



The theory of interval-probability as a unifying concept for uncertainty

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Abstract

The concept of interval-probability is motivated by the goal to generalize classical probability so that it can be used for describing uncertainty in general. The foundations of the theory are based on a system of three axioms – in addition to Kolmogorov's axioms – and definitions of independence as well as of conditional-probability. The resulting theory does not depend upon interpretations of the probability concept. As an example of generalising classical results Bayes' theorem is described – other theorems are only mentioned. © 2000 Elsevier Science Inc. All rights reserved.

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1. The scope of the theory

The theory of interval-probability, as developed in Munich over many years, is motivated by the following goals:

1. Different kinds of uncertainty should be treated by the same concept. This applies to:
 - (a) imprecise probability and uncertain knowledge;
 - (b) imprecise data;
 - (c) the use of capacities;

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- (d) the concept of ambiguity and its employment in decision theory;
 - (e) belief functions and related concepts;
 - (f) interpretation of interval-estimates in classical theory;
 - (g) the study of experiments with possibly diverging relative frequencies;
 - (h) non-additive measures (fuzzy measures).
2. As a special case, classical probability must fit into this theory.
 3. A simple system of axioms must describe the fundamentals of the theory.
 4. All statements of the theory must be derivable from the given axioms and appropriate definitions.
 5. The domain of application must neither be limited to purely formal aspects nor be bound by a certain interpretation of probability.

In classical probability one system of axioms exists not being restricted to a certain type of interpretation: Kolmogorov's axioms. Therefore, the concept of interval-probability is directly related to this system of axioms.

There is one obvious limitation for any theory of interval-probability: only those assessments assigning intervals to random events qualify as genuine subjects of the theory. The benefits and the power of the theory are due to the duality between a set of interval-limits and the corresponding set of classical probabilities. These qualities distinguish the approach described in the following chapters from those admitting more general types of probability assignments, e.g., [3] or [4].

Since the theory of interval-probability is independent on the kind of interpretation it suits for fields of application, where probability is understood as means of argumentation without relation either to betting or to large series of random experiments.

Also it produces freedom in describing behaviour in a very general way: Ellsberg's remark that everyone will switch to a favourable event with probability $[0; 1]$ instead of an equally favourable with probability p , provided that p is small enough, can be taken into consideration adequately.

Altogether, theory of interval-probability comes nearer to the classical understanding of probability assignment than those approaches relying on more general types of assessment.

2. Basic concepts

2.1. *The axioms*

In a slightly specialised version of the axioms all closed intervals in $[0; 1]$ are admitted as components of interval-probability. In this case the following definitions may be understood as describing the system of axioms for interval-probability [7, pp. 49–51].

Definition 2.1. Given a sample-space Ω and a σ -field \mathcal{A} of random events in Ω , a set function $p(\cdot)$ defined on \mathcal{A} is named a **K-function**, if it obeys the axioms of Kolmogorov (I–III).

Since K-functions have the same properties as classical probabilities, sometimes they are named *K-probabilities*.

Definition 2.2. An interval-valued set function $P(\cdot)$ on \mathcal{A} is called an **R-probability** if it obeys the following two axioms:

IV.

$$P(A) = [L(A); U(A)] \quad \forall A \in \mathcal{A}, \quad (1)$$

with

$$0 \leq L(A) \leq U(A) \leq 1 \quad \forall A \in \mathcal{A}. \quad (2)$$

V. The set \mathcal{M} of K-functions $p(\cdot)$ on \mathcal{A} with

$$L(A) \leq p(A) \leq U(A) \quad \forall A \in \mathcal{A}, \quad (3)$$

is not empty.

The name “R-probability” may be related to the word “reasonable”. A quadruple consisting of a sample-space Ω , a σ -field \mathcal{A} of random events and a certain R-probability on $(\Omega; \mathcal{A})$ will be called an **R-(probability) field** $(\Omega; \mathcal{A}; L(\cdot), U(\cdot))$. An important concept is introduced by:

Definition 2.3. Let $\mathcal{R} = (\Omega; \mathcal{A}; L(\cdot), U(\cdot))$ be an R-probability field. Then the non-empty set of K-functions,

$$\mathcal{M} = \{p(\cdot) \mid L(A) \leq p(A) \leq U(A) \quad \forall A \in \mathcal{A}\}, \quad (4)$$

is named the structure of \mathcal{R} .

Therefore, the existence of a non-empty structure is a sufficient condition for any R-field. It is obvious that

$$L(\emptyset) = 0, \quad U(\Omega) = 1 \quad (5)$$

are among the necessary conditions for R-probability.

Definition 2.4. An R-probability obeying the following axiom is named an **F-probability**:

VI.

$$\left. \begin{array}{l} \inf_{p \in \mathcal{M}} p(A) = L(A) \\ \sup_{p \in \mathcal{M}} p(A) = U(A) \end{array} \right\} \quad \forall A \in \mathcal{A}. \quad (6)$$

The letter F may be connected with the word “feasible”. In any F-probability field none of the limits $L(\cdot)$ and $U(\cdot)$ are too wide, while this may be the

case for an R-field. Furthermore, the property of F-probability implies the validity of

$$U(A) = 1 - L(\neg A) \quad \forall A \in \mathcal{A}, \quad (7)$$

and of

$$U(\emptyset) = 0, \quad L(\Omega) = 1, \quad (8)$$

which by use of the symbol

$$[a] := [a; a] \quad (9)$$

together with (5) read as

$$P(\emptyset) = [0], \quad P(\Omega) = [1]. \quad (10)$$

A triple consisting of a sample-space Ω , a σ -field \mathcal{A} of random events and a given F-probability is understood to be an F-(probability) field $(\Omega; \mathcal{A}; L(\cdot))$.

The concept of structure is fundamental for the theory of interval-probability. Most definitions and proofs are directly or indirectly related to it. Another important concept is that of *prestructure*.

