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ON THE CORES OF SCALAR MEASURE GAMES

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Abstract. A $CVM(k)$ game is a game of the form $f \circ \lambda$, where λ is a k-dimensional non-atomic measure and f is a continuously differentiable function on R^k. For a convex $CVM(1)$ game, we characterize the "least upper bound" and "greatest lower bound" of the core elements in terms of the distribution function. We also show that the core of a convex $CVM(1)$ game expands as the underlying measure λ changes in a "convex manner". These results provide a partial geometric picture for the core and its variations of a convex $CVM(1)$ game.

1. Introduction

A game V with a *continuum* of players is a bounded real-valued function defined on Σ, the set of all Borel subsets of $I = [0, 1]$, such that $V(\emptyset) = 0$. The elements of Σ are interpreted as *coalitions* of players; for each coalition S, $V(S)$ gives the maximum payoff achieved by the efforts of all members in the coalition. With this interpretation, we shall assume that V is non-negative and that V is not identically zero throughout this paper. For $S \in \Sigma$, we denote by S^* the indicator function of S. Let $\mathbf{B}$ be the Banach space spanned by the set $\Sigma^* = \{S^* : S \in \Sigma\}$ with the sup norm. Then the space of all bounded additive functions on Σ, denoted by $\mathbf{BA}$, is isometrically isomorphic to the norm-dual of $\mathbf{B}$. A *payoff* of the game V is an element $\mu \in \mathbf{BA}$ with $\mu(I) = V(I)$. The *core* C_V of the game V consists of all payoffs μ such that no coalition can improve upon, i.e., $\mu(S) \geq V(S)$ for each $S \in \Sigma$. The core is a convex and compact set (in the $\mathbf{B}$ topology). The assumption that V is

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non-negative implies that any element in the core is also non-negative. In fact, $\mu \in C_V$ must be monotonic in the sense that $\mu(S) \leq \mu(T)$ whenever $S \subseteq T$.

Since Σ is a σ-algebra, we can endow a measure λ with it[1]. Then a special class of symmetric games (with respect to the measure λ) can be defined as follows:

Definition 1. V is called *symmetric* (with respect to λ) if $V(S) = V(T)$ for all S and T such that $\lambda(S) = \lambda(T)$.

It is clear that V is symmetric if and only if there is a unique real-valued function $f : [0, \lambda(I)] \longrightarrow R$ such that $V(S) = f \circ \lambda(S)$ for all $S \in \Sigma$, and $f(0) = 0$. We shall follow Aumann and Shapley [1974] and call such V a *scalar measure game* (See Aumann and Shapley [1974], p. 14). More generally, we can define:

Definition 2. Let $\mathbf{0}$ be the origin in R^k and λ be a k-dimensional vector measure on (I, Σ) such that its range, denoted by Range(λ), is of full dimension in R^k. The class of all *k-dimensional vector measure games* $VM(k)$ (with respect to λ) consists of all games V of the form: $V(S) = f \circ \lambda(S)$, where f is a non-negative real-valued function defined on R^k with $f(\mathbf{0}) = 0$.

Let $CVM(k)$ be the class of all games $V = f \circ \lambda \in VM(k)$ such that f is continuously differentiable on R^k, and that λ is a non-atomic vector measure[2] on I. The *Shapley value* of a game $V = f \circ \lambda \in CVM(k)$ is a payoff ϕ_V of V given by the following formula (Aumann and Shapley [1974], p. 23)

$$\phi_V(S) = \int_0^1 f_{\lambda(S)}(t\lambda(I))dt, \tag{1}$$

where $f_{\lambda(S)}$ is the derivative of f in the direction $\lambda(S) \in R^k$. In the case $k = 1$, the Shapley value is in the core if and only if for each $S \in \Sigma$ such that $\lambda(S) > 0$, $\frac{v(S)}{\lambda(S)}$ is bounded above by $\frac{v(I)}{\lambda(I)}$.

