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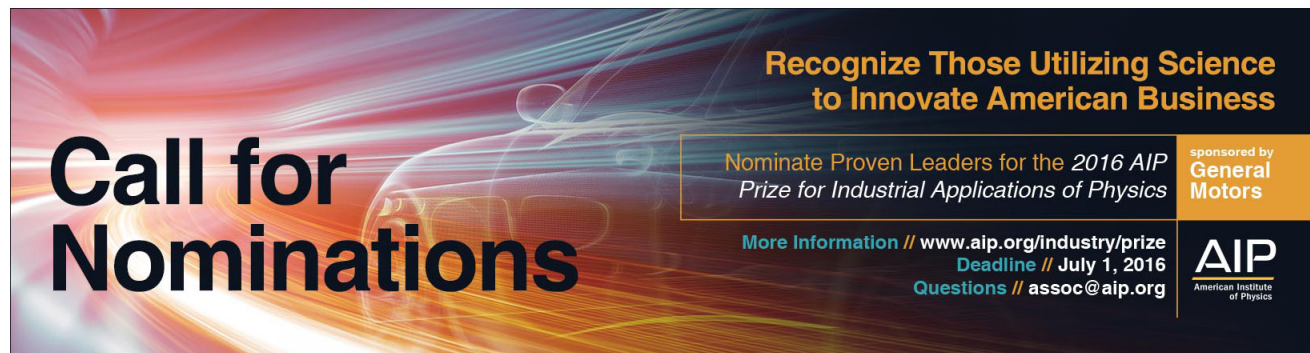
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Quantum probabilities as Dempster-Shafer probabilities in the lattice of subspaces

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The orthocomplemented modular lattice of subspaces $\mathcal{L}[H(d)]$, of a quantum system with d -dimensional Hilbert space $H(d)$, is considered. A generalized additivity relation which holds for Kolmogorov probabilities is violated by quantum probabilities in the full lattice $\mathcal{L}[H(d)]$ (it is only valid within the Boolean subalgebras of $\mathcal{L}[H(d)]$). This suggests the use of more general (than Kolmogorov) probability theories, and here the Dempster-Shafer probability theory is adopted. An operator $\mathfrak{D}(H_1, H_2)$, which quantifies deviations from Kolmogorov probability theory is introduced, and it is shown to be intimately related to the commutator of the projectors $\mathfrak{P}(H_1), \mathfrak{P}(H_2)$, to the subspaces H_1, H_2 . As an application, it is shown that the proof of the inequalities of Clauser, Horne, Shimony, and Holt for a system of two spin $1/2$ particles is valid for Kolmogorov probabilities, but it is not valid for Dempster-Shafer probabilities. The violation of these inequalities in experiments supports the interpretation of quantum probabilities as Dempster-Shafer probabilities. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4891972>]

I. INTRODUCTION

Probability theory needs for its axioms, the concepts of conjunction, disjunction, and negation, and in this sense it is tacitly defined with respect to a lattice. Kolmogorov's probability theory is intimately connected to Boolean algebras. It is defined on a powerset 2^Ω of some set Ω , and subset (\subseteq), intersection (\cap) union (\cup), complement ($\bar{A} = \Omega - A$) are the logical connectives partial order ($<$), conjunction (\wedge), disjunction (\vee), negation (\neg), correspondingly. A basic relation of Kolmogorov probabilities $q(A)$ is that

$$\delta(A, B) = 0; \quad \delta(A, B) = q(A \vee B) - q(A) - q(B) + q(A \wedge B); \quad A, B \in 2^\Omega. \quad (1)$$

This is a “generalized additivity relation” (the term “generalized” refers to the $q(A \wedge B)$ for non-exclusive events).

Quantum logic^{1–6} is based on the orthomodular lattice of closed subspaces of a Hilbert space, which has various Boolean algebras as sublattices. We show that Eq. (1) is valid for quantum probabilities only within a Boolean subalgebra, but it is violated in the full orthomodular lattice. This leads to more general (than Kolmogorov) probability theories, which violate the additivity relation of Eq. (1).^{7–10} Non-additive probabilities are used in cases where there is added value in a coalition. In everyday language this is described with the expression “the whole is greater than the sum of its parts.” One such theory is the Dempster-Shafer theory, which has been used extensively in Artificial Intelligence, Operations Research, Economics, etc.^{11–14}

The original problem that Dempster considered¹¹ was that of a sample space X and a multivalued map Γ from X to another sample space Ω . He then carried Kolmogorov probabilities on subsets of X into lower and upper probabilities on subsets of Ω . For lower probabilities $\delta(A, B) \geq 0$ and for upper probabilities $\delta(A, B) \leq 0$. The need for lower and upper probabilities is intimately related to the multivaluedness of the map Γ . The Dempster multivaluedness is similar to the passage from classical physics to quantum physics, where classical quantities become operators, and we have the problem of ordering in products of these operators (see below).

We introduce an operator $\mathfrak{D}(H_1, H_2)$ (where H_1, H_2 are subspaces of the Hilbert space of the system $H(d)$), which is analogous to $\delta(A, B)$. We show that this operator is related to the commutator of the projectors $\mathfrak{P}(H_1), \mathfrak{P}(H_2)$ into H_1, H_2 (Proposition 3.3 below). Also let ρ be a density matrix, $p(H_i|\rho) = \text{Tr}[\rho\mathfrak{P}(H_i)]$ and $\mathfrak{d}(H_1, H_2|\rho) = \text{Tr}[\rho\mathfrak{D}(H_1, H_2)]$ (this is analogous to $\delta(A, B)$). The $p(H_1|\rho), p(H_2|\rho)$ can be viewed as lower or upper Dempster-Shafer probabilities according to whether $\mathfrak{d}(H_1, H_2|\rho) \geq 0$ or $\mathfrak{d}(H_1, H_2|\rho) \leq 0$, correspondingly. Therefore, the $\mathfrak{D}(H_1, H_2)$ characterizes the nature of a pair of quantum probabilities $p(H_1|\rho), p(H_2|\rho)$.

One application of these ideas is in the general area of Bell inequalities and contextuality,^{15,16} which has been studied extensively for a long time.^{17–29} In order to prove inequalities that involve quantum probabilities, it is important to understand the properties of these probabilities. We show that the CHSH (Clauser, Horne, Shimony, and Holt¹⁷) inequalities, which are proved using the properties of Kolmogorov probabilities, do not hold for Dempster-Shafer probabilities. Their violation in experiments supports the interpretation of quantum probabilities as Dempster-Shafer probabilities.

General probabilistic theories have been used in quantum mechanics by various authors.^{30–36} Operational approaches and convex geometry methods have been studied in Refs. 37–41. Test spaces have been studied in Refs. 42 and 43. Fuzzy phase spaces have been studied in Refs. 44–46. Category theory methods have been studied in Refs. 47 and 48. In this paper we show that the use of Dempster-Shafer probability theory alleviates the difficulties that appear when we use Kolmogorov's theory for quantum probabilities. An example of these difficulties is that they lead to various Bell-like inequalities, which are violated by experiment.

In Sec. II we introduce the operator $\mathfrak{D}(H_1, H_2)$ as a quantum mechanical analogue to $\delta(A, B)$, and we discuss its relationship with the commutator $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$. In Sec. III we state briefly the properties of the Dempster-Shafer (upper and lower) probabilities, and discuss their use as quantum probabilities. In Sec. IV we study CHSH¹⁶ inequalities. We show that their proof is valid for Kolmogorov probabilities but it is invalid for Dempster-Shafer probabilities. These inequalities are violated in experiments, and this supports the view that quantum probabilities are Dempster-Shafer probabilities.

We conclude in Sec. V with a discussion of our results.

II. PRELIMINARIES

We consider a quantum system with position and momenta in $\mathbb{Z}(d)$ (the integers modulo d), described by a d -dimensional Hilbert space $H(d)$. In this space we consider the orthonormal basis of “position states” $|X; n\rangle$, which we represent with the vectors $|X; 0\rangle = (0, \dots, 0, 1)^T$, $|X; 1\rangle = (0, \dots, 1, 0)^T$, etc. We also consider the basis of momentum states

$$|P; m\rangle = d^{-1/2} \sum_m \omega(mn) |X; n\rangle; \quad \omega(\alpha) = \exp\left(\frac{i2\pi\alpha}{d}\right); \quad m, n, \alpha \in \mathbb{Z}(d) \quad (2)$$

and the displacement operators

$$D(\alpha, \beta) = Z^\alpha X^\beta \omega(-2^{-1}\alpha\beta); \quad Z = \sum_m \omega(m) |X; m\rangle \langle X; m|; \quad X = \sum_m |X; m+1\rangle \langle X; m|. \quad (3)$$

Some aspects of the formalism of finite quantum systems are slightly different in the cases of odd or even d . For example, the factor 2^{-1} above is an element of $\mathbb{Z}(d)$, and it exists for odd d . Below, in the formulas that use the displacement operators, we assume that the dimension d is an odd integer.