Definition 2.5. Let $\mathcal{F} = (\Omega; \mathcal{A}; L(\cdot))$ be an F-probability field. Then any set \mathcal{J} of K-probabilities on $(\Omega; \mathcal{A})$ is named a *prestructure* of \mathcal{F} if the following relations hold:

$$\inf_{p \in \mathcal{J}} p(A) = L(A) \quad \forall A \in \mathcal{A}. \quad (11)$$

According to that, every subset of the structure producing the same lower – and as a set of K-probabilities also the same upper – limits constitutes a prestructure. As long as it does not contain all K-functions in accordance with (11), \mathcal{J} is a proper subset of \mathcal{M} .

The concept of prestructure proves important for several types of calculations. Also it shows one way of dealing with situations in which for any reason the state of information is described by a set of classical probabilities which has not the properties of a structure: it cannot be defined by interval-limits (for instance any polyhedron [6, p. 397]). In such situations employment of the theory of interval-probability in the narrow sense may be viewed as a loss of information. In most cases of practical interest linear transformations make it possible to convert the given set into a structure without the use of the concept of prestructure and therefore without any loss of information (see Section 2.4).

The given system of axioms may be applied to finite sample-spaces and to infinite ones, but in the case of F-probability it proves useful to distinguish continuous F-probability, since axioms I–VI do not guarantee the continuity of the set functions $L(\cdot)$ and $U(\cdot)$.

Definition 2.6. An F-probability is called continuous if for any decreasing sequence of events of \mathcal{A} :

$$A_1 \supset A_2 \supset \cdots \supset A_n \dots \quad (12)$$

for which

$$\bigcap_{i=1}^{\infty} A_i =: A \quad (13)$$

is valid, the following equation holds:

$$\lim_{n \rightarrow \infty} U(A_n) = U(A). \quad (14)$$

2.2. R- and F-probabilities

R-probability may be interpreted as “not contradictory, but not necessarily perfect”, since on the one hand it allows the existence of a structure, but on the other hand some of the limits may be not narrow enough with respect to this structure. The concept is used by Huber [2, p. 257], and materially it is related to Walley’s concept of “avoiding sure loss” [4, pp. 67–72, 135]. For R-probability fields which do not possess the F-property one may use the expression “redundant R-probability fields”.

F-probability may be interpreted as a perfect generalization of classical probability to an interval-valued one. The structure and the set of interval-limits imply each other. Huber [2, p. 255] calls probability of this nature “representable”, materially it corresponds to Walley’s “coherent probability” [4, pp. 72–86, 135].

Since the probabilist must expect to be confronted with redundant R-probability, he should be prepared to “improve” such an assessment. There are two possible standpoints concerning his attitude towards a certain redundant R-probability field.

1. He may use the interval-limits to derive the structure of the R-probability field and pass over to the limits of that F-probability field, which is in accordance with this structure. In this way an F-probability field can uniquely be *derived* from every redundant R-probability without violating any of the interval-limits. This is called the *rigid standpoint*: it reduces the original interval-length for every redundant R-probability field. The procedure of constructing the *derivable* F-probability field is similar to Walley’s concept of natural extension of previsions avoiding sure loss [4, pp. 122–127].
2. It may – at least in some cases – be argued that, after adjustment to (10), any of the remaining limits should necessarily describe the outcome of the probability component $p(A)$ for at least one element $p(\cdot)$ of the structure. None of the values contained in such an interval therefore must be excluded: the structure has to be enlarged in order to include at least one K-probability

$p(\cdot)$ for which $p(A) = L(A)$ is true and one $p'(\cdot)$ for which $p'(A) = U(A)$ holds. There is no unique way of enlarging the structure for this purpose, but there exist criteria to distinguish “minimum enlargements”. If no information in favour of a certain kind of minimum enlargement is provided, the union of all F-probability fields produced in this way may be used. It is an F-probability field itself and is named the *F-cover* of the given redundant R-probability field. The standpoint producing this type of procedure may be called the *cautious standpoint*, because it leads to larger intervals and therefore weaker statements. It seems that there is no counterpart to the cautious standpoint in Walley’s theory of imprecise probabilities.

2.3. Partially determinate probability

Definition 2.7. Let Ω be a sample-space and \mathcal{A} be the σ -field of random events. Furthermore, let

$$\mathcal{A}' = \mathcal{A} \setminus \{\Omega, \emptyset\} \quad (15)$$

and

$$\mathcal{A}_L \subseteq \mathcal{A}', \quad \mathcal{A}_U \subseteq \mathcal{A}'. \quad (16)$$

Then an assessment is called a partially determinate R-probability if (10) holds, and for each $A \in \mathcal{A}_L$ a lower limit $L(A)$ is given, as well as for each $A \in \mathcal{A}_U$ an upper limit $U(A)$, so that there exists a non-empty structure \mathcal{M} of K-probabilities $p(\cdot)$, for which the following inequalities hold:

$$\begin{aligned} L(A) &\leq p(A) \quad \forall A \in \mathcal{A}_L, \\ p(A) &\leq U(A) \quad \forall A \in \mathcal{A}_U. \end{aligned} \quad (17)$$

Definition 2.8. If for a partially determinate R-probability the conditions

$$\inf_{p \in \mathcal{M}} p(A) = L(A) \quad \forall A \in \mathcal{A}_L, \quad (18)$$

$$\sup_{p \in \mathcal{M}} p(A) = U(A) \quad \forall A \in \mathcal{A}_U \quad (19)$$

are fulfilled, it is called a partially determinate F-probability.

For a partially determinate F-probability there exists a rather simple way of constructing the complete F-probability field: The use of (6) produces all originally lacking limits. This procedure is called *normal completion*. The procedure of normal completion is based on the properties of the structure and therefore closely related to the calculation of the derivable F-field. It shows similarity to certain aspects of Walley’s concept of natural extension.

Definition 2.9. Let $\mathcal{F} = (\Omega; \mathcal{A}; L(\cdot))$ be an F-probability field. Then $(\mathcal{A}_L, \mathcal{A}_U)$ is named a support of \mathcal{F} if there exists a partially determinate F-probability according to (17) together with (18) and (19) which produces \mathcal{F} via normal completion.

Interpretation of this concept is obvious: information about limits $L(A) \forall A \in \mathcal{A}_L$, and $U(A) \forall A \in \mathcal{A}_U$, is sufficient for constructing \mathcal{F} .

