Discrete Quantum Theories and Computing

Yu-Tsung Tai

Department of Mathematics and Department of Computer Science
Indiana University, Bloomington

Dilemma of quantum computing?

- Textbook quantum mechanics is correct.
- There does not exist an efficient classical factoring algorithm.
- The extended Church-Turing thesis —that probabilistic Turing machines can efficiently simulate any physically realizable model of computation —is correct.

Check the compatibility of Quantum Mechanics and Computer Science.

Quantum Mechanics is based on continuous. How about Computer Science?

	Discrete	Continuum
Theoretical Model	Turing machine	BCSS machine
Physical Realization	Digital Computer	Analog Computer
How the models realize?	Reliably	Not Reliably: 1. The quality might be quantized.. 2. The precision of an analog computer is low.

Build a more faithful Quantum Computing model?

Our Quantum Models	Quantum Theories and Computing over Finite Fields	Quantum Interval-Valued Probability Measures

Conventional Quantum Theory

Conventional Quantum Theory

i. D orthonormal basis vectors for a Hilbert space of dimension D.

ii. D complex probability amplitude coefficients describing the contribution of each basis vector.

iii. A set of probability-conserving unitary matrix operators that suffice to describe all required state transformations of a quantum circuit.

iv. A measurement framework.

Pure State

- A pure state can be represented as a D-dimensional vector, $|\Psi\rangle = \sum_{i=0}^{D-1} \alpha_i |i\rangle$, where $\{|0\rangle, |1\rangle \dots, |D-1\rangle\}$ form an orthonormal basis.
- Given two states $|\Psi\rangle = \sum_{i=0}^{D-1} \alpha_i |i\rangle$ and $|\Phi\rangle = \sum_{i=0}^{D-1} \beta_i |i\rangle$, their inner product

$$\langle\Phi|\Psi\rangle = \sum_{i=0}^{D-1} \beta_i^* \alpha_i$$

  satisfying the following properties:

  A. $\langle\Phi|\Psi\rangle$ is the complex conjugate of $\langle\Psi|\Phi\rangle$;

  B. $\langle\Phi|\Psi\rangle$ is conjugate linear in its first argument and linear in its second argument;

  C. $\langle\Psi|\Psi\rangle$ is always non-negative and is equal to 0 only if $|\Psi\rangle$ is the zero vector.

Mixed State

- A mixed state is the weighted average of the density matrices of pure states

$$\rho = \sum_{j=1}^{N} q_j |\Phi_i\rangle\langle\Phi_i| \ ,$$

where $|\Phi_i\rangle$ are normalized, $q_j > 0$, and $\sum_{j=1}^{N} q_j = 1$.

Probability Space

Abstraction

- Sample space Ω.
- Event Space 2^{Ω}.
- Probability measure $\mu: 2^{\Omega} \to [0,1]$
  - $\mu(\emptyset) = 0$.
  - $\mu(\Omega) = 1$.
  - For any event E,
    $$\mu(\bar{E}) = 1 - \mu(E)\ .$$
  - For disjoint events E_0 and E_1,
    $\mu(E_0 \cup E_1) = \mu(E_0) + \mu(E_1)$.

Example

- Sending a particle to a beam splitter with the split beams $|0\rangle$, $|1\rangle$, and $|2\rangle$.
- Sample space $\Omega_0 = \{|0\rangle, |1\rangle, |2\rangle\}$.
- Event Space 2^{Ω_0}.
- Probability measure $\mu_0: 2^{\Omega_0} \to [0,1]$.

Probability Space

Example

- Sending a particle to a beam splitter with the split beams $|0\rangle$, $|1\rangle$, and $|2\rangle$.
- Sample space $\Omega_0 = \{|0\rangle, |1\rangle, |2\rangle\}$.
- Event Space 2^{Ω_0}.
- Probability measure $\mu_0: 2^{\Omega_0} \to [0,1]$.

Another Example

- Sending the same particle to a beam splitter with the split beams $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$, and $|2\rangle$.
- Sample space $\Omega_1 = \{|+\rangle, |-\rangle, |2\rangle\}$.
- Event Space 2^{Ω_1}.
- Probability measure $\mu_1: 2^{\Omega_1} \to [0,1]$.

When the particle is the same, the probability of the same event is the same: $\mu_0(\{|2\rangle\}) = \mu_1(\{|2\rangle\})$.
So does their complement: $\mu_0(\{|0\rangle, |1\rangle\}) = \mu_1(\{|+\rangle, |-\rangle\})$

Glue Classical Event Spaces to a Quantum Event Space

- When the particle is the same, the probability of the same event is the same: $\mu_0(\{|2\rangle\}) = \mu_1(\{|2\rangle\})$.
  So does their complement: $\mu_0(\{|0\rangle, |1\rangle\}) = \mu_1(\{|+\rangle, |-\rangle\})$
- Consider
  $$\varphi(E) = \sum_{|j\rangle \in E} |j\rangle\langle j|$$
  Then,
  $\varphi(\{|0\rangle, |1\rangle\}) = |0\rangle\langle 0| + |1\rangle\langle 1| = |+\rangle\langle +| + |-\rangle\langle -| = \varphi(\{|+\rangle, |-\rangle\})$
- The quantum event of a classical event E is the projector $\varphi(E)$, and the set of all projectors on a given Hilbert space is called a quantum event space $\mathcal{E}$.

