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Limit Laws for Non-additive Probabilities and Their Frequentist Interpretation*

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In this paper we prove several limit laws for non-additive probabilities. In particular, we prove that, under a multiplicative notion of independence and a regularity condition, if the elements of a sequence $\{X_k\}_{k\geqslant 1}$ are i.i.d. random variables relative to a totally monotone and continuous capacity v, then

$$v\left(\left\{\int X_1\,dv\leqslant \liminf_n \frac{1}{n}\sum_{k=1}^n X_k\leqslant \limsup_n \frac{1}{n}\sum_{k=1}^n X_k\leqslant -\int -X_1\,dv\right\}\right)=1.$$

Since in the additive case $\int X_1\,dv=-\int -X_1\,dv$, this is an extension of the classic Kolmogorov's Strong Law of Large Numbers to the non-additive case. We argue that this result suggests a frequentist perspective on non-additive probabilities. *Journal of Economic Literature* Classification Numbers: C60, D81. © 1999 Academic Press

1. INTRODUCTION

The literature on limit theorems in Probability Theory has always considered additive probabilities. In both objective and subjective settings, additivity has been generally considered a fairly natural assumption. In a frequentist set-up, the dominant objective interpretation of probability, additivity is obviously compelling. But, in the subjective interpretation as well, additivity has been for a long time an uncontroversial assumption, justified as a rationality requirement (e.g., de Finetti's notion of coherence; see de Finetti [2]).

However, in some areas this apparently quite natural property has been abandoned in favor of non-additive probabilities. In subjective probability

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this happened because additivity prevents an effective analysis of the degree of confidence that decision makers have in their probability assessments. A number of papers in artificial intelligence, mathematical economics, and statistics have used non-additive probabilities to study the implications of this crucial feature of subjective assessments (see, e.g., Dempster [4, 5], Gilboa [13], Huber [18, 19], Shafer [30], Schmeidler [26, 28], Wasserman and Kadane [36], Seidenfeld and Wasserman [29], Wakker [33], and Walley [34]).

A bit more surprisingly, non-additive probabilities have been used also in a frequentist setting, especially in Quantum Mechanics. Indeed, as a consequence of the famous wave–particle duality, the probabilities that describe quantum phenomena are generally non-additive, even though a frequentist interpretation is usually attached to them (cf. Feynman and Hibbs [11]). This has often been regarded as a puzzling feature of the subject.

Our study of limit laws for non-additive probabilities has been motivated by their importance in both subjective and objective settings. Here we focus on the subjective side. The standard example used to explain the importance of non-additivity in subjective probability is the classic Ellsberg Paradox, due to Ellsberg [10]. There are two urns: urn I, of which both composition and proportion are known, say, 50 red and 50 black balls; and urn II, in which the proportion of red and black balls is not known. In urn I it is natural to consider as equiprobable both the drawing of a red ball (event R_I) and that of a black ball (event B_I). Even in urn II, though, considerations of symmetry suggest that the events R_{II} and B_{II} are equally likely. In classic subjective probability for both urns we would end up with

$$P(R_I) = P(B_I) = P(R_{II}) = P(B_{II}) = \tfrac{1}{2}.$$

However, it is natural to expect that the confidence which the decision maker (DM) has in the two assignments is different, as he might well prefer to base a decision on the probabilities $P(R_I)$ and $P(B_I)$ rather than on $P(R_{II})$ and $P(B_{II})$. With non-additive probabilities it is possible to take care of this problem in a rather simple way. For example, the decision maker may retain the symmetry based equiprobability judgment he has for urn II, so that $P(R_{II}) = P(B_{II})$; however, he can now put $P(R_{II}) = P(B_{II}) = 0.3$ and the difference $1 - P(R_{II}) - P(B_{II}) = 0.4$ expresses the degree of confidence he has in his own probability judgment. Additive beliefs represent the special case of full confidence. For instance, this is the case for urn I, in which the assignment $P(R_I) = P(B_I) = \frac{1}{2}$ not only expresses an equiprobability judgement, but also a full confidence in it.

1.1. *Independence and the Frequentist Interpretation of Subjective Probabilities*

We adopt a multiplicative notion of independence, that is, two events A and B are independent whenever $P(A \cap B) = P(A)P(B)$, and we consider sequences of random variables that are i.i.d. according to such a rule.[1]

Both in the standard theory and in this paper, the most important class of phenomena that the i.i.d. case models are those exemplified by drawings from a sequence of urns with the same balls' composition. In the standard setting, the DM has an unambiguous (i.e., additive) subjective assessment over the balls' proportions in the urns. In particular, in the i.i.d. case such an assessment is identical for all urns. For example, in a sequence of urns containing black and red balls, a possible probability assignment is $P(R) = \frac{1}{3}$ and $P(B) = \frac{2}{3}$ for all urns.

In our more general setting, however, the DM may have different levels of confidence in his beliefs over the balls' proportions, so that his overall subjective assessment is represented by a non-additive probability. Again, such an assessment is identical for all urns in the i.i.d. case. For instance, in the above sequence of urns with black and red balls, a possible non-additive probability assignment is $P(R) = \frac{1}{4}$ and $P(B) = \frac{1}{2}$ for all urns.

For our more general setting, we prove that, in contrast to the additive case, the DM believes that the limit frequency of red balls belongs to some interval, but is not able to pin down its value to a single number.

Specifically, let us consider an i.i.d. sequence $\{X_n\}_{n \geqslant 1}$ of $\{0, 1\}$-valued random variables, where $\{X_n = 0\}$ and $\{X_n = 1\}$ are, respectively, the events "draw a red ball from urn n" and "draw a black ball from urn n." Suppose $P(\{X_n = 0\}) = P(\{X_n = 1\}) = \frac{1}{4}$ for all $n \geqslant 1$. We prove that:

$$P\left(\frac{1}{4} \leqslant \liminf_n \frac{1}{n} \sum_{k=1}^{n} X_k \leqslant \limsup_n \frac{1}{n} \sum_{k=1}^{n} X_k \leqslant \frac{3}{4}\right) = 1.$$

This shows that the DM has an unambiguous probability 1 belief that the proportion of times a black ball is drawn is in the interval $[\frac{1}{4}, \frac{3}{4}]$. However, unlike the additive case, the DM is not able to further pin down its value.[2]

[1] Parallel results can be proved if $\bar{P}(A \cap B) = \bar{P}(A)\, \bar{P}(B)$, where $\bar{P}$ is the dual probability defined by $\bar{P}(A) = 1 - P(A^c)$.

[2] We will prove that, under additional conditions, it also holds that

$$P\left(\frac{1}{4} < \liminf_n \frac{1}{n} \sum_{k=1}^{n} X_k \leqslant \limsup_n \frac{1}{n} \sum_{k=1}^{n} X_k < \frac{3}{4}\right) = 0.$$

This is a natural and fairly neat extension of the standard limit laws. In fact, if the DM were fully confident in his beliefs, say $P(\{X_n = 0\}) = P(\{X_n = 1\}) = \frac{1}{2}$ for all $n \geqslant 1$, then we would get back to the standard result

$$P(\liminf_n S_n = \limsup_n S_n = \tfrac{1}{2}) = 1.$$

In standard additive subjective probability, an important feature of limit laws is to provide a frequentist perspective on subjective probabilities. Specifically, let us consider drawings from a sequence of urns with the same balls' composition and suppose that the DM has unambiguous (i.e., additive) subjective probability assessments over the ball's proportions in the urns. In particular, suppose that these assessments are identical for all urns, so that we are in an i.i.d. setting. Then, by the classic limit laws, such DM's subjective probability assessments are equal to his subjective assessment of the long run frequency with which the different balls' colors will appear in a sequence of drawings, one for each urn in the sequence. Consequently, one can think of his subjective probabilities as his assessment of such long run frequencies. This gives an interesting connection between two different views of probability (cf., e.g., Kreps [21] chap. 11).

In a similar vein, our limit laws for non-additive probabilities suggest a frequentist perspective on subjective non-additive probabilities. Let us consider a sequence of Ellsberg urns containing red and black balls and assume that the DM has the same non-additive subjective probability assessment $P(R)$ on all urns. The limit laws we will prove in the paper suggest that the non-additive probabilities $P(R)$ and $\overline{P(R)} = 1 - P(B)$ (the dual of $P(R)$) may be thought of as the DM's assessment of, respectively, the lim inf and lim sup of the long run frequencies with which the red balls will appear in a sequence of independent drawings from the given sequence of urns.

As non-additive probabilities have often been seen as a further departure of subjective probability from the frequentist tradition, this connection seems especially valuable.

1.2. *Related Literature and Organization*

Related papers on limit laws for non-additive probabilities include Walley and Fine [35] and Dow and Werlang [7]. In particular, a frequentist twist of non-additive probabilities was already noticed by Walley and Fine [35]. However, our results are more general. Partly, this is the case because our main results (sections 5 and 6) are based on a much weaker notion of independence, which only requires multiplicativity of the marginals. Indeed, in the non-additive case different probabilities on the product space may correspond to a given set of independent marginals, and we therefore have an indeterminacy problem (see Section 7.1). Both Walley and Fine [35] and Dow and Werlang [7] consider stronger notions of independence that select a particular probability on the product space for a given set of independent marginals.[3] In contrast, our definition of independence requires only that the probability of a product set equals the product of the marginal probabilities of the sets.

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 introduces the notion of independence used in our main results. Section 4 presents the result on which our derivation hinges (Theorem 6). Section 5 contains a non-additive version of Kolmogorov's strong law. This is the main result of the paper. Section 6 presents a weak law, while Section 7 discusses some extensions. In particular, it shows which form our results take for the stronger notion of independence based on the representation of non-additive probabilities as lower envelopes of sets of additive probabilities. The proofs and the more technical material are in the appendix.

2. PRELIMINARIES

Let $\mathscr{S}$ be an algebra of subsets of a space Ω. A real-valued function $X\colon \Omega \to \mathbb{R}$ is said to be $\mathscr{S}$-measurable if for every $\alpha \in \mathbb{R}$ the upper sets $\{\omega\colon X(\omega) \geqslant \alpha\}$ and $\{\omega\colon X(\omega) > \alpha\}$ belong to $\mathscr{S}$. If $\mathscr{S}$ is a σ-algebra, this is the standard notion of measurability. We denote by $\mathscr{M}$ the set of all $\mathscr{S}$-measurable functions.

A set function $v\colon \mathscr{S} \to [0, 1]$ is called a capacity if it satisfies the following three properties:

1. $v(\varnothing) = 0$,
2. $v(A) \leqslant v(B)$ whenever $A \subseteq B$ and $A, B \in \mathscr{S}$,
3. $v(\Omega) = 1$.

A capacity v is inner continuous if

4. $\lim_n v(A_n) = v(A)$ for every nondecreasing monotone sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{S}$ with $\bigcup_{n=1}^{\infty} A_n = A$.

It is outer continuous if

5. $\lim_n v(A_n) = v(A)$ for every nonincreasing monotone sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{S}$ with $\bigcap_{n=1}^{\infty} A_n = A$.

A capacity which is both inner and outer continuous is called continuous.

[3] One of these notions will be discussed in section 7.1.

A capacity v is convex if

6. $v(A)+v(B)\leqslant v(A\cup B)+v(A\cap B)$ for all $A, B\in\mathscr{S}$.

Convex capacities are often used in mathematical economics to model decision makers who do not like ambiguity (uncertainty aversion; cf. Schmeidler [28]).

A capacity v is totally monotone if

7. $v(A_1\cup\cdots\cup A_n)\geqslant\sum_{\{I\subseteq\{1,...,n\}\}}(-1)^{|I|+1}\,v(\bigcap_{i\in I}A_i)$ for every collection of subsets $\{A_1,...,A_n\}\subseteq\mathscr{S}$.

Totally monotone capacities are often called belief functions and play an important role in the theory of non-additive beliefs.[4] Inner measures are an example of inner continuous and totally monotone capacities. In the appendix we will provide an example of a continuous and totally monotone capacity.

A capacity v is a mass if

8. $v(A\cup B)=v(A)+v(B)$ for any $A, B\in\mathscr{S}$ such that $A\cap B=\varnothing$.

Every mass is a totally monotone capacity, with

$$v(A_1\cup\cdots\cup A_n)=\sum_{\{I\subseteq\{1,...,n\}\}}(-1)^{|I|+1}\,v\left(\bigcap_{i\in I}A_i\right)$$

for every collection of subsets $\{A_1,...,A_n\}\subseteq\mathscr{S}$ (inclusion-exclusion formula).

A capacity v is a measure if

9. $v(\bigcup_{n=1}^\infty A_n)=\sum_{n=1}^\infty v(A_n)$ for any sequence $\{A_n\}_{n=1}^\infty$ in $\mathscr{S}$ of pairwise disjoint sets such that $\bigcup_{n=1}^\infty A_n\in\mathscr{S}$.

As is well-known, a set function $v\colon\mathscr{S}\to[0,1]$ is a measure if and only if it is a continuous mass.

Finally, a capacity v is null-additive if

10. $v(A\cup B)=v(A)$ for any $A, B\in\mathscr{S}$ such that $A\cap B=\varnothing$ and $v(B)=0$.

It is worth noting that a convex capacity is null-additive if and only if $v(A)=0$ implies $v(A^c)=1$ for every $A\in\mathscr{S}$.

Null-additive capacities are an important class of capacities and Pap [24] is mostly devoted to their study. From a decision-theoretic standpoint null-additivity is closely connected to Savage-nullity. In the choice model based on non-additive probabilities axiomatized by Schmeidler [28], a set A is Savage-null if and only if $v(A\cup B)=v(B)$ for all $B\in\mathscr{S}$ such that

[4] See Dempster [4, 5], Shafer [30]. For recent work on totally monotone capacities, see Gilboa and Schmeidler [15], Marinacci [22], as well as the references they contain.

$A \cap B = \varnothing$ (see Schmeidler [28] p. 586). It is then easy to check that a capacity is null-additive if and only if all its null sets are Savage-null.

We now introduce the core, an important notion in the theory of capacities. The core $\mathscr{C}(v)$ of a capacity $v: \mathscr{S} \to [0, 1]$ is the set

$$\{\mu: \mathscr{S} \to [0, 1]: \mu \text{ is a mass and } \mu(A) \geqslant v(A) \text{ for all } A \in \mathscr{S}\}.$$

In other words, $\mathscr{C}(v)$ consists of all masses that dominate setwise the capacity v.

For convex capacities we always have $\mathscr{C}(v) \neq \varnothing$ (see Kelley [20] and Shapley [31]). Moreover, for capacities with nonempty cores there is a simple characterization of continuity. In fact, it is easy to see that a capacity $v: \mathscr{S} \to [0, 1]$ with $\mathscr{C}(v) \neq \varnothing$ is continuous if and only if $\lim_n v(A_n) = v(\Omega)$ for any nondecreasing monotone sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{S}$ with $\bigcup_{n=1}^{\infty} A_n = \Omega$.

The following known result will play an important role in what follows (see Delbaen [3]).

Proposition 1. *Let $\mathscr{S}$ be an algebra of subsets of a given space Ω, and v a convex capacity defined on $\mathscr{S}$. Let C be a chain of subsets in $\mathscr{S}$. There exists a mass $\mu \in \mathscr{C}(v)$ such that $\mu(A) = v(A)$ for all $A \in C$.*

We close this section by introducing the notion of integral which we will use in the sequel, developed in the seminal contribution of Choquet [1]. For the special case of inner and outer measures this integral was studied by Vitali [32] (see Marinacci [23]).

Definition 2. Let $v: \mathscr{S} \to [0, 1]$ be a capacity, and let $X \in \mathscr{M}$. The Choquet integral of X relative to v is defined as follows:

$$\int X\, dv = \int_0^{\infty} v(\{X \geqslant \alpha\})\, d\alpha + \int_{-\infty}^{0} [v(\{X \geqslant \alpha\}) - 1]\, d\alpha.$$

If v is additive, $\int X\, dv$ coincides with standard notions of integral.

A few papers have recently studied the Choquet integral.[5] In the sequel, however, we only need the following elementary properties.

Proposition 3. *Let $\mathscr{S}$ be an algebra of subsets of a given space Ω, v a capacity on $\mathscr{S}$, and X_n, X, Y elements of $\mathscr{M}$. Then*

[5] See, e.g., Greco [16, 17], Schmeidler [27], and Zhou [37].

1. *if* $X(\omega) \leqslant Y(\omega)$ *for all* $\omega \in \Omega$, *then* $\int X\, dv \leqslant \int Y\, dv$;
2. *if* $\alpha \geqslant 0$, *then* $\int \alpha X\, dv = \alpha \int X\, dv$;
3. *if* v *is convex, then* $\int (X+Y)\, dv \geqslant \int X\, dv + \int Y\, dv$.

Finally, Dennenberg [6] provides a comprehensive introduction to non-additive set functions and integrals, and we refer the interested reader to this book for more material on the subject.

3. INDEPENDENCE AND REGULARITY

In this section we will introduce independence in our set-up and present a technical condition, regularity, used in the sequel.

Let $\mathscr{S}$ be an algebra. The function X is called a random variable if it is $\mathscr{S}$-measurable, i.e., $X \in \mathscr{M}$. We denote by $\mathscr{B}_0$ the algebra generated by the open sets of the real line. We set

$$\mathscr{A}(X) = \{\{\omega : X(\omega) \in B\} : B \in \mathscr{B}_0\},$$

that is, $\mathscr{A}(X)$ is the smallest algebra in Ω that makes X a random variable.

Next we introduce independence in our set-up.

