

Non-contextuality, finite precision measurement and the Kochen–Specker theorem

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Abstract

Meyer originally raised the question of whether non-contextual hidden variable models can, despite the Kochen–Specker theorem, simulate the predictions of quantum mechanics to within any fixed finite experimental precision (Phys. Rev. Lett. 83 (1999) 3751). Meyer's result was extended by Kent (Phys. Rev. Lett. 83 (1999) 3755). Clifton and Kent later presented constructions of non-contextual hidden variable theories which, they argued, indeed simulate quantum mechanics in this way (Proc. Roy. Soc. Lond. A 456 (2000) 2101).

These arguments have evoked some controversy. Among other things, it has been suggested that the Clifton–Kent models do not in fact reproduce correctly the predictions of quantum mechanics, even when finite precision is taken into account. It has also been suggested that careful analysis of the notion of contextuality in the context of finite precision measurement motivates definitions which imply that the Clifton–Kent models are in fact contextual. Several critics have also argued that the issue can be definitively resolved by experimental tests of the Kochen–Specker theorem or experimental demonstrations of the contextuality of Nature.

One aim of this paper is to respond to and rebut criticisms of the Meyer–Clifton–Kent papers. We thus elaborate in a little more detail how the Clifton–Kent models can reproduce the predictions of quantum mechanics to arbitrary precision. We analyse in more detail the relationship between classicality, finite precision measurement and contextuality, and defend the claims that the Clifton–Kent models are both essentially classical and non-contextual. We also examine in more detail the senses in which a theory can be said to be contextual or non-contextual, and in which an experiment can be said to provide evidence on the point. In

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particular, we criticise the suggestion that a decisive experimental verification of contextuality is possible, arguing that the idea rests on a conceptual confusion.

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1. Introduction

1.1. The Kochen–Specker theorem

Consider a set \mathcal{K} of Hermitian operators that act on an n -dimensional Hilbert space. Suppose that V is a map that takes a Hermitian operator in \mathcal{K} to a real number in its spectrum. We call such a map a *colouring* of \mathcal{K} . If the following conditions are satisfied:

$$\begin{aligned} V(\hat{A} + \hat{B}) &= V(\hat{A}) + V(\hat{B}), \\ V(\hat{A}\hat{B}) &= V(\hat{A})V(\hat{B}), \\ \forall \hat{A}, \hat{B} \in \mathcal{K} \text{ such that } [\hat{A}, \hat{B}] &= 0, \end{aligned} \tag{1}$$

then the map is a *KS-colouring* of \mathcal{K} . We call these conditions the KS criteria. Kochen and Specker’s celebrated theorem (Specker, 1960; Kochen & Specker, 1967) states that if $n > 2$ there are *KS-uncolourable* sets, i.e., sets \mathcal{K} for which no KS-colouring exists. It follows trivially that the set of all Hermitian operators acting on a Hilbert space of dimension > 2 is KS-uncolourable.

The fact that the set of all Hermitian operators in dimension > 2 is KS-uncolourable is a corollary of Gleason’s theorem (Gleason, 1957). This was first pointed out in Bell (1966), where an independent proof was also given. Kochen and Specker constructed the first finite KS-uncolourable set. Many proofs along the lines of Kochen and Specker’s have since been produced by constructing demonstrably KS-uncolourable sets (see, e.g., Peres, 1995, Chapter 7; Zimba & Penrose, 1993; Conway & Kochen, unpublished; Cabello, Estebaranz, & García-Alcaine, 1996). The most common type of proof describes a set of one-dimensional projection operators in n dimensions that is KS-uncolourable. If we represent one-dimensional projections by vectors onto which they project, and colour the corresponding set of vectors with a 1 or a 0, the KS criteria would imply that for each orthogonal n -tuple of vectors, exactly one must be coloured 1, and all the rest 0. The Kochen–Specker theorem can then be proved by showing that the colouring condition cannot be satisfied. In their original proof, Kochen and Specker describe a set of 117 vectors in three dimensions that is KS-uncolourable.¹

Of course, the well-known proofs of the Kochen–Specker theorem referred to above are logically correct. Moreover, the Kochen–Specker theorem undeniably says

¹How small can a KS-uncolourable set of vectors be? The current records stand at 31 vectors in three dimensions (Conway & Kochen) and 18 in four dimensions (Cabello et al., 1996).

something very important and interesting about fundamental physics: it shows that the predictions of quantum theory for the outcomes of measurements of Hermitian operators belonging to a KS-uncolourable set cannot be precisely reproduced by any hidden variable theory that assigns real values to these operators in a way that respects the KS criteria, since no such hidden variable theory exists. However, debate continues over the extent to which Kochen and Specker succeeded in their stated aim, “to give a proof of the non-existence of hidden variables” (Kochen & Specker, 1967, p. 59), even when this is qualified (as it must be) by restricting attention to non-contextual hidden variables. Before summarising and continuing this debate, we review why one might be interested in hidden variable theories in the first place.

Consider a system in a state $|\psi\rangle$ and a set of observables A, B, C, \dots such that $|\psi\rangle$ is not an eigenstate of $\hat{A}, \hat{B}, \hat{C}, \dots$; here we use capital letters with hats to denote Hermitian operators and capital letters without hats to denote the corresponding observables. Orthodox quantum mechanics leads us to say something like this: if we measure A , we will obtain the result a with probability p_a , if we measure B , we will obtain the result b with probability p_b , and so on. With an ease born of familiarity, the well-trained quantum mechanic will not bat an eyelid at such statements. But, one might well ask: why are they so oddly phrased? Could this just be a rather awkward way of saying that with probability p_a , the value of A is a , or with probability p_b , the value of B is b , and so on?

Suppose that the set A, B, C, \dots corresponds to a KS-uncolourable set of operators $\hat{A}, \hat{B}, \hat{C}, \dots$. The suggestion is that at a given time, each observable in the set has some definite value associated with it, defined by some “hidden” variables of the system. The significance of the KS criteria is that if the Hermitian operators associated with two observables commute, then according to quantum mechanics, the observables can be simultaneously measured, and the values obtained will satisfy the KS criteria (and in general will satisfy any functional relationships that hold between the operators themselves). We are not logically compelled to assume that any hidden variable theory shares these properties. However, the standard motivation for considering hidden variables is to examine the possibility that quantum theory, while not incorrect, is incomplete. Thus motivated, it seems natural to assume that the colouring defined by the hidden variables must also satisfy the KS criteria. But given this assumption, since there is no such colouring, the original supposition that the observables have definite values must be wrong.

The contradiction obtained in the Kochen–Specker theorem is avoided if, instead of defining a map V , we assign values to Hermitian operators in such a way that the value assigned to a particular Hermitian operator depends on which commuting set we are considering that operator to be part of. Such a value assignment is called *contextual*. Hidden variable interpretations of quantum theory based on contextual value assignments can be defined. In such contextual hidden variable (CHV) interpretations, the outcome obtained on measuring a certain quantum mechanical observable is indeed pre-defined, but depends in general on which other quantum mechanical observables are measured at the same time. Thus, if we take the

KS criteria for granted, Kochen and Specker's results show that there are no *non-contextual hidden variable* (NCHV) interpretations of the standard quantum mechanical formalism.

It may seem tempting to phrase this more directly, concluding that the Kochen–Specker theorem shows that Nature cannot be described by any non-contextual hidden variable theory. Another possible conclusion is that the KS theorem implies that we could exclude non-contextual hidden variable theories if the predictions of quantum theory were confirmed in a suitably designed experiment. We will argue below that neither conclusion is correct.

1.2. *Querying the scope of the KS theorem*

We next review some earlier discussions that suggest limitations on what can be inferred from the Kochen–Specker theorem.

