Probability

April 18, 2016

1 Probability Spaces

1.1 Classical Probability Spaces

Textbook probability theory [1, 2, 4] is defined using the notions of a *sample space* Ω, a space of *events* $\mathcal{F}$, and a *probability measure* μ. In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events $\mathcal{F}$ as, 2^Ω, the powerset of Ω. A *probability measure* is a function $\mu : \mathcal{F} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i, we have $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two coin experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rclcrcl}
\mu(\emptyset) & = & 0 & \quad & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

1.2 Quantum Probability Spaces

A classical model decides the occurrence or non-occurence of all events simultaneously which is inconsistent with quantum mechanics. Indeed, in the quantum world, there are (non-commuting) events which cannot happen simultaneously. To accommodate this situation, we completely abandon the sample space Ω and define and reason directly about events. Thus a quantum probability space will consist of just two components: a set of events $\mathcal{A}$ and a probability measure $\phi : \mathcal{A} \to [0, 1]$. These components are defined as follows [3, 5].

We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{A}$ as *projections* on $\mathcal{H}$. Similarly to the classical case, a probability measure is a function $\phi : \mathcal{A} \to [0, 1]$ satisfying:

- $\phi(\mathbb{1}) = 1$, and
- for all $A \in \mathcal{A}$, we have $\phi(A^*A) \geq 0$.

Yu-Tsung says: If we follow [3, 5], then we also need

- ϕ can be extended to a linear functional $\phi : alg(\mathcal{A}) \to \mathbb{C}$, where $alg(\mathcal{A})$ is the minimal $*$-algebra generated by $\mathcal{A}$.

Also,

- for all $A \in \mathcal{A}$, we have $\phi(A^*A) \geq 0$.

should be

- for all $A \in alg(\mathcal{A})$, we have $\phi(A^*A) \geq 0$.

because we have $\phi(A^*A) = \phi(A^2) = \phi(A)$ if A is a projection.

As an example, let $P_1, P_2, \ldots, P_k$ be mutually orthogonal projections on $\mathcal{H}$ with sum $\mathbb{1}$ and define the event space $\mathcal{A}$ to be the linear span of these operators:

$$\mathcal{A} = \left\{ \sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{C} \right\}.$$

Yu-Tsung says: $\left\{ \sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{C} \right\}$ is the minimal $*$-algebra generated by $P_1, P_2, \ldots, P_k$, but it contains all possible observables $P_1, P_2, \ldots, P_k$ can generate (and something more) not just projections. For example, $2\mathbb{1} \in \left\{ \sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{C} \right\}$.

Each state $|\psi\rangle$ of the Hilbert space induces a probability measure $\phi_\psi : \mathcal{A} \to [0, 1]$ defined as follows:

$$\phi_\psi(A) = \langle \psi | A \psi \rangle$$

Yu-Tsung says: So ϕ_ψ maps the projections generated by $P_1, P_2, \ldots, P_k$ to $[0, 1]$, and maps $\left\{ \sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{C} \right\}$ to $\mathbb{C}$...

Concrete example: consider the two qubit Hilbert space with computational bases $|0\rangle$ and $|1\rangle$. First, consider the two states

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

and consider the following families of projections:

- Family I: $|0\rangle\langle 0|$, $|1\rangle\langle 1|$
- Family II: $|+\rangle\langle +|$, $|-\rangle\langle -|$

In family I, all operators can be expressed as $\{\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1| \mid \lambda_1, \lambda_2 \in \mathbb{C}\}$. In order to identify projectors among them, we need to solve the following two equations.

$$\begin{aligned}
\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1| &= (\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1|)^* = \lambda_1^* |0\rangle\langle 0| + \lambda_2^* |1\rangle\langle 1| \\
\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1| &= (\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1|)^2 = \lambda_1^2 |0\rangle\langle 0| + \lambda_2^2 |1\rangle\langle 1|
\end{aligned}$$

Therefore, we actually have $\lambda_1, \lambda_2 \in \{0, 1\}$, and there are only four projections: 0, $|0\rangle\langle 0|$, $|1\rangle\langle 1|$, and $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$. Then, $\phi_{|+\rangle}$ and $\phi_{|-\rangle}$ of each projections are

$$\begin{aligned}
\phi_{|+\rangle}(0) &= 0 & \qquad \phi_{|-\rangle}(0) &= 0 \\
\phi_{|+\rangle}(|0\rangle\langle 0|) &= 1/2 & \qquad \phi_{|-\rangle}(|0\rangle\langle 0|) &= 1/2 \\
\phi_{|+\rangle}(|1\rangle\langle 1|) &= 1/2 & \qquad \phi_{|-\rangle}(|1\rangle\langle 1|) &= 1/2 \\
\phi_{|+\rangle}(\mathbb{1}) &= 1 & \qquad \phi_{|-\rangle}(\mathbb{1}) &= 1
\end{aligned}$$

