Kochen–Specker theorem revisited

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Abstract:
The Kochen–Specker theorem is a basic and fundamental 50 year old non-existence result affecting the foundations of quantum mechanix, strongly implying the lack of any meaningful notion of "quantum realism", and typically leading to discussions of "contextuality" in quantum physics. Original proofs of the Kochen–Specker theorem proceeded via brute force counter-examples; often quite complicated and subtle (albeit mathematically "elementary") counter-examples. Only more recently have somewhat more "geometrical" proofs been developed. We present herein yet another simplified geometrical proof of the Kochen–Specker theorem, one that is valid for any number of dimensions, that minimizes the technical machinery involved, and makes the seriousness of the issues raised manifest.

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Contents

1 Introduction

The Kochen–Specker theorem [1, 2], (sometimes also called the Bell–Kochen–Specker theorem), was originally proved 50 years ago by explicitly finding a set of 117 distinct projection operators on 3-dimensional Hilbert space [3, 4], and then showing that there was no way to consistently assign values in $\{0, 1\}$ to these projection operators. (That is, these 117 "quantum questions" that one might ask could not be consistently assigned yes-no answers.) A later version of the Kochen–Specker theorem reduced the number of projection operators to 33 [5]. This was further reduced to 24 [5], to 20 [6], and to 18 [7, 8], at the cost of slightly increasing the dimension of the Hilbert space to 4. The number of projection operators was further reduced to 13 in an 8-dimensional Hilbert space in reference [9]. Interest in these foundational issues has continued unabated [10, 11], with at least two "geometrical" proofs that avoid explicit construction of sets of projection operators [12, 13].

We shall provide below yet another "geometrical" proof of the Kochen–Specker theorem, which, while it is non-constructive, (proceeding by establishing an inconsistency), is utterly minimal in its technical requirements, and so hopefully instructive.

2 Statement of the Kochen–Specker theorem

An explicit statement of the Kochen–Specker theorem, (based on the discussion in the Stanford encyclopaedia of philosophy [2]), runs thus:

Theorem: *Let H be a Hilbert space of QM state vectors of dimension $d \geq 3$. There is a set M of observables on H, containing n elements, such that the following two assumptions are contradictory:*

KS1: *All n members of M simultaneously have values, that is, they are unambiguously mapped onto real numbers (designated, for observables A, B, C, ..., by $v(A)$, $v(B)$, $v(C)$, ...).*

KS2: *Values of observables conform to the following constraints:*

- **(a)** *If A, B, C are all compatible and $C = A + B$, then $v(C) = v(A) + v(B)$.*
- **(b)** *If A, B, C are all compatible and $C = AB$, then $v(C) = v(A)v(B)$.*
- **(c)** *$\exists$ at least one observable X with $v(X) \neq 0$.*

There are several issues with this presentation. Without condition **KS2c** the theorem is actually false — the trivial valuation where for all observables X one sets $v(X) = 0$ provides an explicit counter-example. Without condition **KS2c**, $v(I) = v(I^2) = v(I)^2$ only implies $v(I) \in \{0, 1\}$. With condition **KS2c** we have the stronger statement that $v(X) = v(IX) = v(I)v(X)$ implies $v(I) = 1$.

A more subtle issue is this: Physically, we would like to have $v(zA) = z\, v(A)$, for any $z \in \mathbb{C}$. But using the conditions **KS2a** and **KS2b** we could only deduce this for rational numbers. Extending this to the complex numbers requires us to first construct the real numbers "on the fly" using Dedekind cuts, and then to formally construct the complex numbers as an algebraic extension of the field of real numbers — while this is certainly possible, in a physics context it is rather pointless — it would seem more reasonable to start with the complex numbers as being given, even if you then need slightly stronger axioms.

Improved KS2 axioms:

- If $[A, B] = 0$ and $a, b \in \mathbb{C}$, then $v(aA + bB) = av(A) + bv(B)$.
- If $[A, B] = 0$ then $v(AB) = v(A)v(B)$.
- $\exists$ at least one observable X with $v(X) \neq 0$.

