Big toy models

Representing physical systems as Chu spaces

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Abstract   We pursue a model-oriented rather than axiomatic approach to the foundations of Quantum Mechanics, with the idea that new models can often suggest new axioms. This approach has often been fruitful in Logic and Theoretical Computer Science. Rather than seeking to construct a simplified toy model, we aim for a 'big toy model', in which both quantum and classical systems can be faithfully represented— as well as, possibly, more exotic kinds of systems. To this end, we show how Chu spaces can be used to represent physical systems of various kinds. In particular, we show how quantum systems can be represented as Chu spaces over the unit interval in such a way that the Chu morphisms correspond exactly to the physically meaningful symmetries of the systems—the unitaries and antiunitaries. In this way we obtain a full and faithful functor from the groupoid of Hilbert spaces and their symmetries to Chu spaces. We also consider whether it is possible to use a finite value set rather than the unit interval; we show that three values suffice, while the two standard possibilistic reductions to two values both fail to preserve fullness.

Keywords   Quantum mechanics · Chu spaces · Representation of physical systems · Wigner's theorem

1 Introduction

1.1 Models vs. axioms

The main method pursued in the foundations of quantum mechanics has been *axiomatic*; one seeks conceptually primitive and clearly motivated axioms, shows that

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quantum systems satisfy these axioms, and then, often, aims for a *representation theorem* showing that the axioms essentially determine the "standard model" of Quantum Mechanics. Or one may admit non-standard interpretations, and seek to locate Quantum Mechanics in a larger "space" of theories.

There is an alternative and complementary approach, which has been less explored in the foundations of Quantum Mechanics, although it has proved very fruitful in mathematics, logic and theoretical computer science. Namely, one looks for conceptually natural *constructions of models*. Often a new model construction can suggest new axioms, articulated in terms of new forms of structure. There are many examples of this phenomenon, sheaves and topos theory being one case in point (MacLane and Moerdijk 1992), and domain-theoretic models of the λ-calculus another (Scott 1970).

A successful recent example of gaining insight by model construction is the well-known paper by Rob Spekkens on a toy model for Quantum Mechanics (Spekkens 2007), which has led to novel ideas on the connections between phase groups and non-locality (Coecke et al. 2009).

1.2 Big toy models

We shall also, in a sense, be concerned with "toy models" in the present article; with building models which exhibit "quantum-like" features without necessarily exactly corresponding to the standard formalism of Quantum Mechanics. Indeed, the more different the model construction can be to the usual formalism, while still reproducing many quantum-like features, the more interesting it will be from this perspective. However, there will be an important difference between the kind of model we shall study, and the usual idea of a "toy model". Usually, a toy model will be a small, simplified gadget, which gives a picture of Quantum Mechanics in some "collapsed" form, with much detail thrown away. By contrast, we are aiming for a *big* toy model, in which *both quantum and classical systems can be faithfully represented*—as well as, possibly, many more exotic kinds of systems.

1.3 Results

More precisely, we shall see how the simple, discrete notions of Chu spaces suffice to determine the appropriate notions of state equivalence, and to pick out the physically significant symmetries on Hilbert space in a very striking fashion. This leads to a full and faithful representation of the category of quantum systems, with the groupoid structure of their physical symmetries, in the category of Chu spaces valued in the unit interval. The arguments here make use of Wigner's theorem and the dualities of projective geometry, in the modern form developed by Faure and Frölicher (Faure and Frölicher 2000; Stubbe and van Steirteghem 2007). The surprising point is that unitarity/anitunitarity is essentially *forced* by the mere requirement of being a Chu morphism. This even extends to surjectivity, which here is *derived* rather than assumed.

We also consider the question of whether we can obtain a natural representation of this form in Chu spaces over a finite value set. We show that this can be done with just

three values. By contrast, the two standard possibilistic reductions to two values both *fail to preserve fullness*.

The use of Chu spaces for representing physical systems as initiated in this article seems quite promising; a number of further topics immediately suggest themselves, including mixed states, universal models, the representation of convex theories, linear types, and local logics for quantum systems.

The plan of the remainder of the article is as follows. In Sect. 2, we shall provide a brief overview of Chu spaces. Section 3 contains the main technical results, leading to a full and faithful representation of quantum systems and their symmetries as Chu spaces and morphisms of Chu spaces. Section 4 presents the results on finite value sets. Finally, Sect. 5 contains a discussion of conceptual and methodological issues.

2 Chu spaces

We shall assume that the reader is familiar with a few basic notions of category theory.[1] The bare definitions of category and functor will suffice for the most part.

Chu spaces are a special case of a construction which originally appeared in (Chu 1979), written by Po-Hsiang Chu as an appendix to Michael Barr's monograph on *-autonomous categories (Barr 1979).

Interest in *-autonomous categories increased with the advent of Linear Logic (Girard 1987), since *-autonomous categories provide models for Classical Multiplicative Linear Logic (and with additional assumptions, for the whole of Classical Linear Logic) (Seely 1989). The Chu construction applied to the category **Set** of sets and functions was independently introduced (under the name of 'games') by Yves Lafont and Thomas Streicher (Lafont and Streicher 1991), and subsequently (under the name of *Chu spaces*) formed the subject of a series of papers by Vaughan Pratt and his collaborators, e.g. (Devarajan et al. 1999; Pratt 1995, 1999). Recent papers on Chu spaces include (Droste and Zhang 2007; Palmigiano and Venema 2007).

Chu spaces have several interesting aspects:

– They have a rich type structure, and in particular form models of Linear Logic.
– They have a rich representation theory; many concrete categories of interest can be fully embedded into Chu spaces.
– There is a natural notion of 'local logic' on Chu spaces (Barwise and Seligman 1997), and an interesting characterization of information transfer across Chu morphisms (van Benthem 2000).

Applications of Chu spaces have been proposed in a number of areas, including concurrency (Pratt 2003), hardware verification (Ivanov 2008), game theory (Vannucci 2007) and fuzzy systems (Papadopoulos and Syropoulos 2000; Nguyen et al. 2001). Mathematical studies concerning the general Chu construction include (Barr 1998; Giuli and Tholen 2007).

We briefly review the basic definitions.

Fix a set K. A Chu space over K is a structure (X, A, e), where X is a set of 'points' or 'objects', A is a set of 'attributes', and $e : X \times A \to K$ is an evaluation function.

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[1] The charming introductory text (Pierce 1991) should be more than sufficient.

A morphism of Chu spaces

$$f : (X, A, e) \to (X', A', e')$$

is a pair of functions

$$f = (f_* : X \to X', f^* : A' \to A)$$

such that, for all $x \in X$ and $a' \in A'$:

$$e(x, f^*(a')) = e'(f_*(x), a').$$

Chu morphisms compose componentwise: if $f : (X_1, A_1, e_1) \to (X_2, A_2, e_2)$ and $g : (X_2, A_2, e_2) \to (X_3, A_3, e_3)$, then

$$(g \circ f)_* = g_* \circ f_*, \qquad (g \circ f)^* = f^* \circ g^*.$$

Chu spaces over K and their morphisms form a category $\mathbf{Chu}_K$.

