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ON INNER PRODUCTS IN LINEAR, METRIC SPACES

BY P. JORDAN AND J. V. NEUMANN

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1. In his foregoing paper[1] Mr. M. Fréchet discussed the following question: When is a linear, metric space L isometric with a generalized Hilbert space?[2] In other words: When can one define in it a bilinear symmetric inner product, from which its given metric can be derived in the customary way? Mr. Fréchet discovered a necessary and sufficient algebraic condition, from which he derived this more abstract criterium: This is the case if and only if every $\leqq$ 3-dimensional linear subspace L' of L is isometric with a Euclidean space.

On the pages which follow we will derive another necessary and sufficient algebraic condition, which implies that Mr. Fréchet's abstract criterium can be weakened as follows: The answer to the above question is affirmative if and only if every $\leqq$ 2-dimensional linear subspace L' of L is isometric with a Euclidean space.

The criterium which we will derive has some further interest, because it shows that the linear spaces with a bilinear symmetric inner product are in a certain sense limiting cases of the general linear metric spaces.

We will only consider complex linear spaces. Real linear spaces could be discussed along the same lines, even with some simplifications. Mr. Fréchet showed, loc. cit., how the two types of linearity are connected.

2. We repeat the customary definitions of linearity, metricity, and of the bilinear symmetric inner products. As we consider complex linearity, the symmetry of inner products will be interpreted as Hermitian symmetry.

The definitions in question are:

DEFINITION 1.[3] A space L is (complex) linear and metric, if for all $f, g \in L$ and

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[1] Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert, Ann. of Math. (2) 36 (1935) pp. 705–718.

[2] Hilbert space is uniquely characterized (up to an isomorphism) by five postulates A–E, which have been formulated by one of us, Math. Ann. vol. 102 (1929), pp. 63–66. Of these postulates A, B are the really essential ones: The omission of C would only add the (finite dimensional) Euclidean spaces; the omission of D could be compensated by the standard manipulation of "completion;" in the absence of E essentially new hyper-Hilbert spaces arise, but they are nevertheless similar to Hilbert space under most aspects.

The hyper-Hilbert spaces (without E) have been first discussed by H. Löwig, Acta Szeged, vol. 7 (1934), pp. 1–33. The "completion" of spaces without D plays a fundamental rôle in the work of K. Friedrichs, Math. Ann., vol. 109 (1933), pp. 472–476. A connected discussion of the respective rôles of the conditions A-E is to be found in a series of mimeographed lectures on operator theory, given by one of us in Princeton in the year 1933–34.

[3] Cf. S. Banach, Fund. Math., Vol. 3, pp. 134–136 (1922); and H. Hahn, Monatshefte für Math. und Phys., Vol. 92, pp. 1–4 (1922).

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all complex numbers α the quantities $\alpha \cdot f$, $f + g \in L$ and the real number $\| f \|$ are defined, with the following properties:

1. $\alpha \cdot f$, $f + g$ obey the rules of the vector-calculus:

   11. $1 \cdot f = f$, $0 \cdot f = 0$ (independent of f),

   12. $(\alpha + \beta) \cdot f = \alpha \cdot f + \beta \cdot f$, $\alpha \cdot (f + g) = \alpha \cdot f + \alpha \cdot g$,

   13. $\alpha\beta \cdot f = \alpha \cdot (\beta \cdot f)$,

   14. $f + g = g + f$, $(f + g) + h = f + (g + h)$.

   (We use the customary notations $(-1) \cdot f = -f$, $f + (-g) = f - g$.)

2. $\| f \|$ obeys the rules for an absolute value:

   21. $\| f \| > 0$ if $f \neq 0$,

   22. $\| f + g \| \leqq \| f \| + \| g \|$,

   23. $\| \alpha \cdot f \| = | \alpha | \cdot \| f \|$.

DEFINITION 2. A space L is generalized (complex) linear and metric, if it fulfills all conditions of the preceding definition, except for 23, which is replaced by the weaker condition:

23′. $\| \alpha \cdot f \| \to 0$ if $\alpha \to 0$ and $\| if \| = \| f \|$.

DEFINITION 3.[4] A space L is linear with a (bilinear and symmetric) inner product, if for all $f, g \in L$ and all complex numbers α the quantities $\alpha \cdot f$, $f + g \in L$ and the complex number (f, g) are defined, with the following properties:

1. As in Definition 1.
2. (f, g) is linear in f, conjugate linear in g, Hermitian-symmetric, and definite:

   21. $(\alpha \cdot f, g) = \alpha \cdot (f, g)$,

   22. $(f' + f'', g) = (f', g) + (f'', g)$,

   23. $(f, g) = \overline{(g, f)}$.

