# *The Wave Mechanics of α-Ray Tracks.*

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The present note is suggested by a recent paper by Prof. Darwin,* and is intended to show how one of the most typically particle-like properties of matter can be derived from the wave mechanics. In the theory of radioactive disintegration, as presented by Gamow, the α-particle is represented by a spherical wave which slowly leaks out of the nucleus. On the other hand, the α-particle, once emerged, has particle-like properties, the most striking being the ray tracks that it forms in a Wilson cloud chamber. It is a little difficult to picture how it is that an outgoing spherical wave can produce a straight track; we think intuitively that it should ionise atoms at random throughout space. We could consider that Gamow's outgoing spherical wave should give the probability of disintegration, but that, when the particle is outside the nucleus, it should be represented by a wave packet moving in a definite direction, so as to produce a straight track. But it ought not to be necessary to do this. The wave mechanics unaided ought to be able to predict the possible results of any observation that we could make on a system, without invoking, until the moment at which the observation is made, the classical particle-like properties of the electrons or α-particles forming that system. If we consider the α-ray alone as the system under consideration, then the gas of the Wilson chamber must be considered as the means by which we observe the particle; so in this case we must consider the α-ray as a particle as soon as it is outside the nucleus, because that is the moment at which the observation is made. If, however, we consider the α-particle and the gas together as one system, then it is ionised atoms that we observe; interpreting the wave function should give us simply the probability that such and such an atom is ionised. Until this final interpretation is made, no mention should be made of the α-ray being a particle at all.

The difficulty that we have in picturing how it is that a spherical wave can produce a straight track arises from our tendency to picture the wave as existing in ordinary three dimensional space, whereas we are really dealing with wave functions in the multispace formed by the co-ordinates both of the α-particle and of every atom in the Wilson chamber.

* 'Roy. Soc. Proc.,' A, vol. 124, p. 375 (1929).

For our purpose it will be sufficient to consider only two atoms ; for simplicity we shall suppose that they are hydrogen atoms. The position of the nuclei of the atoms we shall treat as parameters ; this is legitimate, since the nuclei are many times heavier than the electrons, and move very much more slowly than the $\alpha$-particles ; therefore, during the whole time of formation of a track, they may be considered effectively at rest.* We shall then show that the atoms cannot both be ionised unless they lie in a straight line with the radioactive nucleus.

Let $\psi_J(\mathbf{r})$ be the wave function of an excited hydrogen atom, referred to axes that pass through its nucleus. We shall denote by $\psi_0(\mathbf{r})$ the wave function corresponding to the normal state. We shall take axes such that the nucleus of the radioactive atom lies at the origin, and the two hydrogen atoms at the points $\mathbf{a}_1$, $\mathbf{a}_2$. Then the wave functions of the two hydrogen atoms are, in these co-ordinates

$$\Psi^{\mathrm{I}}(\mathbf{r}) = \psi(\mathbf{r} - \mathbf{a}_1)$$
$$\Psi^{\mathrm{II}}(\mathbf{r}) = \psi(\mathbf{r} - \mathbf{a}_2).$$

Let $\mathbf{R}$ be the co-ordinate of the $\alpha$-particle and $\mathbf{r}_1$, $\mathbf{r}_2$ the co-ordinates of the two electrons. Let $F(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2)\, e^{iEt/h}$ be a periodic wave function of the $\alpha$-particle and of the two atomic electrons. We can expand F in a series of wave functions of the two atoms, of the form

$$F(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) = \sum_{J_1 J_2} f_{J_1 J_2}(\mathbf{R})\ \Psi_{J_1}^{\mathrm{I}}(\mathbf{r}_1)\ \Psi_{J_2}^{\mathrm{II}}(\mathbf{r}_2). \qquad (1)$$

We can now see what form the wave function must have, in order that we shall obtain straight tracks. Interpreting our wave function, we see that $|f_{J_1 J_2}(\mathbf{R})|^2\, dV$ is the probability that we shall find the $\alpha$-particle in the volume element $dV$, and at the same time the first atom in the excited (or ionised) state $J_1$ and the second in the state $J_2$. To obtain a consistent theory of the straight tracks, we must have $f_{00}(\mathbf{R})$ representing an outgoing spherical wave, at any rate for $|\mathbf{R}|$ less than either $|\mathbf{a}_1|$ or $|\mathbf{a}_2|$. $f_{J,0}(\mathbf{R})$ will represent the probable positions of an $\alpha$-particle that has excited the first atom, but not the second. It should therefore be independent of $\mathbf{a}_2$, and should represent a

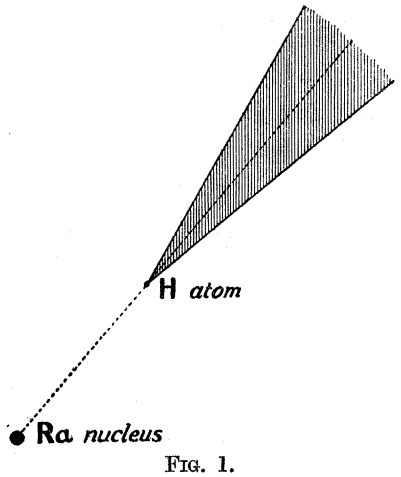

FIG. 1.