Due to space limitation, the focus of this paper is mostly on the case $k = 1$, which carries quite strong geometric intuition. In the second section, we begin with Schmeidler's [1972] result that the core of any convex game consists

[1] Of particular interest is the Lebsegue measure on Σ. We also recall that a measure is a non-negative countably additive function in **BA**.

[2] We recall that an *atom* of a vector measure λ is a coalition $S \in \Sigma$ such that $\lambda(S) \neq \mathbf{0}$, and for each proper subset T of S, either $\lambda(T) = \mathbf{0}$ or $\lambda(S - T) = \mathbf{0}$. If λ has no atom, then it is called a non-atomic measure. It is clear that any measure which is absolutely continuous with respect to the k-dimensional Lebsegue measure on I must be non-atomic.

entirely of countably additive elements if the game satisfies some type of continuity condition. It is quite obvious that every convex game in $CVM(1)$ satisfies this condition, so that each core element can be represented by a distribution function. In Section 3, for a convex game in $CVM(1)$, we characterize the "least" upper bound, and the "greatest" lower bound for all the distribution functions in the core. This shows a clear geometric picture for the "shape" of the core. Finally, we prove a theorem on expanding core as we change the underlying measure λ.

2. Games in CVM

To every game V, we can define an extended real number $|V|$, called the norm of V, by

$$|V| = \sup\Big\{\sum_i a_i V(S_i) : (a_i,\, S_i) \text{ is a finite sequence in } R_+ \times \Sigma \text{ such that } \sum_i a_i S^* \leq I^*\Big\}. \tag{2}$$

For each game V with finite norm, the *exact envelope* of V is a game defined by

$$\overline{V}(S) = \min\{\mu(S) : \mu \in \mathbf{BA}, \mu \geq V \text{ and } \mu(I) = |V|\} \quad \text{for all } S \in \Sigma. \tag{3}$$

Definition 3. A game V is called *balanced* if $|V| = V(I)$. It is called *exact* if $\overline{V} = V$. It is called *convex* if $V(S) + V(T) \leq V(S \cup T) + V(S \cup T)$ for all $S, T \in \Sigma$.

It is clear that for any convex game $V = f \circ \lambda \in CVM(1)$, the Shapley value $\phi_V(S) = \lambda(S)\frac{V(I)}{\lambda(I)} \geq V(S)$ by convexity of f for each $S \in \Sigma$. Hence, the Shapley value of a convex game in $CVM(1)$ always lies in its core.

Definition 4. A game V is *continuous* at $S \in \Sigma$ if $V(S_n) \longrightarrow V(S)$ for any monotone sequence $\{S_n\}$ in Σ such that $\bigcup_n S_n = S$. If V is continuous at each $S \in \Sigma$, then V is said to be continuous.

Proposition 1. (Shapley [1971], Schmeidler [1972]) *A game V has a non-empty core if and only if it is balanced. Every convex game is exact, and every exact game is balanced.*

Proposition 2. (Schmeidler [1972]) *Let V be an exact game. Then every element in the core of V is countably additive if and only if V is continuous at I.*

Theorem 1. *Every exact (hence convex) game in $CVM(k)$ has a non-empty core with countably additive elements only.*

Proof. We first prove that every game in $CVM(k)$ is continuous at I for all k. Let $V = f \circ \lambda \in CVM(k)$, and $\{S_n\}$ be any monotone sequence in Σ such that $\bigcup_n S_n = I$. Countable additivity of λ implies that $\lambda(I) = \lim_n \lambda(S_n)$. Continuity (continuous differentiability is not required here) of f implies that $f \circ \lambda(I) = \lim_n f \circ \lambda(S_n)$. Hence, the assertion of Theorem 1 follows immediately from Propositions 1 and 2. ∎

It is easy to show that a game $V = f \circ \lambda \in CVM(1)$ is convex if and only if the derivative f' is non-decreasing on Range(λ).[3] The implication of Theorem 1 is that there is a measure on Σ such that every element of an exact game in $CVM(1)$ is absolutely continuous with respect to it. We can, then, via the Radon-Nikodym theorem, represent every element in the core by a real-valued function on I. In fact, the following theorem shows that for a game $V = f \circ \lambda$, λ is exactly the measure we are seeking.

