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Generalizations of Kochen and Specker's theorem and the effectiveness of Gleason's theorem

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Abstract

Kochen and Specker's theorem can be seen as a consequence of Gleason's theorem and logical compactness. Similar compactness arguments lead to stronger results about finite sets of rays in Hilbert space, which we also prove by a direct construction. Finally, we demonstrate that Gleason's theorem itself has a constructive proof, based on a generic, finite, effectively generated set of rays, on which every quantum state can be approximated.

Keywords: Gleason's theorem; Logical compactness; Constructive mathematics

1. Introduction

Gleason's theorem (Gleason, 1957) plays an important role in the foundations of quantum mechanics. On the positive side it demonstrates how the probabilistic structure of quantum theory follows from its logical structure, that is, the geometry of Hilbert spaces. On the negative side, the theorem puts a severe constraint on possible hidden-variable interpretations of quantum mechanics. The present paper deals with two seemingly distinct aspects of Gleason's theorem. Firstly, the question whether it can be proved by constructive mathematical principles; and secondly, the

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relations between Gleason's theorem and the Kochen and Specker (KS) theorem. We shall see that these two issues are mathematically and conceptually related.

On the first issue, Hellman (1993) argued that physics in general, and quantum mechanics in particular, require the application of non-constructive mathematics, such as the axiom of choice. Gleason's theorem was one of the examples he chose to demonstrate this point. Indeed, among the principles underlying Gleason's proof is the assertion that every continuous function on a compact space obtains its minimum. This principle is non-constructive because we cannot always explicitly show where the minimum is obtained. However, Billinge (1997) pointed out that there is an alternative constructive formulation of Gleason's theorem, and a constructive proof soon followed (Richman & Bridges, 1999).

Apart from its intrinsic value this issue should be of particular interest to those who hold that quantum mechanics is a theory of information. Since all the information that we shall ever possess about physical systems is finite, the fully fledged Hilbert space structure must be an idealization. There is nothing wrong with idealizations, of course, it is only that one would like to know what exactly is being idealized, that is, what is the real skeleton of the theory. The application of non-constructive mathematics in a theorem that concerns the structure of *states* (which encode our information) only magnifies this problem.

The first to notice the importance of Gleason's theorem to the hidden-variables question was Bell. Bell (1966) even proved a simple version of Gleason's theorem. It states that there does not exist a bi-valued probability function on the rays (one-dimensional subspaces) of a Hilbert space of dimension $\geqslant 3$. A later improvement (Kochen & Specker, 1967) showed that there is a *finite* subset of rays on which no bi-valued probability function exists. A natural question to ask is why KS is considered an improvement. If, for example, contextuality follows from the infinite case, what advantage could we gain from the finite version? A possible answer: the finite version shows that contextuality is not a by-product of our idealizations. Rather, it is a property of a finite set of observables, each observable having a finite set of possible values. In other words, contextuality can be expressed in terms of a simple proposition about finitely many rays (or projections).

The purpose of this paper is to demonstrate that all conceptually interesting aspects of (finite-dimensional) quantum mechanics share this characteristic. Indeed, our main result (Theorem 3) implies that Gleason's theorem, and all the peculiarities of quantum probability originating from it, are finitely generated in a very precise sense. Given any margin of error $\varepsilon > 0$, and any finite set of rays Γ_0 in a Hilbert space, we can find a finite set of rays Γ, $\Gamma \supset \Gamma_0$, with the following property: For any probability function p defined on the rays of Γ there is a (fully defined) quantum state which is ε close to p on the elements of Γ_0. This means that the orthogonality relations among the rays in Γ already force a (near) quantum behavior of the probabilities assigned to the elements of Γ_0. Moreover, the set Γ can be effectively generated by an algorithm.

A good intuitive tool for understanding the issues raised in this paper is the compactness theorem of logic. It states that a set of first-order propositions has a model if, and only if, each of its finite subsets has a model. In the next section we shall show how to use the compactness theorem, in conjunction with Gleason's theorem, to prove results about finite sets of rays, without actually constructing them explicitly. In Section 3, we give explicit constructive proofs to some of these results. In the last section there is the proof of the main result.

Our purpose here is to state and prove the theorems which, we hope, help to clarify the logical relations between various aspects of quantum theory. We do not present arguments in favor or against any particular interpretation of quantum mechanics. However, it is not difficult to trace the quantum logical motivation behind the results. Elsewhere (Pitowsky, 2003) it is argued that quantum probability can be seen as Bayesian betting ratios in finite gambles *provided* that we take the logical relations among the observables seriously. Here we complete this picture. As a consequence of our theorems one can consistently maintain that "quantum gambles", that is, finite sets of observables together with their logical relations, form the real skeleton of quantum theory. The rest can be seen as an idealization.

2. Gleason's theorem and logical compactness

The Kochen and Specker's theorem (Kochen & Specker, 1967) is often regarded as an improvement on the infinite argument based on Gleason's theorem (see, for example, Held, 2000). It is less often noted that KS can actually be derived from Gleason's theorem. More precisely, the fact that *there is* a finite set of rays on which no bi-valued probability function exists follows from the application of the compactness theorem to Gleason's theorem (Pitowsky, 1998; a similar point is made in Bell, 1996). Of course, KS provided an explicit construction of the set and not just an abstract proof of its existence. However, the chain of reasoning which begins with the compactness theorem has the advantage that it is easily generalizable. We can use logical compactness to prove the existence of finite sets of rays with interesting features. Subsequently, we can attempt to construct them explicitly. In Pitowsky (1998) some such results are pointed out, and some are also given a constructive proof. In this section we formulate this procedure rigorously, and apply it to increasingly complex propositions.

Let $\mathbb{H}$ be a Hilbert space of a finite dimension $n \geqslant 3$ over the complex or real field. A non-negative real function p defined on the unit vectors in $\mathbb{H}$ is called *a state on* $\mathbb{H}$ if the following conditions hold:

(1) $p(\alpha x) = p(x)$ for every scalar $\alpha, |\alpha| = 1$, and every unit vector $x \in \mathbb{H}$.
(2) If $x_1, x_2, \ldots, x_n$ is an orthonormal basis in $\mathbb{H}$ then $\sum_{j=1}^{n} p(x_j) = 1$.

Gleason's theorem characterizes all states:

Theorem 1. *Given a state p, there is a Hermitian, non-negative operator W on* $\mathbb{H}$, *whose trace is unity, such that* $p(x) = (x, Wx)$ *for all unit vectors* $x \in \mathbb{H}$, *where* (,) *is the inner product*.

