A Constructive Formulation of Gleason's Theorem

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ABSTRACT. In this paper I wish to show that we can give a statement of a restricted form of Gleason's Theorem that is classically equivalent to the standard formulation, but that avoids the counterexample that Hellman gives in "Gleason's Theorem is not Constructively Provable".

1. Introduction

All proofs of Gleason's Theorem have used essentially non-constructive results (for example, the Bolzano–Weierstrass theorem and its corollaries). This does not mean that it is not possible to prove it using constructive methods, merely that no one has done so.[1] However, Hellman, in his paper "Gleason's Theorem is not Constructively Provable" (Hellman [2]) showed that the theorem restricted to the three-dimensional real Hilbert space $\Re^3$ (we will call this 'Gleason (restricted)' following Hellman) is not constructively provable. This works as a constructive proof that the theorem itself is not constructively provable because $A \rightarrow B$ is equivalent to $\neg B \rightarrow \neg A$ constructively as well as classically. Thus, if we cannot prove a consequence of a theorem, then we cannot prove the theorem itself.

Hellman uses constructive reasoning to show that we cannot constructively prove Gleason (restricted), so if he is correct a constructive mathematian must accept his argument. He uses the method of weak counterexamples to show that if we assume that we can constructively prove Gleason (restricted), then we can also decide whether there is a sequence of nine 1's in the decimal expansion of π, for example; that is, we are led to saying that either there is a sequence of nine 1's in the decimal expansion of π or there is no such sequence. It is a well-known fact that the constructivist would not accept that we are justified in asserting this since we do not have a proof of either disjunct. We are led by the assumption that we can constructively prove Gleason (restricted) to asserting something that we are not justified in asserting. Thus, we can say that we will never be able to prove Gleason (restricted) using constructive methods. This is not the same as saying that the theorem is false, because we are not saying that if we assume that it is true we

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derive a contradiction (e.g., $2 = 1$). In fact, it is a constructive theorem that if we can prove a proposition classically, then we can prove the double negation of that theorem constructively (this means that we could derive a contradiction from assuming that Gleason (restricted) was false). Of course, the double-negation version of the standard formulation of Gleason's Theorem would only be constructively acceptable if the double-negation translations of the relevant mathematical axioms were constructively acceptable.[2]

If we really cannot prove Gleason's Theorem this has very serious consequences for constructive mathematics. The theorem is considered to be one of the fundamental theorems of the foundations of Quantum Mechanics, and says that all probability measures over the projection lattice of the Hilbert space are of a certain form; that is, they can all be represented as density operators, and there are no others. One illustration of the importance of Gleason's Theorem is its corollary, the Kochen-Specker result. This result means that we cannot have sharp values assigned to all variables in all states. This denies the possibility of non-contextual hidden variables in the quantum context.[3]

The problem here is that if we cannot prove some form of Gleason's Theorem using constructive mathematics, it looks like we have lost a very important theorem of quantum mechanics. It would seem that classical mathematics is indispensible for modern science. This does not undermine the value of constructive mathematics in other contexts, but it would, I think, be a psychological blow to the constructive mathematician. It does not matter, however, if we cannot get *exactly* the same mathematical results as classical mathematics as long as we can get the same physical results with the mathematics that we do have.[4]

This paper tries to open a door to providing a constructive way of getting around Hellman's damning results. Hellman shows in his paper that we cannot constructively prove Gleason (restricted) *as it is usually stated*. In this work I have tried to give a version of Gleason (restricted) that is classically equivalent to the standard formulation, but which is constructively immune to Hellman's counterexample. This would not, of course, show that we can prove Gleason's Theorem constructively. The next step would be to provide a constructive formulation of the full theorem and then to prove it using constructive methods. Hellman wants his technical results to motivate a wider philosophical thesis that non-constructive mathematics is indispensible for certain important physical applications. If the work here is correct then he has not succeeded, but has merely shown that constructive mathematics cannot prove the same theorems as classical mathematics, which is hardly news to the constructivist.

The remainder of this paper assumes only basic knowledge of classical undergraduate analysis (see Lang [6]) and some familiarity with intuitionistic logic (see volume one of Troelstra and van Dalen [1]).

