A Radon-Nikodym theorem for capacities

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1. Introduction

In many problems of mathematical statistics and geometric probability it seems natural to use capacities instead of probability measures (cf. Huber [5], Matheron [7]). But, until now, the probabilistic aspects of the theory of capacities have not been developed to a level comparable with the standards of measure theory. Professor Kölzow, referring to the central role of the Radon-Nikodym theorem in measure theory, probability theory, and mathematical statistics (cf. Halmos-Savage [3]), asked the question whether a Radon-Nikodym type theorem could be proved in the context of capacity theory.

A Radon-Nikodym type theorem for capacities, suitable for the construction of minimax tests, has been proved by Huber and Strassen [4]. They showed that, for two strongly subadditive capacities u and v on the Borel field $\mathfrak{B}(X)$ of a Polish space X, there is always a measurable function $f\colon X \to \bar{\mathbb{R}}_+$ such that

$$\alpha v(\{f \geqq \alpha\}) + u(\{f < \alpha\}) = \inf\{\alpha v(A) + u(A^c) \mid A \in \mathfrak{B}(X)\}$$

for all $\alpha \in \mathbb{R}_+$. If u and v are measures such that u is absolutely continuous with respect to v then f is the ordinary Radon-Nikodym derivative of u w.r.t. v, i.e. u is the indefinite integral fv of v. For more general capacities, however, this is not true, even if u is absolutely continuous w.r.t. v (written $u \ll v$).

It is the purpose of this note to prove a Radon-Nikodym type theorem for capacities in the classical integral form by characterizing the indefinite integrals of a given capacity v. As it turns out the relation $u = fv$ holds for some measurable f if and only if the pair (u, v) satisfies not only $u \ll v$ but, in addition, a certain Hahn-decomposition property. Contrary to the case of measures not every pair of capacities possesses this Hahn-decomposition property.

In section 2 we summarize some of the results concerning integration with respect to a capacity. We will use the Choquet integral for capacities (cf. Choquet [2]) which, for a large class of capacities, agrees with the integral in the sense of Lumer [6].

In section 3 we characterize those pairs of capacities which possess a certain Hahn-decomposition property.

In section 4 the Radon-Nikodym theorem, mentioned above, is proved.

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In section 5 we consider the problem of existence for conditional expectations of capacities. We prove that, excluding pathological cases, measures are exactly those capacities for which the conditional expectation with respect to every sub-σ-field exists. This result indicates that many of the classical consequences of the Radon-Nikodym theorem cannot be extended to any class of capacities enlarging the class of all measures significantly.

In section 6 we show, among other things, that the usual Radon-Nikodym theorem for measures generalizes to the corresponding outer measures.

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2. Preliminaries. Integrals for capacities

We shall use the term "capacity" in the sense of Mokobodzki [8].

2. 1. Definition. Let $(X, \mathfrak{A})$ be a measurable space, i.e. X is a set and $\mathfrak{A}$ a σ-field of subsets of X. A map $v\colon \mathfrak{A} \to \mathbb{R}_+$ is called a *capacity* if the following conditions hold:

(i) $v(\emptyset)=0$

(ii) $\forall\, A, B \in \mathfrak{A}\colon\ v(A \cup B) \leqq v(A)+v(B)$ (*subadditivity*)

(iii) $\forall\, A, B \in \mathfrak{A}\colon\ A \subset B \Rightarrow v(A) \leqq v(B)$ (*monotonicity*)

(iv) For every increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}$ the equality

$v(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} v(A_n)$ holds. (*continuity*)

(We will write $A_n \uparrow A \Rightarrow v(A_n) \uparrow v(A)$.)

A capacity is called *strongly subadditive* (or *2-alternating*) if

(v) $\forall\, A, B \in \mathfrak{A}\colon\ v(A \cap B)+v(A \cup B) \leqq v(A)+v(B)$.

For a capacity $v\colon \mathfrak{A} \to \mathbb{R}_+$ and a set $A \in \mathfrak{A}$ let v_A denote the restriction $v|_{\mathfrak{A} \cap A}$ of v to $\mathfrak{A} \cap A$, where $\mathfrak{A} \cap A = \{B \in \mathfrak{A} \mid B \subset A\}$. Then v_A is a capacity on A.

We denote by $\leqq$ the canonical order relation on the set of all capacities on $\mathfrak{A}$. Besides that $\leqq$ also stands for the usual ordering on $\mathbb{R}$.

It should be mentioned that conditions (i) to (iv) imply the *countable subadditivity* of v, i.e.

(vi) $v(\bigcup_{n\in\mathbb{N}} A_n) \leqq \sum_{n=1}^{\infty} v(A_n)$ for every sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

In particular, the sets of capacity zero form a σ-ideal in $\mathfrak{A}$.

Mappings on $\mathfrak{A}$ satisfying conditions (i) to (iii) are often called submeasures.

The main examples of capacities are the upper envelopes of families of measures. Let $\mathbf{M}$ be a family of finite measures on $\mathfrak{A}$ with $\sup\{\mu(X) \mid \mu \in \mathbf{M}\} < +\infty$. Then $v\colon \mathfrak{A} \to \mathbb{R}_+$ defined by $v(A) = \sup\{\mu(A) \mid \mu \in \mathbf{M}\}$ is a capacity. For instance, the classical Newtonian capacity (when restricted to a bounded region of $\mathbb{R}^n$) is a capacity of this type. In particular every finite measure is a capacity.

Choquet [2] introduced the following integral for capacities. Given a measurable function $f\colon X \to \bar{\mathbb{R}}_+ = [0, +\infty]$ let

$$\int f\,dv := \int_0^\infty v(\{f \geqq \alpha\})\,d\alpha,$$

where integration is with respect to Lebesgue measure on $\mathbb{R}$ and, for $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) \geqq \alpha\}$ is denoted by $\{f \geqq \alpha\}$. As usual we will write $\int_A f\,dv$ for $\int 1_A \cdot f\,dv$.

If v is the upper envelope of a family of measures then there exists another canonical definition of an integral due to Lumer [6], namely

$${}^{(L)}\!\!\int f\,dv := \sup\{\int f\,d\mu \mid \mu \in \mathbf{M}_v\},$$

where $\mathbf{M}_v = \{\mu \mid \mu$ measure on $\mathfrak{A}$ and $\mu \leqq v\}$.

If v is a measure then both definitions of an integral agree with the standard Lebesgue integral w.r.t. v. Our first aim is to show that the two definitions of an integral agree for a larger class of capacities on topological spaces. For upper semi-continuous (u.s.c.) integrands this fact is well-known (cf. Anger [1]). We will deduce the general case from this special case. But before we can formulate the result we need another definition.

2. 2. Definition. Let X be a Hausdorff topological space and let $\mathfrak{A}$ be the Borel field $\mathfrak{B}(X)$ of X. A capacity $v\colon \mathfrak{A} \to \mathbb{R}_+$ is called *regular* if

$$v(B) = \sup\{v(K) \mid K \subset B,\ K \text{ compact}\} = \inf\{v(U) \mid B \subset U,\ U \text{ open}\}$$

for all $B \in \mathfrak{A}$.

2. 3. Proposition. *Let X be a Hausdorff topological space and $v\colon \mathfrak{B}(X) \to \mathbb{R}_+$ a regular, strongly subadditive capacity. If $f\colon X \to \mathbb{R}_+$ is $\mathfrak{B}(X)$-measurable then*

$$\int f\,dv = {}^{(L)}\!\!\int f\,dv.$$

For the proof of the proposition the following lemma will be useful.

2. 4. Lemma. *Let $\mathfrak{G}$ be a (pointwise) filtering increasing collection of decreasing real-valued functions on $\mathbb{R}_+$. If the pointwise supremum g of $\mathfrak{G}$ is still real-valued then there exists a (pointwise) increasing sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathfrak{G}$ with*

$$\lim_{n \to \infty} g_n(x) = g(x)$$

for all $x \in \mathbb{R}_+$.

Proof. Let P be the set of all points at which g is not continuous. Since g is decreasing the set P is at most countable. Let Q be a countable dense subset of $\mathbb{R}_+ \setminus P$. Moreover let $(s_n)_{n \in \mathbb{N}}$ be an enumeration of $P \cup Q$. — For each $n \in \mathbb{N}$ there exists a (pointwise) increasing sequence $(g_{n,k})_{k \in \mathbb{N}}$ in $\mathfrak{G}$ with $\lim_{k \to \infty} g_{n,k}(s_n) = g(s_n)$. Define $g_1 := g_{1,1}$. Suppose $g_1, \ldots, g_m$ have already been defined. Then there exists a $g_{m+1} \in \mathfrak{G}$ which dominates g_m and each of the functions $g_{i,m+1}$ $(i = 1, \ldots, m+1)$. The sequence $(g_m)_{m \in \mathbb{N}}$ defined in this way is obviously (pointwise) increasing and its supremum h is decreasing and dominated by g. We will show $h = g$. Let $m \in \mathbb{N}$ be fixed. For arbitrary $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a $k_\varepsilon \in \mathbb{N}$ such that

$$g(s_m) - g_{m,k}(s_m) < \varepsilon$$

for all $k \geqq k_\varepsilon$. Hence we deduce

$$g_k(s_m) \geqq g_{m,k}(s_m) \geqq g(s_m) - \varepsilon$$

for all $k \geqq \max(k_\varepsilon, m)$. This implies

$$\lim_{k \to \infty} g_k(s_m) = g(s_m).$$

Since $m \in \mathbb{N}$ was arbitrary we obtain

$$g|_{P \cup Q} = h|_{P \cup Q}.$$

Now let $s \in \mathbb{R}_+ \setminus P$ be arbitrary. Assume $h(s) < g(s)$. Because g is continuous in s and Q is dense in $\mathbb{R}_+ \setminus P$ there exists a $t \in Q$ with $t > s$ and

$$h(s) < g(t) \leqq g(s).$$

Since $g(t) = h(t)$ this last inequality contradicts the fact that h is decreasing. Thus the proof of the lemma is completed.