Acting with $D(\alpha, \beta)$ on a (normalized) fiducial vector

$$|f\rangle = \sum_m f_m |X; m\rangle, \quad (4)$$

which should not be a position state, or a momentum state, we get the following d^2 states which we call coherent states:^{49,50}

$$|C; \alpha, \beta\rangle = D(\alpha, \beta) |f\rangle; \quad \alpha, \beta \in \mathbb{Z}(d). \quad (5)$$

The X, P, C in the notation are not variables, but they simply indicate position states, momentum states, and coherent states. The coherent states obey the resolution of the identity equation:

$$\frac{1}{d} \sum_{\alpha, \beta} |C; \alpha, \beta\rangle \langle C, \alpha, \beta| = \mathbf{1}. \quad (6)$$

For later use we give the formula

$$\lambda(\alpha, \beta; \gamma, \delta) = \langle C; \alpha, \beta | C; \gamma, \delta \rangle = \omega[2^{-1}(\alpha\beta + \gamma\delta) - \alpha\delta] \sum_n f_{n+\delta-\beta}^* f_n \omega[n(\gamma - \alpha)]. \quad (7)$$

We next consider a spin 1/2 particle described with the Hilbert space $H(2)$. S_x, S_y, S_z are spin operators in the directions x, y, z , correspondingly. The vectors $|\frac{1}{2}, \frac{1}{2}\rangle = (1, 0)^T$ and $|\frac{1}{2}, -\frac{1}{2}\rangle = (0, 1)^T$ represent spin up and spin down states in the z -direction. Also

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad S_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (8)$$

We express S_x in terms of projectors as

$$S_x = \frac{1}{2} \Pi(x, 1) - \frac{1}{2} \Pi(x, 0); \quad \Pi(x, 1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad \Pi(x, 0) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{1}_2 - \Pi(x, 1). \quad (9)$$

With the $SU(2)$ rotation

$$U(a, b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}; \quad |a|^2 + |b|^2 = 1, \quad (10)$$

we get

$$S_{a,b} = U(a, b) S_x [U(a, b)]^\dagger = \frac{1}{2} \Pi(a, b; 1) - \frac{1}{2} \Pi(a, b; 0),$$

$$\Pi(a, b; 1) = \frac{1}{2} \begin{pmatrix} |a+b|^2 & a^2 - b^2 \\ (a^*)^2 - (b^*)^2 & |a-b|^2 \end{pmatrix}; \quad \Pi(a, b; 0) = \frac{1}{2} \begin{pmatrix} |a-b|^2 & -a^2 + b^2 \\ -(a^*)^2 + (b^*)^2 & |a+b|^2 \end{pmatrix}. \quad (11)$$

Clearly $S_{0,0} = S_x$.

For a system of two spin 1/2 particles, described with the Hilbert space $H(2) \otimes H(2) = H(4)$, we use analogous notation.

III. THE ORTHOCOMPLEMENTED MODULAR LATTICE $\mathcal{L}[H(d)]$ AND THE GENERALIZED ADDITIVITY OPERATOR $\mathfrak{D}(H_1, H_2)$

The closed subspaces of a Hilbert space are themselves Hilbert spaces, and they form an orthomodular lattice, which has been studied extensively after the work of Birkhoff and von Neumann on quantum logic.¹⁻⁵ General references on lattices are Refs. 51–53 and on orthomodular lattices are Refs. 54–57. Let $\mathcal{L}[H(d)]$ be the orthomodular lattice of subspaces of the d -dimensional Hilbert space $H(d)$. In this case all subspaces are closed, and for this reason we omit the term “closed.” Another consequence of the fact that our Hilbert spaces are finite dimensional is that the orthomodular lattice $\mathcal{L}[H(d)]$ is modular. Orthomodularity is weaker concept than orthocomplemented modularity and the $\mathcal{L}[H(d)]$ is orthocomplemented modular lattice.

The logical connectives in the lattice $\mathcal{L}[H(d)]$ are defined as follows. If H_1, H_2 are subspaces of $H(d)$, then

$$H_1 \wedge H_2 = H_1 \cap H_2; \quad H_1 \vee H_2 = \text{span}(H_1 \cup H_2). \quad (12)$$

We use the notation \mathcal{O} and \mathcal{I} for the smallest and greatest elements in the lattice. Then $\mathcal{O} = H(0)$ is the zero-dimensional subspace that contains only the zero vector, and $\mathcal{I} = H(d)$. $H_1 \prec H_2$ means

that H_1 is a subspace of H_2 . The orthocomplement H_1^\perp is a subspace of $H(d)$, orthogonal to H_1 , such that

$$H_1 \wedge H_1^\perp = \mathcal{O}; \quad H_1 \vee H_1^\perp = \mathcal{I}. \quad (13)$$

We denote as $\mathfrak{P}(H_1)$ the projector to the subspace H_1 . In the following lemma we give, without proof, some elementary properties which are needed later.

Lemma 3.1.

(1)

$$\begin{aligned} \mathfrak{P}(H_1 \wedge H_2)\mathfrak{P}(H_1) &= \mathfrak{P}(H_1)\mathfrak{P}(H_1 \wedge H_2) = \mathfrak{P}(H_1 \wedge H_2), \\ \mathfrak{P}(H_1 \vee H_2)\mathfrak{P}(H_1) &= \mathfrak{P}(H_1)\mathfrak{P}(H_1 \vee H_2) = \mathfrak{P}(H_1). \end{aligned} \quad (14)$$

(2)

$$\mathfrak{P}(H_1)\mathfrak{P}(H_1^\perp) = 0; \quad \mathfrak{P}(H_1) + \mathfrak{P}(H_1^\perp) = \mathbf{1}; \quad \mathfrak{P}(H_1 \wedge H_2)\mathfrak{P}(H_1^\perp \wedge H_3) = 0. \quad (15)$$

(3) If $\mathfrak{P}(H_1)\mathfrak{P}(H_2) = 0$ then $\mathfrak{P}(H_2)\mathfrak{P}(H_1) = 0$, $H_1 \wedge H_2 = \mathcal{O}$ and $\mathfrak{P}(H_1) + \mathfrak{P}(H_2) = \mathfrak{P}(H_1 \vee H_2)$. The $H_1 \wedge H_2 = \mathcal{O}$ does not necessarily imply the $\mathfrak{P}(H_1)\mathfrak{P}(H_2) = 0$. The $\mathfrak{P}(H_1)\mathfrak{P}(H_2) = 0$ is equivalent to $H_1 \prec H_2^\perp$, and it is denoted as $H_1 \perp H_2$.

Definition 3.2. H_1 commutes with H_2 (the standard notation for this is $H_1 \mathcal{C} H_2$) if

$$H_1 = (H_1 \wedge H_2) \vee (H_1 \wedge H_2^\perp). \quad (16)$$

It is easily seen that $H_1 \prec H_2$ implies that $H_1 \mathcal{C} H_2$. Also, since $\mathcal{L}[H(d)]$ is an orthomodular lattice, if $H_1 \mathcal{C} H_2$, then $H_2 \mathcal{C} H_1$ and also $H_1 \mathcal{C} H_2^\perp$, $H_1^\perp \mathcal{C} H_2$. Furthermore, if $H_1 \mathcal{C} H_2$ then

$$(H_1 \wedge H_2) \vee (H_1 \wedge H_2^\perp) \vee (H_1^\perp \wedge H_2) \vee (H_1^\perp \wedge H_2^\perp) = \mathcal{I}, \quad (17)$$

Proposition 3.3. In the lattice $\mathcal{L}[H(d)]$, let

$$\mathfrak{D}(H_1, H_2) = \mathfrak{P}(H_1 \vee H_2) + \mathfrak{P}(H_1 \wedge H_2) - \mathfrak{P}(H_1) - \mathfrak{P}(H_2). \quad (18)$$

Then

(1) The fact that $\mathcal{L}[H(d)]$ is a modular lattice implies that $\text{Tr}[\mathfrak{D}(H_1, H_2)] = 0$.

(2) The $\mathfrak{P}(H_1 \wedge H_2)$ is in general different from $\mathfrak{P}(H_1)\mathfrak{P}(H_2)$ and their difference is given by

$$\mathfrak{P}(H_1 \wedge H_2) - \mathfrak{P}(H_1)\mathfrak{P}(H_2) = \mathfrak{P}(H_1)\mathfrak{D}(H_1, H_2) = \mathfrak{D}(H_1, H_2)\mathfrak{P}(H_2). \quad (19)$$

(3) The commutator $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$ is related to $\mathfrak{D}(H_1, H_2)$ through the relation:

$$[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = \mathfrak{D}(H_1, H_2)[\mathfrak{P}(H_1) - \mathfrak{P}(H_2)] = -[\mathfrak{P}(H_1) - \mathfrak{P}(H_2)]\mathfrak{D}(H_1, H_2). \quad (20)$$

(4) The following are equivalent:

- (a) $\mathfrak{D}(H_1, H_2) = 0$,
- (b) $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$,
- (c) $\mathfrak{P}(H_1 \wedge H_2) = \mathfrak{P}(H_1)\mathfrak{P}(H_2)$,
- (d) $H_1 \mathcal{C} H_2$.

(5)

$$[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = -[\mathfrak{P}(H_1), \mathfrak{D}(H_1, H_2)]. \quad (21)$$

(6)

$$\mathfrak{D}(H_1^\perp, H_2^\perp) = -\mathfrak{D}(H_1, H_2). \quad (22)$$

(7) If $H_1 \perp H_2$ or $H_1 < H_2$, then $\mathfrak{D}(H_1, H_2) = 0$.*Proof.*(1) Let $\dim(H_1)$ be the dimension of H_1 , and $\mathfrak{h}(H_1)$ the height of the element H_1 in the lattice $\mathcal{L}[H(d)]$. Then

$$\mathfrak{h}(H_1) = \dim(H_1) = \text{Tr}[\mathfrak{P}(H_1)]. \quad (23)$$

 $\mathcal{L}[H(d)]$ is a modular lattice and therefore the height obeys the relation (e.g., Ref. 51, p. 41)

$$\mathfrak{h}(H_1 \vee H_2) + \mathfrak{h}(H_1 \wedge H_2) - \mathfrak{h}(H_1) - \mathfrak{h}(H_2) = 0. \quad (24)$$

From this follows that

$$\text{Tr}[\mathfrak{D}(H_1, H_2)] = \text{Tr}[\mathfrak{P}(H_1 \vee H_2)] + \text{Tr}[\mathfrak{P}(H_1 \wedge H_2)] - \text{Tr}[\mathfrak{P}(H_1)] - \text{Tr}[\mathfrak{P}(H_2)] = 0. \quad (25)$$

(2) In order to prove this, we multiply Eq. (18) on the left hand side with $\mathfrak{P}(H_1)$, using Eq. (14). We also multiply Eq. (18) on the right hand side with $\mathfrak{P}(H_2)$, using Eq. (14).

