Statistics and Probability Letters 82 (2012) 1043–1048

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Set valued probability and its connection with set valued measure

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A R T I C L E I N F O

Article history:
Received 16 October 2011
Accepted 23 February 2012
Available online 1 March 2012

Keywords:
Set valued probability
Set valued measure
Atom

A B S T R A C T

Set valued probability theory is used to analyze and model highly uncertain probability systems. In this work a set valued probability is defined over the measurable space. The range of set valued probability is the set of subsets of the unit interval. Some basic properties and the connection with set valued measures are discussed.

1. Introduction

Uncertainty is usually modeled by a probability distribution, and treated using techniques from probability theory. Such an uncertainty model will often be inadequate in cases where insufficient information is available to identify a unique probability distribution. In that case, imprecise probabilities aim to represent and manipulate the really available knowledge about the system. Imprecise probability is a generic term for covering all mathematical models which measure chance or uncertainty without sharp numerical probabilities. It includes both qualitative (comparative probability, partial preference orderings, ...) and quantitative modes (interval probabilities, possibility theory, belief functions, upper and lower previsions, upper and lower probabilities, ...).

A highly imprecise probabilistic system could be formalized using the theory of set valued random variables—random sets, or using the theory of imprecise probability. Some (very few) of the results can be found in the papers Akbari and Rezaei (2009), Augustin and Coolen (2004), Choquet (1953–1954), Coolen and Coolen-Schrijner (2006), Dempster (1967), Fishburn (1986), Manski (2003), Stojaković (2011, 1994a,b), Troffaes (2007), Walley (1991), Weichselberger (2000), Zadeh (2002).

In this work, a new concept of imprecise probability—set valued probability—is introduced. Set valued probability is defined over the measurable space with values in the set of subsets of the unit interval. The method of restricted set arithmetics (Klir and Pan, 1998) is used to treat the probabilities which are set valued but in spite of that the sum of all the individual probabilities is 1. That kind of set valued probability still has some nice properties—it is normed and *additive, where *additivity is the additivity with respect to addition in restricted arithmetics. One can consider this concept as the extension and generalization of the classical model of probability theory. This model is suitable for generalizing the single, interval valued model and the model which is based on distributions with parameters which are set valued, which is the general idea in statistical interval estimation. It turns out that the set valued probability is strongly connected with set valued measure. In this work the connections in both ways (set valued probability $\leftrightarrow$ set valued measure) are investigated. Since there are no assumptions about convexity, this theory can be used to model and analyze probabilistic systems where the values of probability are highly imprecise but discrete.

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0167-7152/$ – see front matter
doi:10.1016/j.spl.2012.02.021

2. Preliminaries

For the convenience of the reader, we give a list of symbols used in this work: By $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ we denote the set of reals, nonnegative reals, natural numbers. $(\Omega, \mathcal{A})$ is a measurable space where $\mathcal{A}$ is a σ-algebra on Ω. $\mathcal{K}_{(f)\,(k)\,(c)}(\mathbb{R})$ are sets of nonempty ((closed) (compact) (convex)) subsets of $\mathbb{R}$. If $A \subset \Omega$, A' is the complement of A with respect to Ω. By h we denote Hausdorff metric on $\mathcal{K}_f(\mathbb{R})$ defined by $h(A, B) = \max\{\sup_{y\in B} \inf_{x\in A} |x-y|, \sup_{x\in A} \inf_{y\in B} |x-y|\}$, $A, B \in \mathcal{K}_f(\mathbb{R})$. For $A \subset \mathbb{R}$, $|A| = h(A, \{0\}) = \sup_{x\in A} |x|$. $\bar{A}$ and $\bar{\text{co}}A$ denote the closure and convex closure of A.

