Measurement and Probability in Fuzzy Quantum Theories

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1 Introduction

This is a *theoretical* investigation of *experimental* physics using *computational* methods. All experiments and computations are processes bounded in space, time, energy, and other resources [**Jaeger2007**, **Piccinini2015**]. Yet, for centuries, the mathematical formalization of such processes has been founded on the infinitely precise real or complex numbers [**Ziegler2007**, **weihrauch2012computable**, **blum2012complexity**]. Our purpose here is to exploit the consequences of replacing infinitely-specified quantum probabilities by finite number of intervals used in interval-valued probability measures (IVPM); in particular, we show that a quantum interval-valued probability measure may not be induced by a infinitely-precise state, but it might be more and more likely to be identified to a particular state when the measurement resources increase.

Indeed, almost every description of quantum mechanics, quantum computation, or quantum experiments refers to entities such as e, π, $\sqrt{2}$, etc (see, e.g., [**Redhead1987-REDINA**, **544199**, **Mermin2007**]). From a computational perspective, such numbers do not exist in their entirety "for free [**Kent1999**, **CliftonKent2000**]." For example, the state of the art algorithms for computing the nth binary digit of π require on the order of $O\left(n \log^{O(1)}(n)\right)$ operations [**journals/moc/BaileyBP97**]. In other words, simply referring to the nth digit of π requires more and more resources as n gets larger. Taking such resource bounds into consideration is what founded computer science as a discipline and is crucial for understanding the very nature of computation and, following Feynman [**Feynman1982Simulating**], Landauer [**Landauer1996188**], and others, for understanding the very nature of physical processes.

We have been revisiting quantum mechanics, quantum information, and quantum computation from this resource-aware perspective. Our initial results in that domain showed how subtle the issues can be [**usat**, **geometry2013**, **DQT2014**]: a straightforward replacement of the complex numbers by a finite field yields a variant of quantum mechanics in which computationally hard problems like UNIQUE-SAT (which decides whether a given Boolean formula is unsatisfiable or has exactly one satisfying assignment [**Valiant198685**, **Papadimitriou1993**, **AroraBarak2009**]) can be deterministically solved in constant time. To eliminate such unrealistic theories requires delicate analysis of the structure of the Hilbert space, the process of observation, and the notion of probability teasing apart their reliance on the infinitely precise real numbers [**geometry2013**, **DQT2014**].

In this paper, we shift focus from the infinitely-specified but not directly observable quantum states, to observable measurable properties of quantum systems and their probabilities. Furthermore, we insist that our theories of measurement and probability only refer to finitely communicable evidence within feasible computational bounds. It follows that states, observations, and probabilities all become "fuzzy", i.e., specified by intervals of confidence that can only increase in precision if the available resources increase proportionally. Our notion of "fuzzy quantum mechanics" is related to existing work [**GranikCaulfield1996**, **Pykacz2013**, **SNL2009**, **Gudder2005**, **aerts1993physical**] but, as will be explained in more detail, is distinguished by its unique computational character.

We will begin by reviewing existing work that recasts classical probability spaces in a resource-aware setting and move to aim at recasting quantum probability and hence quantum measurement to a corresponding resource-aware setting. In particular, we develop a measurement framework based on quantum IVPM. Surprisingly, we found a quantum IVPM which can not be induced by a state, while Shapley proved a classical convex IVPM can always be induced by a classical "state" [**Shapley1971**, **Grabisch2016**], and Gleason proved a infinitely-specified quantum probabilities measure can always be induced by a quantum state [**gleason1957**, **Redhead1987-REDINA**, **peres1995quantum**]. Nevertheless, a quantum IVPM could still correspond to many possible states if we break it into pieces. Our examples also suggest that a quantum IVPM could be more and more likely induced by a state while the intervals become sharper, i.e., the measurement resources increase.

Since Gleason's theorem supports the idea of von Neumann to formulate quantum states as points in a projective Hilbert space, a point in a projective Hilbert space might not be the adequate imprecise quantum "state." Instead, a quantum state might be more like an interactive system suggested by Quantum Bayesianism or QBism [**Fuchs2010**, **VonBaeyer2016**, **Fuchs2012**], which would be different for different interactor. Furthermore, the validity the fundamental theorems of quantum mechanics followed up by Gleason's theorem, such as Bell [**BellBook1987**, **Redhead1987-REDINA**, **peres1995quantum**, **Jaeger2007**] and Kochen-Specker [**kochenspecker1967**, **Redhead1987-REDINA**, **peres1995quantum**, **Jaeger2007**], might need to be reassessed based on our imprecise measurement. Our research then might provide a new insight to the old debate among Meyer, Mermin, and others about whether finite precision measurement would nullify the Kochen-Specker theorem or not [**PhysRevLett.83.3751**, **Mermin1999**, **BarrettKent2004**].

2 Classical Probability

A *probability space* specifies the necessary conditions for reasoning coherently about collections of uncertain events [**Kolmogorov1950**, **Shafer1976**, **Griffiths2003**, **Swart2013**]. We review the conventional presentation of probability spaces and then discuss the computational resources needed to estimate probabilities.