Theorem 2. *Let $V = f \circ \lambda \in CVM(1)$ be an exact game. Then for each $\mu \in C_V$, there is a non-negative Borel measurable function g_μ (unique up to the measure λ) such that*

$$\mu(S) = \int_S g_\mu d\lambda .$$

Proof. Because of Theorem 1 and the Radon-Nikodym theorem, we are left with proving that μ is absolutely continuous with respect to λ. We first recall that any element in the core is non-negative since $\mu \geq V \geq 0$. Now, let $S \in \Sigma$ with $\lambda(S) = 0$. Then

$$\mu(I) - \mu(S) = \mu(I - S) \geq V(I - S) = V(I) = \mu(I),$$

which implies $\mu(S) \leq 0$. It follows that $\mu(S) = 0$, and hence μ is absolutely continuous with respect to λ. ∎

According to Theorem 2, for an exact game in $CVM(1)$, we may identify each $\mu \in C_V$ with its *density function* g_μ. Two measures μ_1 and μ_2 in the core will always be regarded as equivalent if $\mu_1 = \mu_2$ except on a measurable set E, with $\lambda(E) = 0$.

Definition 5. A measurable auto-map π on I is *measure-preserving* for λ if it is one-one, onto, and $\lambda(S) = \lambda(\pi(S))$ for each $S \in \Sigma$. (Note we have used the notation $\pi(S) = \{\pi(x) : x \in S\}$.)

[3] Curiously, for $k > 1$, there is no corresponding statement in either directions.

It is clear that the set of all measure-preserving maps for λ forms a group consisting of idempotent elements only. An easy example of a measure-preserving map for the Lebesgue measure on I is the function $\pi(x) = 1 - x$. We note that $\pi^{-1} = \pi$.

Theorem 3. *Let π be any measure-preserving map for λ. Then for any game $V \in CVM(1)$, $\mu \in C_V$ implies $\mu \circ \pi \in C_V$. In fact, $g_{\mu\circ\pi} = g_\mu \circ \pi$.*

Proof. Let $\mu \in C_V$ and π be a measure-preserving map for λ. Then for each $S \in \Sigma$, we have

$$(\mu \circ \pi)(S) = \mu(\pi(S)) \geq V(\pi(S)) = V(S).$$

Hence, $\mu \circ \pi \in C_V$. The second statement follows from:

$$\int_S g_\mu \circ \pi d\lambda = \int_{\pi(S)} g_\mu d(\lambda \circ \pi^{-1}) = \int_{\pi(S)} g_\mu d\lambda,$$

where $\pi^{-1}(S) = \{x \in I : \pi(x) \in S\}$. ∎

Theorem 3 has quite concrete geometric interpretation. Furthermore, for any convex game in $CVM(1)$, we can classify the "shape" of the core in terms of *distribution functions.* This is the main purpose of the next section.

3. Convex Games in $CVM(1)$

Definition 6. For any measure μ on Σ, one can define the *distribution function* of μ on I by $F_\mu(x) = \mu([0,x])$ for $x \in I$.

It can be easily seen that F_μ has the following properties:

(P1) F_μ is a nondecreasing function;

(P2) F_μ is right continuous and has a left-hand limit everywhere;

(P3) $F_\mu(0) = 0$ and $F_\mu(1) = \mu(I)$.

It turns out the converse is also true. Given a distribution function F on I with properties (P1), (P2), $F(0) = 0$, and $F(1)$ finite, one can define[4] a unique measure γ_F on Σ such that $\gamma_F((x,y]) = F(y) - F(x)$ for any $0 \leq x < y \leq 1$.