Proof of Proposition 2.3. (1) Since for every measure $\mu \leqq v$ the inequality

$$\int f\,d\mu = \int_0^\infty \mu(\{f \geqq \alpha\})\,d\alpha \leqq \int_0^\infty v(\{f \geqq \alpha\})\,d\alpha = \int f\,dv$$

holds it follows that ${}^{(L)}\!\int f\,dv \leqq \int f\,dv$.

(2) To prove the converse inequality we will first show that there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of measures on $\mathfrak{A}$ dominated by v and satisfying

$$\lim_{n \to \infty} \mu_n(\{f \geqq \alpha\}) = v(\{f \geqq \alpha\})$$

for all $\alpha \in \mathbb{R}_+ = [0, +\infty[$. Since v is regular it is easy to see that

$$v(\{f \geqq \alpha\}) = \sup\{v(\{h \geqq \alpha\}) \mid 0 \leqq h \leqq f,\ h \text{ u.s.c.}\}.$$

Let $\mathfrak{H}$ be the set of all nonnegative u.s.c. real-valued functions on X which are dominated by f. For $h \in \mathfrak{H}$ define $\varphi_h: \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi_h(\alpha) = v(\{h \geqq \alpha\})$. Then $(\varphi_h)_{h \in \mathfrak{H}}$ is a filtering increasing family of decreasing real-valued functions. Moreover we have

$$v(\{f \geqq \alpha\}) = \sup\{\varphi_h(\alpha) \mid h \in \mathfrak{H}\}$$

for every $\alpha \in \mathbb{R}_+$. It follows from Lemma 2. 4 that there exists an increasing sequence $(h_n)_{n \in \mathbb{N}}$ in $\mathfrak{H}$ such that

$$\lim_{n \to \infty} \varphi_{h_n}(\alpha) = v(\{f \geqq \alpha\})$$

According to Anger [1], p. 250 (19) (cf. also Huber-Strassen [4], p. 253, Lemma 2. 4) there exists, for each $n \in \mathbb{N}$, a measure $\mu_n \leqq v$ with

$$\mu_n(\{h_n \geqq \alpha\}) = v(\{h_n \geqq \alpha\})$$

for all $\alpha \in \mathbb{R}_+$. Since

$$\mu_n(\{h_n \geqq \alpha\}) \leqq \mu_n(\{f \geqq \alpha\}) \leqq v(\{f \geqq \alpha\})$$

this implies

$$\lim_{n\to\infty} \mu_n(\{f\geqq\alpha\}) = v(\{f\geqq\alpha\})$$

for all $\alpha\in\mathbb{R}_+$. Applying the theorem on monotone convergence we, therefore, deduce ${}^{(L)}\int f\,dv \geqq \int f\,dv$.

3. Hahn-decompositions for capacities

One of the main tools in measure theory is the Hahn-decomposition of a signed measure. It can be reformulated in the following way: Given two finite (positive) measures v and μ there exists a measurable set A such that $v_A\leqq\mu_A$ and $v_{A^c}\geqq\mu_{A^c}$. In the context of capacities this last statement will turn out to be one of the necessary conditions for a Radon-Nikodym theorem to hold.

Let us make the following definition.

3. 1. Definition. Let $(X, \mathfrak{A})$ be a measurable space and let $u, v\colon \mathfrak{A}\to\mathbb{R}_+$ be capacities.

(a) The pair (u, v) is said to possess the *weak decomposition property* (W.D.P.) if, for every $\alpha\in\mathbb{R}_+$, there exists a set $A_\alpha\in\mathfrak{A}$ such that

$$\alpha v_{A_\alpha}\leqq u_{A_\alpha} \qquad \text{and} \qquad \alpha v_{A_\alpha^c}\geqq u_{A_\alpha^c}.$$

(b) The pair (u, v) is said to possess the *strong decomposition property* (S.D.P.) if, for every $\alpha\in\mathbb{R}_+$, there exists a set $A_\alpha\in\mathfrak{A}$ such that the following conditions hold:

(i) $\forall\, A, B\in\mathfrak{A}\colon B\subset A\subset A_\alpha \Rightarrow \alpha\big(v(A)-v(B)\big)\leqq u(A)-u(B)$

(ii) $\forall\, A\in\mathfrak{A}\colon \alpha\big(v(A)-v(A\cap A_\alpha)\big)\geqq u(A)-u(A\cap A_\alpha)$.

3. 2. Remarks. 1) It is obvious that for a pair of measures (u, v) the two decomposition properties are equivalent. Moreover, any two measures satisfy conditions (a) and (b).

2) The strong decomposition property always implies the weak decomposition property. To see this let A_α be as in the definition of the S.D.P. For $A\in\mathfrak{A}$ with $A\subset A_\alpha$ we obtain

$$\alpha v(A)=\alpha\big(v(A)-v(\emptyset)\big)\leqq u(A)-u(\emptyset)=u(A);$$

hence

$$\alpha v_{A_\alpha}\leqq u_{A_\alpha}.$$

For $A\in\mathfrak{A}$ with $A\subset A_\alpha^c$ we deduce

$$\alpha v(A)=\alpha\big(v(A)-v(A\cap A_\alpha)\big)\geqq u(A)-u(A\cap A_\alpha)=u(A);$$

hence

$$\alpha v_{A^c}\geqq u_{A^c}.$$

Our next aim is to characterize those pairs (u, v) of capacities on $\mathfrak{A}$ which possess the weak decomposition property. To simplify the statement of the result we will use the following definitions.

3. 3. Definition. Let $(X, \mathfrak{A})$ be a measurable space and let $u, v\colon \mathfrak{A}\to\mathbb{R}_+$ be capacities.

(a) *Domination of v over u* is said to be *stable* if, for all $A, B\in\mathfrak{A}$, $u_A\leqq v_A$ and $u_B\leqq v_B$ imply $u_{A\cup B}\leqq v_{A\cup B}$.

(b) v is said to have the *strict domination property* w.r.t. u if, for every $A \in \mathfrak{A}$ with $u(A) < v(A)$, there exists a set $A_0 \in \mathfrak{A}$ such that $A_0 \subset A$, $v(A \setminus A_0) < v(A)$, and $u_{A_0} \leqq v_{A_0}$.

To show that domination of v over u is stable it is obviously enough to prove

$$u_A \leqq v_A \quad \text{and} \quad u_B \leqq v_B \Rightarrow u(A \cup B) \leqq v(A \cup B)$$

for all $A, B \in \mathfrak{A}$.

3. 4. Proposition. *Let $(X, \mathfrak{A})$ be a measurable space and let $u, v \colon \mathfrak{A} \to \mathbb{R}_+$ be capacities. Then the following conditions are equivalent:*

(a) *(u, v) has the* W.D.P.

(b) *For every $\alpha \in \mathbb{R}_+$ domination of u over αv and of αv over u is stable; moreover, αv has the strict domination property* w.r.t. *u and vice versa.*

(c) *There exists an $\mathfrak{A}$-measurable map $f \colon X \to \bar{\mathbb{R}}_+ = [0, +\infty]$ such that*

$$\alpha v_{\{f \geqq \alpha\}} \leqq u_{\{f \geqq \alpha\}} \quad \textit{and} \quad \alpha v_{\{f < \alpha\}} \geqq u_{\{f < \alpha\}}$$

for every $\alpha \in \mathbb{R}_+$.

Proof. (a) $\Rightarrow$ (c). For each $\alpha \in \mathbb{R}_+$ let A_α be as in the definition of the W.D.P. Given $\beta \in]0, +\infty[$ define

$$E_\beta = \bigcap \{A_\alpha \mid \alpha \in \mathbb{Q}, \beta > \alpha \geqq 0\} \quad (\mathbb{Q} = \text{set of all rational numbers}).$$

To simplify notation set $E_0 = X$. Define $f \colon X \to \bar{\mathbb{R}}_+$ by

$$f(x) = \sup \{\beta \in \mathbb{R}_+ \mid x \in E_\beta\}.$$

(1) We will show $E_\beta = \{f \geqq \beta\}$. If $\beta > 0$ then, by definition, $x \in E_\beta$ implies $f(x) \geqq \beta$. Conversely, if $x \in \{f \geqq \beta\}$ then, for every $\alpha \in \mathbb{Q}$ with $0 \leqq \alpha < \beta$, there exists a $\beta' \in]\alpha, \beta[\cap \mathbb{Q}$ with $x \in E_{\beta'}$. Thus we deduce $x \in A_\alpha$. Since $\alpha \in [0, \beta[\cap \mathbb{Q}$ was arbitrary we obtain

$$x \in \bigcap \{A_\alpha \mid \alpha \in \mathbb{Q}, 0 \leqq \alpha < \beta\} = E_\beta.$$

Since $E_0 = \{f \geqq 0\}$ is obviously true we have proved our claim. Because the sets E_β are measurable we know that f is also measurable.

(2) Next we will prove that $v(A_\alpha \setminus A_\beta) = 0 = u(A_\alpha \setminus A_\beta)$ for all $\alpha, \beta \in \mathbb{R}_+$ with $\beta < \alpha$. From the definition of A_α and A_β we deduce

$$\alpha v(A_\alpha \setminus A_\beta) \leqq u(A_\alpha \setminus A_\beta) \leqq \beta v(A_\alpha \setminus A_\beta).$$

This inequality implies

$$(\beta - \alpha)\, v(A_\alpha \setminus A_\beta) = 0$$

and, therefore,

$$v(A_\alpha \setminus A_\beta) = 0.$$

Since $\beta v(A_\alpha \setminus A_\beta) \geqq u(A_\alpha \setminus A_\beta)$ this leads to $u(A_\alpha \setminus A_\beta) = 0$.