Set valued measure is a natural generalization of the single-valued measure. Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ a σ-algebra of measurable subsets of the set Ω. If $M : \mathcal{A} \to \mathcal{K}(\mathbb{R})$ is a mapping such that for every sequence $\{A_i\}_{i\in N}$ of pairwise disjoint elements of $\mathcal{A}$ the following equality is satisfied: $M(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} M(A_i)$, where $\sum_{i=1}^{\infty} M(A_i) = \{x \in \mathbb{R}, x_i \in M(A_i) : x = \sum_{i=1}^{\infty} x_i(\text{uncond.conv.})\}$, and $M(\emptyset) = \{0\}$, then M is a set valued measure. By $|M| : \mathcal{A} \to \mathbb{R}_+$ we denote the single-valued positive measure (called the variation) defined by $|M|(A) = \sup \sum_{i=1}^{n} |M(A_i)|$, $A \in \mathcal{A}$, where the supremum is taken over all finite measurable partitions $\{A_i\}_{i=1}^{n}$ of A. If M is a positive set valued measure, $|M|(A) = |M(A)|$. A set valued measure M is of bounded variation iff $|M|(\Omega) < \infty$. A set valued measure M is σ-finite if there exists a countable measurable partition $\{A_k\}_{k\in\mathbb{N}}$ of Ω such that for every $k \in \mathbb{N}$, $|M|(A_k) < \infty$. M is a complete set valued measure if $\mathcal{A}$ contains all subsets of measure zero. With S_M we denote the set of measure selectors $m : \mathcal{A} \to \mathbb{R}$ of M.

3. Set valued probability

Definition 3.1. Let $(\Omega, \mathcal{A})$ be a measurable space and let $P : \mathcal{A} \to 2^{[0,1]}$ be a set valued set function such that:

1. for every $A \in \mathcal{A}$, $P(A) \neq \emptyset$,
2. for every $A \in \mathcal{A}$ and every $x \in P(A)$, there exists a probability selector $p : \mathcal{A} \to [0, 1]$ of P, such that $p(A) = x$.

Then P is a set valued probability.

If P is a set valued probability, by S_P we denote the set of all probability selectors of P.

Example. Let $(\Omega, \mathcal{A})$ be a measurable space and let $\{p_h\}_{h\in H}$, $H \subseteq \mathbb{R}$, be a family (collection) of probabilities, $p_h : \mathcal{A} \to [0, 1]$. The mapping $P : \mathcal{A} \to 2^{[0,1]}$ defined by $P(A) = \{p_h(A),\ h \in H\}$ is a set valued probability and $\{p_h\}_{h\in H} \subseteq S_P$.

Definition 3.2. Let $P : \mathcal{A} \to 2^{[0,1]}$ be a set valued probability. The positive valued set function $|P| : \mathcal{A} \to \mathbb{R}_+$, called the variation of P, is defined by $|P|(A) = \sup \sum_{i=1}^{n} |P(A_i)|$, where the supremum is taken over all finite measurable partitions $\{A_i\}_{i=1}^{n}$ of $A \in \mathcal{A}$.

Notice that $|P|$ is not a probability function and that $|P|(A) \neq |P(A)|$.

Definition 3.3. Set valued probability P is σ-finite iff there exists a countable measurable partition $\{A_k\}_{k\in\mathbb{N}}$ of Ω such that for every $k \in \mathbb{N}$, $|P|(A_k) < \infty$.

Here is an example of a set valued probability which is not σ-finite.

Example. Let $\Omega = [0, 1]$ and $\mathcal{A} = \mathcal{B}([0, 1])$, where $\mathcal{B}$ denotes a Borel σ-algebra. A set valued probability is defined by

$$P(A) = \left\{ k\,\mathcal{L}\left(A \cap \left[0, \frac{1}{k}\right]\right) : k \in \mathbb{N} \right\}, \quad A \in \mathcal{A},$$

where $\mathcal{L}$ is the Lebesgue measure. For every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. A countable partition of $(0, \varepsilon)$ is made up of the set of points $\{\frac{1}{n} : n = n_0, n_0 + 1, \ldots\}$. Since $|P(\frac{1}{n+1}, \frac{1}{n}]| = \frac{1}{n}$, we get $|P|(0, \varepsilon) > \sum_{n=n_0}^{\infty} \frac{1}{n} = \infty$.

Any countable partition of $[0, 1]$ must contain an interval $(0, \varepsilon)$ for some $\varepsilon > 0$, which implies that P is not σ-finite.

Definition 3.4. A set valued probability P is of bounded variation iff $|P|(\Omega) < \infty$.

It is obvious that bounded variation implies σ-finiteness. The next example shows that the converse statement is not true. In this example a set valued probability that is σ-finite but not of bounded variation is presented.