DEFINITION 4. *Let* $\{X_k\}_{k=1}^n$ *be a finite sequence in* $\mathscr{M}$, *and let* v *be a capacity on* $\mathscr{S}$. *We say that the random variables* $\{X_k\}_{k=1}^n$ *are independent relative to* v *if*

$$v\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n v(A_k)$$

whenever $A_k \in \mathscr{A}(X_k)$ *for* $1 \leqslant k \leqslant n$. *Moreover, if* $\{X_k\}_{k \geqslant 1}$ *is an infinite sequence of random variables, we say that the random variables* $\{X_k\}_{k \geqslant 1}$ *are independent if every finite subclass of* $\{X_k\}_{k \geqslant 1}$ *consists of independent random variables.*

In the sequel we will often need the following technical assumption:

DEFINITION 5. *Let* $\{X_k\}_{k \geqslant 1}$ *be a sequence in* $\mathscr{M}$, *and* v *a capacity on* $\mathscr{S}$. *Let* v_k *be the restriction of* v *on* $\mathscr{A}(X_k)$. *The random variables* $\{X_k\}_{k \geqslant 1}$ *are regular relative to* v *if, for any* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha < \beta$, *we have*

$$v_k(\{\omega : X_k(\omega) \geqslant \beta\}) = v_k(\{\omega : X_k(\omega) \geqslant \alpha\})$$

whenever $\nu_k(\{\omega: \alpha \leqslant X_k(\omega) < \beta\}) = 0$, *and*

$$\nu_k(\{\omega: X_k(\omega) \geqslant \alpha\}) = \nu_k(\{\omega: X_k(\omega) > \alpha\})$$

whenever $\nu_k(\{X_k(\omega) = \alpha\}) = 0$.

If each ν_k were additive, then every sequence of random variables would be regular. It is easy to check that this is also the case if each ν_k were null-additive, a property introduced in Section 2. This is a very convenient case because regularity is a condition on the random variables and, in general, one has to check if a given sequence of random variables satisfies the condition. It is also worth noting that monotonicity of ν_k already implies that $\nu_k(\{\omega: X_k(\omega) \geqslant \alpha\}) \neq \nu_k(\{\omega: X_k(\omega) > \alpha\})$ for at most a countable number of $\alpha \in \mathbb{R}$.

The fact that regularity always holds whenever the marginals ν_k are null-additive capacities is especially interesting. Capacities that are strictly positive on every non-empty subset of $\mathscr{A}(X_k)$, i.e., $\nu_k(A) > 0$ for all $A \in \mathscr{A}(X_k)$, are a simple, but important, example of null-additive capacities ν_k. For instance, in Bernoulli trials we have $\mathscr{A}(X_k) = \{\emptyset, A, A^c, \Omega\}$, where $A = \{\omega: X_k(\omega) = 0\}$ and $A^c = \{\omega: X_k(\omega) = 1\}$. Here it is natural to assume that $\nu_k(A) > 0$ and $\nu_k(A^c) > 0$. Such a ν_k is null-additive. Distortions are another example of null-additive capacities. In fact, let μ_k be a mass defined on $\mathscr{A}(X_k)$ and $f: [0, 1] \to [0, 1]$ a strictly increasing function such that $f(0) = 0$ and $f(1) = 1$. Suppose that ν on $\mathscr{S}$ is such that $\nu_k(A) = f(\mu_k(A))$ for all $A \in \mathscr{A}(X_k)$ and all $k \geqslant 1$ (e.g., this is the case if ν itself is defined by $f(\mu(A))$ for all $A \in \mathscr{S}$, where μ is a mass on $\mathscr{S}$). It is easy to see that ν_k is null-additive.

Finally, it is easy to check that in finite algebras every convex capacity can be approximated setwise, as closely as wanted, with a convex null-additive capacity. This shows that regularity is a mild condition on finite algebras (for example, $\mathscr{A}(X_k)$ is finite whenever X_k is finite-valued).

4. A KEY THEOREM

The next theorem will be a key ingredient in the proofs of our limit results and it is also of interest in itself. It says that for any sequence of regular and independent random variables, we can always find an additive set function that preserves independence and expected value.

This result is important for our purposes because through it one may hope to obtain in a non-additive setting several results from classical probability theory, such as the limit laws proved later. Let $\sigma(X_k)$ denote the σ-algebra $\{\{\omega: X_k(\omega) \in B\} : B \in \mathscr{B}\}$, where $\mathscr{B}$ is the Borel σ-algebra of the real line. That is, $\sigma(X_k)$ is the smallest σ-algebra in Ω that makes X_k a random variable.

THEOREM 6. *Let Ω be a compact space, and v a convex and continuous capacity on $\mathcal{S}$. Let $\{X_k\}_{k \geqslant 1}$ be a sequence of regular and independent random variables relative to v. Suppose each random variable is continuous on Ω. Then there exists a measure μ defined on $\sigma(X_1, ..., X_k, ...)$ such that*

(i) *$\{X_k\}_{k \geqslant 1}$ is a sequence of independent random variables relative to μ;*

(ii) *$\int X_k \, dv = \int X_k \, d\mu$ for all $k \geqslant 1$.*

Remark. We need the topological assumptions in order to establish the countable additivity of μ. In the appendix we prove a weaker form of Theorem 6 in which no topological assumptions are made, but μ is then only finitely additive (Lemma 18).

4.1. *Example: Bernoullian Sequences*

In this subsection we study an interesting case in which the topological assumptions of Theorem 6 become superfluous. To this end we introduce the following definition, in which we consider only finite-valued random variables (that is, the range $R(X_k)$ of each X_k is finite).

DEFINITION 7. *A sequence $\{X_k\}_{k \geqslant 1}$ of finite-valued, independent random variables on $(\Omega, \mathcal{S}, v)$ is called Bernoullian if for each sequence $\{\alpha_k\}_{k \geqslant 1}$, with $\alpha_k \in R(X_k)$ for all $k \geqslant 1$, we have*

$$\bigcap_{k \geqslant 1} \{\omega : X_k(\omega) = \alpha_k\} \neq \varnothing.$$

This assumption just requires that each sequence of values that the random variables $\{X_k\}_{k \geqslant 1}$ may take on is possible. For example, let $\{X_k\}_{k \geqslant 1}$ be a sequence of two-valued random variables, with $X_k(\omega) \in \{0, 1\}$ for every $\omega \in \Omega$. For $i_k \in \{0, 1\}$, set $A_k^{i_k} = \{\omega : X_k(\omega) = i_k\}$. According to this notation, $A_k^0 = \{\omega : X_k(\omega) = 0\}$ and $A_k^1 = \{\omega : X_k(\omega) = 1\}$. The algebra $\mathcal{A}(X_k)$ consists of $\{\varnothing, A_k^0, A_k^1, \Omega\}$. The sequence is Bernoullian if $\bigcap_{k=1}^{\infty} A_k^{i_k} \neq \varnothing$ for each infinite sequence $\{i_k\}_{k \geqslant 1} \in \{0, 1\}^{\mathbb{N}}$. In other words, every sequence of successes and failures is possible.

For Bernoullian sequences the following nontopological version of Theorem 6 holds.[6]

[6] The continuity of the capacity v is not needed here because Bernoullian sequences consist of finite-valued random variables. A similar observation applies to Corollary 12 below.

COROLLARY 8. *Let v be a convex capacity on $\mathscr{S}$. Let $\{X_k\}_{k\geqslant 1}$ be a Bernoullian sequence of regular and independent random variables relative to v. Then there exists a measure μ defined on $\sigma(X_1, ..., X_k, ...)$ such that*

(i) $\{X_k\}_{k\geqslant 1}$ *is a sequence of independent random variables relative to μ;*

(ii) $\int X_k \, dv = \int X_k \, d\mu$ *for all $k \geqslant 1$.*

This result is a corollary of Theorem 6. This might seem a bit surprising because no topology is involved in Corollary 8. However, we can endow the space Ω with an auxiliary compact topology under which every random variable in a Bernoullian sequence is continuous. This topology, denoted by $\tau(\mathscr{A})$, has as basis the algebra $\mathscr{A}(X_1, ..., X_n, ...)$ itself, that is, $\tau(\mathscr{A})$ consists of all sets that are unions (finite or infinite) of subsets belonging to $\mathscr{A}(X_1, ..., X_n, ...)$. Using $\tau(\mathscr{A})$ we can prove the following interesting lemma. Corollary 8 is then a simple consequence of it and Theorem 6.

LEMMA 9. *Let $\{X_k\}_{k\geqslant 1}$ be a Bernoullian sequence in $(\Omega, \mathscr{S}, v)$. The topology $\tau(\mathscr{A})$ is compact and every random variable in the sequence is continuous with respect to $\tau(\mathscr{A})$.*

5. STRONG LAW OF LARGE NUMBERS

Theorem 6 is a powerful result. Using it we can obtain in the non-additive case results analogous to the standard limit theorems for independent random variables. Among them, we look at the Strong Law of Large Numbers of Kolmogorov, the most famous limit law.

DEFINITION 10. Let v be a capacity defined on a σ-algebra $\mathscr{S}$. A sequence $\{X_k\}_{k\geqslant 1}$ is a sequence of r.i.i.d. random variables relative to v if they are independent, identically distributed, and if both $\{X_k\}_{k\geqslant 1}$ and $\{-X_k\}_{k\geqslant 1}$ are regular relative to v.

It is easy to check that if each v_k is null-additive on $\mathscr{A}(X_k)$ then both $\{X_k\}_{k\geqslant 1}$ and $\{-X_k\}_{k\geqslant 1}$ are regular relative to v.

We can now state our main result.

THEOREM 11. *Let Ω be a compact space, and v a totally monotone and continuous capacity on a σ-algebra $\mathscr{S}$ of subsets of Ω. Let $\{X_k\}_{k\geqslant 1}$ be a sequence of continuous r.i.i.d. random variables relative to v such that both $E_v(X_1)$ and $E_v(-X_1)$ exist. Set $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then*

(i) $\nu(\{\omega : E_\nu(X_1) \leqslant \liminf_n S_n(\omega) \leqslant \limsup_n S_n(\omega) \leqslant -E_\nu(-X_1)\}) = 1$,

(ii) $\nu(\{\omega : E_\nu(X_1) > \liminf_n S_n(\omega)\}) = 0$,

(iii) $\nu(\{\omega : -E_\nu(-X_1) < \limsup_n S_n(\omega)\}) = 0$.

Remark. In the additive case $E_\nu(X_1) = -E_\nu(-X_1)$, and every sequence of random variables is regular.

The important case of Bernoullian sequences is considered in the following corollary, whose simple proof is omitted. For this case the topological assumptions are superfluous.

COROLLARY 12. *Let ν a totally monotone capacity on a σ-algebra $\mathscr{S}$ of subsets of Ω. Let $\{X_k\}_{k\geqslant 1}$ be a Bernoullian sequence of r.i.i.d. random variables relative to ν such that both $E_\nu(X_1)$ and $E_\nu(-X_1)$ exist. Then*

(i) $\nu(\{\omega : E_\nu(X_1) \leqslant \liminf_n S_n(\omega) \leqslant \limsup_n S_n(\omega) \leqslant -E_\nu(-X_1)\}) = 1$,

(ii) $\nu(\{\omega : E_\nu(X_1) > \liminf_n S_n(\omega)\}) = 0$,

(iii) $\nu(\{\omega : -E_\nu(-X_1) < \limsup_n S_n(\omega)\}) = 0$.

6. A WEAK LAW

We now prove a version of the weak law of large numbers. This weaker result requires a weaker condition on ν: we no longer need total monotonicity, but convexity is enough.

THEOREM 13. *Let Ω be a compact space, and ν a convex and continuous capacity on a σ-algebra $\mathscr{S}$ of subsets of Ω. Let $\{X_k\}_{k\geqslant 1}$ be a sequence of continuous r.i.i.d. random variables relative to ν such that both $E_\nu(X_1)$ and $E_\nu(-X_1)$ exist. Then, for every $\varepsilon > 0$ we have*

(i) $\lim_n \nu(\{\omega : E_\nu(X_1) - \varepsilon \leqslant S_n(\omega) \leqslant -E_\nu(-X_1) + \varepsilon\}) = 1$,

(ii) $\lim_n \nu(\{\omega : S_n(\omega) \geqslant E_\nu(X_1) + \varepsilon\}) = 0$,

(iii) $\lim_n \nu(\{\omega : S_n(\omega) \leqslant -E_\nu(-X_1) - \varepsilon\}) = 0$.

7. EXTENSIONS

7.1. *Controlled Capacities*

The notion of independence we have been using is rather weak. To see why, let us consider two algebras $\mathscr{A}(X_k)$ and $\mathscr{A}(X_{k'})$. Our independence condition simply requires multiplicativity on the sets $A_k \cap A_{k'}$, i.e., $\nu(A_k \cap A_{k'}) = \nu(A_k)\nu(A_{k'})$ for $A_k \in \mathscr{A}(X_k)$ and $A_{k'} \in \mathscr{A}(X_{k'})$. We do not ask anything on the sets of $\mathscr{A}(X_k, X_{k'})$ which are not of the form $A_k \cap A_{k'}$. This is an especially attractive feature of our results. In fact, while in the additive case there is a unique mass compatible with given values of $\nu(A_k \cap A_{k'})$, this is no longer true for capacities. In other words, in the additive case once we know the values of ν on $\mathscr{A}(X_k)$ and $\mathscr{A}(X_{k'})$ (i.e., its marginals on $\mathscr{A}(X_k)$ and $\mathscr{A}(X_{k'})$), by independence it is then possible to determine uniquely its value on the entire algebra $\mathscr{A}(X_k, X_{k'})$. This is not true for capacities, where different values of ν on $\mathscr{A}(X_k, X_{k'})$ may be compatible with given independent marginals on $\mathscr{A}(X_k)$ and $\mathscr{A}(X_{k'})$. Thus, an important feature of our results is their robustness to this indeterminacy.

Still, one might wonder if we can strenghten our results by imposing some specific link between the marginals and the overall capacity on $\mathscr{A}(X_k, X_{k'})$. In the literature several possible links have been discussed.[7] Among them, we focus on a quite popular alternative, motivated by the characterization of convex capacities as lower envelopes of their cores. Specifically, for every convex capacity ν defined on an algebra it holds that

$$\nu(A) = \min_{\mu \in \mathscr{C}(\nu)} \mu(A) \tag{1}$$

for all sets A in the given algebra. This result provides an interesting perspective on ν. In fact, as we discussed in the introduction, capacities are used to model ambiguity in beliefs. Equality (1) shows that we can also model ambiguity using sets of additive probabilities. The difference between a standard decision maker and one that perceives ambiguity in his beliefs is that while the former represents his beliefs with a single additive probability, the latter uses a set of them. In particular, the more ambiguous his beliefs are, the larger such a set is. This view has been axiomatized by Gilboa and Schmeidler [14] in the so-called multiple priors model.

As to the indeterminacy issue we were discussing, the multiple priors approach suggests to look at all uniquely determined masses μ generated by independent marginals $\mu_k \in \mathscr{C}(\nu_k)$ and $\mu_{k'} \in \mathscr{C}(\nu_{k'})$ and then to require that the value assigned by ν to a set $A \in \mathscr{A}(X_k, X_{k'})$ be equal to their lower envelope. Formally,

Definition 14. *Let ν be a capacity on $\mathscr{S}$, and $\{X_k\}_{k \geq 1}$ a sequence of random variables on $\mathscr{S}$. Let*

[7] See Eichberger and Kelsey [9] and Ghirardato [12] for recent work on this and related subjects, as well as the references they contain.

$$\Gamma = \Big\{\mu : \mu \text{ is a mass on } \mathscr{S} \text{ such that } \mu\Big(\bigcap_{k \in I} A_k\Big) = \prod_{k \in I} \mu_k(A_k) \text{ for all finite index sets } I, \text{ where } \mu_k \in \mathscr{C}(v_k) \text{ and } A_k \in \mathscr{A}(X_k) \text{ for each } k \in I\Big\}. \quad (2)$$

The capacity v is said to be controlled if for each $A \in \mathscr{S}$ we have

$$v(A) = \min_{\mu \in \Gamma} \mu(A).$$

This is, for example, the approach taken by Walley and Fine [35], who mostly study controlled capacities.

With controlled capacities it is possible to obtain interesting strengthenings of our results. For example, the strong law of large numbers takes the following form.

THEOREM 15. *Let Ω be a compact space, and v a convex, continuous, and controlled capacity on a σ-algebra $\mathscr{S}$ of subsets of Ω. Let $\{X_k\}_{k \geqslant 1}$ be a sequence of continuous and r.i.i.d. random variables relative to v such that both $E_v(X_1)$ and $E_v(-X_1)$ exist. Then*

(i) $v(\{\omega : E_v(X_1) \leqslant \liminf_n S_n(\omega) \leqslant \limsup_n S_n(\omega) \leqslant -E_v(-X_1)\}) = 1$;

(ii) $v(\{\omega : \liminf_n S_n(\omega) < E_v(X_1)\})$
$= v(\{\omega : -E_v(-X_1) < \limsup_n S_n(\omega)\}) = 0$;

(iii) $v(\{\omega : E_v(X_1) < \liminf_n S_n(\omega) \leqslant \limsup_n S_n(\omega) < -E_v(-X_1)\}) = 0$;

(iv) $v(\{\omega : E_v(X_1) \neq \liminf_n S_n(\omega)\}) = 0$,
$v(\{\omega : -E_v(-X_1) \neq \limsup_n S_n(\omega)\}) = 0$;

(v) *if v is null-additive, then*

$$v(\{\omega : E_v(X_1) = \liminf_n S_n(\omega)\}) = 1$$

$$v(\{\omega : -E_v(-X_1) = \limsup_n S_n(\omega)\}) = 1;$$

The novel parts relative to Theorem 11 are (iii)–(v). It is worth noting that only convexity is used, not any more total monotonicity.

More interestingly, however, with controlled capacities it is possible to establish a central limit theorem. To state and prove it we need some notation. In the standard additive set-up, we look at the limit behavior of the random variable $(nS_n - nE_\mu(X_n))/\sigma n^{1/2}$, where σ is the standard deviation. In our context, we set:

1. $\gamma_n = E_v(X_n^2) - E_v(X_n)^2$,
2. $\rho_n = E_v(-X_n^2) - E_v(-X_n)^2$,

3. $Y_n = (nS_n - nE_\nu(X_n))/\gamma_n n^{1/2}$,

4 $Z_n = (nS_n - n(-E_\nu(-X_n)))/\rho_n n^{1/2}$.

Notice that if ν were additive, then $\gamma_n^2 = \rho_n^2 = Var_\nu(X_n)$, $E_\nu(X_n) = -E_\nu(-X_n)$ and so $Y_n = Z_n = (nS_n - nE_\nu(X_n))/\sigma n^{1/2}$. Furthermore, here in general $\gamma_n \neq Var_\nu(X_n)$.