Gleason's original proof (Gleason, 1957) of the theorem has three parts: The first is to show that every state p on $\mathbb{R}^3$ is continuous. The second part is a proof of the theorem in the case of $\mathbb{R}^3$, and the third part is a reduction of the general theorem to $\mathbb{R}^3$. The theorem is also valid in the infinite-dimensional case which we shall not consider.

Let us make more precise what are the formal logical assumptions underlying the proof of Gleason's theorem. For simplicity, we shall concentrate on the three-dimensional real case which comprises the first two parts of Gleason's proof. All our results are extendable to any real or complex Hilbert space of a finite dimension $n \geqslant 3$.

Consider the first-order formal theory of the real numbers (that is, a first-order theory of some standard model $\mathbb{R}$ of the real numbers). This induces a theory of $\mathbb{R}^3$, together with the inner product, and the unit sphere $\mathbb{S}^2$. Add to this first-order theory a function symbol $p : \mathbb{S}^2 \to \mathbb{R}$. Let the Greek letters α, β, γ denote variables ranging over the reals and x, y, z be variables ranging over $\mathbb{S}^2$. Now add the axioms:

G1. $\forall x\, p(x) \geqslant 0$.
G2. $\forall x\, p(-x) = p(x)$.
G3. *For each orthonormal triple* $x, y, z \in \mathbb{S}^2$ *an axiom*: $p(x) + p(y) + p(z) = 1$.

Note that in G3 we do not use the universal quantifier. Instead G3 is an axiom schema with a continuum of propositions. The next axiom is just the statement that every set of reals which is bounded from below has a greatest lower bound. However, since we want to use only first-order formulae we write it as an axiom schema:

G4. *For every one place predicate of reals* $A(.)$ *expressible in our language an axiom*

$$\exists\beta\forall\alpha(A(\alpha)\to\alpha\geqslant\beta)\to\exists\beta[\forall\alpha(A(\alpha)\to\alpha\geqslant\beta)\wedge\forall\varepsilon>0\exists\gamma(A(\gamma)\wedge\beta>\gamma-\varepsilon)].$$

Thus, for example, the claim that p itself has an infimum will follow from G1 and the application of G4 to the predicate $\exists x(p(x) = \alpha)$. As a matter of fact, the proof of Gleason's theorem requires twice the application of G4, that is, for two predicates $A(.)$.

The proof that p is continuous depends on the axioms G1–G4. One can see from Gleason's original proof (1957), or more directly from Pitowsky (1998) that only a finite number of applications of the schema G3 are required.

To prove the second part, that every state on $\mathbb{R}^3$ is given by a self-adjoint, non-negative, trace one operator, another axiom is needed:

G5. *If p is continuous then its minimum and maximum are obtained*:

$$\forall\varepsilon>0\exists\delta>0\forall x,y(\|x-y\|<\delta\to|p(x)-p(y)|<\varepsilon)$$
$$\to\exists x,y\forall z(p(x)\leqslant p(z)\leqslant p(y)).$$

An elementary way to complete the proof of Gleason's theorem on the basis of G5 is in Cooke, Keane, and Moran (1984) or Richman and Bridges (1999). Here the proof is based on a limiting process. Using the continuity of p, which has been proved from G1–G4, the claim that p obtains its minimum and maximum follows from G5. Using the minimum and maximum points of p, one determines the operator W, which is the candidate to represent it. Then one proves that for all $\varepsilon > 0$ the proposition $\forall x(|p(x) - (x, Wx)| < \varepsilon)$ holds, which completes the proof.

With these observations it is easy to see how Kochen and Specker's theorem follows from Gleason's theorem. Consider the proposition:

F1. *There is a state p such that* $\forall x(p(x) = 0 \vee p(x) = 1)$.

Now, the conjunction of F1 with G1–G4 is inconsistent, since the latter imply that p is continuous. Hence, there is a proof of a contradiction from G1–G4 + F1. The proof of that contradiction uses only finitely many cases of schema G3 (since any proof is finite). If one collects the directions $x \in \mathbb{S}^2$ which appear in that proof one gets a finite set of directions on which no two-valued probability function exists. To put it directly in terms of the compactness theorem: Since G1–G4 + F1 do not have a model, there is a finite subset of G3 which together with G1, G2, G4, and F1 fails to have a model. Of course this argument does not yield an explicit set, but it may serve as an incentive to look for one, which might have been Kochen and Specker's motivation. A similar argument was explicitly used in Clifton (1993). He simply lifted the vectors which appear in Bell's (1966) simplified version of Gleason's theorem to obtain a KS theorem. See also, Fine and Teller (1978) and Pitowsky (1982).

The argument just presented can be easily generalized to include many more propositions which contradict Gleason's theorem. Let $\Gamma \subset \mathbb{S}^2$ be a finite set such that $x \in \Gamma \rightarrow -x \in \Gamma$. We shall say that $p : \Gamma \rightarrow \mathbb{R}$ *is a state on* Γ if p satisfies G1–G3 for all directions *in* Γ. Now, consider the statement

F2. *There is a state p that has exactly k values* ($k \geqslant 2$). *In other words, p satisfies the proposition*:

$$\mathscr{A}_k = \exists x_1, x_2, \ldots, x_k \bigwedge_{i \neq j} (p(x_i) \neq p(x_j)) \\ \wedge \forall y(p(y) = p(x_1)) \vee \cdots \vee (p(y) = p(x_k)).$$

This contradicts Gleason's theorem since, again by continuity, if p has two or more values it has infinitely many. Hence, for all $k \geqslant 2$ there is a finite set Γ which contains elements $x_1, x_2, \ldots, x_k, y$ among others, and such that any state p on Γ which assigns k distinct values to $x_1, x_2, \ldots, x_k$ assigns a different value to y. Also, taking the disjunction $\bigvee_{k=2}^{n} \mathscr{A}_k$, we obtain by the same method that *for each $n \geqslant 2$ there is a finite set Γ_n such that every non-constant state p on Γ_n has at least n values*. We shall give below an explicit construction of Γ_n in a somewhat more restricted context.[1]

So far we have used only the continuity of p, which is proved by G1–G4, but Gleason's theorem puts more severe restrictions on states than continuity. Conceptually, one of the important outcomes of Gleason's theorem is the

[1] This result has been used in Breuer (2002) to give an argument against the "nullification" of KS (Meyer, 1999; Clifton & Kent, 2000; see also, Pitowsky, 1983, 1985; Appleby, 2002).