Some time ago, Pitowsky (1983, 1985) devised models that assign values non-contextually to the orthogonal projections in three dimensions and nonetheless satisfy (1) “almost everywhere” (Pitowsky, 1983, p. 2317). The models are non-constructive, requiring the axiom of choice and the continuum hypothesis (or some suitable weaker assumption) for their definition. Another complication is that the term “almost everywhere” is not meant in the standard sense, but with respect to a non-standard version of measure theory proposed by Pitowsky (1983) which, among other disconcerting features, allows the intersection of two sets of probability measure 1 to have probability measure 0 (Pitowsky, 1982a).

Pitowsky's models disagree with quantum mechanics for some measurement choices, as the KS theorem shows they must. They thus do not per se seem to pose an insuperable obstacle to arguments that—either directly from the theorem or with the aid of suitable experiments—purport to demonstrate the contextuality of Nature.² After all, either the demonstration of a finite non-colourable set of projectors is sufficient to run an argument, or it is not. If it is, Pitowsky's models are irrelevant to the point; if it is not, it is not obvious that the models, equipped as they are with an entirely novel version of probability theory, are either necessary or sufficient for a refutation.

A more direct challenge to the possibility of theoretical or experimental refutations of non-contextual hidden variables was presented in Meyer (1999), where the implications of finite experimental precision are emphasised: “Only finite precision measurements are experimentally reasonable, and they cannot distinguish a dense subset from its closure” (Meyer, 1999, p. 3751). Meyer identified a particularly simple and elegant construction, originally described in Godsil and Zaks (1988), of a KS-colourable dense subset of the set of projectors in three dimensions.³ His conclusion was that, at least in three dimensions, the Kochen–Specker theorem could

² Nor, it should be stressed, did Pitowsky suggest that they do.

³ As Pitowsky has since noted, Meyer's argument could also be framed using one of Pitowsky's constructions of dense KS-colourable sets of projectors rather than Godsil and Zaks'.

be “nullified”.⁴ As a corollary, Meyer argued that the KS theorem alone cannot discriminate between quantum and classical (therefore non-contextual) information processing systems.

Meyer left open the question of whether static non-contextual hidden variable theories reproducing the predictions of quantum theory for three-dimensional systems actually exist: his point was that, contrary to most previous expectations, the Kochen–Specker theorem does not preclude such hidden variable theories.

Meyer’s result was subsequently extended by a construction of KS-colourable dense sets of projectors in complex Hilbert spaces of arbitrary dimension (Kent, 1999). Clifton and Kent (CK) extended the result further by demonstrating the existence of dense sets of projection operators, in complex Hilbert spaces of arbitrary dimension, with the property that no two compatible projectors are members of incompatible resolutions of the identity (Clifton & Kent, 2000). The significance of this property is that it makes it trivial to construct a distribution over different hidden states that recovers the quantum mechanical expectation values. Such a distribution is, of course, necessary for a static hidden variable theory to reproduce the predictions of quantum theory. Similar constructions of dense subsets of the sets of all positive operators were also demonstrated (Kent, 1999; Clifton & Kent, 2000). CK presented their constructions as non-contextual hidden variable theories that can indeed simulate the predictions of quantum mechanics in the sense that the theories are indistinguishable in real experiments in which the measurement operators are defined with finite precision.

The arguments set out by Meyer, Kent, and Clifton and Kent (MKC) have evoked some controversy (see, e.g., Mermin, 1999; Cabello, 1999; Basu, Bandyopadhyay, Kar, & Home, 2001; Simon, Brukner, & Zeilinger, 2001; Larsson, 2002; Peres, 2003a; Appleby, 2000, 2001, 2002, 2003; Havlicek, Krenn, Summhammer, & Svozil, 2001; Cabello, 2002; Breuer, 2002) and even a parody (Peres, 2003b). Among other things, it has been suggested (Cabello, 2002) that the CK models do not in fact reproduce correctly the predictions of quantum mechanics, even when finite precision is taken into account. It has also been suggested (Simon et al., 2001; Larsson, 2002; Appleby, 2002) that careful analysis of the notion of contextuality in the context of finite precision measurement motivates definitions which imply that the CK models are in fact contextual.

Several of these critiques raise novel and interesting points, which have advanced our understanding of the Kochen–Specker theorem and its implications. Nonetheless, we remain convinced that the essential insight of Meyer (1999) and all the substantial points made in Kent (1999) and Clifton and Kent (2000) are valid. One aim of this paper is thus to respond to and rebut MKC’s critics.

Perhaps unsurprisingly, quite a few critics have made similar points. Also, some purportedly critical arguments make points irrelevant to the arguments of the MKC

⁴It should perhaps be emphasised that the sense of “nullify” intended here is “counteract the force or effectiveness of”, not “invalidate”. Neither Meyer nor anyone else has suggested that the proofs of the Kochen–Specker theorem are flawed.

papers (which were carefully limited in their scope). Rather than producing a comprehensive—but, we fear, unreadable—collection of counter-critiques of each critical article, we have tried in this paper to summarise and comment on the most interesting new lines of argument.

Among other things, we explain here in a little more detail how the CK models can reproduce the predictions of quantum mechanics to arbitrary precision, both for single measurements and for sequences. We point out a conceptual confusion among critics who suggest that the models are contextual, noting that the arguments used would (incorrectly) suggest that Newtonian physics and other classical theories are contextual. We also defend the claim that the CK models are essentially classical. Indeed, as we explain, the models show in principle that one can construct classical devices that assign measurement outcomes non-contextually and yet simulate quantum mechanics to any given fixed non-zero precision. In summary, we reiterate the original claim of MKC that the models, via finite precision, provide a loophole—which is physically implausible but logically possible—in the Kochen–Specker argument.

Running through these debates is another theme: the alleged possibility of experimental tests of the Kochen–Specker theorem, or experimental demonstrations of the contextuality of Nature. Quite a few experiments purporting to test contextuality have recently been proposed (Cabello & García-Alcaine, 1998; Simon, Żukowski, Weinfurter, & Zeilinger, 2000; Basu et al., 2001) and performed (Huang, Li, Zhang, Pan, & Guo, 2003; Hasegawa, Loidl, Badurek, Baron, & Rauch, 2003). Several authors have suggested an analogy between these purported experimental tests of contextuality and Bell experiments testing local causality.

Another aim of this paper is to go beyond previous discussions in examining in detail the senses in which a theory can be said to be contextual or non-contextual, and in which an experiment can be said to provide evidence for these. Broadly, we are critical of the idea of an experimental test of non-contextuality, arguing that the idea rests on conceptual confusion. The experiments that have been performed test predictions of quantum mechanics which certainly conflict with some classical intuitions, and which might indeed raise questions about the contextuality of measurements to someone familiar only with certain aspects of quantum theory. But, as we re-emphasize in this paper, they certainly do not provide loophole-free demonstrations of the contextuality of Nature, since the CK models can reproduce the experimental data.

There is also a more basic problem. Interpreting the experiments in a way which raises the question of contextuality at all requires assuming significant parts of the formalism of quantum theory. On the other hand, if we simply assume quantum theory is valid, without any qualification, we need no experiment: the Kochen–Specker theorem already excludes non-contextual hidden variable theories. It is thus quite hard to pin down what exactly a purported experimental test of contextuality proves, or could prove, that we do not know already. In our opinion, this key point is not adequately addressed in the papers under discussion (Cabello & García-Alcaine, 1998; Simon et al., 2000; Basu et al., 2001; Huang et al., 2003; Hasegawa et al., 2003).

2. MKC models

Kochen and Specker's declared motivation for constructing finite uncolourable sets is interesting, both because it partly anticipates the point made a third of a century later by Meyer and because its implications seem to have been largely ignored in the period intervening:

It seems to us important in the demonstration of the non-existence of hidden variables that we deal with a small finite partial Boolean algebra. For otherwise a reasonable objection can be raised that in fact it is not physically meaningful to assume that there are a continuum number of quantum mechanical propositions (Kochen & Specker, 1967, p. 70).