Example. Let $\mathcal{A} = \mathcal{B}(\mathbb{R}_+)$ be a Borel σ-algebra of $\Omega = \mathbb{R}_+$ and let

$$P(A) = \left\{ p_k(A) = \frac{\mathcal{L}(A \cap [0, k])}{k} : k \in \mathbb{N} \right\},$$

where $\mathcal{L}$ is the Lebesgue measure. Then

$$|P|(\Omega) = |P|(\mathbb{R}_+) = \sup\left\{\sum_{i=1}^{n} \sup|P(A_i)|, \{A_i\}_{i=1}^{n} \text{ meas.part.of } \Omega\right\}$$
$$= \sup\left\{\sum_{i=1}^{n} \sup_{k\in\mathbb{N}} p_k(A_i)\right\} = \sup\left\{\sum_{i=1}^{n} |P|(A_i),\ \{A_i\}_{i=1}^{n} \text{ meas.part.of } \Omega\right\}.$$

There exists a partition $\cup_{i=0}^{n}[i, i+1) \cup [n+1, \infty)$ of Ω such that $|P|(A_i) > \frac{1}{i+2}$, where $A_i = [i, i+1)$. Since the sequence $\sum \frac{1}{n}$ diverges, for every $M > 0$ there exists a $k \in \mathbb{N}$ such that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{k+1} > M$. This means that for every $M > 0$, $|P|(\Omega) > M$, implying that P is not of bounded variation.

But P is σ-finite. If we consider a countable partition $\Omega = \cup_{i=0}^{\infty}[i, i+1)$, then for every element $[i, i+1)$, $i \in \mathbb{N}$, of that partition, we get

$$|P|([i, i+1)) = \sup\left\{\sum_{k=1}^{n} |P|(A_k), \{A_k\}_{k=1}^{n} \text{ meas.part.of } [i, i+1)\right\} = \frac{1}{i+1} < \infty,$$

meaning that there exists a partition $\{A_i\}_{i\in\mathbb{N}}$ of Ω such that for every $i \in \mathbb{N}$, $|P|(A_i) < \infty$.

Definition 3.5. Let $P : \mathcal{A} \to 2^{[0,1]}$ be a set valued probability and S_P be related set of probability selectors. Then for all $A, B \in \mathcal{A}$,

$$P(A) \overset{*}{+} P(B) \overset{\text{def}}{=} \{\, p(A) + p(B),\ p \in S_P \,\}$$

and for all $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}$,

$$* \sum_{k=1}^{\infty} P(A_k) \overset{\text{def}}{=} \left\{\sum_{k=1}^{\infty} p(A_k),\ p \in S_P\right\}.$$

Theorem 1. *If P is a set valued probability, then:*

1. $P(\emptyset) = 0,\ P(\Omega) = 1$.
2. $P(A + B) \subseteq P(A) + P(B)$.
3. $P(A \cup B) \subseteq P(A) + P(B) - P(A \cap B)$.
4. $1 \in P(A) + P(A')$.
5. $A \subseteq B \Rightarrow P(A) \subseteq P(B)$.
6. $P(A + B) = P(A) \overset{*}{+} P(B)$.
7. $P(\sum_{k=1}^{\infty} A_k) = * \sum_{k=1}^{\infty} P(A_k)$.
8. *If P is a set valued probability, then $\bar{P} : \mathcal{A} \to \mathcal{K}_k([0, 1])$ defined by $\bar{P}(A) = \mathrm{cl}\, P(A)$ and $\bar{\mathrm{co}}\, P : \mathcal{A} \to \mathcal{K}_{kc}([0, 1])$ defined by $(\bar{\mathrm{co}}\, P)(A) = \bar{\mathrm{co}}\,(P(A))$ are set valued probabilities.*
9. *If P is σ-finite (of bounded variation), then $\bar{P}$ and $\bar{\mathrm{co}}\, P$ are σ-finite (of bounded variation) too.*

Proof. 1–7, 9. The proofs are elementary, so they are omitted.

