Probability

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1 Classical Probability Spaces

A *probability space* [7, 9, 19] collects necessary conditions for reasoning coherently about collections of uncertain events. It consists of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ. In this paper, we will only consider *finite* sets of events: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events $\mathcal{E}$ as 2^Ω, the powerset of Ω. Given the set of events $\mathcal{E}$, the *probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection E_i, of pairwise disjoint events, $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two-coin probability space). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rcl@{\qquad}rcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

The assignment satisfies the two constraints for probability measures: the probability of the entire sample space is 1, and the probability of every collection of disjoint events (e.g., $\{HT\} \cup \{TH\}$) is the sum of the individual probabilities.

In a strict computational or experimental setting, one may question the reliance of the definition of probability space on the uncountable and uncomputable real interval $[0, 1]$. This interval includes numbers like $0.h_1h_2h_3\ldots$ where h_i is 1 or 0 depending on whether Turing machine M_i halts or not. Such numbers cannot be computed. This interval also includes numbers like $\frac{\pi}{4}$ which can be computed but with increasingly large resources as the precision increases. Therefore it is more appropriate in such situations to consider probability measures mapping events to a finite set of elements computable with a fixed set of resources [2, 21, 16, 6]. The simplest such set, and the one we will consider exclusively in this paper, is the set $\mathscr{L}_2 = \{\text{impossible}, \text{possible}\}$ together with the operation $\vee$ where $x \vee y = \text{impossible}$ if and only if $x = y = \text{impossible}$. In relation to the first definition, one can interpret impossible as the closed interval [0], possible as the open interval $(0, 1]$, and $\vee$ as the union of intervals. The definition of a probability measure in this case is modifed as being a function $\mu : \mathcal{E} \to \mathscr{L}_2$ such that:

- $\mu(\Omega) = \text{possible}$, and

- for a collection E_i, of pairwise disjoint events, $\mu(\bigcup E_i) = \bigvee \mu(E_i)$.

Example 2 (Two-coin probability space with finite set-valued probability measure.). Under the new set-valued requirement, the probability measure in the first example becomes:

$$\begin{array}{rcl} \mu(\emptyset) & = & \text{impossible} \\ \mu(\{HH\}) & = & \text{possible} \\ \mu(\{HT\}) & = & \text{impossible} \\ \mu(\{TH\}) & = & \text{possible} \\ \mu(\{TT\}) & = & \text{impossible} \\ \mu(\{HH, HT\}) & = & \text{possible} \\ \mu(\{HH, TH\}) & = & \text{possible} \\ \mu(\{HH, TT\}) & = & \text{possible} \end{array} \qquad \begin{array}{rcl} \mu(\{HT, TH\}) & = & \text{possible} \\ \mu(\{HT, TT\}) & = & \text{impossible} \\ \mu(\{TH, TT\}) & = & \text{possible} \\ \mu(\{HH, HT, TH\}) & = & \text{possible} \\ \mu(\{HH, HT, TT\}) & = & \text{possible} \\ \mu(\{HH, TH, TT\}) & = & \text{possible} \\ \mu(\{HT, TH, TT\}) & = & \text{possible} \\ \mu(\{HH, HT, TH, TT\}) & = & \text{possible} \end{array}$$

2 Quantum Probability Spaces

The mathematical framework above assumes that one has complete knowledge of the events and their relationships. But even in many classical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot be determined with certainty. In the quantum case, this partial knowledge is compounded by the fact that not all quantum events can be observed simultaneously. Indeed, in the quantum world, there are non-commuting events which cannot even happen simultaneously. To accommodate these more complex situations, we abandon the sample space Ω and define and reason directly about events. A quantum probability space consist of just two components: a set of events $\mathcal{E}$ and a probability measure $\mu : \mathcal{E} \to [0, 1]$. We give an example before giving the formal definition.

Consider the two-qubit Hilbert space with computational basis $|0\rangle$ and $|1\rangle$ and states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}\,.$$

The set of events associated with this Hilbert space consists of all projections including the empty projection $\mathbb{0}$ and the unit projection $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$:

$$\{\mathbb{0}, |0\rangle\langle 0|, |1\rangle\langle 1|, |+\rangle\langle +|, |-\rangle\langle -|, \ldots, \mathbb{1}\}$$

Each event is interpreted as a possible post post-measurement state of a quantum system as follows: given some arbitrary current quantum state $|\psi\rangle$ to be measured, the event $|0\rangle\langle 0|$ states that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ states that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ states that the post-measurement state will be $|+\rangle$; the event $|-\rangle\langle -|$ states that the post-measurement state will be $|-\rangle$; the event $\mathbb{1}$ states that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the event $\mathbb{0}$ states that the post-measurement state will be the empty state.

Irrespective of the current state $|\psi\rangle$ and irrespective of the particular experiment, the probability of event $\mathbb{0}$ will always be 0 (it is an impossible event) and the probability of event $\mathbb{1}$ will always be 1 (it is a certain event). The probabilities attached to other events will depend on the particular state in question. If the state is $|0\rangle$, the probability of event $|0\rangle\langle 0|$ is 1; the probability of event $|1\rangle\langle 1|$ is 0; the probability of event $|+\rangle\langle +|$ is $\frac{1}{2}$; and the probability of event $|-\rangle\langle -|$ is $\frac{1}{2}$. If the state is $|+\rangle$, the probability of each event $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ will be $\frac{1}{2}$; the probability of event $|+\rangle\langle +|$ is 1; and the probability of event $|-\rangle\langle -|$ is 0.