What Kochen and Specker neglected to consider is that the objection might be sharpened: it could be that in fact only a specified countable set of quantum mechanical propositions exist, and it could be that this set has no KS-uncolourable subsets (finite or otherwise). This is the possibility that the MKC models exploit.

Before discussing these models, we wish to reemphasise the disclaimers made in Clifton and Kent (2000). MKC models describe a type of hidden variable theory that is a logically possible alternative to standard quantum theory, but not, in our view, a very plausible one. The CK constructions in particular, are ugly and contrived models, produced merely to make a logical point. One might hope to devise prettier hidden variable models which do the same job, using a colouring scheme as natural and elegant as Godsil and Zaks'. Even if such models were devised, though, we would not be inclined to take them too seriously as scientific theories.

However, we think it important to distinguish between scientific implausibility and logical impossibility. The models show that only the former prevents us from adopting a non-contextual interpretation of any real physical experiment. Another reason for studying the models—in fact, Meyer's main original motivation (Meyer, 1999)—is to glean insights into the possible rôle of contextuality in quantum information theory.

2.1. Projective measurements

The argument of Kochen and Specker (1967), and most later discussions until recently, including Meyer (1999), assume that the quantum theory of measurement can be framed entirely in terms of projective measurements. This remains a tenable view, so long as one is willing to accept that the experimental configuration defines the quantum system being measured.⁵ We adopt it here, postponing discussion of positive operator valued (POV) measurements to the next subsection.

Meyer identified a KS-colouring, originally described in Godsil and Zaks (1988), of the set $S^2 \cap Q^3$ of unit vectors in R^3 with rational components, or equivalently of

⁵For instance, a projective measurement on a quantum system S together with an ancillary quantum system A requires us, on this view, to take $S + A$ as the system being measured, rather than speaking of a positive operator valued measurement being carried out on S .

the projectors onto these vectors. As he pointed out, not only is this set of projectors dense in the set of all projectors in R^3 , but the corresponding set of projective decompositions of the identity is dense in the space of all projective decompositions of the identity.

Meyer's result is enough to show that an NCHV theory along these lines is not ruled out by the Kochen–Specker theorem. It does not show that such a theory exists. For this we need there to be KS-colourable dense sets of projectors in complex Hilbert spaces of arbitrary dimension. Further, it is not enough for each set to admit at least one KS-colouring. For each quantum state, one must be able to define a distribution over different KS-colourings such that the correct quantum expectation values are obtained. For these reasons, Kent (1999) extended Meyer's result by constructing KS-colourable dense sets of projectors in complex Hilbert spaces of arbitrary dimension. Clifton and Kent (2000) extended the result further by demonstrating the existence of dense sets of projection operators, in complex Hilbert spaces of arbitrary dimension, with the property that no two compatible projectors are members of incompatible resolutions of the identity. The significance of this property is that it makes it trivial to construct a distribution over different hidden states that recovers the quantum mechanical expectation values.

CK argue that this construction allows us to define a non-contextual hidden variable theory that simulates quantum mechanics, by the following reasoning. First, let us suppose that, as in the standard von Neumann formulation of quantum mechanics, every measurement corresponds to a projective decomposition of the identity. However, because any experimental specification of a measurement has finite precision, we need *not* suppose that every projective decomposition corresponds to a possible measurement. Having defined a dense set of projectors \mathcal{P} that gives rise to a dense set of projective decompositions of the identity \mathcal{D} , we may stipulate that every possible measurement corresponds to a decomposition of the identity in \mathcal{D} . The result of any measurement is determined by hidden variables that assign a definite value to each operator in \mathcal{P} in a non-contextual manner. Via the spectral decomposition theorem, those Hermitian operators whose eigenvectors correspond to projectors in \mathcal{P} are also assigned values. If measurements could be specified with infinite precision, then it would be easy to distinguish this alternative theory from standard quantum mechanics. We could simply ensure that our measurements correspond exactly to the projectors featured in some KS-uncolourable set. If they in fact corresponded to slightly different projectors, we would detect the difference.

Now, for any finite precision, and any KS-uncolourable set of projectors, there will be projectors from \mathcal{P} sufficiently close that the supposition that our measurements correspond to those from \mathcal{P} will not make a detectable difference. So, which particular element of \mathcal{D} does this measurement correspond to? CK propose that the answer to this question is determined algorithmically by the hidden variable theory.

Let us illustrate how this could work by fleshing out, with more detail than was given in Clifton and Kent (2000), one way in which a CK model could work. Consider some ordering $\{d^1, d^2, \dots\}$ of the countable set \mathcal{D} . Let ε be a parameter

much smaller than the precision attainable in any current or foreseeable experiment. More precisely, ε is sufficiently small that it will be impossible to tell from the outcome statistics if a measurement attempts to measure a decomposition $d = \{P_1, \dots, P_n\}$ and actually measures a decomposition $d' = \{P'_1, \dots, P'_n\}$, provided $|P_i - P'_i| < \varepsilon$ for all i . Suppose now we design a quantum experiment which would, if quantum theory were precisely correct, measure the projective decomposition d . (Of course, the experimenter can only identify d to within the limits of experimental precision, but, on the hypothesis that all measurements are fundamentally projective, we suppose that in reality the value of d is an objective fact.) We could imagine that the hidden variable theory uses the following algorithm: first, it identifies the first decomposition $d^i = \{P^i_1, \dots, P^i_n\}$ in the sequence such that $|P_j - P^i_j| < \varepsilon$ for all j from 1 to n . Then, it reports the outcome of the experiment as that defined by the hidden variables for d^i : in other words, it reports outcome j if the hidden variable theory ascribes value 1 to P^i_j (and hence 0 to the other projectors in d).

It may be helpful to visualise this sort of model applied to projectors in three real dimensions. The system to be measured can be pictured as a sphere with (infinitesimally thin) spines of some fixed length sticking out along all the vectors corresponding to projectors in D , coloured with 1 or 0 at their endpoint. A quantum measurement defines an orthogonal triple of vectors, which in general is not aligned with an orthogonal triple of spines. Applying the measurement causes the sphere to rotate slightly, so that a nearby orthogonal triple of spines becomes aligned with the measurement vectors. The measurement outcome is then defined by the spine colourings.

Some points are worth emphasising here. First, the algorithm we have just described obviously *cannot* be obtained from standard quantum theory. It is the hidden variable theory that decides which projective decomposition is actually measured. Some critics have implicitly (or explicitly) assumed that the measured decomposition must be precisely identified by standard quantum theoretic calculations.⁶ But finite precision hidden variable models need not be so constrained: all they need to do is simulate quantum theory to within finite precision.

Second, as the algorithm above suggests, any given CK model actually contains an infinite collection of sub-models defined by finite subsets $\{d^1, d^2, \dots, d^r\}$ of D with the property that they are able to reproduce quantum theory to within some finite precision ε_r , where $\varepsilon_r \rightarrow 0$ as $r \rightarrow \infty$. At any given point in time, there is a lower bound on the precision actually attainable in any feasible experiment. Hence, at any given point in time, one (in fact infinitely many) of the finite sub-models suffices to reproduce quantum theory to within attainable experimental precision. In other words, at any given point in time, MKC's argument can be run without using infinite dense subsets of the sets of projectors and projective decompositions.

Third, we recall that the models CK originally defined are not complete hidden variable models, since no dynamics was defined for the hidden variables. As CK

⁶See, for example, Peres (2003a), where the "challenge" seems to be based on a misunderstanding of this point and on neglect of the POV models defined in the next section, and also Appleby (2001).

noted, the models can be extended to cover sequential measurements simply by assuming that the hidden variables undergo a discontinuous change after a measurement, so that the probability distribution of the post-measurement hidden variables corresponds to that defined by the post-measurement quantum mechanical state vectors. A complete dynamical non-contextual hidden variable theory needs to describe successive measurements in which the intervening evolution of the quantum state is non-trivial. In fact (though CK did not note it), this could easily be done, by working in the Heisenberg rather than the Schrödinger picture, and applying the CK rules to measurements of Heisenberg operators. In this version of the CK model, the hidden variables define outcomes for measurements, change discontinuously so as to reproduce the probability distributions for the transformed quantum state, and then remain constant until the next measurement.

