Contextuality and Nonlocality in Discrete Quantum Theory

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1. Introduction

Theories of quantum mechanics based on finite complexified fields can provide novel insights into subtle issues of nonclassical theories. While the study of such discrete quantum theories is motivated by the desire to restrict our physical theories to computable numbers, all such theories are handicapped by the absence of naturally ordered quantities that can be associated with traditional probabilities.

Gerardo says: This paper is about the measurement process in Discrete Quantum Theories, that is, the process by which one extracts information from a physical system in a hypothetical world characterized by a finite number of resources, albeit arbitrarily large.

Amr says: I would also start with a statement along the following lines and then using Gerardo's paragraph: Whether Nature is continuous or discrete, any *representation* of a physical system in a computer or in a laboratory must use finite resources. In the realm of quantum computing, these finite resources must arguably be founded on vector spaces over finite fields that have as many properties of the conventional Hilbert spaces. This paper is about the measurement process in Discrete Quantum Theories, that is, the process by which one extracts information from a physical system in a hypothetical world characterized by a finite number of resources, albeit arbitrarily large.

Having investigated the geometrical implications and properties of discrete quantum theories in previous work [1, 2], we turn here to the study of the measurement process within such theories. Our goal is to rediscover properties of the foundations of standard axiomatic quantum mechanics, including Gleason's theorem, the Kochen-Specker theorem, and Bell's theorem, consistent within the discrete-field context, even while lacking ordered probabilities in the number system. We find for example, that the discrete version of Gleason's theorem motivates a corresponding Born rule.

2. Fundamentals

Measurement, considered as the process by which one extracts information from a physical system in a world with finite resources, is a fundamental property of quantum mechanics in our discrete field context that we will study in this paper.

Andy says: This section needs a clear treatment of *contextuality*, and a mention of the issues for measurement with possibly illegal intermediate states, and degenerate eigenvalues as troublesome special cases.

Gerardo says: Add a statement "Here we concentrate on a discrete quantum theory that uses the number systems, but not the cardinal probability adaptation, that we introduced in our previous paper [2], that used partially transitively-ordered sets to address the question of probabilities."

In quantum mechanics, measurement is contextual, meaning that the outcome of an

experiment depends not only on the system being measured and the choice of the measured observable, but also on the details of the measurement process. It is also nonlocal, meaning that correlations exist that are inconsistent with causal influences. Classical physics, on the other hand, seems to be local and independent of the context. The introduction of hidden variables, along with several important results proving their inconsistency, have been motivated from the very early days of quantum mechanics by the classical intuition that some (partially unknown) local context might influence the outcome of experiments and thus invalidate traditional quantum theory.

The Concept of Contextuality and its Role in the Properties of Candidates for Quantum Theories. Quantum Physics is contextual, meaning that the outcome of a measurement (or a set of measurements) depends not only on the choice of the measured observable(s) and on the system being measured, but also on the details of the measurement process. To be more precise, as noted by Peres [3] and Mermin [4], the result of measuring an observable in a physical system must not depend on whether we measure the observable alone, or along with arbitrary sets of mutually commuting observables.

> Yu-Tsung says: According to the postulates in page 207 in Peres [3], the last sentence should be "To be more precise, as noted by Peres [3] and Mermin [4], it is impossible to predetermine definite measurement outcomes for all observables on a system, such that the predetermined outcome of any observable does not depend on whether we measure the observable alone, or along with arbitrary sets of mutually commuting observables."

Classical Physics, on the other hand, is usually agreed to be independent of the context; that is, we presume that every physical variable has a predetermined value. The motivation for studying the possibility that the probabilistic features of quantum mechanics can be explained by hidden variables is the possibility that a (partially unknown but complete) context influences the outcome of experiments.

> Yu-Tsung says: I don't understand the last sentence...

Basic Features We Expect in Quantum Theories. To understand how much of conventional measurement theory we can maintain in discrete quantum theories without ordered probabilities, along with the possibility of non-trivial null-norm states, we must critically address three pillars at the foundation of traditional quantum theories. Many of these issues are treated in detail, e.g., in Peres [3]. The issues we shall consider are:

- *Gleason's theorem* tells us that for Hilbert spaces of dimension > 2, the Born Rule is the *only* way to assign consistent probabilities to the statistical prediction of a quantum measurement [5]. This result excludes a very general class of hidden variable theories using continuity and probabilities. In the conventional quantum formalism the Born rule provides the link between the formalism of quantum mechanics and the potential results of experimental observations. In order to achieve similar results with finite fields, we must find

a way around the dependence on continuity and ordered probabilities. Replacing continuous-valued probabilities with discrete interval-valued probabilities (e.g., possible and impossible measurement outcomes) provides one route to the desired outcomes.

- *The Kochen-Specker theorem* in its classical form employs a complicated array of measurements on a single system along with geometric arguments to exclude additional wide classes of hidden variable theories [6]. The Kochen-Specker theorem does not depend on statistical averages, and it is geometric in spirit. (More succinct approaches can be found, e.g., in Peres [3].) The freedom from statistical averages in the Kochen-Specker methods suggests that important parts of the result could be independent of ordered probabilities and thus in fact might possess suitable analogs in finite fields. The key is to understand the implications of contextuality, and here we find that the possibility of valid null intermediate states in discrete quantum theory can disrupt what would seem to be trivial contextuality. Yu-Tsung says: I don't understand this sentence... To deal with this, we are compelled to explicitly consider, not only the initial and final resources needed to represent the system and the final measurement outcome, but also the resources needed to represent the intermediate steps in a series of experiments. We find that, to preserve the spirit of the Kochen-Specker theorem, we have to explicitly introduce and propagate *error states* whose representation requires more than the available resources.
- *Bell's theorem* reveals the intrinsic non-local nature of correlations in quantum physics, i.e., the nature of entanglement, which has no classical analog [7]. Bell's original arguments depended on continuity and probability to produce inequalities satisfied by any local realistic theories, as broadly defined by Einstein, thus quantifying nonlocality. Since then, many Bell-type inequalities have been derived for different observable quantities, refined by the incorporation of particular experimental setups. In addition, it was later realized [8, 9] that one can derive Bell-type equalities encoding non-local correlations and, as we will see, it is this version of Bell's theorem that is generalizable to Discrete Quantum Theories.

All of these basic insights into the structure of traditional quantum mechanics rely on *contextuality* and *nonlocality*: We will argue in our conclusions that, in discrete quantum theory, we can consistently employ both contextuality, the statement that measurement of a quantum observable does not depend on any given second observable provided it commutes with the first observable, Yu-Tsung says: Should be "contextuality, the statement that the predetermined measurement outcome of a quantum observable depends on any given second observable although it may commute with the first observable" and, non-locality, which says that spatially separated measurements in a quantum theory can in fact be correlated, counter to classical intuitions.

Andy says:

Internal note: Structure Outline

- **Critical analysis of measurement in conventional theory.**
- **Discrete Vector Space.** $\mathbb{F}_{p^2}^d \rightarrow \mathbb{F}_{p^2}^{d\,*}$.
- **No go theorem.** Cannot use Real numbers in $[0, 1]$.
- **Post measurement issues.** Requires explicit error states.
- **Show contextuality works.** Prove Gleason, Bell, and Kochen-Specker to conclude that DQT with interval-valued probabilities and explicit error states is sufficient to retain a quantum theory with contextuality and absence of local hidden variables.

3. Critical Analysis of the Measurement Process in CQT

Before discussing the measurement process and the manifestations of nonlocality and contextuality in discrete quantum mechanics, we first review various notations brought forward from conventional quantum theory. Assume that a physical system is represented by a state $|\Psi\rangle$, and that we wish to measure a physical quantity $\mathcal{O}$ of this system, with corresponding observable $\mathbf{O}$, whose eigenspectrum is given by

$$\mathbf{O}\,|i\rangle = \lambda_i\,|i\rangle \ .$$

Then Born's rule of CQT tells us that the probability of obtaining the eigenvalue λ_i as an outcome of the measurement is given by

$$\mathcal{P}_\Psi\left(\lambda_i\right) \ = \ \frac{\langle\Psi|P_i|\Psi\rangle}{\langle\Psi\mid\Psi\rangle} \ = \ \frac{\langle\Psi\mid i\rangle\langle i\mid\Psi\rangle}{\langle\Psi\mid\Psi\rangle\langle i\mid i\rangle} \ , \tag{1}$$

where P_i is the projector onto the subspace generated by the eigenvector $|i\rangle$. We have until now assumed non-degenerate eigenvalues. If, for some set of states $\{k\}$, the eigenvalue λ is degenerate, we account for that by replacing the simple expression for the projector P_i by

$$P_\lambda = \sum_{k=1}^{K_\lambda} \frac{|k\rangle\,\langle k|}{\langle k\mid k\rangle} \ ,$$

where K_λ is the degeneracy of the eigenvalue λ.