We can now state the theorem. $N(\cdot)$ is the cumulative distribution function of a standard normal distribution.

THEOREM 16. *Let Ω be a compact space, and ν a convex, continuous, and controlled capacity on a σ-algebra $\mathcal{S}$ of subsets of Ω. Let $\{X_k\}_{k \geqslant 1}$ be a sequence of continuous and r.i.i.d. nonnegative random variables relative to ν such that both $E_\nu(X_1^2)$ and $E_\nu(-X_1^2)$ exist. Then*

(i) $\lim_n \nu(\{\omega: Z_n \leqslant \alpha\}) = N(\alpha)$,

(ii) $\lim_n \nu(\{\omega: Y_n > \alpha\}) = 1 - N(\alpha)$.

Remark. If ν were additive, this result would reduce to a standard Central Limit Theorem. However, the result is weaker than its additive counterpart because, given a random variable X, the knowledge of the value of ν on the sets $\{\omega: X(\omega) \leqslant \alpha\}$ and $\{\omega: X(\omega) > \alpha\}$ is not enough to characterize the distribution of ν on the entire σ-algebra $\sigma(X)$.

We close this subsection with an interesting property of sums and products of random variables that are independent relative to a controlled capacities. We omit the proof of this result, which is a simple consequence of Lemma 22.

PROPOSITION 17. *Let ν be a controlled capacity on a σ-algebra $\mathcal{S}$ of subsets of Ω, and $\{X_k\}_{k=1}^n$ a finite sequence of bounded, independent, and regular random variables relative to ν. If each ν_k is convex on $\mathcal{A}(X_k)$, then*

(i) $E_\nu(X_1 \cdots X_n) = E_\nu(X_1) \cdots E_\nu(X_n)$,

(ii) $E_\nu(X_1 + \cdots + X_n) = E_\nu(X_1) + \cdots + E_\nu(X_n)$.

Notice, in particular, how additivity arises under independence, which is the polar case of comonotonicity. Interestingly, Ghirardato [12] has showed that a similar result can be derived in a rather different approach.

7.2. *Product Spaces*

In this paper we have been working in a setting with a fixed probability space, whose underlying probability was non-additive. Walley and Fine

[35] and Dow and Werlang [7] have, in contrast, used an alternative setting in which one starts from sequences of random variables defined on distinct probability spaces. We preferred to work with a given probability space because it is, in our opinion, a setting better suited to obtain general results that hold regardless of which particular procedure one may choose to deal with the indeterminacy problem mentioned before.

In any event, all the results we have proved in our setting with a fixed probability space hold in the alternative setting with distinct probability spaces. We omit their statements in this different setting as they are basically translations of the original results we stated before. The only significant difference is that, in the setting with distinct probability spaces, the assumption of regularity becomes superfluous.

8. APPENDIX: PROOFS AND RELATED ANALYSIS

8.1. *Section* 4

In order to prove Theorem 6, we first prove the following lemma which is also of interest in itself. Recall that by a mass we mean a finitely additive set function.

LEMMA 18. *Let v be a convex capacity on $\mathscr{S}$. Let $\{X_k\}_{k \geqslant 1}$ be a sequence of regular and independent random variables relative to v. Then there exists a mass μ defined on $\mathscr{A}(X_1, ..., X_k, ...)$ such that*

(i) *$\{X_k\}_{k \geqslant 1}$ is a sequence of independent random variables relative to μ;*

(ii) *$\int X_k \, dv = \int X_k \, d\mu$ for all $k \geqslant 1$.*

Remark. For this theorem we need only the first part of the definition of regularity, i.e., that for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we have

$$v_k(\{\omega : X_k(\omega) \geqslant \beta\}) = v_k(\{\omega : X_k(\omega) \geqslant \alpha\})$$

whenever $v_k(\{\omega : \alpha \leqslant X_k(\omega) < \beta\}) = 0$.

Proof. We prove (i) and (ii) for a given finite sequence $\{X_1, ..., X_n\}$. For $k \geqslant 1$, let C_k be the chain formed by the upper sets $\{X_k \geqslant \alpha\}$, where $\alpha \in \mathbb{R}$. The algebra $\mathscr{A}(X_k)$ coincides with the algebra generated by C_k. This can be proved with a kind of $\pi - \lambda$ argument. Let v_k be the restriction of v on $\mathscr{A}(X_k)$. By Proposition 1, there exists a mass $\mu_k \in \mathscr{C}(v_k)$ such that $\mu_k(A) = v(A)$ for all $A \in C_k$. Define a set function μ' on $\mathscr{A}(X_1) \cup \mathscr{A}(X_2) \cup \cdots \cup \mathscr{A}(X_n)$ as follows:

$$\mu'(A) = \mu_k(A) \qquad \text{if} \quad A \in \mathscr{A}(X_k).$$

We now show that μ' is well defined. Let $A \subseteq B$ with $A \in \mathscr{A}(X_k)$ and $B \in \mathscr{A}(X_{k'})$ where $1 \leqslant k,\ k' \leqslant n$. By independence, $v(A) = v(A \cap B) = v(A)\, v(B)$. Therefore, $v(A) = 0$ or $v(B) = 1$, or both. Suppose $v(B) = 1$, then $\mu_{k'}(B) = 1$ because $\mu_{k'} \in \mathscr{C}(v)$. In turn, this implies $\mu'(B) = 1$, so that $\mu'(A) \leqslant \mu'(B)$, as desired. Suppose $v(A) = 0$. Since $A \in \mathscr{A}(X_k)$ we can write $A = \bigcup_{i=1}^{n} (E_i \cap F_i)$, where $E_i \in C_k$, $F_i^c \in C_k$ and $E_i \supseteq F_i^c$. The sets $(E_i \cap F_i)$ are pairwise disjoint. Since $v(A) = 0$, we have $v(E_i \cap F_i) = 0$ for each $1 \leqslant i \leqslant n$. By construction, $\mu_k(E_i \cap F_i) = v(E_i) - v(F_i^c)$. Since $v(E_i \cap F_i) = 0$, regularity implies $v(E_i) = v(F_i^c)$, so that $\mu_k(E_i \cap F_i) = 0$. Since $\mu_k(A) = \sum_{i=1}^{n} \mu_k(E_i \cap F_i) = 0$, we conclude $\mu'(A) \leqslant \mu'(B)$, as desired. This proves that μ' is a well defined mass on $\mathscr{A}(X_1) \cup \mathscr{A}(X_2) \cup \cdots \cup \mathscr{A}(X_n)$. Let $A \in \mathscr{A}(X_1, ..., X_n)$. It is easy to check that A has the form $\bigcup_{i=1}^{m} (A_i^1 \cap \cdots \cap A_i^n)$ where $A_i^k \in \mathscr{A}(X_k)$. Moreover, the sets $(A_i^1 \cap \cdots \cap A_i^n)$ can be taken pairwise disjoint. Define μ on $\mathscr{A}(X_1, ..., X_n)$ as follows

$$\mu(A) = \sum_{i=1}^{m} \mu'(A_i^1) \cdots \mu'(A_i^n).$$

We now check that μ is well defined. Suppose

$$A = \bigcup_{i=1}^{m} (A_i^1 \cap \cdots \cap A_i^n) = \bigcup_{j=1}^{m'} (B_j^1 \cap \cdots \cap B_j^n),$$

where $B_j^k \in \mathscr{A}(X_1)$ and the sets $(B_i^1 \cap \cdots \cap B_i^n)$ are pairwise disjoint. Let $\Omega^n = \Omega_1 \times \cdots \times \Omega_n$, where the index in Ω_k denotes only the position in the Cartesian product. Let π_k be a map from $\mathscr{A}(X_k)$ to the power set of Ω_k defined as follows

$$\pi_k(A) = A \qquad \text{for each} \quad A \in \mathscr{A}(X_k).$$

Let $\mathscr{A}^*(X_k)$ be the projection of $\mathscr{A}(X_k)$ through π_k. Let $\mathscr{A}^*(X_1, ..., X_n)$ denote the algebra generated by the measurable rectangles of the algebras $\mathscr{A}^*(X_1), ..., \mathscr{A}^*(X_n)$. Define a map $\pi_{1, ..., n}$ from $\mathscr{A}(X_1, ..., X_n)$ to $\mathscr{A}^*(X_1, ..., X_n)$ as

$$\pi_{1, ..., n}\left(\bigcup_{i=1}^{m} (A_i^1 \cap \cdots \cap A_i^n)\right) = \bigcup_{i=1}^{m} (A_i^1 \times \cdots \times A_i^n)$$

for each $A = \bigcup_{i=1}^{m} (A_i^1 \cap \cdots \cap A_i^n)$. Set $\mu_k^*(\pi_{1, ..., n}(A)) = \mu_k'(A)$ for each $A \in \mathscr{A}(X_k)$. Since each μ_k^* is additive on $\mathscr{A}^*(X_k)$, it is known there exists a mass $\mu_{1, ..., n}^*$ on $\mathscr{A}^*(X_1, ..., X_n)$ defined as

$$\mu_{1, ..., n}^*(A) = \sum_{i=1}^{m} [\mu_1^*(A_i^1) \cdots \mu_n^*(A_i^n)]$$

for each $A \in \mathscr{A}^*(X_1, ..., X_n)$. Its projection through $\pi_{1, ..., n}$ is

$$\mu_{1, ..., n}(A) = \sum_{i=1}^{m} [\mu_1(A_i^1) \cdots \mu_n(A_i^n)]$$

for each $A \in \mathscr{A}(X_1, ..., X_n)$. Clearly, $\mu_{1, ..., n}(A) = \mu(A)$ for each $A \in \mathscr{A}(X_1, ..., X_n)$. Since $\mu^*_{1, ..., n}$ is well defined, if $\bigcup_{i=1}^m (A_i^1 \cap \cdots \cap A_i^n) = \bigcup_{j=1}^{m'} (B_j^1 \cap \cdots \cap B_j^n) \neq \varnothing$ we get:

$$\mu^*_{1, ..., n}\left(\pi_{1, ..., n}\left(\bigcup_{i=1}^{m} (A_i^1 \cap \cdots \cap A_i^n)\right)\right)$$

$$= \sum_{i=1}^{m} [\mu_1^*(A_i^1) \cdots \mu_n^*(A_i^n)] = \sum_{j=1}^{m'} [\mu_1^*(B_j^1) \cdots \mu_n^*(B_j^n)]$$

$$= \mu^*_{1, ..., n}\left(\pi_{1, ..., n}\left(\bigcup_{j=1}^{m'} (B_j^1 \cap \cdots \cap B_j^n)\right)\right).$$

Therefore, $\sum_{i=1}^m \mu'(A_i^1) \cdots \mu'(A_i^n) = \sum_{j=1}^{m'} \mu'(B_j^1) \cdots \mu'(B_j^n)$, and this proves that μ is well defined. Clearly, under μ the random variables $\{X_1, ..., X_n\}$ are independent. Since, by construction, $\mu(A) = v(A)$ for $A \in C_1 \cup \cdots \cup C_n$, it follows that $\int X_k \, dv = \int X_k \, d\mu$ for all $1 \leqslant k \leqslant n$.

Let I be a finite subset of $\{1, ..., n, ...\}$. Let $\{X_k\}_{k \in I} \subseteq \{X_k\}_{k \geqslant 1}$. By what has been proved above, there exists a mass μ_I that satisfies (i) and (ii) on $\{X_k\}_{k \in I}$. Let I' be another finite subset of $\{1, ..., n, ...\}$ such that $I \subseteq I'$. If $A \in \mathscr{A}(\{X_k\}_{k \in I})$, then

$$\mu_I(A) = \mu_{I'}(A). \tag{2}$$

Equality (2) is easy to check. For, if $A \in \mathscr{A}(\{X_k\}_{k \in I})$, then $A = \bigcup_{i=1}^m \bigcap_{k \in I} A_i^k$ for some $A_i^k \in \mathscr{A}(X_k)$. Consequently, by construction $\mu_I(A) = \sum_{i=1}^m \prod_{k \in I} \mu_k(A_i^k) = \mu_{I'}(A)$.

Suppose now that I' neither contains nor is contained in I. Set $I'' = I' \cup I$. Using (2) on $\mathscr{A}(\{X_k\}_{k \in I''})$ it is easy to check that $\mu_I(A) = \mu_{I''}(A) = \mu_{I'}(A)$ if $A \in \mathscr{A}(\{X_k\}_{k \in I}) \cap \mathscr{A}(\{X_k\}_{k \in I'})$. This allows us to define a set function μ on $\mathscr{A}(X_1, ..., X_k, ...)$ as follows: $\mu(A) = \mu_I(A)$ if $A \in \mathscr{A}(\{X_k\}_{k \in I})$. The set function μ is well defined by what proved above. Moreover, since $\mathscr{A}(X_1, ..., X_k, ...) = \bigcup_I \mathscr{A}(\{X_k\}_{k \in I})$, if A, B are disjoint subsets of $\mathscr{A}(X_1, ..., X_k, ...)$, with $A \in \mathscr{A}(\{X_k\}_{k \in I})$ and $B \in \mathscr{A}(\{X_k\}_{k \in I'})$, then there exists a finite subset I^* such that $A \cup B \in A(\{X_k\}_{k \in I^*})$. Set $I'' = I \cup I' \cup I^*$. Using equality (2) on $\mathscr{A}(\{X_k\}_{k \in I''})$ it is then easy to check additivity of μ. This completes the proof of parts (i) and (ii). ∎

Proof of Theorem 6. Before starting, we remind that a set function μ is regular on a collection of sets $\mathscr{F}$ if $\mu(A)=\sup\{\mu(F)\colon F\subseteq A,\ F\in\mathscr{F},\ F$ closed$\}$. Let C_k be the chain in $\mathscr{S}$ formed by the sets $\{\omega\colon X_k(\omega)\geqslant\alpha\}$ and $\{\omega\colon X_k(\omega)>\alpha\}$ for each $\alpha\in\mathbb{R}$. Let μ_k be the mass on $\mathscr{A}(X_k)$ such that $v(A)=\mu_k(A)$ for every $A\in C_k$. Since μ_k is in the core of v_k, μ_k is a measure on $\mathscr{A}(X_k)$. Regularity of $\{X_k\}_{k\geqslant 1}$ implies the following equalities for any $\alpha,\beta\in\mathbb{R}$ with $\alpha<\beta$:

$$v_k\{\omega\colon X_k(\omega)>\beta\}=v_k\{\omega\colon X_k(\omega)\geqslant\alpha\}\qquad\text{if}\quad v_k\{\omega\colon\alpha\leqslant X_k(\omega)\leqslant\beta\}=0,$$

$$v_k\{\omega\colon X_k(\omega)>\beta\}=v_k\{\omega\colon X_k(\omega)>\alpha\}\qquad\text{if}\quad v_k\{\omega\colon\alpha< X_k(\omega)\leqslant\beta\}=0,$$

$$v_k\{\omega\colon X_k(\omega)\geqslant\beta\}=v_k\{\omega\colon X_k(\omega)>\alpha\}\qquad\text{if}\quad v_k\{\omega\colon\alpha< X_k(\omega)<\beta\}=0.$$

We prove the first equality. From $v_k\{\omega\colon\alpha\leqslant X_k(\omega)\leqslant\beta\}=0$ it follows that $v_k\{\omega\colon\alpha\leqslant X_k(\omega)<\beta\}=0$, and so $v_k\{\omega\colon X_k(\omega)\geqslant\beta\}=v_k\{\omega\colon X_k(\omega)\geqslant\alpha\}$ by regularity. Since $v_k\{\omega\colon X_k(\omega)=\beta\}=0$, by regularity $v_k\{\omega\colon X_k(\omega)\geqslant\beta\}=v_k\{\omega\colon X_k(\omega)>\beta\}$, and this proves the equality. As to the second one, $v_k\{\omega\colon\alpha<X_k(\omega)\leqslant\beta\}=0$ implies $v_k\{\omega\colon\alpha+1/n\leqslant X_k(\omega)\leqslant\beta\}=0$ for all n large enough. Then, by the previous equality, $v_k\{\omega\colon X_k(\omega)>\beta\}=v_k\{\omega\colon X_k(\omega)\geqslant\alpha+1/n\}$ for all n large enough. Since $(\alpha,\infty)=\bigcup_n(\alpha+1/n,\infty)$, continuity implies

$$\begin{aligned}v_k\{\omega\colon X_k(\omega)>\alpha\}&=\lim_n v_k\left\{\omega\colon X_k(\omega)\geqslant\alpha+\frac{1}{n}\right\}\\&=v_k\{\omega\colon X(\omega)>\beta\}.\end{aligned}$$

Finally, let us consider the last equality. From $v_k\{\omega\colon\alpha<X_k(\omega)<\beta\}=0$ it follows that $v_k\{\omega\colon\alpha+1/n\leqslant X_k(\omega)<\beta\}=0$ for n large enough. Then, by regularity, $v_k\{\omega\colon X_k(\omega)\geqslant\alpha+1/n\}=v_k\{\omega\colon X_k(\omega)\geqslant\beta\}$ for all n large enough, so that continuity of v implies $v_k\{\omega\colon X_k(\omega)\geqslant\beta\}=v_k\{\omega\colon X_k(\omega)>\alpha\}$.

We can now proceed as in the proof of Lemma 18 to construct a mass μ on $\mathscr{A}(X_1,\ldots,X_k,\ldots)$ whose restriction on $\mathscr{A}(X_k)$ coincides with μ_k and which satisfies points (i) and (ii) in Lemma 18. We want to show that μ_k is countably additive. Continuity of μ_k on $\mathscr{A}(X_k)$ implies

$$\mu_k\{\omega\colon X_k(\omega)\geqslant\alpha\}=\mu_k\left(\bigcap_n\left\{\omega\colon X_k(\omega)>\alpha-\frac{1}{n}\right\}\right)$$

$$=\lim_n \mu_k\left\{\omega\colon X_k(\omega)>\alpha-\frac{1}{n}\right\},$$

$$\mu_k\{\omega\colon X_k(\omega)>\alpha\}=\mu_k\left(\bigcup_n\left\{\omega\colon X_k(\omega)\geqslant\alpha+\frac{1}{n}\right\}\right)$$

$$=\lim_n \mu_k\left\{\omega\colon X_k(\omega)\geqslant\alpha+\frac{1}{n}\right\}.$$

Since X_k is continuous, the sets $\{\omega\colon X_k(\omega)\geqslant\alpha\}$ and $\nu_k\{\omega\colon X_k(\omega)>\alpha\}$ are, respectively, closed and open for each $\alpha\in\mathbb{R}$. For each $A\in C_k$ this implies

$$\mu_k(A)=\sup\{\mu_k(F)\colon F\subseteq A,\, F\in C_k,\, F \text{ closed}\},$$

$$\mu_k(A)=\inf\{\mu_k(G)\colon G\subseteq A,\, G\in C_k,\, G \text{ open}\}.$$

Let $A\in\mathscr{A}(X_k)$. Then we can write $A=\bigcup_{i=1}^n (E_i\cap F_i)$ where $E_i\in C_k$ and $F_i^c\in C_k$. The sets $E_i\cap F_i$ are pairwise disjoint. A standard procedure ensures that μ_k is also regular on the entire $\mathscr{A}(X_k)$.