(3) We claim that $\beta v_{E_\beta} \leqq u_{E_\beta}$ for every $\beta \in \mathbb{R}_+$. If $\beta = 0$ then there is nothing to show. For $\beta > 0$ consider an arbitrary $B \in \mathfrak{A}$ with $B \subset E_\beta$. For every $\alpha \in [0, \beta[\cap \mathbb{Q}$ we have $B \subset A_\alpha$ and, therefore, $\alpha v(B) \leqq u(B)$. Since $\alpha \in [0, \beta[\cap \mathbb{Q}$ was arbitrary this implies $\beta v(B) \leqq u(B)$ and proves the claim.

(4) To complete the proof of (a) $\Rightarrow$ (c) we will show that $\beta v_{E_\beta^c} \geqq u_{E_\beta^c}$ for each $\beta \in \mathbb{R}_+$. If $\beta = 0$ this inequality is satisfied by the definition of E_0. For $\beta > 0$ let $B \in \mathfrak{A}$ with $B \cap E_\beta = \emptyset$ be arbitrary. We claim that

$$u(B) = \sup\{u(B \cap A_\alpha^c) \mid \alpha \in [0, \beta[\cap \mathbb{Q}\}.$$

Clearly the term on the right is dominated by $u(B)$. To prove the converse let $(\alpha_n)_{n \in \mathbb{N}}$ be an enumeration of $[0, \beta[\cap \mathbb{Q}$. Then

$$\left(\bigcup_{k=1}^{n} B \cap A_{\alpha_k}^c\right) \uparrow B \quad \text{implies} \quad u(B) = \lim_{n \to \infty} u\left(\bigcup_{k=1}^{n} B \cap A_{\alpha_k}^c\right).$$

For $n \in \mathbb{N}$ let α'_n denote the maximum of $\{\alpha_1, \ldots, \alpha_n\}$. Using (2) we obtain

$$\begin{aligned} u\left(\bigcup_{k=1}^{n} B \cap A_{\alpha_k}^c\right) &\leqq u\left(B \cap A_{\alpha'_n}^c \cap \bigcup_{k=1}^{n} A_{\alpha_k}^c\right) + u\left(B \cap \bigcup_{k=1}^{n} (A_{\alpha'_n} \setminus A_{\alpha_k})\right) \\ &\leqq u(B \cap A_{\alpha'_n}^c) + \sum_{k=1}^{n} u(A_{\alpha'_n} \setminus A_{\alpha_k}) = u(B \cap A_{\alpha'_n}^c). \end{aligned}$$

Hence we deduce

$$u(B) = \sup\{u(A_\alpha^c \cap B) \mid \alpha \in [0, \beta[\cap \mathbb{Q}\}.$$

By the definition of the sets A_α this implies

$$\begin{aligned} \beta v(B) &\geqq \sup\{\alpha v(B \cap A_\alpha^c) \mid \alpha \in [0, \beta[\cap \mathbb{Q}\} \\ &\geqq \sup\{u(B \cap A_\alpha^c) \mid \alpha \in [0, \beta[\cap \mathbb{Q}\} = u(B). \end{aligned}$$

Combining (1) to (4) finishes the proof of (a) $\Rightarrow$ (c).

(c) $\Rightarrow$ (b). Let $\alpha > 0$ and $A, B \in \mathfrak{A}$ such that $\alpha v_A \leqq u_A$ and $\alpha v_B \leqq u_B$. Let $0 < \beta < \alpha$. Then

$$u(A \cap \{f < \beta\}) \leqq \beta v(A \cap \{f < \beta\}) \leqq \frac{\beta}{\alpha} u(A \cap \{f < \beta\})$$

which shows that $u(A \cap \{f < \beta\}) = 0$ and hence as well $v(A \cap \{f < \beta\}) = 0$. Similarly $u(B \cap \{f < \beta\}) = 0 = v(B \cap \{f < \beta\})$.

Hence we get

$$\beta v(A \cup B) = \beta v((A \cup B) \cap \{f \geqq \beta\}) \leqq u((A \cup B) \cap \{f \geqq \beta\}) \leqq u(A \cup B)$$

for all such β. This leads to $\alpha v(A \cup B) \leqq u(A \cup B)$.

For $\alpha = 0$ the last inequality is trivially satisfied. Thus we have proved that domination of u over αv is stable. Similarly one can show that domination of αv over u is stable.

Next we will prove that u has the strict domination property w.r.t. αv. To this end let $A \in \mathfrak{A}$ be such that $\alpha v(A) < u(A)$. Define $A_0 = A \cap \{f \geqq \alpha\}$. Then, obviously, $\alpha v_{A_0} \leqq u_{A_0}$ and

$$u(A \setminus A_0) = u(A \cap \{f < \alpha\}) \leqq \alpha v(A \cap \{f < \alpha\}) \leqq \alpha v(A) < u(A).$$

This proves our claim.

In a similar way one can show that αv has the strict domination property w.r.t. u.

(b) $\Rightarrow$ (a). For $\alpha, \beta \in \mathbb{R}_+$ with $0<\alpha<\beta$ let $\mathfrak{J}=\{A \in \mathfrak{A} \mid \alpha v_A \leqq u_A\}$ and $\mathfrak{H}=\{B \in \mathfrak{A} \mid \beta v_B \geqq u_B\}$. Using the fact that domination of u over αv and of αv over u is stable it is easy to check that $\mathfrak{J}$ and $\mathfrak{H}$ are σ-ideals in $\mathfrak{A}$. This implies that there exist $A \in \mathfrak{J}$ and $B \in \mathfrak{H}$ with

$$v\big((A \cup B)^c\big)=\inf\{v\big((A' \cup B')^c\big) \mid A' \in \mathfrak{J},\, B' \in \mathfrak{H}\}$$

and
$$u\big((A \cup B)^c\big)=\inf\{u\big((A' \cup B')^c\big) \mid A' \in \mathfrak{J},\, B' \in \mathfrak{H}\}.$$

Define
$$C:=(A \cup B)^c.$$

We claim that $v(C)=0=u(C)$. — Suppose $v(C)>0$.

i) If $\alpha v(C)<u(C)$ then the strict domination property of u w.r.t. αv implies that there exists an $A' \in \mathfrak{J}$ with

$$A' \subset C \qquad \text{and} \qquad u(C \setminus A')<u(C).$$

Then we have

$$A \cup A' \in \mathfrak{J} \qquad \text{and} \qquad u\big((A \cup A' \cup B)^c\big)=u(C \setminus A')<u(C),$$

a contradiction.

ii) If $\alpha v(C) \geqq u(C)$ then $\beta v(C)>u(C)$ and the strict domination property of βv w.r.t. u implies that there is a $B' \in \mathfrak{H}$ with

$$B' \subset C \qquad \text{and} \qquad v(C \setminus B')<v(C).$$

Then we have

$$B \cup B' \in \mathfrak{H} \qquad \text{and} \qquad v\big((A \cup B \cup B')^c\big)=v(C \setminus B')<v(C),$$

a contradiction.

Thus we deduce that $v(C)=0$. — In a similar way one can show $u(C)=0$. It, therefore, follows that $\alpha v_A \leqq u_A$ and $\beta v_{A^c} \geqq u_{A^c}$.

Now let $(\beta_n)_{n \in \mathbb{N}}$ be a decreasing sequence with $\beta_n>\alpha$ and $\lim\limits_{n \to \infty} \beta_n=\alpha$. According to the preceding considerations there exists a sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{A}$ with

$$\alpha v_{A_n} \leqq u_{A_n} \qquad \text{and} \qquad \beta_n v_{A_n^c} \geqq u_{A_n^c}.$$

Define $A_\alpha:=\bigcup\limits_{n \in \mathbb{N}} A_n$. Then it is easy to deduce

$$\alpha v_{A_\alpha} \leqq u_{A_\alpha} \qquad \text{and} \qquad \alpha v_{A_\alpha^c} \geqq u_{A_\alpha^c}.$$

Thus, if we define $A_0=X$, then $(A_\alpha)_{\alpha \in \mathbb{R}_+}$ has the properties required in the definition of the W.D.P.

3. 5. Definition. Let $(X, \mathfrak{A})$ be a measurable space and let $u, v\colon \mathfrak{A} \to \mathbb{R}_+$ be capacities.

(i) If (u, v) has the W.D.P. then every $\mathfrak{A}$-measurable function $f\colon X \to \bar{\mathbb{R}}_+$ with

$$\alpha v_{\{f \geqq \alpha\}} \leqq u_{\{f \geqq \alpha\}} \quad \text{and} \quad \alpha v_{\{f<\alpha\}} \geqq u_{\{f<\alpha\}} \quad \text{for all} \quad \alpha \in \mathbb{R}_+$$

is called a *decomposition function* of (u, v).

(ii) Two $\mathfrak{A}$-measurable functions $f, g: X \to \bar{\mathbb{R}}_+$ are called *v-equivalent* if $v(\{f \neq g\}) = 0$.

(iii) u is said to be *absolutely continuous* w.r.t. v and we write $u \ll v$ if, for every $A \in \mathfrak{A}$, $v(A) = 0$ implies $u(A) = 0$.

3. 6. Proposition. *Let $(X, \mathfrak{A})$ be a measurable space and let $u, v: \mathfrak{A} \to \mathbb{R}_+$ be capacities such that (u, v) has the* W.D.P. *Then any two decomposition functions of (u, v) are u- and v-equivalent.*

Proof. Let $f, g: X \to \bar{\mathbb{R}}_+$ be decomposition functions of (u, v). For $p, q \in \mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$ with $p < q$ define $A_{p,q} = \{f < p\} \cap \{g \geqq q\}$. Then we obtain

$$\{f < g\} = \bigcup \{A_{p,q} \mid p, q \in \mathbb{Q}_+, p < q\}.$$

Because

$$q \cdot v(A_{p,q}) \leqq u(A_{p,q}) \leqq p \cdot v(A_{p,q})$$

it follows that $v(A_{p,q}) = 0$.