One other issue that will confront us in the transition from CQT to DQT is the presence of post-measurement states. In CQT, the Von Neumann projective measurement state, immediately after the measurement of $\mathcal{O}$, is given in general by

$$\frac{P_\lambda|\Psi\rangle}{\sqrt{\langle\Psi|P_\lambda|\Psi\rangle}} \ ,$$

where for the non-degenerate case, $P_\lambda \rightarrow P_i$, and so the resulting state is just $|i\rangle$.

4. States, Observables, and Probability in DQT

Following the development in the previous section, we need to carefully define a few notions in order to describe the measurement process in DQT. In the following, we will briefly introduce discrete quantum theory (DQT), as defined in [1, 2], and will highlight the main differences and contrasts with conventional quantum theory (CQT). We will start by defininining states and observables and then spend the remainder of the section investigating the proper notion of probability in DQT.

4.1. States in DQT

Our fundamental divergence from CQT is to replace continuous complex amplitudes in the field of complex numbers $\mathbb{C}$ by discrete fields $\mathbb{F}_{p^2}$. If we restrict ourselves to odd primes with $p \equiv 3 \bmod 4$, the fields $\mathbb{F}_p$ are exactly those for which $x^2 + 1$ has no root in the field, and we can adjoin i as a root of $x^2 + 1$ to the field $\mathbb{F}_p$. For a given element $\alpha \in \mathbb{F}_{p^2}$, we define the equivalent of complex conjugation as the map $\alpha \mapsto \alpha^p$, which we may write as $\alpha^* = \alpha^p$. The resulting extended field is isomorphic to $\mathbb{F}_{p^2}$ and is a realization of discrete complex numbers suitable for quantum mechanical calculations.

Let us start by defining a vector space $\mathbb{F}_{p^2}^d$ of d-dimensional vectors with elements in $\mathbb{F}_{p^2}$, where the conjugate transpose is defined in the natural way giving us the notion of a dual vector $\langle\Psi|$ for every vector $|\Psi\rangle \in \mathbb{F}_{p^2}^d$ as well as unitary operators. Next we propose the Hermitian dot product defined as the map from $\mathbb{F}_{p^2}^d \times \mathbb{F}_{p^2}^d$ to $\mathbb{F}_{p^2}$ given by $(|\Psi\rangle, |\Phi\rangle) \longmapsto \langle\Phi|\Psi\rangle$. Notice that this product is technically not an inner product because there are vectors $|\Psi\rangle$ other than the zero vector whose "norm" $\langle\Psi|\Psi\rangle$ vanishes. We therefore exclude such cases by moving to the domain $\mathbb{F}_{p^2}^{d\,*}$, where the * indicates the exclusion of null vectors; that is we consider $|\Psi\rangle \in \mathbb{F}_{p^2}^{d\,*}$, where

$$\mathbb{F}_{p^2}^{d\,*} = \left\{ |\Psi\rangle \in \mathbb{F}_{p^2}^d \mid \langle\Psi \mid \Psi\rangle \neq 0 \right\} .$$

Amr says: we want to emphasize that it is not sufficient to remove these null vectors once and for all at the beginning of time; it is possible to wrap around during calculation (unitary evolution as well as projection to post-measurement state) and produce a null state from a well-defined state. In the rest of the paper (say where exactly) we will arrange for such invalid operations to be detected and dealt with properly.

Gerardo says: 1) When the eigenvalues are non-degenerate there is never a problem; 2) The problem is a manifestation of the lack of resources. Had we used a larger p from the start, this situation would have been avoided.

Andy says: I definitely agree. Gerardo and I discussed at length whether the decision to "fix the p in advance" could be relaxed, and, though that is very complicated, I think that we make a very good connection to the previous paper if we emphasize the role of the size of p and the RESOURCES. Without changing p in the measurement PROCESS, it is still legitimate to state that a glitch in the PROCESS can arguably come directly from the RESOURCES allocated. Then our overall story is that DQT permits the splitting-apart of "computability" in quantum theories into state-representation and measurement resources. So I'd say, if anything, such an interpretation needs to be expanded and made very clear. Then the readers of the sequence of papers know exactly who we are and what our narrative implies about computability and quantum theory in general.

Amr says: Long discussion about whether to allow zero norm states as intermediate values. Also discussion about whether we are computing with states or equivalence classes of states; in the latter case, all zero-norm states should be equivalent.

4.2. Unitary and Hermitian operators in DQT

After we define the state vectors and the Hermitian dot product, we are ready to define unitary and Hermitian operators representing time evolution and observables, respectively. All unitary operators can represent time-evolution operators. Because a unitary operator $\mathbf{U}$ preserves norm, a physical state $|\Psi\rangle$ always evolves to another physical state $\mathbf{U}|\Psi\rangle$ according to $\mathbf{U}$.‡

The situation for observables is more delicate. As discussed in section 3, the measurement result of an observable $\mathbf{O}$ is a real eigenvalue of $\mathbf{O}$; the post-measurment state is a physical eigenstate of $\mathbf{O}$. However, a Hermitian operator $\mathbf{H}$ over $\mathbb{F}_{p^2}$ may not be qualified for an observable because $\mathbf{H}$ may not have suitable eigenvalues and eigenvectors to define the measurement process, or equivalently $\mathbf{H}$ is not unitarily diagonalizable, for the following reasons:

(i) $\mathbf{H}$ may lack eigenvalues. More precisely, the total multiplicity of eigenvalues of $\mathbf{H}$ in $\mathbb{F}_{p^2}$ may be less than d, where K_λ is the multiplicity of an eigenvalue λ means that $(x-\lambda)^{K_\lambda}$ is a factor of the characteristic polynomial

$$\det\left(\mathbf{H} - x\mathbb{1}\right) , \tag{2}$$

while $(x-\lambda)^{K_\lambda+1}$ is not a factor of the characteristic polynomial. In another word, $\mathbb{F}_{p^2}$ is not algebraically closed [10]. For example, consider a Hermitian matrix

$$\mathbf{H}_1 = \begin{pmatrix} -1 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & -\mathrm{i} \\ 0 & \mathrm{i} & 1 \end{pmatrix}$$

whose characteristic polynomials over $\mathbb{F}_9$ are

$$\det\left(\mathbf{H}_1 - x\mathbb{1}\right) = 2x^3 + x^2 + 2$$

‡ Although some unitary operators might not be induced by a Hamiltonian, we might discuss further in our upcoming papers.

which have no solution in $\mathbb{F}_9$ so that the total multiplicities is 0 less than 3. Hence, $\mathbf{H}_1$ should not be an observable because there is no eigenvalue as any measurement result.

(ii) The dimension of an eigenspace may be less than the multiplicity of its eigenvalue. For example, consider a Hermitian matrix

$$\mathbf{H}_2 = \begin{pmatrix} -1 & -1-\mathrm{i} \\ -1+\mathrm{i} & 0 \end{pmatrix}$$

Over $\mathbb{F}_9$, its only eigenvalue is 1 of multiplicity 2. The only eigenspace of $\mathbf{H}_2$ is spanned by $\begin{pmatrix} 1 & -1+\mathrm{i} \end{pmatrix}^T$. The dimension of the eigenspace is 1 less than 2 so that $\mathbf{H}_2$ are not diagonalizable over $\mathbb{F}_9$. If $\mathbf{H}_2$ were an observable, $\mathbf{H}_2$ would always have the measurement result 1 with the post-measurement state $\begin{pmatrix} 1 & -1+\mathrm{i} \end{pmatrix}^T$. In another word, if we measured $\mathbf{H}_2$ by the Stern-Gerlach experiment, we would always produce a $\begin{pmatrix} 1 & -1+\mathrm{i} \end{pmatrix}^T$ beam without even splitting the input beam. This violates our intuitively of quantum measurement. Therefore, $\mathbf{H}_2$ should not be an observable.