Given a Chu space $C = (X, A, e)$, we say that C is:

– *extensional* if for all $a_1, a_2 \in A$:

$$[\forall x \in X. \, e(x, a_1) = e(x, a_2)] \implies a_1 = a_2$$

– *separated* if for all $x_1, x_2 \in X$:

$$[\forall a \in A. \, e(x_1, a) = e(x_2, a)] \implies x_1 = x_2$$

– *biextensional* if it is extensional and separated.

We define a relation on X by:

$$x_1 \sim x_2 \iff \forall a \in A. \, e(x_1, a) = e(x_2, a).$$

This is evidently an equivalence relation: C is separated exactly when this relation is the identity. There is a Chu morphism

$$(q, \mathsf{id}_A) : (X, A, e) \to (X/\sim, A, e')$$

where $e'([x], a) = e(x, a)$ and $q : X \to X/\sim$ is the quotient map. The space $(X/\sim, A, e')$ is separated; if (X, A, e) is extensional, it is biextensional.

Proposition 2.1 *If* $f : (X, A, e) \to (X', A', e')$ *is a Chu morphism, then* f_* *preserves* $\sim$. *That is, for all* $x_1, x_2 \in X$,

$$x_1 \sim x_2 \implies f_*(x_1) \sim f_*(x_2).$$

Proof For any $a' \in A'$:

$$e'(f_*(x_1), a') = e(x_1, f^*(a')) = e(x_2, f^*(a')) = e'(f_*(x_2), a'). \qquad \square$$

We shall write $\mathbf{eChu}_K$, $\mathbf{sChu}_K$ and $\mathbf{bChu}_K$ for the full subcategories of $\mathbf{Chu}_K$ determined by the extensional, separated and biextensional Chu spaces.

We shall mainly work with extensional and biextensional Chu spaces. Obviously $\mathbf{bChu}_K$ is a full sub-category of $\mathbf{eChu}_K$.

Proposition 2.2 *The inclusion* $\mathbf{bChu}_K \hookrightarrow \mathbf{eChu}_K$ *has a left adjoint.*

Proof The unit of the adjunction is the Chu morphism

$$(q, \mathsf{id}_A) : (X, A, e) \to (X/\sim, A, e')$$

we have already described, while Proposition 2.1 guarantees that given a Chu morphism

$$f : (X, A, e) \to (Y, B, r)$$

to a biextensional Chu space, we can factor it through the quotient space $(X/\sim, A, e')$.

The functor $Q : \mathbf{eChu}_K \to \mathbf{bChu}_K$ provided by this adjunction sends morphisms

$$(f_*, f^*) : (X, A, e_1) \to (X', A', e_2)$$

to

$$(f_*/\sim, f^*) : (X/\sim, A, e_1') \to (X'/\sim, A', e_2')$$

where $f_*/\sim([x]) = [f_*(x)]$. $\qquad \square$

We refer to the functor Q as the *biextensional collapse*.

We can define an equivalence relation on the Chu morphisms in each hom-set in $\mathbf{eChu}_K$ by:

$$f \sim g \iff \forall x. f_*(x) \sim g_*(x).$$

Then $Qf = Qg \Leftrightarrow f \sim g$.

2.1 Representations

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if for each pair of objects A, B of $\mathcal{C}$, the induced map $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is injective; it is *full* if each F_{AB} is surjective; and it is an *embedding* if F is faithful and injective on objects. We refer to a full and faithful functor as a *representation*, and to a full embedding as a *strict representation*. Note that if F is a representation, it can only identify isomorphic

objects. If F is a representation, then $\mathcal{C}$ is equivalent to a full sub-category of $\mathcal{D}$, while if F is a strict representation, then $\mathcal{C}$ is isomorphic to a full sub-category of $\mathcal{D}$.

As a first example of the representational capacity of Chu spaces, suppose that $\{0, 1\} \subseteq K$. For any set X, define the following Chu space on K: $(X, \mathcal{P}X, e_X)$, where:

$$e_X(x, S) = \begin{cases} 1, & x \in S \\ 0 & \text{otherwise} \end{cases}$$

Given a function $f : X \to Y$, we send it to the Chu space morphism

$$(f, f^{-1}) : (X, \mathcal{P}X, e_X) \to (Y, \mathcal{P}Y, e_Y).$$

It is easy to see that this defines a full embedding of **Set** into $\mathbf{Chu}_K$.

3 Representation of quantum systems

Our point of view in modelling physical systems as Chu spaces will be as follows. We take a system to be specified by its set of *states* S, and the set of *questions* Q which can be 'asked' of the system. We shall consider only 'yes/no' questions; however, the result of asking a question in a given state will in general be *probabilistic*. This will be represented by an evaluation function

$$e : S \times Q \to [0, 1]$$

where $e(s, q)$ is the probability that the question q will receive the answer 'yes' when the system is in state s. This is essentially the point of view taken by Mackey in his classic pioneering work on the foundations of Quantum Mechanics (Mackey 1963). Note that, following Mackey, we prefer the term 'question' to 'property', since in the case of Quantum Mechanics we cannot think in terms of static properties which are determinately possessed by a given state; questions imply a dynamic *act* of asking.

It is standard in the foundational literature on quantum mechanics to focus on yes/no questions. However, the usual approaches to quantum logic avoid the direct introduction of probabilities. We shall return to the issue of whether it is necessary to take probabilities as our value set in Sect. 4.

We can take the category **Set** itself as a crude version of discrete deterministic classical systems, with arbitrary irreversible transformations allowed. We now consider the quantum case, in the pure state formulation. Mixed states will be considered in a sequel to the present article.

Let $\mathcal{H}$ be a complex Hilbert space.[2] We define the following Chu space over $[0, 1]$:

$$(\mathcal{H}_\circ, \mathsf{L}(\mathcal{H}), e_\mathcal{H})$$

where:

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[2] A useful reference for the mathematical background is (Jordan 1969).

- $\mathcal{H}_\circ = \mathcal{H} \setminus \{\mathbf{0}\}$, the set of non-zero vectors. We shall regard all such vectors, not necessarily normalized, as representations of states of the system. Note that the zero vector is *not* a legitimate state; its rôle in Quantum Mechanics proper (as opposed to linear-algebraic calculations) is largely as an 'error element' when operations cannot legitimately be performed.
- $\mathsf{L}(\mathcal{H})$ is the lattice of closed subspaces of $\mathcal{H}$. This is the standard notion of yes/no questions in Quantum Mechanics. The observable corresponding to the subspace S is the self-adjoint operator whose spectral decomposition is $S \oplus S^\perp \cong \mathcal{H}$. To each subspace S there corresponds the projector P_S.
- The evaluation $e_\mathcal{H}$ is the fundamental formula or 'statistical algorithm' (Redhead 1987) giving the basic predictive content of Quantum Mechanics:

$$e_\mathcal{H}(\psi, S) = \frac{\langle \psi \mid P_S \psi \rangle}{\langle \psi \mid \psi \rangle} = \frac{\langle P_S \psi \mid P_S \psi \rangle}{\langle \psi \mid \psi \rangle} = \frac{\| P_S \psi \|^2}{\| \psi \|^2}.$$

Note that $e_\mathcal{H}(\psi, S) = e_\mathcal{H}(\frac{\psi}{\|\psi\|}, S)$, so this is equivalent to working with normalized vectors only.