* We do not consider the possibility of a collision between the $\alpha$-particle and the nucleus.

wave diverging from the point $\mathbf{a}_1$, whose amplitude vanishes except inside a small cone, pointing away from the origin. Finally, $f_{J_1J_2}(\mathbf{R})$ will give the probability that the α-particle excites both atoms.

$f_{J_1J_2}(\mathbf{R})$ should vanish, therefore, unless the line joining $\mathbf{a}_1$ and $\mathbf{a}_2$ passes near the origin. We shall obtain a solution with these properties.

The wave equation is

$$\left\{\frac{h^2}{2M}\nabla_R^2+\frac{h^2}{2m}(\nabla_1^2+\nabla_2^2)+E+\frac{\varepsilon^2}{|\mathbf{r}_1-\mathbf{a}_1|}+\frac{\varepsilon^2}{|\mathbf{r}_2-\mathbf{a}_2|}-\frac{\varepsilon^2}{|\mathbf{R}-\mathbf{r}_1|}-\frac{\varepsilon^2}{|\mathbf{R}-\mathbf{r}_2|}\right\}F=0. \quad (2)$$

where $\nabla_R^2$ refers to the co-ordinates of the α-particle, and $\nabla_1^2$, $\nabla_2^2$ to the co-ordinates of the electrons. We have left out the terms giving the interaction between the α-particle and the nuclei of the atoms, which would in fact produce deviations from the straight track, and are irrelevant to our purpose. We treat the interaction of the atoms and the α-particle as a perturbation, and solve by a method of successive approximations, similar to that used by Born* in his work on collisions. We set

$$F=F^{(0)}+F^{(1)}+F^{(2)}+\ldots$$

with

$$F^{(0)}=e^{ik|\mathbf{R}|}/|\mathbf{R}|\,.\,\Psi_0^{\mathrm{I}}(\mathbf{r}_1)\,\Psi_0^{\mathrm{II}}(\mathbf{r}_2),$$

where

$$k=\sqrt{2M(E-2E_0)}/h,$$

representing an outgoing wave for the α-particle, and the normal states of both atoms. Then F is a solution of (2) if

$$\left\{\frac{h^2}{2M}\nabla_R^2+\frac{h^2}{2m}(\nabla_1^2+\nabla_2^2)+E+\frac{\varepsilon^2}{|\mathbf{r}_1-\mathbf{a}_1|}+\frac{\varepsilon^2}{|\mathbf{r}_2-\mathbf{a}_2|}\right\}F^{(n)}=\left\{\frac{2\varepsilon^2}{|\mathbf{R}-\mathbf{r}_1|}+\frac{2\varepsilon^2}{|\mathbf{R}-\mathbf{r}_2|}\right\}F^{(n-1)}. \quad (3)$$

Let us solve first for $F^{(1)}$. We expand $F^{(1)}$ as a series of the form (1), namely

$$F^{(1)}(\mathbf{R},\mathbf{r}_1,\mathbf{r}_2)=\sum_{J_1J_2}f^{(1)}_{J_1J_2}(\mathbf{R})\,\Psi^{\mathrm{I}}(\mathbf{r}_1)\,\Psi^{\mathrm{II}}(\mathbf{r}_2). \quad (4)$$

If we insert (4) in (3), multiply by $\Psi_{J_1}^{\mathrm{I}}(\mathbf{r}_1)\,\Psi_{J_2}^{\mathrm{II}}(\mathbf{r}_2)$ and integrate over all $\mathbf{r}_1\,\mathbf{r}_2$, we obtain a differential equation satisfied by $f^{(1)}_{J_1J_2}(\mathbf{R})$, namely

$$\left\{\frac{h^2}{2M}\nabla^2+E-E_{J_1}-E_{J_2}\right\}f^{(1)}_{J_1J_2}(\mathbf{R})=K(\mathbf{R}), \quad (5)$$