2.1 Classical Probability Spaces

The conventional definition of a probability space builds upon the field of real numbers. In more detail, a probability space consists of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ mapping events in $\mathcal{E}$ to the real interval $[0, 1]$. We will only consider *finite* sets of events and restrict our attention to non-empty finite sets Ω as the sample space. The space of events $\mathcal{E}$ includes every possible subset of Ω: it is the powerset $2^\Omega = \{E \mid E \subseteq \Omega\}$. For future reference, we emphasize that events are the primary notion of interest and that the sample space is a convenient artifact that allows us to treat events as sets obeying the laws of Boolean algebra [**Boole1948**, **Redhead1987-REDINA**, **Griffiths2003**].

Definition 1 (Probability Measure). Given the set of events $\mathcal{E}$, a *probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(\Omega) = 1$,
- for every event E, $\mu(\Omega \backslash E) = 1 - \mu(E)$ where $\Omega \backslash E$ is the complement event of E, and
- for every collection $\{E_i\}_{i=1}^N$ of pairwise disjoint events, $\mu\left(\bigcup_{i=1}^N E_i\right) = \sum_{i=1}^N \mu(E_i)$.

There is some redundancy in the definition that will be useful when moving to quantum probability spaces.

Example 1 (Two-coins experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. There are 16 total events including the event $\{HH, HT\}$ that the first coin lands heads up, the event $\{HT, TH\}$ that the two coins land on opposite sides, and the event $\{HT, TH, TT\}$ that at least one coin lands tails up. Here is a possible probability measure for these events:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}$$

It is useful to think that this probability measure is completely determined by the "reality" of the two coins in question and their characteristics, in the sense that each pair of coins induces a measure, and each measure must correspond to some pair of coins. The measure above would be induced by two particular coins such that the first coin is twice as likely to land tails up than heads up and the second coin is double-headed. In a strict computational or experimental setting, one should question the reliance of the definition of probability space on the uncountable and uncomputable real interval $[0, 1]$ [**Turing˙1937**, **Ziegler2007**, **weihrauch2012computable**]. This interval includes numbers like $0.h_1h_2h_3\ldots$ where h_i is 1 or 0 depending on whether Turing machine TM_i halts or not. Such numbers cannot be computed. This interval also includes numbers like $\frac{\pi}{4}$ which can only be computed with increasingly large resources as the precision increases. Therefore, in a resource-aware computational or experimental setting, it is more appropriate to consider probability measures that map events to a set of elements computable with a fixed set of resources. We expand on this observation in the next section and then recall its formalization using interval-valued probability measures [**Weichselberger2000**, **JamisonLodwick2004**].

2.2 Measuring Probabilities: Buffon's Needle Problem

In the previous example, we assumed the probability $\mu(E)$ of each event E is known a priori. In reality, although each event is assumed to have a probability, the exact value of $\mu(E)$ may not be known. According to the *frequency interpretation of probability* (which we will revisit when moving to the quantum case) [**Venn1876**, **Hajek2012**], to determine the probability of an event, we run M independent trials which gives us an approximation of the (assumed) "true" or "real" probability. Let x_i be 1 or 0 depending on whether the event E occurs in the i-th trial or not, then $\mu(E)$ could be approximated to given accuracy $\epsilon > 0$ by the relative frequency $\frac{1}{M}\sum_{i=1}^{M} x_i$ with the probability converging to one as M goes to infinity, i.e.,

$$\forall \epsilon > 0, \lim_{M\to\infty} \mu\left(\left|\mu(E) - \frac{1}{M}\sum_{i=1}^{M} x_i\right| < \epsilon\right) = 1 \ .$$

This fact is called the law of large numbers [**Bernoulli2006**, **Kolmogorov1950**, **Uspensky1937**, **Shafer1976**, **544199**].

Let's look at a concrete example. Suppose we drop a needle of length ℓ onto a floor made of equally spaced parallel lines a distance h apart, where $\ell < h$. It is a known fact that the probability of the needle crossing a line is $\frac{2\ell}{\pi h}$ [**Buffon1777**, **DeMorgan1872**, **Hall1873**, **Uspensky1937**]. Consider an experimental setup consisting of a collection of M identical needles of length ℓ. We throw the M needles one needle at a time, and observe the number M_c of needles that cross a line, thus estimating the probability of a needle crossing a line to be $\frac{M_c}{M}$. In an actual experiment with 500 needles and the ratio $\frac{\ell}{h} = 0.75$ [**Hall1873**], it was found that 236 crossed a line so the relative frequency is 0.472 whereas the idealized mathematical probability is $0.4774\ldots$. In a larger experiment with 5000 needles and the ratio $\frac{\ell}{h} = 0.8$ [**Uspensky1937**], the relative frequency was calculated to be 0.5064 whereas the idealized mathematical probability is $0.5092\ldots$. We see that the observed probability approaches $\frac{2\ell}{\pi h}$ but only if *larger and larger resources* are expended. These resource considerations suggest that it is possible to replace the real interval $[0, 1]$ with rational numbers up to a certain precision related to the particular experiment in question. There is clearly a connection between the number of needles and the achievable precision: in the hypothetical experiment with 3 needles, it is not sensible to retain 100 digits in the expansion of $\frac{2\ell}{\pi h}$.