We have thus directly transcribed the basic ingredients of the Dirac/von Neumann-style formulation of Quantum Mechanics (Dirac 1947; von Neumann 1955) into the definition of the Chu space corresponding to a given Hilbert space.

3.1 Characterizing Chu morphisms on quantum Chu spaces

Recall firstly the following explicit expression for the projection of a vector ψ on a subspace S. Let $\{e_i\}$ be an orthonormal basis for S. Then

$$P_S \psi = \sum_i \langle \psi \mid e_i \rangle e_i.$$

It follows that $\psi \perp S$ if and only if $P_S \psi = \mathbf{0}$.

We begin with a basic fact which we record explicitly.

Lemma 3.1 *For $\psi \in \mathcal{H}_\circ$ and $S \in \mathsf{L}(\mathcal{H})$:*

$$\psi \in S \iff e_\mathcal{H}(\psi, S) = 1.$$

Proof Firstly, if $\psi \in S$, then $P_S(\psi) = \psi$, so $e_\mathcal{H}(\psi, S) = 1$.

Next, we recall that $P_{S^\perp} = I - P_S$. Hence

$$e_\mathcal{H}(\psi, S^\perp) = \frac{1}{\langle \psi \mid \psi \rangle} \langle \psi - P_S \psi \mid \psi - P_S \psi \rangle$$
$$= \frac{1}{\langle \psi \mid \psi \rangle} (\langle \psi \mid \psi \rangle - \langle \psi \mid P_S \psi \rangle - \langle P_S \psi \mid \psi \rangle + \langle P_S \psi \mid P_S \psi \rangle)$$
$$= \frac{1}{\langle \psi \mid \psi \rangle} (\langle \psi \mid \psi \rangle - \langle P_S \psi \mid P_S \psi \rangle).$$

Hence

$$e_{\mathcal{H}}(\psi, S) + e_{\mathcal{H}}(\psi, S^{\perp}) = \frac{1}{\langle \psi | \psi \rangle}(\langle P_S \psi \mid P_S \psi \rangle + \langle \psi \mid \psi \rangle - \langle P_S \psi \mid P_S \psi \rangle)$$
$$= \frac{1}{\langle \psi | \psi \rangle} \langle \psi \mid \psi \rangle = 1.$$

So if $\psi \notin S$, it suffices to show that $e_{\mathcal{H}}(\psi, S^{\perp}) > 0$. In this case, $\psi = \theta + \chi$, where $\theta \in S$ and $\chi \in S^{\perp} \setminus \{\mathbf{0}\}$; so $P_{S^{\perp}}(\theta) = \mathbf{0}$ and $P_{S^{\perp}}(\chi) = \chi$. Then

$$e_{\mathcal{H}}(\psi, S^{\perp}) = \frac{1}{\langle \psi | \psi \rangle} \langle P_{S^{\perp}}(\theta) + P_{S^{\perp}}(\chi) \mid P_{S^{\perp}}(\theta) + P_{S^{\perp}}(\chi) \rangle$$
$$= \frac{1}{\langle \psi | \psi \rangle} \langle \chi \mid \chi \rangle > 0.$$

$\square$

Proposition 3.2 *The Chu space $(\mathcal{H}_{\circ}, \mathsf{L}(\mathcal{H}), e_{\mathcal{H}})$ is extensional but not separated. The equivalence classes of the relation $\sim$ on states are exactly the rays of $\mathcal{H}$. That is:*

$$\phi \sim \psi \iff \exists \lambda \in \mathbb{C}. \phi = \lambda \psi.$$

Proof Extensionality follows directly from Lemma 3.1, since if two subspaces have the same evaluations on all states, they have the same elements.

We have

$$e_{\mathcal{H}}(\lambda \psi, S) = \frac{|\lambda|^2}{|\lambda|^2} \frac{\langle P_S \psi \mid P_S \psi \rangle}{\langle \psi \mid \psi \rangle} = e_{\mathcal{H}}(\psi, S)$$

so $\phi = \lambda \psi \implies \phi \sim \psi$. For the converse, let S be the one-dimensional subspace (ray) spanned by ψ, and suppose that $\phi \notin S$. By Lemma 3.1, $e_{\mathcal{H}}(\psi, S) = 1$, while $e_{\mathcal{H}}(\phi, S) \neq 1$. Hence $\phi \not\sim \psi$. $\square$

Thus we have recovered the standard notion of pure states as the rays of the Hilbert space from the general notion of state equivalence in Chu spaces.

We shall now use some notions and results from projective geometry. We shall use the very nice Handbook article (Stubbe and van Steirteghem 2007) as a convenient reference.

Given a vector $\psi \in \mathcal{H}_{\circ}$, we write $\bar{\psi} = \{\lambda \psi \mid \lambda \in \mathbb{C}\}$ for the ray which it generates. The rays are the *atoms* in the lattice $\mathsf{L}(\mathcal{H})$.

We write $\mathsf{P}(\mathcal{H})$ for the set of rays of $\mathcal{H}$. By virtue of Proposition 3.2, we can write the biextensional collapse of $(\mathcal{H}_{\circ}, \mathsf{L}(\mathcal{H}), e_{\mathcal{H}})$ given by Proposition 2.2 as

$$(\mathsf{P}(\mathcal{H}), \mathsf{L}(\mathcal{H}), \bar{e}_{\mathcal{H}})$$

where $\bar{e}_{\mathcal{H}}(\bar{\psi}, S) = e_{\mathcal{H}}(\psi, S)$.

We restate Lemma 3.1 for the biextensional case.

Lemma 3.3 *For $\psi \in \mathcal{H}_\circ$ and $S \in \mathsf{L}(\mathcal{H})$:*

$$\bar{e}_{\mathcal{H}}(\bar{\psi}, S) = 1 \iff \bar{\psi} \subseteq S.$$

Proof Since S is a subspace, $\bar{\psi} \subseteq S$ iff $\psi \in S$, and the result follows from Lemma 3.1. □

We now turn to the issue of characterizing the Chu morphisms between these biextensional Chu representations of Hilbert spaces. This will lead to our first representation theorem.

To fix notation, suppose we have Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and a Chu morphism

$$(f_*, f^*) : (\mathsf{P}(\mathcal{H}), \mathsf{L}(\mathcal{H}), \bar{e}_{\mathcal{H}}) \to (\mathsf{P}(\mathcal{K}), \mathsf{L}(\mathcal{K}), \bar{e}_{\mathcal{K}}).$$

Proposition 3.4 *For $\psi \in \mathcal{H}_\circ$ and $S \in \mathsf{L}(\mathcal{K})$:*

$$\bar{\psi} \subseteq f^*(S) \iff f_*(\bar{\psi}) \subseteq S.$$

Proof By Lemma 3.3:

$$\bar{\psi} \subseteq f^*(S) \Leftrightarrow \bar{e}_{\mathcal{H}}(\bar{\psi}, f^*(S)) = 1 \Leftrightarrow \bar{e}_{\mathcal{K}}(f_*(\bar{\psi}), S) = 1 \Leftrightarrow f_*(\bar{\psi}) \subseteq S. \quad \square$$

Note that $\mathsf{P}(\mathcal{H}) \subseteq \mathsf{L}(\mathcal{H})$.