Since v is countably subadditive we, therefore, deduce

$$v(\{f < g\}) = 0.$$

Exchanging the role of f and g leads to

$$v(\{g < f\}) = 0.$$

Hence f and g are v-equivalent.

In the same way one can show that f and g are u-equivalent.

3. 7. Remarks. 1) Let (u, v) have the W.D.P. and let f be a decomposition function of (u, v). Then (v, u) has the W.D.P. and $\frac{1}{f}$ is a decomposition function of (v, u).

2) If u and v are measures (with $u \ll v$) then f is a decomposition function of (u, v) if and only if f is a Radon-Nikodym derivative of u w.r.t. v.

The following proposition is a weak form of a Radon-Nikodym theorem which resembles the analogous statement for measures (stated in the preceding remark).

3. 8. Proposition. *Let $(X, \mathfrak{A})$ be a measurable space and let $u, v: \mathfrak{A} \to \mathbb{R}_+$ be capacities such that (u, v) has the* W.D.P. *Moreover let $f: X \to \bar{\mathbb{R}}_+$ be a decomposition function of (u, v). Then the following conditions are equivalent:*

(i) $u \ll v$

(ii) $\forall A \in \mathfrak{A}$: $u(A) = 0 \Leftrightarrow \int_A f\, dv = 0$.

Proof. Clearly (ii) implies (i). To prove the converse let $A \in \mathfrak{A}$ be arbitrary.

(1) If $u(A) = 0$ then we deduce that, for every $\alpha \in \mathbb{R}_+$,

$$\alpha v(A \cap \{f \geqq \alpha\}) \leqq u(A) = 0.$$

This last inequality implies $v(A \cap \{f \geqq \alpha\}) = 0$ for all $\alpha \in \mathbb{R}_+^*$. This, in turn, leads to

$$\int_A f\, dv = \int_0^\infty v(A \cap \{f \geqq \alpha\})\, d\alpha = 0.$$

(2) If $\int_A f\,dv = 0$ then, by the definition of the integral,

$$v(A \cap \{f \geqq \alpha\}) = 0$$

for all $\alpha \in \mathbb{R}_+^*$. Since $u \ll v$ this implies

$$u(A \cap \{f \geqq \alpha\}) = 0$$

for all $\alpha \in \mathbb{R}_+^*$; hence

$$u(A \cap \{f > 0\}) = 0.$$

For all $\alpha \in \mathbb{R}_+^*$ we have

$$\alpha v(A \cap \{f = 0\}) \geqq u(A \cap \{f = 0\}).$$

This implies

$$u(A \cap \{f = 0\}) = 0.$$

Hence we obtain

$$u(A) = 0.$$

In our next proposition and the following examples we will study the connections between the weak and the strong decomposition property.

3. 9. Proposition. *Let $(X, \mathfrak{A})$ be a measurable space and let $u, v\colon \mathfrak{A} \to \mathbb{R}_+$ be capacities. Then (u, v) has the* S.D.P. *if and only if the following holds*:

(i) *(u, v) has the* W.D.P.

(ii) *For every $\alpha \in \mathbb{R}_+$, for every $A \in \mathfrak{A}$ with $\alpha v_A \leqq u_A$, and for every $B \in \mathfrak{A}$ with $B \subset A$ the inequality*

$$\alpha\big(v(A) - v(B)\big) \leqq u(A) - u(B)$$

is satisfied.

(iii) *For every $\alpha \in \mathbb{R}_+$, for every $A \in \mathfrak{A}$, and for every $B \in \mathfrak{A}$ with $B \subset A$, $\alpha v_B \leqq u_B$, and $\alpha v_{A \setminus B} \geqq u_{A \setminus B}$ the inequality*

$$\alpha\big(v(A) - v(B)\big) \geqq u(A) - u(B)$$

is satisfied.

Proof. "$\Rightarrow$". By Remark 3. 2. 2 we know that the S.D.P. implies the W.D.P. Let $(A_\alpha)_{\alpha \in \mathbb{R}_+}$ be as in the definition of the S.D.P.

(1) To prove that the S.D.P. implies (ii) let $\alpha \in \mathbb{R}_+$, $A \in \mathfrak{A}$ with $\alpha v_A \leqq u_A$, and $B \in \mathfrak{A}$ with $B \subset A$ be given. As in the proof of (c) $\Rightarrow$ (b) in Proposition 3. 4 one can see that

$$v(A \cap A_\beta^c) = 0 = u(A \cap A_\beta^c)$$

for all $\beta \in [0, \alpha[$. Hence it follows that

$$\beta\big(v(A) - v(B)\big) = \beta\big(v(A \cap A_\beta) - v(B \cap A_\beta)\big) \leqq u(A \cap A_\beta) - u(B \cap A_\beta) = u(A) - u(B)$$

for arbitrary $\beta \in [0, \alpha[$; which implies

$$\alpha\big(v(A) - v(B)\big) \leqq u(A) - u(B).$$

(2) Next we will show that the S.D.P. implies (iii). Let $\alpha \in \mathbb{R}_+$, $A \in \mathfrak{A}$, and $B \in \mathfrak{A}$ with $B \subset A$, $\alpha v_B \leqq u_B$, and $\alpha v_{A \setminus B} \geqq u_{A \setminus B}$ be arbitrary. For $\alpha = 0$ the condition (iii) is obviously fulfilled. So let us assume $\alpha > 0$. As before we obtain

$$v\big((A \setminus B) \cap A_\gamma\big) = 0 = u\big((A \setminus B) \cap A_\gamma\big)$$

for all $\gamma \in]\alpha, +\infty[$. This equality implies

$$v(A \cap A_\gamma) \leqq v(B \cap A_\gamma) + v\big((A \setminus B) \cap A_\gamma\big) \leqq v(B \cap A_\gamma)$$

and

$$u(A \cap A_\gamma) \leqq u(B \cap A_\gamma) + u\big((A \setminus B) \cap A_\gamma\big) \leqq u(B \cap A_\gamma);$$

hence

$$\text{(I)} \quad \gamma\big(v(A) - v(B \cap A_\gamma)\big) = \gamma\big(v(A) - v(A \cap A_\gamma)\big) \geqq u(A) - u(A \cap A_\gamma) = u(A) - u(B \cap A_\gamma)$$

for all $\gamma \in]\alpha, +\infty[$. Since $\alpha v_B \leqq u_B$ we deduce from (ii) that

$$\text{(II)} \qquad \alpha\big(v(B) - v(B \cap A_\gamma)\big) \leqq u(B) - u(B \cap A_\gamma)$$

for all $\gamma \in]\alpha, +\infty[$. Subtracting (II) multiplied by $\frac{\gamma}{\alpha}$ from (I) yields

$$\begin{aligned}\gamma\big(v(A) - v(B)\big) &\geqq u(A) - u(B \cap A_\gamma) - \frac{\gamma}{\alpha} u(B) + \frac{\gamma}{\alpha} u(B \cap A_\gamma)\\ &\geqq u(A) + \left(\frac{\gamma}{\alpha} - 1\right) u(B \cap A_\gamma) - \frac{\gamma}{\alpha} u(B) \geqq u(A) - \frac{\gamma}{\alpha} u(B)\end{aligned}$$

for all $\gamma \in]\alpha, +\infty[$.

When γ tends to α this inequality becomes

$$\alpha\big(v(A) - v(B)\big) \geqq u(A) - u(B).$$

Thus our claim is proved.

"$\Leftarrow$". Let $(A_\alpha)_{\alpha \in \mathbb{R}_+}$ be as in the definition of the W.D.P.

(1) For arbitrary $A, B \in \mathfrak{A}$ with $B \subset A \subset A_\alpha$ we deduce by means of condition (ii) that

$$\alpha\big(v(A) - v(B)\big) \leqq u(A) - u(B).$$

(2) Let $A \in \mathfrak{A}$ be arbitrary. Then we know that

$$\alpha v_{A \cap A_\alpha} \leqq u_{A \cap A_\alpha} \qquad \text{and} \qquad \alpha v_{A \setminus A_\alpha} \geqq u_{A \setminus A_\alpha}.$$

Hence condition (iii) implies

$$\alpha\big(v(A) - v(A \cap A_\alpha)\big) \geqq u(A) - u(A \cap A_\alpha).$$

Thus (u, v) has the S.D.P.

3. 10. Examples. 1) Here we give an example of a pair of capacities (u, v) such that (u, v) satisfies (ii) and (iii) in the preceding proposition but does not possess the S.D.P.

Let $X=[0,1]$ and let $\mathfrak{A}$ be the σ-field of all $A\subset[0,1]$ such that A or A^c is at most countable. Define $v\colon \mathfrak{A}\to \mathbb{R}_+$ by

$$v(A)=\begin{cases}0, & A=\emptyset\\ 1, & A\neq\emptyset\end{cases}$$

and $u\colon \mathfrak{A}\to \mathbb{R}_+$ by

$$u(A)=\begin{cases}0, & A\cap[0,\,{}^1\!/_2]=\emptyset\\ 1, & A\cap[0,\,{}^1\!/_2]\neq\emptyset.\end{cases}$$

It is easy to check that u and v are capacities on $\mathfrak{A}$ and that (u, v) satisfies conditions (ii) and (iii) in Proposition 3. 9. The pair (u, v) does not posses the W.D.P. since u does not have the strict domination property w.r.t. ${}^1\!/_2\ v$. ($1_{[0,\frac{1}{2}]}$ is no decomposition function of (u, v) because it is not $\mathfrak{A}$-measurable.) For later use let us notice that $u\ll v$.