indeterminacy principle. Casting it in our language it says that any two non-orthogonal, non-opposite directions cannot both have extreme probability values (zero or one) unless they are both zero. To see the finite version consider the opposite statement:

F3. *There is a state p such that*

$$\exists x, y(0 < (x, y) < 1) \wedge ((p(x) = p(y) = 1) \vee (p(x) = 1 \wedge p(y) = 0) \vee (p(x) = 0 \wedge p(y) = 1)).$$

Since F3 is false we can prove the following: *Given any x, y with $0 < (x, y) < 1$ there is a finite set Γ such that $x, y \in \Gamma$, and every state p on Γ satisfies $p(x), p(y) \in \{0, 1\} \leftrightarrow p(x) = p(y) = 0$.* This is the logical indeterminacy principle (Pitowsky, 1998) which has been proved by an explicit construction; a simplified construction is given below. Note that this result is stronger than KS since it is constraining every *probability* distribution on Γ, and not merely the "truth values". It is "logical" in the sense that it follows from the orthogonality relations alone.

We can obtain more dramatic results of this kind, using the fact that by Gleason's theorem $p(x) = (x, Wx)$. However, recall that this consequence is derived in the form $\forall \varepsilon > 0 \forall x(|p(x) - (x, Wx)| < \varepsilon)$. We should therefore be careful when moving to finite subsets. Let us begin with the simple example of a pure state. If we know that $p(z_0) = 1$ then, by Gleason's theorem, $p(x) = |(z_0, x)|^2$ for all x. The statement $(p(z_0) = 1) \wedge \exists x(p(x) \neq |(z_0, x)|^2)$ contradicts Gleason's theorem, but it is refuted by showing that given x, and given $\varepsilon > 0$ the condition $|p(x) - |(z_0, x)|^2| < \varepsilon$ is satisfied. Hence, one cannot expect to be able to fully force the equality relation $p(x) = |(z_0, x)|^2$ in the finite case, but only an approximate relation. Therefore, consider:

F4. *There is a state p that satisfies $(p(z_0) = 1) \wedge \exists x(|p(x) - |(z_0, x)|^2| > \varepsilon)$ for some fixed $\varepsilon > 0$.*

Proposition F4 clearly contradicts Gleason's theorem. Using our method we conclude: *For all $\varepsilon > 0$ and $z_0, x \in \mathbb{S}^2$ there is a finite set of directions Γ such that z_0, $x \in \Gamma$, and every state p on Γ satisfies*: $p(z_0) = 1 \rightarrow |p(x) - |(z_0, x)|^2| < \varepsilon$. Obviously, this is also true for any finite number of directions beside x. The general case, that of a mixture W, follows the same pattern. Here it is not enough to specify the value of p at one point z_0. Rather, five points are needed since, in general, W is a 3×3, self-adjoint, non-negative matrix with trace unity. Given these points $z_1, \ldots, z_5$, and the values $p(z_i) = \alpha_i$, we find for each x and $\varepsilon > 0$ a finite set on which the conditions $p(z_i) = \alpha_i$ imply $|p(x) - (x, Wx)| < \varepsilon$. In order to construct this set of directions explicitly one can painstakingly follow the steps of the constructive proof of Richman and Bridges (1999), and "lift" the vectors in the proof. An alternative to this tedious procedure is presented in Section 4, where it is shown that there is a generic algorithmic way to construct such sets (and on the way demonstrate again that Gleason's theorem has a constructive proof). All these theorems are easily extendable to any real or complex Hilbert space of a finite dimension $n \geqslant 3$.

The inverse of these compactness results is the claim that there are very large subsets $\Omega \subset \mathbb{S}^2$ on which bi-valued states do exist. The "size" of such possible Ω depends on set-theoretic assumptions. For example, if the continuum hypothesis is assumed to hold, there is an Ω whose intersection with every major circle C in $\mathbb{S}^2$ satisfies $|C \backslash \Omega \cap C| \leqslant \aleph_0$. Weaker assumptions lead to "smaller" sets (Pitowsky, 1983, 1985).

3. Some constructions

In this section, we shall be using rays (one-dimensional subspaces) rather than unit vectors, and take states to be defined on them. Given a Hilbert space $\mathbb{H}$, the assumption $p(\alpha x) = p(x)$ for every scalar $\alpha, |\alpha| = 1$, and every unit vector $x \in \mathbb{H}$ implies that p actually depends on the ray and not on the unit vector we choose to represent it. Our first aim is to prove the logical indeterminacy principle. The proof here is simpler than the one given in Pitowsky (1998) and is based on the "lifting" of the vectors in an argument of Piron (1976).

Theorem 2. *Let a and b be two non-orthogonal rays in a Hilbert space* $\mathbb{H}$ *of finite dimension* $\geqslant 3$. *Then there is a finite set of rays* $\Gamma(a, b)$ *such that* $a, b \in \Gamma(a, b)$ *and such that a state p on* $\Gamma(a, b)$ *satisfies* $p(a), p(b) \in \{0, 1\}$ *only if* $p(a) = p(b) = 0$.

Proof. First, consider the three-dimensional real space $\mathbb{R}^3$. If z and q are two rays in that space there is a unique great circle which they determine. Let q' be the ray orthogonal to both z and q and let q'' be the ray orthogonal to both q and q'. Now, consider a great circle through q and q' (Fig. 1).

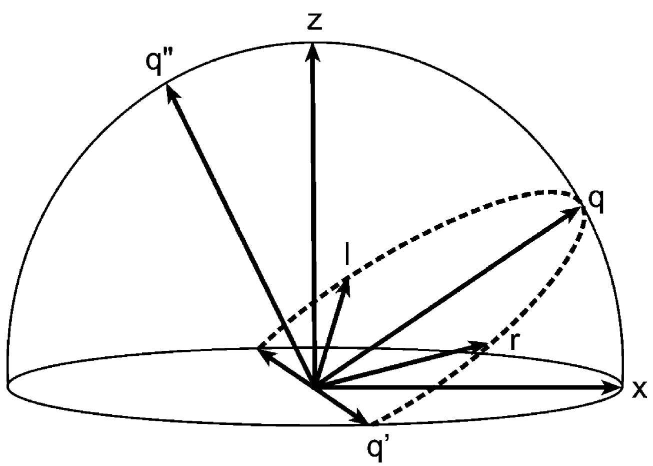

Fig. 1.

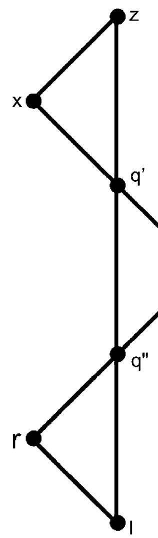

Fig. 2.