Proposition 3.5 *The following are equivalent:*

- *f_* is injective*
- *The following diagram commutes:*

$$
\begin{array}{ccc}
\mathsf{P}(\mathcal{H}) & \xrightarrow{\;f_*\;} & \mathsf{P}(\mathcal{K}) \\
\big\downarrow & & \big\downarrow \\
\mathsf{L}(\mathcal{H}) & \xleftarrow{\;f^*\;} & \mathsf{L}(\mathcal{K})
\end{array}
\tag{1}
$$

That is, for all $\psi \in \mathcal{H}_\circ$:

$$\bar{\psi} = f^*(f_*(\bar{\psi})).$$

Proof Clearly, (1) implies that f_* is injective. For the converse, Proposition 3.4 implies that $\bar{\psi} \subseteq f^*(f_*(\bar{\psi}))$. Now suppose that $\bar{\phi} \subseteq f^*(f_*(\bar{\psi}))$. Applying Proposition 3.4 again, this implies that $f_*(\bar{\phi}) \subseteq f_*(\bar{\psi})$. Since $f_*(\bar{\phi})$ and $f_*(\bar{\psi})$ are atoms, this implies that $f_*(\bar{\phi}) = f_*(\bar{\psi})$, which since f_* is injective implies that $\bar{\phi} = \bar{\psi}$. Thus the only

atom below $f^*(f_*(\bar{\psi}))$ is $\bar{\psi}$. Since $\mathsf{L}(\mathcal{H})$ is *atomistic* (Stubbe and van Steirteghem 2007), this implies that $f^*(f_*(\bar{\psi})) \subseteq \bar{\psi}$.   □

We state another important basic property of the evaluation.

Lemma 3.6 *For any $\phi, \psi \in \mathcal{H}_\circ$:*

$$\bar{e}_\mathcal{H}(\bar{\phi}, \bar{\psi}) = 0 \iff \phi \perp \psi.$$

Proof

$$\bar{e}_\mathcal{H}(\bar{\phi}, \bar{\psi}) = 0 \Leftrightarrow \langle P_{\bar{\psi}}(\phi) \mid P_{\bar{\psi}}(\phi) \rangle = 0 \Leftrightarrow P_{\bar{\psi}}(\phi) = \mathbf{0} \Leftrightarrow \phi \perp \psi.$$

□

Proposition 3.7 *If f_* is injective, it preserves and reflects orthogonality. That is, for all $\phi, \psi \in \mathcal{H}_\circ$:*

$$\phi \perp \psi \iff f_*(\bar{\phi}) \perp f_*(\bar{\psi}).$$

Proof

$$\begin{aligned}
\phi \perp \psi &\iff \bar{e}_\mathcal{H}(\bar{\phi}, \bar{\psi}) = 0 && \text{Lemma 3.6} \\
&\iff \bar{e}_\mathcal{H}(\bar{\phi}, f^*(f_*(\bar{\psi}))) = 0 && \text{Proposition 3.5} \\
&\iff \bar{e}_\mathcal{K}(f_*(\bar{\phi}), f_*(\bar{\psi})) = 0 && \\
&\iff f_*(\bar{\phi}) \perp f_*(\bar{\psi}) && \text{Lemma 3.6}
\end{aligned}$$

□

We define a map $f^\rightarrow : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{K})$:

$$f^\rightarrow(S) = \bigvee \{f_*(\bar{\psi}) \mid \psi \in S_\circ\}$$

where $S_\circ = S \setminus \{\mathbf{0}\}$.

Lemma 3.8 *The map $f^\rightarrow$ is left adjoint to f^*.*

Proof We must show that, for all $S \in \mathsf{L}(\mathcal{H})$ and $T \in \mathsf{L}(\mathcal{K})$:

$$f^\rightarrow(S) \subseteq T \iff S \subseteq f^*(T).$$

Using Proposition 3.4, we have:

$$\begin{aligned}
f^\rightarrow(S) \subseteq T &\iff \forall \psi \in S_\circ.\, f_*(\bar{\psi}) \subseteq T \\
&\iff \forall \psi \in S_\circ.\, \bar{\psi} \subseteq f^*(T) \\
&\iff S \subseteq f^*(T).
\end{aligned}$$

□

We can now extend the diagram (1):

$$
\begin{array}{ccc}
\mathsf{P}(\mathcal{H}) & \xrightarrow{\;f_*\;} & \mathsf{P}(\mathcal{K}) \\[2pt]
\cap\Big\downarrow & & \Big\downarrow\cap \\[2pt]
\mathsf{L}(\mathcal{H}) & \underset{f^*}{\overset{f^{\rightarrow}}{\underset{\perp}{\rightleftarrows}}} & \mathsf{L}(\mathcal{K})
\end{array}
\tag{2}
$$

By construction, $f^{\rightarrow}$ extends f_*: this says that $f^{\rightarrow}$ preserves atoms. Since $f^{\rightarrow}$ is a left adjoint, it preserves sups. Hence $f^{\rightarrow}$ and f_* are paired under the duality of projective lattices and projective geometries, for which see Theorem 16 of (Stubbe and van Steirteghem 2007). In particular, we have the following.

Proposition 3.9 f_* *is a total map of projective geometries* (Stubbe and van Steirteghem 2007).

It follows that we can apply *Wigner's Theorem*, in the form given as Theorem 4.1 in (Faure 2002). In order to state this, we need some additional notions.

Let V_1 be a vector space over the field $\mathbb{F}$ and V_2 a vector space over the field $\mathbb{G}$. A *semilinear map* from V_1 to V_2 is a pair (f, α) where $\alpha : \mathbb{F} \rightarrow \mathbb{G}$ is a field homomorphism, and $f : V_1 \rightarrow V_2$ is an additive map such that, for all $\lambda \in \mathbb{F}$ and $v \in V_1$:

$$
f(\lambda v) = \alpha(\lambda) f(v).
$$

Note that semilinear maps compose: if $(f, \alpha) : V_1 \rightarrow V_2$ and $(g, \beta) : V_2 \rightarrow V_3$, then $(g \circ f, \beta \circ \alpha) : V_1 \rightarrow V_2$ is a semilinear map.

This notion is usually defined in greater generality, for division rings, but we are only concerned with Hilbert spaces over the complex numbers.

Given a semilinear map $g : V_1 \rightarrow V_2$, we define $\mathsf{P}g : \mathsf{P}V_1 \rightarrow \mathsf{P}V_2$ by

$$
\mathsf{P}(g)(\bar{\psi}) = \overline{g(\psi)}.
$$

We can now state Wigner's Theorem in the form we shall use it.