If one accepts these improved **KS2** axioms then immediately

$$v(I) = 1; \qquad v(aI) = a; \qquad v(f(A)) = f(v(A)). \tag{2.1}$$

Note that this last condition, $v(f(A)) = f(v(A))$, is where physics discussions of the Kochen–Specker theorem often *start*. Let us write $A = \sum_i a_i P_i$ where the a_i are real and the P_i are projection operators onto 1-dimensional subspaces; so the $P_i = |\psi_i\rangle \langle\psi_i|$ can be identified with the vectors $|\psi_i\rangle$ which form a basis for the Hilbert space. Then

$$v(A) = v\left(\sum_i a_i P_i\right) = \sum_i a_i \; v(P_i), \tag{2.2}$$

which focusses attention on the valuations $v(P_i)$. Furthermore, since $P_i^2 = P_i$, condition **KS2b** implies that $v(P_i) \in \{0, 1\}$; the valuation must be a yes-no valuation. Now consider the identity operator $I = \sum_i P_i$ and note

$$\sum_i v(P_i) = v(I) = 1. \tag{2.3}$$

It is customary to identify the projectors P_i with the corresponding vectors n_i, with the n_i forming a basis for Hilbert space, and in d dimensions write

$$\sum_{i=1}^{d} v(n_i) = 1; \qquad\qquad v(n_i) \in \{0, 1\}. \tag{2.4}$$

It is the *claimed existence* of this function $v(n)$, having the properties stated above for *any arbitrary* basis of Hilbert space, which is the central point of the **KS1** and **KS2** conditions. This discussion allows us to rephrase the Kochen–Specker theorem in terms of the *non-existence* of such a valuation.

Theorem: *For $d \geq 3$ there is no valuation $v : S^{d-1} \to \{0, 1\}$, where S^{d-1} is the unit hypersphere, such that $f(-n) = f(n)$ for all n and*

$$\sum_{i=1}^{d} v(n_i) = 1, \tag{2.5}$$

for every basis (frame, d-bein) of orthogonal unit vectors n_i.

It is this statement about bases in Hilbert space that is often more practical to work with, rather than the formulation at the start of this section — of course without that initial formulation it would be less than clear why the basis formulation is physically interesting.

3 Yet another proof of the Kochen–Specker theorem

We will start by looking in a non-traditional place, by considering one-dimensional and two-dimensional Hilbert spaces, before dealing with three-dimensional Hilbert space, (which then settles things for any higher dimensionality). Since one is trying to prove an inconsistency result, there will be an infinite number of ways of doing so; the question is whether one learns anything new by coming up with a different proof.

3.1 One dimension

There is no Kochen–Specker no-go result in one dimension, since in one dimension all operators are multiples of the identity, $A = aI$, and then

$$v(f(A)) = v(f(aI)) = v(f(a)I) = f(a)\, v(I) = f(a). \tag{3.1}$$

In particular, for the (unique) normalized basis vector we have $v(n_1) = 1$.

3.2 Two dimensions

There is no Kochen–Specker no-go result in two dimensions, but there are still interesting things to say. Consider the valuation $v : S^1 \to \{0, 1\}$ (where S^1 is the unit circle) such that $v(-n) = v(n)$ for all n and

$$v(n_1) + v(n_2) = 1 \tag{3.2}$$

for every dyad (every pair of orthogonal unit vectors) n_1, n_2. Indeed in two dimensions we *can* construct such a valuation. Re-characterize n_1 and n_2 in terms of the angle they make with (say) the x axis; then the constraints we want to impose are

$$v(\theta) = v(\theta + \pi); \qquad\qquad v(\theta) + v(\theta \pm \tfrac{\pi}{2}) = 1. \tag{3.3}$$

But these conditions are easily solved: Let $g(\theta)$ be an arbitrary function mapping the interval $[0, \frac{\pi}{2}) \to \{0, 1\}$, and define

$$v(\theta) = \begin{cases} g(\theta) & \text{for } \theta \in [0, \frac{\pi}{2}); \\ 1 - g(\theta - \frac{\pi}{2}) & \text{for } \theta \in [\frac{\pi}{2}, \pi); \\ g(\theta - \pi) & \text{for } \theta \in [\pi, \frac{3\pi}{2}); \\ 1 - g(\theta - \frac{3\pi}{2}) & \text{for } \theta \in [\frac{3\pi}{2}, 2\pi). \end{cases} \tag{3.4}$$

So the existence of a Kochen–Specker valuation is easily verified in two dimensions, and because points separated by $\pi/2$ radians must be given opposite valuations, the image $v(S^1)$ is automatically 50%–50% zero-one. That is, average value of $v(n)$ over the unit circle is exactly one half;

$$\overline{v(n)} = \frac{1}{2}. \tag{3.5}$$

(We will recycle this result repeatedly when we turn to three and higher dimensions.) Note in particular that the function $v(\theta)$ *cannot* be everywhere continuous.