2) Our next example shows that there are capacities u and v such that (u, v) has the W.D.P. and, moreover, satisfies condition (iii) in Proposition 3. 9 but nevertheless does not possess the S.D.P. Let $X=\{1,2\}$ and let $\mathfrak{A}$ be the power set $\mathfrak{P}(X)$ of X. Define $v\colon \mathfrak{A}\to\mathbb{R}_+$ by

$$v(A)=\begin{cases}0, & A=\emptyset\\ 1, & A=\{1\}\\ 2, & A=\{2\}\text{ or } A=X\end{cases}$$

and define $u\colon \mathfrak{A}\to\mathbb{R}_+$ by

$$u(A)=\begin{cases}0, & A=\emptyset\\ 1, & A\neq\emptyset.\end{cases}$$

Obviously u and v are capacities. It is easy to verify that $f\colon X\to\mathbb{R}_+$ defined by $f(x)=\dfrac{1}{x}$ is a decomposition function of (u, v) and that (u, v) satisfies condition (iii) in Proposition 3. 9. Since $X\subset\{f\geqq {}^1\!/_2\}$ but ${}^1\!/_2\ (v(X)-v(\{1\}))>u(X)-u(\{1\})$ the pair (u, v) does not satisfy condition (ii) in Proposition 3. 9; hence (u, v) does not possess the S.D.P. Note that again $u\ll v$.

3) Our last example shows that there exists a pair (u, v) of capacities with the W.D.P. which also satisfies condition (ii) in Proposition 3. 9 but which does not possess the S.D.P.

Let X, $\mathfrak{A}$, and v be defined as in the preceding example. Define $u\colon \mathfrak{A}\to\mathbb{R}_+$ by

$$u(A)=\begin{cases}0, & A=\emptyset\\ 1, & A=\{1\}\text{ or } A=\{2\}\\ 2, & A=X.\end{cases}$$

Then u is a measure on $\mathfrak{A}$. It is easy to check that $f\colon X\to\mathbb{R}_+$ defined by $f(x)=\dfrac{1}{x}$ is a decomposition function for (u, v). A direct computation shows that (u, v) satisfies condition

(ii) in Proposition 3. 9. For $\alpha = {}^3/_4$, $A = X$, and $B = \{1\}$ we have

$$\alpha v_B \leqq u_B \qquad \text{and} \qquad \alpha v_{A \setminus B} \geqq u_{A \setminus B}.$$

But since $\alpha(v(A) - v(B)) < u(A) - u(B)$ condition (iii) in Proposition 3. 9 is not satisfied.

Note that again $u \ll v$.

4. A Radon-Nikodym theorem for capacities

Some proofs of the usual Radon-Nikodym theorem for measures use the Hahn-decomposition of a measure. For capacities we will proceed in a similar way. — Before we can state our main result we need one more definition.

4. 1. Definition. Let $(X, \mathfrak{A})$ be a measurable space and let $u, v: \mathfrak{A} \to \mathbb{R}_+$ be capacities. We say that u is an *indefinite integral* of v if there exists a measurable function $f: X \to \mathbb{R}_+$ with $u(A) = \int_A f\,dv$ for all $A \in \mathfrak{A}$. In this case we write $u = fv$.

4. 2. Remark. Let $f: X \to \mathbb{R}_+$ be a measurable function with $\int_X f\,dv < +\infty$. Then $A \mapsto \int_A f\,dv$ defines a capacity on $\mathfrak{A}$ which is absolutely continuous w.r.t. v.

4. 3. Theorem. *Let $(X, \mathfrak{A})$ be a measurable space and let $u, v: \mathfrak{A} \to \mathbb{R}_+$ be capacities. Then u is an indefinite integral of v if and only if (u, v) has the* S.D.P. *and $u \ll v$.*

Proof. "$\Rightarrow$". Let $f: X \to \mathbb{R}_+$ be measurable with $u = fv$. Then u is obviously absolutely continuous w.r.t. v. Let $\alpha \in \mathbb{R}_+$ and $A, B \in \mathfrak{A}$ with $B \subset A \subset \{f \geqq \alpha\}$ be given. Since $u = fv$ we have

$$\begin{aligned} u(A) - u(B) &= \int_A f\,dv - \int_B f\,dv = \int_0^\infty [v(A \cap \{f \geqq t\}) - v(B \cap \{f \geqq t\})]\,dt \\ &= \int_0^\alpha (v(A) - v(B))\,dt + \int_\alpha^\infty [v(A \cap \{f \geqq t\}) - v(B \cap \{f \geqq t\})]\,dt \geqq \alpha(v(A) - v(B)). \end{aligned}$$

Now let $\alpha \in \mathbb{R}_+$ and $A \in \mathfrak{A}$ be arbitrary. Then we deduce

$$\begin{aligned} u(A) - u(A \cap \{f \geqq \alpha\}) &= \int_0^\infty [v(A \cap \{f \geqq t\}) - v(A \cap \{f \geqq \alpha\} \cap \{f \geqq t\})]\,dt \\ &= \int_0^\alpha [v(A \cap \{f \geqq t\}) - v(A \cap \{f \geqq \alpha\})]\,dt \leqq \alpha(v(A) - v(A \cap \{f \geqq \alpha\})). \end{aligned}$$

Thus we have shown that (u, v) has the S.D.P.

"$\Leftarrow$". Let (u, v) have the S.D.P. and let u be absolutely continuous w.r.t. v. Let $\tilde{f}: X \to \bar{\mathbb{R}}_+$ be a decomposition function of (u, v).

(1) We claim that $v(\{\tilde{f} = +\infty\}) = 0$.

Since, for every $n \in \mathbb{N}$, $n v_{\{\tilde{f} = \infty\}} \leqq u_{\{\tilde{f} = \infty\}}$ the claim follows immediately from $u(\{\tilde{f} = +\infty\}) < \infty$.

(2) Now define $f\colon X \to \mathbb{R}_+$ by

$$f(x)=\begin{cases}\tilde{f}(x), & \tilde{f}(x)<\infty\\ 0\quad, & \tilde{f}(x)=\infty.\end{cases}$$

Then f is measurable and we will show that f is again a decomposition function of (u,v). For $\alpha \in \mathbb{R}_+$ and $B \subset \{f \geqq \alpha\}$ we also have $B \subset \{\tilde{f} \geqq \alpha\}$. Thus we obtain

$$\alpha v(B) \leqq u(B).$$

For $\alpha \in \mathbb{R}_+$ and $B \in \mathfrak{A}$ with $B \subset \{f<\alpha\}$ the inclusion $B \subset \{\tilde{f}<\alpha\} \cup \{\tilde{f}=\infty\}$ holds; hence

$$\alpha v(B)=\alpha v(B \cap \{\tilde{f}<\alpha\}) \geqq u(B \cap \{\tilde{f}<\alpha\}).$$

Since $u \ll v$ and, therefore, $u(\{\tilde{f}=\infty\})=0$ we get

$$u(B \cap \{\tilde{f}<\alpha\})=u(B).$$

(3) Next we will prove $u=fv$.

a) Let $A \in \mathfrak{A}$ be such that there exists an $a \in \mathbb{R}_+$ with $A \subset \{f<a\}$.

Let $0=\alpha_0<\cdots<\alpha_n=a$ be arbitrary. Then we have

$$\begin{aligned}\sum_{i=1}^{n} (\alpha_i-\alpha_{i-1})\, v(A \cap \{f \geqq \alpha_i\}) &= \sum_{i=1}^{n} \alpha_i v(A \cap \{f \geqq \alpha_i\}) - \sum_{i=1}^{n} \alpha_{i-1} v(A \cap \{f \geqq \alpha_i\})\\ &= \sum_{i=1}^{n-1} \alpha_i [v(A \cap \{f \geqq \alpha_i\}) - v(A \cap \{f \geqq \alpha_{i+1}\})].\end{aligned}$$

Since $\alpha_i v_{A \cap \{f \geqq \alpha_i\}} \leqq u_{A \cap \{f \geqq \alpha_i\}}$ and $A \cap \{f \geqq \alpha_{i+1}\} \subset A \cap \{f \geqq \alpha_i\}$ it follows from Proposition 3.9 that

$$\alpha_i [v(A \cap \{f \geqq \alpha_i\}) - v(A \cap \{f \geqq \alpha_{i+1}\})] \leqq u(A \cap \{f \geqq \alpha_i\}) - u(A \cap \{f \geqq \alpha_{i+1}\}).$$

Hence we obtain

$$\begin{aligned}\sum_{i=1}^{n} (\alpha_i-\alpha_{i-1})\, v(A \cap \{f \geqq \alpha_i\}) &\leqq \sum_{i=1}^{n-1} [u(A \cap \{f \geqq \alpha_i\}) - u(A \cap \{f \geqq \alpha_{i+1}\})]\\ &= u(A \cap \{f \geqq \alpha_1\}) - u(A \cap \{f \geqq \alpha_n\}) = u(A \cap \{f \geqq \alpha_1\})\\ &\leqq u(A).\end{aligned}$$

Because the decreasing function $\alpha \mapsto v(A \cap \{f \geqq \alpha\})$ is Riemann-integrable we get

$$\int_A f\, dv = \int_0^a v(A \cap \{f \geqq \alpha\})\, d\alpha \leqq u(A).$$

b) Let $A \in \mathfrak{A}$ be arbitrary. Then we know that, for every $\alpha \in \mathbb{R}_+$,

$$v(A \cap \{f \geqq \alpha\}) = \lim_{m \to \infty} v(A \cap \{f<m\} \cap \{f \geqq \alpha\}).$$