In order to understand the reason why $\mathbf{H}_2$ is not diagonalizable, we can diagonalize $\mathbf{H}_2$ in $\mathbb{C}$. The eigenvalues of $\mathbf{H}_2$ in $\mathbb{C}$ are 1 and -2.

$$-2 \equiv 1 \,(\mathrm{mod} 3)$$

In another word, two eigenvalues in $\mathbb{C}$ map to only one eigenvalue of multiplicity 2 in $\mathbb{F}_9$, and the corresponding two eigenspaces with dimension 1 in $\mathbb{C}$ map to the same eigenspace still with dimension 1 in $\mathbb{F}_9$.

(iii) Some eigenvectors are zero-norm so that $\mathbf{H}$ may be diagonalizable, but not unitarily diagonalizable. In this case, the corresponding eigenvalue λ of a Hermitian matrix may not be in $\mathbb{F}_p$.

For example, consider a Hermitian matrix $\mathbf{H}_3$

$$\mathbf{H}_3 = \begin{pmatrix} -1 & -1-\mathrm{i} \\ -1+\mathrm{i} & -1 \end{pmatrix}$$

Over $\mathbb{F}_9$, the eigenvalues and eigenvectors of $\mathbf{H}_3$ are $-1+\mathrm{i}$ with $\begin{pmatrix} 1 & 1+\mathrm{i} \end{pmatrix}^T$ and $-1-\mathrm{i}$ with $\begin{pmatrix} 1 & -1-\mathrm{i} \end{pmatrix}^T$. It is easy to verify the eigenvalues are not in $\mathbb{F}_3$, and the eigenvectors are all zero-normed.

If we diagonalize $\mathbf{H}_3$ in $\mathbb{C}$, $\mathbf{H}_3$ has the eigenvalues $-1 \pm \sqrt{2}$. If we extend the modulo p to some algebraic numbers by

$$a \equiv b \,(\mathrm{mod} p) \Leftrightarrow \pm\sqrt{a} \equiv \pm\sqrt{b} \,(\mathrm{mod} p) \ ,$$

then

$$-1 \pm \sqrt{2} \equiv \ -1 \pm \mathrm{i} \,(\mathrm{mod} 3) \ ,$$

This means the real eigenvalues in $\mathbb{C}$ may be corresponding to the complex eigenvalues in $\mathbb{F}_{p^2}$. The complex eigenvalues in $\mathbb{F}_{p^2}$ is further related to the zero-norm eigenstates because if an eigenstate is not zero-normed, its corresponding eigenvalue must be in $\mathbb{F}_p$.

Yu-Tsung says: Although may not need to type it down, need to check the definition of the map, passing congruence into square root, and partial homomorphism... etc.

As long as there exists a non-zero-norm eigenbasis of $\mathbf{H}$, $\mathbf{H}$ has an orthonormal eigenbasis [11]. Therefore, these are the only reasons that $\mathbf{H}$ may not be unitarily diagonalizable, or an observable. Besides, some unitary operators are not unitarily diagonalizable as well although they are still time-evolution operators.

Andy says: We want to see where the argument of *compatible observables* fits in.

4.3. Considering Continuous Probabilities

As seen in the previous section, the CQT Born rule gives a physical meaning to the state vectors, which can be seen as holding the information regarding outcomes of measurements. Given a state $|\Psi\rangle$, the measurement of a physical quantity will result in one of some number of mutually exclusive outcomes λ.

It is worth asking whether the state vectors in DQT can contain probabilistic information about measurement outcomes. When the Born rule is literally translated directly into the finite field framework, i.e., so the quantity $\mathcal{P}_\Psi(\lambda)$ is in the field $\mathbb{F}_p$, then $\mathcal{P}$ can no longer be considered as a probability because it cannot be interpreted as a real number in the interval $[0, 1]$, or as any other transitively ordered set. Thus in order to get a rule in DQT for extracting information from the state vectors, the Born rule must be modified.

Thus we are led first to consider the most obvious extension, which is to postulate the existence of a consistent function $\tilde{\pi}$ with outcomes compatible with CQT, that is

$$\tilde{\pi}\colon \mathbb{F}_{p^2}^{d\,*} \times \mathbb{F}_{p^2}^{d\,*} \longrightarrow [0, 1]\ .$$

For the outcomes of this new rule to truly be interpreted as probabilities, the rule $\tilde{\pi}$ must satisfy certain properties. First, probabilities must satisfy a normalization requirement:

(i) For any orthonormal basis $\{|i\rangle\}$ and any state $|\Psi\rangle$, $\displaystyle\sum_{i=0}^{d-1} \tilde{\pi}(|\Psi\rangle, |i\rangle) = 1$.

That is, the sum of probabilities for a complete set of mutually exclusive outcomes must be 1. To motivate the next requirement, consider the state $\alpha|0\rangle + \beta|1\rangle$ in the vector space spanned by $|0\rangle$ and $|1\rangle$. We can embed this vector subspace in the larger d-dimensional space spanned by $|0\rangle$, $|1\rangle$, ..., $|d-1\rangle$ and view our state in the larger vector space, i.e., view $\alpha|0\rangle + \beta|1\rangle$ as $\alpha|0\rangle + \beta|1\rangle + 0|2\rangle + \cdots + 0|d-1\rangle$. The probabilities corresponding to outcomes $|0\rangle$ and $|1\rangle$ should not depend on whether we are viewing the state in the original two-dimensional subspace or the larger one. Thus, in light of property **(i)**, the probabilities corresponding to outcomes $|2\rangle$, ..., $|d-1\rangle$ should all be zero. This suggests the following necessary condition:

(ii) If $\langle\Phi|\Psi\rangle = 0$ then $\tilde{\pi}(|\Psi\rangle, |\Phi\rangle) = 0$.

The final property we require of a rule for deriving probabilities is that it be basis independent:

(iii) $\tilde{\pi}(U|\Phi\rangle, U|\Psi\rangle) = \tilde{\pi}(|\Phi\rangle, |\Psi\rangle)$ where U is any unitary map.

In CQT, Gleason's theorem [5] shows that properties similar to these are sufficient to uniquely identify the Born rule as the only consistent rule for defining probabilities from states when $d > 2$. If we adopt the natural conditions (**i**), (**ii**), and (**iii**) in DQT, we find from the following theorem that a map $\tilde{\pi}$ satisfying these conditions does not exist.

4.4. A No-Go Theorem

We will fill out our arguments by introducing a no-go theorem that explains why it is impossible in DQT to incorporate into Born's rule a continuous real probability in the interval $[0, 1]$.

Theorem 1. *There is no map* $\tilde{\pi}\colon \mathbb{F}_{p^2}^{d\,*} \times \mathbb{F}_{p^2}^{d\,*} \longrightarrow [0, 1]$ *satisfying properties (***i***), (***ii***), and (***iii***) when* $d > 3$.

