Probability

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1 Probability Spaces

1.1 Classical Probability Spaces

Textbook probability theory [1, 2, 4] is defined using the notions of a *sample space* Ω, a space of *events* $\mathcal{F}$, and a *probability measure* μ. In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events $\mathcal{F}$ as, 2^{Ω}, the powerset of Ω. A *probability measure* is a function $\mu : \mathcal{F} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i, we have $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two coin experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

1.2 Quantum Probability Spaces

A classical model decides the occurrence or non-occurence of all events simultaneously which is inconsistent with quantum mechanics. Indeed, in the quantum world, there are (non-commuting) events which cannot happen simultaneously. To accommodate this situation, we completely abandon the sample space Ω and define and reason directly about events. Thus a quantum probability space will consist of just two components: a set of events $\mathcal{A}$ and a probability measure $\phi : \mathcal{A} \to [0, 1]$. These components are defined as follows [3, 5].

We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{A}$ as *projections* on $\mathcal{H}$. Similarly to the classical case, a probability measure is a function $\phi : \mathcal{A} \to [0, 1]$ satisfying:

- $\phi(\mathbb{1}) = 1$, and
- for all $A \in \mathcal{A}$, we have $\phi(A^* A) \geq 0$.

Yu-Tsung says: If we follow [3, 5], then we also need

- ϕ can be extended to a linear functional $\phi : alg(\mathcal{A}) \to \mathbb{C}$, where $alg(\mathcal{A})$ is the minimal $*$-algebra generated by $\mathcal{A}$.

As an example, let $P_1, P_2, \ldots, P_k$ be mutually orthogonal projections on $\mathcal{H}$ with sum $\mathbb{1}$ and define the event space $\mathcal{A}$ to be the linear span of these operators:

$$\mathcal{A} = \{\sum_{j=1}^{k} \lambda_j P_j \mid \lambda_i, \ldots, \lambda_k \in \mathbb{C}\}.$$

Yu-Tsung says: $\left\{\sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_i, \ldots, \lambda_k \in \mathbb{C}\right\}$ is the minimal $*$-algebra generated by $P_1, P_2, \ldots, P_k$, but it contains all possible observables $P_1, P_2, \ldots, P_k$ can generate (and something more) not just projections. For example, $2\mathbb{1} \in \left\{\sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_i, \ldots, \lambda_k \in \mathbb{C}\right\}$.

Each state $|\psi\rangle$ of the Hilbert space induces a probability measure $\phi_\psi : \mathcal{A} \to [0, 1]$ defined as follows:

$$\phi_\psi(A) = \langle\psi|A\psi\rangle$$

Yu-Tsung says: So ϕ_ψ maps the projections generated by $P_1, P_2, \ldots, P_k$ to $[0, 1]$, and maps $\left\{\sum_{j=1}^{k} \lambda_j P_j \;\middle|\; \lambda_i, \ldots, \lambda_k \in \mathbb{C}\right\}$ to $\mathbb{C}$...

Concrete example: consider the two qubit Hilbert space with computational bases $|0\rangle$ and $|1\rangle$ and consider the following families of projections:

- Family I: $|0\rangle\langle 0|$, $|1\rangle\langle 1|$
- Family II: $|+\rangle\langle +|$, $|-\rangle\langle -|$

and consider the two states ...

1.3 Plan

Several assumptions are woven in the definition of a quantum probability space:

- the Hilbert space $\mathcal{H}$;
- the real interval $[0, 1]$;
- the fact that each state induces a probability measure, i.e., the Born rule;
- the fact that every probability measure is induced by a state, i.e., Gleason's theorem

In the remainder of the paper, we examine each of these assumptions and consider variations motivated by computation in a world with limited resources. In particular, we will consider a variant of the Hilbert space over finite fields; we will consider set-valued probability measures; we will consider ways other than the Born rule in which a state can induce a probability measure, and we will consider probability measures that may come from information beyond the quantum states.

References

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