2. Suggested constructive versions of Gleason (restricted)

We can now try and formulate a version of Gleason (restricted) that is constructive, but classically equivalent to the standard formulation,[5]

> Gleason (restricted). Let f be a bounded frame function on S, and define $\gamma \equiv \gamma_f \equiv w_f - M_f - m_f$. There exists a frame (p, q, r) such that, for any $s \in S$, if the coordinates of s with respect to (p, q, r) are (x, y, z) then,
>
> $$f(s) = Mx^2 + \gamma y^2 + mz^2.$$

Gleason (restricted) says that all frame functions on the unit sphere of real three-dimensional Hilbert space have the same form. A constructive version must say something like 'we can show that all frame functions have a certain form to within epsilon'.

To my mind there are three possible formulations that a constructive version of Gleason (restricted) could take.

a) Let f be a bounded frame function on S (the unit sphere of the real three-dimensional Hilbert space $\Re^3$). Define $\gamma \equiv w - M - m$. Then there exists a frame (p, q, r) such that, for any $s \in S$, if the coordinates of s with respect to (p, q, r) are (x, y, z) then,

$$\forall k \in Z^+ \ |f(s) - (Mx^2 + \gamma y^2 + mx^2)| < 2^{-k}.$$

b) Let f be a bounded frame function on S. Define $\gamma \equiv w - M - m$. Then $\forall k \in Z^+$, there exists a frame (p, q, r) such that, for any $s \in S$, if the coordinates of s with respect to (p, q, r) are (x, y, z) then,

$$|f(s) - (Mx^2 + \gamma y^2 + mz^2)| < 2^{-k}.$$

c) Let f be a bounded frame function on S. Define $\gamma \equiv w - M - m$. Then there exists a frame (p, q, r) such that,

$\forall k \in Z^+$, there is an $s \in S$ with coordinates (x, y, z) with respect to (p, q, r) such that,

$$|f(s) - (Mx^2 + \gamma y^2 + mz^2)| < 2^{-k}.$$

2.1. *Option c*

Option c does not seem to be saying what we want Gleason (restricted) to say. It is merely saying that we can always find some point on the sphere such that the value of f at that point is as close as we like. This is not saying anything about the form of f, merely that we can represent it in a certain way at a certain point. So we discount c as a possibility.

2.2. *Option a*

Next we shall look at a. This seems to be what we want Gleason (restricted) to say; we can find a frame such that we can get as close as we like to the representation we want.[6] We have to check if it avoids Hellman's counterexample.

Define the real generator as in Hellman's paper;

$$\beta(n) = \begin{cases} 0, & \text{if } \forall m \leqslant n \neg A(m) \\ 2^{-k}, & \text{if } A(k) \,\&\, n \leqslant m \,\&\, \forall m < k(\neg A(m)) \,\&\, B(k) \\ -2^{-k}, & \text{if } A(k) \,\&\, n \leqslant m \,\&\, \forall m < k(\neg A(m)) \,\&\, \neg B(k) \end{cases}$$

Here $A(k)$ and $B(k)$ are both decidable, but it is known neither whether $\exists n A(n)$ nor whether the least n, if it exists, is such that it satisfies B or not.

Then put β as the real number generated by $\beta(n)$.

Throughout his paper Hellman uses five 'facts' about constructive functions,

1. If f is a (constructive) bounded frame function, so is αf for any (constructive) real number α.
2. For any real α and bounded frame function f, if $\alpha > 0, \alpha f$ attains its supremum at the same points as f, and αf attains its infimum at the same points as f ....
3. Under the same conditions as 2, if $\alpha < 0$, αf attains its supremum at the points where f attains its infimum and αf attains its infimum at the points where f attains its supremum ....
4. Under the hypothesis that Gleason (restricted) is constructively provable, if f is a (constructive) bounded frame function, there exist (i.e., can be found) at least one point p such that f attains its supremum at p and a point $r \perp p$ at which f attains its infimum ....

5. Fix a vector $p \in S$ (think of p as the North pole); the function

$$f(s) = \cos^2 \varphi(p, s)$$

is a constructive bounded frame function (of weight 1) such that,

i) f attains its supremum, $M_f = 1$, uniquely at p and $-p$,
ii) for any r, if f attains its infimum at r, $f(r) = m_f = 0$, then $r \perp p$, i.e., r lies on the equator, $E \equiv \{s \in S\colon\ s \perp p\}$. (Hellman [2, p. 197].)

All of these still hold with our new version of Gleason (restricted) except for fact 4; under the hypothesis of our new Gleason (restricted) f only gets close as you like to its supremum, but does not necessarily attain it.