Theorem 3.10 *Let* $f : \mathsf{P}(\mathcal{H}) \rightarrow \mathsf{P}(\mathcal{K})$ *be a total map of projective geometries, where* $\dim \mathcal{H} > 2$. *If* f *preserves orthogonality, meaning that*

$$
\bar{\phi} \perp \bar{\psi} \;\Rightarrow\; f(\bar{\phi}) \perp f(\bar{\psi})
$$

then there is a semilinear map $g : \mathcal{H} \rightarrow \mathcal{K}$ *such that* $\mathsf{P}(g) = f$, *and*

$$
\langle g(\phi) \mid g(\psi) \rangle = \sigma(\langle \phi \mid \psi \rangle),
$$

where σ is the homomorphism associated with g. Moreover, this homomorphism is either the identity or complex conjugation, so g is either linear or antilinear. The map g is unique up to a phase, i.e. a scalar of modulus 1.

The final statement follows from the Second Fundamental Theorem of Projective Geometry, Theorem 3.1 in (Faure 2002) or Theorem 46 in (Stubbe and van Steirteghem 2007).

Note that in our case, taking $f_* = f$, $\mathsf{P}g$ is just the action of the biextensional collapse functor on Chu morphisms.

Note that a total map of projective geometries must necessarily come from an *injective* map g on the underlying vector spaces, since $\mathsf{P}(g)$ maps rays to rays, and hence g must have trivial kernel. For this reason, partial maps of projective geometries are considered in the Faure-Frölicher approach (Faure and Frölicher 2000; Stubbe and van Steirteghem 2007). However, we are simply following the 'logic' of Chu space morphisms here.

Proposition 3.11 *Let $g : \mathcal{H} \to \mathcal{K}$ be a semilinear morphism such that $\mathsf{P}(g) = f_*$ where f is a Chu space morphism, and $\dim(\mathcal{H}) > 0$. Suppose that the endomorphism $\sigma : \mathbb{C} \to \mathbb{C}$ associated with g is surjective, and hence an automorphism. Then g is surjective.*

Proof We write $\mathrm{Im}\, g$ for the set-theoretic direct image of g, which is a linear subspace of $\mathcal{K}$, since σ is an automorphism. In particular, g carries rays to rays, since $\lambda g(\phi) = g(\sigma^{-1}(\lambda)\phi)$.

We claim that for any vector $\psi \in \mathcal{K}_\circ$ which is not in the image of g, $\psi \perp \mathrm{Im}\, g$. Given such a ψ, for any $\phi \in \mathcal{H}_\circ$ it is not the case that $f_*(\bar{\phi}) \subseteq \bar{\psi}$; for otherwise, for some λ, $g(\phi) = \lambda\psi$, and hence $g(\sigma^{-1}(\lambda^{-1})\phi) = \psi$. Then by Proposition 3.4, $f^*(\bar{\psi}) = \{\mathbf{0}\}$. It follows that for all $\phi \in \mathcal{H}_\circ$,

$$\bar{e}_\mathcal{K}(f_*(\bar{\phi}), \bar{\psi}) = \bar{e}_\mathcal{H}(\bar{\phi}, \{\mathbf{0}\}) = 0,$$

and hence by Lemma 3.6 that $\psi \perp \mathrm{Im}\, g$.

Now suppose for a contradiction that such a ψ exists. Consider the vector $\psi + \chi$ where χ is a non-zero vector in $\mathrm{Im}\, g$, which must exist since g is injective and $\mathcal{H}$ has positive dimension. This vector is not in $\mathrm{Im}\, g$, nor is it orthogonal to $\mathrm{Im}\, g$, since e.g. $\langle \psi + \chi \mid \chi \rangle = \langle \chi \mid \chi \rangle \neq 0$. This yields the required contradiction. □

We can now put the pieces together to obtain the main result of this section. We say that a map $U : \mathcal{H} \to \mathcal{K}$ is *semiunitary* if it is either unitary or antiunitary; that is, if it is a bijective map satisfying

$$U(\phi + \psi) = U\phi + U\psi, \qquad U(\lambda\phi) = \sigma(\lambda)U\phi, \quad \langle U\phi \mid U\psi \rangle = \sigma(\langle \phi \mid \psi \rangle)$$

where σ is the identity if U is unitary, and complex conjugation if U is antiunitary. Note that semiunitaries preserve norm, so if U and V are semiunitaries and $U = \lambda V$, then $|\lambda| = 1$.

Theorem 3.12 *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces of dimension greater than 2. Consider a Chu morphism*

$$(f_*, f^*) : (\mathsf{P}(\mathcal{H}), \mathsf{L}(\mathcal{H}), \bar{e}_{\mathcal{H}}) \to (\mathsf{P}(\mathcal{K}), \mathsf{L}(\mathcal{K}), \bar{e}_{\mathcal{K}}).$$

where f_* is injective. Then there is a semiunitary $U : \mathcal{H} \to \mathcal{K}$ such that $f_* = \mathsf{P}(U)$. U is unique up to a phase.

Proof By the proviso on injectivity, we can apply Proposition 3.7. By this and Proposition 3.9, together with the proviso on dimension, we can apply Wigner's Theorem 3.10. Since the semilinear map in Wigner's Theorem has an associated automorphism, we can apply Proposition 3.11. □

3.2 The representation theorem

We now turn to the big picture. We define a category **SymmH** as follows:

– The objects are Hilbert spaces of dimension > 2.
– Morphisms $U : \mathcal{H} \to \mathcal{K}$ are semiunitary (i.e. unitary or antiunitary) maps.
– Semiunitaries compose as explained more generally for semilinear maps in the previous subsection. Since complex conjugation is an involution, semiunitaries compose according to the rule of signs: two antiunitaries or two unitaries compose to form a unitary, while a unitary and an antiunitary compose to form an antiunitary.

This category is a groupoid, i.e. every arrow is an isomorphism.

The semiunitaries are the *physically significant symmetries of Hilbert space* from the point of view of Quantum Mechanics. The usual dynamics according to the Schrödinger equation is given by a continuous one-parameter group $\{U(t)\}$ of these symmetries; the requirement of continuity forces the $U(t)$ to be unitaries.[3] However, some important physical symmetries are represented by antiunitaries, e.g. *time reversal* and *charge conjugation*.

By the results of the previous subsection, Chu morphisms essentially force us to consider the symmetries on Hilbert space. As pointed out there, linear maps which can be represented as Chu morphisms in the biextensional category must be injective; and if $g : \mathcal{H} \to \mathcal{K}$ is an injective linear or antilinear map, then $\mathsf{P}(g)$ is injective. Our results then show that if g can be represented as a Chu morphism, it must in fact be semiunitary. This delineation of the physically significant symmetries by the logic of Chu morphisms should be seen as a strong point in favour of this representation by Chu spaces.

We define a functor $R : \mathbf{SymmH} \to \mathbf{eChu}_{[0,1]}$:

$$R : U : \mathcal{H} \to \mathcal{K} \; \longmapsto \; (U_\circ, U^{-1}) : (\mathcal{H}_\circ, \mathsf{L}(\mathcal{H}), e_{\mathcal{H}}) \to (\mathcal{K}_\circ, \mathsf{L}(\mathcal{K}), e_{\mathcal{K}})$$

where $U_\circ$ is the restriction of U to $\mathcal{H}_\circ$.