By the theorem of monotone convergence for the Lebesgue integral this implies

$$\begin{aligned}\int_A f\, dv &= \int_0^\infty v(A \cap \{f \geqq \alpha\})\, d\alpha\\ &= \lim_{m \to \infty} \int_0^\infty v(A \cap \{f<m\} \cap \{f \geqq \alpha\})\, d\alpha \leqq \lim_{m \to \infty} u(A \cap \{f<m\}) \leqq u(A).\end{aligned}$$

c) Let $A \in \mathfrak{A}$ be such that there exists an $a \in \mathbb{R}_+$ with $A \subset \{f<a\}$. Let $0=\alpha_0<\cdots<\alpha_{n-1}=a<\alpha_n$ be arbitrary. Then, using the summation-by-parts-formula, we get

$$\sum_{i=1}^{n} (\alpha_i-\alpha_{i-1})\, v(A \cap \{f \geqq \alpha_{i-1}\}) = \sum_{i=1}^{n-1} \alpha_i [v(A \cap \{f \geqq \alpha_{i-1}\}) - v(A \cap \{f \geqq \alpha_i\})].$$

Since $A \cap \{f \geqq \alpha_i\} \subset A \cap \{f \geqq \alpha_{i-1}\}$, $\alpha_i v_{A \cap \{f \geqq \alpha_i\}} \leqq u_{A \cap \{f \geqq \alpha_i\}}$, and $\alpha_i v_{A \cap \{f<\alpha_i\}} \geqq u_{A \cap \{f<\alpha_i\}}$ it follows from Proposition 3. 9 that

$$\alpha_i [v(A \cap \{f \geqq \alpha_{i-1}\}) - v(A \cap \{f \geqq \alpha_i\})] \geqq u(A \cap \{f \geqq \alpha_{i-1}\}) - u(A \cap \{f \geqq \alpha_i\});$$

hence

$$\begin{aligned}\sum_{i=1}^{n} (\alpha_i-\alpha_{i-1})\, v(A \cap \{f \geqq \alpha_{i-1}\}) &\geqq \sum_{i=1}^{n-1} [u(A \cap \{f \geqq \alpha_{i-1}\}) - u(A \cap \{f \geqq \alpha_i\})] \\ &= u(A) - u(A \cap \{f \geqq \alpha_{n-1}\}) = u(A).\end{aligned}$$

Again the Riemann-integrability of $\alpha \mapsto v(A \cap \{f \geqq \alpha\})$ implies

$$\int_A f\, dv = \int_0^\infty v(A \cap \{f \geqq \alpha\})\, d\alpha \geqq u(A).$$

d) For $A \in \mathfrak{A}$ arbitrary we have

$$\int_A f\, dv \geqq \int_{A \cap \{f<m\}} f\, dv \geqq u(A \cap \{f<m\})$$

for all $m \in \mathbb{N}$. Because of

$$\lim_{m \to \infty} u(A \cap \{f<m\}) = u(A)$$

we obtain

$$\int_A f\, dv \geqq u(A).$$

Thus the proof of the theorem is completed.

4. 4. Remarks. 1) The proof of the preceding theorem shows that, if (u, v) has the S.D.P., then $u=fv$ holds for every decomposition function f of (u, v). Conversely, if $u=fv$ then f is a decomposition function of (u, v).

2) If $u=fv$ and $u=gv$ then f and g are v-equivalent. — This follows immediately from Proposition 3. 6.

3) Examples 3. 10, 1)—3) show that in Theorem 4. 3 the assumption of (u, v) having the S.D.P. can essentially not be weakened.

4) Huber-Strassen [4] showed that, given two regular strongly subadditive capacities u and v on the Borel field $\mathfrak{B}(X)$ of a Polish space X, there is always a measurable function $f\colon X \to \mathbb{R}_+$ with

$$v(\{f \geqq \alpha\}) + u(\{f<\alpha\}) = \inf\{v(A) + u(A^c) \mid A \in \mathfrak{B}(X)\}$$

for all $\alpha \in \mathbb{R}_+$. For measures u and v this function agrees with the ordinary Radon-Nikodym derivative. The following example shows that, in general, $u \neq fv$ even if u is an indefinite integral of v. Let $X = [0, 1]$ and define $v: \mathfrak{B}(X) \to \mathbb{R}_+$ by

$$v(A) = \begin{cases} 0, & A = \emptyset \\ 1, & A \neq \emptyset \end{cases}$$

and $u = \mathrm{id}_X\, v$.

Then we obtain

$$u(A) = \begin{cases} 0, & A = \emptyset \\ \sup A, & A \neq \emptyset. \end{cases}$$

Let $h: X \to \mathbb{R}_+$ be any measurable function with $u = hv$. Then, according to Remark 2 above, $h = \mathrm{id}_X$. This implies

$$\alpha v(\{h \geqq \alpha\}) + u(\{h < \alpha\}) = \begin{cases} 2\alpha, & 0 \leqq \alpha \leqq 1 \\ 1, & 1 < \alpha \end{cases}.$$

Since

$$\inf \{\alpha v(A) + u(A^c) \mid A \in \mathfrak{B}(X)\} = \begin{cases} \alpha, & 0 \leqq \alpha < 1 \\ 1, & 1 \leqq \alpha \end{cases}$$

we deduce $u \neq fv$.

5) If (u, v) has the S.D.P. and f is a decomposition function for (u, v) then the following condition is satisfied:

$$\alpha v(\{f \geqq \alpha\}) - u(\{f \geqq \alpha\}) = \inf \{\alpha v(A) - u(A) \mid A \in \mathfrak{A}\}$$

for all $\alpha \in \mathbb{R}_+$. For measures u and v this condition agrees with that of Huber-Strassen. It would be interesting to have a characterization of those pairs (u, v) of capacities for which a function satisfying the above condition exists.

5. Characterizations of measures among capacities

5. 1. Definition. Let $(X, \mathfrak{A})$ be a measurable space, $\mathfrak{B}$ a sub-σ-field of $\mathfrak{A}$, and $v: \mathfrak{A} \to \mathbb{R}_+$ a capacity.

a) We say that the *conditional expectation* of v w.r.t. $\mathfrak{B}$ exists if, for every bounded $\mathfrak{A}$-measurable function $f: X \to \mathbb{R}_+$, there is a $\mathfrak{B}$-measurable function $g: X \to \mathbb{R}_+$ with

$$\int_B f\, dv = \int_B g\, dv$$

for all $B \in \mathfrak{B}$.

b) By $\mathfrak{A}/v$ we denote the quotient of the σ-field $\mathfrak{A}$ w.r.t. the σ-ideal of all $A \in \mathfrak{A}$ with $v(A) = 0$.

The following theorem shows that the property of having a conditional expectation w.r.t. every sub-σ-field essentially characterizes measures among capacities.

5. 2. Theorem. *Let* $(X, \mathfrak{A})$ *be a measurable space and let* $v: \mathfrak{A} \to \mathbb{R}_+$ *be a capacity such that* card $\mathfrak{A}/v > 4$ *and* card $v(\mathfrak{A}) > 2$. *Then the following conditions are equivalent*:

(i) v *is a measure.*

(ii) *For every sub-σ-field* $\mathfrak{B}$ *of* $\mathfrak{A}$ *the conditional expectation of* v w.r.t. $\mathfrak{B}$ *exists.*

Proof. The implication (i) ⇒ (ii) is a well-known consequence of the Radon-Nikodym theorem for measures.

To prove (ii) ⇒ (i) it is enough to show $v(A_1 \cup A_2) = v(A_1) + v(A_2)$ for all disjoint $A_1, A_2 \in \mathfrak{A}$. That this is so follows directly from the continuity property of capacities. If $v(A_1) = 0$ or $v(A_2) = 0$ then the above identity is obviously satisfied. Thus we may assume $v(A_1) > 0$ and $v(A_2) > 0$.

a) First let us consider the case that $v(A_1^c \setminus A_2) > 0$. Define $A_3 = A_1^c \setminus A_2$. Let $\mathfrak{B}$ be the σ-field generated by A_1, i.e. $\mathfrak{B} = \{\emptyset, A_1, A_1^c, X\}$, and let $f: X \to \mathbb{R}_+$ be an arbitrary bounded $\mathfrak{A}$-measurable function. According to our assumptions there exists a $\mathfrak{B}$-measurable function $g: X \to \mathbb{R}_+$ such that

$$\int_A f\,dv = \int_B g\,dv \tag{1}$$

for all $B \in \mathfrak{B}$. Since g is $\mathfrak{B}$-measurable we have

$$g = a_1 1_{A_1} + a_2 1_{A_1^c}$$

for suitable $a_1, a_2 \in \mathbb{R}_+$. Hence we deduce

$$\int_{A_1} g\,dv = a_1 v(A_1), \qquad \int_{A_1^c} g\,dv = a_2 v(A_1^c),$$

and

$$\int_X g\,dv = \begin{cases} a_2(v(X) - v(A_1)) + a_1 v(A_1), & a_1 \geqq a_2 \\ a_1(v(X) - v(A_1^c)) + a_2 v(A_1^c), & a_1 < a_2. \end{cases}$$

Using (1) we obtain

$$a_1 = \frac{1}{v(A_1)} \int_{A_1} f\,dv, \tag{2}$$

$$a_2 = \frac{1}{v(A_1^c)} \int_{A_1^c} f\,dv, \tag{3}$$

and

$$\int_X f\,dv = \begin{cases} a_2(v(X) - v(A_1)) + a_1 v(A_1), & a_1 \geqq a_2 \\ a_1(v(X) - v(A_1^c)) + a_2 v(A_1^c), & a_1 < a_2. \end{cases} \tag{4}$$