Let $f(s) = \cos^2 \varphi(p, s)$. By fact 5 this is a constructive function. Now assume our new version of Gleason (restricted) is constructively provable, then $\exists p' \ |\beta f(p') - M_{\beta f}| < 2^{-k}$ for all k. Unfortunately, we can still say that either $p' \perp E$ fails or $p' \in E$ fails. This is because the proof that Hellman gives for this in Hellman [2] holds for any p' in S, it does not depend on the fact that $\beta f(p') = M_{\beta f}$. This means that we still have $\neg(p' \perp E) \vee \neg(p' \in E)$.

Now we still also have $\beta > 0 \rightarrow p' \perp E$ and $\beta < 0 \rightarrow p' \in E$ since $\beta > 0 \rightarrow \beta(m) = 2^{-k}$ for all $m \geqslant$ some n and similarly for $\beta < 0$. Hence we have a definite value for β in each case and so βf does attain its supremum either at the same point as f or at a point where f attains its infimum by facts 2 and 3 (see above).

Thus putting $\neg(p' \perp E) \vee \neg(p' \in E)$ and $\beta > 0 \rightarrow p' \perp E$ and $\beta < 0 \rightarrow p' \in E$ together we can say that either $\neg\beta < 0$ or $\neg\beta > 0$ which we are not in a position to do constructively.

Hence a is subject to the same counterexample as the standard formulation of Gleason (restricted), so we cannot use it as the desired constructive version (as one might expect given that, according to Bridges, they are the same).

2.3. *Option b*

We now need to look at version b of Gleason (restricted). There are two questions that we need to ask;

1. Is version b subject to Hellman's counterexample?
2. Is version b classically equivalent to the standard formulation of Gleason (restricted)?

2.3.1. *Question 1*

Suppose we have the same set up as for Hellman's original counterexample. So we have $f(s) = \cos^2 \varphi(p, s)$, where $M_f = f(\pm p) = 1$. We also have the real number β generated by $\beta(n)$ and we know that βf is a bounded frame function. Now suppose that we assume that version b of Gleason (restricted) is constructively provable. Then $\forall k \exists (p_k, q_k, r_k)$ such that $\forall s$ with coordinates (x_k, y_k, z_k) with respect to (p_k, q_k, r_k) then

$$|\beta f(s) - (M x_k^2 + \gamma y_k^2 + m z_k^2)| < 2^{-k}.$$

This means that we have $\forall k \exists p_k \; |\beta f(p_k) - M| < 2^{-k}$. Hence we have a sequence $(\beta f(p_k))_k$ such that $\lim_{k\to\infty} \beta f(p_k) = M$.

At this point in his argument Hellman says that we have a point p' such that $\beta f(p') = M$. He then goes on to say that we can assert $\neg(p' \perp E) \vee \neg(p' \in E)$. But with our version all we can say is that we have a sequence of points that give a sequence of values of βf that tend to M. We can, though, say for each p_k that $\neg(p_k \perp E) \vee \neg(p_k \in E)$.

Hellman then notes that $\beta > 0 \to p' \perp E$ and $\beta < 0 \to p' \in E$ and hence we have a proof of $\neg(\beta > 0) \vee \neg(\beta < 0)$ which we cannot constructively assert. But in this case we have:

$$\begin{aligned} \beta > 0 &\Rightarrow \beta = 2^{-k} \text{ for some } k \\ &\Rightarrow \beta f = 2^{-k} \cos^2 \varphi(p, s) \\ &\Rightarrow M_{\beta f} = \beta f(p) = 2^{-k}. \end{aligned}$$

Hence $p \perp E$.

But we have not said that βf attains its supremum, only that $\forall k \exists p_k$ $|\beta f(p_k) - M| < 2^{-k}$. So the counterexample does not apply and we are not led to saying $\neg(\beta > 0) \vee \neg(\beta < 0)$.

It could be argued that since we have a sequence $(\beta f(p_k))_k$ such that $\lim_{k\to\infty} \beta f(p_k) = M$, the sequence of points $(p_k)_k$ must have a limit p and this must give us $\beta f(p) = M$. However, constructively, we cannot assume that this will be the case. We cannot assume that $(p_k)_k$ will converge.