Proof: Assume such a map $\tilde{\pi}$ satisfying all three properties exists. Consider the state

$$|\Psi\rangle = \alpha(|0\rangle + |1\rangle + |2\rangle + \beta|3\rangle)$$

expressed in terms of orthonormal vectors, where $\beta \in \mathbb{F}_{p^2}$ is chosen so that $|\beta|^2 = p - 1$ and $\alpha \in \mathbb{F}_{p^2}$ is a constant such that $|\Psi\rangle$ is normalized. Let $|\Phi\rangle = |1\rangle + |2\rangle + \beta|3\rangle$. Note that $|\Phi\rangle$ is normalized in $\mathbb{F}_p$. We can write our original state $|\Psi\rangle$ as $|\Psi\rangle = \alpha(|0\rangle + |\Phi\rangle)$ The vectors $|0\rangle$ and $|\Phi\rangle$ are orthonormal and so are part of some larger orthonormal basis. Thus by properties (**i**) and (**ii**) we have

$$1 = \tilde{\pi}(|\Psi\rangle, |0\rangle) + \tilde{\pi}(|\Psi\rangle, |\Phi\rangle). \tag{3}$$

Let U be the unitary map that simply permutes the orthonormal vectors $|0\rangle$ and $|\Phi\rangle$. Note that $|\Psi\rangle$ is invariant under U, so invoking property (**iii**), we have

$$\tilde{\pi}(|\Psi\rangle, |0\rangle) = \tilde{\pi}(U|\Psi\rangle, U|0\rangle) = \tilde{\pi}(|\Psi\rangle, |\Phi\rangle)\ .$$

This along with equation (3) gives

$$1 = \tilde{\pi}(|\Psi\rangle, |0\rangle) + \tilde{\pi}(|\Psi\rangle, |0\rangle),$$

so

$$\tilde{\pi}(|\Psi\rangle, |0\rangle) = \tilde{\pi}(|\Psi\rangle, |\Phi\rangle) = \frac{1}{2} \in [0, 1]\ . \tag{4}$$

For $i = 0, 1, 2$ let U_i be the unitary map that permutes the basis vectors $|0\rangle$ and $|i\rangle$ and acts as the identity for the rest. Note that $|\Psi\rangle$ is invariant under U_i. So again by property (**iii**) we have

$$\tilde{\pi}(|\Psi\rangle, |i\rangle) = \tilde{\pi}(U_i|\Psi\rangle, U_i|0\rangle) = \tilde{\pi}(|\Psi\rangle, |0\rangle) = \frac{1}{2}$$

for $i = 0, 1, 2$. These probability assignments are inconsistent with the requirement that the probabilities for orthogonal outcomes add up to 1. By property (**i**)

$$\begin{aligned} 1 &= \tilde{\pi}(|\Psi\rangle, |0\rangle) + \tilde{\pi}(|\Psi\rangle, |1\rangle) + \tilde{\pi}(|\Psi\rangle, |2\rangle) + \tilde{\pi}(|\Psi\rangle, |3\rangle) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \tilde{\pi}(|\Psi\rangle, |3\rangle) \\ &\geq \frac{3}{2}, \end{aligned}$$

so we have a contradiction. The assumption of a satisfactory map $\tilde{\pi}$ is thus false.

(It should be pointed out that the argument above explicitly uses states from a vector space of four dimensions or larger, and thus does not hold if the largest vector space one can consider has dimension $d \leq 3$. But it does hold for any system that, for example, contains more than a single qubit.)

Yu-Tsung says: I am going to prove a no-go theorem for $d = 3$ here with an extra property

(iv) $\tilde{\pi}(|\Phi\rangle, |\Psi\rangle) = \tilde{\pi}(\gamma|\Phi\rangle, |\Psi\rangle) = \tilde{\pi}(|\Phi\rangle, \gamma|\Psi\rangle)$ where $\gamma \in \mathbb{F}_{p^2}^*$.

This property means that the phase difference and normalization will not effect the probability assignment. Although this property will not be used explicitly, considering non-normalized states may simplify the proof.

Theorem 2. *There is no map* $\tilde{\pi}: \mathbb{F}_{p^2}^{d\,*} \times \mathbb{F}_{p^2}^{d\,*} \longrightarrow [0, 1]$ *satisfying properties (***i***), (***ii***), (***iii***), and (***iv***) when* $d \geq 3$ *and* $p > 3$.

Yu-Tsung says: *Proof:* Assume such a map $\tilde{\pi}$ satisfying all four properties exists. Consider the orthonormal basis and the non-normalized state

$$\begin{aligned}
|0\rangle &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \\
|1\rangle &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T \\
|2\rangle &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \\
|+\rangle &= |0\rangle + |1\rangle + |2\rangle \,,
\end{aligned}$$

where $|+\rangle$ is not zero-normed when $p > 3$. For $i = 1, 2$, let $\mathbf{U}_i$ be the unitary map that permutes the basis vectors $|0\rangle$ and $|i\rangle$ and acts as the identity for the rest. Note that $|+\rangle$ is invariant under $\mathbf{U}_i$. So by property (**iii**) we have

$$\tilde{\pi}(|+\rangle, |i\rangle) = \tilde{\pi}(\mathbf{U}_i|+\rangle, \mathbf{U}_i|0\rangle) = \tilde{\pi}(|+\rangle, |0\rangle)$$

for $i = 1, 2$. By property (**i**)

$$\tilde{\pi}(|+\rangle, |0\rangle) + \tilde{\pi}(|+\rangle, |1\rangle) + \tilde{\pi}(|+\rangle, |2\rangle) = 1$$

so

$$\tilde{\pi}(|+\rangle, |i\rangle) = \frac{1}{3} \in [0, 1] \,. \tag{5}$$

Let

$$\begin{aligned}
\mathbf{U}' &= \begin{pmatrix} \alpha & \alpha & 0 \\ -\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
|\Phi\rangle &= 2\alpha|0\rangle + |2\rangle = \mathbf{U}'|+\rangle
\end{aligned}$$

where $\alpha \in \mathbb{F}_{p^2}$ is chosen so that $2\alpha^*\alpha = 1$. Thus by property (**iii**), we have

$$\tilde{\pi}(|\Phi\rangle, |2\rangle) = \tilde{\pi}(\mathbf{U}'|+\rangle, \mathbf{U}'|2\rangle) = \tilde{\pi}(|+\rangle, |2\rangle) \,.$$

Also, properties (**i**) and (**ii**) imply

$$1 = \tilde{\pi}(|\Phi\rangle, |0\rangle) + \tilde{\pi}(|\Phi\rangle, |2\rangle) \,.$$

This along with equation (5) gives

$$\tilde{\pi}(|\Phi\rangle, |0\rangle) = \frac{2}{3} \in [0, 1] \,. \tag{6}$$

Yu-Tsung says: *Proof:* Finally, consider

$$\mathbf{U}'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^* & 2\alpha \\ 0 & 2\alpha^* & \beta \end{pmatrix}$$

$$|\Psi\rangle = 2\alpha\,(|0\rangle + |1\rangle) + \beta|2\rangle = \mathbf{U}''|\Phi\rangle\ ,$$

where $\beta \in \mathbb{F}_{p^2}$ is chosen so that $\beta^*\beta = -1$. Because $|\Psi\rangle$ is invariant under $\mathbf{U}_1$, invoking property (**iii**) with equation (6), we have

$$\begin{aligned} &\tilde{\pi}(|\Psi\rangle, |1\rangle) = \tilde{\pi}(\mathbf{U}_1|\Psi\rangle, \mathbf{U}_1|1\rangle) \\ &= \tilde{\pi}(|\Psi\rangle, |0\rangle) = \tilde{\pi}(\mathbf{U}''|\Phi\rangle, \mathbf{U}''|0\rangle) \\ &= \tilde{\pi}(|\Phi\rangle, |0\rangle) = \frac{2}{3} \end{aligned}$$

These probability assignments are inconsistent with the requirement that the probabilities for orthogonal outcomes add up to 1. By property (**i**)

$$\begin{aligned} 1 &= \tilde{\pi}(|\Psi\rangle, |0\rangle) + \tilde{\pi}(|\Psi\rangle, |1\rangle) + \tilde{\pi}(|\Psi\rangle, |2\rangle) \\ &= \frac{2}{3} + \frac{2}{3} + \tilde{\pi}(|\Psi\rangle, |2\rangle) \\ &\geq \frac{4}{3}\ , \end{aligned}$$

so we have a contradiction. The assumption of a satisfactory map $\tilde{\pi}$ is thus false.

What are the implications of this theorem? Intuitively one might think that the lack of an order on the field elements is the main obstacle to extracting probabilities from DQT, but the above result shows that this is not the case. Even supplementing the field elements with an ordering does not allow for a rule that satisfies the three properties above. And any other scheme for extracting probabilities from quantum mechanics over a finite field will violate one of the three properties above.

We next observe that one way out of this dilemma is to define a framework for quantum measurement in DQT that is based not upon probabilities but upon discrete interval-valued probabilities.

Gerardo says: There is a paper by Mermin, Quantum Mysteries Revisited [4], that could be used profitably in the next arguments. See also John's paper, section "GHZ Experiment."