We now formalize a *quantum probability space* as follows [4, 8, 18, 1, 13]. We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{E}$ as all *projections* on $\mathcal{H}$. Each quantum state $|\psi\rangle \in \mathcal{H} \setminus \{0\}$ induces a probability measure $\mu_\psi : \mathcal{E} \to [0, 1]$ on the space of events defined for any event $E \in \mathcal{E}$ as follows[1]:

$$\mu_\psi(E) = \langle \psi | E\psi \rangle \tag{1}$$

[1]Recently, people extend the domain of μ_ψ to all operators $\mathcal{A}$ on $\mathcal{H}$ and consider $\mu_\psi : \mathcal{A} \to \mathbb{C}$ [13, 20]. When an operator $A \in \mathcal{A}$ is Hermitian, $\mu_\psi(A)$ is the expectation value of A. We does not take this approach because we want to focus only on probability.

which is called the Born rule [5, 14].

Yu-Tsung says: One small thing is whether we should say

for each normalized state $|\psi\rangle ... \mu_\psi(E) = \langle\psi|E\psi\rangle$

as in Mermin [14] and Maassen [13] or

for each state $|\psi\rangle \in \mathcal{H}\backslash\{0\} ... \mu_\psi(E) = \frac{\langle\psi|E\psi\rangle}{\langle\psi|\psi\rangle}$

as in Abramsky [1]. Personally, I prefer Abramsky's approach, because normalizing a state in DQT is messier than in CQT, and we have already considered states whose norm is not one in our previous paper [11].

Our previous examples can be verified in terms of μ_ψ. Because $|0\rangle\langle 0|$ and $|-\rangle\langle -|$ are non-commutative, their total probabilities are not a probability of an event

$$\mu_0(|0\rangle\langle 0|) + \mu_0(|-\rangle\langle -|) \quad = \quad \frac{3}{2} > 1 \ .$$

When events are orthogonal, i.e., commutative, they can be measured together and their probability is additive:

$$\mu_0(|0\rangle\langle 0|) + \mu_0(|1\rangle\langle 1|) = \mu_0(|+\rangle\langle +|) + \mu_0(|-\rangle\langle -|) = 1 = \mu_0(\mathbb{1}) \ .$$

Similarly to the classical case, the total probability is one and the orthogonal additivity can be summarized into the following two rules:

- $\mu(\mathbb{1}) = 1$, and
- for a collection of pairwise orthogonal E_i, we have $\mu(\sum_i E_i) = \sum_i \mu(E_i)$.

These two conditions define quantum probability measures. Recall that the Born rule (1) induces a probability measure from a state. Naturally, we may ask: is every probability measure induced from a state by the Born rule? The answer is yes by Gleason's theorem when the dimension ≥ 3 and extend the domain of the Born rule to mixed state [8, 15, 18].

Yu-Tsung says: There are three ways to charactize a quantum probability:

1. The above two conditions as Gleason did.
2. As a linear functional on a * algebra.
3. As induced by a state vector by the Born rule (1).

According to equation (37) in section 4 Quantum Probability in [13], the equalivance between 1. and 2. only hold for $d \geq 3$ because of Gleason' theorem. Therefore, it seems hard to get insight for how to handle Gleason's theorem by *-algebra formulism...
The equalivance between 2. and 3. can be found in Proposition 4.1.1 in [20]. Notice that this proposition holds for every dimension... So it is not exactly Gleason's theorem, or at least not the main part of the Gleason's theorem...

2.1 Plan

In the remainder of the paper, we consider variations of quantum probability spaces motivated by computation of numerical quantities in a world with limited resources:

- Instead of the Hilbert space $\mathcal{H}$ (constructed over the uncountable and uncomputable complex numbers $\mathbb{C}$), we will consider variants constructed over finite fields [12, 11, 10].
- Instead of real-valued probability measures producing results in the uncountable and uncomputable interval $[0, 1]$, we will consider finite set-valued probablitity measures [3, 17].

We will then ask if it is possible to construct variants of quantum probability spaces under these conditions. The main question is related to the definition of probability measures: is it possible to still define a probability measure as a function that depends on a single state? Specifically,

- given a state $|\psi\rangle$, is there a probability measure mapping events to probabilities that only depends on $|\psi\rangle$? In the conventional quantum probability space, the answer is yes by the Born rule [5, 14] and the map is given by: $E \mapsto \langle\psi|E\psi\rangle$.
- given a probability measure μ mapping each event E to a probability, is there a *unique* state ψ such that $\mu(E) = \langle\psi|E\psi\rangle$? In the conventional case, the answer is yes by Gleason's theorem [8, 15, 18].

3 All Continuous or All Discrete

Before we turn to the main part of the paper, we quickly dismiss the possibility of having one but not the other of the discrete variations. Speficially, it is impossible to maintain the Hllbert space and have a finite set-valued probability measure and it is also impossible to have a vector space constructed over a finite field with a real-valued probability measure.

3.1 Hilbert Space with Finite Set-Valued Probability Measure

However, there is a $\mathscr{L}_2$-valued probability measure

$$\hat{\mu}_1(E) = \begin{cases} \text{impossible} & \text{, if } E = |+\rangle\langle+|; \\ \bar{\mu}_0(E) & \text{, otherwise.} \end{cases}$$

such that $\hat{\mu}_1 \neq \bar{\mu}_\psi$ for all mixed state $|\psi\rangle$.

3.2 Discrete Vector Space with Real-Valued Probability Measure

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