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Note on Convex Functions

Author(s): Hewitt Kenyon

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$U$ , and hence is bounded above on a translate  $V$  of  $U$  which includes the origin. But then  $\tau^{-1}$  is bounded on the set  $V \cap (-V)$ , which is a neighborhood of the origin. Since  $\tau^{-1}$  is rationally homogeneous, it follows that  $\tau^{-1}$  is continuous at the origin, and hence by additivity at each point of  $R^2$ . Since, however,  $\tau^{-1}$  maps  $R^2$  biuniquely onto  $R$ , it cannot be continuous, and the contradiction completes the proof.

#### References

1. Reinhold Baer, *Linear Algebra and Projective Geometry*, New York, 1952.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1934.
3. E. M. Wright, An inequality for convex functions, this MONTHLY, vol. 61, 1954, pp. 620-622.

#### NOTE ON CONVEX FUNCTIONS

HEWITT KENYON, University of Rochester

E. M. Wright in an interesting note [1] on a convex inequality states that it is unknown whether functions exist which satisfy condition (i) below and not condition (ii). It is not difficult to construct such a function, making use of a Hamel basis. (This makes use of the axiom of choice. See [2] or [3].) The conditions are as follows:

$$(i) \quad f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} \quad \text{for real } a \text{ and } b.$$

$$(ii) \quad \text{If } a \leq b \text{ and } \delta > 0 \text{ then } f(a+\delta) - f(a) \leq f(b+\delta) - f(b).$$

Let  $H$  be a Hamel basis for the real numbers over the rationals. Suppose without loss of generality that 1 and  $\pi$  belong to  $H$ . Then each real number  $x$  has the unique representation  $x = \sum_{h \in H} r_{x,h} \cdot h$ , where the  $r_{x,h}$  are rational numbers, only a finite number of which are not zero. Let

$$f(x) = \sum_{h \in H} r_{x,h}^2 \quad \text{for each real } x.$$

It is easy to check that (i) holds. To see that (ii) does not hold, let  $a=1$ ,  $b=\pi$ , and  $\delta=1$ . Then  $a < b$ ,  $\delta > 0$ , and  $f(a+\delta) - f(a) = 4 - 1 > 2 - 1 = f(b+\delta) - f(b)$ .

The function  $f$  may be modified so that (i) is still satisfied; and so that (ii) is satisfied for any preassigned set of values of  $\delta > 0$  of power less than the continuum, but not for all  $\delta > 0$ .

#### References

1. E. M. Wright, An inequality for convex functions, this MONTHLY, vol. 61, 1954, pp. 620-622.
2. G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung:  $f(x+y) = f(x) + f(y)$ , Math. Ann., vol. 60, 1905, pp. 459-462.
3. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1934, p. 96.