4.5. Interval-Valued Probabilities for DQT

Yu-Tsung says: It seems that almost everything currently in this section should move to Section 5, because they are already focus on the measurement process in DQT, especially $\pi\left(|\Psi\rangle, |\Phi\rangle\right)$ should be intuitively corresponding to $\mathcal{P}_{\Psi}\left(\lambda_i\right)$ in CQT. Or we may combine Section 4.5 and Section 5.1 into one section.
If we want to keep Section 4.5 and Section 5.1 separately, we may try to focus on the definition of Interval-Valued Probabilities and these algebraic relation, like =, 0, 1, and +. However, the readers may not know why they need these algebraic relation before we define the measurement process...

In order to write down the analog of the CQT Born rule and quantum measurement process for DQT, we must explicitly define a corresponding discrete measurement framework. To this end, consider an orthogonal basis $\{|i\rangle\}$ of $\mathbb{F}_{p^2}^{d\,*}$. In this basis, two arbitrary vectors can be written in the form

$$|\Psi\rangle = \sum_{i=0}^{d-1} \alpha_i \, |i\rangle \;\; , \;\; |\Phi\rangle = \sum_{i=0}^{d-1} \beta_i \, |i\rangle \, , \tag{7}$$

with α_i and β_i elements of the field $\mathbb{F}_{p^2}$. The Hermitian product in our discrete number system is thus reducible to the form

$$\langle \Phi \mid \Psi \rangle = \sum_{i=0}^{d-1} \beta_i^{\,p} \, \alpha_i \, . \tag{8}$$

These issues are all crucial for DQT because in the discrete domain we do not have transitive ordered probabilities defined on the real numbers. Furthermore, although the physical state $|\Psi\rangle$ before the measurement belongs to $\mathbb{F}_{p^2}^{d\,*}$, the post-measurement state may not belong to $\mathbb{F}_{p^2}^{d\,*}$ in the case of *degenerate eigenvalues*.

This leads us to introduce a modified Born's rule, with values in discrete interval-valued probabilities, for the study of the properties that DQT shares with CQT but not with classical theories. It will turn out that in order to accommodate the anomalous situation where post-measurement states do not belong to $\mathbb{F}_{p^2}^{d\,*}$, we explicitly detect such invalid intermediate states and cause them to halt the entire computation immediately.

Amr says: I think the previous two paragraphs could be even more clear if we said the following. First we talk about the no-go theorem and say that we must go to "interval-valued probabilities." Then say we will use two intervals *(possible, impossible)* . Then explain that a measurement operation is a *partial function* that takes a current state (non-null vector) and an observable $\mathcal{O}$ and as explained next produces *three* things when defined. Generally if we measure observable $\mathcal{O}$ of quantum state $|\phi\rangle$, then (i) with interval-value probability p_i, (ii) the result of the measurement will be eigenvalue λ_i, and (iii) the quantum state will collapse to the state $|\psi\rangle_i$. In the case the post-measurement state would be null because of wrapping around that might occur with degenerate observables the entire measurement operation is *undefined*. Otherwise the result is the triple of probability, eigenvalue, and post-measurement state. Note that composition of measurement operations is only defined if both measurement operations are defined. In particular, if is possible that two commuting observables $\mathcal{O}_1$ and $\mathcal{O}_2$ could be measured in one order but not the order if one of the intermediate steps is undefined.

Since discrete quantum theories do not have orderable fields, and thus lack orderable probabilities, we must discover a way to understand how Born's rule could manifest itself in a consistent discrete theory.

To begin our study of discrete theory, we define a map π that will play the role of a discrete Born rule, with the replacement of real-valued probabilities by the two tags 'impossible' and 'possible':

$$\pi : \mathbb{F}_{p^2}^{d\,*} \times \mathbb{F}_{p^2}^{d\,*} \to \mathscr{L}_2 = \{\text{impossible}, \text{possible}\}\,,$$

Thus π can be defined in the possibility field $\mathscr{L}_2$ as

$$\pi\left(|\Psi\rangle, |\Phi\rangle\right) = \begin{cases} \text{possible, if } \langle\Psi \mid \Phi\rangle \neq 0 \\ \text{impossible, if } \langle\Psi \mid \Phi\rangle = 0 \end{cases} . \tag{9}$$

5. Measurement Process in DQT

5.1. Basic Measurement

Amr says: Adapt the following to DQT

Assume that a physical system is represented by a state $|\Psi\rangle$, and that we wish to measure a physical quantity $\mathcal{O}$ of this system, with corresponding observable $\mathbf{O}$, whose eigenspectrum is given by

$$\mathbf{O}\,|i\rangle = \lambda_i\,|i\rangle\ .$$

Then Born's rule of CQT tells us that the probability of obtaining the eigenvalue λ_i as an outcome of the measurement is given by

$$\mathcal{P}_\Psi\left(\lambda_i\right) \;=\; \frac{\langle\Psi|P_i|\Psi\rangle}{\langle\Psi \mid \Psi\rangle} \;=\; \frac{\langle\Psi \mid i\rangle\langle i \mid \Psi\rangle}{\langle\Psi \mid \Psi\rangle\langle i \mid i\rangle}\ , \tag{10}$$

where P_i is the projector onto the subspace generated by the eigenvector $|i\rangle$. We have until now assumed non-degenerate eigenvalues. If, for some set of states $\{k\}$, the eigenvalue λ is degenerate, we account for that by replacing the simple expression for the projector P_i by

$$P_\lambda = \sum_{k=1}^{K_\lambda} \frac{|k\rangle \langle k|}{\langle k \mid k\rangle} ,$$

where K_λ is the degeneracy of the eigenvalue λ.

5.2. Degeneracy and Post-Measurement States

Amr says: Explain that problem and need to deal with errors

One other issue that will confront us in the transition from CQT to DQT is the presence of post-measurement states. In CQT, the Von Neumann projective measurement state, immediately after the measurement of $\mathcal{O}$, is given in general by

$$\frac{P_\lambda|\Psi\rangle}{\sqrt{\langle\Psi|P_\lambda|\Psi\rangle}} ,$$

where for the non-degenerate case, $P_\lambda \to P_i$, and so the resulting state is just $|i\rangle$.

Amr says: Assume O1 and O2 are compatible: f([O1.O2],psi) = f([O2.O1],psi)
In CQT, f(O1,f(O2,psi)) = f([O1.O2],psi) = f([O2.O1],psi) = f(O2,f(O1,psi))
In DQT, it is possible that: f(O1,f(O2,psi)) /= f(O2,f(O1,psi)) because one of them, say f(O1,psi) wraps in the field

6. Contextuality and Non-Locality in DQT

6.1. Gleason's Theorem in DQT

Recall the measurement process discussed in section 3. Given a physical system represented by a state $|\Psi\rangle$, and an observable $\mathbf{O}$, whose eigenspectrum is given by $\mathbf{O}\,|i\rangle = \lambda_i\,|i\rangle$. Then the Born rule of CQT tells us that the probability of obtaining the eigenvalue λ_i as an outcome of the measurement is given by

$$\mathcal{P}_\Psi\left(|i\rangle\right) \;=\; \mathcal{P}_\Psi\left(\lambda_i\right) \;=\; \frac{\langle\Psi \mid i\rangle\langle i \mid \Psi\rangle}{\langle\Psi \mid \Psi\rangle\langle i \mid i\rangle} \quad , \tag{11}$$

If the system is a mixed state represented by a density matrix

$$\rho = \sum_j p_j \frac{|\Psi_j\rangle\langle\Psi_j|}{\langle\Psi_j \mid \Psi_j\rangle} ,$$

we can extend the Born rule (11) and get

$$\mathcal{P}_\rho\left(|i\rangle\right) = \mathcal{P}_\rho\left(\lambda_i\right) \;=\; \sum_i p_i \frac{\langle\Psi_j \mid i\rangle\langle i \mid \Psi_j\rangle}{\langle\Psi_j \mid \Psi_j\rangle\langle i \mid i\rangle} \tag{12}$$

Let $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\mathbb{C}^{d*} = \left\{|\Psi\rangle \in \mathbb{C}^d \mid \langle\Psi \mid \Psi\rangle \neq 0\right\}$ be the set of non-zero vectors. Because normalization and global phase gives the same state, and $\mathcal{P}_\rho : \mathbb{C}^{d*} \to [0,1]$ should be a state-function, we should have

$$\text{For any vector } |i\rangle \in \mathbb{C}^{d*}, \text{ and } \gamma \in \mathbb{C}^* \,,\, \mathcal{P}_\rho(|i\rangle) = \mathcal{P}_\rho(\gamma|i\rangle) \,. \tag{13}$$

More importantly, $\mathcal{P}_\rho$ defines a probability which means

$$\text{For any orthonormal basis } \{|i\rangle\} \,,\, \sum_{i=0}^{d-1} \mathcal{P}_\rho\left(|i\rangle\right) = 1 \,. \tag{14}$$

Conversely, if a state-function $\mathcal{P} : \mathbb{C}^{d*} \to [0,1]$ is a probability, i.e., satisfying equation (13) and (14), we can identify $\mathcal{P}$ to a mixed state ρ by Gleason's theorem.