We now prove that μ is regular on each $\mathscr{A}(X_{t_1}, ..., X_{t_n})$. W.l.o.g., we consider $\mathscr{A}(X_1, ..., X_n)$. The mass μ has been defined as

$$\mu(A)=\sum_{i=1}^m \mu_1(A_i^1)\cdots\mu_n(A_i^n),$$

where $A\in\mathscr{A}(X_1, ..., X_n)$ has the form $\bigcup_{i=1}^m (A_i^1\cap\cdots\cap A_i^n)$, with $A_i^k\in\mathscr{A}(X_k)$. The sets $(A_i^1\cap\cdots\cap A_i^n)$ are pairwise disjoint. Let $A=\bigcup_{i=1}^m (A_i^1\cap\cdots\cap A_i^n)$ and $\varepsilon>0$. Let $F_i^j\in\mathscr{A}(X_j)$ for $1\leqslant j\leqslant n$ be closed subsets such that $\mu(A_i^j)-\mu(F_i^j)\leqslant\delta$ for $1\leqslant j\leqslant n$, where δ is a given positive quantity. Then

$$\sum_{i=1}^m \mu(A_i^1)\cdots\mu(A_i^n)-\sum_{i=1}^m \mu(F_i^1)\cdots\mu(F_i^n)$$

$$=\sum_{i=1}^m \mu(A_i^1)\cdots\mu(A_i^n)-\sum_{i=1}^m \mu(A_i^1)\cdots\mu(A_i^{n-1})\,\mu(F_i^n)$$

$$+\sum_{i=1}^m \mu(A_i^1)\cdots\mu(A_i^{n-1})\,\mu(F_i^n)-\cdots+\sum_{i=1}^m \mu(A_i^1)\,\mu(F_i^2)\cdots\mu(F_i^2)$$

$$-\sum_{i=1}^m \mu(F_i^1)\cdots\mu(F_i^n)$$

$$= \sum_{i=1}^{m} \mu(A_i^1) \cdots \mu(A_i^{n-1})[\mu(A_i^n) - \mu(F_i^n)] + \cdots$$

$$+ \sum_{i=1}^{m} \mu(F_i^2) \cdots \mu(F_i^n)[\mu(A_i^1) - \mu(F_i^1)]$$

$$\leqslant \sum_{i=1}^{m} \{[\mu(A_i^n) - \mu(F_i^n)] + \cdots + [\mu(A_i^1) - \mu(F_i^1)]\} \leqslant mn\delta.$$

As n and m are fixed for a given A, taking δ small enough we get

$$\sum_{i=1}^{m} \mu(A_i^1) \cdots \mu(A_i^n) - \sum_{i=1}^{m} \mu(F_i^1) \cdots \mu(F_i^n) \leqslant \varepsilon.$$

Since $F = \bigcup_{i=1}^{m} (F_i^1 \cap \cdots \cap F_i^n)$ is a closed subset of $\mathscr{A}(X_1, ..., X_n)$, we are done. We conclude that μ is regular on $\mathscr{A}(X_1, ..., X_n)$. Since Ω is compact, the set of closed subsets of $\mathscr{A}(X_1, ..., X_n)$ form a compact class. By a well-known result, μ is then countably additive on $\mathscr{A}(X_1, ..., X_n)$.

Let us denote by $\mu_{t_1, ..., t_n}$ the regular measure on $\mathscr{A}(X_{t_1}, ..., X_{t_n})$ whose existence we have just proved. As we know from the proof of Lemma 18, μ is defined on $\bigcup_n \mathscr{A}(X_1, ..., X_n)$ as follows:

$$\mu(A) = \mu_{t_1, ..., t_n}(A) \qquad \text{if} \quad A \in \mathscr{A}(X_{t_1}, ..., X_{t_n}).$$

The mass μ is then regular on $\bigcup_n \mathscr{A}(X_1, ..., X_n)$. Since Ω is compact, it follows that μ is countably additive (see, e.g., Dunford and Schwartz [8] p. 138). Therefore, μ is a measure on $\bigcup_n \mathscr{A}(X_1, ..., X_n)$. On the other hand, $\sigma(X_1, ..., X_n, ...)$ is the σ-algebra generated by $\bigcup_n \mathscr{A}(X_1, ..., X_n)$. Then a standard application of Caratheodory extension theorem ensures the existence of a unique measure on $\sigma(X_1, ..., X_n, ...)$ that extends μ. This is the measure on $\sigma(X_1, ..., X_n, ...)$ we were looking for. ∎

Proof of Lemma 9. Let p be a filter in $\mathscr{A}(X_1, ..., X_n, ...)$. Let us consider the set $B = \bigcap \{A : A \in p\}$. Fix a $k \geqslant 1$, and let $\{A_i^k\}$ be the collection of all subsets in p such that each $A_i^k \in \mathscr{A}(X_k)$. Since p is a filter, and the collection $\{A_i^k\}$ is finite, we have $\bigcap_i A_i^k \neq \varnothing$ and $\bigcap_i A_i^k \in \mathscr{A}(X_k)$. Since each X_k is finite-valued, the algebra $\mathscr{A}(X_k)$ is generated by the finite partition $\{\omega : X_k(\omega) = \alpha\}_{\alpha \in \mathbb{R}}$. Therefore, there exists a real number α_k such that $\{\omega : X_k(\omega) = \alpha_k\} \subseteq \bigcap_i A_i^k$. Consequently,

$$\bigcap_{k \geqslant 1} \{\omega : X_k(\omega) = \alpha_k\} \subseteq \bigcap_{k \geqslant 1} \left(\bigcap_i A_i^k \right) = \bigcap \{A : A \in p\},$$

which implies

$$\bigcap \{A : A \in p\} \neq \varnothing \tag{3}$$

because the sequence is Bernoullian.

Now let $\{G_i\}_{i \in I}$ be an open cover of Ω, i.e., each $G_i \in \tau(\mathscr{A})$ and $\bigcup_{i \in I} G_i = \Omega$. W.l.o.g., suppose each $G_i \in \mathscr{A}(X_1, ..., X_n, ...)$. Suppose $\tau(\mathscr{A})$ is not a compact topology. Then $\bigcap_{i \in I'} G_i^c \neq \varnothing$ for each finite subset I' of I, and this implies the existence of a proper filter p in $\mathscr{A}(X_1, ..., X_n, ...)$ generated by the sequence $\{G_i^c\}_{i \in I}$. By (3), there exists a set $\varnothing \neq B \subseteq \Omega$ such that $p \subseteq \{A \in \mathscr{A}(X_1, ..., X_n, ...) : B \subseteq A\}$. Then $B \subseteq G_i^c$ for each $i \in I$, so that $B^c \supseteq G_i$ for each $i \in I$. But, this implies $\bigcup_{i \in I} G_i \subseteq B^c \subsetneq \Omega$, which contradicts $\bigcup_{i \in I} G_i = \Omega$.

Finally, it is clear that each random variable is continuous relative to $\tau(A)$. ∎

8.2. *Section* 5

The proof of Theorem 11 rests on the following lemma. A nontopological version of the lemma holds for Bernoullian sequences.

LEMMA 19. *Let Ω be a compact space, and v a convex and continuous capacity on a σ-algebra $\mathscr{S}$ of subsets of Ω. Let $\{X_k\}_{k \geqslant 1}$ be a sequence of regular, independent and continuous random variables relative to v. Set $S_n = 1/n \sum_{i=1}^n X_i$, and let μ be the measure provided by Theorem* 6. *We have*

$$\mu(\omega : S_n \geqslant \alpha) \leqslant v(\omega : S_n \geqslant \alpha) \qquad \textit{for all} \quad \alpha \in \mathbb{R}.$$

Moreover, if v is totally monotone we also have

$$\mu(\omega : \liminf_n S_n \geqslant \alpha) \leqslant v(\omega : \liminf_n S_n \geqslant \alpha) \qquad \textit{for all} \quad \alpha \in \mathbb{R}.$$

Proof. As is well known, we have

$$\{\omega : X_1(\omega) + X_2(\omega) \geqslant \alpha\} = \bigcup_r \{\omega : X_1(\omega) \geqslant r\} \cap \{\omega : X_2(\omega) \geqslant \alpha - r\},$$

where r goes over the rational numbers. Let $\{r^1, ..., r^n, ...\}$ be a ranking of rational numbers, unrelated to the natural ordering $\leqslant$. Consider $I = \{r^1, ..., r^n\}$. For convenience, set

$$A_r = \{\omega : X_1(\omega) \geqslant r\} \qquad \text{and} \qquad A_{\alpha - r} = \{\omega : X_2(\omega) \geqslant \alpha - r\}.$$

For convenience, suppose $r^1 \leqslant \cdots \leqslant r^n$. This is without loss of generality because the set $\{r^1, ..., r^n\}$ is always linearly ordered. Consequently, we have

$$A_{r(1)} \supseteq \cdots \supseteq A_{r(n)} \quad \text{and} \quad A_{\alpha - r(1)} \subseteq \cdots \subseteq A_{\alpha - r(n)}.$$

We claim that

$$\begin{aligned} &v\left\{\bigcup_{r \in I} (A_r \cap A_{\alpha - r})\right\} \\ &\qquad \geqslant v(A_{r(1)} \cap A_{\alpha - r(1)}) + \sum_{i=2}^{n} [v(A_{r(i)} \cap A_{\alpha - r(i)}) - v(A_{r(i)} \cap A_{\alpha - r(i-1)})]. \end{aligned}$$

We prove it by induction. Let $n = 2$. Using convexity we get:

$$\begin{aligned} &v((A_{r(1)} \cap A_{\alpha - r(1)}) \cup (A_{r(2)} \cap A_{\alpha - r(2)})) \cup (A_{r(3)} \cap A_{\alpha - r(3)})) \\ &\quad \geqslant v((A_{r(1)} \cap A_{\alpha - r(1)}) \cup (A_{r(2)} \cap A_{\alpha - r(2)})) + v(A_{r(3)} \cap A_{\alpha - r(3)}) \\ &\qquad - v[((A_{r(1)} \cap A_{\alpha - r(1)}) \cap (A_{r(3)} \cap A_{\alpha - r(3)})) \\ &\qquad \cup ((A_{r(2)} \cap A_{\alpha - r(2)}) \cap (A_{r(3)} \cap A_{\alpha - r(3)}))] \\ &\quad = v((A_{r(1)} \cap A_{\alpha - r(1)}) \cup (A_{r(2)} \cap A_{\alpha - r(2)})) + v(A_{r(3)} \cap A_{\alpha - r(3)}) \\ &\qquad - v((A_{r(3)} \cap A_{\alpha - r(1)}) \cup (A_{r(3)} \cap A_{\alpha - r(2)})) \\ &\quad = v((A_{r(1)} \cap A_{\alpha - r(1)}) \cup (A_{r(2)} \cap A_{\alpha - r(2)})) \\ &\qquad + v(A_{r(3)} \cap A_{\alpha - r(3)}) - v(A_{r(3)} \cap A_{\alpha - r(2)}) \\ &\quad \geqslant v(A_{r(1)} \cap A_{\alpha - r(1)}) + v(A_{r(2)} \cap A_{\alpha - r(2)}) - v(A_{r(2)} \cap A_{\alpha - r(1)}) \\ &\qquad + v(A_{r(3)} \cap A_{\alpha - r(3)}) - v(A_{r(3)} \cap A_{\alpha - r(2)}). \end{aligned}$$

Suppose the result is true for $n - 1$. Then, using convexity and the induction hypothesis, we get

$$\begin{aligned} &v\left(\bigcup_{i=1}^{n} (A_{r(i)} \cap A_{\alpha - r(i)})\right) \\ &\qquad \geqslant v\left(\bigcup_{i=1}^{n-1} (A_{r(i)} \cap A_{\alpha - r(i)})\right) + v(A_{r(n)} \cap A_{\alpha - r(n)}) \\ &\qquad\quad - v\left(\left(\bigcup_{i=1}^{n-1} (A_{r(i)} \cap A_{\alpha - r(i)})\right) \cap (A_{r(n)} \cap A_{\alpha - r(n)})\right) \end{aligned}$$

$$
\begin{aligned}
&= \nu\left(\bigcup_{i=1}^{n-1} (A_{r(i)} \cap A_{\alpha - r(i)})\right) + \nu(A_{r(n)} \cap A_{\alpha - r(n)}) \\
&\quad - \nu\left(\bigcup_{i=1}^{n-1} (A_{r(i)} \cap A_{\alpha - r(i)} \cap A_{r(n)} \cap A_{\alpha - r(n)})\right) \\
&= \nu\left(\bigcup_{i=1}^{n-1} (A_{r(i)} \cap A_{\alpha - r(i)})\right) + \nu(A_{r(n)} \cap A_{\alpha - r(n)}) \\
&\quad - \nu\left(\bigcup_{i=1}^{n-1} (A_{r(n)} \cap A_{\alpha - r(i)})\right) \\
&= \nu\left(\bigcup_{i=1}^{n-1} (A_{r(i)} \cap A_{\alpha - r(i)})\right) + \nu(A_{r(n)} \cap A_{\alpha - r(n)}) - \nu(A_{r(n)} \cap A_{\alpha - r(n-1)}) \\
&= \nu(A_{r(1)} \cap A_{\alpha - r(1)}) + \sum_{i=2}^{n-1} [\nu(A_{r(i)} \cap A_{\alpha - r(i)}) - \nu(A_{r(i)} \cap A_{\alpha - r(i-1)})] \\
&\quad + \nu(A_{r(n)} \cap A_{\alpha - r(n)}) - \nu(A_{r(n)} \cap A_{\alpha - r(n-1)}) \\
&= \nu(A_{r(1)} \cap A_{\alpha - r(1)}) + \sum_{i=2}^{n} [\nu(A_{r(i)} \cap A_{\alpha - r(i)}) - \nu(A_{r(i)} \cap A_{\alpha - r(i-1)})].
\end{aligned}
$$

This proves the claim. It is clear from the proof of the claim that for an additive set function like μ it holds

$$
\begin{aligned}
&\mu\left\{\bigcup_{r \in I} (A_r \cap A_{\alpha - r})\right\} \\
&\qquad = \mu(A_{r(1)} \cap A_{\alpha - r(1)}) + \sum_{i=2}^{n} [\mu(A_{r(i)} \cap A_{\alpha - r(i)}) - \mu(A_{r(i)} \cap A_{\alpha - r(i-1)})].
\end{aligned}
$$

On the other hand, using independence and the fact that $A_{r(i)} \in C_1$ and $A_{\alpha - r(i)} \in C_2$ we obtain

$$
\begin{aligned}
\nu(A_{r(i)} \cap A_{\alpha - r(i)}) &= \nu(A_{r(i)})\, \nu(A_{\alpha - r(i)}) = \mu(A_{r(i)})\, \mu(A_{\alpha - r(i)}) \\
&= \mu(A_{r(i)} \cap A_{\alpha - r(i)}).
\end{aligned}
$$

Similarly, $\nu(A_{r(i)} \cap A_{\alpha - r(i-1)}) = \mu(A_{r(i)} \cap A_{\alpha - r(i-1)})$. We conclude that

$$
\nu\left\{\bigcup_{r \in I} (A_r \cap A_{\alpha - r})\right\} \geqslant \mu\left\{\bigcup_{r \in I} (A_r \cap A_{\alpha - r})\right\}. \tag{4}
$$

Set $A(I_{1,\ldots,n}) = \bigcup_{r \in I} (A_r \cap A_{\alpha - r})$. Following the ordering $\{r^1, \ldots, r^n, \ldots\}$, we get a nondecreasing sequence $\{A(I_{1,\ldots,n})\}_{n \geqslant 1}$ with, by (4), $v\{A(I_{1,\ldots,n})\} \geqslant \mu\{A(I_{1,\ldots,n})\}$ for all $n \geqslant 1$. Since the continuity of v implies that of μ, from

$$\lim_n A(I_{1,\ldots,n}) = \{\omega\colon X_1(\omega) + X_2(\omega) \geqslant \alpha\}$$

it follows

$$v\{\omega\colon X_1(\omega) + X_2(\omega) \geqslant \alpha\} \geqslant \mu\{\omega\colon X_1(\omega) + X_2(\omega) \geqslant \alpha\}.$$

We now extend the previous argument to the case $\{\omega\colon \sum_{i=1}^n X_i(\omega) \geqslant \alpha\}$. We have

$$\left\{\sum_{i=1}^n X_i \geqslant \alpha\right\} = \bigcup_{r_1} \cdots \bigcup_{r_{n-1}} \{X_1 \geqslant r_1\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}\} \cap \left\{X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j\right\}.$$

Let $I = \{r_1(i_1)\}_{i_1 \in \{1,\ldots,k\}} \cup \cdots \cup \{r_{n-1}(i_{n-1})\}_{i_{n-1} \in \{1,\ldots,k\}}$. W.l.o.g. suppose $r_j(i_j) \leqslant r_j(i_j + 1)$ for all $1 \leqslant j \leqslant n-1$. Therefore,

$$\{X_j \geqslant r_j(1)\} \supseteq \cdots \supseteq \{X_j \geqslant r_j(k)\} \qquad \text{for all} \quad 1 \leqslant j \leqslant n-1 \tag{5}$$

and

$$\begin{aligned} &\{X_n \geqslant \alpha - r_1(1) - \cdots - r_{n-1}(1)\} \\ &\quad \subseteq \{X_n \geqslant \alpha - r_1(2) - \cdots - r_{n-1}(1)\} \subseteq \cdots \\ &\quad \subseteq \{X_n \geqslant \alpha - r_1(k-1) - \cdots - r_{n-1}(k-1)\}. \end{aligned} \tag{6}$$