3.3 Three dimensions

Now things get interesting. We are interested in valuations $v : S^2 \to \{0, 1\}$, (where S^2 is the unit 2-sphere), such that $v(-n) = v(n)$ for all n and

$$v(n_1) + v(n_2) + v(n_3) = 1 \tag{3.6}$$

for every triad (every triplet of orthogonal unit vectors) n_1, n_2, n_3. In the argument below we shall make extensive use of the great circles S^1 in the unit 2-sphere S^2.

Lemma 0A: On any great circle in S^2, under the conditions given above, the valuation is either 50%–50% zero-one (as in two dimensions), or is 100% zero (identically zero).
Proof:
Pick any great circle and for convenience align it with the equator. Now look at the poles.

- If $v(\text{poles}) = 1$, then $v(\text{equator}) \equiv 0$ is identically zero.
  (Since points on the equator will be part of some triad that includes the unit vector pointing to the poles.)
- If $v(\text{poles}) = 0$, then any dyad lying in the equator will satisfy the conditions of the two dimensional argument given above, and so will be 50%–50% zero-one.

$\square$

Assuming the valuation exists, we now prove two mutually contradictory lemmata:

$$\overline{v(n)} = \frac{1}{2}; \qquad \overline{v(n)} = \frac{1}{3}. \tag{3.7}$$

Lemma 1A:
The image of the valuation on the 2-sphere $v(S^2)$ is 50%–50% zero-one.

(So the average value of $v(n)$ over the 2-sphere is one half; $\overline{v(n)} = \frac{1}{2}$.)

Proof:
Take any point such that $f(\text{point}) = 1$, and for convenience rotate to put it at the pole: so $f(\text{pole}) = 1$. Now consider the *longitudinal* great circles: Each of these longitudinal great circles has *at least* two places where the valuation is 1, at the poles. Therefore by Lemma 0A, each of these longitudinal great circles has valuations 50%–50% zero-one. Since these longitudinal great circles cover the entire 2-sphere, and do not intersect except for the set of measure zero at the poles, the entire 2-sphere has valuation 50%–50% zero-one; so $\overline{v(n)} = \frac{1}{2}$. □

Lemma 2A:
The image of the valuation on the 2-sphere $v(S^2)$ is (2/3)–(1/3) zero-one.

(So the average value of $v(n)$ over the 2-sphere is one third; $\overline{v(n)} = \frac{1}{3}$.)

Proof:
From $v(n_1) + v(n_2) + v(n_3) = 1$, we have $\overline{v(n_1) + v(n_2) + v(n_3)} = 1$.

So by permutation symmetry $3\,\overline{v(n)} = 1$, whence $\overline{v(n)} = \frac{1}{3}$. □

(With some hindsight, an argument somewhat along these lines is presented in [14], along with some hints that it might be possible to expand this into a proof of the Kochen–Specker theorem. With further hindsight, the current analysis can be seen to be somewhat distantly related to "colouring" arguments; as presented for example in references [15–19]. Note particularly that we are avoiding any notion of "great circle descents".)

Finally, note that Lemmata 1A, and 2A are in blatant contradiction with each other — the correct deduction to take from this is that the assumed existence of the valuation function $v(n)$ is inconsistent with the properties we have assigned it.

This completes the proof of Kochen–Specker in three dimensions. We feel that this is a nice simple proof of Kochen–Specker that does not rely on finding explicit bases for the Hilbert space — it also seems to us to be considerably simpler than the other geometric or colouring arguments.

3.4 $d > 3$ dimensions

What happens in a $d > 3$-dimensional Hilbert space? The 3 dimensional logic carries over with minimal modifications. One only has to deal with S^{d-1} instead of S^2, and slightly rephrase the logic.

Lemma 0B: On any great circle in S^{d-1}, the valuation is either 50%–50% zero-one, (as in two dimensions), or is 100% zero (identically zero).

Proof:
Pick any great circle and pick two orthogonal unit vectors lying on that great circle. Now complete the basis by picking another $d-2$ mutually orthogonal unit vectors that are orthogonal to the first two that lie on the great circle.