If r is any ray through this great circle then $r \perp q''$. Let l be the ray orthogonal to both r and q''. The orthogonality relations between z, q, q', q'', l, and r are given in the graph $G = G(z, q, r)$ (Fig. 2).

Subsequently, we shall loosely identify sets of rays with their orthogonality graphs. If p is a state defined on the rays in the graph G then $p(z) = 1$ entails $p(q) \geqslant p(r)$. Indeed, $p(q) + p(q') + p(q'') = p(r) + p(l) + p(q'') = 1$. Also, since $p(z) = 1$ we have $p(q') = 0$. Hence $p(q) = p(r) + p(l) \geqslant p(r)$.

The relation between the points z, q, and r can be best depicted on the projective plane, where z is taken as the pole of projection (Fig. 3).

In the projective plane great circles appear as straight lines, and latitudes (relative to z as the pole) appear as concentric circles. In the projective plane q is on a line through z, call this line $\mathscr{L}(z, q)$, and r is on the line through q which is perpendicular to $\mathscr{L}(z, q)$ at q.

Next, consider three points z, q, r which do not necessarily have that relation. Assume only that $\measuredangle zq < \measuredangle zr$ so that if z is the north pole, then r is more to the south than q. In this case we can find a finite sequence of points $q_1, q_2, \ldots, q_m$ with $q_1 = q$ and $q_m = r$ and such that q_{k+1} is on the line perpendicular to $\mathscr{L}(z, q_k)$ for $k = 1, 2, \ldots, m-1$. A case with $m = 5$ is considered in Fig. 4.

The number m of intermediate points depends on the difference $\measuredangle zr - \measuredangle zq$ and on the respective longitude of q and r.

Given this set of vectors we can construct for each k a graph $G_k = G(z, q_k, q_{k+1})$ in which z, q_k, q_{k+1} play the role of z, q, r, respectively (note, z is the same throughout). Let $G'(z, q, r) = \bigcup_{k=1}^{m-1} G_k$ then any state on G' that satisfies $p(z) = 1$ also satisfies

$$p(q) = p(q_1) \geqslant p(q_2) \geqslant \cdots \geqslant p(q_n) = p(r).$$

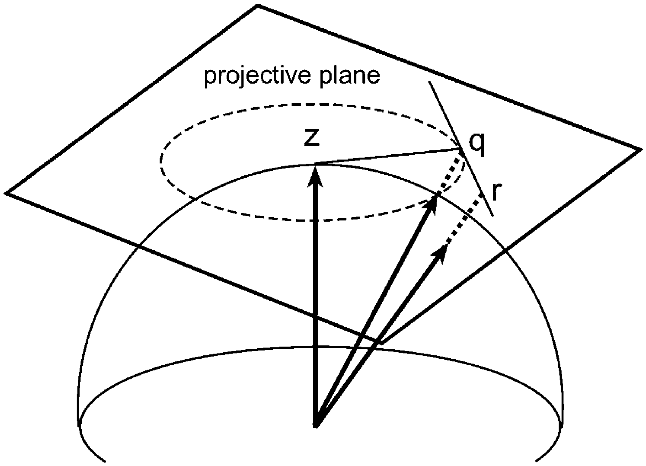

Fig. 3.

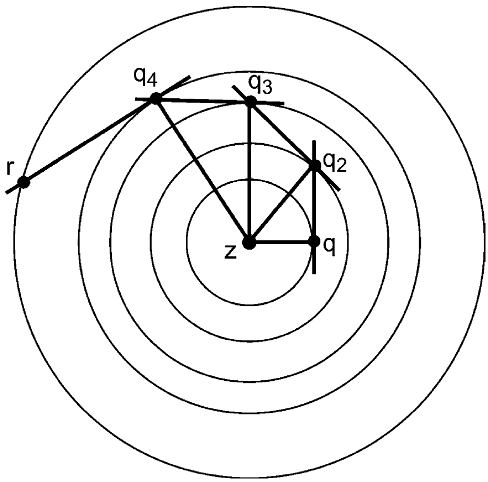

Fig. 4.

To finish the proof for $\mathbb{R}^3$ let a, b be two non-orthogonal rays. We can always choose a sequence of rays $c_1, c_2, \ldots, c_n$ such that $\measuredangle ab > \measuredangle bc_1 > \measuredangle c_1c_2 > \cdots > \measuredangle c_{n-1}c_n$ and such that $a \perp c_n$. A case with $n = 4$ is depicted in Fig. 5.

Consider b as a pole (projection point) and construct a graph $G'_0 = G'(b, c_1, a)$ which is like $G'(z, q, r)$ with b, c_1, a play the role of z, q, r, respectively. If p is a state on the rays in this graph with $p(b) = 1$, then $p(c_1) \geqslant p(a)$. Consider c_1 as a pole and construct the graph $G'_1 = G'(c_1, c_2, b)$, which is like $G'(z, q, r)$, with c_1, c_2, b play the role of z, q, r, respectively. For a probability function p on G'_1 which satisfies $p(c_1) = 1$ we get $p(c_2) \geqslant p(b)$. Now construct the graph G'_2 with c_2 as pole and c_3, c_1 play the role of q, r, respectively, etc. Suppose that p is a state on $G''(a, b) =$

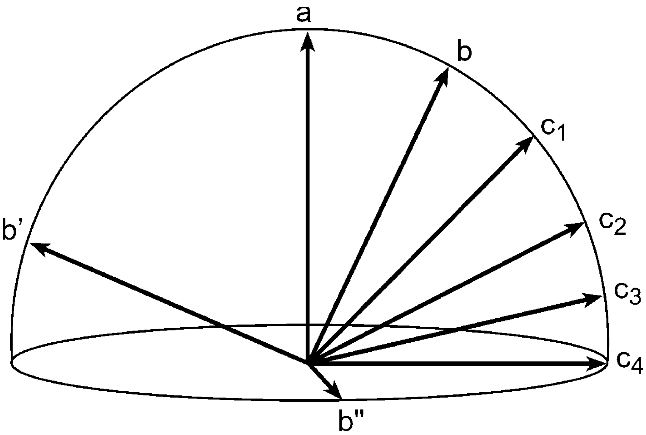

Fig. 5.