We can write

$$\bigcup_{i_1=1}^k \cdots \bigcup_{i_{n-1}=1}^k \{X_1 \geqslant r_1(i_1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(i_{n-1})\} \cap \left\{X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(i_j)\right\}$$

$$
\begin{aligned}
&= \left[\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(1)\} \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1) \right\} \right] \\
&\quad \cup \Bigg[\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(2)\} \\
&\quad \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-2} r_j(1) - r_{n-1}(2) \right\} \Bigg] \\
&\quad \cup \cdots \cup \Bigg[\{X_1 \geqslant r_1(k)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\} \\
&\quad \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(k) \right\} \Bigg].
\end{aligned}
$$

Using convexity, (5), and (6) we get:

$$
\begin{aligned}
&v \Bigg\{ \bigcup_{i_1=1}^{k} \cdots \bigcup_{i_{n-1}=1}^{k} \{X_1 \geqslant r_1(i_1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(i_{n-1})\} \\
&\qquad \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(i_j) \right\} \Bigg\} \\
&\quad \geqslant v \Bigg\{ \left[\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(1)\} \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1) \right\} \right] \\
&\qquad \cup \Bigg[\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(2)\} \\
&\qquad \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-2} r_j(1) - r_{n-1}(2) \right\} \Bigg] \cup \cdots \\
&\qquad \cup \Bigg[\{X_1 \geqslant r_1(k-1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\} \\
&\qquad \cap \left\{ X_n \geqslant \alpha - \sum_{j=2}^{n-1} r_j(k) - r_1(k-1) \right\} \Bigg] \Bigg\} \\
&\qquad + v \Bigg\{ \{X_1 \geqslant r_1(k)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\} \\
&\qquad \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(k) \right\} \Bigg\} \\
&\qquad - v \Bigg\{ \{X_1 \geqslant r_1(k)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\} \\
&\qquad \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1) \right\} \Bigg\}.
\end{aligned}
$$

Now, repeat the same argument, using convexity, (5), and (6), for the sets

$$\{r_1(k-2), r_2(k-1), ..., r_{n-1}(k-1)\}$$

$$\{r_1(k-2), r_2(k-2), ..., r_{n-1}(k-1)\}$$

and so on until $\{r_1(1), r_2(1), ..., r_{n-1}(2)\}$. We eventually get that

$$v\left\{\bigcup_{i_1=1}^{k} \cdots \bigcup_{i_{n-1}=1}^{k} \{X_1 \geqslant r_1(i_1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(i_{n-1})\}\right.$$
$$\left.\cap\left\{X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(i_j)\right\}\right\}$$

is larger than the expression

$$v\left\{\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(1)\} \cap \left\{X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1)\right\}\right\}$$
$$+ v\left\{\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(2)\}\right.$$
$$\left.\cap\left\{X_n \geqslant \alpha - \sum_{j=1}^{n-2} r_j(1) - r_{n-1}(2)\right\}\right\}$$
$$- v\left\{\{X_1 \geqslant r_1(1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(2)\}\right.$$
$$\left.\cap\left\{X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1)\right\}\right\} + \cdots + v\left\{\{X_1 \geqslant r_1(k-1)\} \cap \cdots\right.$$
$$\left.\cap \{X_{n-1} \geqslant r_{n-1}(k)\} \cap \left\{X_n \geqslant \alpha - \sum_{j=2}^{n-1} r_j(k) - r_1(k-1)\right\}\right\}$$
$$- v\left\{\left\{X_1 \geqslant r_1(k-1)\right\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\}\right.$$
$$\left.\cap\left\{X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1)\right\}\right\}$$
$$+ v\left\{\left\{X_1 \geqslant r_1(k)\right\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\}\right.$$

$$\cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(k) \right\}\Bigg\}$$

$$- v \left\{ \{X_1 \geqslant r_1(k)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(k)\} \right.$$

$$\left. \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(1) \right\} \right\},$$

whose components are positive or negative elements of the form

$$v\{\{X_1 \geqslant r_1(i)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(i)\} \cap \{X_n \geqslant \alpha - \beta\}\},$$

where β is a sum of elements in the set I, like for instance $\sum_{j=1}^{n-1} r_j(1)$. As before, we have equality if we consider the additive μ. Using independence we conclude that

$$v\left(\bigcup_{i_1=1}^{k} \cdots \bigcup_{i_{n-1}=1}^{k} \{X_1 \geqslant r_1(i_1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(i_{n-1})\} \right.$$

$$\left. \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(i_j) \right\} \right)$$

$$\geqslant \mu\left(\bigcup_{i_1=1}^{k} \cdots \bigcup_{i_{n-1}=1}^{k} \{X_1 \geqslant r_1(i_1)\} \cap \cdots \cap \{X_{n-1} \geqslant r_{n-1}(i_{n-1})\} \right.$$

$$\left. \cap \left\{ X_n \geqslant \alpha - \sum_{j=1}^{n-1} r_j(i_j) \right\} \right).$$

Continuity of μ leads to

$$v \left\{ \omega \colon \sum_{i=1}^{n} X_i(\omega) \geqslant \alpha \right\} \geqslant \mu \left\{ \omega \colon \sum_{i=1}^{n} X_i(\omega) \geqslant \alpha \right\}.$$

In particular, since the argument is not based on n, this implies

$$v\{\omega \colon S_n \geqslant \alpha\} \geqslant \mu\{\omega \colon S_n \geqslant \alpha\}.$$

This completes the proof of part (i).

As to point (ii), set $S_n^- = \inf_{k \geqslant n} \{S_k\}$. We have $\{\omega \colon S_n^- \geqslant \alpha\} = \bigcap_{k \geqslant n} \{\omega \colon S_k \geqslant \alpha\}$. We want to show that $v\{\omega \colon S_n^- \geqslant \alpha\} \geqslant \mu\{\omega \colon S_n^- \geqslant \alpha\}$. By continuity of v, it suffices to prove that for any $\{\omega \colon S_{i_1}, ..., S_{i_n}\}$ of arbitrary finite length it holds $v\{\omega \colon S_{i_1}, ..., S_{i_n}\} \geqslant \mu\{\omega \colon S_{i_1}, ..., S_{i_n}\}$. Suppose $m > n$. Then

$$S_m = \frac{1}{m} \sum_{i=n+1}^{m} X_i + \frac{n}{m} S_n.$$

Set $\gamma_{n,m} = (n/m)$. We can write

$$
\begin{aligned}
&\{\omega\colon S_m \geqslant \alpha, S_n \geqslant \alpha\}\\
&\quad = \{\omega\colon S_m \geqslant \alpha\} \cap \{\omega\colon S_n \geqslant \alpha\}\\
&\quad = \left\{\omega\colon \frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i + S_n \geqslant \frac{\alpha}{\gamma_{n,m}}\right\} \cap \{\omega\colon S_n \geqslant \alpha\}\\
&\quad = \left\{\bigcup_r \left[\left\{\omega\colon \frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i \geqslant \frac{\alpha}{\gamma_{n,m}} - r\right\} \cap \{\omega\colon S_n \geqslant r\}\right]\right\}\\
&\qquad \cap \{\omega\colon S_n \geqslant \alpha\}\\
&\quad = \bigcup_r \left[\left\{\omega\colon \frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i \geqslant \frac{\alpha}{\gamma_{n,m}} - r\right\} \cap \{S_n \geqslant r\} \cap \{\omega\colon S_n \geqslant \alpha\}\right]\\
&\quad = \left[\bigcup_{r \leqslant \alpha} \left[\left\{\omega\colon \frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i \geqslant \frac{\alpha}{\gamma_{n,m}} - r\right\} \cap \{S_n \geqslant \alpha\}\right]\right]\\
&\qquad \cup \left[\bigcup_{r > \alpha} \left[\left\{\omega\colon \frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i \geqslant \frac{\alpha}{\gamma_{n,m}} - r\right\} \cap \{S_n \geqslant r\}\right]\right].
\end{aligned}
$$

We know that

$$
\begin{aligned}
&\left\{\frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i \geqslant \frac{\alpha}{\gamma_{n,m}} - r\right\}\\
&\qquad = \bigcup_{s_1} \cdots \bigcup_{s_{m-n-1}} \{Y_1 \geqslant s_1\} \cap \cdots \cap \{Y_{m-n-1} \geqslant s_{m-n-1}\}\\
&\qquad\quad \cap \left\{Y_{m-n} \geqslant \left[\frac{\alpha}{\gamma_{n,m}} - r\right] - \sum_{j=1}^{m-n-1} s_j\right\}
\end{aligned}
$$

and

$$
\{S_n \geqslant r\} = \bigcup_{t_1} \cdots \bigcup_{t_{n-1}} \{Z_1 \geqslant t_1\} \cap \cdots \cap \{Z_{n-1} \geqslant t_{n-1}\} \cap \left\{Z_n \geqslant r - \sum_{j=1}^{n-1} t_j\right\},
$$

where both the s_i and the t_i go over the rational numbers, and we have set $Y_i = (1/m\gamma_{n,m})\, X_{i+n+1}$ and $Z_i = (1/n)\, X_i$. Then

$$
\begin{aligned}
&\left[\bigcup_{r\leqslant\alpha}\left[\left\{\omega:\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\right\}\cap\{S_n\geqslant\alpha\}\right]\right]\\
&\qquad\cup\left[\bigcup_{r>\alpha}\left[\left\{\omega:\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\right\}\cap\{S_n\geqslant r\}\right]\right]\\
&\qquad\cup\left[\bigcup_{r\leqslant\alpha}\left[\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\{Y_1\geqslant s_1\}\cap\cdots\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}\right.\right.\\
&\qquad\left.\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-r\right]-\sum_{j=1}^{m-n-1}s_j\right\}\right]\\
&\qquad\cap\left[\bigcup_{t_1}\cdots\bigcup_{t_{n-1}}\{Z_1\geqslant t_1\}\cap\cdots\cap\{Z_{n-1}\geqslant t_{n-1}\}\right.\\
&\qquad\left.\left.\cap\left\{Z_n\geqslant r-\sum_{j=1}^{n-1}t_j\right\}\right]\right]\\
&\qquad\cup\left[\bigcup_{r>\alpha}\left[\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\{Y_1\geqslant s_1\}\cap\cdots\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}\right.\right.\\
&\qquad\left.\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-r\right]-\sum_{j=1}^{m-n-1}s_j\right\}\right]\\
&\qquad\cap\left[\bigcup_{t_1}\cdots\bigcup_{t_{n-1}}\{Z_1\geqslant t_1\}\cap\cdots\cap\{Z_{n-1}\geqslant t_{n-1}\}\right.\\
&\qquad\left.\left.\cap\left\{Z_n\geqslant\alpha-\sum_{j=1}^{n-1}t_j\right\}\right]\right]\\
&=\left[\bigcup_{r\leqslant\alpha}\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\bigcup_{t_1}\cdots\bigcup_{t_{n-1}}\{Y_1\geqslant s_1\}\right.\\
&\qquad\cap\cdots\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}\\
&\qquad\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-r\right]-\sum_{j=1}^{m-n-1}s_j\right\}\\
&\qquad\left.\cap\left\{\{Z_1\geqslant t_1\}\cap\cdots\cap\{Z_{n-1}\geqslant t_{n-1}\}\cap\left\{Z_n\geqslant r-\sum_{j=1}^{n-1}t_j\right\}\right\}\right]\\
&\qquad\cup\left[\bigcup_{r>\alpha}\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\bigcup_{t_1}\cdots\bigcup_{t_{n-1}}\{X_1\geqslant s_1\}\cap\cdots\cap\{X_{n-1}\geqslant s_{m-n-1}\}\right.\\
&\qquad\cap\left\{X_n\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-r\right]-\sum_{j=1}^{m-n-1}s_j\right\}\cap\{Z_1\geqslant t_1\}\\
&\qquad\left.\cap\cdots\cap\{Z_{n-1}\geqslant t_{n-1}\}\cap\left\{Z_n\geqslant\alpha-\sum_{j=1}^{n-1}t_j\right\}\right].
\end{aligned}
$$

Set

$$I=\{r(i)\}_{i\in\{1,\dots,k\}}\cup\{s_1(i_1)\}_{i_1\in\{1,\dots,k\}}\cdots$$
$$\cup\{s_{m-n-1}(i_{m-n-1})\}_{i_{m-n-1}\in\{1,\dots,k\}}$$
$$\cup\{t_1(i'_1)\}_{i'_1\in\{1,\dots,k\}}\cdots\cup\{t_{n-1}(i'_{n-1})\}_{i'_{n-1}\in\{1,\dots,k\}}.$$

W.l.o.g. suppose

$$s_j(i_j)\leqslant s_j(i_j+1)\qquad\text{for all}\quad 1\leqslant j\leqslant m-n-1$$
$$t_j(i)\leqslant t_j(i+1)\qquad\text{for all}\quad 1\leqslant j\leqslant n-1$$
$$\alpha\leqslant r(i)\leqslant r(i+1).$$

Therefore,

$$\{Y_j\geqslant s_j(1)\}\supseteq\cdots\supseteq\{Y_j\geqslant s_j(k)\}\qquad\text{for all}\quad 1\leqslant j\leqslant m-n-1$$
$$\{Z_j\geqslant t_j(1)\}\supseteq\cdots\supseteq\{Z_j\geqslant t_j(k)\}\qquad\text{for all}\quad 1\leqslant j\leqslant n-1$$
$$\{Y_{m-n}\geqslant\alpha-s_1(1)-\cdots-s_{n-1}(1)-r(1)\}$$
$$\subseteq\{Y_{m-n}\geqslant\alpha-s_1(2)-\cdots-s_{n-1}(1)-r(1)\}\cdots$$
$$\subseteq\{Y_{m-n}\geqslant\alpha-s_1(k-1)-\cdots-s_{n-1}(k-1)-r(k-1)\}$$
$$\{Z_n\geqslant r-t_1(1)-\cdots-t_{n-1}(1)\}$$
$$\subseteq\{Z_n\geqslant r-t_1(2)-\cdots-t_{n-1}(1)\}\cdots$$
$$\subseteq\{Z_n\geqslant r-t_1(k-1)-\cdots-t_{n-1}(k-1)\}.\tag{7}$$

Moreover,

$$v\{Y_j\geqslant s_j(i)\}=\mu\{Y_j\geqslant s_j(1)\}\qquad\text{for all}\quad 1\leqslant j\leqslant m-n-1$$
$$\text{and}\quad 1\leqslant i\leqslant k;$$
$$v\{Z_j\geqslant t_j(i)\}=\mu\{Z_j\geqslant t_j(i)\}\qquad\text{for all}\quad 1\leqslant j\leqslant n-1$$
$$\text{and}\quad 1\leqslant i\leqslant k;$$

and

$$v\{Y_{m-n} \geqslant \alpha - s_1(1) - \cdots - s_{n-1}(1) - r(1)\}$$
$$= \mu\{Y_{m-n} \geqslant \alpha - s_1(1) - \cdots - s_{n-1}(1) - r(1)\};$$
$$v\{Y_{m-n} \geqslant \alpha - s_1(2) - \cdots - s_{n-1}(1) - r(1)\}$$
$$= \mu\{Y_{m-n} \geqslant \alpha - s_1(2) - \cdots - s_{n-1}(1) - r(1)\};$$

and so forth until

$$v\{Y_{m-n} \geqslant \alpha - s_1(k-1) - \cdots - s_{n-1}(k-1) - r(k-1)\}$$
$$= \mu\{Y_{m-n} \geqslant \alpha - s_1(k-1) - \cdots - s_{n-1}(k-1) - r(k-1)\};$$
$$v\{Z_n \geqslant r - t_1(1) - \cdots - t_{n-1}(1)\} = \mu\{Z_n \geqslant r - t_1(1) - \cdots - t_{n-1}(1)\};$$

and so forth until

$$v\{Z_n \geqslant r - t_1(k-1) - \cdots - t_{n-1}(k-1)\}$$
$$= \mu\{Z_n \geqslant r - t_1(k-1) - \cdots - t_{n-1}(k-1)\}.$$

Using the fact that v is totally monotone, (7), the above equalities, and independence, we get

$$\begin{aligned} v\Bigg(\bigcup_{r \in I} \bigcup_{t_1 \in I} \cdots \bigcup_{t_{n-1}} \bigcup_{s_1 \in I} \cdots \bigcup_{s_{m-n-1} \in I} &\{Z_1 \geqslant t_1\} \cap \cdots \cap \{Z_{n-1} \geqslant t_{n-1}\} \\ &\cap \left\{Z_n \geqslant r - \sum_{j=1}^{n-1} t_j\right\} \cap \{Y_1 \geqslant s_1\} \cap \cdots \\ &\cap \{Y_{m-n-1} \geqslant s_{m-n-1}\} \cap \left\{Y_{m-n} \geqslant \frac{\alpha}{\gamma_{n,m}} - r - \sum_{j=1}^{m-n-1} s_j\right\}\Bigg) \\ \geqslant \mu\Bigg(\bigcup_{r \in I} \bigcup_{t_1 \in I} \cdots \bigcup_{t_{n-1}} \bigcup_{s_1 \in I} \cdots \bigcup_{s_{m-n-1} \in I} &\{Z_1 \geqslant t_1\} \cap \cdots \cap \{Z_{n-1} \geqslant t_{n-1}\} \\ &\cap \left\{Z_n \geqslant r - \sum_{j=1}^{n-1} t_j\right\} \cap \{Y_1 \geqslant s_1\} \cap \cdots \cap \{Y_{m-n-1} \geqslant s_{m-n-1}\} \\ &\cap \left\{Y_{m-n} \geqslant \frac{\alpha}{\gamma_{n,m}} - r - \sum_{j=1}^{m-n-1} s_j\right\}\Bigg). \end{aligned}$$

Finally, by continuity, we get

$$v\{\omega \colon S_m \geqslant \alpha, S_n \geqslant \alpha\} \geqslant \mu\{\omega \colon S_m \geqslant \alpha, S_n \geqslant \alpha\}.$$