Since λ is a non-atomic measure on I, F_λ is a continuous function on I. Assuming that F_λ is strictly increasing, we can define the following auto map on I:

(4) $$\hat{\pi}(x) = \{y \in I : F_\lambda(x) + F_\lambda(y) = \lambda(I)\}, \quad x \in I.$$

[4] This is a standard fact in probability theory. See, e.g., Shiryayev [1984].

We now show that $\hat{\pi}$ is a measure-preserving map for λ. Since $F_\lambda(x) = \lambda(I) - F_\lambda(\hat{\pi}(x))$, we have $\lambda([0,x]) = \lambda([\hat{\pi}(x),1])$. It can be readily seen that $\lambda(S) = \lambda \circ \hat{\pi}(S)$ for all open intervals $S \in \Sigma$. The collection Σ is the smallest σ-algebra generated by all the open intervals in I. By Carathéodory's extension theorem, $\lambda(S) = \lambda \circ \hat{\pi}(S)$ for all $S \in \Sigma$. Since F_λ is strictly increasing, $\hat{\pi}$ is one-one. Now, $\hat{\pi}(0) = 1$, $\hat{\pi}(1) = 0$ and so by the Intermediate Value Theorem, $\hat{\pi}$ is onto as well. It follows that $\hat{\pi}$ is a measure-preserving map for λ.

For each monotonic game $V \in CVM(1)$, we can define $F_V(x) = V([0,x])$ for each $x \in I$. Clearly, F_V satisfies properties (P1) to (P3), and there is a unique measure γ_{F_V} associated with it. Note that γ_{F_V} is different from V itself unless V is countably additive.

Theorem 4. *Suppose that $V = f \circ \lambda$ is a convex game in $CVM(1)$ and that F_λ is strictly increasing. Then for any $\mu \in C_V$, we have*

$$F_V(x) \leq F_\mu(x) \leq V(I) - F_V(\hat{\pi}(x)) \quad \text{for each } x \in I. \tag{5}$$

Furthermore, the unique measures associated with $F_V(x)$ and $V(I) - F_V(\hat{\pi}(x))$, respectively, are elements of the core.

Proof. Since a convex game must be monotonic, we know that F_V is well-defined. We first prove $\gamma_{F_V} \in C_V$. It is clear that for each S, $\gamma_{F_V}(S) = \int_S f' d\lambda$. Thus, $\gamma_{F_V} \in C_V$ if and only if

$$\int_S f' d\lambda \geq f(\lambda(S)) \quad \text{for all } \ S \in \Sigma. \tag{6}$$

We observe that the collection of all subsets in Σ satisfying inequality (6) is closed under finite disjoint union. Now suppose that $S = (x,y]$ for some $0 \leq x \leq y \leq 1$. Then

$$\begin{aligned} \int_x^y f' d\lambda &= f \circ F_\lambda(y) - f \circ F_\lambda(x) \\ &\geq f(F_\lambda(y) - F_\lambda(x)) \quad \text{(by convexity and } \ f(0) = 0) \\ &= f \circ \lambda((x,y]), \end{aligned}$$

which is exactly inequality (6). Since λ is a non-atomic measure, and f' is bounded on $[0, \lambda(I)]$, inequality (6) holds for all closed intervals contained in I as well. Hence, inequality (6) is true for any finite disjoint union of closed intervals in I. Suppose that for some $S \in \Sigma$, inequality (6) is false. Let

$$\epsilon = f \circ \lambda(S) - \int_S f' d\lambda > 0.$$

Using the facts that λ is a non-atomic measure on Σ, f' is bounded on I, and f is continuous, we can find a set T which is a finite disjoint union of closed intervals in I such that

$$\max\left\{|f\circ\lambda(S)-f\circ\lambda(T)|,\left|\int_S f'd\lambda-\int_T f'd\lambda\right|\right\}<\frac{\epsilon}{2}.$$

This is a contradiction since $f\circ\lambda(T)=\int_T f'd\lambda$. Hence we have proved that $\gamma_{F_V}\in C_V$.