   23 transforms 21, 22 into 21* $(f, \alpha \cdot g) = \bar{\alpha} \cdot (f, g)$ and

   22* $(f, g' + g'') = (f, g') + (f, g'')$.

   24. (f, f) (which is real by 23) > 0 if $f \neq 0$.

Well known considerations show that $\| f \| = \sqrt{(f, f)}$ fulfills the conditions 2 of Definition 1.[5] In this sense every linear space L with an inner product is at the same time a linear space with a metric. We call $\| f \| = \sqrt{(f, f)}$ the metric derived from the inner product (f, g).

[4] Cf. footnote 2, we are using the conditions A, B.

[5] Cf. pp. 64–65 in the Math. Ann. paper quoted in footnote 2.

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3. We will now determine all generalized linear, metric spaces L, which can be considered at the same time as linear spaces with an inner product. This is our criterium:

THEOREM I. *Let L be a generalized linear, metric space, with the operations* $\alpha \cdot f$, $f + g$, $\| f \|$. *In order that it be possible to define an operation* (f, g) *in L, so that L with the original* $\alpha \cdot f$, $f + g$ *and this* (f, g) *is a linear space with an inner product, and that the original* $\| f \|$ *is the metric derived from this* (f, g), *the following condition is necessary and sufficient:*

$$(*) \qquad \| f + g \|^2 + \| f - g \|^2 = 2 \, (\| f \|^2 + \| g \|^2) \text{ for all } f, g \in L.$$ [6]

If this is the case, then (f, g) *is uniquely determined by the above requirements.*

PROOF: Assume first that an (f, g) of the desired sort exists. Relations 21, 21* (for $\alpha = -1$) and 22, 22* from Definition 3 give immediately:

$$(f + g, f + g) + (f - g, f - g) = 2 \, ((f, f) + (g, g)),$$

$$(f + g, f + g) - (f - g, f - g) = 2 \, ((f, g) + (g, f)).$$

The first equation coincides with (*), thus (*) is necessary. The second equation gives, considering 23, $\mathfrak{R}(f, g) = \frac{1}{4}(\| f + g \|^2 - \| f - g \|^2)$. 21 (for $\alpha = i$) gives $(i \cdot f, g) = i \cdot (f, g)$, $\mathfrak{R}(i \cdot f, g) = - \mathfrak{I}(f, g)$. Thus $\| f \|$ (for all $f \in L$) determines $\mathfrak{R}(f, g)$ and $\mathfrak{I}(f, g)$ uniquely, so that (f, g) itself is unique. So we need only to prove the sufficiency of (*).

Assume therefore that (*) is fulfilled. We must define (f, g), but we know that the only possibility is

$$(\#) \qquad \begin{cases} \mathfrak{R}(f, g) = \frac{1}{4}(\| f + g \|^2 - \| f - g \|^2), \\ (f, g) = \mathfrak{R}(f, g) - i \cdot \mathfrak{R}(i \cdot f, g). \end{cases}$$

So our task is to verify 21–23 (Definition 3) and $\| f \| = \sqrt{(f, f)}$.

Replace f, g in (*) by $f' \pm g$, f'', and subtract. Then

$$\| f' + f'' + g \|^2 - \| f' + f'' - g \|^2 + \| f' - f'' + g \|^2 - \| f' - f'' - g \|^2 = 2 \, (\| f' + g \|^2 - \| f' - g \|^2)$$

obtains, that is (considering (#)):

$$(\S) \qquad \mathfrak{R}(f' + f'', g) + \mathfrak{R}(f' - f'', g) = 2 \, \mathfrak{R}(f', g).$$

(*) with $f = 0$ proves $\| -g \|^2 = \| g \|^2$, and so (#) with $f = 0$ gives $\mathfrak{R}(0, g) = 0$. Therefore (§) with $f' = f''$ is $\mathfrak{R}(2f', g) = 2\mathfrak{R}(f', g)$. Thus (§) itself becomes

$$\mathfrak{R}(f' + f'', g) + \mathfrak{R}(f' - f'', g) = \mathfrak{R}(2f', g),$$

[6] This equation occurred in a paper of E. Wigner and the authors, Annals of Math., vol. 35 (1934), p. 32. Its importance in generalized Hilbert space has been pointed out by F. Riess, Acta Szeged, vol. 7 (1934), p. 36, cf. equation (6), loc. cit.

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or if we replace f', f'' by $\frac{1}{2}(f' + f'')$, $\frac{1}{2}(f' - f'')$:

$$\mathfrak{R}\,(f', g) + \mathfrak{R}\,(f'', g) = \mathfrak{R}\,(f' + f'', g).$$

Now (※) gives immediately

$$(f', g) + (f'', g) = (f' + f'', g)$$

proving 22 (Definition 3).