2.2. *Positive operator valued measurements*

Dealing with projective measurements is arguably not enough. One quite popular view of quantum theory holds that a correct version of the measurement rules would take POV measurements as fundamental, with projective measurements either as special cases or as idealisations which are never precisely realised in practice. In order to define an NCHV theory catering for this line of thought, Kent constructed a KS-colourable dense set of positive operators in a complex Hilbert space of arbitrary dimension, with the feature that it gives rise to a dense set of POV decompositions of the identity (Kent, 1999). Clifton and Kent constructed a dense set of positive operators in complex Hilbert space of arbitrary dimension with the special feature that no positive operator in the set belongs to more than one decomposition of the identity (Clifton & Kent, 2000). Again, the resulting set of POV decompositions is dense, and the special feature ensures that one can average over hidden states to recover quantum predictions. Each of the three points made at the end of the last section applies equally well to the POV models.

We should stress that the projective and POV hidden variable models defined in Kent (1999) and Clifton and Kent (2000) are separate theories. One can consider whichever model one prefers, depending whether one is most interested in simulating projective or POV quantum measurements, but they are not meant to be combined. The POV hidden variable model does, as of course it must, define outcomes for projective measurements considered as particular cases of POV measurements—but not in the same way that the projective hidden variable model does.

The CK models for POV measurements have, surprisingly, been neglected by some critics (e.g., Peres, 2003a), who object to the CK projective models on the grounds that they unrealistically describe outcomes of ideal but imprecisely specified projective measurements. As we noted above, this objection is indeed reasonable if one takes the view that one should define the measured quantum system in advance, independent of the details of the measurement apparatus, or if one regards POV measurements as fundamental for any other reason. The POV measurement models were devised precisely to cover these points.

3. Some criticisms of the MKC models

3.1. Are the CK models classical?

Clifton and Kent claimed that the CK models show “there is no truly compelling argument establishing that non-relativistic quantum mechanics describes classically inexplicable physics” (Clifton & Kent, 2000, p. 2113). Some (Appleby, 2000, 2001, 2003; Havlicek et al., 2001) have queried whether the models can, in fact, properly be described as classical, given that they define values on dense subsets of the set of measurements in such a way that every neighbourhood contains operators with both truth values. This feature implies that the models do not satisfy what we call the *faithful measurement condition*: that one can in general ascribe a value to an operator P , such that this value, or one close to it, is obtained with high probability when a high precision measurement of P is performed.

Appleby (2000, 2001, 2003) has discussed the faithful measurement condition at some length, arguing that it is a necessary property of measurements in classical models. Appleby notes that a classical measurement tells us some definite fact about the system as it was before measurement, and goes on to argue that the dense—in Appleby’s words, “radical” or “pathological” (Appleby, 2001, 2003)—discontinuities of truth values in the CK models mean that they cannot satisfy this epistemological criterion: let us call it the *definite revelation criterion*.

Before considering Appleby’s argument, one might first ask whether dense discontinuities are actually necessarily a feature of any CK-type model that simulates quantum mechanics. As Appleby (2001, 2003) and Cabello (2002) show, they are.⁷ Appleby’s argument thus cannot be sidestepped.

However, in our opinion, while the CK models clearly do not satisfy the faithful measurement criterion, they *do* satisfy the definite revelation criterion, in the same sense that standard models in classical mechanics do. The CK models can thus indeed properly be described as classical.

We believe this claim is ultimately justified by virtue of the phase space structure and the logical structure of the CK models, both of which are classical. However, since discussion has focussed on the discontinuity of the CK models, it is worth considering this point in more detail.

Note, first, that discontinuity per se is clearly not an obstacle to classicality, according to standard definitions. Point particles and finite extended objects with boundary discontinuities are routinely studied in classical physics. Moreover, if the mere existence of discontinuities in the truth values assigned to operators were the crucial issue, the KS theorem would be redundant—it is immediately obvious that

⁷In Cabello (2002), it is ostensibly argued that any model of the type constructed by CK must lead to experimental predictions that differ from those of quantum mechanics. This is clearly not correct. However, an examination of Cabello’s argument reveals a technical assumption that is not true of the CK models, as noticed by Clifton in a private communication to Cabello, reported by Cabello in a footnote. Cabello’s reply to Clifton is essentially an attempt to justify the assumption by appeal to something like the faithful measurement condition. Thus his argument is best viewed as a demonstration that CK-type models cannot have this feature. Appleby (2001) offers a similar analysis of Cabello’s argument.

any truth values assigned by hidden variables must be discontinuous, since the only possible truth values are 0 and 1, and both must be realised. Any argument against the classicality of the CK models must, then, stem from the fact that their discontinuities are dense.

One possible argument against the classicality of models with dense discontinuities might be that, if the faithful measurement condition is not satisfied, then little sense can be given to the notion of one finite precision measurement being more “precise” than another. If one is not able to compare degrees of precision, it might be argued, one has not recovered the classical concept of measurement at all. In reply, we note that there is in fact a clear definition of the precision of measurement devices within CK models. For example, if a high precision device is supposed to measure z -spin, then it will with high probability return a value of $+1$ whenever a measurement is performed on a particle prepared (by another high precision device) in the corresponding eigenstate. The precision of the relevant devices is then calibrated by the difference between the actual outcome probability and 1, which would represent perfect precision. This feature of CK models seems to have been neglected: for instance, it is simply not true that, as Appleby suggests (Appleby, 2001, p. 6), in CK models, the outcome of a measurement of an observable P “does not reveal any more information ... [about the pre-existing value of P] ... than could be obtained by tossing a coin”. If an unknown quantum state drawn from a known ensemble is measured, then obtaining a valuation for the actually measured observable P' generally *does* give some statistical information about the pre-measurement valuation of the target observable P , whenever P is one of the observables to which the model assigns a valuation.

Another possible argument might begin from the observation that systems that are actually studied in the context of classical mechanics generally satisfy the faithful measurement condition, which might suggest that the condition is implied by some part of classical intuition. But induction based merely on familiarity is a dangerous exercise. (Three-legged dogs are still canine, for example.) It is, admittedly, rare to consider classical systems which have dense discontinuities, but it does not contradict any standard definition of classicality of which we are aware. Given consistent evolution laws, one can sensibly study the behaviour of a classical system in which point particles are initially sited at every rational vector in R^3 , for instance.

To address Appleby’s point directly, we note that according to the CK models a measurement *does*, in fact, reveal the pre-existing valuation of an observable. Consider again a CK model defined by the algorithm given in Section 2.1. It is true that finite experimental precision makes it impossible for a human experimenter to identify precisely either the quantum observable which any given experiment would end up measuring if quantum theory is correct, or the CK observable which it would end up measuring if a CK model were correct. Nonetheless there is, according to the CK models, a fact of the matter about the identity of both observables. The process runs thus: some definite quantum observable is defined by the experimental configuration; some definite CK observable, related to the quantum observable by some definite algorithm, is thus also indirectly defined by the experimental configuration; the pre-existing valuation of this CK observable is revealed by the

experiment. An omniscient deity viewing the whole process “from the outside” could verify the action of the CK model, following (for example) the algorithm discussed in Section 2.1, and predict in advance precisely which CK observable will be addressed and the valuation that will be revealed. In other words, the CK models *do* satisfy the definite revelation criterion, as we understand it.

