Probability

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1 Classical Conventional Probability Spaces

Textbook probability theory is defined using the notions of *sample spaces*, *events*, and *measures*.

1.1 Sample Space Ω

In this paper, we will only consider **finite** sample spaces. We therefore define a sample space Ω as a non-empty finite set.

Example 1 (A Classical Sample Space.). Consider an experiment that tosses three coins. A possible outcome of the experiment is HHT which means that the fist and second coins landed with "heads" as the face-up side and that the third coin landed with "tails" as the face-up side. There are clearly a total of eight possible outcomes, and this collection constitutes the sample space:

$$\Omega_C = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Example 2 (A Quantum Sample Space.). Consider a quantum system composed of three electrons. By the postulates of quantum mechanics, an experiment designed to measure whether the spin of each electron along the x axis is left (L) or right (R) can only result in one of eight outcomes:

$$\Omega_H = \{LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR\}$$

1.2 Events $\mathcal{F}$

The space of events $\mathcal{F}$ associated with a sample space Ω is 2^Ω, the powerset of Ω. In other words, every subset of Ω is a possible event.

Example 3 (Some classical events.). The following are events associated with Ω_C:

- E_0, exactly zero coins are H, is the set $\{TTT\}$.
- E_1, exactly one coin is H, is the set $\{HTT, THT, TTH\}$.
- E_2, exactly two coins are H, is the set $\{HHT, HTH, THH\}$.
- E_3, exactly three coins are H, is the set $\{HHH\}$.
- $E_{>0}$, at least one coin is H, is the set $\{HHH, HHT, HTH, HTT, THH, THT, TTH\}$.

As the examples illustrate, events are *indirect* questions built from elementary elements of the sample space using logical connectives. Also note that some events may be disjoint and that some events may be expressed as combinations of other events. For example, we have $E_{>0} = E_1 \cup E_2 \cup E_3$ and each of these four events is disjoint from event E_0.

Example 4 (Some quantum events.). The following are events associated with Ω_H:

- F_0, exactly zero electrons are spinning L, is the set $\{RRR\}$.
- F_1, exactly one electron is spinning L, is the set $\{LRR, RLR, RRL\}$.
- F_2, exactly two electrons are spinning L, is the set $\{LLR, LRL, RLL\}$.
- F_3, exactly three electrons are spinning L, is the set $\{LLL\}$.
- $F_{>0}$, at least one electron is spinning L, is the set $\{LLL, LLR, LRL, LRR, RLL, RLR, RRL\}$.

As the examples illustrate, quantum events are, at first glance, similar to classical events. There are however some subtle differences that we point out in the next section.

1.3 Measures $\mathbb{P}$

The last ingredient of a probability space is a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns to each event a real number in the closed interval $[0, 1]$ subject to the following conditions:

- $\mathbb{P}(\Omega) = 1$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \Sigma_i\ \mathbb{P}(A_i)$.

Example 5 (Classical probability measure). There are 2^8 events associated with Ω_C. A possible probability measure for these events is:

$$\mathbb{P}(E) = \begin{cases} 1 & \text{if } E = \Omega \\ 0 & \text{otherwise} \end{cases}$$

Yu-Tsung says: Actually, the above $\mathbb{P}$ is not a probability because

$$\mathbb{P}(\Omega) = 1 \neq 0 + 0 = \mathbb{P}(\{HHH\}) + \mathbb{P}(\Omega \backslash \{HHH\})$$

A more interesting measure is defined recursively as follows:

$$\begin{aligned}
\mathbb{P}(\emptyset) &= 0 \\
\mathbb{P}(\{HHH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E) \\
\mathbb{P}(\{HHT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{HTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E) \\
\mathbb{P}(\{HTT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{THH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E) \\
\mathbb{P}(\{THT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{TTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E) \\
\mathbb{P}(\{TTT\} \cup E) &= \mathbb{P}(E)
\end{aligned}$$

Yu-Tsung says: It seems more clear to write like this:

$$\begin{aligned}
\mathbb{P}(\{HHH\}) &= \tfrac{1}{5}\\
\mathbb{P}(\{HHT\}) &= 0\\
\mathbb{P}(\{HTH\}) &= \tfrac{3}{10}\\
\mathbb{P}(\{HTT\}) &= 0\\
\mathbb{P}(\{THH\}) &= \tfrac{1}{5}\\
\mathbb{P}(\{THT\}) &= 0\\
\mathbb{P}(\{TTH\}) &= \tfrac{3}{10}\\
\mathbb{P}(\{TTT\}) &= 0\\
\mathbb{P}(E) &= \textstyle\sum_{\omega\in E}\mathbb{P}(\{\omega\})
\end{aligned}$$