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[3] Indeed, the Schrödinger equation can actually be recovered from this group via Stone's Theorem (Simon 1976).

As noted in Proposition 2.2, the inclusion $\mathbf{bChu}_{[0,1]} \hookrightarrow \mathbf{eChu}_{[0,1]}$ has a left adjoint, which we name Q. By Proposition 3.2, this can be defined on the image of R as follows:

$$Q : (\mathcal{H}_\circ, \mathsf{L}(\mathcal{H}), e_\mathcal{H}) \mapsto (\mathsf{P}\mathcal{H}, \mathsf{L}(\mathcal{H}), \bar{e}_\mathcal{H})$$

and for $(U_\circ, U^{-1}) : (\mathcal{H}_\circ, \mathsf{L}(\mathcal{H}), e_\mathcal{H}) \to (\mathcal{K}_\circ, \mathsf{L}(\mathcal{K}), e_\mathcal{K})$,

$$Q : (U_\circ, U^{-1}) \longmapsto (\mathsf{P}U, U^{-1}).$$

We write **emChu**, **bmChu** for the subcategories of $\mathbf{eChu}_{[0,1]}$ and $\mathbf{bChu}_{[0,1]}$ obtained by restricting to Chu morphisms f for which f_* is injective. The functors R and Q factor through these subcategories.

Proposition 3.13 $R : \mathbf{SymmH} \to \mathbf{emChu}$ *and* $Q : \mathbf{emChu} \to \mathbf{bmChu}$ *are functors. R is faithful but not full; Q is full but not faithful.*

Proof We verify that if $U : \mathcal{H} \to \mathcal{K}$ is semiunitary, RU is a well-defined morphism in **emChu**. Firstly, we verify the Chu morphism condition. Since U is semiunitary, for $\psi \in \mathcal{H}_\circ$ and $S \in \mathsf{L}(\mathcal{K})$:

$$P_S(U\psi) = U(P_{U^{-1}(S)}\psi).$$

Indeed, if U is unitary, let $\{e_i\}$ be an orthonormal basis for S. Then $\{U^{-1}e_i\}$ is an orthonormal basis for $U^{-1}S$. Now

$$U(P_{U^{-1}(S)}\psi) = U\left(\sum_i \langle \psi \mid U^{-1}e_i\rangle U^{-1}e_i\right)$$

$$= \sum_i \langle \psi \mid U^{-1}e_i\rangle e_i$$

$$= \sum_i \langle U\psi \mid e_i\rangle e_i$$

$$= P_S U\psi$$

where the third equation holds because $U^{-1} = U^\dagger$. A similar calculation holds if U is antiunitary. In this case, the inner product is commuted when we apply conjugate linearity in the second equation, and commuted back in the third, since for an antiunitary we have

$$\langle U^{-1}e_i \mid \psi \rangle = \langle U^{-1}e_i \mid U^{-1}U\psi \rangle = \langle U\psi \mid e_i \rangle,$$

leading to the same result.

Moreover, U preserves norms, so $\|U\psi\| = \|\psi\|$. Now

$$\langle P_S U\psi \mid P_S U\psi \rangle = \langle U(P_{U^{-1}(S)}\psi) \mid U(P_{U^{-1}(S)}\psi) \rangle$$

$$= \langle P_{U^{-1}(S)}\psi \mid P_{U^{-1}(S)}\psi \rangle.$$

Hence $e_{\mathcal{H}}(\psi, U^{-1}(S)) = e_{\mathcal{K}}(U\psi, S)$, so $(U_\circ, U^{-1})$ is a Chu morphism. Finally, U is bijective, so $U_\circ$ is injective.   □

We can analyze the non-fullness of R more precisely as follows.

Proposition 3.14 *Let* $(U_\circ, U^{-1}) : (\mathcal{H}_\circ, \mathsf{L}(\mathcal{H}), e_{\mathcal{H}}) \to (\mathcal{K}_\circ, \mathsf{L}(\mathcal{K}), e_{\mathcal{K}})$ *be a Chu morphism in the image of R. Given an arbitrary function* $f : \mathcal{H} \to \mathbb{C} \setminus \{0\}$, *define* $fU : \mathcal{H}_\circ \to \mathcal{K}_\circ$ *by:*

$$fU(\psi) = f(\psi)U(\psi).$$

Then $(fU, U^{-1}) \sim (U_\circ, U^{-1})$. *Moreover, the* $\sim$-*equivalence class of U is exactly the set of functions of this form.*

Thus before biextensional collapse, Chu morphisms can introduce arbitrary scalar factors pointwise. Once we move to the biextensional category, we know by Theorem 3.12 that our representation will be full, and essentially faithful—up to a global phase. This points to the need for a projective version of the symmetry groupoid.

The mathematical object underlying phases is the *circle group* $\mathsf{U}(1)$:

$$\mathsf{U}(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} = \left\{ e^{i\theta} \mid \theta \in \mathbb{R} \right\}$$

Since λ has modulus 1 if and only if $\lambda\bar{\lambda} = 1$, $\mathsf{U}(1)$ is the unitary group on the one-dimensional Hilbert space.

The circle group acts on the symmetry groupoid **SymmH** by scalar multiplication. For Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ we can define

$$\mathsf{U}(1) \times \mathbf{SymmH}(\mathcal{H}, \mathcal{K}) \to \mathbf{SymmH}(\mathcal{H}, \mathcal{K}) :: (\lambda, U) \mapsto \lambda U.$$

Moreover, this is a category action, meaning that

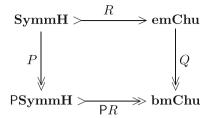$$(\lambda U) \circ V = U \circ (\lambda V) = \lambda (U \circ V).$$

It follows that we can form a quotient category (in fact again a groupoid) with the same objects as **SymmH**, and in which the morphisms will be the orbits of this group action:

$$U \sim V \iff \exists \lambda \in \mathsf{U}(1). U = \lambda V.$$

We call the resulting category **PSymmH**, the *projective quantum symmetry groupoid*. It is a natural generalization of the standard notion of the *projective unitary group* on Hilbert space. There is a quotient functor $P : \mathbf{SymmH} \to \mathbf{PSymmH}$, and by virtue of Theorem 3.12, we can factor $Q \circ R$ through this quotient to obtain a functor $PR : \mathbf{PSymmH} \to \mathbf{bmChu}$.

The situation can be summarized by the following diagram:

$$\begin{CD} \mathbf{SymmH} @>{R}>> \mathbf{emChu} \\ @V{P}VV @VV{Q}V \\ \mathbf{PSymmH} @>>{\mathsf{P}R}> \mathbf{bmChu} \end{CD}$$

Theorem 3.15 *The functor* $\mathsf{P}R : \mathbf{PSymmH} \to \mathbf{bmChu}$ *is a representation.*

Proof This follows from Theorem 3.12. To see that $\mathsf{P}R$ is essentially injective on objects, we can use the representation theorems of Piron and Solèr (Stubbe and van Steirteghem 2007), which tell us that we can reconstruct $\mathcal{H}$ as a Hilbert space from $\mathsf{L}(\mathcal{H})$. This reconstruction will give us a Hilbert space $\mathcal{H}'$ such that $\mathsf{L}(\mathcal{H}) \cong \mathsf{L}(\mathcal{H}')$, and $\mathsf{P}(\mathcal{H}) \cong \mathsf{P}(\mathcal{H}')$. We can apply Wigner's theorem to this isomorphism to obtain a semiunitary $U : \mathcal{H} \cong \mathcal{H}'$ from which we can recover the Hilbert space structure on $\mathcal{H}$. This means that we have recovered $\mathcal{H}$ uniquely to within the coset of $\mathsf{id}_{\mathcal{H}}$ in **PSymmH**.   □

4 Reducing the value set

We now return to the issue of whether it is necessary to use the full unit interval as the value set for our Chu spaces.