2.4. Extensions

A probability assignment not defined by interval-limits and therefore not describing a structure directly can be understood as a prestructure of an F-probability field. Since the structure of this field includes K-probabilities which are not elements of the prestructure some loss of information is produced. Provided the set of all admissible K-probabilities consists of a union of polyhedra, alternatively the assignment may be transformed into a model which can be analysed using F-probability without any loss of information. The methodology to be applied is described in [11, Section 4.5].

3. Conditional probability

3.1. General remarks

Following Kolmogorov's procedure, the system of axioms has to be completed by two definitions: the definition of conditional probability and the definition of independence. Defining conditional probability affords a series of considerations both on principle and of a technical kind and can be referred to here only in a highly abridged version.

The concepts of conditional probability are applied to F-probability fields only, since it may be assumed that redundant R-probability fields are transferred into F-probability fields by one of the ways described in Section 2.

Generally conditional probability affords the existence of a partition \mathcal{C} of Ω :

$$\begin{aligned} \mathcal{C} &= \{C_1, C_2, \dots, C_r\}, \\ C_i &\in \mathcal{A}', \quad C_i \cap C_k = \emptyset, \quad i \neq k, \\ \bigcup_{i=1}^r C_i &= \Omega. \end{aligned} \tag{20}$$

If $p(C) > 0$, it produces an assessment of conditional K-probability:

$$p_{\mathcal{C}}(A|C) \quad \forall A \in \mathcal{A}, \quad \forall C \in \mathcal{C}. \tag{21}$$

It should be stressed that (21) is applied to all conditioning events in \mathcal{C} , but not to conditioning events which belong to the field produced by \mathcal{C} and not to \mathcal{C}

itself: with respect to K-probability this restriction serves to avoid paradoxical results.

3.2. The intuitive concept

Concerning conditional F-probability there is one first concept promising a simple solution. It is described by the following definitions:

Definition 3.1. Let \mathcal{M} be the structure of the F-probability field $(\Omega; \mathcal{A}; L(\cdot))$, $C \in \mathcal{A}$, and $U(C) > 0$. Then

$$\mathcal{M}_C := \{p(\cdot) \in \mathcal{M} \mid p(C) > 0\} \quad (22)$$

is called the *C-docked structure*.

Definition 3.2. Under the requirements of Definition 3.1 the equations

$$iL_{\mathcal{C}}(A \mid C) := \inf_{p \in \mathcal{M}_C} \frac{p(A \cap C)}{p(C)} \quad \forall A \in \mathcal{A}, \quad \forall C \in \mathcal{C}, \quad (23)$$

together with

$$\begin{aligned} iU_{\mathcal{C}}(A \mid C) &:= 1 - iL_{\mathcal{C}}(\neg A \mid C) = 1 - \inf_{p \in \mathcal{M}_C} \frac{p(\neg A \cap C)}{p(C)} \\ &= \sup_{p \in \mathcal{M}_C} \frac{p(A \cap C)}{p(C)} \end{aligned} \quad (24)$$

produce the intuitive concept of conditional probability.

This concept has a considerable number of pleasant properties. For each $C \in \mathcal{C}$ mit $U(C) > 0$ a conditional F-field is generated. The result $iP_{\mathcal{C}}(A \mid C) = [0; 1]$ is possible not only in cases where $L(C) = 0$. The concept is easy to understand and easy to use. Nevertheless, it is of limited interest, because it does not allow to reconstruct the F-probability field from which it is gained.

This may be demonstrated by the following two examples.

Example 3.1. The following assessment produces an F-probability field on the sample-space $\Omega = E_1 \cup E_2 \cup E_3$:

$$\begin{aligned} P(E_1) &= [0.10; 0.25], & P(E_1 \cup E_2) &= [0.40; 0.60], \\ P(E_2) &= [0.20; 0.40], & P(E_1 \cup E_3) &= [0.60; 0.80], \\ P(E_3) &= [0.40; 0.60], & P(E_2 \cup E_3) &= [0.75; 0.90]. \end{aligned}$$

A partition \mathcal{C} of Ω be given by

$$\begin{aligned}\mathcal{C} &= \{C_1, C_2\}, \quad C_1 = E_1 \cup E_2, \quad C_2 = E_3 \\ \Rightarrow \quad P(C_1) &= [0.40; 0.60], \quad P(C_2) = [0.40; 0.60].\end{aligned}$$

Application of (23) leads to

$$iL_{\mathcal{C}}(E_1 | C_1) = \inf_{p \in \mathcal{M}} \frac{p(E_1)}{p(C_1)} = \frac{0.10}{0.10 + 0.40} = 0.20,$$

$$iL_{\mathcal{C}}(E_2 | C_1) = \inf_{p \in \mathcal{M}} \frac{p(E_2)}{p(C_1)} = \frac{0.20}{0.20 + 0.25} = 0.44.$$

Therefore, the intuitive concept generates the following assessment:

$$\begin{aligned}iP_{\mathcal{C}}(E_1 | C_1) &= [0.20; 0.55], & iP_{\mathcal{C}}(E_1 | C_2) &= [0], \\ iP_{\mathcal{C}}(E_2 | C_1) &= [0.44; 0.80], & iP_{\mathcal{C}}(E_2 | C_2) &= [0], \\ iP_{\mathcal{C}}(E_3 | C_1) &= [0], & iP_{\mathcal{C}}(E_3 | C_2) &= [1].\end{aligned}$$

Example 3.2. Another F-probability field:

$$\begin{aligned}P(E_1) &= [0.11; 0.225], & P(E_1 \cup E_2) &= [0.40; 0.60], \\ P(E_2) &= [0.18; 0.44], & P(E_1 \cup E_3) &= [0.56; 0.82], \\ P(E_3) &= [0.40; 0.60], & P(E_2 \cup E_3) &= [0.775; 0.89]\end{aligned}$$

with the same partition as in Example 3.1 and the same marginal probability produces:

$$\begin{aligned}iL_{\mathcal{C}}(E_1 | C_1) &= \frac{0.11}{0.11 + 0.44} = 0.20, \\ iL_{\mathcal{C}}(E_2 | C_1) &= \frac{0.18}{0.225 + 0.18} = 0.44 \\ \Rightarrow \quad iP_{\mathcal{C}}(E_1 | C_1) &= [0.20; 0.55], & iP_{\mathcal{C}}(E_1 | C_2) &= [0], \\ iP_{\mathcal{C}}(E_2 | C_1) &= [0.44; 0.80], & iP_{\mathcal{C}}(E_2 | C_2) &= [0], \\ iP_{\mathcal{C}}(E_3 | C_1) &= [0], & iP_{\mathcal{C}}(E_3 | C_2) &= [1].\end{aligned}$$

As Examples 3.1 and 3.2 demonstrate, it is possible that two different F-probability fields lead to the same marginal probability and to the same conditional probability, if the intuitive concept is used. For that reason it is impossible to reconstruct the given F-probability field from marginal probability together with conditional-probability. Therefore, conditional probability according to this concept can never contain the type of information one wants to transfer from one model to the other.