Because this is a *classical* situation, the probability assignments can be understood *locally* and *non-contextually*. In other words, we can reason about each coin separately and perform experiments on it ignoring the rest of the context. If we were to perform such experiments we may find that for the first coin, the probability of either outcome H or T is $\frac{1}{2}$; for coin two, the probabilities are skewed a little with the probability of outcome H being $\frac{2}{5}$ and the probability of outcome T being $\frac{3}{5}$; and that coin 3 is a fake double-headed coin where the probability of outcome H is 1 and the probability of outcome T is 0. The reader may check that these local observations are consistent with the probability measure above.

Example 6. [Quantum probability measure] Like in the classical case, there are 2^8 events. But as Mermin explains in a simple example [4], here is a possible probability measure:

$$\begin{aligned}
\mathbb{P}(\{LLL\}) &= \tfrac{1}{4}\\
\mathbb{P}(\{LLR\}) &= 0\\
\mathbb{P}(\{LRL\}) &= 0\\
\mathbb{P}(\{LRR\}) &= \tfrac{1}{4}\\
\mathbb{P}(\{RLL\}) &= 0\\
\mathbb{P}(\{RLR\}) &= \tfrac{1}{4}\\
\mathbb{P}(\{RRL\}) &= \tfrac{1}{4}\\
\mathbb{P}(\{RRR\}) &= 0\\
\mathbb{P}(E) &= \textstyle\sum_{\omega\in E}\mathbb{P}(\{\omega\})
\end{aligned}$$

In contrast with the classical example previously, the probability of each electron is not independent [3, 2, 5], i.e., we cannot find three probability space for each subsystem such that

$$\mathbb{P}(\{abc\}) = \mathbb{P}(\{a\})\mathbb{P}(\{b\})\mathbb{P}(\{c\})\ .$$

Correlated variables are also common in the classical world. For example, if the first coin is head or tail decided by whether the temperature is higher than a particular degree and the second coin is decided by whether Coca Cola is sold more than a particular amount, then we know these two coins are correlated. Einstein, Podolsky, and Rosen [1] suggested any correlated quantum probability results may be interpreted classically. For example, maybe the natural actually rolls an tetrahedron die with $\{LLL, LRR, RLR, RRL\}$ in its four faces, and this tetrahedron die could not be accessed by human for some reasons so that we think we are handling three electrons.

The interesting part is that three electrons cannot be measured the spin only along the x axis, but also along the y axis with the result down (D) or up (U).

Interestingly and paradoxically, it is *not* possible to analyze each electron independently and assign a consistent probability to each one locally and non-contextually.

1.4 Finite Precision of Measurements

In a laboratory setting or a computational setting, there are neither uncountable entities nor uncomputable entities. We are thus looking at alternative probability spaces which do not depend on the real numbers and revisit the mysteries of quantum mechanics in that setting. In other words, is it possible that at least part of the quantum mysteries related to probability and measurement are due to the reliance on uncomputable probability values?

Following previous work on probability, we will replace the closed interval $[0, 1]$ by the *finite set* $S = \{\textbf{possible}, \textbf{impossible}\}$ and adapt the definition of probability measure as follows.

A set-valued probability measure $\mathbb{P} : \mathcal{F} \to S$ assigns to each event either the tag **possible** or the tag **impossible** subject to the following conditions:

- $\mathbb{P}(\Omega) = \textbf{possible}$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \textbf{possible}$ if any event A_i is **possible** and **impossible** otherwise.

2 Conventional Quantum Mechanics

Attempting to modify the probability measure to be set-valued, while keeping the rest of the mathematical framework of quantum mechanics intact leads to a contradiction. More precisely, it is not possible to maintain infinite precision probability amplitudes in the presence of set-valued probabilities without violating essential aspects of quantum theory.

. . . explain and give theorem

3 Discrete Quantum Theory

The next question to ask is therefore whether the infinite precision of probability amplitudes is itself justified. If all measurements are finite and all probabilities are computable, then it is plausible that the internal mathematical representation of quantum states should also be based on countable computable entities.

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References

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