AN INDUCTIVE METHOD FOR CONSTRUCTING MINIMAL BALANCED COLLECTIONS OF FINITE SETS*

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INTRODUCTION

In this paper we give a method for obtaining the minimal balanced collections of order $n+1$ from those of order n. Balanced and minimal balanced collections of finite sets were defined by L. S. Shapley. Shapley also found important applications of these concepts to game theory; in particular he found, using the theory of balanced sets, a necessary and a sufficient condition for the non-emptiness of the core† of a characteristic function n-person game [3]. The theory of balanced sets plays an important role in the works of H. Scarf [1,2] and is one of the subjects of Thompson's work [4].

1. DEFINITIONS

Let N be the set of the first n natural numbers. If $b = \{B_1, \ldots, B_k\}$ is a set‡ of subsets of N, the incidence matrix that corresponds to b is the matrix

$$Y = (y_{ij}), \quad i = 1, \ldots, k, \quad j = 1, \ldots, n,$$

where

$$y_{ij} = \begin{cases} 1, & j \in B_i \\ 0, & j \notin B_i \end{cases}$$

A balancing vector for b is a vector $c = (c_1, \ldots, c_k)$ that satisfies:

$$\sum_{i=1}^{k} c_i \, y_{ij} = 1, \quad j = 1, \ldots, n,$$

and

$$c_i > 0, \quad i = 1, \ldots, k.$$

We denote by $C(b)$ the set of the balancing vectors of b. b is balanced, if it has a balancing vector. b is a minimal balanced collection if it does not contain a proper subset b^* which is

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†This problem was also solved by Bondareva, Vestnik Leningrad Univ. 17, (1962) No. 13, 141-142. Shapley's result is more complete; it enables us, when we know the minimal balanced collections of order n, to decide whether the core of a given n-person game is empty or not.

‡We shall deal only with indexed sets of subsets of N.

balanced. We remark that if b is balanced then a necessary and a sufficient condition that it will be a minimal balanced collection, is that the rows of the incidence matrix are independent. So, b is a minimal balanced collection if, and only if, it has a unique balancing vector.

An extension vector for b is a vector $x = (x_1, x_2, \ldots x_{2k+1})$ that satisfies:

$$x_i = 0 \text{ or } 1,$$

and

$$x_{2i-1} \geq x_{2i}, \quad i = 1, \ldots, k.$$

With the pair (b, x) we associate a list $\bar{B}_1, \ldots, \bar{B}_{2k+1}$ of subsets of $\bar{N} = \{1, \ldots, n+1\}$, defined as follows:

$$\bar{B}_j = \begin{cases} B_{\left[\frac{j+1}{2}\right]} \cup \{n+1\}, & x_j = 1 \\ B_{\left[\frac{j+1}{2}\right]} \quad , & x_j = 0 \end{cases} \qquad j = 1, \ldots, 2k,$$

and

$$\bar{B}_{2k+1} = \begin{cases} \{n+1\}, & x_{2k+1} = 1 \\ \phi \quad , & x_{2k+1} = 0 \end{cases}.$$

The extension of the pair (b, x) is the set $\bar{b}$ of the non-empty subsets that appear in the above list.* We remark that to a set $\bar{b}$ of subsets of $\bar{N}$ there corresponds no more than one pair (b, x) where b is a set of subsets of N and x is an extension vector for b, such that $\bar{b}$ is the extension of (b, x).

For a pair (b, x), where $b = \{B_1, \ldots, B_k\}$ is a set of subsets of N and x is an extension for b, we define the sets: $S = \{i : x_{2i-1} = x_{2i},\ 1 \leq i \leq k\}$, $T = \{1, \ldots, k\} - S$, and the vector $z = (z_i)_{i \in S}$, $z_i = x_{2i-1}$. In what follows we always assume that the members of b are indexed such that if $i \in S$ and $j \in T$ then $j > i$. We denote by s the number of members of S.

2. BALANCED COLLECTIONS

In this section we describe the connections between the balanced collections of $N = \{1, 2, \ldots, n\}$ and those of $\bar{N} = \{1, 2, \ldots, n, n+1\}$.