By Theorem 3, $\gamma_{F_V}\circ\hat{\pi}\in C_V$. It is easy to check that its distribution function is given by $V(I)-F_V\circ\hat{\pi}$.

For the first part of the theorem, let μ be an element in C_V. By definition, $F_\mu(x)\geq F_V(x)$ for each $x\in I$. From Theorem 3, $\mu\circ\hat{\pi}\in C_V$, and hence

$$F_\mu(x)=V(I)-F_{\mu\circ\hat{\pi}}(\hat{\pi}(x))\leq V(I)-F_V(\hat{\pi}(x))\quad\text{for each } x\in I. \qquad \blacksquare$$

Definition 7. An element in the core of a game V is called a *vertex* if it cannot be written as the convex combination of two distinct elements in the core.

According to the above definition, if $V=f\circ\lambda\in CVM(1)$, then both γ_{F_V} and $\gamma_{F_V}\circ\hat{\pi}$ are vertices. The convex hull of these two elements forms part of C_V. There are also other vertices. In the example below we show a method of constructing some vertices.

Example. Consider the convex game $V=f\circ\lambda$, where $f(x)=x^2$ and λ is the Lebesgue measure. Then $\hat{\pi}(x)=1-x$ in Theorem 4. All distribution functions in the core are bounded by $y=x^2$ and $y=1-f\circ\hat{\pi}(x)=1-(1-x)^2$. Now for each $0<a\in I$, let us consider the measure-preserving map $\hat{\pi}_a$ for λ defined by

$$\hat{\pi}_a(x)=\begin{cases} a-x, & \text{if } x\in[0,a];\\ x, & \text{if } x\in(a,1].\end{cases}$$

Through $\hat{\pi}_a$, we obtain the following density function for a distribution function which we call F_a:

$$g_{F_a}(x)=\begin{cases} 2(a-x), & \text{if } x\in[0,a];\\ 2x, & \text{if } x\in(a,1].\end{cases}$$

We claim that F_a is a vertex of the core. Suppose that there are two elements F_1 and F_2 in the core such that $F_a=\alpha F_1+(1-\alpha)F_2$ for some $\alpha\in(0,1)$. Since F_1 and F_2 are bounded below by F_V, we must have $F_1(x)=F_2(x)=F_V(x)$ for all $x\geq a$ (since all distribution functions are right continuous). Since

$F_1 \in C_V$, the distribution defined by $\overline{F}_1 = F_1 \circ \hat{\pi}_a \in C_V$. Then for each $x \leq a$, we have $\overline{F}_1(x) = V(a) - F_1(\hat{\pi}_a(x))$ and hence

$$F_1(x) = V(a) - \overline{F}_1(\hat{\pi}_a(x)) \leq V(a) - F_V(\hat{\pi}_a(x)) = F_a(x).$$

Similarly, $F_2(x)$ is bounded above by F_a on the range $[0, a]$. It follows that $F_1 = F_2 = F_a$, a contradiction. Hence, F_a is a vertex of the core (c.f. Figure 1). ∎

Remark. As regards the preceding example, let us consider a finite union of disjoint intervals $E_i \subseteq I$, $i = 1, 2, \cdots, n$. On each E_i, one can define a measure-preserving map π_i by renaming the players of E_i in the "opposite" direction. Then define a measure-preserving map π on I by letting $\pi = \pi_i$ on each E_i, and π be the identity map elsewhere. It is clear that the distribution function with density function $f \circ \pi$ is a vertex. ∎

For each $V = f \circ \lambda$, and each $0 < y \in I$, we can define a subgame V_y on $[0, y]$ by $V_y(S) = V(S)$ for all $S \subseteq [0, y]$ and $S \in \Sigma$. Then, we can easily prove that any element in C_{V_y} can be extended to some element in C_{V_x} whenever $y \leq x$. The ideas of the proof are clear from Example 1 and the proof of Theorem 4. We now state another theorem in which we change the underlying measure rather than the underlying subspace $[0, y]$.