Similarly, $\phi_{|+\rangle}$ and $\phi_{|-\rangle}$ gives two probability for family II as well.

$$\begin{aligned} \phi_{|+\rangle}(0) &= 0 \\ \phi_{|+\rangle}(|+\rangle\langle+|) &= 1 \\ \phi_{|+\rangle}(|-\rangle\langle-|) &= 0 \\ \phi_{|+\rangle}(\mathbb{1}) &= 1 \end{aligned} \qquad \begin{aligned} \phi_{|-\rangle}(0) &= 0 \\ \phi_{|-\rangle}(|+\rangle\langle+|) &= 0 \\ \phi_{|-\rangle}(|-\rangle\langle-|) &= 1 \\ \phi_{|-\rangle}(\mathbb{1}) &= 1 \end{aligned}$$

1.3 Plan

Several assumptions are woven in the definition of a quantum probability space:

- the Hilbert space $\mathcal{H}$;
- the real interval $[0, 1]$;
- the fact that each state induces a probability measure, i.e., the Born rule;
- the fact that every probability measure is induced by a state, i.e., Gleason's theorem

In the remainder of the paper, we examine each of these assumptions and consider variations motivated by computation in a world with limited resources. In particular, we will consider a variant of the Hilbert space over finite fields $\mathbb{F}_{p^2}^d$. Instead of $[0, 1]$, we will consider set-valued probability measures, in particular $\{0\}$, impossible, and $(0, \infty)$, possible. Surprisingly, some combinations of space and probability will result in no probability measure or an unique probability measure. In these cases, there is no need to discuss whether there is a Born rule, because we do not have enough probability to correspond to every state.

If there may be more than one probability measure, we will discuss whether there is a Born rule to generate a probability measure from a state. When the space is $\mathbb{C}^d$, we will try to induced a Born rule from the conventional Born rule; when the space is $\mathbb{F}_{p^2}^d$, there is no natural way to induce a probability measure from a state, so we will set some conditions a Born $\tilde{\pi}$ should have:

- Given a pure state $|\Psi\rangle \in \mathbb{F}_{p^2}^{d\,*}$, a Born-rule $\tilde{\pi}$ should give a probability $\tilde{\pi}_\Psi$;
- $\langle\Psi|\Phi\rangle = 0 \Leftrightarrow \tilde{\pi}_\Psi\left(|\Phi\rangle\right) = \tilde{0}$, where $\tilde{0}$ is 0 while considering $[0, 1]$ and $\tilde{0}$ is impossible while considering $\{\text{impossible}, \text{possible}\}$.
- $\tilde{\pi}_\Psi\left(|\Phi\rangle\right) = \tilde{\pi}_{\mathbf{U}|\Psi\rangle}\left(\mathbf{U}|\Phi\rangle\right)$,where $|\Psi\rangle, |\Phi\rangle \in \mathbb{F}_{p^2}^{d\,*}$ and $\mathbf{U}$ is any unitary map, i.e., $\mathbf{U}^\dagger\mathbf{U} = \mathbb{1}$.

Notice that when the space is $\mathbb{C}^d$, every Born rule we consider will satisfy these three conditions.

Finally, if there is a Born rule, we will see whether every probability measure is induced by a state, and establish Gleason's theorem.

References

[1] William G. Faris. Appendix: Probability in quantum mechanics. In *The infamous boundary : seven decades of controversy in quantum physics.* Boston : Birkhauser, 1995.

[2] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* A foundation for computer science. Addison-Wesley, 1994.

[3] Hans Maassen. Quantum probability and quantum information theory. In *Quantum information, computation and cryptography*, pages 65–108. Springer, 2010.

[4] V.K. Rohatgi and A.K.M.E. Saleh. *An Introduction to Probability and Statistics.* Wiley Series in Probability and Statistics. Wiley, 2011.

[5] JM Swart. Introduction to quantum probability. *Lecture Notes*, 2013.