THEOREM 1: Let $b = \{B_1, \ldots, B_k\}$ be a set of subsets of N and x an extension vector for b. $\bar{b}$, the extension of the pair (b, x) is balanced if, and only if, b is balanced and one of the following conditions is satisfied:

1.1. $x_{2k+1} = 0$, $T = \phi$ and there is a $c \in C(b)$ such that $\Sigma_S\, c_i z_i = 1$.

1.2. $x_{2k+1} = 0$, $T \neq \phi$, $S = \phi$ and there is a $c \in C(b)$ such that $\Sigma_T\, c_i > 1$.

*We assume, say, that the members of b are indexed according to their order in the list.

1.3. $x_{2k+1} = 0$, $T \neq \phi$, $S \neq \phi$ and there is a $c \in C(b)$ such that $1 > \Sigma_S c_i z_i > 1 - \Sigma_T c_i$.

1.4. $x_{2k+1} = 1$, $T = \phi$ and there is a $c \in C(b)$ such that $\Sigma_S c_i z_i < 1$.

1.5. $x_{2k+1} = 1$, $T \neq \phi$, $S = \phi$.

1.6. $x_{2k+1} = 1$, $T \neq \phi$, $S \neq \phi$ and there is a $c \in C(b)$ such that $1 > \Sigma_S c_i z_i$.

PROOF:

NECESSITY: Suppose $\bar{b}$ is balanced. Let $d \in C(\bar{b})$. Define a vector $c = (c_1, \ldots, c_k)$ by $c_i = d_i$, $i \in S$, and $c_i = d_{2i-s-1} + d_{2i-s}$, for $i \in T$. Let $\bar{Y}$ be the incidence matrix that corresponds to $\bar{b}$, and Y the incidence matrix that corresponds to b.

$$\sum_{i=1}^{k} c_i y_{ij} = \sum_{i=1}^{2k-s} d_i \bar{y}_{ij} = 1, \quad j = 1, \ldots, n,$$

so $c \in C(b)$ and b is balanced. If $x_{2k+1} = 0$ and $T = \phi$ then

$$\sum_S c_i z_i = \sum_{i=1}^{k} d_i \bar{y}_{i,n+1} = 1.$$

If $x_{2k+1} = 0$, $T \neq \phi$, and $S = \phi$, then

$$\sum_T c_i > \sum_{i=1}^{2k} d_i \bar{y}_{i,n+1} = 1.$$

If $x_{2k+1} = 0$, $T \neq \phi$ and $S \neq \phi$, then

$$1 = \sum_{i=1}^{2k-s} d_i \bar{y}_{i,n+1} = \sum_S c_i z_i + \sum_{i=s+1}^{2k-s} d_i \bar{y}_{i,n+1}.$$

Since

$$0 < \sum_{i=s+1}^{2k-s} d_i \bar{y}_{i,n+1} < \sum_T c_i,$$

we have that $1 > \Sigma_S c_i z_i > 1 - \Sigma_T c_i$. If $x_{2k+1} = 1$ and $T = \phi$, then

$$\sum_S c_i z_i < \sum_{i=1}^{k+1} d_i \bar{y}_{i,n+1} = 1.$$

If $x_{2k+1} = 1$, $T \neq \phi$, and $S \neq \phi$, then

$$\sum_S c_i z_i < \sum_{i=1}^{2k-s} d_i \bar{y}_{i,n+1} < 1.$$

So we proved that if $\bar{b}$ is balanced then b is balanced and one of the conditions 1.1 – 1.6 is satisfied.