8. If $x \in \bar{P}(A)$, then there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subset S_P$ such that $\lim_{n\to\infty} p_n(A) = x$. We shall show that there exists a probability measure $p : \mathcal{A} \to [0, 1]$ such $p(A) = x$ and that for all $B \in \mathcal{A}$, $p(B) \in \bar{P}(B)$. Since $\{p_n\}_{n\in\mathbb{N}}$ is a sequence in the product space $\Pi_{A\in\mathcal{A}} \mathrm{cl}\, P(A)$, which is compact in the product topology, the cluster point p of $\{p_n\}_{n\in\mathbb{N}}$ can be selected. Then there exists a subsequence $\{p_m\}_{m\in\mathbb{N}} \subset \{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p_m(B) = p(B)$, for all $B \in \mathcal{A}$. Obviously, $p(A) = x$, $p(\Omega) = 1$ and $p(B) \in \bar{P}(B)$ for all $B \in \mathcal{A}$. Since $\{p_m\}_{m\in\mathbb{N}}$ is a sequence of probabilities on $\mathcal{A}$ and for every $B \in \mathcal{A}$, $\lim_{m\to\infty} p_m(B) = p(B)$, by the Nikodym theorem we get that p is σ-additive on $\mathcal{A}$ and that σ-additivity is uniform on $\mathcal{A}$ for $m \in \mathbb{N}$.

Further, if $x \in (\mathrm{co}\, P)(A) = \{\lambda p(A) + (1 - \lambda) q(A),\ p, q \in S_P, \lambda \in [0, 1]\}$, then for every $B \in \mathcal{A}$ there exists a $\lambda \in [0, 1]$ such that $\lambda p(B) + (1 - \lambda) q(B) \in (\mathrm{co}\, P)(B)$. Now, applying the previous proof, we can conclude that $(\bar{\mathrm{co}}\, P)(A)$ is a set valued probability. $\square$

4. Generated set valued measure

Theorem 2. *Let $P : \mathcal{A} \to 2^{[0,1]}$ be a set valued probability. If $M : \mathcal{A} \to 2^{\mathbb{R}^+}$ is a mapping defined by*

$$M(A) = \left\{\sum_{i=1}^{n} p_i(A_i), \{A_i\}_{i=1}^{n} \subset \mathcal{A},\ \textit{meas.fin.part.of } A, \{p_i\}_{i=1}^{n} \subseteq S_P, n \in \mathbb{N}\right\},$$

1. *then M is the minimal set valued measure containing P,*
2. *if P is σ-finite set valued probability, then M is σ-finite too,*
3. *if P is a set valued probability of bounded variation, then M is finite set valued measure,*
4. *if $|M|$ is variation of M, then $|M|(A) = \sup_{x\in M(A)} x$ and $|M|$ is a positive measure on $\mathcal{A}$,*
5. *if $P(A)$ is convex for all $A \in \mathcal{A}$, then $M(A)$ is convex too,*
6. *$\bar{M} : \mathcal{A} \to \mathcal{K}_f(\mathbb{R}^+)$ defined by $\bar{M}(A) = \mathrm{cl}\,(M(A))$ and $\bar{\mathrm{co}}\, M : \mathcal{A} \to \mathcal{K}_c(\mathbb{R}^+)$ defined by $(\bar{\mathrm{co}}\, M)(A) = \bar{\mathrm{co}}\,(M(A))$ are set valued measures.*

Proof. 1. To show that M is a set valued measure, we prove countable additivity. Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable partition of Ω. Then

$$\begin{aligned}
M\left(\bigcup_{k=1}^{\infty} A_k\right) &= M(\Omega) = \left\{\sum_{i=1}^{n} p_i(\Omega_i),\, \Omega = \bigcup_{i=1}^{n} \Omega_i,\, \{p_i\}_{i=1}^{n} \subseteq S_P,\, n \in \mathbb{N}\right\}\\
&= \left\{\sum_{k=1}^{\infty}\sum_{i=1}^{n} p_i(A_k \cap \Omega_i),\, \Omega = \bigcup_{i=1}^{n} \Omega_i,\, \{p_i\}_{i=1}^{n} \subseteq S_P,\, n \in \mathbb{N}\right\}\\
&= \left\{\sum_{k=1}^{\infty}\sum_{i=1}^{n} p_i(A_{ki}),\, A_k = \bigcup_{i=1}^{n} A_{ki},\, \{p_i\}_{i=1}^{n} \subseteq S_P,\, n \in \mathbb{N}\right\}\\
&= \sum_{k=1}^{\infty}\left\{\sum_{i=1}^{n} p_i(A_{ki}),\, A_k = \bigcup_{i=1}^{n} A_{ki},\, \{p_i\}_{i=1}^{n} \subseteq S_P,\, n \in \mathbb{N}\right\} = \sum_{k=1}^{\infty} M(A_k).
\end{aligned}$$