   We begin with some generalities. Given a function $v : K \to L$, we define a functor $F_v : \mathbf{Chu}_K \to \mathbf{Chu}_L$:

$$F_v : (X, A, e) \mapsto (X, A, v \circ e)$$

and $F_v f = f$ for Chu morphisms f.

Proposition 4.1 F_v *is a faithful functor. If* v *is injective, it is full.*

Proof This is easily verified. The Chu morphism condition is preserved by composing with any function on values, while F_v is evidently faithful. For fullness, note that the only values in L relevant to whether a pair of functions

$$(f, g) : (X, A, v \circ e) \to (X', A', v \circ e')$$

satisfies the Chu morphism condition are those in the ranges of $v \circ e$ and $v \circ e'$, which if v is injective are in bijection with those in the ranges of e and e'.   □

   We can now state the question we wish to pose more precisely:

Is there a mapping $v : [0, 1] \to K$ from the unit interval to some finite set K such that the restriction of the functor F_v to the image of $\mathsf{P}R$ is full, and thus the composition

$$F_v \circ PR : \mathbf{PSymmH} \to \mathbf{bmChu}_K$$

is a representation?

There is no *general* reason to suppose that this is possible; in fact, we shall show that it is, although not quite in the obvious fashion.

We shall write $\mathbf{n} = \{0, \ldots, n-1\}$ for the finite ordinals. The most popular choice of value set for Chu spaces, by far, has been $\mathbf{2}$, and indeed many interesting categories can be strictly (and even concretely) represented in $\mathbf{Chu_2}$ (Pratt 1995). This makes the following question natural:

Question 4.2 *Is there a function $v : [0, 1] \to \mathbf{2}$ such that $F_v \circ PR$ is full and faithful?*

What we can show is that the most plausible candidates for such functions, yielding the two canonical forms of *possibilistic semantics* as a coarse-graining of probabilistic semantics, both in fact *fail*.

Note that any function $v : [0, 1] \to \{0, 1\}$ must lose information either on 0 or on 1 – or both. In this sense, the two 'sharpest' mappings[4] will be:

$$v_0 : 0 \mapsto 0, (0, 1] \mapsto 1 \qquad v_1 : [0, 1) \mapsto 0, 1 \mapsto 1.$$

These are the two canonical reductions of probabilistic to possibilistic information: the first maps 'definitely not' to 'no', and anything else to 'yes', which is to be read as 'possibly yes'; the second maps 'definitely yes' to 'yes', and anything else to 'no', to be read as 'possibly no'. Note that, under the first of these, Lemma 3.1 will no longer hold, while under the second, Lemma 3.6 will fail.

Proposition 4.3 *For neither $v = v_0$ nor $v = v_1$ is $F_v \circ PR$ full.*

Proof Let $\mathcal{H}$ be a Hilbert space with $2 < \dim \mathcal{H} < \infty$, and let (g, σ) be any semilinear automorphism of $\mathcal{H}$, where σ can be any automorphism of the complex field.[5] For each of the above two mappings of the unit interval to $\mathbf{2}$, we shall construct a $\mathbf{Chu_2}$ endomorphism $f : F_v \circ PR(\mathcal{H}) \to F_v \circ PR(\mathcal{H})$ with $f_* = P(g)$. This will show the non-fullness of F_v.

Case 1 Here we consider the mapping v_1 which sends $[0, 1)$ to 0 and fixes 1. In this case:

$$\bar{e}_{\mathcal{H}}(\bar{\psi}, S) = 1 \iff \psi \in S$$

and hence the Chu morphism condition on (f_*, f^*), where $f_* = P(g)$, is:

$$\psi \in f^*(S) \iff g(\psi) \in S.$$

Taking $f^* = g^{-1}$ obviously fulfills this condition. Note that, since g is a semilinear automorphism, and $\mathcal{H}$ is finite-dimensional, $g^{-1} : L(\mathcal{H}) \to L(\mathcal{H})$ is well-defined.

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[4] We consider only functions which fix 0 and 1, to exclude irrelevant permutations and the trivial case of constant maps.

[5] We can extend the argument to infinite-dimensional Hilbert space by requiring g to be continuous.

Case 2 Now consider the mapping v_0 keeping 0 fixed and sending $(0, 1]$ to 1. In this case:

$$\bar{e}_{\mathcal{H}}(\bar{\psi}, S) = 0 \iff \psi \perp S$$

and hence the Chu morphism condition on (f_*, f^*), where $f_* = \mathsf{P}(g)$, is:

$$\psi \perp f^*(S) \iff g(\psi) \perp S.$$

We define $f^*(S) = g^{-1}(S^\perp)^\perp$. Note that $f^* : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ is well defined, and also that $g^{-1}(S^\perp)$ is a subspace of $\mathcal{H}$; hence $g^{-1}(S^\perp)^{\perp\perp} = g^{-1}(S^\perp)$. Now:

$$\begin{aligned}
\bar{e}_{\mathcal{H}}(\bar{\psi}, f^*S) = 0 &\iff \psi \perp f^*S \\
&\iff \psi \in g^{-1}(S^\perp)^{\perp\perp} = g^{-1}(S^\perp) \\
&\iff g(\psi) \in S^\perp \\
&\iff g(\psi) \perp S \\
&\iff \bar{e}_{\mathcal{H}}(f_*(\bar{\psi}), S) = 0
\end{aligned}$$

and hence (f_*, f^*) is a Chu morphism as required. $\square$

However, this negative result immediately suggests a remedy: *to keep the interpretations of* **both** *0 and 1 sharp*. We can do this with three values! Namely, we define $v : [0, 1] \to \mathbf{3}$ by

$$0 \mapsto 0, \quad (0, 1) \mapsto 2, \quad 1 \mapsto 1$$

Thus we lose information only on the probabilities strictly between 0 and 1, which are lumped together as 'maybe'—represented here, by arbitrary convention, by 2.