SUFFICIENCY: Let b be balanced. If $c \in C(b)$ denote by $D(c)$ the set of the vectors $d = (d_1, \ldots, d_{2k-s})$ that satisfy $d_i = c_i$, $i \in S$, and $d_{2i-s-1} + d_{2i-s} = c_i$, $i \in T$. We remark that if $d \in D(c)$ then

$$\sum_{i=1}^{2k-s} d_i \bar{y}_{ij} = 1,$$

for $j = 1, \ldots, n$. Suppose that 1.1 is satisfied. There is a $c \in C(b)$ such that $\Sigma_S c_i z_i = 1$. Since

$$\sum_S c_i z_i = \sum_{i=1}^{k} c_i \bar{y}_{i,n+1},$$

c is a balancing vector for $\bar{b}$. If 1.2 is satisfied then $x_{2k+1} = 0$, $T \neq \phi$, $S = \phi$, and there is a $c \in C(b)$ such that $\Sigma_T c_i > 1$. Define $d \in D(c)$ by

$$d_{2i-1} = \frac{c_i}{\Sigma_T c_j}$$

and $d_{2i} = c_i - d_{2i-1}$, $i \in T$. Since

$$\sum_{i=1}^{2k} d_i \bar{y}_{i,n+1} = \sum_{i=1}^{k} d_{2i-1} = \sum_T \frac{c_j}{\Sigma_T c_i} = 1,$$

d is a balancing vector for $\bar{b}$. If $x_{2k+1} = 0$, $T \neq \phi$, $S \neq \phi$ and there is a $c \in C(b)$ such that $1 > \Sigma_S c_i z_i > 1 - \Sigma_T c_i$, define $d \in D(c)$ by $d_i = c_i$, $i \in S$, $d_{2i-s-1} = \bar{k} c_i$, and $d_{2i-s} = (1-\bar{k}) c_i$ for $i \in T$, where

$$\bar{k} = \frac{1 - \Sigma_S c_i z_i}{\Sigma_T c_i}.$$

Since

$$\sum_{i=1}^{2k-s} d_i \bar{y}_{i,n+1} = \sum_S c_i z_i + \sum_{i=s+1}^{k} d_{2i-s-1} = \sum_S c_i z_i + \bar{k} \sum_T c_i = 1,$$

d is a balancing vector for $\bar{b}$. If $x_{2k+1} = 1$, $T = \phi$, and there is a $c \in C(b)$ such that $\Sigma_S c_i z_i < 1$, define $d = (d_1, \ldots, d_{k+1})$ by $d_i = c_i$, $i \in S$, and $d_{k+1} = 1 - \Sigma_S c_i z_i$. Since

$$\sum_{i=1}^{k+1} d_i \bar{y}_{i,n+1} = \sum_S c_i z_i + d_{k+1} = 1,$$

d is a balancing vector for $\bar{b}$. If $x_{2k+1} = 1$, $T \neq \phi$, and $S = \phi$, let $c \in C(b)$. Define $d \in D(c)$ by

$$d_{2i-1} = \frac{c_i}{2\,\Sigma_T c_i}$$

and $d_{2i} = c_i - d_{2i-1}$, $i \in T$. Define also $d_{2k+1} = 1/2$ and $d^* = (d_1, \ldots, d_{2k}, d_{2k+1})$. Since

$$\sum_{i=1}^{2k+1} d_i \bar{y}_{i,n+1} = \sum_{i=1}^{k} d_{2i-1} + d_{2k+1} = 1,$$

d^* is a balancing vector for $\bar{b}$. If $x_{2k+1} = 1$, $T \neq \phi$, $S \neq \phi$ and there is a $c \in C(b)$ such that $1 > \Sigma_S c_i z_i$, define $d \in D(c)$ by $d_i = c_i$, $i \in S$, $d_{2i-s-1} = \bar{k} c_i$, and $d_{2i-s} = (1-\bar{k}) c_i$ for $i \in T$, where

$$\bar{k} = \frac{1 - \Sigma_S c_i z_i}{t\,\Sigma_T c_i}$$

and t is chosen such that $0 < \bar{k} < 1$.

Define also $d_{2k-s+1} = \frac{t-1}{t}(1 - \Sigma_S c_i z_i)$, and $d^* = (d_1, \ldots, d_{2k-s}, d_{2k-s+1})$. Since

$$\sum_{i=1}^{2k-s+1} d_i \bar{y}_{i,n+1} = \sum_S c_i z_i + \sum_{i=s+1}^{k} d_{2i-s-1} + d_{2k-s+1} = 1,$$

d^* is a balancing vector for $\bar{b}$. This completes the proof of Theorem 1.