There is another more subtle assumption of unbounded computational power in the experiment. We are assuming that we can always determine with certainty whether a needle is crossing a line. But "lines" on the the floor have thickness, their distance apart is not exactly h, and the needles' lengths are not all absolutely equal to ℓ. These perturbations make the events "fuzzy." Thus, in an experiment with limited resources, it is not possible to talk about the idealized event that exactly M_c needles cross lines as this would require the most expensive needles built to the most precise accuracy, laser precision for drawing lines on the floor, and the most powerful microscopes to determine if a needle does cross a line. Instead we might talk about the event that $M_c - \delta$ needles evidently cross lines and $M_c + \delta'$ needles plausibly cross lines where δ and δ' are resource-dependent approximations. This fuzzy notion of events leads to probabilities being only calculable within intervals of confidence reflecting the certainty of events and their plausibility. This is indeed consistent with published experiments: in an experiment with 3204 needles and the ratio $\frac{\ell}{h} = 0.6$ [**DeMorgan1872**], 1213 needles clearly crossed a line and 11 needles were close enough to plausibly be considered as crossing the line: we would express the probability in this case as the interval $\left[\frac{1213}{3204}, \frac{1224}{3204}\right]$ expressing that we are certain that the event has probability at least $\frac{1213}{3204}$ but it is possible that it would have probability $\frac{1224}{3204}$.

2.3 Classical Interval-Valued Probability Measures

As motivated above, an event E_1 may have an interval of probability $[l_1, r_1]$. Assume that another disjoint event E_2 has an interval of probability $[l_2, r_2]$, what is the interval of probability for the event $E_1 \cup E_2$? The answer is somewhat subtle: although it is possible to use the sum of the intervals $[l_1 + l_2, r_1 + r_2]$ as the combined probability, one can find a much tighter interval if information *against* the event (i.e., information about the complement event) is also taken into consideration. Formally, for a general event E with probability $[l, r]$, the evidence that contradicts E is evidence supporting the complement of E. The complement of E must therefore have probability $[1 - r, 1 - l]$ which we abbreviate $[1, 1] - [l, r]$. Given a sample space Ω and its set of events $\mathcal{E}$, a function $\bar{\mu} : \mathcal{E} \to [0, 1]$ is a classical interval-valued probability measure if and only if $\bar{\mu}$ satisfies the following conditions [**JamisonLodwick2004**] where the last line uses $\subseteq$ to allow for tighter intervals that exploit the complement event:

- $\bar{\mu}(\emptyset) = [0, 0]$.
- $\bar{\mu}(\Omega) = [1, 1]$.
- For any event E, $\bar{\mu}(\Omega \backslash E) = [1, 1] - \bar{\mu}(E)$
- For a collection $\{E_i\}_{i=1}^{M}$ of pairwise disjoint events, we have $\bar{\mu}\left(\bigcup_{i=1}^{M} E_i\right) \subseteq \sum_{i=1}^{M} \bar{\mu}(E_i)$.

Example 2 (Two-coin experiment with interval probability)*.* We split the unit interval $[0, 1]$ in the following four closed sub-intervals: $[0, 0]$ which we call $\mathscr{impossible}$, $[0, \frac{1}{2}]$ which we call $\mathscr{unlikely}$, $[\frac{1}{2}, 1]$ which we call $\mathscr{likely}$, and $[1, 1]$ which we call $\mathscr{certain}$. Using these new values, we can modify the probability measure of Ex. 1 by mapping each numeric value to the smallest sub-interval containing it to get the following:

$$
\begin{array}{rclrcl}
\bar{\mu}(\emptyset) & = & \mathscr{impossible} & \bar{\mu}(\{HT, TH\}) & = & \mathscr{likely} \\
\bar{\mu}(\{HH\}) & = & \mathscr{unlikely} & \bar{\mu}(\{HT, TT\}) & = & \mathscr{impossible} \\
\bar{\mu}(\{HT\}) & = & \mathscr{impossible} & \bar{\mu}(\{TH, TT\}) & = & \mathscr{likely} \\
\bar{\mu}(\{TH\}) & = & \mathscr{likely} & \bar{\mu}(\{HH, HT, TH\}) & = & \mathscr{certain} \\
\bar{\mu}(\{TT\}) & = & \mathscr{impossible} & \bar{\mu}(\{HH, HT, TT\}) & = & \mathscr{unlikely} \\
\bar{\mu}(\{HH, HT\}) & = & \mathscr{unlikely} & \bar{\mu}(\{HH, TH, TT\}) & = & \mathscr{certain} \\
\bar{\mu}(\{HH, TH\}) & = & \mathscr{certain} & \bar{\mu}(\{HT, TH, TT\}) & = & \mathscr{likely} \\
\bar{\mu}(\{HH, TT\}) & = & \mathscr{unlikely} & \bar{\mu}(\{HH, HT, TH, TT\}) & = & \mathscr{certain}
\end{array}
$$