Let $k<n$. We can write

$$\begin{aligned}
&\{\omega\colon S_m\geqslant\alpha,\, S_n\geqslant\alpha,\, S_k\geqslant\alpha\}\\
&\quad=\Bigg[\bigcup_{r\leqslant\alpha}\Bigg[\Bigg\{\omega\colon\frac{1}{\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\Bigg\}\cap\{S_n\geqslant\alpha\}\Bigg]\cap\{\omega\colon S_k\geqslant\alpha\}\Bigg]\\
&\qquad\cup\Bigg[\bigcup_{r>\alpha}\Bigg[\Bigg\{\omega\colon\frac{1}{\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\Bigg\}\cap\{S_n\geqslant r\}\Bigg]\\
&\qquad\cap\{\omega\colon S_k\geqslant\alpha\}\Bigg].
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\{\omega\colon S_n\geqslant r,\, S_k\geqslant\alpha\}\\
&\quad=\Bigg[\bigcup_{t\leqslant\alpha}\Bigg[\Bigg\{\omega\colon\frac{1}{n\gamma_{n,m}}\sum_{i=k+1}^{n}X_i\geqslant\frac{r}{\gamma_{k,n}}-t\Bigg\}\cap\{S_k\geqslant t\}\Bigg]\Bigg]\\
&\qquad\cup\Bigg[\bigcup_{t>\alpha}\Bigg[\Bigg\{\omega\colon\frac{1}{n\gamma_{k,n}}\sum_{i=k+1}^{n}X_i\geqslant\frac{r}{\gamma_{k,n}}-t\Bigg\}\cap\{S_k\geqslant\alpha\}\Bigg]\Bigg].
\end{aligned}$$

We finally get the following equality, which we denote by (*)

$$\begin{aligned}
&\{\omega\colon S_m\geqslant\alpha,\, S_n\geqslant\alpha,\, S_k\geqslant\alpha\}\\
&\quad=\Bigg[\bigcup_{r\leqslant\alpha}\bigcup_{t>\alpha}\Bigg\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-t\Bigg\}\\
&\qquad\cap\Bigg\{\omega\colon\frac{1}{n\gamma_{k,n}}\sum_{i=k+1}^{n}X_i\geqslant\frac{r}{\gamma_{k,n}}-t\Bigg\}\cap\{S_k\geqslant t\}\Bigg]\\
&\qquad\cup\Bigg[\bigcup_{r\leqslant\alpha}\bigcup_{t\leqslant\alpha}\Bigg\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-t\Bigg\}\\
&\qquad\cap\Bigg\{\omega\colon\frac{1}{n\gamma_{k,n}}\sum_{i=k+1}^{n}X_i\geqslant\frac{r}{\gamma_{k,n}}-t\Bigg\}\cap\{S_k\geqslant\alpha\}\Bigg]\\
&\qquad\cup\Bigg[\bigcup_{r>\alpha}\bigcup_{t>\alpha}\Bigg\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-t\Bigg\}\\
&\qquad\cap\Bigg\{\omega\colon\frac{1}{n\gamma_{k,n}}\sum_{i=k+1}^{n}X_i\geqslant\frac{\alpha}{\gamma_{k,n}}-t\Bigg\}\cap\{S_k\geqslant t\}\Bigg]\\
&\qquad\cup\Bigg[\bigcup_{r>\alpha}\bigcup_{t\leqslant\alpha}\Bigg\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m}X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-t\Bigg\}\\
&\qquad\cap\Bigg\{\omega\colon\frac{1}{n\gamma_{k,n}}\sum_{i=k+1}^{n}X_i\geqslant\frac{\alpha}{\gamma_{k,n}}-t\Bigg\}\cap\{S_k\geqslant\alpha\}\Bigg].
\end{aligned}$$

We know that

$$\left\{\frac{1}{m\gamma_{n,m}} \sum_{i=n+1}^{m} X_i \geqslant \frac{\alpha}{\gamma_{n,m}} - t\right\}$$

$$= \bigcup_{s_1} \cdots \bigcup_{s_{m-n-1}} \{Y_1 \geqslant s_1\} \cap \cdots \cap \{Y_{m-n-1} \geqslant s_{m-n-1}\}$$

$$\cap \left\{Y_{m-n} \geqslant \left[\frac{\alpha}{\gamma_{n,m}} - t\right] - \sum_{j=1}^{m-n-1} s_j\right\};$$

$$\left\{\omega : \frac{1}{n\gamma_{k,n}} \sum_{i=k+1}^{n} X_i \geqslant \frac{r}{\gamma_{k,n}} - t\right\}$$

$$= \bigcup_{h_1} \cdots \bigcup_{h_{n-k-1}} \{T_1 \geqslant h_1\} \cap \cdots \cap \{T_{n-k-1} \geqslant h_{n-k-1}\}$$

$$\cap \left\{T_{n-k} \geqslant \left[\frac{r}{\gamma_{k,n}} - t\right] - \sum_{j=1}^{n-k-1} h_j\right\};$$

$$\{S_k \geqslant t\} = \bigcup_{d_1} \cdots \bigcup_{d_{k-1}} \{Z_1 \geqslant d_1\} \cap \cdots \cap \{Z_{k-1} \geqslant d_{k-1}\}$$

$$\cap \left\{Z_k \geqslant t - \sum_{j=1}^{k-1} d_j\right\},$$

where $Y_i = (1/m\gamma_{n,m})\, X_{i+n+1}$, $Z_i = (1/k)\, X_i$, and $T_i = (1/n\gamma_{k,n})\, X_{i+k+1}$. Putting these equalities in $(*)$, we get:

$$\{\omega : S_m \geqslant \alpha, S_n \geqslant \alpha, S_k \geqslant \alpha\}$$

$$= \left\{\bigcup_{r \leqslant \alpha} \bigcup_{t > \alpha} \left[\bigcup_{s_1} \cdots \bigcup_{s_{m-n-1}} \{Y_1 \geqslant s_1\} \cap \cdots \cap \{Y_{m-n-1} \geqslant s_{m-n-1}\}\right.\right.$$

$$\left.\cap \left\{Y_{m-n} \geqslant \left[\frac{\alpha}{\gamma_{n,m}} - t\right] - \sum_{j=1}^{m-n-1} s_j\right\}\right]$$

$$\cap \left[\bigcup_{h_1} \cdots \bigcup_{h_{n-k-1}} \{T_1 \geqslant h_1\} \cap \cdots \cap \{T_{n-k-1} \geqslant h_{n-k-1}\}\right.$$

$$\left.\cap \left\{T_{n-k} \geqslant \left[\frac{r}{\gamma_{k,n}} - t\right] - \sum_{j=1}^{n-k-1} h_j\right\}\right]$$

$$\cap\Bigg[\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Z_1\geqslant d_1\}\cap\cdots\cap\{Z_{k-1}\geqslant d_{k-1}\}$$

$$\cap\left\{Z_k\geqslant t-\sum_{j=1}^{k-1}d_j\right\}\Bigg]\Bigg\}$$

$$\cup\Bigg\{\bigcup_{r\leqslant\alpha}\bigcup_{t\leqslant\alpha}\Bigg[\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\{Y_1\geqslant s_1\}\cap\cdots\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}$$

$$\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-t\right]-\sum_{j=1}^{m-n-1}s_j\right\}\Bigg]$$

$$\cap\Bigg[\bigcup_{h_1}\cdots\bigcup_{h_{n-k-1}}\{T_1\geqslant h_1\}\cap\cdots\cap\{T_{n-k-1}\geqslant h_{n-k-1}\}$$

$$\cap\left\{T_{n-k}\geqslant\left[\frac{r}{\gamma_{k,n}}-t\right]-\sum_{j=1}^{n-k-1}h_j\right\}\Bigg]$$

$$\cap\Bigg[\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Z_1\geqslant d_1\}\cap\cdots\cap\{Z_{k-1}\geqslant d_{k-1}\}$$

$$\cap\left\{Z_k\geqslant\alpha-\sum_{j=1}^{k-1}d_j\right\}\Bigg]\Bigg\}$$

$$\cup\Bigg\{\bigcup_{r>\alpha}\bigcup_{t>\alpha}\Bigg[\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\{Y_1\geqslant s_1\}\cap\cdots\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}$$

$$\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-t\right]-\sum_{j=1}^{m-n-1}s_j\right\}\Bigg]$$

$$\cap\Bigg[\bigcup_{h_1}\cdots\bigcup_{h_{n-k-1}}\{T_1\geqslant h_1\}\cap\cdots\cap\{T_{n-k-1}\geqslant h_{n-k-1}\}$$

$$\cap\left\{T_{n-k}\geqslant\left[\frac{\alpha}{\gamma_{k,n}}-t\right]-\sum_{j=1}^{n-k-1}h_j\right\}\Bigg]$$

$$\cap\Bigg[\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Z_1\geqslant d_1\}\cap\cdots\cap\{Z_{k-1}\geqslant d_{k-1}\}$$

$$\cap\left\{Z_k\geqslant t-\sum_{j=1}^{k-1}d_j\right\}\Bigg]\Bigg\}$$

$$\cup\Bigg\{\bigcup_{r>\alpha}\bigcup_{t\leqslant\alpha}\Bigg[\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\{Y_1\geqslant s_1\}\cap\cdots\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}$$

$$\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-t\right]-\sum_{j=1}^{m-n-1}s_j\right\}\Bigg]$$

$$\cap\left[\bigcup_{h_1}\cdots\bigcup_{h_{n-k-1}}\{T_1\geqslant h_1\}\cap\cdots\cap\{T_{n-k-1}\geqslant h_{n-k-1}\}\right.$$

$$\left.\cap\left\{T_{n-k}\geqslant\left[\frac{\alpha}{\gamma_{k,n}}-t\right]-\sum_{j=1}^{n-k-1}h_j\right\}\right]$$

$$\cap\left[\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Z_1\geqslant d_1\}\cap\cdots\cap\{Z_{k-1}\geqslant d_{k-1}\}\right.$$

$$\left.\left.\cap\left\{Z_k\geqslant\alpha-\sum_{j=1}^{k-1}d_j\right\}\right]\right\}.$$

This implies

$$\{\omega\colon S_m\geqslant\alpha,\,S_n\geqslant\alpha,\,S_k\geqslant\alpha\}$$

$$=\left\{\bigcup_{r\leqslant\alpha}\bigcup_{t>\alpha}\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\bigcup_{h_1}\cdots\bigcup_{h_{n-k-1}}\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Y_1\geqslant s_1\}\cap\cdots\right.$$

$$\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-t\right]-\sum_{j=1}^{m-n-1}s_j\right\}$$

$$\cap\{T_1\geqslant h_1\}\cap\cdots\cap\{T_{n-k-1}\geqslant h_{n-k-1}\}$$

$$\cap\left\{T_{n-k}\geqslant\left[\frac{r}{\gamma_{k,n}}-t\right]-\sum_{j=1}^{n-k-1}h_j\right\}$$

$$\left.\cap\left\{\{Z_1\geqslant d_1\}\cap\cdots\cap\{Z_{k-1}\geqslant d_{k-1}\}\cap\left\{Z_k\geqslant t-\sum_{j=1}^{k-1}d_j\right\}\right\}\right\}$$

$$\cup\left\{\bigcup_{r\leqslant\alpha}\bigcup_{t\leqslant\alpha}\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\bigcup_{h_1}\cdots\bigcup_{h_{n-k-1}}\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Y_1\geqslant s_1\}\cap\cdots\right.$$

$$\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-t\right]-\sum_{j=1}^{m-n-1}s_j\right\}$$

$$\cap\{T_1\geqslant h_1\}\cap\cdots\cap\{T_{n-k-1}\geqslant h_{n-k-1}\}$$

$$\cap\left\{T_{n-k}\geqslant\left[\frac{r}{\gamma_{k,n}}-t\right]-\sum_{j=1}^{n-k-1}h_j\right\}\cap\{Z_1\geqslant d_1\}\cap\cdots$$

$$\left.\cap\{Z_{k-1}\geqslant d_{k-1}\}\cap\left\{Z_k\geqslant\alpha-\sum_{j=1}^{k-1}d_j\right\}\right\}$$

$$\cup\left\{\bigcup_{r>\alpha}\bigcup_{t>\alpha}\bigcup_{s_1}\cdots\bigcup_{s_{m-n-1}}\bigcup_{h_1}\cdots\bigcup_{h_{n-k-1}}\bigcup_{d_1}\cdots\bigcup_{d_{k-1}}\{Y_1\geqslant s_1\}\cap\cdots\right.$$

$$\cap\{Y_{m-n-1}\geqslant s_{m-n-1}\}\cap\left\{Y_{m-n}\geqslant\left[\frac{\alpha}{\gamma_{n,m}}-t\right]-\sum_{j=1}^{m-n-1}s_j\right\}$$

$$\cap \{T_1 \geqslant h_1\} \cap \cdots \cap \{T_{n-k-1} \geqslant h_{n-k-1}\}$$

$$\cap \left\{T_{n-k} \geqslant \left[\frac{\alpha}{\gamma_{k,n}} - t\right] - \sum_{j=1}^{n-k-1} h_j\right\}$$

$$\cap \{Z_1 \geqslant d_1\} \cap \cdots \cap \{Z_{k-1} \geqslant d_{k-1}\} \cap \left\{Z_k \geqslant t - \sum_{j=1}^{k-1} d_j\right\}\Bigg\}$$

$$\cup \Bigg\{\bigcup_{r>\alpha} \bigcup_{t \leqslant \alpha} \bigcup_{s_1} \cdots \bigcup_{s_{m-n-1}} \bigcup_{h_1} \cdots \bigcup_{h_{n-k-1}} \bigcup_{d_1} \cdots \bigcup_{d_{k-1}} \{Y_1 \geqslant s_1\} \cap \cdots$$

$$\cap \{Y_{m-n-1} \geqslant s_{m-n-1}\} \cap \left\{Y_{m-n} \geqslant \left[\frac{\alpha}{\gamma_{n,m}} - t\right] - \sum_{j=1}^{m-n-1} s_j\right\}$$

$$\cap \{T_1 \geqslant h_1\} \cap \cdots \cap \{T_{n-k-1} \geqslant h_{n-k-1}\}$$

$$\cap \left\{T_{n-k} \geqslant \left[\frac{\alpha}{\gamma_{k,n}} - t\right] - \sum_{j=1}^{n-k-1} h_j\right\} \cap \{Z_1 \geqslant d_1\} \cap \cdots$$

$$\cap \{Z_{k-1} \geqslant d_{k-1}\} \cap \left\{Z_k \geqslant \alpha - \sum_{j=1}^{k-1} d_j\right\}\Bigg\}.$$

Now we can take a finite index set I as we did before to prove part (i). Proceeding then in the same way, by using total monotonicity of v, independence, and this last set of equalities, we get that v is larger than μ when we consider the above unions with indexes on I. By continuity of v, we finally obtain

$$v\{\omega: S_m \geqslant \alpha, S_n \geqslant \alpha, S_k \geqslant \alpha\} \geqslant \mu\{\omega: S_m \geqslant \alpha, S_n \geqslant \alpha, S_k \geqslant \alpha\} \tag{8}$$

as wanted. Now we complete the proof by induction. By (8), there are sets $A_1 \subseteq \mathbb{R}, ..., A_n \subseteq \mathbb{R}$ such that

$$\{\omega: S_{i_1} \geqslant \alpha, ..., S_{i_n} \geqslant \alpha\} = \bigcup_{\alpha_{i_1} \in A_{i_1}} \cdots \bigcup_{\alpha_{i_n} \in A_{i_n}} (X_{i_1} \geqslant \alpha_{i_n}) \cap \cdots \cap (X_{i_n} \geqslant \alpha_{i_n}).$$

Therefore

$$\{\omega: S_{i_1} \geqslant \alpha, ..., S_{i_n} \geqslant \alpha, S_{i_{n+1}} \geqslant \alpha\}$$

$$= \left[\bigcup_{\alpha_{i_1}} \cdots \bigcup_{\alpha_{i_n}} (X_{i_1} \geqslant \alpha_{i_n}) \cap \cdots \cap (X_{i_n} \geqslant \alpha_{i_n})\right] \cap [S_{i_{n+1}} \geqslant \alpha]$$

$$= \left[\bigcup_{\alpha_{i_1}} \cdots \bigcup_{\alpha_{i_n}} (X_{i_1} \geqslant \alpha_{i_n}) \cap \cdots \cap (X_{i_n} \geqslant \alpha_{i_n})\right]$$

$$\cap \left[\bigcup_{r_1} \cdots \bigcup_{r_n} (X_1 \geqslant r_1) \cap \cdots \cap (X_{i_n} \geqslant r_{i_n}) \cap \left(X_{i_{n+1}} \geqslant \alpha - \sum_{j=1}^{n-1} r_j\right)\right].$$

Consequently, there exist sets $A_1^* \subseteq \mathbb{R}, ..., A_n^* \subseteq \mathbb{R}, A_{n+1}^* \subseteq \mathbb{R}$ such that

$$\{\omega: S_{i_1} \geqslant \alpha, ..., S_{i_n} \geqslant \alpha, S_{i_{n+1}} \geqslant \alpha\}$$
$$= \bigcup_{\alpha_{i_1} \in A_{i_1}^*} \cdots \bigcup_{\alpha_{i_n} \in A_{i_n}^*} \bigcup_{\alpha_{i_{n+1}} \in A_{i_{n+1}}^*} (X_{i_1} \geqslant \alpha_{i_n}) \cap \cdots \cap (X_{i_n} \geqslant \alpha_{i_n}) \cap (X_{i_{n+1}} \geqslant \alpha_{i_{n+1}}).$$

Using this equality, independence, and total monotonicity, we conclude that

$$v\{\omega: S_{i_1}, ..., S_{i_n}\} \geqslant \mu\{\omega: S_{i_1}, ..., S_{i_n}\}$$

as wanted.