(3) This is proved using Eq. (19).

(4) We prove the four parts of the “loop” $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$.(i) If $\mathfrak{D}(H_1, H_2) = 0$ then $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$ according to Eq. (20).(ii) Let $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$. From this follows that $\mathfrak{P}(H_1)\mathfrak{P}(H_2)$ is a projector. The $\mathfrak{P}(H_1)\mathfrak{P}(H_2)$ projects into a space with vectors which belong to both H_1 and H_2 . Therefore, these vectors belong to $H_1 \cap H_2$, and this proves that $\mathfrak{P}(H_1)\mathfrak{P}(H_2)$ is the same projector as $\mathfrak{P}(H_1 \wedge H_2)$.(iii) We assume that $\mathfrak{P}(H_1 \wedge H_2) = \mathfrak{P}(H_1)\mathfrak{P}(H_2)$. Since the $\mathfrak{P}(H_1 \wedge H_2)$ is symmetric (i.e., $\mathfrak{P}(H_1 \wedge H_2) = \mathfrak{P}(H_2 \wedge H_1)$), it follows that $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$. From this we prove that $[\mathfrak{P}(H_1), \mathfrak{P}(H_2^\perp)] = 0$, and therefore $\mathfrak{P}(H_1 \wedge H_2^\perp) = \mathfrak{P}(H_1)\mathfrak{P}(H_2^\perp)$.From $\mathfrak{P}(H_1 \wedge H_2)\mathfrak{P}(H_1 \wedge H_2^\perp) = 0$, it follows that

$$\begin{aligned} \mathfrak{P}[(H_1 \wedge H_2) \vee (H_1 \wedge H_2^\perp)] &= \mathfrak{P}(H_1 \wedge H_2) + \mathfrak{P}(H_1 \wedge H_2^\perp) \\ &= \mathfrak{P}(H_1)\mathfrak{P}(H_2) + \mathfrak{P}(H_1)\mathfrak{P}(H_2^\perp) = \mathfrak{P}(H_1). \end{aligned} \quad (26)$$

Therefore, $H_1 = (H_1 \wedge H_2) \vee (H_1 \wedge H_2^\perp)$ which proves that $H_1 \mathcal{C} H_2$.(iv) If $H_1 \mathcal{C} H_2$, then Eq. (16), and the fact that $\mathfrak{P}(H_1 \wedge H_2)\mathfrak{P}(H_1 \wedge H_2^\perp) = 0$, leads to

$$\mathfrak{P}(H_1) = \mathfrak{P}(H_1 \wedge H_2) + \mathfrak{P}(H_1 \wedge H_2^\perp). \quad (27)$$

In a similar way we get

$$\mathfrak{P}(H_2) = \mathfrak{P}(H_1 \wedge H_2) + \mathfrak{P}(H_1^\perp \wedge H_2). \quad (28)$$

We note that

$$\begin{aligned} \mathfrak{P}(A_i)\mathfrak{P}(A_j) &= 0; \quad A_1 = H_1 \wedge H_2; \quad A_2 = H_1^\perp \wedge H_2; \quad A_3 = H_1 \wedge H_2^\perp; \\ A_4 &= H_1^\perp \wedge H_2^\perp, \end{aligned} \quad (29)$$

and therefore Eq. (17) gives

$$\mathfrak{P}(H_1 \wedge H_2) + \mathfrak{P}(H_1 \wedge H_2^\perp) + \mathfrak{P}(H_1^\perp \wedge H_2) + \mathfrak{P}(H_1^\perp \wedge H_2^\perp) = \mathbf{1}. \quad (30)$$

From this follows that

$$\mathfrak{P}(H_1 \wedge H_2) + \mathfrak{P}(H_1 \wedge H_2^\perp) + \mathfrak{P}(H_1^\perp \wedge H_2) = \mathbf{1} - \mathfrak{P}(H_1^\perp \wedge H_2^\perp) = \mathfrak{P}(H_1 \vee H_2). \quad (31)$$

We now add Eqs. (27)(28), taking into account Eq. (31), and we prove that $\mathfrak{D}(H_1, H_2) = 0$.

- (5) The proof of this is straightforward (using Eq. (14)).
- (6) The proof of this is straightforward (using Eq. (15)).
- (7) The proof of this is straightforward.

□

Remark 3.4. The operator $\mathfrak{D}(H_1, H_2)$ is analogous to $\delta(A, B)$ for Kolmogorov probabilities. Therefore, the $\mathfrak{D}(H_1, H_2)$, which in general is non-zero, quantifies how different quantum probabilities are from Kolmogorov probabilities. Equation (20) shows that the $\mathfrak{D}(H_1, H_2)$ is also a measure of non-commutativity.

Let ρ is a density matrix of a system described by the space $H(d)$. An important theorem by Gleason⁵⁸ shows that quantum probabilities associated with the projectors $\mathfrak{P}(H_1)$ are given by $p(H_1|\rho) = \text{Tr}[\rho\mathfrak{P}(H_1)]$ (in spaces with dimension greater than 2). Then

$$p(H_1^\perp|\rho) = 1 - p(H_1|\rho). \quad (32)$$

Let

$$\mathfrak{d}(H_1, H_2|\rho) = \text{Tr}[\rho\mathfrak{D}(H_1, H_2)] = p(H_1 \vee H_2|\rho) + p(H_1 \wedge H_2|\rho) - p(H_1|\rho) - p(H_2|\rho). \quad (33)$$

Then $H_1 \mathcal{C} H_2$ or equivalently $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$, if and only if $\mathfrak{d}(H_1, H_2|\rho) = 0$, for all density matrices. For a particular density matrix it may be that $\mathfrak{d}(H_1, H_2|\rho) = 0$, although $\mathfrak{D}(H_1, H_2)$ is non-zero.

Let λ_i and $|v_i\rangle$ (with $i = 1, \dots, d$) be the eigenvalues and eigenvectors of $\mathfrak{D}(H_1, H_2)$. We order the eigenvalues as $\lambda_1 \leq \dots \leq \lambda_d$. Then

$$\begin{aligned} \text{Tr}[\mathfrak{D}(H_1, H_2)] &= \sum_i \lambda_i = 0; \quad \lambda_1 \leq 0 \leq \lambda_d, \\ \mathfrak{d}(H_1, H_2|\rho) &= \sum \lambda_i \langle v_i | \rho | v_i \rangle; \quad \lambda_1 \leq \mathfrak{d}(H_1, H_2|\rho) \leq \lambda_d. \end{aligned} \quad (34)$$

For various density matrices, the $\mathfrak{d}(H_1, H_2|\rho)$ takes both positive and negative values. For example,

$$\mathfrak{d}(H_1, H_2||v_1\rangle\langle v_1|) = \lambda_1 \leq 0; \quad \mathfrak{d}(H_1, H_2||v_d\rangle\langle v_d|) = \lambda_d \geq 0; \quad \mathfrak{d}(H_1, H_2|\frac{1}{d}\mathbf{1}) = 0. \quad (35)$$

Example 3.5. In the space $H(3)$ we consider the one-dimensional subspaces

$$H_1 = \{a(1, 0, 0)^T\}; \quad H_2 = \{a(1, 1, 0)^T\}. \quad (36)$$

Here we give the general vector that belongs to each of these spaces. In this case

$$H_1 \vee H_2 = \{(a, b, 0)^T\}; \quad H_1 \wedge H_2 = \mathcal{O}. \quad (37)$$

The corresponding projectors are

$$\mathfrak{P}(H_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathfrak{P}(H_2) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathfrak{P}(H_1 \vee H_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (38)$$

In this example, the H_1, H_2 do not commute. We find that

$$\mathfrak{D}(H_1, H_2) = \frac{1}{2} \begin{pmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad [\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (39)$$

and we can confirm Eq. (20).

Remark 3.6. For non-commuting projectors $\mathfrak{P}(H_1), \mathfrak{P}(H_2)$, Ref. 59 has considered the set of states $|s\rangle$ in $H(d)$ for which $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]|s\rangle = 0$. They form a subspace of $H(d)$ which we denote as $\mathcal{H}[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$. More generally let A, B be the non-commuting observables

$$A = \sum_i \lambda_i \mathfrak{P}_{A_i}; \quad B = \sum_j \mu_j \mathfrak{P}_{B_j}, \quad (40)$$

where λ_i, μ_j are their eigenvalues, and $\mathfrak{P}_{A_i}, \mathfrak{P}_{B_j}$ are their eigenprojectors. The space $\mathcal{H}(A, B)$ contains all states $|s\rangle$ in $H(d)$ for which $[\mathfrak{P}_{A_i}, \mathfrak{P}_{B_j}]|s\rangle = 0$ for all i, j . Although A, B do not commute, they are “weakly compatible,” i.e., compatible with respect to the states in $\mathcal{H}(A, B)$ (Ref. 59 uses the term “relative compatibility”). It will be interesting to extend these ideas in our context, i.e., for $\mathfrak{D}(H_1, H_2) \neq 0$, to consider the set \mathcal{R} of density matrices for which $\mathfrak{D}(H_1, H_2|\rho) = \text{Tr}[\mathfrak{D}(H_1, H_2)\rho] = 0$. Then $\mathfrak{P}(H_1), \mathfrak{P}(H_2)$ can be associated to Kolmogorov probabilities, only for density matrices in the set \mathcal{R} . We do not study further this idea in this paper.