Finally, but importantly, we can offer an alternative response to those unpersuaded by any of the above arguments. As we noted earlier, one can define CK models that simulate quantum mechanics adequately (given any specific attainable experimental precision) using finite collections of projections and projective decompositions. These models are still discontinuous, but they have only finitely many discontinuities, rather than a dense set. They therefore satisfy the faithful measurement condition (this being possible because any particular such model will make different predictions from quantum mechanics once a certain precision is exceeded). As above, one can visualise such a model, in R^3 , as defined by a sphere with finitely many spines projecting from it. In terms of its discontinuities (which are finite in number) and its dynamics (which could be precisely defined by a sufficiently complex force law) such an object is analogous to a finite set of point particles. There is no sensible definition of classicality that renders it (or analogues with more degrees of freedom) non-classical.

3.2. Are the CK models consistent with quantum probabilities?

In Appleby (2000), it is argued that any model of a certain type must either be contextual or violate the predictions of quantum mechanics. In Breuer (2002), it is argued that NCHV models of yet a different type make different predictions from quantum mechanics. Appleby and Breuer both make assumptions that are not true of the CK constructions.

In Appleby (2000), it is argued that any non-contextual model of a certain kind makes different predictions from quantum theory. Appleby assumes that in an imprecise measurement of observables corresponding to three projectors, the three projectors actually measured are not exactly commuting, but are picked out via independent probability distributions. However, this is not how CK models work. For example, in a CK model for projective measurements, the projectors actually measured are always commuting (assuming that they are measured simultaneously)—this is one of the axioms of the theory that relate its mathematical structure to the world, i.e., it is not some kind of miraculous coincidence. If the projectors are measured sequentially, then the rules of the model stipulate that the hidden state changes discontinuously after each measurement and Appleby’s analysis no longer applies. Similar remarks apply to the POV version.

In Breuer (2002) it is shown that any finite precision NCHV model that assigns values to a dense subset of projection operators, and also satisfies a certain extra assumption, must make different predictions from quantum mechanics. Suppose that a spin measurement is performed on a spin-1 particle and that the measurement direction desired by the experimenter (the target direction) is \vec{n} . The assumption is that the actual measurement direction is in a random direction \vec{m} , and that the

distribution $\omega_{\vec{n},\varepsilon}(\vec{m})$ over possible actual directions, given \vec{n} and the experimental precision ε , satisfies

$$\omega_{R\vec{n},\varepsilon}(R\vec{m}) = \omega_{\vec{n},\varepsilon}(\vec{m})$$

for all rotations R . Of course, the CK models do not satisfy this condition, and Breuer notes this. In fact, it is clear that no model that colours only a countable set of vectors could satisfy the condition. To those who regard Breuer's condition as desirable on aesthetic grounds, we need offer no counter-argument: it was conceded from the beginning that the CK models are unaesthetic.

3.3. *Non-locality and quantum logic*

Any hidden variable theory that reproduces the predictions of quantum mechanics must be non-local, by Bell's theorem. The CK models are no exception. Some have argued (Appleby, 2002; Boyle & Schafir, 2001), however, that non-locality is itself a kind of contextuality, and that any theory that is non-local must also, therefore, be contextual. Indeed, it is relatively common to read in the literature the claim that non-locality is a special case of contextuality. Here, we simply wish to point out that non-locality and contextuality are logically independent concepts. Newtonian gravity provides an example of a theory that is non-contextual and non-local. One can also imagine theories that are contextual and local—for example, a sort of modified quantum mechanics, in which wave function collapse propagates at the speed of light (Kent, 2002). Appleby notes the example of Newtonian gravity himself, but states that “in the framework of quantum mechanics the phenomena of contextuality and non-locality are closely connected” (Appleby, 2002, p. 1). This is true, but it is not necessarily the case that what is true in the framework of quantum mechanics is still true when we take the point of view of the hidden variables—and when assessing hidden variable models, it is the hidden variables' point of view that is important. Appleby (2002) concludes, based on a Greenberger–Horne–Zeilinger-type (GHZ-type) example, that the CK models display “existential contextuality”. It seems to us that, considered from the proper hidden variable model theoretic rather than quantum theoretic perspective, Appleby's argument simply demonstrates the non-locality of the CK models—which were, of course, explicitly presented as non-relativistic and necessarily non-local.

Finally, some have objected to the MKC models on the grounds that elements of the quantum formalism, for example the superposition principle (Cabello, 1999) or the quantum logical relations between projectors (Havlicek et al., 2001; Busch, 2003), are not preserved. We note that this is of no importance from the point of view of the hidden variables. The whole point is that they have their own classical logical structure.

4. Experimental tests of contextuality?

Another issue that has arisen, both prior to and during the course of these debates, is that of an experimental test of contextuality. Some experiments have actually been

performed. An examination of this issue, in particular of what the experiments can really tell us, is of interest independently from the MKC models and will improve our understanding of the Kochen–Specker theorem. But the issue is also relevant for MKC models. Indeed if it were possible to rule out non-contextual theories via a decisive experimental test, this would seem to contradict the claim that the CK models reproduce the predictions of quantum mechanics to arbitrary precision and are non-contextual. In Section 4.1 we argue that, quite independently of the issue of finite precision, the idea of an experimental refutation of non-contextuality is based on conceptual confusion, and that the experiments that have actually been carried out are, as far as contextuality goes, not of major significance. We examine in particular an experiment that has actually been performed, [Huang et al. \(2003\)](#) inspired by a proposal of [Simon et al. \(2000\)](#), in turn based on a scheme of [Cabello and García-Alcaine \(1998\)](#). (Another recent experiment is that of [Hasegawa et al. \(2003\)](#), which is similar to a proposal of [Basu et al. \(2001\)](#)—we do not discuss this in detail, since the same arguments apply.) In Section 4.2, we argue that in addition, the MKC finite precision loophole does apply, in the sense that any experiment can be simulated by the CK models. Finally, in Section 4.3 we discuss the operational approach of [Simon et al. \(2001\)](#) and [Larsson \(2002\)](#).

4.1. What can an experiment tell us about contextuality?

We begin by discussing the possibility of an experimental test of contextuality in the absence of finite precision considerations. It is easiest to do this with a particular example in mind, so we make particular reference to the scheme which [Simon et al. \(SZWZ\)](#) proposed and which inspired the experiments reported by [Huang et al. \(HLZPG\)](#). Consider a four-dimensional Hilbert space, which we can think of as representing two two-dimensional subsystems. The two subsystems are associated with the path and polarisation degrees of freedom of a single photon. Define the subsystem observables Z_1, X_1, Z_2, X_2 , where subscript 1 indicates the path degree of freedom and subscript 2 the polarisation degree of freedom. Suppose that $\hat{Z}_i = \sigma_{zi}$ and $\hat{X}_i = \sigma_{xi}$, where σ_{zi} and σ_{xi} are Pauli operators acting on subsystem i . Each of these observables can take the values $+1, -1$. In an NCHV interpretation, a hidden state must assign a value to each of these observables that would simply be revealed on measurement. This in turn defines a colouring of the corresponding set of operators, $V(\hat{Z}_1), V(\hat{X}_1), V(\hat{Z}_2), V(\hat{X}_2)$.