Set $S_\infty^- = \liminf_n S_n$, $A_n = \{\omega: S_n^-(\omega) \geqslant \alpha\}$, and $A = \{\omega: S_\infty^-(\omega) \geqslant \alpha\}$. The sequence $\{A_n\}_{n \geqslant 1}$ is nondecreasing because the S_n^- form a nondecreasing sequence. Moreover, $\bigcup_n A_n = A$. Since we have already proved that $v(A_n) \geqslant \mu(A_n)$ for each n, we get $v(A) \geqslant \mu(A)$ from the continuity of v and μ. This completes the proof of part (ii). ∎

Proof of Theorem 11. By lemma 19(ii), we have

$$\mu(\omega: \liminf_n S_n \geqslant \alpha) \leqslant v(\omega: \liminf_n S_n \geqslant \alpha).$$

On the other hand, the sequence $\{-X_k\}_{k \geqslant 1}$ is also made of independent and identically distributed random variables. Since we have also assumed that each $-X_k$ is regular, there exists a measure m (in general, $\mu \neq m$) on $\sigma(X_1, ..., X_k, ...)$ such that under m the sequence $\{-X_k\}_{k \geqslant 1}$ is made up of independent and identically distributed random variables, with $E_m(-X_k) = E_v(-X_k)$. By lemma 19(ii), we have

$$m(\omega: \liminf_n - S_n \geqslant \alpha) \leqslant v(\omega: \liminf_n - S_n \geqslant \alpha).$$

By the classic Kolmogorov's Strong Law of Large Numbers, we have

$$\mu(\omega: \liminf_n S_n \geqslant E_v(X_1)) = m(\omega: \liminf_n - S_n \geqslant E_v(-X_1)) = 1.$$

Together with

$$\{\omega: \liminf_n - S_n \geqslant E_v(-X_1)\} = \{\omega: \limsup_n S_n \leqslant -E_v(-X_1)\}$$

this implies

$$v(\omega: \liminf_n S_n \geqslant E_v(X_1)) = v(\omega: \limsup_n S_n \leqslant -E_v(-X_1)) = 1.$$

Convexity of v implies

$$v(\{\omega: E_v(X_1) \leqslant \liminf_n S_n(\omega) \leqslant \limsup_n S_n(\omega) \leqslant -E_v(-X_1)\}) = 1$$

as wanted. ∎

Theorem 11 holds for a continuous and totally monotone capacity. For example, if μ is a measure, then μ^2 is a simple example of a continuous and totally monotone capacity. We now provide a more interesting example.

Example. Let Ω_1 and Ω_2 be two infinite spaces, and $\Gamma: \Omega_1 \to 2^{\Omega_2}$ a correspondence mapping points of Ω_1 to subsets of Ω_2. For each $A \subseteq \Omega_2$, set $A^* = \{\omega \in \Omega_1 : \Gamma(\omega) \subseteq A\}$. Let μ be a measure on the power set 2^{Ω_1}. Define a set function v on 2^{Ω_2} as follows:

$$v(A) = \mu(A^*) \qquad \text{for all} \quad A \subseteq \Omega_2.$$

As is well-known, v is totally monotone (cf. Dempster [4]). We prove that if $|\Gamma(\omega)|$ is finite for all $\omega \in \Omega_1$, then v is continuous. Let $\{A_n\}_{n \geqslant 1}$ be a nondecreasing sequence in Ω_2 such that $\bigcup_{n \geqslant 1} A_n = \Omega_2$. It easy to check that the sequence $\{A_n^*\}_{n \geqslant 1}$ is nondecreasing in Ω_1. Moreover, for each $\omega \in \Omega_1$ there exists n_ω such that $\Gamma(\omega) \subseteq A_n$ for all $n \geqslant n_\omega$. For, suppose to the contrary that for all $n \geqslant 1$ it holds $\Gamma(\omega) \nsubseteq A_n$ (this is the correct negation because $A_n \subseteq A_{n+1}$ for all $n \geqslant 1$). Since $|\Gamma(\omega)|$ is finite, there exists at least one $\hat{\omega} \in \Gamma(\omega)$ such that for all $n \geqslant 1$ there exists $\hat{n}_n \geqslant n$ with $\hat{\omega} \notin A_{\hat{n}_n}$. Otherwise for all $\omega' \in \Gamma(\omega)$ we would have eventually $\omega' \in A_n$. All this implies the existence of a subsequence $\{A_{n_j}\}_{j \geqslant 1} \subseteq \{A_n\}_{n \geqslant 1}$ and a $\hat{\omega} \in \Gamma(\omega)$ such that $\bigcup_{j \geqslant 1} A_{n_j} = \Omega_2$ and $\hat{\omega} \notin A_{n_j}$ for all $j \geqslant 1$. But this is impossible. Therefore, for each $\omega \in \Omega_1$ we have eventually $\Gamma(\omega) \subseteq A_n$. This implies $\bigcup_{n \geqslant 1} A_n^* = \Omega_1$. Using the continuity of μ we conclude

$$\lim_n v(A_n) = \lim_n \mu(A_n^*) = \mu(\Omega_1) = 1$$

as wanted.

Remark. In sections 4 and 5 we have assumed continuity of the capacity v on $\mathscr{S}$, except when all random variables are finite-valued. However, all our results can be proved under the following somewhat weaker condition.

Definition 20. *Let $\{X_k\}_{k \geqslant 1}$ be a sequence of random variables defined on a σ-algebra $\mathscr{S}$. An $\mathscr{A}(X_1, ..., X_n, ...)$-capacity is an outer continuous capacity v on $\mathscr{S}$ such that for every nondecreasing sequence $\{A_n\}_{n \geqslant 1}$ of subsets of $\mathscr{A}(X_k)$ with $\bigcup_{n \geqslant 1} A_n = \Omega$ we have $\lim_n v(A_n) = v(\Omega)$.*

Here we ask full continuity only on the algebras $\mathscr{A}(X_k)$ and only outer continuity on the whole σ-algebra $\mathscr{S}$.

8.3. *Section* 6

We prove Theorem 13. Notation is as in the proof of Theorem 11. By the standard weak law, for every $\varepsilon > 0$ it holds

$$\lim_n \mu(\omega\colon S_n \geqslant E_\nu(X_1) - \varepsilon) = 1 \qquad \text{and}$$

$$\lim_n m(\omega\colon -S_n \geqslant E_\nu(-X_1) - \varepsilon) = 1.$$

By Lemma 19,

$$\mu(\omega\colon S_n \geqslant E_\nu(X_1) - \varepsilon) \leqslant \nu(\omega\colon S_n \geqslant E_\nu(X_1) - \varepsilon),$$
$$m(\omega\colon -S_n \geqslant E_\nu(-X_1) - \varepsilon) \leqslant \nu(\omega\colon -S_n \geqslant E_\nu(-X_1) - \varepsilon).$$

Therefore

$$\lim_n \nu(\omega\colon S_n \geqslant E_\nu(X_1) - \varepsilon) = \lim_n \nu(\omega\colon -S_n \geqslant E_\nu(-X_1) - \varepsilon) = 1$$

and, by monotonicity,

$$\lim_n \nu(\{\omega\colon E_\nu(X_1) - \varepsilon \leqslant S_n(\omega)\} \cup \{\omega\colon S_n(\omega) \leqslant -E_\nu(-X_1) + \varepsilon\}) = 1.$$

Convexity of ν then implies

$$\begin{aligned}
1 &\geqslant \lim_n \nu(\{\omega\colon E_\nu(X_1) - \varepsilon \leqslant S_n(\omega) \leqslant -E_\nu(-X_1) + \varepsilon\}) \\
&\geqslant \lim_n \nu(\omega\colon S_n \geqslant E_\nu(X_1) - \varepsilon) + \lim_n \nu(\omega\colon S_n \leqslant -E_\nu(-X_1) + \varepsilon) \\
&\quad - \lim_n \nu(\{\omega\colon E_\nu(X_1) - \varepsilon \leqslant S_n(\omega)\} \cup \{\omega\colon S_n(\omega) \leqslant -E_\nu(-X_1) + \varepsilon\}) \\
&= 1 + 1 - 1 = 1
\end{aligned}$$

as wanted. Parts (i) and (ii) are proved in a similar way. ∎

8.4. *Section* 7.1

In order to prove Theorem 15, we first derive the following simple, but important, property of controlled capacities.

Proposition 21. *Let ν be a controlled capacity on $\mathscr{S}$, and $\{X_k\}_{k \geqslant 1}$ a sequence of regular and independent random variables on $\mathscr{S}$. Let μ be the mass provided by Lemma 18. For every finite subclass $\{X_k\}_{k \in I}$ we have $\mu(A) \geqslant \nu(A)$ for each $A \in \mathscr{A}(\{X_k\}_{k \in I})$.*

Proof. Using the notation of the proof of Lemma 18, we have $\mu_k \in \mathscr{C}(v_k)$, where v_k is the restriction of v on $\mathscr{A}(X_k)$. Recall that for each set $A \in \mathscr{A}(\{X_k\}_{k \in I})$ there exists an integer n such that $A = \bigcup_{i=1}^n \bigcap_{k \in I} A_i^k$, where $A_i^k \in \mathscr{A}(X_k)$ for each $k \in I$, and the sets $\bigcap_{k \in I} A_i^k$ are pairwise disjoint. Let $m \in ba(\Omega, \mathscr{S})$, and let m_k be the restriction of m on $\mathscr{A}(X_k)$. By Definition 14,

$$v\left(\bigcup_{i=1}^{n} \bigcap_{k \in I} A_i^k\right) = \min\left\{\sum_{i=1}^{n} \prod_{k \in I} m_k(A_i^k) \colon m \in ba(\Omega, \mathscr{S})\right\}.$$

The result is now an easy consequence of the definition of μ. ∎

We now state and prove a version of lemma 19 for controlled capacities. The improvement is twofold. On the one hand, we get equalities where we had before only inequalities. On the other hand, to prove (iii) below we do not need total monotonicity; for controlled capacities the convexity of the restrictions v_k is enough.

LEMMA 22. *Let Ω be a compact space, and v a continuous capacity on the σ-algebra $\mathscr{S}$ of subsets of Ω. Let $\{X_k\}_{k \geqslant 1}$ be a sequence of continuous, regular and independent random variables relative to v. Set $S_n = (1/n) \sum_{i=1}^n X_i$, and let μ be the measure provided by Theorem 6. If v is controlled and if each v_k is convex on $\mathscr{A}(X_k)$, then*

(i) $\mu(\omega \colon S_n(\omega) \geqslant \alpha) = v(\omega \colon S_n(\omega) \geqslant \alpha)$ *for all* $\alpha \in \mathbb{R}$,

(ii) $\mu(\omega \colon S_n(\omega) > \alpha) = v(\omega \colon S_n(\omega) > \alpha)$ *for all* $\alpha \in \mathbb{R}$,

(iii) $\mu(\omega \colon \liminf_n S_n \geqslant \alpha) = v(\omega \colon \liminf_n S_n \geqslant \alpha)$ *for all* $\alpha \in \mathbb{R}$.

Proof. Consider the index set $I = \{r^1, ..., r^n\}$. As in Lemma 19, set

$$A_r = \{\omega \colon X_1(\omega) \geqslant r\} \qquad \text{and} \qquad A_{\alpha - r} = \{\omega \colon X_2(\omega) \geqslant \alpha - r\}.$$

Without loss, suppose $r^1 \leqslant \cdots \leqslant r^n$. We have

$$A_{r(1)} \supseteq \cdots \supseteq A_{r(n)} \quad \text{and} \quad A_{\alpha - r(1)} \subseteq \cdots \subseteq A_{\alpha - r(n)}.$$

This implies that for any $T \subseteq I$ the set $\bigcap_{r \in T}(A_r \cap A_{\alpha - r})$ has the form $A_{r'} \cap A_{\alpha - r}$ for some $r, r' \in I$, so that $v(\bigcap_{r \in T}(A_r \cap A_{\alpha - r})) = \mu(\bigcap_{r \in T}(A_r \cap A_{\alpha - r}))$. Therefore,

$$v\left(\bigcup_{r\in I}(A_r\cap A_{\alpha-r})\right)$$

$$=\int 1_{\bigcup_{r\in I}(A_r\cap A_{\alpha-r})}\,dv=\int \sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,1_{\bigcap_{r\in T}A_r\cap A_{\alpha-r}}\,dv$$

$$=\min_{\mu}\left\{\sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,1_{\bigcap_{r\in T}A_r\cap A_{\alpha-r}}\,d\mu:\mu(A_{r(i)})=\mu_1(A_{r(i)}),\right.$$

$$\left.\mu(A_{\alpha-r(i)})=\mu_1(A_{\alpha-r(i)})\text{ for all }i\in I\right\}$$

$$=\sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,\mu\left(\bigcap_{r\in T}(A_r\cap A_{\alpha-r})\right)$$

$$=\sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,v\left(\bigcap_{r\in T}(A_r\cap A_{\alpha-r})\right).$$

Therefore

$$v\left(\bigcup_{r\in I}(A_r\cap A_{\alpha-r})\right)=\sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,v\left(\bigcap_{r\in T}(A_r\cap A_{\alpha-r})\right)\tag{9}$$

and

$$v\left\{\bigcup_{r\in I}(A_r\cap A_{\alpha-r})\right\}=\mu\left\{\bigcup_{r\in I}(A_r\cap A_{\alpha-r})\right\}.$$

Continuity of μ and v then implies $v(\{\omega: X_1+X_2\geqslant\alpha\})=\mu(\{\omega: X_1+X_2\geqslant\alpha\})$.

The above argument can be extended to the sum of n random variables proceeding, mutatis mutandis, as we did in Lemma 19. The proof of point (ii) is similar. It suffices to consider the chain of upper sets $\{\{X_k>\alpha\}:\alpha\in\mathbb{R}\}$.

We finally prove (iii). Suppose $m>n$. We claim that for any $\alpha,\beta\in\mathbb{R}$ it holds

$$v\left\{\omega:\sum_{i=1}^{n}X_i(\omega)\geqslant\alpha,\ \sum_{i=n+1}^{m}X_i(\omega)\geqslant\beta\right\}$$
$$=\mu\left\{\omega:\sum_{i=1}^{n}X_i(\omega)\geqslant\alpha,\ \sum_{i=n+1}^{m}X_i(\omega)\geqslant\beta\right\}.$$

We have

$$\left\{\omega\colon \sum_{i=1}^{n} X_i \geqslant \beta\right\} = \bigcup_{t_1} \cdots \bigcup_{t_{n-1}} \{X_1 \geqslant t_1\} \cap \cdots \cap \{X_{n-1} \geqslant t_{n-1}\}$$
$$\cap \left\{X_n \geqslant \beta - \sum_{j=1}^{n-1} t_j\right\}$$

and

$$\left\{\omega\colon \sum_{i=n+1}^{m} X_i \geqslant \alpha\right\} = \bigcup_{s_n} \cdots \bigcup_{s_{m-1}} \{X_{n+1} \geqslant s_{n+1}\} \cap \cdots \cap \{X_{m-1} \geqslant s_{m-1}\}$$
$$\cap \left\{X_m \geqslant \alpha - \sum_{j=n+1}^{m-1} s_j\right\},$$

where both the s_i and the t_i go over the rational numbers. Then

$$\left\{\omega\colon \sum_{i=1}^{n} X_i(\omega) \geqslant \alpha, \sum_{i=n+1}^{m} X_i(\omega) \geqslant \beta\right\}$$
$$= \left[\bigcup_{t_1} \cdots \bigcup_{t_{n-1}} \{X_1 \geqslant t_1\} \cap \cdots \cap \{X_{n-1} \geqslant t_{n-1}\} \cap \left\{X_n \geqslant \beta - \sum_{j=1}^{n-1} t_j\right\}\right]$$
$$\cap \left[\bigcup_{s_n} \cdots \bigcup_{s_{m-1}} \{X_{n+1} \geqslant s_{n+1}\} \cap \cdots \cap \{X_{m-1} \geqslant s_{m-1}\}\right.$$
$$\left.\cap \left\{X_m \geqslant \alpha - \sum_{j=n+1}^{m-1} s_j\right\}\right]$$
$$= \bigcup_{t_1} \cdots \bigcup_{t_{n-1}} \bigcup_{s_n} \cdots \bigcup_{s_{m-1}} \left[\{X_1 \geqslant t_1\} \cap \cdots \cap \{X_{n-1} \geqslant t_{n-1}\}\right.$$
$$\left.\cap \left\{X_n \geqslant \beta - \sum_{j=1}^{n-1} t_j\right\}\right]$$
$$\cap \left[\{X_{n+1} \geqslant s_{n+1}\} \cap \cdots \cdots \cap \{X_{m-1} \geqslant s_{m-1}\}\right.$$
$$\left.\cap \left\{X_m \geqslant \alpha - \sum_{j=n+1}^{m-1} s_j\right\}\right].$$

Set

$$I = \{t_1(i_1)\}_{i_1=1}^{k} \cup \cdots \cup \{t_{n-1}(i_{n-1})\}_{i_{n-1}}^{k} \cup \{s_{n+1}(i_{n+1})\}_{i_{n+1}}^{k}$$
$$\cup \cdots \{s_{m-1}(i_{m-1})\}_{i_{m-1}=1}^{k}.$$

W.l.o.g. suppose $s_j(i_j) \leqslant s_j(i_j+1)$ for all $n+1 \leqslant j \leqslant m-1$, and $t_j(i_j) \leqslant t_j(i_j+1)$ for all $1 \leqslant j \leqslant n-1$. Therefore:

$$\{X_j \geqslant s_j(1)\} \supseteq \cdots \supseteq \{X_j \geqslant s_j(k)\} \qquad \text{for all} \quad n+1 \leqslant j \leqslant m-1,$$

$$\{X_j \geqslant t_j(1)\} \supseteq \cdots \supseteq \{X_j \geqslant t_j(k)\} \qquad \text{for all} \quad 1 \leqslant j \leqslant n-1,$$

$$\begin{aligned}&\{X_m \geqslant \beta - s_{n+1}(1) - \cdots - s_{m-1}(1)\}\\ &\quad \subseteq \{X_m \geqslant \beta - s_{n+1}(2) - \cdots - s_{m-1}(1)\} \cdots\\ &\quad \subseteq \{X_m \geqslant \beta - s_{n+1}(k-1) - \cdots - s_{m-1}(k-1)\},\end{aligned}$$

$$\begin{aligned}&\{X_n \geqslant \alpha - t_1(1) - \cdots - t_{n-1}(1)\}\\ &\quad \subseteq \{X_n \geqslant \alpha - t_1(2) - \cdots - t_{n-1}(1)\} \cdots\\ &\quad \subseteq \{X_n \geqslant \alpha - t_1(k-1) - \cdots - t_{n-1}(k-1)\}.\end{aligned}$$