A. Boolean algebras associated with orthonormal bases

If B is a sublattice of $\mathcal{L}[H(d)]$ and $H_1 \mathcal{C} H_2$ for any two elements $H_1, H_2 \in B$, then B is a Boolean subalgebra of $\mathcal{L}[H(d)]$ (the negation $\neg H_1$ is the H_1^\perp).⁵¹ Maximal Boolean algebras in $\mathcal{L}[H(d)]$ are sometimes called Boolean blocks. In order to construct one, we consider a basis of d vectors in $H(d)$, which are orthogonal to each other. We then consider the e -dimensional space spanned by e of these vectors. There are

$$n_d(e) = \binom{d}{e}; \quad 0 \leq e \leq d \quad (41)$$

such spaces. Therefore, the total number of spaces in the Boolean algebra is $\sum_e n_d(e) = 2^d$. Acting with a unitary transformation on a Boolean algebra B , we get another Boolean algebra. $\mathcal{L}[H(d)]$ is the union of its Boolean algebras, which are “pasted” with rules discussed in Refs. 54–56, and 57.

For any two elements H_1, H_2 in a Boolean algebra, $\mathfrak{D}(H_1, H_2) = 0$ and $\mathfrak{D}(H_1, H_2|\rho) = 0$. In this case the corresponding quantum probabilities $p(H_1|\rho), p(H_2|\rho)$ are Kolmogorov probabilities.

Boolean algebras within $\mathcal{L}[H(d)]$ play the role of quantum contexts in the sense of Kochen and Specker.¹⁶ Examples of Boolean algebras within $\mathcal{L}[H(4)]$ are given later.

B. Projectors associated to coherent states

We consider the one-dimensional space $H_{\alpha, \beta}$ that contains the coherent state $|C; \alpha, \beta\rangle$, defined in Eq. (6). The projector to this space is

$$\mathfrak{P}(H_{\alpha, \beta}) = |C; \alpha, \beta\rangle\langle C; \alpha, \beta|; \quad \frac{1}{d} \sum_{\alpha, \beta} \mathfrak{P}(H_{\alpha, \beta}) = \mathbf{1}. \quad (42)$$

For $(\alpha, \beta) \neq (\gamma, \delta)$ we get $H_{\alpha, \beta} \wedge H_{\gamma, \delta} = \mathcal{O}$. Also $H_{\alpha, \beta} \vee H_{\gamma, \delta}$ is the two-dimensional space that contains the vectors $\kappa|C; \alpha, \beta\rangle + \mu|C; \gamma, \delta\rangle$, and the corresponding projector is

$$\begin{aligned} \mathfrak{P}(H_{\alpha, \beta} \vee H_{\gamma, \delta}) &= |C; \alpha, \beta\rangle\langle C; \alpha, \beta| + |s\rangle\langle s|, \\ |s\rangle &= (1 - |\lambda(\alpha, \beta; \gamma, \delta)|^2)^{-1/2} [|C; \gamma, \delta\rangle - \lambda(\alpha, \beta; \gamma, \delta)|C; \alpha, \beta\rangle], \\ \langle C; \alpha, \beta|s\rangle &= 0, \end{aligned} \quad (43)$$

where $\lambda(\alpha, \beta; \gamma, \delta)$ has been given in Eq. (7). We note that

$$\begin{aligned} \mathfrak{P}(H_{\alpha, \beta})\mathfrak{P}(H_{\gamma, \delta}) &= \lambda(\alpha, \beta; \gamma, \delta)|C; \alpha, \beta\rangle\langle C; \gamma, \delta|, \\ \mathfrak{P}(H_{\gamma, \delta})\mathfrak{P}(H_{\alpha, \beta}) &= [\lambda(\alpha, \beta; \gamma, \delta)]^*|C; \gamma, \delta\rangle\langle C; \alpha, \beta|, \\ |\lambda(\alpha, \beta; \gamma, \delta)|^2 &= \text{Tr}[\mathfrak{P}(H_{\alpha, \beta})\mathfrak{P}(H_{\gamma, \delta})]. \end{aligned} \quad (44)$$

Therefore,

$$\begin{aligned} \mathfrak{P}(H_{\alpha,\beta} \vee H_{\gamma,\delta}) &= \frac{1}{1 - |\lambda(\alpha, \beta; \gamma, \delta)|^2} \\ &\times [\mathfrak{P}(H_{\alpha,\beta}) + \mathfrak{P}(H_{\gamma,\delta}) - \mathfrak{P}(H_{\alpha,\beta})\mathfrak{P}(H_{\gamma,\delta}) - \mathfrak{P}(H_{\gamma,\delta})\mathfrak{P}(H_{\alpha,\beta})] \end{aligned}$$

and

$$\begin{aligned} \mathfrak{D}(H_{\alpha,\beta}, H_{\gamma,\delta}) &= \frac{|\lambda(\alpha, \beta; \gamma, \delta)|^2}{1 - |\lambda(\alpha, \beta; \gamma, \delta)|^2} [\mathfrak{P}(H_{\alpha,\beta}) + \mathfrak{P}(H_{\gamma,\delta})] \\ &- \frac{1}{1 - |\lambda(\alpha, \beta; \gamma, \delta)|^2} [\mathfrak{P}(H_{\alpha,\beta})\mathfrak{P}(H_{\gamma,\delta}) + \mathfrak{P}(H_{\gamma,\delta})\mathfrak{P}(H_{\alpha,\beta})]. \end{aligned} \quad (46)$$

We can now confirm Eq. (20) for this example using the relations

$$\begin{aligned} \mathfrak{P}(H_{\alpha,\beta})\mathfrak{P}(H_{\gamma,\delta})\mathfrak{P}(H_{\alpha,\beta}) &= |\lambda(\alpha, \beta; \gamma, \delta)|^2 \mathfrak{P}(H_{\alpha,\beta}), \\ \mathfrak{P}(H_{\gamma,\delta})\mathfrak{P}(H_{\alpha,\beta})\mathfrak{P}(H_{\gamma,\delta}) &= |\lambda(\alpha, \beta; \gamma, \delta)|^2 \mathfrak{P}(H_{\gamma,\delta}). \end{aligned} \quad (47)$$

The above example shows that if $H_1 \wedge H_2 = \mathcal{O}$ then $\mathfrak{D}(H_1, H_2) = \mathfrak{P}(H_1 \vee H_2) - \mathfrak{P}(H_1) - \mathfrak{P}(H_2)$ is in general non-zero. The stronger assumption $\mathfrak{P}(H_1)\mathfrak{P}(H_2) = 0$ implies that $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$, and this leads to $\mathfrak{D}(H_1, H_2) = 0$.

IV. THE DEMPSTER-SHAFFER THEORY: MULTIVALUEDNESS AND LOWER AND UPPER PROBABILITIES

Dempster¹¹ studied a multivalued map from a sample space X to another space Ω . He has shown that due to the multivaluedness of the map Kolmogorov probabilities related to subsets of X become lower and upper probabilities on subsets of Ω . We summarize very briefly the basics of the Dempster-Shafer formalism starting with a simple example, which is similar to one in Ref. 60.

A company has $n = n_1 + n_2 + n_3$ employees. The age of n_1 employees is known to be under 30, n_2 employees is known to be over 50, and the rest n_3 employees is known to be between 25 and 45. The Dempster multivaluedness is here the fact that our knowledge about the age of each employee is an interval of values, rather than a single value. We want to find the probability that a random employee is under 35. Let S be the set of employees under 35. We are certain that n_1 employees belong to S , and that n_2 employees do not belong to S . The n_3 employees belong to the “don’t know” category (Dempster¹² emphasized the importance of this category). The Dempster-Shafer theory introduces the lower probability or belief ℓ , and the upper probability or plausibility u , given by

$$\ell = \frac{n_1}{n}; \quad u = \frac{n_1 + n_2}{n}. \quad (48)$$

They quantify what in everyday language is called “worst case scenario” and “best case scenario.” We express these concepts in a more formal way.

Given a set Ω , let A, B be elements of the powerset 2^Ω (i.e., subsets of Ω). The lower probability or belief $\ell(A)$ is a monotone function from 2^Ω to $[0, 1]$, i.e.,

$$A \subseteq B \rightarrow \ell(A) \leq \ell(B), \quad (49)$$

The $1 - \ell(\bar{A})$ is in general different from $\ell(A)$, and it is the upper probability or plausibility $u(A)$ (where $\bar{A} = \Omega - A$). In the above example, $\ell(\bar{A})$ is the lower probability that the age of the average employee is over 35, and it is equal to n_2/n .

The difference between the upper and lower probabilities describes the “don’t know” case:

$$u(A) - \ell(A) = 1 - \ell(A) - \ell(\bar{A}). \quad (50)$$

The upper probability combines the “true” and the “don’t know.”

TABLE I. Properties of the lower and upper probabilities in the Dempster-Shafer theory, and also of the Kolmogorov probabilities. A, B are subsets of Ω .