$\bigcup_{j=0}^{n-1} G'_j$ the union of all these graphs. We shall show that if $p(a) = 1$ then $p(b) < 1$. Assume, by negation, that $p(a) = p(b) = 1$ then, by construction, $p(c_1) \geqslant p(a)$ so that $p(c_1) = 1$. But then $p(c_2) \geqslant p(b)$ so that $p(c_2) = 1$, and $p(c_3) \geqslant p(c_1)$ so that $p(c_3) = 1$, and so on and we finally obtain $p(c_n) = 1$. This is a contradiction since $a \perp c_n$. Hence $p(b) < 1$.

Now, consider the ray b' which is orthogonal to b in the plane spanned by a and b, and let b'' be the ray orthogonal to both b and b'. Repeat the construction of the graph with b' instead of b to obtain the graph $G''(a, b')$, and add this to the previous graph. Let p be a state on the graph $G''(a, b) \cup G''(a, b')$ with $p(a) = 1$, then $p(b) < 1$ and $p(b') < 1$. But then we also have $p(b) > 0$. Otherwise, $p(b) = 0$ together with $p(b'') = 0$ (as $b'' \perp a$ and $p(a) = 1$) entail $p(b') = 1$. This is a contradiction. Hence $p(a) = 1$ entails $0 < p(b) < 1$. Inverting the roles of a and b we construct a graph

$$\Gamma(a, b) = G''(a, b) \cup G''(a, b') \cup G''(b, a) \cup G''(b, a').$$

With a' a vector orthogonal to a in the plane spanned by a and b. Let p be a state on $\Gamma(a, b)$. If $p(b) = 1$ then $0 < p(a) < 1$, and if $p(a) = 1$ then $0 < p(b) < 1$. Therefore $\Gamma(a, b)$ is the required set of rays in $\mathbb{R}^3$.

In the general case of a finite-dimensional Hilbert space $\mathbb{H}$ we do the following: Given rays a, b in $\mathbb{H}$, we consider them first as rays in a three-dimensional subspace $\mathbb{H}'$of $\mathbb{H}$ and complete the construction there. Then we add to the finite set of rays in $\mathbb{H}'$ additional $\dim \mathbb{H} - 3$ orthogonal rays in the orthocomplement of $\mathbb{H}'$. This completes the proof. □

The above construction entails that pure states should have strictly monotone behavior.

Lemma 1. *Given a ray z in a Hilbert space $\mathbb{H}$ of a finite dimension $\geqslant 3$, and rays a and b such that $0 < \measuredangle(a, z) < \measuredangle(b, z)$ then there is a finite set of rays $D(z, a, b)$, which contains z, a, and b, such that every state p on $D(z, a, b)$ for which $p(z) = 1$ also satisfies $p(a) > p(b)$.*

Proof. Consider first $\mathbb{R}^3$. Given a ray z, let q be any ray different from z and not orthogonal to it. Consider once more the vectors in Fig. 1 and their orthogonality graph $G = G(z,q,r)$ in Fig. 2. Now denote

$$D_1(z,q,r) = G(z,q,r) \cup \Gamma(z,l),$$

where $\Gamma(z,l)$ is the set of rays in Theorem 2 with z,l fulfilling the role of a,b. The situation is depicted in Fig. 6.

If p is a probability function on $D_1(z,q,r)$ with $p(z)=1$ then, by construction, $p(q) = p(r) + p(l)$, and also $0 < p(l) < 1$, hence $p(q) > p(r)$. Since $\measuredangle(a,z) < \measuredangle(b,z)$ we can find a sequence $q_0, q_1, \ldots, q_m$ such that $q_0 = a$ and $q_m = b$ and for all $k = 1,2,\ldots,m$ the ray q_k is on the great circle through q_{k-1} and q'_{k-1}, the ray orthogonal to z and q_{k-1}. Putting

$$D(z,a,b) = \bigcup_{k=1}^{m} D_1(z,q_{k-1},q_k)$$

we get that if $p(z)=1$ then

$$p(a) = p(q_0) > p(q_1) > \cdots > p(q_m) = p(b).$$

If $\mathbb{H}$ is a Hilbert space with $3 \leqslant \dim \mathbb{H} < \infty$, complete the construction first on the three-dimensional space $\mathbb{H}_1$ spanned by z, a, and b (in case they all lie in the same plane form $\mathbb{H}_1$ by adding any ray orthogonal to them). Subsequently add a set of orthogonal rays in $\mathbb{H}_1^{\perp}$ to complete the construction. $\square$

An immediate consequence of this lemma is the construction of finite sets of rays on which states must take many values.

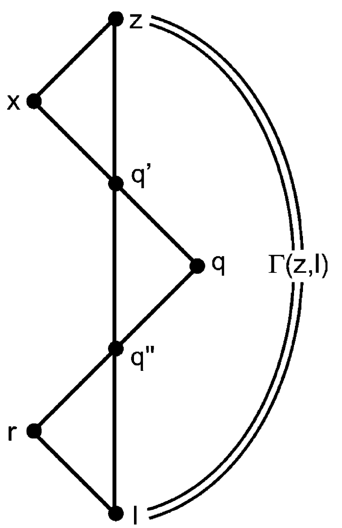

Fig. 6.

Corollary 1. *Given a ray z in a Hilbert space $\mathbb{H}$ of a finite dimension $\geqslant 3$, and an integer k, there is a finite set of rays $\Lambda_k(z)$ such that every state p on $\Lambda_k(z)$ for which $p(z) = 1$ has at least k distinct values.*

4. The effectiveness of Gleason's theorem

Recently, there has been an interesting discussion on the question whether Gleason's theorem has a proof which is acceptable by the standards of constructive mathematics (Hellman, 1993; Billinge, 1997). The discussion culminated in Richman and Bridges (1999), who gave a constructive formulation and proof of the theorem. Our aim is to give a (much shorter) proof of the conditional statement: *If Gleason's theorem is true then it must have an effective proof.*

The result follows from a generic sequence of approximations of states on finite sets. As will become clear subsequently the approximations in question are determined by an algorithm. Again, we shall consider the three-dimensional case but the results generalize immediately. To simplify matters, we shall work with general *frame functions*, not (positive normalized) states. This means that we replace G1 and G3, respectively, by the axioms:

G′1. *There is a real constant γ such that $\forall x p(x) \geqslant \gamma$.*
G′3. *There is a real constant δ such that for each orthonormal triple $x, y, z \in \mathbb{S}^2 : p(x) + p(y) + p(z) = \delta$.*

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ be the standard basis in $\mathbb{R}^3$ (or $\mathbb{C}^3$) and $b_{ij} = \frac{1}{\sqrt{2}}(e_i + e_j)$, $1 \leqslant i < j \leqslant 3$.