Now, using the last set of inclusions, we can proceed as we did in part (i) to get equality (9). We then get

$$\begin{aligned}&v\Bigg(\bigcup_{t_1 \in I} \cdots \bigcup_{t_{n-1} \in I} \bigcup_{s_{n+1} \in I} \cdots \bigcup_{s_{m-1} \in I} \Bigg[\{X_1 \geqslant t_1\} \cap \cdots \cap \{X_{n-1} \geqslant t_{n-1}\}\\ &\qquad \cap \Bigg\{X_n \geqslant \beta - \sum_{j=1}^{n-1} t_j\Bigg\} \cap \{X_{n+1} \geqslant s_{n+1}\} \cap \cdots \cap \{X_{m-1} \geqslant s_{m-1}\}\\ &\qquad \cap \Bigg\{X_m \geqslant \alpha - \sum_{j=n+1}^{m-1} s_j\Bigg\}\Bigg]\Bigg)\\ &\quad = \mu\Bigg(\bigcup_{t_1 \in I} \cdots \bigcup_{t_{n-1} \in I} \bigcup_{s_{n+1} \in I} \cdots \bigcup_{s_{m-1} \in I} \Bigg[\{X_1 \geqslant t_1\} \cap \cdots \cap \{X_{n-1} \geqslant t_{n-1}\}\\ &\qquad \cap \Bigg\{X_n \geqslant \beta - \sum_{j=1}^{n-1} t_j\Bigg\} \cap \{X_{n+1} \geqslant s_{n+1}\} \cap \cdots \cap \{X_{m-1} \geqslant s_{m-1}\}\\ &\qquad \cap \Bigg\{X_m \geqslant \alpha - \sum_{j=n+1}^{m-1} s_j\Bigg\}\Bigg]\Bigg).\end{aligned}$$

By continuity we finally obtain

$$\begin{aligned}&v\left\{\omega \colon \sum_{i=1}^{n} X_i(\omega) \geqslant \alpha,\ \sum_{i=n+1}^{m} X_i(\omega) \geqslant \beta\right\}\\ &\qquad = \mu\left\{\omega \colon \sum_{i=1}^{n} X_i(\omega) \geqslant \alpha,\ \sum_{i=n+1}^{m} X_i(\omega) \geqslant \beta\right\}.\end{aligned}$$

Since v is controlled, this implies the claim. Then

$$\begin{aligned} v&\left\{\omega\colon \sum_{i=1}^{n} X_i(\omega)\geqslant\alpha,\ \sum_{i=n+1}^{m} X_i(\omega)\geqslant\beta\right\}\\ &=\mu\left\{\omega\colon \sum_{i=1}^{n} X_i(\omega)\geqslant\alpha,\ \sum_{i=n+1}^{m} X_i(\omega)\geqslant\beta\right\}\\ &=\mu\left\{\omega\colon \sum_{i=1}^{n} X_i(\omega)\geqslant\alpha\right\}\mu\left\{\omega\colon \sum_{i=n+1}^{m} X_i(\omega)\geqslant\beta\right\}\\ &=v\left\{\omega\colon \sum_{i=1}^{n} X_i(\omega)\geqslant\alpha\right\}v\left\{\omega\colon \sum_{i=n+1}^{m} X_i(\omega)\geqslant\beta\right\}. \end{aligned} \tag{10}$$

In part (ii) of Lemma 19 we have proved that

$$\begin{aligned} &\{\omega\colon S_m\geqslant\alpha, S_n\geqslant\beta\}\\ &\quad=\left[\bigcup_{r\leqslant\beta}\left[\left\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m} X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\right\}\cap\{\omega\colon S_n\geqslant\beta\}\right]\right]\\ &\qquad\cup\left[\bigcup_{r>\beta}\left[\left\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m} X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\right\}\cap\{\omega\colon S_n\geqslant r\}\right]\right]. \end{aligned}$$

Let $I=\{r^1, ..., r^n\}$ be a finite sequence or rationals. W.l.o.g., suppose $r^1\leqslant\cdots\leqslant r^n$. Moreover, for convenience, suppose $\alpha\leqslant r^1\leqslant\cdots\leqslant r^n$. Set

$$A_r=\begin{cases}\{\omega\colon S_n\geqslant r\} & \text{if}\quad r>\beta\\ \{\omega\colon S_n\geqslant\beta\} & \text{if}\quad r\leqslant\beta.\end{cases}$$

$$A_{\alpha-r}=\left\{\omega\colon\frac{1}{m\gamma_{n,m}}\sum_{i=n+1}^{m} X_i\geqslant\frac{\alpha}{\gamma_{n,m}}-r\right\}.$$

By the claim we have just proved, we have

$$v(A_r\cap A_{\alpha-r})=\mu(A_r\cap A_{\alpha-r})$$

Moreover,

$$A_{r(1)}\supseteq\cdots\supseteq A_{r(n)}\qquad\text{and}\qquad A_{\alpha-r(1)}\subseteq\cdots\subseteq A_{\alpha-r(n)} \tag{11}$$

and

$$v(A_r)=\mu(A_r)\qquad\text{and}\qquad v(A_{\alpha-r})=\mu(A_{\alpha-r}).$$

Therefore, proceeding as we did in part (i) we get

$$\begin{aligned} v\left\{\bigcup_{r\in I}(A_r\cap A_{\alpha-r})\right\} &= \sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,v\left(\bigcap_{r\in T}(A_r\cap A_{\alpha-r})\right)\\ &= \sum_{\{T:\,\varnothing\neq T\subseteq I\}}(-1)^{|T|+1}\,\mu\left(\bigcap_{r\in T}(A_r\cap A_{\alpha-r})\right)\\ &= \mu\left\{\bigcup_{r\in I}(A_r\cap A_{\alpha-r})\right\}. \end{aligned}$$

Using continuity of v and μ, we conclude that

$$v\{\omega\colon S_m\geqslant\alpha,\ S_n\geqslant\beta\}=\mu\{\omega\colon S_m\geqslant\alpha,\ S_n\geqslant\beta\}.$$

Before going on we prove the following claim: Suppose (i) $A\in C_{n+1}$, (ii) $B\in\mathscr{A}(X_1, ..., X_n)$, and (iii) $v(B)=\mu(B)$; then $v(A\cap B)=\mu(A\cap B)$. In fact, we have $v(A)=\mu(A)$. Moreover, since v is controlled, it holds

$$\begin{aligned} v(A\cap B) &= \min\{\mu'(A\cap B)\colon \mu'_k\in C(v_k)\}\\ &= \min\{\mu'(A)\cap\mu'(B)\colon \mu'_k\in C(v_k)\}=\mu(A)\mu(B)=\mu(A\cap B). \end{aligned}$$

The second equality follows form the definition of controlled capacity, and from (i) and (ii). The third one follows from (iii) and $v(A)=\mu(A)$. We now prove by induction that

$$v\{\omega\colon S_1\geqslant\alpha_1, ..., S_n\geqslant\alpha_n\}=\mu\{\omega\colon S_1\geqslant\alpha_1, ..., S_n\geqslant\alpha_n\}.$$

The first step of the induction has already been proved. We can write

$$\begin{aligned} &\{\omega\colon S_1\geqslant\alpha_1, ..., S_n\geqslant\alpha_n\}\\ &\quad=\left\{\omega\colon S_1\geqslant\alpha_1, ..., S_{n-1}\geqslant\alpha_{n-1}, \frac{1}{n}\sum_{i=1}^{n}X_i\geqslant\alpha_n\right\}\\ &\quad=\left\{\omega\colon S_1\geqslant\alpha_1, ..., S_{n-1}\geqslant\alpha_{n-1}, \frac{n-1}{n}S_{n-1}+\frac{1}{n}X_n\geqslant\alpha_n\right\}\\ &\quad=\left\{\omega\colon S_1\geqslant\alpha_1, ..., S_{n-1}\geqslant\alpha_{n-1}, \bigcup_r\left(\frac{n-1}{n}S_{n-1}\geqslant r\right)\right.\\ &\qquad\left.\cap\left(\frac{1}{n}X_n\geqslant\alpha_n-r\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \{\omega\colon S_1 \geqslant \alpha_1\} \cap \cdots \cap \{\omega\colon S_{n-2} \geqslant \alpha_{n-2}\} \\
&\quad \cap \left\{ \bigcup_{r > ((n-1)/n)\,\alpha_{n-1}} \left(\omega\colon \frac{n-1}{n} S_{n-1} \geqslant r\right) \cap \left(\omega\colon \frac{1}{n} X_n \geqslant \alpha_n - r\right) \right\} \\
&\quad \cup \left\{ \bigcup_{r \leqslant ((n-1)/n)\,\alpha_{n-1}} (\omega\colon S_{n-1} \geqslant \alpha_{n-1}) \cap \left(\omega\colon \frac{1}{n} X_n \geqslant \alpha_n - r\right) \right\} \\
&= \left\{ \bigcup_{r > ((n-1)/n)\,\alpha_{n-1}} \left(\omega\colon \frac{n-1}{n} S_{n-1} \geqslant r\right) \cap \left(\omega\colon \frac{1}{n} X_n \geqslant \alpha_n - r\right) \right. \\
&\quad \left. \cap\, (\omega\colon S_1 \geqslant \alpha_1) \cdots \cap (\omega\colon S_{n-2} \geqslant \alpha_{n-2}) \right\} \\
&\quad \cup \left\{ \bigcup_{r \leqslant ((n-1)/n)\,\alpha_{n-1}} (S_{n-1} \geqslant \alpha_{n-1}) \cdots \cap \left(\frac{1}{n} X_n \geqslant \alpha_n - r\right) \right. \\
&\quad \left. \cap\, (\omega\colon S_1 \geqslant \alpha_1) \cap \cdots \cap (\omega\colon S_{n-2} \geqslant \alpha) \right\}.
\end{aligned}
$$

Let $I = \{r^1, ..., r^n\}$ be a finite sequence or rationals. W.l.o.g. suppose $r^1 \leqslant \cdots \leqslant r^n$. Set

$$
\begin{aligned}
A_{\alpha - r} &= \{\omega\colon X_n \geqslant \alpha_n - r\} \\
B &= (\omega\colon S_1 \geqslant \alpha_1) \cap \cdots \cap (\omega\colon S_{n-2} \geqslant \alpha_{n-2}) \\
A_r &= \begin{cases} \left(\dfrac{n-1}{n} S_{n-1} \geqslant r\right) \cap B & \text{if} \quad r < \dfrac{n-1}{n} \alpha_{n-1} \\ (S_{n-1} \geqslant \alpha) \cap B & \text{if} \quad r \geqslant \dfrac{n-1}{n} \alpha_{n-1}. \end{cases}
\end{aligned}
$$

We have

$$
A_{r(1)} \supseteq \cdots \supseteq A_{r(n)} \qquad \text{and} \qquad A_{\alpha - r(1)} \subseteq \cdots \subseteq A_{\alpha - r(n)}. \tag{12}
$$

The induction hypothesis says that $\nu\{\omega\colon S_1 \geqslant \beta_1, ..., S_{n-1} \geqslant \beta_{n-1}\}$ is equal to $\mu\{\omega\colon S_1 \geqslant \beta_1, ..., S_{n-1} \geqslant \beta_{n-1}\}$ for any finite sequence $\{\beta_i\}_{i=1}^{n-1}$ of real numbers. Hence $\nu(A_r) = \mu(A_r)$. Clearly, $\nu(A_{\alpha - r}) = \mu(A_{\alpha - r})$. Therefore, by the claim we have proved, $\nu(A_r \cap A_{\alpha - r}) = \mu(A_r \cap A_{\alpha - r})$. Hence

$$
\begin{aligned}
\nu(A_{r(i)} \cap A_{\alpha - r(i)}) &= \mu(A_{r(i)} \cap A_{\alpha - r(i)}) = \mu(A_{r(i)})\, \mu(A_{\alpha - r(i)}) \\
&= \nu(A_{r(i)})\, \nu(A_{\alpha - r(i)}) \\
\nu(A_{r(i)} \cap A_{\alpha - r(i-1)}) &= \mu(A_{r(i)} \cap A_{\alpha - r(i-1)}) = \mu(A_{r(i)})\, \mu(A_{\alpha - r(i-1)}) \\
&= \nu(A_{r(i)})\, \nu(A_{\alpha - r(i-1)})
\end{aligned}
$$

for all $1 \leqslant i \leqslant n$. Consequently,

$$\begin{aligned} v\left\{\bigcup_{r \in I} (A_r \cap A_{\alpha-r})\right\} &= \sum_{\{T:\, \varnothing \neq T \subseteq I\}} (-1)^{|T|+1}\, v\left(\bigcap_{r \in T} (A_r \cap A_{\alpha-r})\right) \\ &= \sum_{\{T:\, \varnothing \neq T \subseteq I\}} (-1)^{|T|+1}\, \mu\left(\bigcap_{r \in T} (A_r \cap A_{\alpha-r})\right) \\ &= \mu\left\{\bigcup_{r \in I} (A_r \cap A_{\alpha-r})\right\}. \end{aligned}$$

Using continuity of v and μ, we conclude that

$$v\{\omega: S_1 \geqslant \alpha_1, ..., S_n \geqslant \alpha_n\} = \mu\{\omega: S_1 \geqslant \alpha_1, ..., S_n \geqslant \alpha_n\}.$$

To complete the proof it suffices to use continuity of v and μ. ∎

Proof of Theorem 15. It is easy to see that

$$\begin{aligned} &\int_0^\infty v\{X \geqslant \alpha\}\, d\alpha + \int_{-\infty}^0 [v\{X \geqslant \alpha\} - 1]\, d\alpha \\ &\quad = \int_0^\infty v\{X > \alpha\}\, d\alpha + \int_{-\infty}^0 [v\{X > \alpha\} - 1]\, d\alpha. \end{aligned}$$

By Lemma 22,

$$\mu(\omega: S_n > \alpha) = v(\omega: S_n > \alpha) \qquad \text{for all} \quad \alpha \in \mathbb{R}.$$

By the classic Kolmogorov's Law of Large Numbers,

$$\lim_n \mu(\omega: S_n > E_v(X_1)) = 0.$$

Therefore,

$$\lim_n v(\omega: S_n > E_v(X_1)) = 0,$$

which implies, by continuity of v,

$$v(\omega: \liminf_n S_n(\omega) > E_v(X_1)) = 0.$$

A similar argument for the sequence $\{-X_k\}_{k \geqslant 1}$ implies

$$v(\omega: \liminf_n S_n(\omega) < -E_v - (X_1)) = 0.$$

This completes the proof. ∎

8.5. *Section* 7.2

In this subsection we prove the classic Central Limit Theorem for capacities. We first need a stronger version of Lemma 18, in which the mass μ preserves all moments $E_\nu(X_k^n)$ of the random variables, and not just the means $E_\nu(X_k)$.

LEMMA 23. *Let ν be a convex capacity on $\mathcal{S}$. Let $\{X_k\}_{k\geqslant 1}$ be a sequence of regular, independent, and nonnegative random variables relative to ν. Then there exists a mass μ defined on $\mathcal{A}(X_1, ..., X_k, ...)$ such that*

(i) $\{X_k\}_{k\geqslant 1}$ *is a sequence of independent random variables relative to* μ;

(ii) $E_\nu(X_k^n) = E_\mu(X_k^n)$ *for all* $k \geqslant 1$ *and* $n > 0$.

Proof. Since the random variables are nonnegative, we have $\{\omega: X_k^n \geqslant \alpha\} = \{\omega: X_k \geqslant \alpha^{1/n}\}$ for each $\alpha \in \mathbb{R}$. Therefore,

$$\begin{aligned} \nu\{\omega: X_k^n \geqslant \alpha\} &= \nu\{\omega: X_k \geqslant \alpha^{1/n}\} \\ &= \mu\{\omega: X_k \geqslant \alpha^{1/n}\} = \mu\{\omega: X_k^n \geqslant \alpha\}. \end{aligned}$$

This implies $E_\nu(X_k^n) = E_\mu(X_k^n)$, as wanted. ∎

The following lemma, whose simple proof is omitted, is the version of Theorem 6 that we need for the sequel.

LEMMA 24. *Let Ω be a compact space, and ν a convex and continuous capacity on the σ-algebra $\mathcal{S}$ of subsets of Ω. Let $\{X_k\}_{k\geqslant 1}$ be a sequence of continuous, regular, independent, and nonnegative random variables relative to ν. Then there exists a measure μ defined on $\sigma(X_1, ..., X_k, ...)$ such that*

(1) $\{X_k\}_{k\geqslant 1}$ *is a sequence of independent random variables relative to* μ;

(2) $E_\nu(X_k^n) = E_\mu(X_k^n)$ *for all* $k \geqslant 1$ *and* $n > 0$.

Proof of Theorem 16. Let m be the measure that by Corollary 24 can be associated to the sequence $\{-X_k\}_{k\geqslant 1}$. Since ν is controlled, a slight modification of the proof of Lemma 22 leads to

$$\nu(\{\omega: Z_n \leqslant \alpha\}) = m(\{\omega: Z_n \leqslant \alpha\}).$$

By the classic Central Limit Theorem,

$$\lim_n m(\{\omega: Z_n \leqslant \alpha\}) = N(\alpha)$$

and we conclude that $\lim_n \nu(\{\omega: Z_n \leqslant \alpha\}) = N(\alpha)$, as wanted.

Let μ be the measure that by Corollary 24 can be associated to the sequence $\{X_k\}_{k \geqslant 1}$. Reasoning as before, we get

$$\nu(\{\omega: Y_n > \alpha\}) = \mu(\{\omega: Y_n > \alpha\}).$$

By the classic Central Limit Theorem, $\lim_n \mu(\{\omega: Y_n \leqslant \alpha\}) = N(\alpha)$. Consequently, $\lim_n \mu(\{\omega: Y_n > \alpha\}) = 1 - N(\alpha)$, so that $\lim_n \nu(\{\omega: Y_n > \alpha\}) = 1 - N(\alpha)$, as wanted. ∎

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