Lower probabilities $\ell(A)$	Upper probabilities $u(A) = 1 - \ell(\bar{A})$
$A \subseteq B \rightarrow \ell(A) \leq \ell(B)$	$A \subseteq B \rightarrow u(A) \leq u(B)$
$\ell(\emptyset) = 0; \ell(\Omega) = 1$	$u(\emptyset) = 0; u(\Omega) = 1$
$\ell(A \cup B) - \ell(A) - \ell(B) + \ell(A \cap B) \geq 0$	$u(A \cup B) - u(A) - u(B) + u(A \cap B) \leq 0$
$\ell(\bar{A}) + \ell(A) \leq 1$	$u(\bar{A}) + u(A) \geq 1$
$\ell(A) + \ell(B) - \ell(A \cup B)$ might be negative	$u(A) + u(B) - u(A \cup B) \geq 0$ Boole's inequality
Kolmogorov probabilities $q(A)$	
$A \subseteq B \rightarrow q(A) \leq q(B)$	
$q(\emptyset) = 0; q(\Omega) = 1$	
$q(A \cup B) - q(A) - q(B) + q(A \cap B) = 0$	
$q(\bar{A}) + q(A) = 1$	
$q(A) + q(B) - q(A \cup B) \geq 0$	
Boole's inequality	

For Kolmogorov probabilities $1 - q(\bar{A}) = q(A)$ which means that the statement “belongs to A ” is equivalent with the statement “does not belong to \bar{A} .” In the Dempster-Shafer theory, due to the “don’t know” cases, this is not true in general.

The properties of lower and upper probabilities and also for comparison, of Kolmogorov probabilities, are summarized in Table I. Kolmogorov theory can be viewed as the case where all upper probabilities are equal to the corresponding lower probabilities.

Remark 4.1. From Eq. (1) follows the additivity property of Kolmogorov probabilities

$$A \cap B = \emptyset \rightarrow q(A \cup B) = q(A) + q(B). \quad (51)$$

This is important for integration. Dempster-Shafer probabilities are capacities, i.e., they obey the weaker property of Eq. (49). A different integration concept, known as Choquet integrals,^{63,64} is applicable to them, but we do not use it in this paper.

V. QUANTUM PROBABILITIES AS DEMPSTER-SHAFFER PROBABILITIES DUE TO NON-COMMUTATIVITY

We interpret a pair $p(H_1|\rho)$ and $p(H_2|\rho)$ of quantum probabilities as upper or lower probabilities as follows:

- If $\mathfrak{D}(H_1, H_2|\rho) \geq 0$, then $p(H_1|\rho)$ and $p(H_2|\rho)$ are lower probabilities. Also, according to Eq. (22), in this case $\mathfrak{D}(H_1^\perp, H_2^\perp|\rho) \leq 0$, and therefore the $p(H_1^\perp|\rho)$ and $p(H_2^\perp|\rho)$ are upper probabilities.
- If $\mathfrak{D}(H_1, H_2|\rho) \leq 0$, then $p(H_1|\rho)$ and $p(H_2|\rho)$ are upper probabilities, and the $p(H_1^\perp|\rho)$ and $p(H_2^\perp|\rho)$ are lower probabilities.
- If $\mathfrak{D}(H_1, H_2|\rho) = 0$, then $p(H_1|\rho)$ and $p(H_2|\rho)$ are Kolmogorov probabilities. Also the $p(H_1^\perp|\rho)$ and $p(H_2^\perp|\rho)$ are Kolmogorov probabilities.
- The characterization of lower or upper probabilities refers to a particular pair. It may be that $p(H_1|\rho)$ is a lower probability when paired with $p(H_2|\rho)$, and upper probability when paired with $p(H_3|\rho)$.

It is clear that if H_1, H_2 are elements of a Boolean algebra, then $p(H_1|\rho), p(H_2|\rho)$ are Kolmogorov probabilities, and a relation analogous to Eq. (1) holds. But for general elements H_1, H_2 in $\mathcal{L}[H(d)]$ the $p(H_1|\rho), p(H_2|\rho)$ are Dempster-Shafer probabilities, and a relation analogous to Eq. (1) does not hold. Non-zero value of the commutator $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$ implies non-zero value of $\mathfrak{D}(H_1, H_2)$, and this leads to the adoption of Dempster-Shafer probabilities, which do not obey Eq. (1).

We summarize our motivation for using the Dempster-Shafer theory in a quantum context:

- (1) Kolmogorov probabilities obey Eq. (1), while Dempster-Shafer probabilities might violate it (see Table I). Quantum probabilities violate the analogue of Eq. (1), and this is directly linked to non-commutativity. Equation (20) shows that the commutator $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$ is intimately related to $\mathfrak{D}(H_1, H_2)$, which quantifies deviations from Kolmogorov probabilities. Therefore, in quantum mechanics we need a more general (than Kolmogorov) probability theory, and we propose the Dempster-Shafer theory.
- (2) Kolmogorov probabilities are intimately connected to Boolean algebras. Within the $\mathcal{L}[H(d)]$ there are Boolean algebras, and in these “islands” quantum probabilities behave like Kolmogorov probabilities. But in the full lattice, we need a more general probability theory, and we propose the Dempster-Shafer theory.
- (3) The passage from classical to quantum mechanics can be viewed as a type of Dempster multivaluedness. The product of two classical quantities becomes a product of two operators, which can be ordered in many ways (as $\mathfrak{P}(H_1)\mathfrak{P}(H_2)$ or as $\mathfrak{P}(H_2)\mathfrak{P}(H_1)$ or in many other intermediate ways). This can be interpreted as a type of Dempster multivaluedness, the “spread” of which is quantified with the commutator $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$. Equation (20) shows that this “spread” is intimately related to $\mathfrak{D}(H_1, H_2)$, which quantifies deviations from Kolmogorov probabilities. Dempster’s “don’t know” becomes here “don’t know which ordering rule to use.” So the motivation for introducing an interval of probabilities (from the lower to the upper one) is because there are many products of the operators $\mathfrak{P}(H_1), \mathfrak{P}(H_2)$. We point out here the analogy with the Q -function, Wigner function, and P -function, which are part of a continuum of quantities (not probabilities) related to the ordering of operators (e.g., Refs. 61 and 62). The lower and upper probabilities are another languages for these problems, which might provide a deeper insight to phase space methods in quantum mechanics.
- (4) The Dempster-Shafer theory uses non-additive probabilities (i.e., it violates Eq. (1)). Non-additive probabilities are used in areas like Game theory or Operations Research to describe coalitions (e.g., the merger of two companies). In everyday language this non-additivity is described with the expression “the whole is greater than the sum of its parts.” Non-additive probabilities might be better in describing problems like contextuality than additive (Kolmogorov) probabilities. A measurement M performed in conjunction with the measurements A_1, A_2, \dots might give a different result from the measurement M performed in conjunction with the measurements B_1, B_2, \dots .¹⁸ Here the M, A_1, A_2, \dots commute with each other and the M, B_1, B_2, \dots also commute with each other, but the A_1, A_2, \dots might not commute with the B_1, B_2, \dots . So the measurement M behaves in a different way, when it is in a “coalition” with the A_1, A_2, \dots than when it is in a coalition with the B_1, B_2, \dots . In Sec. VI we make a small step in this direction, and we show that Dempster-Shafer probabilities violate Bell inequalities. The fact that experiment also violates these inequalities supports the adoption of the Dempster-Shafer theory.

VI. DEMPSTER-SHAFFER PROBABILITIES VIOLATE BELL INEQUALITIES

The purpose of this section is to show that the proof of Bell inequalities relies on the properties of Kolmogorov probabilities, and it is not valid for Dempster-Shafer probabilities. The violation of these inequalities in experiments supports the interpretation of quantum probabilities as Dempster-Shafer probabilities.

A. Boolean algebras within $\mathcal{L}[H(4)]$

We consider the subspaces of $H(4)$ shown in Table II. Also $\mathcal{I} = H(4)$ and \mathcal{O} is the zero-dimensional subspace that contains only the zero vector. The

$$B_A = \{\mathcal{I}, \mathcal{O}, H_{iA}, H_{iA}^\perp \mid i = 1, \dots, 7\} \quad (52)$$

TABLE II. The Hilbert spaces in the Boolean algebra B_A within $\mathcal{L}[H(4)]$, and the corresponding projectors. The $\mathcal{I} = H(4)$ and \mathcal{O} also belong to B_A . The general vector that belongs to each of these spaces is shown.