Theorem 3. *(Gleason's Theorem)*

(i) For $d \geq 3$, for any state-function $\mathcal{P} : \mathbb{C}^{d*} \to [0,1]$, if $\mathcal{P}$ is a probability, there is a density matrix ρ such that $\mathcal{P} = \mathcal{P}_\rho$ [12, 5].§

(ii) Consequently, the Born rule (11) is the only reasonable probability assignment, and can be deduced by other reasonable axioms [3, 13].

An important meaning of Gleason's theorem is that the state space is sufficient enough to produce every possible probability we get from an experiment. Hence, we don't have to worry about whether there is a chance that the probability got from a measurement cannot be explained by a state, and we need to introduce a new state. This may be analog to the fact that every algebraic equation over $\mathbb{C}$ can be solved in $\mathbb{C}$ so that we don't have to introduce new numbers to solve an equation. The other important meaning of Gleason's theorem is that it implies the Born rule (11). This is even more important for building a discrete quantum theory. If we plug in vectors over $\mathbb{F}_{p^2}$ into the usual Born rule (11), it gives an elements in $\mathbb{F}_p$ which has no way interpreted as a probability. Everyone who tried to build a quantum theory over finite fields struggled on how to define a reasonable probability. If Gleason's theorem works in any sense over finite fields, we can have a reasonable way to induce the Born rule over finite fields.

A naive way to extend Gleason's Theorem in DQT may be as follow.

Problem 4. Given a state-function $\hat{\pi} : \mathbb{F}_{p^2}^{d*} \to \mathscr{L}_2$ which is also a probability, i.e.,

$$\text{For any vector } |i\rangle \in \mathbb{F}_{p^2}^{d*}, \text{ and } \gamma \in \mathbb{F}_{p^2}^* \,,\, \hat{\pi}(|i\rangle) = \hat{\pi}(\gamma|i\rangle) \,. \tag{15}$$

$$\text{For any orthonormal basis } \{|i\rangle\} \,,\, \bigvee_{i=0}^{d-1} \hat{\pi}\left(|i\rangle\right) = \text{possible} \,. \tag{16}$$

Can we say something about $\hat{\pi}$?

§ In order to simplify the statement, we consider consider the probability on the state vectors instead of the projectors or closed subspaces as in the Gleason's original paper.

The answer is no, and the reasons are simple. First, compare the following arithmetic equation and Boolean equation.

$$p_1 + p_2 + p_3 = 1 \tag{17}$$

$$\pi_1 \vee \pi_2 \vee \pi_3 = \text{possible} \tag{18}$$

When $p_1 = \frac{1}{2}$, we still have an equality $p_2 + p_3 = 1$ for p_2 and p_3 because $(\mathbb{R}, 0, +)$ is a group. In contrast, when $\pi_1 = $ possible, we know nothing about π_2 and π_3 because $(\mathscr{L}_2, \text{impossible}, \vee)$ is a semigroup without cancellative, i.e., $a \vee c = b \vee c$ does not imply $a = b$. The difference between $(\mathbb{R}, 0, +)$ and $(\mathscr{L}_2, \text{impossible}, \vee)$ suggests us to study the range of the probability. A reasonable range should have the structure $(\mathscr{L}, 0, 1, \oplus, <)$, where $(\mathscr{L}, 0, \oplus, <)$ is the underlying commutative ordered semigroup [14] because the condition (14) and (16) suggests the value of the range should be able to add together. $1 \in \mathscr{L}$ is another special value to indicate the total probability. For example, $(\mathscr{L}_2, \text{impossible}, \text{possible}, \vee, <)$ and $(\mathbb{R}, 0, 1, +, <)$ are two possible range of the probability, but we cannot pick $\mathscr{L} = \mathbb{F}_{p^2}$ because we cannot define an inequality such that $(\mathbb{F}_{p^2}, 0, +, <)$ be an ordered semigroup.

Second, Gleason's theorem relies a lot on the continuous probability [12, 5, 3]. However, continuity doesn't even make sense in DQT. This fact implies that the structure of the domain of the probability, $\mathbb{F}_{p^2}^{d*}$, is also too weak. In order to think which structure we need, we can check which the reasonable property $\mathcal{P}_\Psi : \mathbb{C}^{d*} \to [0, 1]$ should have. Despite the total probability is one, $\mathcal{P}_\Psi$ must have

$$\mathcal{P}_\Psi\left(|i\rangle\right) = 0 \text{ if } \langle \Psi \mid i \rangle = 0 \tag{19}$$

where $\gamma \in \mathbb{C}^*$. Moreover, the probability must be invariant under any rotation, i.e.,

$$\mathcal{P}_{U\Psi}\left(U|i\rangle\right) = \mathcal{P}_\Psi\left(|i\rangle\right) \tag{20}$$

for any unitary matrix U. Because we consider the probability of two different state $\mathcal{P}_\Psi$ and $\mathcal{P}_{U\Psi}$ together, it may be more reasonable to ask whether there is an unique "Born rule" satisfying all the conditions, and our problem becomes

Problem 5. Is there a structure $(\mathscr{L}, 0, 1, \oplus, <)$ such that $\tilde{\pi} : \mathbb{F}_{p^2}^{d*} \to \left(\mathbb{F}_{p^2}^{d*} \to \{x \in \mathscr{L} | 0 \le x \le 1\}\right)$ satisfying the following conditions is unique for $d \ge 3$ and infinitely many for $d = 2$?‖

- For any orthonormal basis $\{|i\rangle\}_{n=0}^{d-1} \subset \mathbb{F}_{p^2}^{d*}$, (21)
  $\bigoplus_{i=0}^{d-1} \tilde{\pi}_\Psi\left(|i\rangle\right) = 1$.
- If $\langle \Psi | i \rangle = 0$,then $\tilde{\pi}_\Psi\left(|i\rangle\right) = 0$. (22)
- For any vector $|\Psi\rangle, |i\rangle \in \mathbb{F}_{p^2}^{d*}$,
  and U is a d-by-d unitary matrix over $\mathbb{F}_{p^2}$, (23)

‖ Instead of equation (22), another reasonable condition is $\langle \Psi | i \rangle = 0 \Leftrightarrow \tilde{\pi}_\Psi\left(|i\rangle\right) = 0$. However, when the underlying group $\mathscr{L} = \mathscr{L}_2$, this condition implies an unique solution for $d = 2$. This is too strong and is not what we want.

Possible	Impossible				The first vector is possible because of
001	100	010	110	$1\bar{1}0$	arbitrary choice of z axis
101	$\bar{1}01$	010			arbitrary choice of x vs $-x$
011	$0\bar{1}1$	100			arbitrary choice of y vs $-y$
$1\bar{1}-$	$\bar{1}1-$	110	$+0\bar{1}$	$0+1$	arbitrary choice of x vs y
$10-$	$+0\bar{1}$	010	$+\bar{1}\bar{1}$		orthogonality to 2nd and 3rd vectors
$+11$	$0\bar{1}1$	$+\bar{1}\bar{1}$	$\bar{1}0-$		orthogonality to 2nd and 3rd vectors
$+01$	010	$\bar{1}0-$	$\bar{1}\bar{1}-$		orthogonality to 2nd and 3rd vectors
$11-$	$1\bar{1}0$	$\bar{1}\bar{1}-$	$0+\bar{1}$		orthogonality to 2nd and 3rd vectors
$01-$	100	$0+\bar{1}$	$\bar{1}+\bar{1}$		orthogonality to 2nd and 3rd vectors
$1+1$	$\bar{1}01$	$\bar{1}+\bar{1}$	$0\bar{1}-$		orthogonality to 2nd and 3rd vectors

Table 1. Proof of the Kochen-Specker theorem in DQT(I)(I) in 3 dimensions.

i.e. , $U^\dagger U = I$, then $\tilde{\pi}_\Psi(|i\rangle) = \tilde{\pi}_{U\Psi}(U|i\rangle)$.