Despite the absence of infinitely precise numeric information, the probability measure is quite informative: it reveals that the second coin is double-headed and that the first coin is biased. To understand the $\subseteq$-condition, consider the following calculation:

$$\begin{aligned}\bar{\mu}(\{HH\})+\bar{\mu}(\{HT\})+\bar{\mu}(\{TH\})+\bar{\mu}(\{TT\}) &= \mathit{impossible}+\mathit{unlikely}+\mathit{impossible}+\mathit{likely}\\ &= [0,0]+\left[0,\frac{1}{2}\right]+[0,0]+\left[\frac{1}{2},1\right]\\ &= \left[\frac{1}{2},\frac{3}{2}\right]\end{aligned}$$

If we were to equate $\bar{\mu}(\Omega)$ with the sum of the individual probabilities, we would get that $\bar{\mu}(\Omega)=\left[\frac{1}{2},\frac{3}{2}\right]$. However, using the fact that $\bar{\mu}(\emptyset)=\mathit{impossible}$, we have $\bar{\mu}(\Omega)=1-\bar{\mu}(\emptyset)=\mathit{certain}=[1,1]$. This interval is tighter and a better estimate for the probability of the event Ω, and of course it is contained in $[\frac{1}{2},\frac{3}{2}]$. However it is only possible to exploit the information about the complement when all four events are combined. Thus the $\subseteq$-condition allows us to get an estimate for the combined event from each of its constituents and then gather more evidence knowing the aggregate event.

3 Quantum Probability

The mathematical framework of classical probability above assumes that there exists a predetermined set of events that are independent of the particular experiment — classical physics is non-contextual [**kochenspecker1967**, **Redhead1987-REDINA**, **peres1995quantum**, **Jaeger2007**]. However, even in classical situations, the structure of the event space is often only partially known and the precise dependence of two events on each other cannot, a priori, be determined with certainty. In the quantum framework, this partial knowledge is compounded by the fact that there exist non-commuting events which cannot happen simultaneously. To accommodate these more complex situations, conventional approaches to quantum probability abandon the sample space Ω and reason directly about events which are generalized from plain sets to projection operators. A quantum probability space therefore consists of just two components: a set of events $\mathcal{E}$ often formalized as projection operators and a probability measure $\mu:\mathcal{E}\rightarrow[0,1]$ formalized using the Born rule [**Born1983**, **Mermin2007**, **Jaeger2007**].

3.1 Quantum Events

Definition 2 (Projection Operators; Orthogonality [**10.2307/2308516**, **Redhead1987-REDINA**, **peres1995quantum**, **Griffiths2003**, **Swart2013**]). Given a Hilbert space $\mathcal{H}$, an event (an experimental proposition [**BirkhoffVonNeumann1936**], a question [**10.2307/2308516**, **Abramsky2012**], or an elementary quantum test [**peres1995quantum**]) is represented as a (self-adjoint or orthogonal [**Griffiths2003**, **Maassen2010**]) projection operator $P:\mathcal{H}\rightarrow\mathcal{H}$ onto a linear subspace of $\mathcal{H}$. The following define projections and list some of their properties:

- $\mathbb{0}$ is a projection.
- For any pure state $|\psi\rangle$, $|\psi\rangle\langle\psi|$ is a projection operator.
- Projection operators P_0 and P_1 are *orthogonal* if $P_0P_1=P_1P_0=\mathbb{0}$. The sum of two projection operators P_0+P_1 is also a projection operator if and only if they are orthogonal.
- Conversely, every projection P can be expressed as $\sum_{j=1}^{N}|\psi_j\rangle\langle\psi_j|$, where P actually projects onto the linear subspace with orthonormal basis $\{|\psi_j\rangle\}_{j=1}^{N}$.
- A set of projections $\{P_i\}_{i=1}^{N}$ is called an *ideal measurement* if it is a partition of the identity, i.e., $\sum_{i=1}^{N}P_i=\mathbb{1}$ [**Swart2013**]. In this case, projections $\{P_i\}_{i=1}^{N}$ must be mutually orthogonal [**Griffiths2003**, **Halmos1957**], and N must be less or equal to the dimension of the Hilbert space.
- If P is a projection operator, then $\mathbb{1}-P$ is also a projection operator, called its *complement*. It is orthogonal to P, and corresponds to the complement event $\Omega\backslash E$ in classical probability [**Griffiths2003**].

- Projection operators P_0 and P_1 *commute* if $P_0 P_1 = P_1 P_0$. The product of two projection operators $P_0 P_1$ is also a projection operator if and only if they commute. This corresponds to the classical intersection between events [**peres1995quantum**, **Griffiths2003**].
- For two commuting projection operators P_0 and P_1, their *disjunction* $P_0 \vee P_1$ is defined to be $P_0 + P_1 - P_0 P_1$ [**Griffiths2003**].

