantum Correction for Thermodynamic Equilibrium. Physical Review E 40, 749–759 (1932)