Denote by $\mathscr{F}$ the set of all functions $p : \mathbb{S}^2 \to \mathbb{R}$ that satisfy G′1, G2 and G′3 and let $\mathscr{F}_0 = \{p \in \mathscr{F};\ p(e_i) = p(b_{ij}) = 0,\ i = 1,2,3,\ 1 \leqslant i < j \leqslant 3\}$. Then

Lemma 2. *The following statements are equivalent:*

(a) *If $p \in \mathscr{F}$ there is a self-adjoint operator W in $\mathbb{R}^3$ such that $p(x) = (x, Wx)$.*
(b) *Every element $p \in \mathscr{F}_0$ vanishes identically.*

Proof. If $p \in \mathscr{F}_0$ then by (a) we have $p(x) = (x, Wx)$. But then $(e_i, We_i) = 0$ and $(b_{ij}, Wb_{ij}) = 0$. The latter equation implies that $(e_i, We_j) = 0$ for $1 \leqslant i < j \leqslant 3$. Hence $W = 0$ and therefore $p = 0$.

Conversely, assume (b) holds, let $p \in \mathscr{F}$ and let W be the symmetric matrix which satisfies the equations $p(e_i) = (e_i, We_i)$ and $p(b_{ij}) = (b_{ij}, Wb_{ij})$ for $i = 1,2,3,\ 1 \leqslant i < j \leqslant 3$. Denote $p_0(x) = p(x) - (x, Wx)$. Then $p_0 \in \mathscr{F}$ and $p_0(e_i) = p_0(b_{ij}) = 0$, and therefore $p_0 \in \mathscr{F}_0$. By (b) $p_0 = 0$ and thus $p(x) = (x, Wx)$. $\square$

By Gleason's theorem (a) is true, so we can take (b) as our formulation of this theorem. In the following, we shall work with the first-order formalization of the field of real numbers, the theory of real closed fields, denoted by **R**, or the formalization of the field of complex numbers, the theory of algebraically closed fields (with zero characteristic), denoted by **C**. In both theories there is an effective elimination of quantifiers. This means that there is a (known) algorithm which, given any well-formed formula as input, produces as output an equivalent formula without the quantifiers $\forall$, $\exists$. Consequently, the theories are decidable: there is an algorithmic method to prove every true proposition in them.[2] Also, we shall denote by $\mathbf{R}_-$ the section of **R** without multiplication (including only the addition operation and the inequality relation).

As a result of the elimination of quantifiers every definable set (without parameters) in **R** is a finite Boolean combination of sets of n-tuples, each defined by a rational polynomial inequality (or equality). Thus, for example, $\mathbb{S}^2$ is the set $\{(\alpha_1, \alpha_2, \alpha_3); \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\}$ and therefore definable in **R**. Similarly $\mathbb{S}^2_+$, "the northern hemisphere", which is like $\mathbb{S}^2$ with the additional condition $(\alpha_3 > 0) \vee (\alpha_3 = 0 \wedge \alpha_1 > 0) \vee (\alpha_3 = 0 \wedge \alpha_1 = 0 \wedge \alpha_2 = 1)$, is also definable in **R**. Similar remarks apply to **C** and the unit sphere in the complex, finite-dimensional Hilbert spaces.

We shall consider the general setup where X is some definable set of n-tuples, and Y is a definable set of m-tuples from X. We let $\mathscr{F}_0$ be the set of real-valued functions p on X, such that $\sum_{i=1}^m p(x_i) = 0$ for all $y = (x_1, \ldots, x_m) \in Y$. Our first aim is to prove the following: *If every bounded function of* $\mathscr{F}_0$ *vanishes identically in every model of* **R** (*or* **C**) *then this fact admits an effective proof.*

The connection with Gleason's theorem is as follows: We let $X = \mathbb{S}^2_+$, and Y is the set of triples from X given by $Y = B \cup \{(e_i, e_i, e_i); i = 1, 2, 3\} \cup \{(b_{ij}, b_{ij}, b_{ij}); 1 \leqslant i < j \leqslant 3\}$. Here B is the set of orthogonal triples from $\mathbb{S}^2_+$ and e_i, b_{ij} as in Lemma 2. B is clearly definable since the inner product of $x_1 = (\alpha_1, \alpha_2, \alpha_3)$, $x_2 = (\beta_1, \beta_2, \beta_3)$ is a polynomial in the α_i's and β_i's. Also, the b_{ij}'s are given by a simple formula (for example, $b_{12} = (\alpha, \alpha, 0)$ with $2\alpha^2 - 1 = 0$). In sum, Y is definable. In this case $\mathscr{F}_0$ is the set of real-valued functions p which satisfy $p(x_1) + p(x_2) + p(x_3) = 0$ for all $(x_1, x_2, x_3) \in Y$. By Lemma 2 the conclusion $p = 0$ for all $p \in \mathscr{F}_0$ is equivalent to Gleason's theorem.

Now for a natural number $k > 1$ let

$$\mathscr{S}_k(n) = \left\{ S \subset \{0, \ldots, n\}^m : \mathbf{R}_- \vdash (\forall\alpha_0) \ldots (\forall\alpha_n) \left(\bigwedge_{s \in S} \left(\sum_{i=1}^m \alpha_{s(i)} = 0 \right) \right. \right.$$
$$\left. \left. \rightarrow \bigvee_{j=1}^n (k|\alpha_0| \leqslant |\alpha_j|) \right) \right\}.$$

Let **T** stand for either **R** or **C** then we have:

Lemma 3. *Let X be a definable set of n-tuples, Y a definable set of m-tuples from X, and $\mathscr{F}_0$ the set of real-valued functions p on X, such that $\sum_{i=1}^m p(x_i) = 0$ for all*

[2] For details on these theories see Shoenfield (1967). For recent results on quantifier elimination see Basu (1999).

$y = (x_1, \ldots, x_m) \in Y$. Then the following are equivalent:

(1) *Every bounded function in $\mathscr{F}_0$ vanishes in every model of* $\mathbf{T}$.
(2) *For some n, $\mathbf{T} \vdash (\forall x_0)(\exists x_1)\ldots(\exists x_n) \bigvee_{S \in \mathscr{S}_k(n)} \bigwedge_{s \in S} (y_s \in Y)$ where $y_s = (x_{s(1)}, \ldots, x_{s(m)})$.*

Proof. Let $p \in \mathscr{F}_0$. If (2) holds for n, then there is $S \in \mathscr{S}_k(n)$ such that $\sum_{i=1}^{m} p(x_{s(i)}) = 0$ for all $s \in S$. By the definition of $\mathscr{S}_k(n)$ we have $|p(x_0)| \leqslant (1/k)|p(x_j)|$ for some $1 \leqslant j \leqslant n$. But by (2) for each $x_0 \in X$ we can find such $x_j \in X$. We conclude therefore that if $\beta > 0$ bounds p, so does $(1/k)\beta$. Thus repeating the argument we have proved $p = 0$.