3. MINIMAL BALANCED COLLECTIONS

In this section we describe the connections between the minimal balanced collections of $N = \{1, \ldots, n\}$ and those of $\bar{N} = \{1, \ldots, n, n+1\}$. Let $b = \{B_1, \ldots, B_k\}$ be a set of subsets of N, x an extension vector for b, and $\bar{b}$ the extension of (b, x).

LEMMA 1: If $\bar{b}$ is a minimal balanced collection then $s \geq k-1$. If also $x_{2k+1} = 1$ then $s = k$.

The proof follows from the uniqueness of the balancing vector of $\bar{b}$.

LEMMA 2: The union of two balanced collections is a balanced collection.

The proof is straightforward.

THEOREM 2: $\bar{b}$ is a minimal balanced collection if, and only if, one of the following conditions is satisfied:

2.1. $x_{2k+1} = 1$, $T = \phi$, b is a minimal balanced collection, and c, the balancing vector of b, satisfies $\Sigma_S c_i z_i < 1$.

2.2. $x_{2k+1} = 0$, $T \neq \phi$, $s = k-1$, b is a minimal balanced collection, and c, the balancing vector of b, satisfies $1 > \Sigma_S c_i z_i > 1 - \Sigma_T c_i$.

2.3. $x_{2k+1} = 0$, $T = \phi$, b is a minimal balanced collection, and c, the balancing vector of b, satisfies $\Sigma_S c_i z_i = 1$.

2.4. $x_{2k+1} = 0$, $T = \phi$, b is a union of two minimal balanced collections, the rank of Y, the incidence matrix that corresponds to b, is $k-1$, and there is a unique $c \in C(b)$ such that $\Sigma_S c_i z_i = 1$.

PROOF:

NECESSITY: Inspecting Theorem 1 and Lemma 1, we see that if $\bar{b}$ is a minimal balanced collection then b is balanced and one of the conditions 1.1, 1.3, or 1.4 is satisfied. If 1.4 is satisfied then, since $\bar{b}$ has a unique balancing vector, b also has only one balancing vector, and so it is a minimal balanced collection and 2.1 is satisfied. If 1.3 is satisfied then, again, the minimality of $\bar{b}$ implies the minimality of b, and 2.2 is satisfied. If 1.1 is satisfied and b is a minimal balanced collection, then 2.3 is satisfied. If b is not a minimal balanced collection then, since the rank of $\bar{Y}$, the incidence matrix that corresponds to $\bar{b}$, is k, the rank of Y is $k-1$. The solutions of the following system of inequalities

$$\text{(I)} \quad \begin{cases} \sum_{i=1}^{k} c_i y_{ij} = 1 & j = 1, \ldots, n \\ c_i \geqq 0 & i = 1, \ldots, k \end{cases}$$

are all of the form $c = c_0 + tc_1$, where c_0 is a balancing vector for b, t a real number and c_1 is a solution of the homogeneous system

$$\sum_{i=1}^{k} c_i y_{ij} = 0, \quad j = 1, \ldots, n.$$

So there is a non-degenerate interval $[\alpha\beta]$ that consists of all the solutions of the above system. Let $U_\alpha = \{i : \alpha_i > 0\}$ and $U_\beta = \{i : \beta_i > 0\}$. Clearly U_α and U_β are proper subsets of $\{1, \ldots, k\}$ and $U_\alpha \cup U_\beta = \{1, \ldots, k\}$. Denote $b_1 = \{B_i : i \in U_\alpha\}$ and $b_2 = \{B_i : i \in U_\beta\}$. Clearly α^*, the restriction of α to U_α, belongs to $C(b_1)$, and β^*, the restriction of β to U_β, belongs to $C(b_2)$. Since α and β are the extremal solutions of (I), b_1 and b_2 must be minimal balanced collections. $b = b_1 \cup b_2$. Since every $c \in C(b)$ that satisfies $\Sigma_S c_i z_i = 1$ belongs to $C(\bar{b})$, there is only one such a vector; so we proved that 2.4 is satisfied.