One can also consider observables that are products of these observables, for example, $Z_1 X_2$. Product observables also take the values $+1, -1$, and from the KS criteria we have

$$\begin{aligned} V(\hat{Z}_1 \hat{Z}_2) &= V(\hat{Z}_1) V(\hat{Z}_2), \\ V(\hat{Z}_1 \hat{X}_2) &= V(\hat{Z}_1) V(\hat{X}_2), \\ V(\hat{X}_1 \hat{Z}_2) &= V(\hat{X}_1) V(\hat{Z}_2), \\ V(\hat{X}_1 \hat{X}_2) &= V(\hat{X}_1) V(\hat{X}_2). \end{aligned} \tag{2}$$

Finally, the contradiction arises on consideration of the quantum state

$$\begin{aligned} |\phi_+\rangle &= \frac{1}{\sqrt{2}}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+x\rangle|+x\rangle + |-x\rangle|-x\rangle), \end{aligned}$$

where $|+z\rangle$ is an eigenstate of \hat{Z}_i with eigenvalue $+1$, and so on. This state has the property that measurement of the product Z_1Z_2 always returns 1, as does measurement of X_1X_2 . If $V(\hat{Z}_1) = V(\hat{Z}_2)$, $V(\hat{X}_1) = V(\hat{X}_2)$, and equations (2) are satisfied, then it follows logically that $V(\hat{Z}_1\hat{X}_2) = V(\hat{X}_1\hat{Z}_2)$. Yet in quantum mechanics, one can measure Z_1X_2 and X_1Z_2 simultaneously, and if the state is $|\phi_+\rangle$, then one will get opposite results with certainty. Hence we have a contradiction.⁸

In principle, a laboratory implementation could use a network of beam splitters, polarising beam splitters and half-wave plates in order to prepare a single photon in the state $|\phi_+\rangle$ and perform each of the joint measurements

$$(Z_1, Z_2), (Z_1, X_2), (X_1, Z_2), (X_1, X_2), (Z_1X_2, X_1Z_2).$$

In the experiment of HLZPG, only the (X_1, X_2) and (Z_1X_2, X_1Z_2) measurements were actually performed, with the outcome of a potential (Z_1, Z_2) measurement being assumed from the method of state preparation. Though a detailed critique of HLZPG's experiment is beyond our scope here, we should note that it deviates in various ways from the ideal version proposed by SZWZ, and add that we find their discussion hard to follow at various points: for example, they appear to interpret one of their settings (their setup 2) as performing a simultaneous measurement of X_1 , Z_2 and Z_1X_2 .

What, in any case, *could* an experiment along the lines suggested by SZWZ show? In each of Cabello and García-Alcaine (1998), Simon et al. (2000) and Huang et al. (2003), the work is motivated via an analogy with Bell's theorem. Bell's theorem tells us that locally causal theories are incompatible with quantum mechanics, according to Bell's precise definition (Bell, 1985) of "locally causal". The associated experimental tests have strongly confirmed quantum mechanics. Then it is claimed, for example, that

"The Kochen–Specker theorem states that *non-contextual* theories are incompatible with quantum mechanics" (Simon et al., 2000, p. 1783).

If one takes this at face value, it seems easy to accept that a Kochen–Specker experiment to test non-contextuality would be of similar interest and fundamental importance to a Bell experiment that tests local causality.

However, there is a key point, not noted by these authors, where the analogy breaks down. A Bell experiment allows us to test the predictions of quantum

⁸This argument differs from standard Kochen–Specker-style proofs (and from Cabello and García-Alcaine's argument) in that the predictions from a particular quantum state are used to obtain a contradiction.

mechanics against those of locally causal theories because a definition of all the terms used in a derivation of Bell's theorem (in particular the term “locally causal” itself) can be given that is *theory-independent*. Yet in the Kochen–Specker scheme above, the observables have not been defined in a manner that is theory-independent, but have instead been defined with respect to the quantum mechanical operators. When a simultaneous measurement of $Z_1 X_2$ and $X_1 Z_2$ is performed, the experimental setup as a whole looks different from that employed in a simultaneous measurement of, say, X_1 and X_2 .

For example, HLZPG describe two experimental setups: to get from one to the other one needs to rotate the two half-wave plates they call HWP1 and HWP2. What gives us license to claim that one of these setups really measures two observables, of which one is the product of Z_1 and X_2 and the other is the product of X_1 and Z_2 ? The answer is: our conventional physical understanding of the experiment, *as informed by the quantum formalism*. HLZPG need to assume that the effects of devices such as beam splitters and half-wave plates are well described by the Hilbert space formalism. That they do this implicitly is evident in remarks such as “the interference on a BS [beam splitter] performs a Hadamard transformation of the path qubit” (Huang et al., 2003, p. 2). But there is no reason to assume that such statements will be true (or even meaningful) in a theory that is not quantum mechanics. Thus there is no theory-independent means of knowing that we really are doing a simultaneous measurement of the product of Z_1 and X_2 , and the product of X_1 and Z_2 . But this is crucial if we are to conclude unequivocally that contextuality is being exhibited. Similar comments apply to the experiment of Hasegawa et al. (2003): their spin rotator and phase shifter need to be adjusted to alter their parameters α and χ , and they naturally need to rely on the standard quantum formalism in order to interpret the experiment as carrying out measurements of particular projections onto the path and spin degrees of freedom.

Of course, the mathematical arguments given by these various authors are valid, and offer yet further proofs that there are no NCHV interpretations of the quantum mechanical formalism. And clearly the experiments confirm some predictions of quantum theory. However, Cabello and García-Alcaine's claim that this type of experiment can show that

“NCHV theories, *without* any call to the formal structure of QM, make conflicting predictions with those of QM” (Cabello & García-Alcaine, 1998, p. 1797, *their emphasis*),

which is echoed by HLZPG, is simply not correct.

These remarks apply quite generally to any proposed test of contextuality that involves measuring product observables. Without using locality arguments, there is no way to guarantee that a given measurement is of an observable that is precisely in product form, nor that two different measurements involve products of the same operator. If such an experiment is performed, and results consistent with quantum mechanics obtained, what can we conclude? We have essentially three choices. First, accept the basic quantum formalism and accept also that any underlying hidden variable theory assigning values to Hermitian operators must be contextual. Second,

look for loopholes in our interpretation of the experimental results. Or third, reject the Hilbert space structure and look for an entirely different theory of the experiment that is non-contextual in its own terms.

The second move is exploited by the MKC models. The third move will always be logically possible if non-contextuality is defined (as it often is in the literature) as simply requiring that the value obtained on measuring a given observable does not depend on which other observables are measured at the same time. No mention of Hermitian operators is given in this definition, so it has the appearance of being theory independent. But it is not all that useful. It allows a non-contextual theory of any experiment to be cooked up in a trivial manner, simply by redefining what counts as an observable—for instance, by taking an observable to correspond to the full projective decomposition of the identity defining any given measurement, rather than to a single projection (van Fraassen, 1973).

Note that if a Bell experiment is performed, and the quantum predictions verified, then we have analogues of the first two options above: we can reject local causality, or we can look for loopholes in the experiment. Both options have been much explored. But the analogy breaks down when we consider the third option above, because the Hilbert space structure was not used either in the derivation of Bell's theorem or in the interpretation of the experiment.⁹ It also breaks down when we consider the outcomes of exploring the second option: finite experimental precision poses no fundamental difficulty in the analysis of Bell experiments, but turns out to be an unstoppable loophole in Kochen–Specker experiments.

Granted then, that this type of experiment cannot be of decisive significance, can it have *any* significance? Can it be interpreted as a test between quantum mechanics and a different kind of theory? If it can, then it must be as a test between quantum mechanics and non-contextual theories of a rather restricted kind. Such an experiment, for example, could serve as a test between quantum mechanics and a non-contextual theory that accepts some part of Hilbert space structure (including the operators for path and polarisation degrees of freedom, and the action of devices such as beam splitters), but rejects the KS criteria. Logically, this would be a valid experiment. However, in order to motivate it, one would need to devise an interesting and plausible alternative to quantum theory which retains the features just mentioned but violates (1). Considering such alternatives is beyond our scope here; we only wish to note that the class of such alternatives is not nearly as general and natural as the class of locally causal theories. So far as the project of verifying the contextuality of Nature (as opposed to the contextuality of hidden variable

⁹Of course, even local causality cannot be defined with *no* assumptions about an underlying theory. It requires the notion of a background space–time with a causal structure. Bell's discussion of the implications of local causality for Bell experiments also implicitly requires that the notion of an experimental outcome has its conventional meaning.