Example 3 (One-qubit quantum probability space). Consider a one-qubit Hilbert space with each event interpreted as a possible post-measurement state [**peres1995quantum**, **Mermin2007**, **Jaeger2007**]. For example, the event $|0\rangle\langle 0|$ indicates that the post-measurement state will be $|0\rangle$; the probability of such an event depends on the current state; the event $|1\rangle\langle 1|$ indicates that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ indicates that the post-measurement state will be $|+\rangle$; the event $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ indicates that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$ and is clearly certain; finally the empty event $\mathbb{0}$ states that the post-measurement state will be the empty state and is impossible. As in the classical case, a probability measure is a function that maps events to $[0, 1]$. Here is a partial specification of a possible probability measure that would be induced by a system whose current state is $|0\rangle$, $\mu(\mathbb{0}) = 0$, $\mu(\mathbb{1}) = 1$, $\mu(|0\rangle\langle 0|) = 1$, $\mu(|1\rangle\langle 1|) = 0$, $\mu(|+\rangle\langle +|) = 1/2$, ... Note that, similarly to the classical case, the probability of $\mathbb{1}$ is 1 and the probability of collections of orthogonal events (e.g., $|0\rangle\langle 0| + |1\rangle\langle 1|$) is the sum of the individual probabilities. A collection of non-orthogonal events (e.g., $|0\rangle\langle 0|$ and $|+\rangle\langle +|$) is however not even a valid event. In the classical example, we argued that each probability measure is uniquely determined by two actual coins. A similar (but much more subtle) argument is valid also in the quantum case. By postulates of quantum mechanics and Gleason's theorem, it turns out that for large enough quantum systems, each probability measure is uniquely determined by an actual quantum state as discussed next.

3.2 Quantum Probability Measures

Given our setup, the definition of a quantum probability measure is a small variation on the classical definition.

Definition 3 (Quantum Probability Measure [**10.2307/2308516**, **gleason1957**, **Redhead1987-REDINA**, **Maassen2010**]). Given a Hilbert space $\mathcal{H}$ with its set of events $\mathcal{E}$, a *quantum probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\mathbb{0}) = 0$.
- $\mu(\mathbb{1}) = 1$.
- For any projection P, $\mu(\mathbb{1} - P) = 1 - \mu(P)$.
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have $\mu\left(\sum_{i=1}^N P_i\right) = \sum_{i=1}^N \mu(P_i)$.

A quantum probability measure can be easily constructed if one knows the current state of the quantum system by using the Born rule. Specifically, for each pure normalized quantum state $|\phi\rangle$, the Born rule induces a probability measure μ_ϕ^B defined as $\mu_\phi^B(P) = \langle\phi|P\phi\rangle$. The situation generalizes to mixed states $\rho = \sum_{j=1}^N q_j |\phi_j\rangle\langle\phi_j|$, where $\sum_{j=1}^N q_j = 1$ in which case the generalized Born rule induces a probability measure μ_ρ^B defined as $\mu_\rho^B(P) = \mathrm{Tr}(\rho P) = \sum_{j=1}^N q_j \mu_{\phi_j}^B(P)$ [**peres1995quantum**, **544199**, **Jaeger2007**]. Conversely every probability measure must be of this form.

Theorem 1 (Gleason's theorem [**gleason1957**, **Redhead1987-REDINA**, **peres1995quantum**]). In a Hilbert space $\mathcal{H}$ of dimension $d \geq 3$, given a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$, there exists a unique mixed state ρ such that $\mu = \mu_\rho^B$.

3.3 Measuring Quantum Probabilities

Similarly to the classical case, it is possible to estimate quantum probabilities by utilizing the frequentist approach of the previous section, assuming identical measurements conditions in each repeated experiment [**peres1995quantum**]. For instance, if one wants to determine the probability that the spin of a given silver atom is $+\hbar/2$, a Stern-Gerlach apparatus is built where ideally an inhomogeneous magnetic field is generated along, let's say, the quantization axis z. One then produces a collimated beam of identically prepared (neutral) silver atoms that is directed between the poles of the magnet where a predetermined field-gradient along the z direction has been established. Under appropriate experimental conditions we will observe that the beam, after traversing the magnetic-field region, will be deflected towards two regions identified by distinguished spots on a detector situated behind the apparatus [**Stern1988**, **peres1995quantum**, **544199**, **Griffiths2003**]. Each of the two discrete values is associated to either $+\hbar/2$ or $-\hbar/2$, commonly called "spin up" and "spin down", respectively. By "counting" the number of atoms that are deflected in the "spin up" region one can, in principle, estimate the probability that the prepared state of the silver atom state has spin $+\hbar/2$. Notice that a real experiment does not necessarily represent an "ideal measurement". For example, not all silver atoms will be identically prepared, or the field-gradient could not be large enough to distinguish between the spin up and down situations simply producing a large single blot. In other words, the closer we get to an ideal measurement the better we determine those probabilities at the cost of significantly increasing the number of resources. It is not very well appreciated in the literature that Bohr attempted to argue against the measurability of the spin of a free electron. Essentially, Bohr argued (and Mott later on justified his assertion by an elegant use of uncertainty relations [**10.2307/j.ctt7ztxn5.15**]) that a Stern-Gerlach experiment could not succeed in establishing the spin of an unbound electron because the Lorentz force would blur the detected pattern. This example illustrates the case of a fundamental physical limitation that not even infinite resources could mitigate.