SUFFICIENCY: If one of the conditions 2.1 – 2.4 is satisfied then it follows from Theorem 1 that $\bar{b}$ is balanced. If 2.1, 2.2, or 2.3 is satisfied then the minimality of b implies the minimality of $\bar{b}$. If 2.4 is satisfied then there is a unique $c \in C(b)$ such that $\Sigma_S c_i z_i = 1$.

Since every $c \in C(\bar{b})$ satisfies $c \in C(b)$ and $\Sigma_S c_i z_i = 1$, $\bar{b}$ has only one balancing vector, and therefore it is a minimal balanced collection. This completes the proof of Theorem 2.

4. A METHOD FOR CONSTRUCTING THE MINIMAL BALANCED COLLECTIONS OF $\{1, \ldots, n, n+1\}$ FROM THOSE OF $\{1, \ldots, n\}$

In this section we describe a procedure for obtaining the minimal balanced collections, and their balancing vectors, of $\bar{N} = \{1, \ldots, n, n+1\}$ from those of $N = \{1, \ldots, n\}$. The procedure consists of two steps. The first step is:

For each minimal balanced collection $b = \{B_1, \ldots, B_k\}$ of N, with the balancing vector $c = (c_1, \ldots, c_k)$, do the following checks:

1. For each extension vector x that satisfies $x_{2k+1} = 1$ and $T = \phi$, check if $\Sigma_S c_i z_i < 1$. If the inequality holds then $\bar{b}$, the extension of (b, x), is a minimal balanced collection and $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_{k+1})$, where $\bar{c}_i = c_i$, $1 \le i \le k$, and $\bar{c}_{k+1} = 1 - \Sigma_S c_i z_i$, is the balancing vector of it.

2. For each extension vector x that satisfies $x_{2k+1} = 0$, $T \neq \phi$, and $s = k-1$, check if $1 > \Sigma_S c_i z_i > 1 - \Sigma_T c_i$. If the inequalities hold then $\bar{b}$, the extension of (b, x), is a minimal balanced collection and $\bar{c} = (\bar{c}, \ldots, \bar{c}_s, \bar{c}_{s+1}, \bar{c}_{s+2})$, where $\bar{c}_i = c_i$, $1 \le i \le s$, $\bar{c}_{s+1} = 1 - \Sigma_S c_i z_i$, and $\bar{c}_{s+2} = c_k - \bar{c}_{s+1}$, is the balancing vector of it.

3. For each extension vector x that satisfies $x_{2k+1} = 0$ and $T = \phi$ check if $\Sigma_S c_i z_i = 1$. If the equality holds then $\bar{b}$, the extension of (b, x), is a minimal balanced collection and c is the balancing vector of it.

In the second step, for each pair b_1 and b_2 of minimal balanced collections of N, with the balancing vectors c' and c'', respectively, we form the union $b = b_1 \cup b_2$. If $b = \{B_1, \ldots, B_k\}$ we denote $P = \{i : B_i \in b_1\}$ and $Q = \{i : B_i \in b_2\}$. After forming b we check if the rank of Y, the incidence matrix that corresponds to b, is $k-1$. If the result of the check is positive, then for each vector x for b, that satisfies $x_{2k+1} = 0$ and $T = \phi$, we check if

$$t = \frac{1 - \Sigma_P c_i' z_i}{\Sigma_Q c_i'' z_i - \Sigma_P c_i' z_i}$$

is well defined and if $0 < t < 1$. If $0 < t < 1$ then $\bar{b}$, the extension of (b, x), is a minimal balanced collection and $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_k)$, where $\bar{c}_i = (1-t) c_i'$, $i \in P - Q$, $\bar{c}_i = t c_i''$, $i \in Q - P$, and $\bar{c}_i = t c_i'' + (1-t) c_i'$, $i \in P \cap Q$, is the balancing vector of it.

Theorem 2 shows that by using the above procedure we find all the minimal balanced collections of $\bar{N}$.

Shapley found all the minimal balanced collections for* $n \le 6$. It should be possible to program this method for an electronic computer and continue the enumeration.

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*A list is to appear in a forthcoming Rand Memorandum by Shapley.

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