$H_{1A} = \{a(1, 1, 1, 1)^T\}$	$\mathfrak{P}(H_{1A}) = \Pi(x, 1) \otimes \Pi(x, 1)$
$H_{1A}^\perp = \{(a, b, c, -a - b - c)^T\}$	$\mathfrak{P}(H_{1A}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \Pi(x, 1)$
$H_{2A} = \{a(1, -1, 1, -1)^T\}$	$\mathfrak{P}(H_{2A}) = \Pi(x, 1) \otimes \Pi(x, 0)$
$H_{2A}^\perp = \{(a, b, c, a - b + c)^T\}$	$\mathfrak{P}(H_{2A}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \Pi(x, 0)$
$H_{3A} = \{a(1, 1, -1, -1)^T\}$	$\mathfrak{P}(H_{3A}) = \Pi(x, 0) \otimes \Pi(x, 1)$
$H_{3A}^\perp = \{(a, b, c, a + b - c)^T\}$	$\mathfrak{P}(H_{3A}^\perp) = \mathbf{1}_4 - \Pi(x, 0) \otimes \Pi(x, 1)$
$H_{4A} = \{a(1, -1, -1, 1)^T\}$	$\mathfrak{P}(H_{4A}) = \Pi(x, 0) \otimes \Pi(x, 0)$
$H_{4A}^\perp = \{(a, b, c, -a + b + c)^T\}$	$\mathfrak{P}(H_{4A}^\perp) = \mathbf{1}_4 - \Pi(x, 0) \otimes \Pi(x, 0)$
$H_{5A} = \{(a, b, a, b)^T\}$	$\mathfrak{P}(H_{5A}) = \Pi(x, 1) \otimes \mathbf{1}_2$
$H_{5A}^\perp = \{(a, b, -a, -b)^T\}$	$\mathfrak{P}(H_{5A}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \mathbf{1}_2$
$H_{6A} = \{(a, a, b, b)^T\}$	$\mathfrak{P}(H_{6A}) = \mathbf{1}_2 \otimes \Pi(x, 1)$
$H_{6A}^\perp = \{(a, -a, b, -b)^T\}$	$\mathfrak{P}(H_{6A}^\perp) = \mathbf{1}_4 - \mathbf{1}_2 \otimes \Pi(x, 1)$
$H_{7A} = \{(a, b, b, a)^T\}$	$\mathfrak{P}(H_{7A}) = \Pi(x, 1) \otimes \Pi(x, 1) + \Pi(x, 0) \otimes \Pi(x, 0)$
$H_{7A}^\perp = \{(a, b, -b, -a)^T\}$	$\mathfrak{P}(H_{7A}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \Pi(x, 1) - \Pi(x, 0) \otimes \Pi(x, 0)$

is a sublattice of $\mathcal{L}[H(4)]$. It is easily seen that B_A is closed under the various logical operations. Also any two of its 16 elements commute ($H_{iA}CH_{jA}$). The sublattice B_A is a Boolean algebra within $\mathcal{L}[H(4)]$. Negation is defined as $\neg H_{iA} = H_{iA}^\perp$. Projectors to the spaces in B_A , in terms of the projectors in Eq. (9), are given in Table II.

With any $SU(4)$ transformation, we get another Boolean algebra. For later use we consider the transformation

$$U_B = \mathbf{1}_2 \otimes U(a, b) \quad (53)$$

and we get the Boolean algebra B_B shown in Table III. We use the notation H_{iB} for the Hilbert space that contains the states $U_B|s\rangle$ where $|s\rangle$ is a state in H_{iA} (Table III). In particular we note that $H_{5A} = H_{5B}$ and $H_{5A}^\perp = H_{5B}^\perp$. The Hilbert spaces in

$$B_A \cap B_B = \{\mathcal{I}, \mathcal{O}, H_{5A}, H_{5A}^\perp\} \quad (54)$$

commute with all spaces in both Boolean algebras B_A, B_B . But the spaces in $B_A - (B_A \cap B_B)$, in general, do not commute with the spaces in $B_B - (B_A \cap B_B)$. Similar statement can be made for the projectors to these spaces.

TABLE III. The Hilbert spaces in the Boolean algebra B_B within $\mathcal{L}[H(4)]$, and the corresponding projectors. The $\mathcal{I} = H(4)$ and \mathcal{O} also belong to B_B .

H_{1B}	$\mathfrak{P}(H_{1B}) = \Pi(x, 1) \otimes \Pi(a, b; 1)$
H_{1B}^\perp	$\mathfrak{P}(H_{1B}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \Pi(a, b; 1)$
H_{2B}	$\mathfrak{P}(H_{2B}) = \Pi(x, 1) \otimes \Pi(a, b; 0)$
H_{2B}^\perp	$\mathfrak{P}(H_{2B}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \Pi(a, b; 0)$
H_{3B}	$\mathfrak{P}(H_{3B}) = \Pi(x, 0) \otimes \Pi(a, b; 1)$
H_{3B}^\perp	$\mathfrak{P}(H_{3B}^\perp) = \mathbf{1}_4 - \Pi(x, 0) \otimes \Pi(a, b; 1)$
H_{4B}	$\mathfrak{P}(H_{4B}) = \Pi(x, 0) \otimes \Pi(a, b; 0)$
H_{4B}^\perp	$\mathfrak{P}(H_{4B}^\perp) = \mathbf{1}_4 - \Pi(x, 0) \otimes \Pi(a, b; 0)$
H_{5B}	$\mathfrak{P}(H_{5B}) = \Pi(x, 1) \otimes \mathbf{1}_2$
H_{5B}^\perp	$\mathfrak{P}(H_{5B}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \mathbf{1}_2$
H_{6B}	$\mathfrak{P}(H_{6B}) = \mathbf{1}_2 \otimes \Pi(a, b; 1)$
H_{6B}^\perp	$\mathfrak{P}(H_{6B}^\perp) = \mathbf{1}_4 - \mathbf{1}_2 \otimes \Pi(a, b; 1)$
H_{7B}	$\mathfrak{P}(H_{7B}) = \Pi(x, 1) \otimes \Pi(a, b; 1) + \Pi(x, 0) \otimes \Pi(a, b; 0)$
H_{7B}^\perp	$\mathfrak{P}(H_{7B}^\perp) = \mathbf{1}_4 - \Pi(x, 1) \otimes \Pi(a, b; 1) - \Pi(x, 0) \otimes \Pi(a, b; 0)$

For later use we also consider the transformations

$$U_C = U(a, b) \otimes \mathbf{1}_2; \quad U_D = U(a, b) \otimes U(a, b) \quad (55)$$

and we get the Boolean algebra B_C, B_D correspondingly. We use notation analogous to the above for the Hilbert spaces in these Boolean algebras.

B. CHSH inequalities for a system of two spin 1/2 particles

A logical derivation of Bell-like inequalities for the case of Boolean variables has been presented in Ref. 27 and we have generalized it for Heyting algebras in Ref. 65. We present briefly the derivation for Boolean variables in order to emphasize the crucial role of Boole's inequality, which is valid for Kolmogorov probabilities, but might not be valid for Dempster-Shafer probabilities.

Proposition 6.1. Let H_1, \dots, H_n be elements of $\mathcal{L}[H(d)]$ such that $H_1 \wedge \dots \wedge H_n = \mathcal{O}$. If $p(H_i|\rho)$ have the properties of Kolmogorov probabilities, then

$$\sum_{i=1}^n p(H_i|\rho) \leq n - 1. \quad (56)$$

Proof. We start with the relation

$$H_1^\perp \vee \dots \vee H_n^\perp = (H_1 \wedge \dots \wedge H_n)^\perp = \mathcal{I}. \quad (57)$$

Therefore,

$$p(H_1^\perp \vee \dots \vee H_n^\perp|\rho) = 1. \quad (58)$$

For Kolmogorov probabilities, we can use Boole's inequality (see Table I) to get

$$p(H_1^\perp|\rho) + \dots + p(H_n^\perp|\rho) \geq 1. \quad (59)$$

We then use the relation

$$p(H_i^\perp|\rho) = 1 - p(H_i|\rho), \quad (60)$$

and we get Eq. (56). \square

Remark 6.2. The step from Eq. (58) to Eq. (59) might not be valid for Dempster-Shafer probabilities, and the inequality in Eq. (56) might be violated. Below we give an example of this.

We consider a system of two spin 1/2 particles, described with the Hilbert space $H(4) = H(2) \otimes H(2)$. The system is in the state

$$|s\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right] = \frac{1}{\sqrt{2}} (1, 0, 0, 1)^T. \quad (61)$$

We consider the measurements described with the operators

$$\begin{aligned} A &= S_x \otimes S_x = \frac{1}{4} [\Pi(x, 1) \otimes \Pi(x, 1) + \Pi(x, 0) \otimes \Pi(x, 0)] \\ &\quad - \frac{1}{4} [\Pi(x, 1) \otimes \Pi(x, 0) + \Pi(x, 0) \otimes \Pi(x, 1)] = \frac{1}{4} [\mathfrak{P}(H_{1A}) + \mathfrak{P}(H_{4A})] \\ &\quad - \frac{1}{4} [\mathfrak{P}(H_{2A}) + \mathfrak{P}(H_{3A})], \end{aligned} \quad (62)$$

TABLE IV. Probabilities for the outcome of the measurements A, B, C, D , on a system of two spin $1/2$ particles, in the state $|s\rangle$ of Eq. (61) The values of κ, λ are given in Eq. (68).