4 Quantum Interval-valued Probability Measures

This is the most developed part of our current investigation and is presented in greater detail than the other proposed activities. It relies on current unpublished results. As argued in the previous sections, given fixed finite resources, it is only possible to estimate the quantum probabilities within an interval of confidence. It is therefore natural to propose the notion of a "quantum interval-valued probability measure" that combines the definitions of conventional quantum probability measures with classical interval-probabilities.

Definition 4 (Quantum Interval-valued Probability Measure). Given a Hilbert space $\mathcal{H}$ with quantum events (projections) $\mathcal{E}$, and a collection of intervals $\mathscr{I}$, a *quantum $\mathscr{I}$-interval-valued probability measure* is a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that:

- $\bar{\mu}(\mathbb{0}) = [0, 0]$.
- $\bar{\mu}(\mathbb{1}) = [1, 1]$.
- For any projection P, $\bar{\mu}\left(\mathbb{1} - P\right) = [1, 1] - \bar{\mu}\left(P\right)$.
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^{N}$, we have $\bar{\mu}\left(\sum_{i=1}^{N} P_i\right) \subseteq \sum_{i=1}^{N} \bar{\mu}\left(P_i\right)$.

It is easy to establish that quantum interval-valued probability measures generalize conventional quantum probability measures. In particular, any quantum probability measure can be recast as an interval-valued measure using the three intervals $[0, 0]$, $[1, 1]$ and $[0, 1]$, where $[0, 0]$ and $[1, 1]$ are called $\mathscr{impossible}$ and $\mathscr{certain}$ as before, and $[0, 1]$ is called $\mathscr{unknown}$ because it provides no information. Given a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$, we define a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ by $\bar{\mu}(P) = \iota\left(\mu(P)\right)$, where $\iota : [0, 1] \to \mathscr{I}$ is defined by

$$\iota(x) = \begin{cases} \mathscr{certain} & \text{if } x = 1 \text{ ;} \\ \mathscr{impossible} & \text{if } x = 0 \text{ ;} \\ \mathscr{unknown} & \text{otherwise.} \end{cases}$$

This measure represents the beliefs of an experimenter with no prior knowledge about the particular quantum system in question. Formally, we can ask: what can we deduce about the state of a quantum system given a quantum interval-valued probability measure, i.e., given observations done with finite resources. In the case the intervals are infinitely precise the question reduces to Gleason's theorem which states that the state of the quantum system is uniquely determined by the probability measure. But surely the less resources are available, the less precise the intervals, and the less we expect to know about the state of the system. To formally state and answer this question we begin with defining the *core* and *convexity* of a probability measure as follows.

Definition 5. Given a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$:

- A *core* of $\bar{\mu}$ is the set $\text{core}\,(\bar{\mu}) = \{\mu : \mathcal{E} \to [0,1] \mid \forall E \in \mathcal{E}.\ \mu\,(E) \in \bar{\mu}\,(E)\}$.
- $\bar{\mu}$ is called convex if $\bar{\mu}\,(P_0 \vee P_1) + \bar{\mu}\,(P_0 P_1) \subseteq \bar{\mu}\,(P_0) + \bar{\mu}\,(P_1)$ for all commuting $P_0, P_1 \in \mathcal{E}$.

The core of an interval-valued probability measure is the set of real-valued infinitely-precise probability measures it approximates. An interval-valued probability measure is convex if whenever certain intervals exist then combinations of these intervals must also exist, guaranteeing we can add and manipulate probabilities coherently. In the classical world, every convex measure has a non-empty core, which means that the interval-valued probability measure must approximate at least one infinitely-precise measure.

Theorem 2 (Shapley [**Shapley1971**, **Grabisch2016**]). Every (classical) convex interval-valued probability measure has a non-empty core.

In informal terms, this result states that although measurements done with limited resources and poor precision may not uniquely identify the true state of the system, there is always at least one system that is consistent with the measurements. Surprisingly, we discovered a counterexample to this statement in the quantum case, i.e., we constructed the following convex quantum interval-valued probability measure with an empty core.