This failure rules out the use of the intuitive concept as the only concept of conditional probability. Nevertheless, it remains useful as a means of describing and characterising the phenomenon of conditional probability.

If conditional probability is applied in updating, the intuitive concept gives the proper answer. As far as conditional probability is used in transferring information from one F-field to another, one is led to another concept.

3.3. The canonical concept

Definition 3.3. The subfields with respect to elements of the partition \mathcal{C} are denoted as $\mathcal{A}(C)$:

$$\mathcal{A}(C) := \{C \cap A \mid A \in \mathcal{A}\}, \quad C \in \mathcal{C}. \quad (25)$$

Definition 3.4. An F-probability field $\mathcal{F} = (\Omega; \mathcal{A}; L(\cdot))$ together with a partition \mathcal{C} of Ω is called a laminar constellation $(\mathcal{F}, \mathcal{C})$ if a support $(\mathcal{A}_L, \mathcal{A}_U)$ exists such that for all $A \in \mathcal{A}_L \cup \mathcal{A}_U$ the following holds:

$$A \in \bigcup_{C \in \mathcal{C}} \mathcal{A}(C). \quad (26)$$

The definition of laminar constellation distinguishes constellations in which all information about interval-limits is given by the assessment $P(A)$ for those random events A which are contained in one single element of the partition. The reason for this definition is the following: information with respect to a random event which does not obey (26), can be contained neither in marginal probability nor in conditional probability.

In this article the construction of conditional probability for a laminar constellation is described for the case only, that the conditions

$$L(C) > 0 \quad \forall C \in \mathcal{C} \quad (27)$$

are fulfilled. The concept then requires the calculation of

$$L_{\mathcal{C}}(A \mid C) := \frac{L(A)}{L(C)} \quad \forall A \in \mathcal{A}(C), \quad \forall C \in \mathcal{C}, \quad (28)$$

$$U_{\mathcal{C}}(A \mid C) := \frac{U(A)}{U(C)} \quad \forall A \in \mathcal{A}(C), \quad \forall C \in \mathcal{C}. \quad (29)$$

Concerning this assessment a decisive distinction has to be made.

Definition 3.5. If for all $C \in \mathcal{C}$ the assessment described by (28) and (29) constitutes an F-probability field¹

$$\mathcal{F}_C := (C; \mathcal{A}(C), L_{\mathcal{C}}(\cdot \mid C)), \quad (30)$$

¹ Which means among others that (7) must hold for $L_{\mathcal{C}}(\cdot \mid C)$ and $U_{\mathcal{C}}(\cdot \mid C)$.

then $(\mathcal{F}, \mathcal{C})$ is called an F-laminar constellation. In this situation, each \mathcal{F}_C represents the conditional F-probability with respect to the condition C .

Knowledge of conditional probability and of marginal probability allows reconstruction of the given F-probability field. Eqs. (28) and (29) are converted to

$$L(A) = L_{\mathcal{C}}(A|C) \cdot L(C) \quad \forall A \in \mathcal{A}(C), \quad \forall C \in \mathcal{C}, \quad (31)$$

$$U(A) = U_{\mathcal{C}}(A|C) \cdot U(C) \quad \forall A \in \mathcal{A}(C), \quad \forall C \in \mathcal{C}, \quad (32)$$

and because of laminarity all the rest of the F-probability field may be reconstructed by use of the limits defined by (31) and (32). Furthermore, it may be shown that combination of conditional probability according to (31) and (32) with any marginal F-probability produces an F-probability field. Therefore, (31) together with (32) may be used in order to transfer information to any comparable model.

Example 3.3. The following assessment produces an F-probability field:

$$\begin{aligned} P(E_1) &= [0.10; 0.30], & P(E_1 \cup E_2) &= [0.40; 0.60], \\ P(E_2) &= [0.20; 0.45], & P(E_1 \cup E_3) &= [0.55; 0.80], \\ P(E_3) &= [0.40; 0.60], & P(E_2 \cup E_3) &= [0.70; 0.90]. \end{aligned}$$

With regard to the partition

$$\begin{aligned} \mathcal{C} &= \{C_1, C_2\}, \quad C_1 = E_1 \cup E_2, \quad C_2 = E_3 \\ \Rightarrow \quad P(C_1) &= [0.40; 0.60], \quad P(C_2) = [0.40; 0.60], \end{aligned}$$

the effect of combining in a new model the conditional probability with another marginal probability, namely,

$$P'(C_1) = [0.60; 0.80], \quad P'(C_2) = [0.20; 0.40]$$

is to be determined.