It is worth noting, incidentally, that this last point leaves room for arguing that an Everettian interpretation of quantum theory might be defined so as to be locally causal. We will not pursue this here, since the larger questions of whether a coherent Everettian interpretation exists, and if so on what assumptions, are beyond our present scope.

interpretations of the standard quantum formalism) is concerned, the question is of rather limited relevance and interest.

In conclusion, experiments along the lines of those of [Cabello and García-Alcaine \(1998\)](#), [Simon et al. \(2000\)](#) and [Huang et al. \(2003\)](#), do not and cannot decisively distinguish between contextuality and non-contextuality in Nature. If the quantum formalism of states and operators (and the assignments of states and operators to particular experimental devices) is not assumed, then the experiments tell us little. On the other hand, if the standard quantum formalism is assumed, then we know already from the Kochen–Specker theorem, before we carry out any experiments, that there is no way of assigning values non-contextually to the set of all Hermitian operators. Mermin’s comment that

“the whole notion of an experimental test of [the Kochen–Specker theorem] misses the point” (Mermin, quoted in [Cabello & García-Alcaine, 1998](#))

still seems to us to apply.

4.2. Experiments and finite precision

In addition to the considerations of the last section, it is of course the case that a CK model can simulate any quantum experiment, and this includes so-called tests of the Kochen–Specker theorem. We shall leave it to the reader to examine in detail how a CK model will work when applied to any specific experimental setup. Obviously the fact that beam splitters, half-wave plates and so on, will be constantly shifting in alignment by minute amounts will lead to finite precision in the case of the HLZPG experiment. This means that each time a photon passes through the apparatus, the actual observables measured will be slightly different. The CK models then show us that even if it *is* assumed that the operation of each experimental device is well described by the Hilbert space formalism, a non-contextual, classical simulation of the experiment is possible.¹⁰

We make a brief remark about the experiment of [Hasegawa et al. \(2003\)](#), and the proposal of [Basu et al. \(2001\)](#). In both cases, an inequality is derived, formally identical to the Clauser–Horne–Shimony–Holt inequality ([Clauser, Horne, Shimony, & Holt, 1969](#)), that concerns the spin and path degrees of freedom of a single neutron. It may seem as if this evades the finite precision loophole, since the inequality is violated by an irreducibly finite amount. The derivation of the inequality, however, assumes that all measurements performed are strictly of the form $A \otimes I$, in the case of a path degree of freedom, or $I \otimes B$, in the case of a spin degree of freedom. A CK model, on the other hand, assumes that the actual operators measured are not in fact precisely separable, even in experiments which are

¹⁰At the end of their paper, HLZPG make passing reference to the problem of finite precision, mentioning the work of [Simon et al. \(2001\)](#) and [Cabello \(2002\)](#). The former we discuss below, here noting only that it is not relevant to HLZPG’s experiment, since they do not actually apply the result, nor can it be applied to their data. The latter we have already mentioned in Section 3.1, noting that the faithful measurement condition must be assumed, and that this is not necessary for classicality.

designed to measure separate commuting observables. When arguments based on locality and space-like separation are forbidden—as they are here, since the question is whether quantum contextuality can be demonstrated separately from quantum non-locality—this is not physically implausible. Beam splitters generally have a slight polarising effect, for example. More generally, adjusting any piece of the experimental apparatus slightly influences all the others.

4.3. *Defining observables operationally*

One may try to avoid the above arguments by framing a definition of contextuality that is genuinely independent of Hilbert space structure. This could be done by giving a completely operational definition of “observable” and hence of “contextuality”. This may seem to have the additional advantage of avoiding the issue of finite precision, since operational definitions do not assume infinite precision in the first place. The operational approach is hinted at in Mermin (1999) and worked out explicitly by Simon, Bruckner and Zeilinger (SBZ) and Larsson (Simon et al., 2001; Larsson, 2002). The work of both SBZ and Larsson is motivated by the issue of finite precision and is presented as a riposte to MKC. SBZ, for example, describe their work as showing “how to derive hidden-variable theorems that apply to real experiments, so that non-contextual hidden variables can indeed be experimentally disproved”. This seems to contradict directly the claims of CK, in particular, who say that the CK models are non-contextual and reproduce correctly the quantum predictions for any finite precision experiment. We shall see, however, that there is really no tension here. The apparent contradiction rests on different uses of the word “contextual”. Further, we shall argue that the work of SBZ and Larsson, while interesting, does not have the significance they claim. For definiteness, we discuss the work of SBZ, although Larsson’s is very similar.

SBZ consider a black box with three knobs, each of which has a finite number of different settings. After setting the knobs, an observer presses a “go” button. He then receives an outcome for each knob, which is either a 1 or a 0. As an example of such a box, we can consider one that contains within it a quantum experiment in which the spin squared of a spin-1 particle is measured in three different directions. The directions are determined to some degree of accuracy by the settings of the knobs. However, it will not be the case that a given knob setting corresponds to a measurement of spin squared in precisely the same direction every time the box is used. There will be experimental inaccuracies. In general, we may imagine that there are some hidden variables associated with the measuring apparatus, as well as the quantum system, which determine exactly what measurement is being performed. From the point of view of our observer outside the black box, however, none of this matters. All he has access to are the three knobs and the outcomes. SBZ propose that the observer should simply, by fiat, define observables operationally, with each observable corresponding to a different setting of one of the knobs. He can always be sure which observable he is measuring, according to this operational definition, even though he cannot be sure which observable is actually being measured according to quantum theory.

Not knowing what is happening inside the box, our outside observer can try to formulate a model theory. In a deterministic model theory, the entire inside of the box can be described by some hidden state that predicts what the three outcomes will be for each possible joint setting of the knobs. The model is non-contextual if, for each hidden state, the outcome obtained for each knob depends only on its setting, and not on the settings of the other two knobs. On running the box repeatedly, the observer can build up outcome statistics for each possible joint knob setting. If no non-contextual model of the workings of the box that reproduces these statistics exists, for any deterministic model theory, then, SBZ propose, we should say that the box is “contextual”.

Let us consider the implications of this definition applied to projective measurements on a three-dimensional space. Take a set of three-dimensional vectors that is KS-uncolourable, in the sense that it is impossible to give each vector a 0 or a 1 such that each orthogonal triad consists of one 1 and two 0s. The set of vectors can be written, for example

$$\{\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}, \{\vec{n}_1, \vec{n}_4, \vec{n}_5\}, \dots\}.$$

For the set to be KS-uncolourable, it must be the case that some vectors appear in more than one triad. Suppose that these triads are taken to indicate possible triads of knob settings. Suppose that the experiment is run many times, and it is found that whenever one of these triads is measured, the outcomes consist of one 1 and two 0s. Then we can conclude, from the fact that the set of vectors is KS-uncolourable, that the box is “contextual” according to SBZ’s definition—a property we refer to hereafter as SBZ-contextual.

This, though, is too much of an idealisation. In a real experiment there will be noise, which will sometimes cause non-standard results, for example two 1s and a 0. The core of SBZ’s paper is a proof of the following result. Imagine that the box is run many times, with knob settings corresponding to orthogonal triads, and that the outcomes are one 1 and two 0s in a fraction $1 - \varepsilon$ of cases. Then, the box must be SBZ-contextual if $\varepsilon < 1/N$, where N is the number of orthogonal triads appearing in the set. If the box is in fact a quantum experiment in which the spin squared of a spin-1 particle is measured in different directions, then increasing the accuracy of the experiment will be able to reduce ε below $1/N$. The observer will be able to conclude that the experiment is SBZ-contextual.