Classical and Quantum Probability Measure

Classical Probability measure

- $\mu: 2^{\Omega} \to [0,1]$
- $\mu(\emptyset) = 0.$
- $\mu(\Omega) = 1.$
- For any event E,
$$\mu(\bar{E}) = 1 - \mu(E)\,.$$
- For disjoint events E_0 and E_1 ($E_0 \cap E_1 = \emptyset$),
$$\mu(E_0 \cup E_1) = \mu(E_0) + \mu(E_1)\,.$$

Quantum Probability measure

- $\mu: \mathcal{E} \to [0,1]$
- $\mu(\mathbb{0}) = 0$, where $\mathbb{0}$ is the zero projector.
- $\mu(\mathbb{1}) = 1$, where $\mathbb{1}$ is the identity projector.
- For any projector P,
$$\mu(\mathbb{1} - P) = 1 - \mu(P)\,.$$
- For orthogonal projectors P_0 and P_1 ($P_0 P_1 = \mathbb{0}$),
$$\mu(P_0 + P_1) = \mu(P_0) + \mu(P_1)\,.$$

Fix an orthonormal basis Ω, consider the restricted $\varphi: 2^{\Omega} \to \mathcal{E}$. Then, $\varphi^* \mu: 2^{\Omega} \to [0,1]$ defined by $(\varphi^* \mu)(E) = \mu\big(\varphi(E)\big)$ is a classical probability measure and called the pullback of μ by $\varphi: 2^{\Omega} \to \mathcal{E}$.

Observables and Expectation Values

- A quantum probability measure $\mu : \mathcal{E} \to [0,1]$.
- A observable $\mathbf{O}$ diagonalizable by an orthonormal basis $\Omega = \{|0\rangle, |1\rangle, \dots, |D-1\rangle\}$with spectral decomposition $\mathbf{O} = \sum_{i=1}^{D-1} \lambda_i |i\rangle\langle i|$.
- The expectation value is $\langle \mathbf{O} \rangle_\mu = \sum_{i=1}^{D-1} \lambda_i \mu(|i\rangle\langle i|)$.
- The pullback of $\mathbf{O}$ by $\varphi: 2^\Omega \to \mathcal{E}$ is the random variable $\varphi^* \mathbf{O}: 2^\Omega \to \mathcal{E}$ defined by $\varphi^* \mathbf{O} = \sum_{i=1}^{D-1} \lambda_i \mathbf{1}_{\{|i\rangle\}}$, where $\mathbf{1}_{\{|i\rangle\}}$ is the indicator function.
- The pullback preserves the expectation value

$$\langle \mathbf{O} \rangle_\mu = \int (\varphi^* \mathbf{O}) \, d(\varphi^* \mu)$$

Gleason’s Theorem

Theorem (Gleason’s) When dimension $d \geq 3$, given a quantum probability measure $\mu: \mathcal{E} \to [0,1]$, there exists a unique mixed state ρ such that

$$\mu(P) = \mathrm{Tr}(\rho P)\,.$$

- If we follow the same interpretation that $\mu(P)$ is the probability of the particle in the split beams in P, does ρ represent the state of the particle sending to the beam splitter?

Born Rule

- Let $\mu_{\Phi}^{\mathrm{B}}(P)$ denote the quantum probability measure created by the particle in the normalized pure state $|\Phi\rangle$. It should satisfy:
  - $P|\Phi\rangle = |\Phi\rangle$ if and only if $\mu_{\Phi}^{\mathrm{B}}(P) = 1$.
  - $\mu_{\Phi}^{\mathrm{B}}(P) = \mu_{U|\Phi\rangle}^{\mathrm{B}}(UPU^{\dagger})$ for unitary U.
- Then, $\mu_{\Phi}^{\mathrm{B}}(P) = \langle\, \Phi \mid P \mid \Phi \,\rangle$ is called the Born rule.
- For a mixed state $\rho = \sum_{j=1}^{N} q_j |\Phi_i\rangle\langle\Phi_i|$,

$$\mu_{\rho}^{\mathrm{B}}(P) = \sum_{j=1}^{N} q_j \mu_{\Phi_j}^{\mathrm{B}}(P) \;= \mathrm{Tr}(\rho P)$$

Pauli Operators and the Reduced Density Matrix

- $\sigma_0 = |0\rangle\langle 0| + |0\rangle\langle 0|$, $\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\sigma_y = \mathbb{i}|1\rangle\langle 0| - \mathbb{i}|0\rangle\langle 1|$, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$.
- $\sigma_\eta^j = \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_\eta \otimes \sigma_0 \otimes \cdots \otimes \sigma_0$, where σ_η is the