- If all of the $d-2$ additional unit vectors has valuation 0, then consider the first two that lie on the great circle: one of these must have valuation 0 and the other must have valuation 1. Hold the $d-2$ extra vectors fixed, and consider any dyad of orthogonal unit vectors lying on the great circle — one of these must have valuation 0 and the other must have valuation 1. In this situation the valuation on the great circle is 50%–50% zero-one.
- If any of the $d-2$ additional unit vectors has valuation 1, then the first two unit vectors that lie on the great circle must have valuation zero. Hold the $d-2$ extra vectors fixed, and consider any dyad of orthogonal unit vectors lying on the great circle — they must have valuation 0. In this situation the valuation on the great circle is 100% zero (identically zero). □

Assuming the valuation exists, we now prove two mutually contradictory lemmata:

$$\overline{v(n)} = \frac{1}{2}; \qquad\qquad \overline{v(n)} = \frac{1}{d}. \tag{3.8}$$

Lemma 1B:
The image of the valuation on the 2-sphere $v(S^2)$ is 50%–50% zero-one.

(So the average value of $v(n)$ over the 2-sphere is one half; $\overline{v(n)} = \frac{1}{2}$.)

Proof:
Take any point such that $v(\text{point}) = 1$, and consider the great circles passing through that point. Each of these great circles has *at least* two places where the valuation is 1, the original point and its antipode. Therefore, by Lemma 0B these great circles must have valuations that are 50%–50% zero-one. Since these great circles cover the entire $(d-1)$-sphere, and do not intersect except for the set of measure zero at the original point and its antipode, the entire $(d-1)$-sphere has valuations that are 50%–50% zero-one; so $\overline{v(n)} = \frac{1}{2}$. □

Lemma 2B:
The image of the valuation on the $(d-1)$-sphere $v(S^{d-1})$ is $\left(1-\frac{1}{d}\right)$–$\left(\frac{1}{d}\right)$ zero-one.

(So the average value of $v(n)$ over the $(d-1)$-sphere is $\overline{v(n)} = \frac{1}{d}$.)

Proof:

$$\sum_{i=1}^{d} v(n_i) = 1 \quad \Longrightarrow \quad \overline{\sum_{i=1}^{d} v(n_i)} = 1 \quad \Longrightarrow \quad \sum_{i=1}^{d} \overline{v(n_i)} = 1. \tag{3.9}$$

By permutation symmetry this implies $d\,\overline{v(n)} = 1$ whence $\overline{v(n)} = \frac{1}{d}$. □

Of course, since the whole point is that these two Lemmata, 1B and 2B, are mutually inconsistent, we could continue in this vein and prove $\overline{v(n)} =$ (anything we like); this being the Post criterion for inconsistency. However there is no real purpose served by developing further Lemmata along these lines.

The fact that Lemmata 1B and 2B are mutually inconsistent completes the proof of Kochen–Specker in $d > 3$ dimensions.

Note that if we set $d \to 2$ in the statement of 1B and 2B we do *not* get a contradiction, as both Lemmata collapse to the same result $\left.\overline{v(n)}\right|_{d\to 2} = \frac{1}{2}$. This is another way of seeing that $d = 2$ is special.

4 Discussion

It has been exactly 50 years since the groundbreaking publication of the theorem due to Kochen and Specker [3]. Along with Bell's inequality, their discovery represents one of the two major no-go theorems in the foundations of quantum theory [1–4]. The main implication of the result is that quantum theory fails to allow a non-contextual hidden variable model. More precisely, it states that it is impossible for the predictions of quantum mechanics to be in line with measurement outcomes which are pre-determined in a non-contextual manner. Hence this would rule out a large class of hidden variable models that might otherwise seem at first sight to be intuitive representations of the physical world.

Furthermore, there is increasing support that the notion of contextuality captures the essential difference between the quantum and classical world. A specific case of this would be in the recent evidence that contextuality may be the primary reason for the the speedup of universal quantum computation. This has been shown through 'magic' state injection [20]. Recent work has also connected contextuality with non-locality [21]. With non-locality almost ubiquitous in the field of quantum information, this connection only serves to increase the impact and significance of research into this lesser known property. In addition to theoretical results, recent experimental tests to demonstrate contextuality include a superconducting qutrit implementation [22], as well as a photonic implementation [23].

Simpler proofs of the Kochen–Specker theorem were found in years following the original paper [3], but these mainly involved the mathematically "elementary" (but technically subtle and demanding) machinery of setting up a large number of carefully projection operators or increasing the minimum dimension from three to four or even eight [5–9].

In this article, we have presented a geometric approach where one constructs and exploits the properties of great circles on a n-sphere. This has the power to significantly simplify the argument, while maintaining the validity of the theorem for a minimum dimension of three. We hope that with this simplification, further work might extend the argument to develop newer results in the foundations of quantum physics.

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