	$p(i; 1, 1) = p(H_{1i} s)$	$p(i; 0, 1) = p(H_{3i} s)$	$p(i; 1, 0) = p(H_{2i} s)$	$p(i; 0, 0) = p(H_{4i} s)$
$i = A$	0.5	0	0	0.5
$i = B$	κ	$0.5 - \kappa$	$0.5 - \kappa$	κ
$i = C$	κ	$0.5 - \kappa$	$0.5 - \kappa$	κ
$i = D$	λ	$0.5 - \lambda$	$0.5 - \lambda$	λ

$$\begin{aligned}
B &= S_x \otimes S_{a,b} = \frac{1}{4} [\Pi(x, 1) \otimes \Pi(a, b; 1) + \Pi(x, 0) \otimes \Pi(a, b; 0)] \\
&- \frac{1}{4} [\Pi(x, 1) \otimes \Pi(a, b; 0) + \Pi(x, 0) \otimes \Pi(a, b; 1)] = \frac{1}{4} [\Re(H_{1B}) + \Re(H_{4B})] \\
&- \frac{1}{4} [\Re(H_{2B}) + \Re(H_{3B})], \tag{63}
\end{aligned}$$

$$\begin{aligned}
C &= S_{a,b} \otimes S_x = \frac{1}{4} [\Pi(a, b; 1) \otimes \Pi(x, 1) + \Pi(a, b; 0) \otimes \Pi(x, 0)] \\
&- \frac{1}{4} [\Pi(a, b; 1) \otimes \Pi(x, 0) + \Pi(a, b; 0) \otimes \Pi(x, 1)] = \frac{1}{4} [\Re(H_{1C}) + \Re(H_{4C})] \\
&- \frac{1}{4} [\Re(H_{2C}) + \Re(H_{3C})], \tag{64}
\end{aligned}$$

$$\begin{aligned}
D &= S_{a,b} \otimes S_{a,b} = \frac{1}{4} [\Pi(a, b; 1) \otimes \Pi(a, b; 1) + \Pi(a, b; 0) \otimes \Pi(a, b; 0)] \\
&- \frac{1}{4} [\Pi(a, b; 1) \otimes \Pi(a, b; 0) + \Pi(a, b; 0) \otimes \Pi(a, b; 1)] \\
&= \frac{1}{4} [\Re(H_{1D}) + \Re(H_{4D})] - \frac{1}{4} [\Re(H_{2D}) + \Re(H_{3D})]. \tag{65}
\end{aligned}$$

Each of these measurements gives one of the outcomes $(1, 1), (0, 1), (1, 0), (0, 0)$. We use the notation $p(A; 1, 1)$ for the probability that measurement A will give the outcome $(1, 1)$. From Table I it is seen that this is also the probability $p(H_{1A}||s)$:

$$p(A; 1, 1) = \langle s | \Pi(x, 1) \otimes \Pi(x, 1) | s \rangle = \langle s | \Re(H_{1A}) | s \rangle = p(H_{1A}||s). \tag{66}$$

More generally

$$\begin{aligned}
p(i; 1, 1) &= p(H_{1i}||s); \quad p(i; 0, 1) = p(H_{3i}||s); \quad p(i; 1, 0) = p(H_{2i}||s); \quad p(i; 0, 0) \\
&= p(H_{4i}||s), \quad i = A, B, C, D. \tag{67}
\end{aligned}$$

The values of the probabilities are shown in Table IV, where

$$\kappa = \frac{1}{2}(a_R^2 + b_I^2); \quad \lambda = \frac{1}{8} \{ |a + b|^4 + |a - b|^4 + (a^2 - b^2)^2 + [(a^*)^2 - (b^*)^2]^2 \}. \tag{68}$$

Here the indices R, I indicate the real and imaginary part, correspondingly.

We next prove that

$$[\Re(H_{1A}) + \Re(H_{4A})][\Re(H_{1B}) + \Re(H_{4B})][\Re(H_{1C}) + \Re(H_{4C})][\Re(H_{2D}) + \Re(H_{3D})] = 0. \tag{69}$$

Each of the 16 terms in this product is easily seen to be equal to 0. For example,

$$\begin{aligned} \mathfrak{P}(H_{1A})\mathfrak{P}(H_{1B})\mathfrak{P}(H_{1C})\mathfrak{P}(H_{2D}) &= [\Pi(x, 1) \otimes \Pi(x, 1)][\Pi(x, 1) \otimes \Pi(a, b; 1)] \\ &\times [\Pi(a, b; 1) \otimes \Pi(x, 1)][\Pi(a, b; 1) \otimes \Pi(a, b; 0)] = 0. \end{aligned} \quad (70)$$

We then point out that $\mathfrak{P}(H_{1A})\mathfrak{P}(H_{4A}) = 0$ and therefore $\mathfrak{P}(H_{1A}) + \mathfrak{P}(H_{4A}) = \mathfrak{P}(H_{1A} \vee H_{4A})$. Similar comment can be made for the other factors in Eq. (69), and therefore it can be written as

$$\mathfrak{P}(H_{1A} \vee H_{4A})\mathfrak{P}(H_{1B} \vee H_{4B})\mathfrak{P}(H_{1C} \vee H_{4C})\mathfrak{P}(H_{2D} \vee H_{3D}) = 0. \quad (71)$$

From this follows that

$$[H_{1A} \vee H_{4A}] \wedge [H_{1B} \vee H_{4B}] \wedge [H_{1C} \vee H_{4C}] \wedge [H_{2D} \vee H_{3D}] = \mathcal{O}. \quad (72)$$

Then Proposition VI.1 gives the following CHSH inequality:

$$p(H_{1A} \vee H_{4A} | s) + p(H_{1B} \vee H_{4B} | s) + p(H_{1C} \vee H_{4C} | s) + p(H_{2D} \vee H_{3D} | s) \leq 3. \quad (73)$$

The fact that $\mathfrak{P}(H_{1A}) + \mathfrak{P}(H_{4A}) = \mathfrak{P}(H_{1A} \vee H_{4A})$ implies that $p(H_{1A} \vee H_{4A} | s) = p(H_{1A} | s) + p(H_{4A} | s)$, and similarly for the other terms in Eq. (73), which can be written as

$$\begin{aligned} &p(H_{1A} | s) + p(H_{4A} | s) + p(H_{1B} | s) + p(H_{4B} | s) + p(H_{1C} | s) \\ &+ p(H_{4C} | s) + p(H_{2D} | s) + p(H_{3D} | s) \leq 3. \end{aligned} \quad (74)$$

We insert the values from Table IV and we get

$$1 + 4\kappa + (1 - 2\lambda) \leq 3. \quad (75)$$

As an example, we consider the case with $a = \exp(i\theta)$ and $b = 0$. Then $\kappa = \frac{1}{2}(\cos\theta)^2$ and $\lambda = \frac{1}{8}(2 + 2\cos 4\theta)$. Therefore, the inequality of Eq. (75) becomes

$$2(\cos\theta)^2 - \frac{1}{2}(1 + \cos 4\theta) \leq 1. \quad (76)$$

For $\theta = \pi/8$ this inequality is violated.

Remark 6.3. We have emphasized that the derivation of the inequality in Eq. (56), which is used in Eq. (73), relies on the properties of Kolmogorov probabilities, and in particular on Boole's inequality. We now show explicitly (i) that the probabilities in Eq. (73) are not Kolmogorov probabilities, and (ii) that Boole's inequality is violated.

(i) We consider the $p(H_{1A} \vee H_{4A} | s)$ and $p(H_{1B} \vee H_{4B} | s)$. We use the relations

$$H_{1A} \vee H_{4A} = H_{7A}; \quad H_{1B} \vee H_{4B} = H_{7B} \quad (77)$$

and we find that the commutator of the corresponding projectors is

$$\begin{aligned} &[\mathfrak{P}(H_{1A} \vee H_{4A}), \mathfrak{P}(H_{1B} \vee H_{4B})] = [\mathfrak{P}(H_{7A}), \mathfrak{P}(H_{7B})] \\ &= \Pi(x, 1) \otimes [\Pi(x, 1), \Pi(a, b; 1)] + \Pi(x, 0) \otimes [\Pi(x, 0), \Pi(a, b; 0)]. \end{aligned} \quad (78)$$

According to Eq. (20) in this case $\mathfrak{D}(H_{1A} \vee H_{4A}, H_{1B} \vee H_{4B}) \neq 0$ and the corresponding probabilities are not Kolmogorov probabilities.

(ii) From Eq. (72) it follows that

$$[H_{1A} \vee H_{4A}]^\perp \vee [H_{1B} \vee H_{4B}]^\perp \vee [H_{1C} \vee H_{4C}]^\perp \vee [H_{2D} \vee H_{3D}]^\perp = \mathcal{I}. \quad (79)$$

Therefore, the corresponding probability is

$$p([H_{1A} \vee H_{4A}]^\perp \vee [H_{1B} \vee H_{4B}]^\perp \vee [H_{1C} \vee H_{4C}]^\perp \vee [H_{2D} \vee H_{3D}]^\perp) = 1. \quad (80)$$

On the other hand, from Table IV, it is seen that

$$\begin{aligned} p([H_{1A} \vee H_{4A}]^\perp) &= 1 - p[H_{1A} \vee H_{4A}] = 0, \\ p([H_{1B} \vee H_{4B}]^\perp) &= 1 - p[H_{1B} \vee H_{4B}] = 1 - 2\kappa, \\ p([H_{1C} \vee H_{4C}]^\perp) &= 1 - p[H_{1C} \vee H_{4C}] = 1 - 2\kappa, \\ p([H_{2D} \vee H_{3D}]^\perp) &= 1 - p[H_{2D} \vee H_{3D}] = 1 - (1 - 2\lambda), \end{aligned} \quad (81)$$

and therefore,

$$\begin{aligned} &p([H_{1A} \vee H_{4A}]^\perp) + p([H_{1B} \vee H_{4B}]^\perp) + p([H_{1C} \vee H_{4C}]^\perp) + p([H_{2D} \vee H_{3D}]^\perp) \\ &= 2 - 4\kappa + 2\lambda. \end{aligned} \quad (82)$$

We have seen earlier an example where $2 - 4\kappa + 2\lambda$ is smaller than 1 and therefore the sum of probabilities in Eq. (82) is smaller than the probability in Eq. (80), which shows explicitly that Boole's inequality is violated.