2.3.2. *Question 2*

Now we must look at the second question; is version b classically equivalent to the standard formulation of Gleason (restricted)? We will start by explicitly stating both versions. Let us call version b Gleason (restricted, constructive).

i) GLEASON (restricted). *Let f be a bounded frame function on S, and define $\gamma \equiv \gamma f \equiv w_f - M_f - m_f$. There exists a frame (p, q, r) such* *that, for any $s \in S$, if the coordinates of s with respect to (p, q, r) are (x, y, z) then,*

$$f(s) = Mx^2 + \gamma y^2 + mz^2.$$

ii) GLEASON (restricted, constructive). *Let f be a bounded frame function on S, and define $\gamma \equiv \gamma_f \equiv w_f - M_f - m_f$. For each $k \in Z^+$ there is a frame (p, q, r) such that, for any $s \in S$, if the coordinates of s with respect to (p, q, r) are (x, y, z) then,*

$$|f(s) - (Mx^2 + \gamma y^2 + mz^2)| < 2^{-k}.$$

To show that these are classically equivalent we need to show that i) $\rightarrow$ ii) and that ii) $\rightarrow$ i). It is obvious that i) $\rightarrow$ ii), since if we assume i) we have $|f(s)-(Mx^2+\gamma y^2+mz^2)| = 0$ for some frame (p, q, r). So for each k we can use this frame and we have $|f(s)-(Mx^2+\gamma y^2+mz^2)| < 2^{-k}$.

It is more difficult to show that ii) $\rightarrow$ i). Suppose ii). Then we have $\forall k \exists (p_k, q_k, r_k) \forall s$ with coordinates (x_k, y_k, z_k) with respect to (p_k, q_k, r_k)

$$|f(s) - (Mx_k^2 + \gamma y_k^2 + mz_k^2)| < 2^{-k}.$$

This means that we have a sequence of frames $(p_k, q_k, r_k)_k$ and for each (p_k, q_k, r_k) we have a set of coordinates

$$S_k = \{(x_k, y_k, z_k) : (x_k, y_k, z_k) \text{ are the coordinates of some } s \in S \text{ w.r.t. } (p_k, q_k, r_k)\}$$

corresponding to the elements of S.

Let us introduce some notation here to make things easier to read. Put $\vec{x}_k$ for (x_k, y_k, z_k) and $g(\vec{x}_k)$ for $Mx_k^2+\gamma y_k^2+mz_k^2$. This means that for any s we have $\forall k \exists \vec{x}_k$ $|f(s) - g(\vec{x}_k)| < 2^{-k}$. In turn this means that $\lim_{k\rightarrow\infty} g(\vec{x}_k) = f(s)$.

Now we want to use this to show that classically we can find one frame that gives us $f(s) = g(\vec{x})$. Since S is compact and we are working classically we can say that by passing to a subsequence of the $(\vec{x}_k)_k$ we have $\lim_{k\rightarrow\infty} \vec{x}_k = \vec{x}$ for some $\vec{x} \in S$. Now we need to show that $g(\vec{x}) = f(s)$ by showing that;

a) $\lim_{k\rightarrow\infty} g(\vec{x}_k) = g(\vec{x})$,
b) $\lim_{k\rightarrow\infty} g(\vec{x}_k)$ is unique.

Now g is a uniformly continuous function, since it can be represented as sums and products of very simple uniformly continuous functions. We have the following classical theorem;

Let S be a subset of a normed vector space ($\Re^3$ is a normed vector space with norm $|A| = \sqrt{x_1^2 + x_2^2 + x_3^2}$) and let $f\colon S \rightarrow F$ be a map of

S into a normed vector space F ($\Re$ is also a normed vector space with norm $|A| = |x|$).

Let $v \in S$. The map f is continuous at v if and only if for every sequence (x_k) of elements of S which converge to v we have $\lim_{k\to\infty} f(x_k) = f(v)$.[7]

Now we know that g is uniformly continuous and we know that $\vec{x}_k$ converges to $\vec{x}$ so by the above theorem we have $\lim_{k\to\infty} g(\vec{x}_k) = g(\vec{x})$.

Now we have that $\lim_{k\to\infty} g(\vec{x}_k) = g(\vec{x})$ and $\lim_{k\to\infty} g(\vec{x}_k) = f(s)$ and we need to check that the limit is unique so that we can say $f(s) = g(\vec{x})$.

On page 121 of Lang [6] we have the following passage;

Let E be a normed vector space. A sequence $\{x_n\}$ in E is said to converge if there exists $v \in E$ having the following property: Given ϵ, there exists N such that for all $n \geqslant N$ we have $|x_n - v| < \epsilon$. We then call v the **limit** of the sequence $\{x_n\}$. This limit if it exists, is *uniquely determined*... (my italics).[8]

So classically, the sequence $(g(\vec{x}_k))_k$ has a unique limit and hence we have $f(s) = g(\vec{x})$. Now we need to show that the $\vec{x}$ are all with respect to the same frame.