Conditional probability according to (28) and (29):

$$L_{\mathcal{C}}(E_1|C_1) = \frac{L(E_1)}{L(C_1)} = \frac{0.10}{0.40} = 0.25,$$

$$L_{\mathcal{C}}(E_2|C_1) = \frac{L(E_2)}{L(C_1)} = \frac{0.20}{0.40} = 0.50,$$

$$L_{\mathcal{C}}(E_3|C_2) = \frac{L(E_3)}{L(C_2)} = \frac{0.40}{0.40} = 1,$$

$$U_{\mathcal{C}}(E_1|C_1) = \frac{U(E_1)}{U(C_1)} = \frac{0.30}{0.60} = 0.50,$$

$$U_{\mathcal{C}}(E_2 | C_1) = \frac{U(E_2)}{U(C_1)} = \frac{0.45}{0.60} = 0.75,$$

$$U_{\mathcal{C}}(E_3 | C_2) = \frac{U(E_3)}{U(C_2)} = \frac{0.60}{0.60} = 1$$

$$\Rightarrow P_{\mathcal{C}}(E_1 | C_1) = [0.25; 0.50], \quad P_{\mathcal{C}}(E_1 | C_2) = [0],$$

$$P_{\mathcal{C}}(E_2 | C_1) = [0.50; 0.75], \quad P_{\mathcal{C}}(E_2 | C_2) = [0],$$

$$P_{\mathcal{C}}(E_3 | C_1) = [0], \quad P_{\mathcal{C}}(E_3 | C_2) = [1]$$

proves to constitute an F-probability field as well for C_1 as for C_2 ; therefore, both assessments represent conditional F-probability. The transfer of $P_{\mathcal{C}}(\cdot | C_1)$ to the alternative marginal probability produces by analogy to (31) and (32):

$$L'(E_1) = 0.25 \cdot 0.60 = 0.15,$$

$$L'(E_2) = 0.50 \cdot 0.60 = 0.30,$$

$$U'(E_1) = 0.50 \cdot 0.80 = 0.40,$$

$$U'(E_2) = 0.75 \cdot 0.80 = 0.60$$

$$\Rightarrow P'(E_1) = [0.15; 0.40], \quad P'(E_1 \cup E_2) = [0.60; 0.80],$$

$$P'(E_2) = [0.30; 0.60], \quad P'(E_1 \cup E_3) = [0.40; 0.70],$$

$$P'(E_3) = [0.20; 0.40], \quad P'(E_2 \cup E_3) = [0.60; 0.85].$$

That this assessment represents an F-probability field, is easily proven by the fact that $p_1(\cdot)$ with

$$p_1(E_1) = 0.15, \quad p_1(E_2) = 0.60, \quad p_1(E_3) = 0.25$$

reaches $L'(E_1)$, $U'(E_2)$, $L'(E_1 \cup E_3)$, $U'(E_2 \cup E_3)$, $p_2(\cdot)$ with

$$p_2(E_1) = 0.40, \quad p_2(E_2) = 0.40, \quad p_2(E_3) = 0.20$$

reaches $U'(E_1)$, $L'(E_3)$, $U'(E_1 \cup E_2)$, $L'(E_2 \cup E_3)$ and $p_3(\cdot)$ with

$$p_3(E_1) = 0.30, \quad p_3(E_2) = 0.30, \quad p_3(E_3) = 0.40$$

reaches $L'(E_2)$, $U'(E_3)$, $L'(E_1 \cup E_2)$, $U'(E_1 \cup E_3)$. Therefore, axioms I–VI are fulfilled.

Definition 3.6. If there exists $C \in \mathcal{C}$, so that (28) and (29) violate at least one of the axioms IV and V, then $(\mathcal{F}, \mathcal{C})$ is called a 0-laminar constellation.

In this case, application of the concept of conditional probability according to the canonical concept is not possible.

Example 3.4. For the F-probability field

$$\begin{aligned} P(E_1) &= [0.16; 0.21], & P(E_1 \cup E_2) &= [0.40; 0.60], \\ P(E_2) &= [0.22; 0.42], & P(E_1 \cup E_3) &= [0.58; 0.78], \\ P(E_3) &= [0.40; 0.60], & P(E_2 \cup E_3) &= [0.79; 0.84] \end{aligned}$$

and the partition

$$\mathcal{C} = \{C_1, C_2\}, \quad C_1 = E_1 \cup E_2, \quad C_2 = E_3,$$

the result

$$\begin{aligned} L_{\mathcal{C}}(E_1 | C_1) &= \frac{0.16}{0.40} = 0.40, \\ U_{\mathcal{C}}(E_1 | C_1) &= \frac{0.21}{0.60} = 0.35 \end{aligned}$$

violates axiom IV: the relative length of the interval $P(E_1)$ is too small compared with $P(C_1)$. Consequently, $P(E_1)$ can never be produced by a procedure of the type defined by (31) and (32).

Between F-laminarity and 0-laminarity lies what is called R-laminarity.

Definition 3.7. If for all $C \in \mathcal{C}$ the assessment created by Eqs. (28) and (29) constitutes R-probability, then $(\mathcal{F}, \mathcal{C})$ is called an R-laminar constellation.

R-laminar constellation which is not F-laminar may be described as redundant R-laminar. While the reconstruction of the original F-probability field can be achieved through (31) and (32) also in these situations, the proper way of transferring information contained in (28) and (29) to another model requires a bunch of considerations and decisions. Concerning the conditions under which such transfer can be justified, different views are possible. A comparison of these views and their consequences is beyond the scope of this article and will be given in the volume succeeding [11].

Example 3.5. The constellation described in Example 3.1 proves to be redundant R-laminar.

$$\begin{aligned} L_{\mathcal{C}}(E_1 | C_1) &= \frac{0.10}{0.40} = 0.25, \\ L_{\mathcal{C}}(E_2 | C_1) &= \frac{0.20}{0.40} = 0.50, \\ U_{\mathcal{C}}(E_1 | C_1) &= \frac{0.25}{0.60} = 0.41\dot{6}, \\ U_{\mathcal{C}}(E_2 | C_1) &= \frac{0.40}{0.60} = 0.6\dot{6} \end{aligned}$$

defines a redundant R-probability field: $p(E_1|C_1) = 0.40$, $p(E_2|C_1) = 0.60$ is an element of the structure, therefore $P(\cdot|C_1)$ produces an R-field. Since $p(E_1|C_1) = L_{\mathcal{C}}(E_1|C_1) = 0.25$ is not possible, this R-probability field is redundant.

3.4. The theorem of Bayes

If both concepts of conditional probability are employed in their specific roles, Bayes' theorem for interval-probability can be derived. It is reported here without proof.