* 'Z. Physik,' vol. 38, p. 803 (1926).

where

$$\begin{aligned} \mathrm{K}(\mathbf{R}) &= \mathrm{V}_{0\mathrm{J}}(\mathbf{R}-\mathbf{a}_1)\, e^{ik|\mathbf{R}|}/|\mathbf{R}| && \text{if } \mathrm{J}_1=\mathrm{J},\ \mathrm{J}_2=0 \\ &= \mathrm{V}_{0\mathrm{J}}(\mathbf{R}-\mathbf{a}_2)\, e^{ik|\mathbf{R}|}/|\mathbf{R}| && \text{if } \mathrm{J}_2=\mathrm{J},\ \mathrm{J}_1=0 \\ &= 0. && \text{otherwise.} \end{aligned}$$

In these formulæ

$$\mathrm{V}_{0\mathrm{J}}(\mathbf{R}) = \int 2e^2/|\mathbf{R}-\mathbf{r}| \,.\, \psi_0(\mathbf{r})\, \psi_{\mathrm{J}}(\mathbf{r})\, d\mathbf{r}.$$

Now, the most general solution of (5) is*

$$f^{(1)}_{\mathrm{J}_1\mathrm{J}_2}(\mathbf{R}) = \frac{1}{4\pi}\int \frac{2\mathrm{M}}{h^2}\mathrm{K}(\mathbf{R}')\,.\,\frac{e^{\pm ik'|\mathbf{R}-\mathbf{R}'|}}{|\mathbf{R}-\mathbf{R}'|}\,d\mathbf{R}' + \mathrm{G}(\mathbf{R}), \tag{6}$$

where

$$\sqrt{2\mathrm{M}(\mathrm{E}-\mathrm{E}_{\mathrm{J}_1}-\mathrm{E}_{\mathrm{J}_1})} = k'h$$

and $\mathrm{G}(\mathbf{R})$ is the most general solution of

$$(\nabla^2 + k'^2)\,\mathrm{G} = 0.$$

Now, it is clear that our starting conditions, namely that both atoms are in the normal state before collision, require that

$$\mathrm{G}(\mathbf{R}) = 0\,;$$

for $\mathrm{G}(\mathbf{R})$ represents streams of particles fired at already excited atoms. We see, therefore, that if neither $\mathrm{J}_1$ nor $\mathrm{J}_2$ define the normal state of an atom, $f^{(1)}_{\mathrm{J}_1\mathrm{J}_2}(\mathbf{R})$ vanishes, and therefore, to this approximation, there is *no* probability that both atoms will be excited. This is to be expected; we have treated the probability that one atom will be excited as a small quantity of the first order; the probability of both being excited will therefore be a small quantity of the second order.

If, say, $\mathrm{J}_1$ represents the normal state, then $\mathrm{K}(\mathbf{R})$ vanishes except in the neighbourhood of $\mathbf{a}_2$. Except in the neighbourhood of $\mathbf{a}_2$, therefore, $f^{(1)}_{0\mathrm{J}_2}(\mathbf{R})$ is given by the asymptotic formula

$$f^{(1)}_{0\mathrm{J}_2}(\mathbf{R}) \sim \frac{e^{ik'|\mathbf{R}-\mathbf{a}_2|}}{|\mathbf{R}-\mathbf{a}_2|}\,\Im(\mathbf{l})$$

where

$$\mathbf{l} = (\mathbf{R}-\mathbf{a}_2)/|\mathbf{R}-\mathbf{a}_2|$$

and

$$\Im(\mathbf{l}) = \frac{1}{4\pi}\int \frac{2m}{h^2}\,\mathrm{V}_{0\mathrm{J}_2}(\mathbf{R}')\, e^{-ik'(\mathbf{l}\mathbf{R}')+ik|\mathbf{R}'+\mathbf{a}_2|}\,d\mathbf{R}'. \tag{7}$$

* *Cf.* Courant-Hilbert, "Methoden der Mathematischen Physik," chap. 5, § 10.

We have taken the positive value of $ik'$ in (6), so that $f^{(1)}_{0J_2}$ shall represent a wave diverging from $\mathbf{a}_2$; $\Im(\mathbf{R}-\mathbf{a}_2/|\mathbf{R}-\mathbf{a}_2|)$ gives the amplitude of the wave in any direction.

We can easily see that $\Im(\mathbf{l})$ is very small except in the neighbourhood of

$$\mathbf{l} = \mathbf{a}_2/|\mathbf{a}_2| \tag{8}$$

that is to say, except in a small cone with its vertex at $\mathbf{a}_2$, pointing away from the radioactive nucleus.