We have shown so far that for each s we can determine a subsequence of the frames that converges to a limit, we want to show that this limit is the same in all cases.

Now we determine the coordinates of s in S with respect to a particular frame using some continuous function on the frame and the s, i.e., putting $(p_k, q_k, r_k) = \vec{p}_k$ we have $\vec{x}_k = h(\vec{p}_k, s)$ for some continuous function h. So if we put $g'(\vec{p}_k, s) = g(h(\vec{p}_k, s))$ we have

$$\forall k \exists \vec{p}_k \forall s \in S \ |f(s) - g'(\vec{p}_k, s)| < 2^{-k}.$$

The vector space of triples of elements of $\Re^3$ is a normed vector space and the subset of frames is a compact subset of this space.

This means that any sequence on the subset of frames has a subsequence which is convergent. In particular, our sequence $(\vec{p}_k)_k$ has a subsequence $(\vec{p}_{k_n})_n$ which converges to a limit $\vec{p}$.

We also know that $\lim_{k\to\infty} g'(\vec{p}_k, s) = f(s)$. Now if we consider the function g' as a sequence indexed by the rationals then we can say that $\lim_{k\to\infty} g(\vec{p}_{k_n}, s) = f(s)$ also, since if a sequence has a limit then any subsequence also has the same limit.[9] Now if we follow through the above argument we have $\lim_{k\to\infty} g'(\vec{p}_{k_n}, s) = g'(\vec{p}, s)$ and so $g'(\vec{p}, s) = f(s)$ since continuous functions have unique limits.

This means that for each s we have found a frame such that $f(s) = Mx^2 + \gamma y^2 + mz^2$ where (x, y, z) are the coordinates of s with respect to $\vec{p}$. Hence ii $\rightarrow$ i and so Gleason (restricted) and Gleason (restricted, constructive) are classically equivalent.

Since we do not as yet have a constructive proof of option b, it may be that we can derive the standard formulation from option b using constructive mathematics. This would mean that option b is also constructively unprovable, due to Hellman's results. This is an open question.[10]

3. Conclusion

There are one or two points that must be noted here. It is important to realise that for the above section, the reasoning for Question 1 is constructive and the reasoning for Question 2 is classical. In Question 1 we were trying to show that *constructively* Gleason (restricted, constructive) is not subject to Hellman's counterexample. In Question 2 we were trying to show that Gleason (restricted, constructive) is *classically* equivalent to Gleason (restricted). The notion of a compact set is different in the two cases. Classically a set is compact if every sequence of elements has a point of accumulation in the set. Constructively a metric space, or subset of a metric space, is compact if it is complete and totally bounded. It is this difference that means that the two versions are classically equivalent, but not constructively equivalent.

Finally it must be pointed out that even if this formulation of Gleason (restricted) were to be proved constructively, the constructive mathematician would still have some work to do. The referee points out that the approximate version should suffice for 'no-hidden variables' applications. It is not obvious, however, that such an approximate version of the theorem will be sufficient for all applications. For example, the classical version of Gleason's Theorem tells us that every probability measure on the proposition system based on Hilbert spaces of dimension $\geqslant 3$ is given by a density matrix. It is not clear that the constructive version would be acceptable in this context since it says that the probability measure can be approximated by a density matrix, that matrix depending on the degree of approximation desired. Also, there are applications of Gleason's Theorem that appeal to the uniqueness of the density matrix; for example, there are theorems establishing that a certain set of probabilities for measurable quantities suffice to uniquely determine the state. The uniqueness provided by the classical theorem is clearly lost in the version proposed here. It is an important question whether such an approximate constructive version of Gleason's Theorem can recapture such results and if this would be adequate for quantum mechanics. We leave these issues to future work.[11]

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NOTES

[1] For a classical proof of Gleason's Theorem see: Cooke, Keane and Moran [4].

[2] I would like to thank the anonymous referee for emphasising this point.

[3] For a discussion on the importance of Gleason's Theorem in the foundations of Quantum Mechanics see Redhead [3] or Hughes [5].

[4] It was pointed out by the anonymous referee that there may be constructivists who do not want to claim that *all* legitimate mathematics must be done constructively.

[5] See Hellman [2].

[6] It has been suggested to me by Prof. D. S. Bridges that version a is the same as the standard formulation of Gleason (restricted).

[7] Lang [6, p. 144].

[8] Lang [6, p. 121].

[9] See Lang [6, p. 40].

[10] I would again like to thank the referee for bringing this point to my attention.

[11] This was made clear to me by the anonymous referee.

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