Let $\mathcal{F} = (\Omega; \mathcal{A}; L(\cdot))$ be an F-field and \mathcal{C} a partition of Ω , so that $(\mathcal{F}, \mathcal{C})$ is an F-laminar constellation. Then the following information allows the reconstruction of \mathcal{F} :

1. $F_{\mathcal{C}} = (\mathcal{C}; \mathcal{P}(\mathcal{C}); L(\cdot))$ is the marginal F-field with respect to the partition \mathcal{C} : the “prior probability”.
2. $\{\mathcal{F}_C = (C; \mathcal{A}(C); L_{\mathcal{C}}(\cdot|C)) | C \in \mathcal{C}\}$ is the set of conditional F-probability fields with respect to the canonical concept.

One should remember that because of laminarity each of the – originally not known – true interval-limits $L(\cdot)$ und $U(\cdot)$ of the field \mathcal{F} is produced either directly via

$$P(A) = [L(A|C) \cdot L(C); U(A|C) \cdot U(C)] \quad (33)$$

if $A \subseteq C$ or through normal completion and so is the structure \mathcal{M} . For each $B \in \mathcal{A}$ the intuitive concept of conditional probability creates an F-field

$$iP(A|B) = \left[\inf_{p \in \mathcal{M}} p(A|B); \sup_{p \in \mathcal{M}} p(A|B) \right]. \quad (34)$$

By definition each component $iP(A|B)$ represents the set of all posterior K-probabilities calculated for elements of the structure \mathcal{M} . As a consequence $iP(A|B)$ as posterior F-probability given B possesses the properties of classical posterior probability.

Example 3.6. The (prior) probability of having a certain disease D is known to be between 0.2 and 0.4. A test producing either positive or negative results can be characterised by (D : disease, S : soundness)

$$\begin{aligned} P(+|D) &= [0.6; 0.8], & P(-|D) &= [0.2; 0.4], \\ P(+|S) &= [0.2; 0.3], & P(-|S) &= [0.7; 0.8]. \end{aligned}$$

If this information is understood to be conditional probability due to the canonical concept, application of (31) and (32) and normal completion produce

the F-field \mathcal{F} , the relevant components of which – support and marginal probabilities – are given in Table 1.

Intuitive conditional-probabilities with respect to the outcome of the test can be calculated as

$$\begin{aligned} iP(D|+) &= [0.33; 0.73], & iP(D|-) &= [0.06; 0.28], \\ iP(S|+) &= [0.27; 0.67], & iP(S|-) &= [0.72; 0.94]. \end{aligned}$$

Depend upon the outcome of the test, the posterior F-probability may be used as prior probability for another test.

4. Independence

The second complement of the system of axioms is produced by the definition of independence. It is reported here in an abridged version, since it materially produces the concept already used by Walley and Fine [5, p. 745]. Let, for a sample-space of four elements,

$$\Omega = E_{11} \cup E_{12} \cup E_{21} \cup E_{22}, \quad (35)$$

two partitions consisting of dichotomies be given:

$$\mathcal{C}_A = \{C_1, C_2\}, \quad \mathcal{C}_B = \{C_1, C_2\} \quad (36)$$

with

$$\begin{aligned} C_1 &= E_{11} \cup E_{12}, & C_2 &= E_{21} \cup E_{22}, \\ C_{.1} &= E_{11} \cup E_{21}, & C_{.2} &= E_{12} \cup E_{22}. \end{aligned} \quad (37)$$

This can be represented in a fourfold table:

E_{11}	E_{12}	C_1
E_{21}	E_{22}	C_2
C_1	C_2	Ω

Definition 4.1. A partially determinate F-probability on $(\Omega; \mathcal{P}(\Omega))$ according to (35) is named marginal probability on the four-fold table if

$$\mathcal{A}_L = \mathcal{A}_U = \{C_1, C_{.1}\}. \quad (38)$$

Table 1

The F-field of Example 3.6; components relevant for the intuitive conditional probability with respect to the outcome of the test

$P(+ \cap D) = [0.12; 0.32]$	$P(- \cap D) = [0.04; 0.16]$	$P(D) = [0.2; 0.4]$
$P(+ \cap S) = [0.12; 0.24]$	$P(- \cap S) = [0.42; 0.64]$	$P(S) = [0.6; 0.8]$
$P(+) = [0.24; 0.54]$	$P(-) = [0.46; 0.76]$	[1]

Let

$$\begin{aligned} P(C_{1.}) &:= [L_1; U_1], & P(C_{2.}) &:= [1 - U_1; 1 - L_1], \\ P(C_{.1}) &:= [L_2; U_2], & P(C_{.2}) &:= [1 - U_2; 1 - L_2]. \end{aligned}$$

Then normal completion produces the following lower interval-limits:

$$\begin{aligned} L(E_{11}) &= \text{Max}(0, L_1 + L_2 - 1), \\ L(E_{12}) &= \text{Max}(0, L_1 - U_2), \\ L(E_{21}) &= \text{Max}(0, L_2 - U_1), \\ L(E_{22}) &= \text{Max}(0, 1 - U_1 - U_2), \\ L(E_{11} \cup E_{12}) &= L_1, \\ L(E_{11} \cup E_{21}) &= L_2, \\ L(E_{11} \cup E_{22}) &= \text{Max}(0, 1 - U_1 - U_2, L_1 + L_2 - 1), \\ L(E_{12} \cup E_{21}) &= \text{Max}(0, L_2 - U_1, L_1 - U_2), \\ L(E_{12} \cup E_{22}) &= 1 - U_2, \\ L(E_{21} \cup E_{22}) &= 1 - U_1, \\ L(E_{11} \cup E_{12} \cup E_{21}) &= \text{Max}(L_1, L_2), \\ L(E_{11} \cup E_{12} \cup E_{22}) &= \text{Max}(L_1, 1 - U_2), \\ L(E_{11} \cup E_{21} \cup E_{22}) &= \text{Max}(1 - U_1, L_2), \\ L(E_{12} \cup E_{21} \cup E_{22}) &= \text{Max}(1 - U_1, 1 - U_2) \end{aligned} \tag{39}$$

and the set of conjugate upper interval-limits as defined by (7).

The structure of this F-probability field is denominated by \mathcal{M}_M .

Using the concept of prestructure one may define independence of the two partitions as a property of a certain F-probability field on $(\Omega; \mathcal{P}(\Omega))$ conforming to the marginal probability.

