MATHEMATICAL NOTES

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AN INEQUALITY FOR CONVEX FUNCTIONS

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In what follows, m and n are positive integers, x a non-negative real number and $f(x)$ a real function of x. We call the set of real numbers $a_1, \cdots, a_m$ an m-set if

$$a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq a_m \geqq 0.$$

All our subsequent statements are true if no further restrictions are imposed on x and a's or if, on the other hand, for some fixed $b>0$, we insist that $x \leqq b$ and $a_1 \leqq b$; that is, we may work indifferently in a fixed finite interval or in the semi-infinite interval $x \geqq 0$.

We use $P(m)=P(m, f)$ to denote the proposition (true for some f, false for others):

For every m-set

$$f(a_1) - f(a_2) + \cdots + f(a_m) \geqq f(a_1 - a_2 + \cdots + a_m) \ (m \text{ odd}),$$

$$f(a_1) - f(a_2) + \cdots - f(a_m) \geqq f(a_1 - a_2 + \cdots - a_m) - f(0) \ (m \text{ even}).$$

Payne and Weinstein conjectured and Weinberger [*Proc. Nat. Acad. Sci.*, vol. 38, 1952, pp. 611–613] recently proved the special case of $P(m, f)$ in which $f(x)=x^r$ and $r \geqq 1$. In an abstract of Weinberger's paper [*Math. Reviews*, vol. 14, 1953, p. 24], Bellman supplied a simple proof that $P(m, f)$ (in a slightly weakened form) is true for any continuously differentiable convex f. Both Weinberger's and Bellman's proofs depend on differentiation. In fact, $P(m, f)$ for any continuous convex function f is a special case of Theorem 108 of Hardy, Littlewood and Pólya's *Inequalities* (Cambridge, 1934, hereafter referred to as HLP).

My object here is to show that the $P(m, f)$ for any given f have a very simple logical relationship among themselves, independent of any ideas of differentiation or of continuity. We use capital letters to denote propositions. $A+B \rightarrow C$ means "if A and B are both true, then C is true," and $A \equiv B$ means "$A \rightarrow B$ and $B \rightarrow A$."

All our statements refer to a fixed function f, which need not be continuous. We observe that nothing is changed in $P(m, f)$ if we replace every f by $f-c$ for some c independent of x. Hence, without loss of generality, we may suppose $f(0)=0$. For such an f, $P(m, f)$ reads

For every m-set,

$$\sum_{k=1}^{m} (-1)^{k-1} f(a_k) \geqq f\left(\sum_{k=1}^{m} (-1)^{k-1} a_k \right).$$

$P(1, f)$ is trivially true. If we put $a_{m+1}=0$, we see that

$$P(m+1, f) \rightarrow P(m, f). \tag{1}$$

Next, if we assume $P(m, f)$ and $P(3, f)$ and use them in succession, we have

$$\sum_{k=1}^{m+2} (-1)^{k-1} f(a_k) \geqq f(a_1) - f(a_2) + f\left(\sum_{k=3}^{m+2} (-1)^{k-1} a_k \right)$$
$$\geqq f\left(\sum_{k=1}^{m+2} (-1)^{k-1} a_k \right),$$

and this is $P(m+2, f)$. Hence

$$P(3, f) + P(m, f) \rightarrow P(m+2, f). \tag{2}$$

Using (2) to establish an obvious induction we have $P(3, f) \rightarrow P(2n+1, f)$ and $P(3, f)+P(2, f) \rightarrow P(2n, f)$, for every positive integer n. By (1), however, we see that $P(3, f) \rightarrow P(2, f)$ and so

$$P(3, f) \rightarrow P(m, f), \qquad (m \geqq 1). \tag{3}$$

By repeated use of (1), we see that

$$P(m, f) \rightarrow P(3, f), \qquad (m \geqq 3) \tag{4}$$

and so

$$P(m, f) \equiv P(3, f), \qquad (m \geqq 3). \tag{5}$$

Finally, while $P(3, f) \rightarrow P(2, f)$, the converse is false. For example, let $f(x)$ be defined as

$$0\ (0 \leqq x < 1), \qquad x-1\ (1 \leqq x < 2), \qquad 1\ (2 \leqq x < 3), \qquad x-2\ (x \geqq 3).$$