If f has the special form

$$f = 1_{A_1} + c\,1_{A_j} \qquad (c \in \mathbb{R}_+,\ j \in \{2, 3\})$$

then we get

$$a_1 = 1, \quad a_2 = c\,\frac{v(A_j)}{v(A_1^c)}, \tag{5}$$

and

$$\int_X f\,dv = \begin{cases} cv(A_1 \cup A_j) + (1-c)\,v(A_1), & c \leqq 1 \\ v(A_1 \cup A_j) + (c-1)\,v(A_j), & c > 1. \end{cases} \tag{6}$$

Using (4) and (5) we obtain

$$\int_X f\,dv = \begin{cases} c\,\dfrac{v(A_j)}{v(A_1^c)}\,(v(X) - v(A_1)) + v(A_1), & 1 \geqq c\,\dfrac{v(A_j)}{v(A_1^c)} \\ v(X) - v(A_1^c) + cv(A_j) \qquad , & 1 < c\,\dfrac{v(A_j)}{v(A_1^c)}. \end{cases} \tag{7}$$

Combining (6) and (7) for suitable c yields

$$v(A_1 \cup A_j) - v(A_1) = \frac{v(A_j)}{v(A_1^c)}\,(v(X) - v(A_1)) \tag{8}$$

and

$$v(A_1 \cup A_j) - v(A_j) = v(X) - v(A_1^c). \tag{9}$$

From (8) we deduce

$$v(A_1 \cup A_2) - v(A_1 \cup A_3) = \frac{v(X) - v(A_1)}{v(A_1^c)}\,(v(A_2) - v(A_3)). \tag{10}$$

Similarly (9) implies

$$v(A_1 \cup A_2) - v(A_1 \cup A_3) = v(A_2) - v(A_3). \tag{11}$$

If $v(A_2) \neq v(A_3)$, then by combining (10) and (11) we obtain

$$v(A_1^c) = v(X) - v(A_1). \tag{12}$$

Hence (8) yields

$$v(A_1 \cup A_j) = v(A_1) + v(A_j) \qquad (j = 2, 3). \tag{13}$$

If $v(A_2) = v(A_3)$, then we deduce from (11) that

$$v(A_1 \cup A_2) = v(A_1 \cup A_3). \tag{14}$$

Exchanging the role of A_1 and A_2 in the above considerations leads to

$$v(A_2^c) = v(X) - v(A_2) \tag{15}$$

and

$$v(A_2 \cup A_j) = v(A_2) + v(A_j) \qquad (j = 1, 3) \tag{16}$$

if $v(A_1) \neq v(A_3)$, and to

$$v(A_1 \cup A_2) = v(A_2 \cup A_3) \tag{17}$$

if $v(A_1) = v(A_3)$.

If $v(A_1) \neq v(A_3)$ and $v(A_2) \neq v(A_3)$, then combining (12), (13), (15), and (16) yields

$$v(A_i^c) = v(X) - v(A_i) \qquad (i = 1, 2, 3) \tag{18}$$

and

$$v(A_i \cup A_j) = v(A_i) + v(A_j) \qquad (i \neq j;\ i, j = 1, 2, 3). \tag{19}$$

If $v(A_1)=v(A_3)$ and $v(A_2)\neq v(A_3)$, then combining (12), (13), and (17) leads again to (18) and (19).

Similarly one can deduce (18) and (19) if $v(A_1)\neq v(A_3)$ and $v(A_2)=v(A_3)$.

If $v(A_1)=v(A_2)=v(A_3)=:a$ then (14) and (17) imply

$$v(A_1^c)=v(A_2^c)=v(A_3^c)=:x.$$

Let $y:=v(X)$. From (9) and (8) respectively it follows that

$$x-a=y-x \quad \text{and} \quad x-a=\frac{a}{x}(y-a).$$

Resolving this system of equations yields two pairs of possible solutions, namely,

$$(x,y)=(a,a) \quad \text{or} \quad (x,y)=(2a,3a).$$

If $(x,y)=(2a,3a)$ then again (18) and (19) are true.

If $(x,y)=(a,a)$ then we obtain

$$v(A_1)=v(A_2)=v(A_3)=v(X). \tag{20}$$

Assume there exists an $i\in\{1,2,3\}$ and a set $B\in\mathfrak{A}$ with $B\subset A_i$ and $0<v(B)<v(A_i)$. Define $C=(A_i\setminus B)\cup A_j$ with $j\neq i$. Then we have $v(C)=v(X)$. Since $v(B)\neq v(A_k)$ $(k\in\{i,j\})$ we obtain from our considerations above that

$$v(X)\geqq v(B\cup C)=v(B)+v(C)>v(X),$$

a contradiction.

For $A\in\mathfrak{A}$ with $v(A)>0$ there exists an $i\in\{1,2,3\}$ with $v(A\cap A_i)>0$. This implies $v(A\cap A_i)=v(A_i)$ and, therefore, $v(A)=v(X)$.

Thus (20) leads to card $v(\mathfrak{A})\leqq 2$, which contradicts our assumptions. Hence we have shown

$$v(A_1\cup A_2)=v(A_1)+v(A_2)$$

under the additional assumption of $v(A_1^c\setminus A_2)>0$.

b) Let $v(A_1^c\setminus A_2)=0$. Since we have assumed that card $\mathfrak{A}/v>4$ there exists an $A_3\in\mathfrak{A}$ such that A_3 is not v-equivalent to any of the sets $\emptyset$, A_1, A_2, and X. W.l.o.g. we may assume $A_3\subset A_1$, $0<v(A_3)$ and $v(A_1\setminus A_3)>0$. Define $A_1'=A_1\setminus A_3$. According to our considerations under a) applied to A_1', A_2, A_3 we have

$$v(A_2)+v(A_2^c)=v(X)$$

and, therefore,

$$v(A_1\cup A_2)=v(A_1)+v(A_2).$$

This completes the proof of the theorem.

5. 3. Remark. The condition card $\mathfrak{A}/v>4$ means that $\mathfrak{A}$ is not generated, modulo v, by a single element. It, therefore, excludes a rather trivial case.

There are examples which show that the assumptions in the above theorem cannot be dropped.

5. 4. Proposition. *Let $(X, \mathfrak{A})$ be a measurable space and let $v: \mathfrak{A} \to \mathbb{R}_+$ be a capacity such that $v(A) = \sup\{\nu(A) \mid \nu$ measure, $\nu \leqq v\}$ for all $A \in \mathfrak{A}$. Then the following conditions are equivalent:*

(i) *v is a measure.*

(ii) *For every finite measure μ with $\mu \ll v$ the pair (μ, v) has the* W.D.P.

(iii) *Every finite measure μ with $\mu \ll v$ is an indefinite integral of v.*

Proof. (i) $\Rightarrow$ (iii) follows from the usual Radon-Nikodym theorem for measures.

(iii) $\Rightarrow$ (ii) is a consequence of Theorem 4. 3 combined with Proposition 3. 9.

(ii) $\Rightarrow$ (i). It is enough to show $v(A_1 \cup A_2) \geqq v(A_1) + v(A_2)$ for disjoint $A_1, A_2 \in \mathfrak{A}$. Assume $v(A_1) + v(A_2) > v(A_1 \cup A_2)$. Then there exist measures $\mu_1, \mu_2 \leqq v$ with $\mu_i(A_i^c) = 0$ $(i = 1, 2)$ and

$$\mu_1(A_1) + \mu_2(A_2) > v(A_1 \cup A_2).$$

Since $\mu_1 + \mu_2 \ll v$ it follows from (ii) that $(\mu_1 + \mu_2, v)$ has the W.D.P. Therefore, according to Proposition 3. 4, domination of v over $\mu_1 + \mu_2$ is stable. Hence

$$(\mu_1)_{A_1} = (\mu_1 + \mu_2)_{A_1} \leqq v_{A_1} \quad \text{and} \quad (\mu_2)_{A_2} = (\mu_1 + \mu_2)_{A_2} \leqq v_{A_2}$$

imply $(\mu_1 + \mu_2)(A_1 \cup A_2) \leqq v(A_1 \cup A_2)$. This is a contradiction since

$$(\mu_1 + \mu_2)(A_1 \cup A_2) = \mu_1(A_1) + \mu_2(A_2).$$

6. Some applications

As a first application we characterize the regular capacities which preserve "unions". A part of the following result (namely (i) $\Leftrightarrow$ (iii)) can be found in Choquet [2].

6. 1. Proposition. *Let X be a Hausdorff topological space, $\mathfrak{A}$ the Borel field of X, and $v: \mathfrak{A} \to \mathbb{R}_+$ the capacity defined by*

$$v(A) = \begin{cases} 0, & A = \emptyset \\ 1, & A \neq \emptyset. \end{cases}$$

Moreover, let $u: \mathfrak{A} \to \mathbb{R}_+$ be an arbitrary capacity. Then the following conditions are equivalent:

(i) *u is regular and $u(A \cup B) = \max(u(A), u(B))$ for all $A, B \in \mathfrak{A}$.*

(ii) *There exists an upper semi-continuous function $f: X \to \mathbb{R}_+$ with $u = fv$.*

(iii) *There exists an upper semi-continuous function $f: X \to \mathbb{R}_+$ such that $u(A) = \sup f(A)$ for all $A \in \mathfrak{A}$.*

Proof. (i) $\Rightarrow$ (ii). Define $f: X \to \mathbb{R}_+$ by $f(x) = u(\{x\})$. For $\alpha \in \mathbb{R}_+$ and $x \in X$ with $f(x) < \alpha$ there exists an open neighborhood U of x with $u(U) < \alpha$; hence $f(x') < \alpha$ for all $x' \in U$. Thus we have proved that f is upper semi-continuous.

We claim that f is a decomposition function for (u, v). To see this let $\alpha \in \mathbb{R}_+$ be arbitrary. Then, for every $B \in \mathfrak{A} \setminus \{\emptyset\}$ with $B \subset \{f \geqq \alpha\}$, we obtain

$$\alpha v(B) = \alpha \leqq u(\{x\}) \leqq u(B),$$

where $x \in B$ is arbitrary.