Definition 4.2. If $\mathcal{F} = (\Omega; \mathcal{A}; L(\cdot))$ is F-probability on the sample-space (35) and \mathcal{M}_M is the structure of the marginal F-probability according to Definition 4.1, then the partitions \mathcal{C}_A and \mathcal{C}_B are mutually independent, provided that the set

$$\mathcal{M}_I := \{p(\cdot) \in \mathcal{M}_M \mid p(E_{ij}) = p(C_{i.})p(C_{.j}); i, j = 1, 2\} \tag{40}$$

serves as a prestructure of \mathcal{F} .

This definition requires that for the F-field \mathcal{F} with independence of \mathcal{C}_A and \mathcal{C}_B all interval-limits have to be just wide enough to include all K-functions which qualify for \mathcal{M}_I by:

1. as well being in accordance with the given marginal-probability (i.e., being elements of \mathcal{M}_M);
2. obeying the classical rule of independence.

The lower interval-limits defined by this prestructure are the following:

$$\begin{aligned}
 L(E_{11}) &= L_1 L_2, \\
 L(E_{12}) &= L_1(1 - U_2), \\
 L(E_{21}) &= (1 - U_1)L_2, \\
 L(E_{22}) &= (1 - U_1)(1 - U_2), \\
 L(E_{11} \cup E_{12}) &= L_1, \\
 L(E_{11} \cup E_{21}) &= L_2, \\
 L(E_{11} \cup E_{22}) &= \text{Min}[U_1 U_2 + (1 - U_1)(1 - U_2), U_1 L_2 + (1 - U_1)(1 - L_2), \\
 &\quad L_1 U_2 + (1 - L_1)(1 - U_2), L_1 L_2 + (1 - L_1)(1 - L_2)], \\
 L(E_{12} \cup E_{21}) &= \text{Min}[L_1(1 - L_2) + L_2(1 - L_1), L_1(1 - U_2) + U_2(1 - L_1), \\
 &\quad U_1(1 - L_2) + L_2(1 - U_1), U_1(1 - U_2) + U_2(1 - U_1)], \\
 L(E_{12} \cup E_{22}) &= 1 - U_2, \\
 L(E_{21} \cup E_{22}) &= 1 - U_1, \\
 L(E_{11} \cup E_{12} \cup E_{21}) &= 1 - (1 - L_1)(1 - L_2), \\
 L(E_{11} \cup E_{12} \cup E_{22}) &= 1 - (1 - L_1)U_2, \\
 L(E_{11} \cup E_{21} \cup E_{22}) &= 1 - U_1(1 - L_2), \\
 L(E_{12} \cup E_{21} \cup E_{22}) &= 1 - U_1 U_2.
 \end{aligned} \tag{41}$$

Again the corresponding upper interval-limits are given by Eq. (7).

It must be remembered that \mathcal{M}_I is a prestructure, defining the interval-limits, but in most cases is not the total structure \mathcal{M} of the F-field \mathcal{F} . If \mathcal{M}_I contains more than one K-function, then $\mathcal{M} \setminus \mathcal{M}_I$ is not empty. Therefore, \mathcal{M} includes elements in accordance with (41) but violating the classical multiplication rule for independent K-probabilities: If \mathcal{F} deviates from classical probability, then the interval-limits tolerate K-functions with slight dependence of \mathcal{C}_A and \mathcal{C}_B .

On the other hand all kinds of deviation from the limits given by (41) would violate the concept of independence: either the interval-limits would exclude elements of \mathcal{M}_I or they would include too many dependent K-functions.

Example 4.1. Marginal probability on a fourfold-table is determinate by the following assessment:

$$\begin{aligned} P(C_{1.}) &= [0.3; 0.5], & P(C_{2.}) &= [0.5; 0.7], \\ P(C_{.1}) &= [0.2; 0.4], & P(C_{.2}) &= [0.6; 0.8]. \end{aligned}$$

With normal completion, the marginal F-probability field is derived:

$$\begin{aligned} P(E_{11}) &= [0.0; 0.4], \\ P(E_{12}) &= [0.0; 0.5], \\ P(E_{21}) &= [0.0; 0.4], \\ P(E_{22}) &= [0.1; 0.7], \\ P(E_{11} \cup E_{12}) &= [0.3; 0.5], \\ P(E_{11} \cup E_{21}) &= [0.2; 0.4], \\ P(E_{11} \cup E_{22}) &= [0.1; 1], \\ P(E_{12} \cup E_{21}) &= [0.0; 0.9], \\ P(E_{12} \cup E_{22}) &= [0.6; 0.8], \\ P(E_{21} \cup E_{22}) &= [0.5; 0.7], \\ P(E_{11} \cup E_{12} \cup E_{21}) &= [0.3; 0.9], \\ P(E_{11} \cup E_{12} \cup E_{22}) &= [0.6; 1], \\ P(E_{11} \cup E_{21} \cup E_{22}) &= [0.5; 1], \\ P(E_{12} \cup E_{21} \cup E_{22}) &= [0.6; 1]. \end{aligned}$$

According to (41) and (7) partitions \mathcal{C}_A and \mathcal{C}_B are independent iff

$$\begin{aligned} P(E_{11}) &= [0.06; 0.20], \\ P(E_{12}) &= [0.18; 0.40], \\ P(E_{21}) &= [0.10; 0.28], \\ P(E_{22}) &= [0.30; 0.56], \\ P(E_{11} \cup E_{12}) &= [0.30; 0.50], \\ P(E_{11} \cup E_{21}) &= [0.20; 0.40], \\ P(E_{11} \cup E_{22}) &= [0.50; 0.62], \\ P(E_{12} \cup E_{21}) &= [0.38; 0.50], \\ P(E_{12} \cup E_{22}) &= [0.60; 0.80], \\ P(E_{21} \cup E_{22}) &= [0.50; 0.70], \\ P(E_{11} \cup E_{12} \cup E_{21}) &= [0.44; 0.70], \\ P(E_{11} \cup E_{12} \cup E_{22}) &= [0.72; 0.90], \\ P(E_{11} \cup E_{21} \cup E_{22}) &= [0.60; 0.82], \\ P(E_{12} \cup E_{21} \cup E_{22}) &= [0.80; 0.94] \end{aligned}$$

holds.