Then $P(2, f)$ is true, as may be easily seen from a figure. But $P(3, f)$ is false; for example,

$$f(3) - f(2) + f(1) = 1 - 1 + 0 = 0, \qquad f(3-2+1) = f(2) = 1.$$

Hence $P(3, f)$ is the fundamental inequality. If we put $a_1 = x_1 + \delta$, $a_2 = x_1$, $a_3 = x_2$, we see that $P(3, f)$ is equivalent to

$$f(x_1+\delta) - f(x_1) \geqq f(x_2+\delta) - f(x_2) \tag{6}$$

for all $\delta > 0$ and all $x_1 \geqq x_2 \geqq 0$. This may be taken as a definition of convexity, in which case we have shown that, for any $m \geqq 3$, the truth of $P(m, f)$ is a necessary and sufficient condition for f to be convex.

HLP take

$$f(x_3) + f(x_4) \geqq 2f\left(\frac{x_3 + x_4}{2}\right) \tag{7}$$

for all $x_3 \geqq 0$, $x_4 \geqq 0$ as the definition of convexity. Relation (6) implies (7) and, for continuous $f(x)$, (7) implies (6) (HLP Theorem 86). If there is any $f(x)$ which satisfies (7) and not (6), it must be discontinuous and so (HLP Theorem 111) unbounded in any finite interval. The only known examples of discontinuous convex functions depend for their construction on Zermelo's Axiom of Choice (HLP p. 96). These examples satisfy both (6) and (7), so that whether functions satisfying (7) and not (6) exist is unknown, even if Zermelo's axiom is true.

A NOTE ON COMPLETE RESIDUE SYSTEMS

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The following is known [1]:

THEOREM. *If m is an integer $\geqq 3$, and if $\{a_i\}$, $\{b_i\}$, $i=1, \cdots, m$, are two complete residue systems (mod m), then $\{a_i\, b_i\}$ is not a complete residue system (mod m).*

We here offer a simpler proof.

The theorem is well-known [2] for $m=p$ an odd prime. Indeed, excluding the zero element, by Wilson's Theorem the product of the remaining elements of a complete residue system is congruent to -1 (mod p); since this is true for $\{a_i\}$, $\{b_i\}$, it must fail for $\{a_ib_i\}$.

Evidently the theorem holds for $m=4$. We proceed by multiplicative induction. We assume the theorem for m arbitrary, $m \geqq 3$. Let p be an arbitrary prime; suppose that $\{a_ib_i\}$, $i=1, \cdots, mp$, is a complete residue system (mod mp). Now, every complete residue system (mod mp) contains exactly m multiples of p. Hence $p \mid a_i$ if and only if $p \mid b_i$, else $\{a_ib_i\}$ will contain more than m multiples of p. We may suppose $a_j = a_j' p$, $b_j = b_j' p$, $j=1, \cdots, m$, where $\{a_j'\}$, $\{b_j'\}$ form complete residue systems (mod m). Thus the set $\{a_ib_i\}$ contains the set $\{a_j' b_j' p^2\}$. The incongruence of the elements $a_j' b_j' p^2$ (mod mp) implies the incongruence of the elements $a_j' b_j' p$ (mod m), which implies the incongruence of the elements $a_j' b_j'$ (mod m). This is contrary to our hypothesis on m; hence the supposition that $\{a_ib_i\}$ is a complete residue system (mod mp) is false, and the theorem is established.

References

1. S. Chowla and T. Vijayaraghavan, On complete residue systems, Quart. J. Math., vol. 19, pp. 193–194.

2. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 2, chapter 8, problem 245, pp. 158, 379.