On the sum of projection operators

It is known that a projection operator can be written explicitly as follows:

$$\hat{P} = \sum_{k=1}^{n} \hat{P}_k = \sum_{k=1}^{n} |k\rangle\langle k|$$

where $\{|k\rangle\ , k = 1, \dots, n\}$ are the orthonormal basis.

So it is curious to ask if a projection should always be written as the sum of projection operators to the smaller subspaces, which are orthogonal to each other. This is proved as following.

Given two projection operators $\hat{P}_1$ and $\hat{P}_2$, for the sum $\hat{P} = \hat{P}_1 + \hat{P}_2$ to be also a projection operator:

$$(\hat{P}_1 + \hat{P}_2)^2 = \hat{P}_1 + \hat{P}_2$$
$$\Rightarrow \hat{P}_1^{\ 2} + \hat{P}_2^{\ 2} + \hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1$$
$$= \hat{P}_1 + \hat{P}_2 + \hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1 = \hat{P}_1 + \hat{P}_2$$

Therefore we get

$$\hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1 = 0 \tag{1}$$

(1) left-multiplied by $\hat{P}_1$, we get:

$$\hat{P}_1\hat{P}_2 + \hat{P}_1\hat{P}_2\hat{P}_1 = 0 \tag{2}$$

(1) right-multiplied by $\hat{P}_1$, we get:

$$\hat{P}_1\hat{P}_2\hat{P}_1 + \hat{P}_2\hat{P}_1 = 0 \tag{3}$$

(2) - (3) gives:

$$\hat{P}_1\hat{P}_2 - \hat{P}_2\hat{P}_1 = 0 \tag{4}$$

(1) + (4) eventually gives:

$$\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1 = 0 \tag{5}$$

The above proved that equation (5) is the necessary condition for the sum $\hat{P} = \hat{P}_1 + \hat{P}_2$ to be also a projection operator. It is straightforward to see that (5) is the sufficient condition, too. So we conclude that:

$$(\hat{P}_1 + \hat{P}_2)^2 = \hat{P}_1 + \hat{P}_2$$
$$\iff \hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1 = 0$$

given two projection operators $\hat{P}_1$ and $\hat{P}_2$.

My question is: what is the general theory/theorem that formally addressed the above question and stated the above result?

(linear-algebra) (operator-theory)

edited Feb 13 '15 at 19:41
Michael Hardy
144k 15 133 323

asked Feb 13 '15 at 18:36
WishBeLeibniz
70 4

Out of curiosity, do you know the reason for the convention you're using of putting hats on all the projections? Is that a way to distinguish operators from other objects? – Jonas Meyer Feb 13 '15 at 19:18

I've brought this question much closer to standard MathJax usage conventions. – Michael Hardy Feb 13 '15 at 19:42

@JonasMeyer from my understanding, yes, it is a way to distinguish operators from other objects. – WishBeLeibniz Feb 13 '15 at 19:49

1 Answer

The most straightforward generalization of your result is that if $P_1, P_2, \ldots, P_n$ are self-adjoint projections on a Hilbert space, then $P_1 + P_2 + \cdots + P_n$ is a projection if and only if $P_i P_j = 0$ for $i \neq j$. In an abstract formulation of the same result, we would say that the P_i s are projections in a C*-algebra (i.e., self-adjoint idempotent elements). It is equivalent because every C*-algebra is *-isomorphic to an algebra of operators on Hilbert space.

A proof of the nontrivial implication is the subject of the question Sums of projections in a C*-algebra.

A special case, where the operators sum to the identity, is the subject of the question Orthogonal projections with $\sum P_i = I$, proving that $i \neq j \Rightarrow P_j P_i = 0$.

And in the nonselfadjoint case, the result need not be true, as seen in the question Non-orthogonal projections summing to 1 in infinite-dimensional space.

However, on a finite dimensional space, the special case where they sum to the identity is still true even without selfadjointness, as seen in the question Multiplication of two projection operator is zero.

As you have shown, by not using selfadjointness or finite-dimensionality, the $n = 2$ case depends only on idempotency.

edited Feb 13 '15 at 19:09   answered Feb 13 '15 at 18:53

Jonas Meyer
37.4k 5 115 197

Thanks for the great list of various generalization cases! For n > 2 on a Hilbert space, how should I intuitively (or geometrically) understand the self-adjointness of a projection operator: $\hat{P}^\dagger = \hat{P}$? If I think of a finite-dimensional space, all I could see is that means a symmetric matrix operation. – WishBeLeibniz Feb 13 '15 at 20:18

1 @WishBeLeibniz: In general, corresponding to an idempotent operator A on a vector space V is a decomposition of V into a direct sum of subspaces, $V = V_1 \oplus V_2$, such that each $v \in V$ has a unique representation of the form $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$, and such that $A(v) = A(v_1 + v_2) = v_1$. It is said that A is the projection onto V_1 along V_2 . The subspace V_1 is the range of A , but corresponding to a given subspace, there are many projections having the same range, with many possible V_2 s. (to be continued in next comment) – Jonas Meyer Feb 13 '15 at 20:38

1 (continued) However, if V is a Hilbert space, there is a unique *orthogonal* projection onto V_1 , and this corresponds to the decomposition $V = V_1 \oplus V_1^\perp$, i.e., where $V_2 = V_1^\perp$ is the set of vectors orthogonal to everything in V_1 . For an idempotent (and bounded) operator on Hilbert space, being self-adjoint is equivalent to being an orthogonal projection. – Jonas Meyer Feb 13 '15 at 20:41

(I should have said that V_1 is a *closed* subspace, when talking about the orthogonal projection onto V_1 in a Hilbert space.) – Jonas Meyer Feb 17 '15 at 12:29