- For any vector $|\Psi\rangle, |i\rangle \in \mathbb{F}_{p^2}^{d\,*}$, and $\gamma \in \mathbb{F}_{p^2}^*$,
  $\tilde{\pi}_\Psi(|i\rangle) = \tilde{\pi}_\Psi(\gamma|i\rangle)$. (24)

The no-go theorem in section 4.4 said the if $(\mathscr{L}, 0, 1, \oplus, <) = (\mathbb{R}, 0, 1, +, <)$, there is no $\tilde{\pi} : \mathbb{F}_{p^2}^{d\,*} \to \mathbb{F}_{p^2}^{d\,*} \to \mathscr{L}$ for $d \geq 3$. It is easy to extend the no-go theorem and prove

Theorem 6. *If $(\mathscr{L}, 0, \oplus, <)$ is a ordered group, then there is no $\tilde{\pi} : \mathbb{F}_{p^2}^{d\,*} \to \mathbb{F}_{p^2}^{d\,*} \to \mathscr{L}$ satisfying all conditions in problem 5 for $d \geq 3$.*

When we consider $(\mathscr{L}_2, \text{impossible}, \text{possible}, \vee, <)$, π defined in section 4.5 satisfies all conditions, but π is not the unique solution because of lack of cancellation law. It is still an open question whether problem 5 is true or false in general.

Yu-Tsung says: Is that all probability value should be distinct be an reasonable condition?

6.2. The Kochen-Specker Theorem in DQT

The essence of the Kochen-Specker theorem is to ask if it is possible, for a fixed $|\Psi\rangle \in \mathbb{F}_{p^2}^{d\,*}$, to have another function $v(|\Phi\rangle)$ playing the role of $\pi(|\Psi\rangle, |\Phi\rangle)$ but failing to satisfy any single one of the conditions in the set (21,22, 24,23) and to show that this cannot be true.

The Theorem : For $d \geq 3$, there is no function $v : \mathbb{F}_{p^2}^{d\,*} \to \mathscr{L}_2$ such that

- For any orthogonal basis $\{|n\rangle\}_{n=0}^{d-1} \subset \mathbb{F}_{p^2}^{d\,*}$,
  exactly one of $\{v(|n\rangle)\}_{n=0}^{d-1}$ is possible. (25)

Proof:

Following the general structure in [3], we will prove by contradiction that the theorem holds for primes (with $p \equiv 3 \text{mod } 4$) with $p > 3$ ¶. Suppose v exists, and consider the set of all vectors with entries in $\{0, \pm 1, \pm 1 \pm \mathrm{i}\}$. Then the steps in the proof can be read off from Table 1. In each cell in Table 1, the string xyz represents a state vector, where x, y, and z can be $\{\ 0,\ 1,\ \bar{1} \equiv -1,\ + \equiv (1+\mathrm{i}),\ - \equiv (1-\mathrm{i})\ \}$, and

$$v\,(xyz) = \{\text{possible } \textit{or} \text{ impossible}\ ,$$

depending on its column. Examining the table from top to the bottom, we will eventually get a contradiction. The set of vectors are invariant under interchanges of the x, y, and z axess. Hence, we can assume $v\,(100) = v\,(010) =$ impossible and $v\,(001) =$ possible. Furthermore, 110 and $1\bar{1}0$ are orthogonal to 001, so that $v\,(110) = v\,(1\bar{1}0) =$ impossible. A similar argument may be used to construct the second, third, and fourth rows. After we know enough values of $v\,(|\Psi\rangle)$, we can find that $10-$, $+0\bar{1}$, and 010 is an orthogonal basis, and we already know that $v\,(+0\bar{1}) = v\,(010) =$ impossible, so $v\,(10-) =$ possible. Following a similar procedure, we can continue evaluating the entire content of Table 1. We can see that 100, $0+1$, and $0\bar{1}-$ form an orthogonal basis, but we already know $v\,(100) = v\,(0+1) = v\,(0\bar{1}-) =$ impossible. This contradicts our initial assumptions. Therefore, there is no function v satisfying the conditions.

Notice that this proof applies for almost all p, and, despite the exclusive appearance of integers in $\mathbb{F}_{p^2}$ throughout the computation, there are no occurrences of wrapping back to zero in the number field.

6.3. Bell's Theorem in DQT

We conclude with a discrete quantum theory version of Bell's theorem, showing that non-locality persists in the discrete theory, and no hidden variable theories are consistent with our modal-logic-valued operations. We exploit methods such as those of Greenberger, Horne, and Zeilinger [8] that do not rely on inequalities, and so allow us to escape the disadvantages of lack of orderability of discrete field values (see, e.g, [4, 3]).

First, let $\mathbb{O}^d_{p^2}$ be the set of all observables for $\mathbb{F}^d_{p^2}$, and let $\Delta\,(\mathcal{O})$ be the set of eigenvalues of an observable $\mathcal{O}$. Then, the Born rule in DQT(I) π asserts that given $|\Psi\rangle \in \mathbb{F}^{d\,*}_{p^2}$ and an observable $\mathcal{O} \in \mathbb{O}^d_{p^2}$ whether a eigenvalue $\lambda \in \Delta\,(\mathcal{O})$ is a possible measurement result or not. Hence, a hidden variable theories consistent with π can be described by a function w giving directly the measurement result $\lambda = w(|\Psi\rangle, \mathcal{O})$ and satisfying condition (26).

$$\begin{aligned}
&\bullet \text{ For any } |\Psi\rangle \in \mathbb{F}^{d\,*}_{p^2},\ \mathcal{O} \in \mathbb{O}^d_{p^2},\\
&\quad w(|\Psi\rangle, \mathcal{O}) \in \Delta\,(\mathcal{O})\,. \text{ Moreover,}\\
&\quad \text{if } \mathcal{H} \text{ is an eigenspace of } \mathcal{O} \text{ corresponding to } w\,(|\Psi\rangle, \mathcal{O})\,,\\
&\quad \text{then } \pi\,(|\Psi\rangle, \mathcal{H}) = \text{possible}\ .
\end{aligned} \tag{26}$$

¶ When $p = 3$, the Kochen-Specker theorem still holds, but the proof is longer.

Second, local observables $\mathcal{O}'$ and $\mathcal{O}''$ are represented by the observables in subspace $\mathbb{O}_{p^2}^{d'}$ and $\mathbb{O}_{p^2}^{d''}$ and be considered as observables in the whole space by applying tensor product, $\mathcal{O}' \otimes Id_{d''}$ and $Id_{d'} \otimes \mathcal{O}''$. Then, condition (27)

$$\begin{aligned}
&\bullet \text{ For any } d', d'' \in \mathbb{N},\ d = d'd'',\ \mathcal{O}' \in \mathbb{O}_{p^2}^{d'},\ \mathcal{O}'' \in \mathbb{O}_{p^2}^{d''}. \\
&\text{If } \mathcal{H}' \text{ is an eigenspace of } \mathcal{O}' \otimes Id_{d''}, \\
&\text{and } \mathcal{H}'' \text{ is an eigenspace of } Id_{d'} \otimes \mathcal{O}'', \\
&\text{and } \pi\left(|\Psi\rangle, \mathcal{H}'\right) = \text{possible}, \\
&\text{then } w\left(|\Psi\rangle, \mathcal{H}''\right) = w\left(\hat{P}_{\mathcal{H}'}|\Psi\rangle, \mathcal{H}''\right).
\end{aligned} \tag{27}$$

represents locality because we require the measurement result of $Id_{d'} \otimes \mathcal{O}''$ should be the same no matter whether we apply the other observable $\mathcal{O}' \otimes Id_{d''}$ which may be far away from $Id_{d'} \otimes \mathcal{O}''$.

Theorem For $8 \mid d$, there is no function $w : \mathbb{F}_{p^2}^{d\,*} \times \mathbb{O}_{p^2}^{d} \to \mathbb{F}_p$ satisfying condition (26) and (27).