Why is this adequate? Given a vector ψ and a subspace S, we can write ψ uniquely as $\theta + \chi$, where $\theta \in S$ and $\chi \in S^\perp$. For non-zero ψ, there are only three possibilities: $\theta = \mathbf{0}$ and $\chi \neq \mathbf{0}$, which yields $e_{\mathcal{H}}(\phi, S) = 0$ by Lemma 3.6; $\theta \neq \mathbf{0}$ and $\chi = \mathbf{0}$ which yields $e_{\mathcal{H}}(\phi, S) = 1$ by Lemma 3.1; and $\theta \neq \mathbf{0}$ and $\chi \neq \mathbf{0}$, which yields $e_{\mathcal{H}}(\psi, S) \in (0, 1)$ by these Lemmas again, and hence $v \circ e_{\mathcal{H}}(\psi, S) = 2$. *These are the only case discriminations which are used in our results leading to the Representation Theorem* 3.15. Hence we have:

Theorem 4.4 *The functor $F_v \circ \mathsf{P}R : \mathbf{PSymmH} \to \mathbf{bmChu_3}$ is a representation.*

We note that **Chu$_3$** has found some uses in concurrency and verification (Pratt 2003; Ivanov 2008), under a temporal interpretation: the three values are read as 'before', 'during' and 'after', whereas in our setting the three values represent 'definitely yes', 'definitely no' and 'maybe'.

Theorem 4.4 may suggest some interesting uses for 3-valued 'local logics' in the sense of Jon Barwise (Barwise and Seligman 1997).

5 Discussion

We should understand Chu spaces as providing a very general (and, we might reasonably say, rather simple) 'logic of systems or structures'. Indeed, they have been proposed by Barwise and Seligman as the vehicle for a general logic of 'distributed systems' and information flow (Barwise and Seligman 1997). This logic of Chu spaces was in no way biassed in its conception towards the description of quantum mechanics or any other kind of physical system. Just for this reason, it is interesting to see how much of quantum-mechanical structure and concepts can be absorbed and essentially *determined* by this more general systems logic.

It might be argued that our representation of quantum systems as Chu spaces has already specified the essential ingredients of the quantum structure 'by hand'. The conceptual significance of our technical results is precisely to show that there is a non-trivial 'capturing' of quantum structure by the general notions of Chu spaces:

– Firstly, Proposition 3.2 shows that the general Chu space notion of biextensionality subsumes the standard identification of quantum states with rays in Hilbert space. This is scarcely surprising, but it is a first sign of the proper alignment of concepts.
– The main technical result of the present article is the Representation Theorem 3.15. It is worth spelling out the content of this in more elementary terms. Once we have represented our quantum systems as biextensional Chu spaces $(\mathsf{P}(\mathcal{H}), \mathsf{L}(\mathcal{H}), e_{\mathcal{H}})$, all we have, from the viewpoint 'inside' the category $\mathbf{Chu}_{[0,1]}$, is a pair of sets and an evaluation function, with all information about their provenance lost. A Chu morphism

$$(f_*, f^*) : (\mathsf{P}(\mathcal{H}), \mathsf{L}(\mathcal{H}), e_{\mathcal{H}}) \to (\mathsf{P}(\mathcal{K}), \mathsf{L}(\mathcal{H}), e_{\mathcal{K}})$$

is given by *any* pair of set-theoretic functions (f_*, f^*) satisfying the Chu morphism condition:

$$\bar{e}_{\mathcal{H}} \left(\bar{\psi}, f^*(S) \right) = \bar{e}_{\mathcal{K}} \left(f_*(\bar{\psi}), S \right).$$

The Representation Theorem says that *the logic of this Chu morphism condition is strong enough to guarantee that any such pair of functions must arise from a unitary or antiunitary map $U : \mathcal{H} \to \mathcal{K}$ on the underlying Hilbert spaces*, with the sole proviso of injectivity of f_*.[6] Moreover, U is uniquely determined by f_* up to a phase factor. Of course, we are using one of the 'big guns' of the subject, Wigner's Theorem, to establish this result. It is worth noting, though, that there is some distance to travel between the Chu morphism condition and the hypotheses of Wigner's Theorem; and there are surprises along the way, most notably Proposition 3.11, which *derives* surjectivity from the Chu morphism condition—whereas

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[6] The injectivity assumption on f_* is annoying. It remains unclear if it necessary.

it must invariably be added as a hypothesis to the many versions of Wigner's
Theorem.[7]

– The results on reduction to finite value sets are also intriguing. Not only is the bare
  Chu condition on morphisms sufficient to whittle them down to the semiunitaries,
  this is even the case when the discriminations on which the condition is based
  are reduced to three values. The general case for two values remains open, but
  we have shown that the two standard possibilistic reductions both *fail to preserve
  fullness*. A negative answer for two-valued semantics in general would suggest an
  unexpected rôle for three-valued logic in the foundations of Quantum Mechanics.

5.1 Where next?

Of course, the developments described in the present article are only a first step. We
shall briefly discuss some of the natural continuations of these ideas, several of which
are already in progress.

– There are some interesting and surprising connections between Chu spaces and
  another important paradigm for categorical systems modelling, namely *coalge-
  bra* (Rutten 2000). These connections, which seem not to have been explored
  previously, arise both at the general level, and also with specific reference to the
  representation of physical systems. They are described in a sequel to the present
  article (Abramsky 2010), which lifts the results of the present article to a coalge-
  braic setting. The bivariant nature of Chu spaces is reflected in a novel fibred form
  of coalgebra, in which contravariance is represented as *indexing*.
– A natural next step as regards physical modelling is to consider *mixed states*. There
  is a general construction on Chu spaces which allows mixed states to be studied in
  a uniform fashion, applicable to both classical and quantum systems. This will be
  described in a forthcoming sequel to the present article.
– There are intriguing connections between our approach, and the work of the
  'Geneva School' of Jauch and Piron (Jauch 1968; Piron 1976), particularly (Faure
  et al. 1995). We plan to explore these in a joint paper with Bob Coecke, Isar Stubbe
  and Frank Valckenborgh.
– It is also of interest to consider *universal* Chu spaces; single systems in which all
  Chu spaces of a given class can be embedded, and which therefore provide a single
  model for a large class of systems. We may additionally ask for such systems to
  be *homogeneous*, which means that they exhibit a maximum degree of symme-
  try; such universal, homogeneous spaces are unique up to isomorphism. Universal
  homogeneous Chu spaces have been constructed for *bifinite Chu spaces* in recent
  work by Manfred Droste and Guo-Qiang Zhang (Droste and Zhang 2007). That

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[7] One of the journal referees remarked that surjectivity is not taken as a hypothesis in the version of
Wigner's Theorem due to Wright (Wright 1977). Wright's Corollary 1 does obtain surjectivity from the
assumption that the function $f : \mathsf{P}(\mathcal{H}) \to \mathsf{P}(\mathcal{K})$ on rays can be extended to a *projection-valued state*, i.e. a
map $\mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{K})$ which preserves orthogonal joins and the top element. These are strong assumptions,
from which surjectivity follows immediately, as Wright observes. By contrast, the injectivity of f and the
Chu morphism condition, with respect to an a priori otherwise unspecified map $\mathsf{L}(\mathcal{K}) \to \mathsf{L}(\mathcal{H})$, are much
weaker assumptions, which do not 'build in' surjectivity in any obvious fashion.

context is too limited for our purposes here. It remains to be seen if universal homogeneous models can be constructed for larger subcategories of Chu spaces, encompassing those involved in our representation results.

– The relation of the rich logical and type-theoretic aspects of Chu spaces to quantum and other physical systems should also be investigated.

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