For we can see from (5) that $V_{0J}(\mathbf{R})$ must become very small except in the neighbourhood of $\mathbf{R}=0$. The exponentials oscillate very rapidly in the region where $\mathbf{R}$ does not vanish. $\Im(\mathbf{l})$, therefore, will have a strong maximum when the two exponential terms are in phase at the origin. Since $k-k'$ is negligible compared to $k$, this will be the case when (8) is satisfied.

It is interesting to work out the function $\Im(\mathbf{l})$ for a particular case. Born* has calculated the function $V_{0J}$ for certain simple cases, and also the functions $\Im(\mathbf{l})$ using a plane wave instead of our spherical wave. His results are applicable to our case, however, since the integrand in (7) vanishes except in a small region, in which the spherical wave may be considered plane. We have then†

$$\Sigma\,|\,\Im(\mathbf{l})\,|^2 = 0{\cdot}9866.\quad a_H{}^2\,\frac{M^2}{m^2}\,\frac{k'}{k}\,\frac{1}{z^2(1+z^2)^5}$$

where $a_H$ is the radius of the normal orbit of the hydrogen atom, and

$$z = \tfrac{2}{3}a_H\,|\,k\mathbf{l} - k^1\mathbf{a}_2/|\,\mathbf{a}_2\,|\,|$$

and the summation is over all the three states with principal quantum number two. It is clear that this function is only appreciable in the neighbourhood (8), since $ka_H$ is a very large number.

In order to find the probability that *both* atoms should be excited, we shall now consider $F^{(2)}$. If we expand in a series of the form (4), as before, and insert in (3), then we obtain, analogously to (3), the following differential equation satisfied by $f^{(2)}_{J_1J_2}(\mathbf{R})$

$$\left\{\frac{\hbar^2}{2M}\nabla^2 + E - E_{J_1} - E_{J_2}\right\} f^{(2)}_{J_1J_2}(\mathbf{R})$$

$$= \iint F^{(1)}(\mathbf{R}\,\mathbf{r}_1\mathbf{r}_2)\,\Psi^*_{J_1}{}^{I}(\mathbf{r}_1)\,\Psi^*_{J_2}{}^{II}(\mathbf{r}_2)\,d\mathbf{r}_1\,d\mathbf{r}_2$$

$$= f^{(1)}_{J_1 0}(\mathbf{R})\,V_{0J_2}(\mathbf{R}-\mathbf{a}_2). \tag{9}$$

* 'Gott. Nachr.,' p. 146 (1926).

† *Loc. cit.*, equation (32).

Now, if the point $\mathbf{a}_2$ does not lie very near the straight line joining the origin to $\mathbf{a}_1$, then the right-hand side of (9) will vanish for all $\mathbf{R}$; for $f^{(1)}_{J_1 0}(\mathbf{R})$ vanishes except in the neighbourhood of this line, as we have shown, and $V_{0J_2}(\mathbf{R} - \mathbf{a}_2)$ vanishes except in the neighbourhood of $\mathbf{a}_2$. Therefore in this case, it follows, as before, that the only solution of (9) satisfying our initial conditions is

$$f^{(2)}_{J_1 J_2}(\mathbf{R}) = 0,$$

and there is therefore no probability of both atoms being excited. If, on the other hand, the line joining $\mathbf{a}_1$, $\mathbf{a}_2$ does pass through the origin, then we can obtain as before a solution representing a wave diverging from the point $\mathbf{a}_2$. The amplitude of this wave gives the probability that both atoms are excited, and that the particle is moving in a given direction after exciting both.

In conclusion, the author would like to express his thanks to Prof. Darwin, who has contributed a great deal to the course of the development of this paper.

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## *The Dissociation of Acids in Methyl and in Ethyl Alcohol.*

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A comparison of the electrical conductivities of aqueous solutions of salts and acids reveals a striking contrast between these two classes of substances as regards their electrolytic dissociation. In water almost all salts are largely dissociated into ions and, when no complications are introduced by hydrolysis, the conductivities of dilute salt solutions show remarkably good agreement with the theory of Debye and Hückel as modified by Onsager.* Acids, on the other hand, show a wide variation in their degree of dissociation and very few behave as strong electrolytes. This difference in behaviour between salts and acids is usually explained by the properties of the proton, which, unlike other cations, has no screen of electrons round the nucleus, and therefore is assumed to be incapable of existing uncombined in solution. There is much evidence direct and indirect in support of this view, and it seems probable that

* 'Phys. Z.,' vol. 27, p. 388 (1926) and vol. 28, p. 277 (1927).