Conversely, assume that (2) fails. Note that the formulas

$$\phi_n(x_0) = (\exists x_1)\ldots(\exists x_n) \bigvee_{S \in \mathscr{S}_k(n)} \bigwedge_{s \in S} ((x_{s(1)}, \ldots, x_{s(m)}) \in Y)$$

define an increasing chain of sets $W_k(n) = \{a; \phi_n(a)\}$, so their complements $\overline{W_k}(n) = \{a; \neg\phi_n(a)\}$ form a decreasing chain, and by assumption no element is empty. By the compactness theorem (or Gödel's completeness theorem), there exists a model $\mathscr{M}$ of $\mathbf{T}$ and $a_0 \in |\mathscr{M}|$ such that $a_0 \in \overline{W}_k(n)$ for all n.

We now show that some $p \in \mathscr{F}_0(\mathscr{M})$ is non-zero. For each $a \in X(\mathscr{M})$, let c_a be a new (and different) constant symbol; let $\bar{\mathbf{R}}$ be the theory consisting of the axioms of $\mathbf{R}_-$ and for each $(a_1, \ldots, a_m) \in Y(\mathscr{M})$ the axiom $\sum_{i=1}^{m} c_{a_i} = 0$, and for each $a \in X(\mathscr{M})$ the axiom $|c_a| < k|c_{a_0}|$. Then $\bar{\mathbf{R}}$ is consistent, so it also has a model. Hence, there exists a function g on $X(\mathscr{M})$ into an ordered Abelian group $(B, <)$ with $\sum g(a_i) = 0$ for all $(a_1, \ldots, a_m) \in Y(\mathscr{M})$ and with $|g(a)| < k|g(a_0)|$ for all $a \in X(\mathscr{M})$. In particular, $b_0 := |g(a_0)| > 0$. Also, there exists a unique homomorphism $H : B \to \mathbf{R}$ with $H(z) \geqslant 0$ whenever $z \geqslant 0$, and $H(b_0) = 1$. Now $p = H \circ g$ is a non-zero element of $\mathscr{F}_0(\mathscr{M})$, and therefore (1) fails. $\square$

This establishes the existence of a constructive proof for Gleason's theorem in complex Hilbert spaces. In this case, the condition "every bounded function in $\mathscr{F}_0$ vanishes" holds in every model of $\mathbf{C}$, iff it holds in $\mathbb{C}$.[3] This means that condition (2) of the theorem states that Gleason's theorem is true iff $\forall x_0 \phi_n(x_0)$ is provable in $\mathbf{C}$ for some n. But $\mathbf{C}$ has an effective decision procedure.

In the real case there is some complication because we do not assume Gleason's theorem for every model, but only for the standard model; though a posteriori it will follow for every model. Therefore, we have to bridge the gap between "all models of $\mathbf{R}$" and the standard model $\mathbb{R}$ for which Gleason's theorem is known to be true. From now on we specialize to the concrete Y relevant to Gleason's theorem. More specifically, we shall take X to be the real projective plane, using its identification with the northern hemisphere $\mathbb{S}^2_+$ of the previous section (Fig. 3). With this identification (and with the point at infinity added) X is compact and Y is closed. It follows that the sets $W_k(n)$, being projections of closed sets, are also closed sets, and

[3] Knowing the theorem holds for $\mathbb{C}$ automatically implies it holds for every model, since $\mathbb{C}$ is universal for countable models of $\mathbf{C}$; however, $\mathbb{R}$ does not have that property.

so the sets $\overline{W}_k(n)$ are open; but all we need is their measurability with respect to the uniform measure on the sphere, and the measurability of their intersections with major circles with respect to the uniform measure on the circle. Since $W_k(n)$ are definable sets, the subsets of the circle that will be mentioned below are finite unions of segments (i.e. arcs) and their measure is elementary.

Theorem 3. *The conditions below, referring to the model* $\mathbb{R}$, *are equivalent.*

(1) *Every bounded function in* $\mathscr{F}_0$ *vanishes.*
(2) *For some* n, $\mathbf{T}\vdash(\forall x_0)(\exists x_1)\ldots(\exists x_n)\bigvee_{S\in\mathscr{S}_2(n)}\bigwedge_{s\in S}(y_s\in Y)$ *where* $y_s=(x_{s(1)},\ldots,x_{s(m)})$. (*that is, for some* n, $\overline{W}_2(n)=\emptyset$).
(3) *For some* n, *for every great circle* C, *the set* $C\cap W_4(n)$ *has normalized measure* $>\frac{1}{2}$ *in* C.
(4) $\bigcap_{n\geqslant 1}\overline{W}_8(n)=\emptyset$.

Condition (2), while phrased for $\mathbb{R}$, is a single elementary statement; thus if true it holds in every real closed field, and this fact admits an elementary proof. So in the special case of Gleason's theorem, we can replace "in every model of **R**" by "in $\mathbb{R}$".

Proof. We will use the following kind of Fubini principle: Let C_z be the great circle orthogonal to the point z on the sphere. For $C=C_z$ let μ_C be the normalized uniform measure on C_z; and say that a property P holds "for a majority of points" if $\mu_C(\{x:P(x)\})\geqslant\frac{1}{2}$. Let C be a fixed great circle and suppose that a majority of points $z\in C$ are such that for a majority of points $q\in C_z$, the property $P(q)$ holds. Then, assuming $\{x:P(x)\}$ is measurable on the sphere, it has an area $\geqslant A$, where A is the area of the unit sphere above the 45th northern latitude. The reason is that the region $\{x:P(x)\}$ takes up $\geqslant\frac{1}{4}$ of the area of a Mercator map of the sphere; and the region above the 45th latitude has the greatest distortion under this map. Moving to the projective plane X, we can say that $\{x:P(x)\}$ takes an area $\geqslant\frac{1}{2}$ of X, when we use the area measure induced on X by the Mercator map.

Claim 1. *If* $z\in\overline{W}_k(2n+2)$, *then a majority of* $x\in C_z$ *lie in* $\overline{W}_{2k}(n)$.