The comparison of marginal probability and independent probability shows that a remarkable sharpening of the intervals is caused by independence.

For instance,

$$p_1(\cdot) \text{ with } \begin{array}{ll} p_1(E_{11}) = 0.00, & p_1(E_{12}) = 0.40, \\ p_1(E_{21}) = 0.30, & p_1(E_{22}) = 0.30 \end{array}$$

is an element of \mathcal{M}_M , but not of \mathcal{M} , while

$$p_2(\cdot) \text{ with } \begin{array}{ll} p_2(E_{11}) = 0.12, & p_2(E_{12}) = 0.28, \\ p_2(E_{21}) = 0.18, & p_2(E_{22}) = 0.42 \end{array}$$

represents an independent K-probability in \mathcal{M}_M , therefore being an element of \mathcal{M}_I and consequently of \mathcal{M} . On the other hand

$$p_3(\cdot) \text{ with } \begin{array}{ll} p_3(E_{11}) = 0.10, & p_3(E_{12}) = 0.20, \\ p_3(E_{21}) = 0.20, & p_3(E_{22}) = 0.50 \end{array}$$

shows no independence between lines and columns, consequently being not an element of \mathcal{M}_I , but belonging to \mathcal{M} , since $p_3(\cdot)$ is in accordance with all of the limits (41).

In the theory of interval probability the concept of mutually independent partitions in an F-field among other aspects provides the fundamentals for a *Weak Law of Large Numbers*.

At first it serves to define independent identically F-distributed (i.i.F-d.) samples. This is demonstrated by means of a simple model sufficient for the purpose of studying relative frequencies.

Definition 4.3. Let $\mathcal{F}_n = (\Omega^n; \mathcal{P}(\Omega^n); L_n(\cdot))$ be an F-field with $\Omega^n = \times_{i=1}^n \Omega_i$, $\Omega_i = E_{i,1} \cup E_{i,2}$, $i = 1, \dots, n$. Partitions \mathcal{C}_i are given by $\mathcal{C}_i = \{C_{i,1}, C_{i,2}\}$, $i = 1, \dots, n$, with

$$C_{i,r} = \Omega_1 \times \dots \times \Omega_{i-1} \times E_{i,r} \times \Omega_{i+1} \times \dots \times \Omega_n, \quad r = 1, 2. \quad (42)$$

Then \mathcal{F}_n describes an i.i.F-d. sample of size n , provided that the marginal probabilities are:

1.

$$\left. \begin{array}{l} P(C_{i,1}) = [L; U], \\ P(C_{i,2}) = [1 - U; 1 - L], \end{array} \right\} \quad i = 1, \dots, n, \quad (43)$$

2. \mathcal{C}_i and $\mathcal{C}_{i'}$ are mutually independent, $i, i' = 1, \dots, n$, $i \neq i'$.

The relative frequency of $E_{\cdot 1}$ is defined by the \mathcal{F}_n -random variable

$$T^{(n)} = \frac{1}{n} \sum_{i=1}^n T_i, \quad T_i(E_{i,1}) = 1, \quad T_i(E_{i,2}) = 0. \quad (44)$$

In order to arrive at a Weak Law of Large Numbers the concept of convergence in F-probability has to be introduced.

Definition 4.4. With respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ let $(X_n)_{n \in \mathbb{N}}$ be \mathcal{F}_n -random variables in \mathbb{R}^1 . For $-\infty < \alpha \leq \beta < +\infty$ the random event $A_n(\epsilon)$ is defined by

$$A_n(\epsilon) = \bigcup \{E \subseteq \Omega_n \mid \alpha - \epsilon \leq X_n(E) \leq \beta + \epsilon\}. \quad (45)$$

Then $(X_n)_{n \in \mathbb{N}}$ is convergent in F-probability into $[\alpha, \beta]$ iff for every (ϵ, δ) , $\epsilon > 0$, $0 < \delta < 1$, there exists a $N(\epsilon, \delta)$, such that for all $n \geq N(\epsilon, \delta)$

$$L_n(A_n(\epsilon)) \geq 1 - \delta \quad (46)$$

holds.

Using the concepts of Definitions 4.3 and 4.4 the following statement can be proven:

If for all $n \in \mathbb{N}$ the F-field \mathcal{F}_n describes an i.i.F-d. sample of size n with marginal probabilities given by (43), then the \mathcal{F}_n -random variable $T^{(n)}$ defined by (44) is convergent in F-probability into $[L; U]$.

One proof can be derived from [5, Theorem 4.1, pp. 747–748].

This generalization of Bernoulli's theorem allows a frequency interpretation of interval-probability: for a long i.i.F-d. sample with marginal F-probability $P(\text{"success"}) = [L; U]$ the relative frequency of successes at last almost surely will be found in $[L; U]$. If $L < U$, then it is not possible to know in which part of $[L; U]$ the relative frequency will be found and whether the sequence will be convergent in the classical sense.

5. Conclusions

Despite all differences between the schools concerning the meaning of probability assessments, it might be useful for everybody to consider the logical implications of an assignment containing interval-probability, as described by Bernoulli's theorem which can be supplemented by results concerning the asymptotic behaviour of posterior F-probability generated by Bayes' theorem [11, Section 1.5].

A comprehensive study of the theory is in progress, the first volume will be published in 2000 [11].

An important aspect of the theory not mentioned in the present article is the use of linear optimization to solve fundamental problems on finite sample-

spaces. Some results of this type are already reported in [9]. The concept of uniform F-probability and its use in describing sampling is briefly described in [8] – together with consequences concerning an improvement of the principle of insufficient reason.

Among those aspects not mentioned in the foregoing sections is that of decision theory. A general approach to decision problems with respect to behavioural viewpoints is made possible by the theory. Ellsberg's results and their consequences can be respected. Behaviour under ambiguity can be analysed and classified. A preliminary report is found in [10]. One of the many problems concerning statistical methodology under interval-probability has yet been studied thoroughly: testing statistical hypotheses. Fundamental results are given in [1].

Altogether the unifying concept for uncertainty contained in the theory of interval-probability produces a great number of aspects which deserve intensive research and will create many chances for methodological improvements.

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