Example 4 (Three-dimensional quantum three-interval-valued probability measure). Given a three dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Let $\mathscr{I} = \{\mathit{certain}, \mathit{impossible}, \mathit{unknown}\}$, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|+'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$, and

$$\begin{aligned} \bar{\mu}(\mathbb{0}) = \bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) = \bar{\mu}(|+'\rangle\langle +'|) &= \mathit{impossible} \\ \bar{\mu}(\mathbb{1}) = \bar{\mu}(\mathbb{1} - |0\rangle\langle 0|) = \bar{\mu}(\mathbb{1} - |+\rangle\langle +|) = \bar{\mu}(\mathbb{1} - |+'\rangle\langle +'|) &= \mathit{certain} \\ \bar{\mu}(P) &= \mathit{unknown} \quad \text{otherwise} \end{aligned}$$

It is straightforward to verify that $\bar{\mu}$ is a convex quantum interval-valued probability measure. Now assume there exists a real-valued quantum probability measure μ such that $\mu(P) \in \bar{\mu}(P)$ for every event (projection) P. We derive a contradiction as follows. Suppose $\mu(P) \in \bar{\mu}(P)$ then we must have $\mu(|0\rangle\langle 0|) = \mu(|+\rangle\langle +|) = \mu(|+'\rangle\langle +'|) = 0$. By Gleason's theorem, there is a mixed state $\rho = \sum_{j=1}^{N} q_j\, |\phi_j\rangle\langle\phi_j|$ such that $\mu\,(P) = \sum_{j=1}^{N} q_j\, \langle\phi_j|P\phi_j\rangle$, where $\sum_{j=1}^{N} q_j = 1$ and $q_j > 0$. However, no pure state $|\phi\rangle$ can satisfy $\langle\phi|0\rangle = \langle\phi|+\rangle = \langle\phi|+'\rangle = 0$; a contradiction.

Looking closely at the example above, we might think that the probability measure $\bar{\mu}$ is induced by the state $|2\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) = \mathit{impossible}$, or we might think it is induced by the state $|1\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+'\rangle\langle +'|) = \mathit{impossible}$, or yet we might think it is induced by the mixed state $\frac{\mathbb{1}}{3}$ because $\bar{\mu}(|\phi\rangle\langle\phi|) \neq \mathit{certain}$ for all $|\phi\rangle$. Each of these possibilities is compatible with some but not all of the observations. All is not lost however: if the observations are made more precise, we conjecture that some of the inconsistencies disappear. A partial proof of this conjecture is the following example which refines the previous probability measure by adding more precise intervals. It remains to prove that this refined measure is convex, however.

Example 5 (Three-dimensional quantum four-interval-valued probability measure). Given a three dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Let $\mathscr{I} = \{\mathit{impossible}, \mathit{unlikely}, \mathit{likely}, \mathit{certain}\}$, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. The definition of $\bar{\mu}$ below refers to Fig. 1 which plots the 1-dimensional projectors:

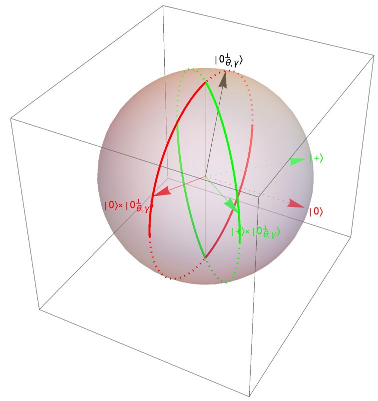

Figure 1: This figure illustrates cases 1 and 2 of example 5 plotted in $\mathbb{R}^3$. The red and green dotted vectors are $|0\rangle$ and $|+\rangle$ respectively. All possible real vectors of the subspaces $\left|0^{\perp}_{\theta,\gamma}\right\rangle$ and $\left|+^{\perp}_{\theta,\gamma}\right\rangle$ are drawn in the red and green circles, respectively. Within the circles, a dotted vector $|\psi\rangle$ means $\bar{\mu}(|\psi\rangle\langle\psi|) = \textit{likely}$; otherwise, $\bar{\mu}(|\psi\rangle\langle\psi|) = \textit{unlikely}$. The gray vector is a generic vector $\left|0^{\perp}_{\theta,\gamma}\right\rangle$, and the red and green solid vectors are normalized $|0\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle$ and $|+\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle$, respectively, where $\times$ is the usual cross product in $\mathbb{R}^3$.

1. Let:

$$\begin{aligned} \bar{\mu}(\mathbb{0}) = \bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) &= \textit{impossible} \\ \bar{\mu}(\mathbb{1}) = \bar{\mu}(\mathbb{1} - |0\rangle\langle 0|) = \bar{\mu}(\mathbb{1} - |+\rangle\langle +|) &= \textit{certain} \end{aligned}$$

   where $|0\rangle$ and $|+\rangle$ are plotted as the red and green dotted vectors, respectively.