5. If $x, y \in M(A)$ one has to prove that for every $\lambda \in [0, 1]$, $\lambda x + (1-\lambda)y \in M(A)$. By the definition of M, $x \in M(A)$ implies that there exists a finite partition $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$ of A such that $x = \sum_{i=1}^{n} p_i(A_i)$, $\{p_i\}_{i=1}^{n} \subseteq S_P$, $n \in \mathbb{N}$, and the same for y: there exists a finite partition $\{B_j\}_{j=1}^{n} \subset \mathcal{A}$ of A such that $y = \sum_{j=1}^{m} p_j(B_j)$, $\{p_j\}_{j=1}^{m} \subseteq S_P$, $n \in \mathbb{N}$. Then $\{A_i \cap B_j\}_{i=1,j=1}^{i=n,j=m}$ is a finite partition of A and the set mapping $\lambda p_i + (1-\lambda)p_j : \mathcal{A} \to [0, 1]$ defined by $(\lambda p_i + (1-\lambda)p_j)(A) = \lambda p_i(A) + (1-\lambda)p_j(A)$ is a probability measure. Convexity of P implies that if $p_i, p_j \in S_P$, then $\lambda p_i + (1-\lambda)p_j \in S_P$. Now, we have

$$\begin{aligned}
(\lambda p_i + (1-\lambda)p_j)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(A_i \cap A_j)\right) &= \sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda p_i + (1-\lambda)p_j)(A_i \cap A_j)\\
&= \lambda\sum_{i=1}^{n}\sum_{j=1}^{m} p_i(A_i \cap A_j) + (1-\lambda)\sum_{j=1}^{m}\sum_{i=1}^{n} p_j(A_i \cap A_j)\\
&= \lambda\sum_{i=1}^{n} p_i \sum_{j=1}^{m}(A_i \cap A_j) + (1-\lambda)\sum_{j=1}^{m} p_j \sum_{i=1}^{n}(A_i \cap A_j)\\
&= \lambda\sum_{i=1}^{n} p_i(A_i) + (1-\lambda)\sum_{j=1}^{m} p_j(A_j) = \lambda x + (1-\lambda)y \in P(A).
\end{aligned}$$

4 and 6 are consequences of 1. □

We say that the set valued measure M mentioned in the last theorem is *generated* by the set valued probability P.

Definition 4.1. $A \in \mathcal{A}$ is an *atom* of set valued probability P, $P(A) \neq \{0\}$, iff for every $B \subset A$, $P(B) = \{0\}$ or $P(A \setminus B) = \{0\}$.

With S_A we denote a subset of S_P such that A is an atom for all $p \in S_A$.

Theorem 3. *Let $P : \mathcal{A} \to 2^{[0,1]}$ be a set valued probability and M be a set valued measure generated by P. If $A \in \mathcal{A}$ is an atom of P, then:*

1. *$S_A \neq \emptyset$.*
2. *For all $q \in S_P \setminus S_A$, $q(A) = 0$.*
3. *A is an atom of M.*
4. *A is an atom for $|P| = |M|$.*
5. *P is nonatomic iff M is nonatomic.*
6. *P is nonatomic iff $|P| = |M|$ is nonatomic.*
7. *If P is a nonatomic set valued probability of bounded variation, then $M(A)$ is convex for every $A \in \mathcal{A}$.*

Proof. 1. If we suppose that $S_A = \emptyset$, then A is not an atom for any $p \in S_P$. But then A cannot be an atom for P, which contradicts the assumption that A is an atom of P.

2. In order to prove that $q(A) = 0$ for all $q \in S_P \setminus S_A$, we assume the opposite: $q(A) > 0$. Since A is not an atom of q, for some $B \subset A$, $q(B) > 0$ and $q(A \setminus B) > 0$. This implies that for this B, $P(B) \neq \{0\}$ and $P(A \setminus B) \neq \{0\}$, which contradicts the assumption that A is an atom of P.
7. By Theorem 3(4), if P is a nonatomic set valued probability of bounded variation, then M is nonatomic too. But all nonatomic set valued measures have convex images. □

5. Derived set valued probability

Let $(\Omega, \mathcal{A})$ be a measurable space and $M : \mathcal{A} \to 2^{\mathbb{R}_+}$ be a finite set valued measure such that $1 \in M(\Omega)$. That kind of set valued measure M we call a *probability adaptive set valued measure*.