Proof: Suppose the function w exists, we are going to construct a contradiction form a (non-normalized) GHZ state $|\Psi_4\rangle = 4|000\rangle - 4|111\rangle$, where $|xyz\rangle$ are a three qubit state $|x\rangle \otimes |y\rangle \otimes |y\rangle$, x, y, $z \in \{0, 1\}$. Then we consider observables representing the j-th experimenter measuring the spin along the μ-axis, the Pauli operators acting on the j-th spin σ_μ^j, where $\mu = x, y$, and

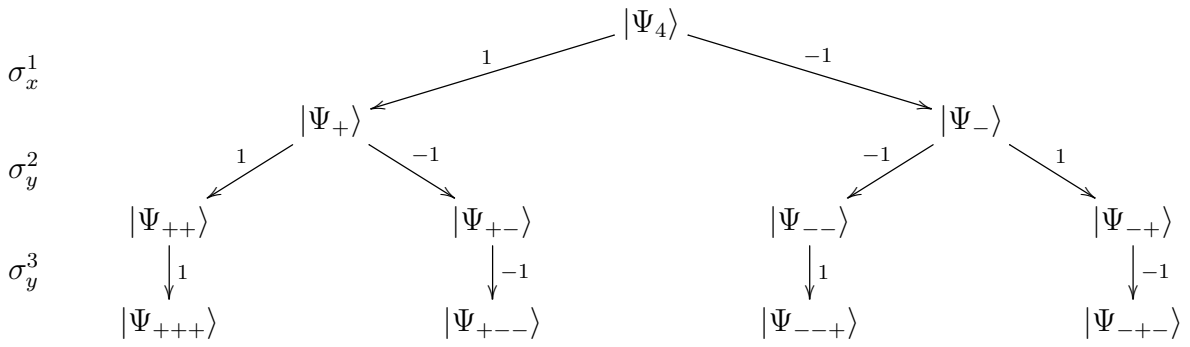
$$\sigma_\mu^j = \overbrace{\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underbrace{\sigma_\mu}_{j^{th}\text{ factor}} \otimes \cdots \otimes \mathbb{1}}^{n\text{ factors}}, \tag{28}$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Suppose the first experimenter perform measurement on the first qubit along the x-axis, and the other experiments perform measurement on the other qubits along the y-axis, their measurements are actually associated with observables σ_x^1, σ_y^2, and σ_y^3. Because σ_x^1, σ_y^2, and σ_y^3 are commuting, the product observables $\sigma_x^1\sigma_y^2\sigma_y^3$ used to be considered in the proof in CQT [4, 3]. However, as we discussed, we may not guarantee commuting observables are compatible and can be measured simultaneously. Therefore, if we want to consider the product of measurement result $\sigma_x^1\sigma_y^2\sigma_y^3$, it is better to consider $3! = 6$ possible orders of measurements, to do these measurements sequentially, and to multiply the measurement results after the measurement process complete.

Assume the first experimenter measures first, and so on. By the definition of π and the von Neumann measurement postulate, we can represent the whole process in Figure 1, where each arrow $|\Psi\rangle \xrightarrow{\lambda} |\Psi'\rangle$ is from a pre-measurement state $|\Psi\rangle$ to a possible post-measurement state $|\Psi'\rangle$, labeled by the measurement result eigenvalue λ on the row labeled by the observable $\mathcal{O}$. By the von Neumann measurement postulate, the possible post-measurement state $|\Psi'\rangle$ is $\hat{P}_{\mathcal{O}=\lambda}|\Psi\rangle$, where $\hat{P}_{\mathcal{O}=\lambda}$ is the projector on the eigenspace of λ. By the condition (26), the



$$
\begin{aligned}
|\Psi_+\rangle &= \hat{P}_{\sigma_x^1=1}|\Psi_4\rangle \\
|\Psi_-\rangle &= \hat{P}_{\sigma_x^1=-1}|\Psi_4\rangle \\
|\Psi_{++}\rangle &= \hat{P}_{\sigma_y^2=1}\hat{P}_{\sigma_x^1=1}|\Psi_4\rangle \\
|\Psi_{+-}\rangle &= \hat{P}_{\sigma_y^2=-1}\hat{P}_{\sigma_x^1=1}|\Psi_4\rangle \\
|\Psi_{-+}\rangle &= \hat{P}_{\sigma_y^2=1}\hat{P}_{\sigma_x^1=-1}|\Psi_4\rangle \\
|\Psi_{--}\rangle &= \hat{P}_{\sigma_y^2=-1}\hat{P}_{\sigma_x^1=-1}|\Psi_4\rangle \\
|\Psi_{+++}\rangle &= \hat{P}_{\sigma_y^3=1}\hat{P}_{\sigma_y^2=1}\hat{P}_{\sigma_x^1=1}|\Psi_4\rangle \\
|\Psi_{+--}\rangle &= \hat{P}_{\sigma_y^3=-1}\hat{P}_{\sigma_y^2=-1}\hat{P}_{\sigma_x^1=1}|\Psi_4\rangle \\
|\Psi_{-+-}\rangle &= \hat{P}_{\sigma_y^3=-1}\hat{P}_{\sigma_y^2=1}\hat{P}_{\sigma_x^1=-1}|\Psi_4\rangle \\
|\Psi_{--+}\rangle &= \hat{P}_{\sigma_y^3=1}\hat{P}_{\sigma_y^2=-1}\hat{P}_{\sigma_x^1=-1}|\Psi_4\rangle
\end{aligned}
$$

Figure 1. A measurement series start from a (non-normalized) GHZ state $|\Psi_4\rangle$, and perform measurement associated with observables σ_x^1, σ_y^2, and σ_y^3 consecutively.

existence of arrow $|\Psi\rangle \xrightarrow{\lambda} |\Psi'\rangle$ implies that $w\left(|\Psi\rangle, \mathcal{O}\right) = \lambda$ is possible; conversely, we must have $w\left(|\Psi\rangle, \mathcal{O}\right) \neq \lambda$ if there is no arrow $|\Psi\rangle \xrightarrow{\lambda} |\Psi'\rangle$.

After we draw Figure 1, by the locality condition (27), we know

$$w\left(|\Psi_4\rangle, \sigma_y^2\right) = w\left(\hat{P}_{\sigma_x^1=1}|\Psi_4\rangle, \sigma_y^2\right)$$

and so on. Therefore, no matter what measurement results are in Figure 1, the product of measurement results always satisfies

$$w\left(|\Psi_4\rangle, \sigma_x^1\right) w\left(|\Psi_4\rangle, \sigma_y^2\right) w\left(|\Psi_4\rangle, \sigma_y^3\right) = 1 \tag{29}$$

By the same reason, no matter which measurement order among σ_x^1, σ_y^2, and σ_y^3, the product of measurement result is always 1. If we apply the similar method on other triad observables, we can get

$$w\left(|\Psi_4\rangle, \sigma_y^1\right) w\left(|\Psi_4\rangle, \sigma_x^2\right) w\left(|\Psi_4\rangle, \sigma_y^3\right) = 1 \tag{30}$$

$$w\left(|\Psi_4\rangle, \sigma_y^1\right) w\left(|\Psi_4\rangle, \sigma_y^2\right) w\left(|\Psi_4\rangle, \sigma_x^3\right) = 1 \quad (31)$$

$$w\left(|\Psi_4\rangle, \sigma_x^1\right) w\left(|\Psi_4\rangle, \sigma_x^2\right) w\left(|\Psi_4\rangle, \sigma_x^3\right) = -1 \quad (32)$$

However, there is no way to assign $w\left(|\Psi_4\rangle, \sigma_\mu^j\right) = 1$ or -1 to satisfying the equation (29), (30), (31), and (32). Therefore, there is no function w.

7. Conclusions

We have shown that by replacing state vectors and continuous probabilities introduced in the conventional quantum mechanical arguments for Gleason's theorem and related theorems invalidating hidden variable theories by vectors in the discrete field $\mathbb{F}_{p^2}^{d*}$ and a modal logic-valued Born Rule, all the basic properties needed for a quantum theory that is distinct from classical theory, forbidding hidden variables, can be recovered.

Andy says: Need to close with an assessment of the Mayer-Mermin discussion about the validity of the KS theorem in a world with finite resources and precision.

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