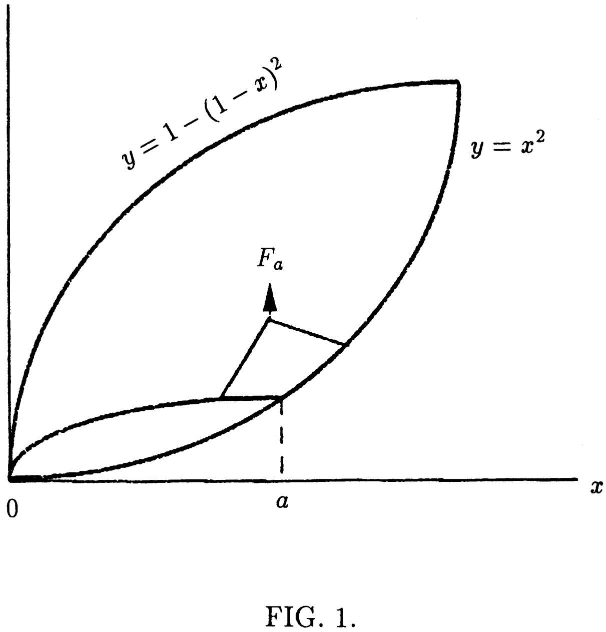

FIG. 1.

Theorem 5. (Core Expansion) *Let* $V_1 = f \circ \lambda$, *and* $V_2 = f \circ \lambda_2$ *be two convex games in* $CVM(1)$. *Assume that* F_{λ_1} *and* F_{λ_2} *are both strictly increasing on* I, *and the function* $h = F_{\lambda_1}^{-1} \circ F_{\lambda_2}$ *is convex with* $\lambda_1(I) \leq \lambda_2(I)$. *Then, there exists a one-one map* $\Phi : C_{V_1} \to C_{V_2}$.

Proof. Let b be the unique point in I satisfying $h(b) = 1$. Take a distribution $F_1 \in C_{V_1}$. Define the following distribution function:

$$F_2(x) = \begin{cases} F_1(h(x)) & \text{if } \ x \leq b; \\ F_V(x) & \text{if } \ x > b. \end{cases}$$

We claim that F_2 is a distribution in C_{V_2}. As in the proof of Theorem 4, observing that the collection of all Borel measurable sets satisfying

$$\gamma_{F_2}(S) \geq V_2(S) \quad \text{for each} \quad S \in \Sigma$$

is closed under finite disjoint union, it suffices to prove that for each $0 < x \leq y \leq 1$,

$$F_2(y) - F_2(x) \geq f(F_{\lambda_2}(y) - F_{\lambda_2}(x)). \tag{7}$$

Inequality (7) is true from Theorem 4 if both $x, y > b$. Now consider the case when both $x, y \leq b$. In this case,

$$\begin{aligned} F_2(y) - F_2(x) &= F_1(h(y)) - F_1(h(x)) \geq f \circ F_{\lambda_1}(h(y) - h(x)) \\ &\geq f \circ F_{\lambda_2}(y - x), \end{aligned}$$

where the last inequality follows from the convexity of h, and the fact that F_{λ_1}, F_{λ_2} and f are increasing. To complete the proof, note that when $x \leq b < y$, $[x, y]$ can be written as the disjoint union of two intervals $[x, b]$, and $(b, y]$. The map Φ, which maps $F_1 \in C_{V_1}$ to $F_2 \in C_{V_2}$ is clearly one-one. ∎

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