We wish to make several related remarks concerning this result. The first thing is to clarify the implication for MKC models. A box with a quantum spin experiment inside is certainly simulable by a CK model, since the models are explicitly constructed to reproduce all the predictions of quantum mechanics for finite precision measurements. How will the simulation work? On each run, the knob settings determine approximately which measurement is performed, but exactly which is determined randomly, or by apparatus hidden variables. The exact measurement corresponds to some Hermitian operator in the CK KS-colourable set. The outcome is determined by a hidden state that assigns a definite value to each operator in the KS-colourable set in a non-contextual manner. Hence if observables are defined by operators, it is true that the value obtained on measuring a given

observable does not depend on which other observables are measured at the same time and in this sense, the CK model is non-contextual. The fact that the black box is SBZ-contextual tells us that the settings of all three knobs together, along with the apparatus hidden state, are needed to determine the Hermitian operators that are in fact being measured. In a way, of course, it couldn't be any different, since one cannot expect an algorithm that chooses three vectors independently generally to produce an orthogonal triad. The SBZ-contextuality of the black box tells us in addition that for at least some apparatus hidden states, whether the measurement corresponds to a triad for which knob i gets outcome 0 or a triad for which knob i gets outcome 1 depends on the settings of knobs j and k .

This should be enough to show that there is no formal contradiction between the CK and the SBZ results. Some may argue, however, that from a physical point of view, the operational definition of SBZ-contextuality is the only interesting one, and that the CK models, therefore, are not non-contextual in any interesting sense—or at least that the operational definition is an interesting one, and the CK models are not non-contextual in this sense. We wish to counter such arguments with some cautionary remarks concerning these black boxes.

First, SBZ, as did the authors of the experiments discussed in Section 4 above, motivate their work via an analogy with Bell's theorem. The disanalogy we mentioned in Section 4 has disappeared now that observables are defined operationally. However, there is another important disanalogy. This is that there is nothing specifically non-classical about a black box that is behaving SBZ-contextually. One could easily construct such a box out of cog-wheels and springs. Thus with no knowledge of or assumptions about the internal workings of the box, one could not use it to distinguish classical from quantum behaviour.

This should be contrasted rigorously with the case of a black box which behaves non-locally in the sense of producing Bell correlations. (Of course, to define this sense of non-locality, we assume that special relativity is approximately correct and an approximately Minkowski causal structure is given.) Then, if we have a black box, large enough to allow space-like separated outputs to be identified, which behaves non-locally, we know that we are in a quantum, and not a classical, universe. Such a box can even be used for information theoretic tasks that cannot be accomplished classically (e.g., [Buhrman, Cleve, & van Dam, 2001](#)). Given a black box that is SBZ-contextual, we have no such guarantees. This seems to us to cast doubt on the use or significance of a purely operational definition of contextuality, as opposed to a theory-relative one.

Second, the fact of the matter is that any realistic experiment, whether carried out in a classical or a quantum universe, will necessarily exhibit SBZ-contextuality to some (possibly tiny) degree. Not only that, the *outcome probabilities* for any given SBZ-observable will depend (at least slightly) on the context of the other knob settings. On moving one knob, for example, its gravitational field will be changed, and this will affect the behaviour of the whole apparatus. This is not a consequence of quantum theory. It would be true of an experiment in which a classical measuring apparatus measures classical observables on a classical system. Yet we would not infer from this SBZ-contextuality of the outcomes that classical physics is (at least

slightly) contextual. We do not take SBZ and Larsson to be advocating otherwise: all sides in the Kochen–Specker debate agree that classical physics is, paradigmatically, non-contextual. Rather, we take the fact that the opposite conclusion follows from SBZ’s and Larsson’s definitions to indicate that the definition of SBZ-contextuality is inherently flawed. Similarly, we take the fact that SBZ’s definition of an observable can in principle empirically be shown to be context-dependent—since the outcome probabilities depend at least slightly on knob settings that are meant to correspond to independent observables—to be a fatal flaw in that definition. An SBZ-observable turns out, under scrutiny, to be a rather complicated construct, with quite different properties from its quantum namesake. A less freighted name—“dial setting”, for instance—would make clearer the obstacles which SBZ would need to surmount in order even to begin a properly founded discussion of finite precision experimental tests of contextuality.

This last point really needs no reinforcement, but it can be reinforced. Consider again the black box that in fact contains a quantum experiment in which the spin squared of a spin-1 particle is measured in different directions. The idea was to run the box repeatedly with certain combinations of knob settings that correspond to the orthogonal triads in a KS-uncolourable set of vectors. However, assuming that they can be moved independently, there is nothing to stop us from setting the knobs in any combination of settings, in particular, in combinations that correspond to triads of non-orthogonal vectors from the KS-uncolourable set. What would happen in this case? The quantum experiment inside the box cannot be effecting a simultaneous measurement of the spin squared in three directions approximating the knob settings, because these spin squared observables will not be co-measurable. Perhaps the box measures spin squared in three orthogonal directions, at least one of which is not close to the corresponding knob setting. Or perhaps the box does some kind of positive operator valued measurement. In either case, it seems that for most quantum experiments, from the observer’s point of view, the outcomes will inevitably be contextual even at the level of the quantum probabilities, and even if we unrealistically neglect the classical perturbations produced on the apparatus by altering any of the knobs. Given that the box is behaving in an overtly contextual manner even at the level of probabilities, one is then again led to ask: why should we be interested in whether the box can be described in a non-contextual fashion in the special case that we carefully restrict our knob settings so that they always correspond to orthogonal triads in the KS-uncolourable set?

Taking these points on board, careful operationalists might try to refine their position by speaking, not of a distinction between SBZ-contextuality and SBZ-non-contextuality, but instead of degrees of SBZ-contextuality. It could be argued that, although classical mechanics is indeed SBZ-contextual, the perturbations that imply SBZ-contextualities in outcome probabilities will generally be very small, and the outcome probability SBZ-contextualities correspondingly hard to detect: indeed, in principle, with sufficient care, the perturbations can be made as small as desired. In contrast, SBZ and Larsson’s results might be interpreted as implying that quantum experiments display an irreducible finite degree of SBZ-contextuality. The difficulty with this line of argument is that, as the CK models illustrate, it is *not* always true in

classical mechanics that small perturbations induce (only) correspondingly subtle effects. Operationalists would need to frame a definition which separates classical mechanics not only from the CK models (defined by infinite subsets of projective or positive operator valued decompositions) but also from infinitely many of their finite sub-models (defined, as above, by finite subsets which reproduce quantum theory to some fixed finite precision ε_r), in order to maintain both that classical mechanics is at least approximately or effectively SBZ-non-contextual and that all finite precision approximations to quantum theory defined by CK models that are precise to within ε , for some $\varepsilon > 0$, are definitively SBZ-contextual. This cannot be done: as we have already noted, in principle the finite sub-models give a prescription for building real classical devices with finitely many degrees of freedom, and these devices are, of course, described by classical mechanics.

In summary, even black box operational definitions do not allow unambiguous experimental discrimination between contextual and non-contextual theories, and thus present no challenge to CK's assertion that non-contextual theories can account for current physics. SBZ's operational definition of contextuality does give us a clear, theory-independent notion of something, but it is not contextuality in any sense consistent with standard usage. In particular, the notion defined is not able to separate the properties of quantum theory and classical mechanics, and so is not of fundamental relevance to the debate over finite precision and the KS theorem. Attractive though it would be to devise a sensible theory-independent definition of (non-)contextuality, we do not believe it is possible. We see no fundamentally satisfactory alternative to restricting ourselves to talking of *theories* as being non-contextual or contextual, and using theory-relative definitions of these terms.

5. A closing comment

We would like to emphasise that neither the preceding discussion nor earlier contributions to this debate (Kent, 1999; Clifton & Kent, 2000) are or were intended to cast doubt on the essential importance and interest of the Kochen–Specker theorem. As we have stressed throughout, our interest in examining the logical possibility of non-contextual hidden variables simulating quantum mechanics is simply that it *is* a logical—if scientifically highly implausible—possibility, which demonstrates interesting limitations on what we can rigorously infer about fundamental physics.

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