Lemma 4. *If M is a probability adaptive set valued measure, then:*

1. *there exists an $r \in \mathbb{R}_+$ such that $\sup_{x \in M(\Omega)} x < r$,*
2. *for every $A \in \mathcal{A}$,*

$$\{(x, x') \in [0,1]^2 : x + x' = 1,\ x \in M(A),\ x' \in M(A'), \} \neq \emptyset,$$

3. *for every countable partition $\{A_1, A_2, \ldots, A_n, \ldots\}$ of Ω,*

$$\{\{x_i\}_{i \in \mathbb{N}} \in l_1([0,1]) : \sum_{i=1}^{\infty} x_i = 1,\ x_i \in M(A_i),\ i \in \mathbb{N}\} \neq \emptyset.$$

Proof. Since M is a finite set valued measure, for every $A \in \mathcal{A}$ and every $x \in M(A)$ there exists a selector m of M such that $m(A) = x$. So, a probability adaptive set valued measure contains at least one measure selector m such that $m(\Omega) = 1$. That m is a probability selector. As a consequence of that fact, we have properties (2) and (3). □

Theorem 5. *Let M be a probability adaptive set valued measure. If a set valued function $P : \mathcal{A} \to 2^{[0,1]}$ is defined by*

$$P(A) = \{x \in [0,1] : x \in M(A), \exists x' \in M(A'),\ x + x' = 1\}, \quad A \in \mathcal{A},$$

then:

1. *P is a set valued probability.*
2. *If $M : \mathcal{A} \to \mathcal{K}_f(R_+)$, then $P(A)$, $A \in \mathcal{A}$, are compact.*
3. *If $M : \mathcal{A} \to \mathcal{K}_{c(f)}(\mathbb{R}_+)$, then $P(A)$, $A \in \mathcal{A}$, are convex (compact).*
4. *If M is a probability adaptive set valued measure, then the derived set valued probability P is unique.*
5. *Different probability adaptive set valued measures could have the same derived set valued probabilities.*
6. *There exists a unique minimal probability adaptive set valued measure Φ for the derived set valued probability $\mathcal{P}$.*
7. *If P is a set valued probability, M is the generated set valued measure and Q is the set valued probability derived from M, then $P = Q$.*
8. *If M is a nonatomic probability adaptive set valued measure, then the derived set valued probability P is nonatomic too.*

Proof. (1) We have to prove that for every $A \in \mathcal{A}$ and for every $x \in P(A)$, there exists a probability selector $p : \mathcal{A} \to [0,1]$ of P such that $p(A) = x$.

Firstly, we assume that M is nonatomic. The dyadic structure of Ω is a collection of measurable sets $A(\epsilon_1 \cdots \epsilon_k)$, $\epsilon_n \in \{0,1\}$, $n \in \mathbb{N}$, such that

$$A(0) \cup A(1) = \Omega, \qquad A(\epsilon_1 \cdots \epsilon_k 0) \cup A(\epsilon_1 \cdots \epsilon_k 1) = A(\epsilon_1 \cdots \epsilon_k),$$
$$A(0) \cap A(1) = \emptyset, \qquad A(\epsilon_1 \cdots \epsilon_k 0) \cap A(\epsilon_1 \cdots \epsilon_k 1) = \emptyset.$$

Further, let $A(0) = A$ and $A(1) = A'$. Since M is nonatomic, the derived finite measure $|M|$ is nonatomic too, implying that there exists a dyadic structure $\{A(\epsilon_1 \cdots \epsilon_k)\} = \mathcal{D}$ of Ω such that $|M|(A(\epsilon_1 \cdots \epsilon_k)) = 2^{-k}|M|(A(\epsilon_1))$. Now, let $p(A) = x \in P(A)$, $p(\bar{A}) = 1 - x \in P(A')$ and $p(\Omega) = 1 \in P(\Omega)$. The additivity of M implies the existence of elements $x(\epsilon_1 \cdots \epsilon_k) \in M(A(\epsilon_1 \cdots \epsilon_k))$, $\epsilon_n \in \{0,1\}$, $n \in \mathbb{N}$, such that

$$x(0) = x, \qquad x(1) = 1 - x, \qquad x(\epsilon_1 \cdots \epsilon_k) = x(\epsilon_1 \cdots \epsilon_k 0) + x(\epsilon_1 \cdots \epsilon_k 1).$$