Indeed, in the projective plane model, a point x on a given circle $C=C_z$, has a unique point $ort(x)$ orthogonal to it on C_z. Suppose the claim fails. So let $z\in\overline{W}_k(2n+2)$ and assume that a set of points of C_z of measure $>\frac{1}{2}$ lies in $W_{2k}(n)$. Since the function ort is a measure-preserving bijection on C_z, the set $\{y:ort(y)\in W_{2k}(n)\}$ also has measure $>\frac{1}{2}$. So these two sets intersect; thus there exist $x,y\in C_z$ such that $x\perp y$ and $x,y\in W_{2k}(n)$. But as x,y,z form an orthogonal triple, $(x,y,z)\in Y$, so it is easy to see that $z\in W_k(2n+2)$; this is a contradiction which proves the claim.

Condition (4) implies (3): Assuming (4), there exists n_4 such that $W_8(n_4)$ has a measure $>A$, that is, Mercator measure $>\frac{1}{4}$. Let $n_3=2n_4+2$. Let C be a great circle. Suppose, by negation, that a majority of points of C lie in $\overline{W}_4(n_3)$. Then, by the claim, for each of these points $z\in\overline{W}_4(n_3)\cap C$, a majority of points x of C_z must lie in

$\overline{W}_8(n_4)$. So this set has (Mercator) area $\geqslant \frac{1}{4}$. Moving to the projective plane we get a contradiction. Thus (3) holds.

Condition (3) implies (2): Assume (3) holds for $n = n_3$, and let $n_2 = 2n_3 + 2$. If $y \in \overline{W}_2(n_2) \neq \phi$, let C_y be the orthogonal great circle. Then by the claim a majority of points on C_y lie in $\overline{W}_4(n_3)$, contradicting (3).

Condition (2) implies (1): The same as in Lemma 3.

Condition (1) implies (4): As soon as there is $a_0 \in \bigcap_{n \geqslant 1} \overline{W}_8(n)$, the proof of Lemma 3 works and provides a non-vanishing bounded function in $\mathscr{F}_0$. $\square$

The theorem provides an effective procedure to prove $p = 0$ in case every $p \in \mathscr{F}_0$ vanishes. The reason is that **R** (and $\mathbf{R}_-$) is decidable, which means that there is a computer program which proves every true propositions of **R**, in particular the proposition $(\forall x_0)\phi_n(x_0)$ for a suitable integer n. Any apparent ineffectiveness in the proof of Theorem 3 is in some sense self-eliminating, as a posteriori one knows the existence of an effective proof too. Nevertheless, it may be worth remarking that all the sets occurring in the proof are definable in **R**. These sets are known to have a simple structure, and there would be no difficulty in formalizing the proof in a very small part of Peano arithmetic.

As a consequence of Theorem 1 we obtain:

Corollary 2. *For any direction x_0 there is a finite, fixed size set $\Gamma \subset \mathbb{S}^2_+$ which include x_0, the e_i's and b_{ij}'s such that any $p \in \mathscr{F}_0(\Gamma)$ satisfy $|p(x_0)| \leqslant \frac{1}{2}|p(x)|$ for some $x \in \Gamma$.*

Here $\mathscr{F}_0(\Gamma)$ is the set of functions $p : \Gamma \to \mathbb{R}$ which satisfy the conditions $p(e_i) = p(b_{ij}) = 0$ and $p(x) + p(y) + p(z) = 0$ for every orthogonal triple $x, y, z \in \Gamma$. Iteration of this process yields a constructive proof of Gleason's theorem. It would be nice to give an explicit construction of the set Γ with this property. As noted, there is a general algorithm to find it, but such algorithm may not be feasible to execute in practice since the decision procedure for **R** has a worst case doubly exponential time lower bound, see Basu (1999).

We shall end with a few comments on the boundedness condition in Gleason's theorem. Suppose that we drop the requirement that the functions $p \in \mathscr{F}_0$ are bounded, then the conclusion $p = 0$ is simply false. One can see this from the following cardinality consideration.

Denote by $\mathscr{G}$ the class of real functions f satisfying $f(x) + f(y) + f(z) = 0$ for every orthogonal triple $x, y, z \in \mathbb{S}^2_+$. By Gleason's theorem the cardinality of all *bounded* functions in $\mathscr{G}$ is $2^{\aleph_0}$. Note that $\mathscr{G}$ (including its unbounded functions) is closed under composition with any additive group homomorphism $\mathbb{R} \to \mathbb{R}$; i.e. if $g \in \mathscr{G}$ and $h : \mathbb{R} \to \mathbb{R}$, $h(x + y) = h(x) + h(y)$, then $h \circ g \in \mathscr{G}$. Let p be a bounded element of $\mathscr{G}$ given by some non-zero matrix with trace zero. If one assumes the axiom of choice, there are $\beth_2$ additive group homomorphisms from $\mathbb{R}$ to $\mathbb{R}$. Composing them with p, we see that $\mathscr{G}$ is enormous, and so certainly Gleason's theorem fails without the boundedness requirement.

On the other hand, consider a model of set theory in which the axiom of choice fails (Solovay, 1970), or a model in which every set has the property of Baire.

Then every function $\mathbb{R} \to \mathbb{R}$ is continuous on a set whose complement is meager; if it is also a group homomorphism, it will be continuous everywhere. Thus, with this kind of negation of the axiom of choice, it is not implausible that the boundedness of frame functions might be automatic. Coming back to the remark at the end of Section 2, we also note that in such models there are no large $\Omega \subset \mathbb{S}^2$ on which bi-valued states exist (Shipman, 1990).

It seems instructive at all events to compare Lemma 3 to what one obtains when Y satisfies the stronger assumption, that *every* function (bounded or otherwise) vanishes. Then one can replace in Lemma 3(2) the reference to $\mathscr{S}_k(n)$ by

$$\mathscr{S}_\infty(n) = \left\{ S \subset \{0, \dots, n\}^m : \right.$$
$$\left. \mathbf{R}_- \vdash (\forall \alpha_0) \dots (\forall \alpha_n) \left(\bigwedge_{s \in S} \left(\sum_{i=1}^{m} \alpha_{s(i)} = 0 \right) \to (\alpha_0 = 0) \right) \right\}.$$

The proof is the same as that of Lemma 3, mutatis mutandis, noting that when B is a divisible Abelian group and $0 \neq b \in B$, there exists a homomorphism $H : B \to \mathbf{R}$ with $H(b) \neq 0$ (not necessarily order preserving).

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