VII. DISCUSSION

An important property of Kolmogorov probabilities is that $\delta(A, B) = 0$ (Eq. (1)). We have introduced the operator $\mathfrak{D}(H_1, H_2)$ of Eq. (18), which is analogous to $\delta(A, B)$ and we have shown that it is related to the commutator $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$ as in Eq. (20). This shows a direct link between commutativity and the Kolmogorov property $\delta(A, B) = 0$. If H_1, H_2 belong to the same Boolean subalgebra of the orthomodular lattice $\mathcal{L}[H(d)]$, then $\mathfrak{D}(H_1, H_2) = [\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = 0$. In this case the corresponding probabilities $p(H_1|\rho)$, $p(H_2|\rho)$, obey Eq. (1), and they are Kolmogorov probabilities. But in general, the $\mathfrak{D}(H_1, H_2)$ and $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$ are non-zero, and then the $p(H_1|\rho)$, $p(H_2|\rho)$ do not obey Eq. (1), and they are not Kolmogorov probabilities.

The Dempster-Shafer theory is designed for “real world” data which have multivaluedness and contradictions. Dempster-Shafer probabilities have the properties shown in Table I, which fit very well with the requirements of quantum probabilities. In particular they violate Eq. (1). The difference between upper and lower probabilities is Dempster's “don't know” cases, which here is related to ambiguity in the ordering of quantum mechanical operators. In the semiclassical limit, the commutators go to zero, and the $\mathfrak{D}(H_1, H_2)$ goes to zero, and lower probabilities become equal to upper probabilities, i.e., they become Kolmogorov probabilities.

As an application we have considered CHSH inequalities and we have stressed that their proof relies on the properties of Kolmogorov probabilities, and it is not valid for Dempster-Shafer probabilities. Their violation in experiments supports the use of the Dempster-Shafer theory for quantum probabilities.

In this work we have considered systems with finite Hilbert space, and then $\mathcal{L}[H(d)]$ is a modular lattice. This was used, for example, in the first part of Proposition III.3. However, our main theme of using Dempster-Shafer probabilities as quantum probabilities could also be applied to systems with infinite-dimensional Hilbert spaces. In this case the lattice is orthomodular but not modular, not every subspace is closed, etc. So there are extra mathematical questions, which need to be considered. Also, the Dempster-Shafer theory for quantum probabilities in a different (distributive) lattice of subsystems, will be discussed elsewhere (Ref. 66).

¹G. Birkhoff and J. von Neumann, *Ann. Math.* **37**, 823 (1936).

²C. Piron, *Foundations of Quantum Physics* (Benjamin, New York, 1976).

³G. W. Mackey, *Mathematical Foundations of Quantum Mechanics* (Benjamin, New York, 1963).

⁴J. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, Reading, 1968).

⁵V. S. Varadarajan, *Geometry of Quantum Theory* (Springer, Berlin, 1985).

⁶E. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics* (Addison-Wesley, Reading, 1981).

⁷P. Walley, *Statistical Reasoning with Imprecise Probabilities* (Chapman and Hall, London, 1991).

⁸C. Camerer and M. Weber, *J. Risk Uncertainty* **5**, 325 (1992).

⁹L. Zadeh, *Fuzzy Sets Syst.* **1**, 3 (1978).

¹⁰D. Dubois and H. Prade, *Ann. Math. Artif. Intel.* **32**, 35 (2001).

- ¹¹ A. P. Dempster, *Ann. Math. Stat.* **38**, 325 (1967).
- ¹² G. Shafer, *A Mathematical Theory of Evidence* (Princeton University Press, Princeton, 1976).
- ¹³ J. Y. Halpern and R. Fagin, *Artif. Intell.* **54**, 275 (1992).
- ¹⁴ *Classic Works of the Dempster-Shafer Theory of Belief Functions*, edited by R. Yager and L. Liu (Springer, Berlin, 2008).
- ¹⁵ J. S. Bell, *Physics* **1**, 195 (1964).
- ¹⁶ S. Kochen and E. P. Specker, *J. Math. Mech.* **17**, 59 (1967).
- ¹⁷ J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys. Rev. Lett.* **23**, 880 (1969).
- ¹⁸ N. D. Mermin, *Rev. Mod. Phys.* **65**, 803 (1993).
- ¹⁹ L. Hardy, *Phys. Rev. Lett.* **71**, 1665 (1993).
- ²⁰ I. Pitowsky, *Br. J. Philos. Sci.* **45**, 95 (1994).
- ²¹ A. A. Klyachko, M. A. Can, S. Binicioglu, and A. S. Shumovsky, *Phys. Rev. Lett.* **101**, 020403 (2008).
- ²² K. Svozil, in *Handbook of Quantum Logic and Quantum Structures*, edited by K. Engesser, D. M. Gabbay, and D. Lehmann (Elsevier, Amsterdam, 2009).
- ²³ H. Bartosik *et al.*, *Phys. Rev. Lett.* **103**, 040403 (2009).
- ²⁴ G. Kirchmair *et al.*, *Nature (London)* **460**, 494 (2009).
- ²⁵ Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, *Phys. Rep.* **506**, 1 (2011).
- ²⁶ S. Abramsky and A. Brandenburger, *New J. Phys.* **13**, 113036 (2011).
- ²⁷ S. Abramsky and L. Hardy, *Phys. Rev. A* **85**, 062114 (2012).
- ²⁸ A. B. Sainz, T. Fritz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acin, *Phys. Rev. A* **89**, 032117 (2014).
- ²⁹ A. Cabello, S. Severini, and A. Winter, *Phys. Rev. Lett.* **112**, 040401 (2014).
- ³⁰ P. Lahti, *Int. J. Theor. Phys.* **19**, 905 (1980).
- ³¹ P. Lahti and J. Maczynski, *J. Math. Phys.* **33**, 4133 (1992).
- ³² A. Caticha, *Phys. Rev. A* **57**, 1572 (1998).
- ³³ L. Hardy, "Quantum theory from five reasonable axioms," e-print [arXiv:quant-ph/0101012](https://arxiv.org/abs/quant-ph/0101012).
- ³⁴ R. Schack, T. A. Brun, and C. M. Caves, *Phys. Rev. A* **64**, 014305 (2001).
- ³⁵ J. Barrett, *Phys. Rev. A* **75**, 032304 (2007).
- ³⁶ G. Chiribella, G. M. D'Ariano, and P. Perinotti, *Phys. Rev. A* **81**, 062348 (2010).
- ³⁷ G. Ludwig, *Commun. Math. Phys.* **4**, 331 (1967).
- ³⁸ E. B. Davies and J. T. Lewis, *Commun. Math. Phys.* **17**, 239 (1970).
- ³⁹ S. Gudder, *Commun. Math. Phys.* **29**, 249 (1973).
- ⁴⁰ B. Mielnik, *Commun. Math. Phys.* **37**, 221 (1974).
- ⁴¹ P. Busch, M. Grabowski, and P. J. Lahti, *Operational Quantum Physics* (Springer, Berlin, 1995).
- ⁴² D. J. Foulis and C. H. Randall, *J. Math. Phys.* **13**, 1667 (1972).
- ⁴³ C. H. Randall and D. J. Foulis, *J. Math. Phys.* **14**, 1472 (1973).
- ⁴⁴ S. Twareque Ali and G. G. Emch, *J. Math. Phys.* **15**, 176 (1974).
- ⁴⁵ S. Twareque Ali and H. D. Doebner, *J. Math. Phys.* **17**, 1105 (1976).
- ⁴⁶ S. Twareque Ali and E. Prugovecki, *J. Math. Phys.* **18**, 219 (1977).
- ⁴⁷ S. Abramsky and B. Coecke, in *Proceedings of 19th Annual IEEE Symposium on Logic in Computer Science* (IEEE, 2004), p. 415.
- ⁴⁸ B. Coecke and D. Ross, *New J. Phys.* **13**, 043016 (2011).
- ⁴⁹ S. Zhang and A. Vourdas, *J. Phys. A* **37**, 8349 (2004).
- ⁵⁰ N. Cotfas, J.-P. Gazeau, and A. Vourdas, *J. Phys. A* **44**, 175303 (2011).
- ⁵¹ G. Birkhoff *Lattice Theory* (American Mathematical Society, Rhode Island, 1995).
- ⁵² G. Szasz, *Introduction to Lattice Theory* (Academic, London, 1963).
- ⁵³ G. A. Grätzer, *General Lattice Theory* (Springer, Berlin, 2003).
- ⁵⁴ G. Kalmbach, *Orthomodular Lattices* (Academic, London, 1983).
- ⁵⁵ P. Ptak and S. Pulmannova, *Orthomodular Lattices as Quantum Logics* (Kluwer, Dordrecht, 1991).
- ⁵⁶ R. J. Greechie, *J. Comb. Theory, Ser. A* **10**, 119 (1971).
- ⁵⁷ S. S. Holland, Jr., *Trans. Am. Math. Soc.* **108**, 66 (1963).
- ⁵⁸ A. Gleason, *J. Math. Mech.* **6**, 885 (1957).
- ⁵⁹ G. M. Hardegree, *Found. Phys.* **7**, 495 (1977).
- ⁶⁰ L. A. Zadeh, *Artif. Intell. Mag.* **7**, 85 (1986).
- ⁶¹ G. S. Agarwal and E. Wolf, *Phys. Rev. D* **2**, 2161 (1970).
- ⁶² G. S. Agarwal and E. Wolf, *Phys. Rev. D* **2**, 2187 (1970).
- ⁶³ G. Choquet, *Ann. Inst. Fourier* **5**, 131 (1954).
- ⁶⁴ M. Grabisch and C. Labreuche, *Ann. Oper. Res.* **175**, 247 (2010).
- ⁶⁵ A. Vourdas, *J. Math. Phys.* **54**, 082105 (2013).
- ⁶⁶ A. Vourdas, *J. Phys. A*, to appear.