Gleason's theorem

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Gleason's theorem (named after Andrew M. Gleason) is a mathematical result which is of particular importance for the field of quantum logic. It proves that the Born rule for the probability of obtaining specific results for a given measurement follows naturally from the structure formed by the lattice of events in a real or complex Hilbert space. The theorem states:

> **Theorem**. Suppose H is a separable Hilbert space of complex dimension at least 3. Then for any quantum probability measure on the lattice Q of self-adjoint projection operators on H there exists a unique trace class operator W such that $P(E) = \mathbf{Tr}(W E)$ for any self-adjoint projection E in Q.

The lattice of projections Q can be interpreted as the set of quantum propositions, each proposition having the form "$a \leq A \leq b$", where A is the measured value of some observable on H (given by a self-adjoint linear operator). The trace-class operator W can be interpreted as the density matrix of a quantum state. Effectively, the theorem says that any legitimate probability measure on the space of allowable propositions is generated by some quantum state. This implies that the Standard Quantum Logic can be viewed as a manifold of interlocking perspectives that cannot be embedded into a single perspective [Edwards]. Hence, the perspectives cannot be viewed as perspectives on one real world. So, even considering one world as a methodological principle breaks down in the quantum micro-domain.

Contents

Context

Quantum logic treats quantum events (or measurement outcomes) as logical propositions, and studies the relationships and structures formed by these events, with specific emphasis on quantum measurement. More formally, a "quantum logic" is a set of events that is closed under the operation of disjunction of countably many mutually exclusive events. The *representation theorem* in quantum logic shows that such a logic forms a lattice which is isomorphic to the lattice of subspaces of a vector space with a scalar product.

It remains an open problem in quantum logic to prove that the field K over which the vector space is defined must be either the real numbers, complex numbers, or the quaternions. This would have negative implications for the possibility of a p-adic quantum mechanics. This is a necessary result for Gleason's theorem to be applicable, since in these three cases (but not for the p-adics) the definition of the inner product of a vector with itself makes the vector space in question into a Hilbert space. Solèr's Theorem, which under certain hypotheses restricts the field to just these three fields [1] (http://golem.ph.utexas.edu/category/2010/12/solers_theorem.html), suggests negative implications for the possibility of a p-adic quantum mechanics.

Application

The representation theorem allows us to treat quantum events as a lattice *L* = *L(H)* of subspaces of a real or complex Hilbert space. Gleason's theorem allows us to "attach" these events to probabilities. This section draws extensively from the analysis presented in Pitowsky (2005).

We let *A* represent an observable with finitely many potential outcomes: the eigenvalues of the Hermitian operator *A*, i.e. $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$. An "event", then, is a proposition x_i, which in natural language can be rendered "the outcome of measuring *A* on the system is α_i". The events x_i generate a sublattice of the Hilbert space which is a finite Boolean algebra, and if *n* is the dimension of the Hilbert space, then each event is an atom.

A *quantum probability function* over *H* is a real function *P* on the atoms in *L* that has the following properties:

1. $P(0) = 0$, and $P(y) \geq 0$ for all $y \in L$
2. $\sum_{j=1}^{n} P(x_j) = 1$, if $x_1, x_2, x_3, ..., x_n$ are orthogonal atoms

This means for every lattice element *y*, the probability of obtaining *y* as a measurement outcome is known, since it may be expressed as the union of the atoms under *y*: $P(y) = \Sigma\{P(x_j)|x_j \leq y\}$

In this context, Gleason's theorem states:

> *Given a quantum probability function P over a space of dimension* ≥ 3*, there is an Hermitian, non-negative operator W on H, whose trace is unity, such that* $P(x) = \langle \mathbf{x}, W\mathbf{x} \rangle$ *for all atoms* $x \in L$*, where* $\langle\ ,\ \rangle$ *is the inner product, and* $\mathbf{x}$ *is a unit vector along* x*.*

As one consequence: if some x_0 satisfies $P(x_0) = 1$, then *W* is the projection onto the complex line spanned by x_0 and $P(x) = |\langle \mathbf{x_0}, \mathbf{x} \rangle|^2$ for all $x \in L$.

The theorem expresses the Born rule for probability in quantum mechanics. The theorem presumes that the underlying set of numbers that the functions are defined over are real numbers or complex numbers. A constructive proof exists.

Implications

Gleason's theorem highlights a number of fundamental issues in quantum measurement theory. The fact that the logical structure of quantum events dictates the probability measure of the formalism is taken by some to demonstrate an inherent stochasticity in the very fabric of the world. To some researchers, such as Pitowski, the result is convincing enough to conclude that quantum mechanics represents a new theory of probability. Alternatively, such approaches as relational quantum mechanics make use of Gleason's theorem as an essential step in deriving the quantum formalism from information-theoretic postulates.

The theorem is often taken to rule out the possibility of hidden variables in quantum mechanics. This is because the theorem implies that there can be no bivalent probability measures, i.e. probability measures having only the values 1 and 0. To see this, note that the mapping $u \to \langle Wu, u \rangle$ is continuous on the unit sphere of the Hilbert space for any density operator *W*. Since this unit sphere is connected, no continuous function on it can take only the value of 0 and 1. (Wilce (2006), pg. 3) But, a hidden variables theory which is deterministic implies that the probability of a given outcome is *always* either 0 or 1: either the electron's spin is up, or it isn't (which accords with classical intuitions). Gleason's theorem therefore seems to hint that quantum theory represents a deep and fundamental departure from the classical way of looking at the world, and that this departure is *logical*, not *interpretational*, in nature.

See also

- Quantum logic
- Born's rule
- Measurement in quantum mechanics
- Hilbert space

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