Now let $B \in \mathfrak{A} \setminus \{\emptyset\}$ with $B \subset \{f < \alpha\}$ be arbitrary. To show $u(B) \leqq \alpha v(B)$ it is enough to prove $u(K) \leqq \alpha v(B)$ for every compact subset K of B (regularity of u). Let $K \subset B$ be an arbitrary compact set. Then the regularity of u implies that, for each $x \in K$, there exists an open neighborhood U_x of x with $u(U_x) < \alpha$. Since K is compact, a finite number of the sets U_x, say $U_{x_1}, \ldots, U_{x_n}$, cover K; hence we obtain

$$u(K) \leqq u(U_{x_1} \cup \cdots \cup U_{x_n}) = \max\big(u(U_{x_1}), \ldots, u(U_{x_n})\big) < \alpha = \alpha v(B).$$

Thus our claim is proved.

Next we will show that (u, v) has the S.D.P. It is easy to check that

$$\forall\, A, B \in \mathfrak{A}\colon B \subset A \subset \{f \geqq \alpha\} \Rightarrow \alpha\big(v(A) - v(B)\big) \leqq u(A) - u(B).$$

For an arbitrary $A \in \mathfrak{A}$ we have

$$u(A) = \max\big(u(A \cap \{f \geqq \alpha\}), u(A \cap \{f < \alpha\})\big).$$

Since $u(A \cap \{f < \alpha\}) \leqq \alpha$ we deduce that $u(A) = u(A \cap \{f \geqq \alpha\})$ if $A \cap \{f \geqq \alpha\} \neq \emptyset$; hence

$$\alpha\big(v(A) - v(A \cap \{f \geqq \alpha\})\big) \geqq u(A) - u(A \cap \{f \geqq \alpha\}).$$

Therefore it follows from Theorem 4. 3 and Remark 4. 4. 1 that $u = fv$.

(ii) $\Rightarrow$ (iii). Let $f\colon X \to \mathbb{R}_+$ be upper semi-continuous and such that $u = fv$. For $A \in \mathfrak{A}$ we obtain

$$u(A) = \int_0^\infty v(A \cap \{f \geqq \alpha\})\, d\alpha = \sup\{\alpha \in \mathbb{R}_+ \mid A \cap \{f \geqq \alpha\} \neq \emptyset\} = \sup f(A).$$

(iii) $\Rightarrow$ (i). If u is of the form described in (iii), then u clearly satisfies

$$u(A \cup B) = \max\big(u(A), u(B)\big)$$

for all $A, B \in \mathfrak{A}$. Let $A \in \mathfrak{A}$ be arbitrary. Then it is obvious that

$$u(A) = \sup\{u(K) \mid K \subset A,\ K \text{ compact}\}.$$

Thus, to prove the regularity of u, it remains to show

$$u(A) = \inf\{u(U) \mid A \subset U,\ U \text{ open}\}.$$

Since for each $\alpha > u(A)$ the set $U = \{f < \alpha\}$ is open, contains A, and satisfies $u(U) \leqq \alpha$ this last requirement is also fulfilled.

The following rather technical result shows that the Radon-Nikodym theorem in its classical form can be extended to certain capacities which are closely related to measures.

6. 2. Proposition. *Let $(X, \mathfrak{A})$ be a measurable space, $\mathfrak{B}$ a Boolean σ-algebra, and $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ a mapping with the following properties:*

(a) $\Phi(\emptyset) = 0$, $\Phi(X) = 1$.

(b) $\Phi(\bigcup_{n \in \mathbb{N}} A_n) = \bigcup_{n \in \mathbb{N}} \Phi(A_n)$ *for each sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{A}$.*

(c) *For every $A \in \mathfrak{A}$ the set $\{\Phi(D) \mid D \in \mathfrak{A},\ D \subset A\}$ is a sub-σ-algebra of $\mathfrak{B}_{\Phi(A)} = \{B \in \mathfrak{B} \mid B \subset \Phi(A)\}$.*

(d) *If $A_1, A_2, A \in \mathfrak{A}$ are such that $\Phi(A_1) \cap \Phi(A_2) = 0$ and $\Phi(A_1) \cup \Phi(A_2) = \Phi(A)$, then there exist disjoint $A'_1, A'_2 \in \mathfrak{A}$ with $A'_1 \cup A'_2 = A$ and $\Phi(A'_i) = \Phi(A_i)$ $(i = 1, 2)$.*

Moreover let ν and μ be finite measures on $\mathfrak{B}$. Then, for the capacities $u = \mu \circ \Phi$ and $v = \nu \circ \Phi$, the following statements are equivalent:

(i) $u \ll v$.

(ii) *u is an indefinite integral of v.*

Proof. Since (ii) $\Rightarrow$ (i) is always true it remains to prove (i) $\Rightarrow$ (ii). Our first step towards the proof of this latter implication is to show that (u, v) has the W.D.P. To see this let $\alpha \in \mathbb{R}_+$ and $A, B \in \mathfrak{A}$ be given. Then

$$\alpha v(A \cup B) = \alpha\big(\nu(\Phi(A) \setminus \Phi(B)) + \nu(\Phi(B))\big).$$

According to assumption (c) there exists a $C \in \mathfrak{A}$ with $C \subset A$ and $\Phi(C) = \Phi(A) \setminus \Phi(B)$. If $\alpha v_A \leqq u_A$ and $\alpha v_B \leqq u_B$ we, therefore, obtain

$$\alpha v(A \cup B) = \alpha\big(v(C) + v(B)\big) \leqq u(C) + u(B) = \mu\big(\Phi(A) \setminus \Phi(B)\big) + \mu\big(\Phi(B)\big) = u(A \cup B).$$

Similarly we obtain

$$u(A \cup B) \leqq \alpha v(A \cup B)$$

if $u_A \leqq \alpha v_A$ and $u_B \leqq \alpha v_B$.

Thus we have shown that domination of u over αv and of αv over u is stable.

If $\alpha v(A) < u(A)$ and, therefore, $\alpha \nu\big(\Phi(A)\big) < \mu\big(\Phi(A)\big)$, then the fact that $\{\Phi(D) \mid D \in \mathfrak{A},\, D \subset A\}$ is a sub-σ-algebra of $\mathfrak{B}_{\Phi(A)}$ implies the existence of an $A'_0 \in \mathfrak{A}$ with the following properties: $A'_0 \subset A$, $\mu\big(\Phi(A'_0)\big) > 0$, and $\alpha \nu\big(\Phi(D)\big) \leqq \mu\big(\Phi(D)\big)$ for all $D \in \mathfrak{A}$ with $D \subset A'_0$. According to assumption (d) there exist $A_0, A_1 \in \mathfrak{A}$ such that $A_0 \cap A_1 = \emptyset$, $A_0 \cup A_1 = A$, $\Phi(A_0) = \Phi(A'_0)$, and $\Phi(A_1) = \Phi(A) \setminus \Phi(A'_0)$. It is easy to check that $\alpha v_{A_0} \leqq u_{A_0}$ and $u(A \setminus A_0) < u(A)$. Thus u has the strict domination property w.r.t. αv. — Similarly one can see that αv has the strict domination property w.r.t. u.

According to Proposition 3. 9 combined with Theorem 4. 3 we have shown that u is an indefinite integral of v if we can show that (u, v) satisfies conditions (ii) and (iii) in Proposition 3. 9. Let $\alpha \in \mathbb{R}_+$ and $A \in \mathfrak{A}$ be given. If $\alpha v_A \leqq u_A$ then, using assumption (c), it is easy to check that $\alpha\big(v(A) - v(B)\big) \leqq u(A) - u(B)$ for all $B \in \mathfrak{A}$ with $B \subset A$. Now let $B \in \mathfrak{A}$ with $B \subset A$ satisfy $\alpha v_B \leqq u_B$ and $\alpha v_{A \setminus B} \geqq u_{A \setminus B}$. According to assumption (c) there exists a $C \in \mathfrak{A}$ with $C \subset A$ and $\Phi(C) = \Phi(A) \setminus \Phi(B)$. Then

$$\Phi(C \cap B) \subset \Phi(C) \cap \Phi(B) = \emptyset$$

and hence

$$v(C \cap B) = 0 = u(C \cap B).$$

Therefore

$$\alpha\big(v(A) - v(B)\big) = \alpha v(C) = \alpha v(C \setminus B) \geqq u(C \setminus B) = u(C) = u(A) - u(B).$$

Thus we have shown that (u, v) satisfies conditions (ii) and (iii) in Proposition 3. 9 and the proof of the proposition is completed.

6. 3. Corollary. *Let $(X, \mathfrak{A})$ be a measurable space, let $\nu, \mu: \mathfrak{A} \to \mathbb{R}_+$ be finite measures, and let ν^* and μ^* be the outer measures on $\mathfrak{P}(X)$ corresponding to ν and μ respectively. Then ν^* and μ^* are capacities and the following statements are equivalent:*

(i) $\nu \ll \mu$.

(ii) $\nu^* \ll \mu^*$.

(iii) *There exists an $\mathfrak{A}$-measurable map $f: X \to \mathbb{R}_+$ with $\nu^* = \int f \, d\mu^*$.*

Proof. It is easy to check that ν^* and μ^* are capacities. Moreover, (iii) $\Rightarrow$ (ii) and (i) $\Leftrightarrow$ (ii) are obviously true. To show (ii) $\Rightarrow$ (iii) consider the map $\Phi: \mathfrak{P}(X) \to \mathfrak{A}/\mu$ which assigns to an $A \subset X$ the equivalence class of any of its measurable covers and apply Proposition 6. 2.

Remark. A direct proof of the preceding corollary is also not very hard.

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