Define $p : \mathcal{D} \to [0,1]$, $p(A(\epsilon_1 \cdots \epsilon_k)) = x(\epsilon_1 \cdots \epsilon_k)$. This set function is additive on the algebra generated by $\mathcal{D}$. Since $p(B) \le |M|(B)$ for all $B \in \mathcal{D}$, p has σ-additive extension on σ-algebra $\mathcal{A}_1$ generated by $\mathcal{D}$. Denote this extension by the same letter p. It is a probability measure on $\mathcal{A}_1 \subseteq \mathcal{A}$. If the set $S \in \mathcal{A}$ is a countable union of the elements from $\mathcal{D}$, from the additivity of M, we get $p(S) \in M(S)$. Further, if $T \in \mathcal{A}$ and $|M|(T \setminus S) = 0$ (where $S \in \mathcal{A}$ is a countable union of the elements from $\mathcal{D}$), then, using the $|M|$-continuity of M, we have $M(T) = M(S)$.

Secondly, if M is not nonatomic, let Ω_0 be the atomic part of Ω. Then for any $x \in P(A)$ there exist $a \in M(A \cap \Omega_0)$ and $b \in M(A \setminus \Omega_0)$, $x = a + b$. Further, if $\{A_n\} \subset \mathcal{A}$ is a family (finite or countable) of all atoms contained in A, then there exists a family $\{a_n\}$, $a_n \in M(A_n)$ such that $a = \sum a_n$. Then for every $T \in \mathcal{A}$, $T \subset A_n$, we define p by

$$p(T) = \begin{cases} 0, & M(T) = \{0\}, \\ a_n, & M(A_n \setminus T) = \{0\}. \end{cases}$$

Now, for all $T \in \mathcal{A}, T \subset A \cap \Omega_0$, let $p(T) = \sum p(T \cap A_n)$. Obviously, p is a selector of M on the atomic part of A and $p(A \cap \Omega_0) = a$. For the nonatomic part of A we repeat the procedure explained at the beginning of the proof. Since $x \in P(A)$, there exists an $x' \in P(\bar{A})$ such that $x + x' = 1$, implying that p is a probability selector of M.

So, we have proved that in both cases (atomic and nonatomic) there exists a probability selector p of M (and, consequently, p is a selector of P) such that $p(A) = x$.

((2), (3)) $M(A) \times M(\bar{A}) \in \mathcal{P}_{f(c)}(\mathbb{R}^2_+)$, so the set $T = \mathcal{M}(A) \times \mathcal{M}(A') \cap \{(x, y) \in [0, 1]^2 : x + y = 1\}$ is also a compact (and convex) subset of $[0, 1]^2$. Since $P(A) = \text{proj}_{M(A)} T$, the set $P(A)$ is a compact (and convex) subset of $[0, 1]$.

(7) It is easy to see that for all $A \in \mathcal{A}, P(A) \subseteq Q(A)$. The assumption that $P \neq Q$ implies that there exist a probability selector p of M (and, consequently, of Q) and $A \in \mathcal{A}$ such that $p(A) \in M(A)$, but $p(A) \notin P(A)$. But by the definition of the measure M, there exists a partition of $\{A_i\}_{i=1}^k$ of Ω and the set of selectors $\{p_i\}_{i=1}^k$ of P such that for some $1 \leq n \leq k, A = \cup_{i=1}^n A_i, \Omega \setminus A = \cup_{i=n+1}^k A_i$, where $p(A) = \sum_{i=1}^n p_i(A_i)$ and $p(\Omega \setminus A) = \sum_{i=n+1}^k p_i(A_i)$. The probability $q : \mathcal{A} \to [0, 1]$ defined by $q(B) = \sum_{i=1}^k p_i(B \cap A_i)$ is a selector of P. Obviously, $q(A) = p(A)$, which contradicts the assumption that $p(A) \notin P(A)$. $\square$

Acknowledgment

This work was partially supported by Ministarstvo nauke i životne sredine Republike Srbije.

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