2. The red and green circles are are the states orthogonal to $|0\rangle$ and $|+\rangle$, respectively, and can be parametrized as $\left|0^{\perp}_{\theta,\gamma}\right\rangle = \mathrm{e}^{\mathrm{i}\gamma}\sin\theta|1\rangle + \cos\theta|2\rangle$ and $\left|+^{\perp}_{\theta,\gamma}\right\rangle = -\mathrm{e}^{\mathrm{i}\gamma}\sin\theta|-\rangle + \cos\theta|2\rangle$, where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \gamma < 2\pi$. The dotted half of those states need special treatment, i.e., whenever $0 \leq \theta < \frac{\pi}{2}$ and $0 \leq \gamma < \pi$, we define

$$\begin{aligned} \bar{\mu}\left(\left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) = \bar{\mu}\left(\left|+^{\perp}_{\theta,\gamma}\right\rangle\left\langle +^{\perp}_{\theta,\gamma}\right|\right) &= \textit{likely} \\ \bar{\mu}\left(\mathbb{1} - \left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) = \bar{\mu}\left(\mathbb{1} - \left|+^{\perp}_{\theta,\gamma}\right\rangle\left\langle +^{\perp}_{\theta,\gamma}\right|\right) &= \textit{unlikely} \end{aligned}$$

3. Otherwise, $\bar{\mu}(|\psi\rangle\langle\psi|) = \textit{unlikely}$ and $\bar{\mu}(\mathbb{1} - |\psi\rangle\langle\psi|) = \textit{likely}$.

It is straightforward but tedious to check that $\bar{\mu}$ is a quantum interval-valued probability measure. Although we have not verified that $\bar{\mu}$ is convex, the following argument establishes that is has an empty core. Assume there is a real-valued probability measure satisfying $\mu^B_\rho(P) \in \bar{\mu}(P)$ for all $P \in \mathcal{E}$. Because $\mu^B_\rho(|0\rangle\langle 0|) \in \bar{\mu}(|0\rangle\langle 0|) = \textit{impossible}$ and $\mu^B_\rho(|+\rangle\langle +|) \in \bar{\mu}(|+\rangle\langle +|) = \textit{impossible}$, we must have $\mu^B_\rho(|0\rangle\langle 0|) = \mu^B_\rho(|+\rangle\langle +|) = 0$ so that $\mu^B_\rho = \mu^B_{|2\rangle}$. However,

$$\mu^B_{|2\rangle}(|2\rangle\langle 2|) = 1 \notin \textit{unlikely} = \bar{\mu}(|2\rangle\langle 2|)\ .$$

This measure however is “better” than the previous one in the sense that it might only be induced by $|2\rangle$ or the density matrix $\frac{\mathbb{1}}{3}$, but not by $|1\rangle$.

In general, we conjecture that if the measurement equipment is made more and more precise, the corresponding interval-valued probability measure will be closer and closer to the Born rule. In the limit case, $\mathscr{I} = \{\{a\} \mid a \in [0,1]\}$ we do indeed recover the conventional Gleason's theorem and the Born rule.

5 Conclusion and Discussion

If we insist that probabilities cannot be computed to infinite precision and are bound to be approximations represented by intervals of confidence, then quantum states themselves can only be discussed within intervals of confidence. Despite the fact that classically a interval-valued probability measure could always correspond to a real-valued probability measure, we found that there may be no "real" infinitely-precise quantum state approximated by a quantum interval-valued probability measure as a whole. However, if a quantum interval-valued probability measure is decomposed into pieces, each piece might still be induced by a quantum state. Moreover, our examples and Gleason's theorem summarized in table 1 suggests that as the measurement resources increase, an entire quantum interval-valued probability measure could more and more likely be identified to a particular state. Additional work is in progress on the following research questions:

Measurement resources	Lowest	$\longleftrightarrow$		Highest
Count of $\mathscr{I}$	3	4	$\dots$	∞
$\sup_{[l,r]\in\mathscr{I}} \lvert r-l\rvert$	1	$\frac{1}{2}$	$\dots$	0
Count of core $(\bar{\mu})$	0	0		1
Count of states might correspond to $\bar{\mu}$	3	2	$\dots$	1
How precise we could identify a state?	Coarse	$\longleftrightarrow$		Precise

Table 1: Relation between measurement resources and interval-valued probability measures

Research Question. Confirm that there is always a quantum interval-valued probability measure with an empty-core if the length of the intervals is bigger than zero.

Research Question. Find a general method to determine how a quantum interval-valued probability measure is close to the Born rule.

Research Question. Confirm that as the number and precision of the intervals increase the quantum interval-valued probability measure converges to the measure induced by the Born rule.

Research Question. Investigate the status of the theorems of Gleason, Bell, and Kochen-Specker for quantum interval-valued probability measures.

Although we discussed how a quantum interval-valued probability measure could be induced from a real quantum state, our investigation leaves an open question of whether there exists a "real" entity that exists independently of measurements and probabilities. The possibility of no "real" underlying state is consistent with the elegantly recent work on Quantum Bayesianism or QBism [**Fuchs2010**, **VonBaeyer2016**, **Fuchs2012**], which suggests that the quantum state is more like an interactive system in computer science parlance. In another word, the quantum state is subjective: each observer has a different view of the quantum system that is consistent with their previous observations and that allows that observer, independently of other observers, to assign beliefs (i.e., probabilities) to possible future interactions.