22 (Definition 1) with $f = f''$, $g = f' - f''$ gives $\|f'\| - \|f''\| \leqq \|f' - f''\|$. Interchanging f', f'' now shows $\|f''\| - \|f'\| \leqq \|f'' - f'\| = \|f' - f''\|$ (remember $\|-g\| = \|g\|$), thus $|\,\|f'\| - \|f''\|\,| \leqq \|f' - f''\|$. Therefore $|\,\|\alpha\cdot f \pm g\| - \|\beta\cdot f \pm g\|\,| \leqq \|(\alpha - \beta)\cdot f\|$. So $\alpha \to \beta$ implies by 23′ (Definition 2) $\|\alpha\cdot f \pm g\| \to \|\beta\cdot f \pm g\|$ that is, $\|\alpha\cdot f \pm g\|$ is continuous in α. Now by (※) $\mathfrak{R}\,(\alpha\cdot f, g)$ and $(\alpha\cdot f, g)$ are also continuous in α.

Consider now the set S of all α's for which 21 (Definition 3) holds. $1 \epsilon S$ is obvious; by 22 (Definition 3) $\alpha, \beta \epsilon S$ imply $\alpha \pm \beta \epsilon S$. So all $\alpha = 0, \pm 1, \pm 2, \cdots$ are in S. Clearly $\alpha, \beta \epsilon S, \beta \neq 0$, imply $\alpha/\beta \epsilon S$, so all rational α's are in S. The above proved continuity of $(\alpha\cdot f, g)$ in α implies that S is closed, so all real α's are in S. Finally (※) gives $i \epsilon S$ (use $(-f, g) = -(f, g)$, as $-1 \epsilon S$!), therefore if α_1, α_2 are real $\alpha_1 - i\alpha_2 = \alpha_1 + \frac{\alpha_2}{i} \epsilon S$. Thus all complex α's are in S, that is 21 holds always.

As $\|if\| = \|f\|$, the first equation in (※) gives $\mathfrak{R}\,(if, ig) = \mathfrak{R}\,(f, g)$. It gives immediately $\mathfrak{R}\,(f, g) = \mathfrak{R}\,(g, f)$ too, and these combined give

$$\mathfrak{R}\,(if, g) = \mathfrak{R}\,(i\cdot if, ig) = \mathfrak{R}\,(-f, ig) = -\mathfrak{R}\,(f, ig) = -\mathfrak{R}\,(ig, f).$$

So $(f, g) = \overline{(g, f)}$, proving 23 (Definition 3). 21 and 23 (all in Definition 3) imply 21*, and thus $(\alpha f, \alpha f) = |\alpha|^2 (f, f)$.

Now (※) gives $\mathfrak{R}\,(f, f) = \frac{1}{4}(\|2\cdot f\|^2 - \|0\cdot f\|^2) = \|f\|^2$, $\mathfrak{R}\,(i\cdot f, f) = \frac{1}{4}(\|(1 + i)\cdot f\|^2 - \|(1 - i)\cdot f\|^2) = 0$ so $(f, f) = \|f\|^2$. This proves 24 (Definition 3), and $\|f\| = \sqrt{(f, f)}$.

Thus the proof is completed.

4. The condition that every $\leqq$ 2-dimensional subspace L' of L be isometric to a Euclidean space, is obviously necessary for the existence of an inner product in the generalized linear, metric space L. It is sufficient, too, because if it is fulfilled, we can argue as follows: If $f_0, g_0 \epsilon L$ the space L' of all $\alpha\cdot f_0 + \beta\cdot g_0$ (α, β arbitrary complex numbers) is $\leqq$ 2 dimensional, thus (*) holds in L' (as in every Euclidean space). Therefore it holds in particular for $f = f_0$, $g = g_0$, and as f_0, g_0 are arbitrary, Theorem I proves the existence of an inner product.

5. The following theorem holds for linear, metric spaces:

THEOREM II. *Let L be a linear, metric space. Define*

$$C_{f,g} = \frac{1}{2}\cdot\frac{\|f + g\|^2 + \|f - g\|^2}{\|f\|^2 + \|g\|^2} \quad \text{for } f, g \epsilon L, \text{ not } f = g = 0$$

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and denote the l.u.b. of the $C_{f,g}$ by b, and their g.l.b. by a. Then we have

$$\tfrac{1}{2} \leqq a \leqq 1 \leqq b \leqq 2, \; a = \frac{1}{b}.$$

The linear spaces with a (bilinear and symmetric) inner product represent the extreme case

$$a = b = 1.$$

PROOF: Clearly $0 \leqq a \leqq b$. 22 (Definition 1) gives:

$$C_{f,g} \leqq \frac{1}{2} \cdot \frac{2 \cdot (\| f \| + \| g \|)^2}{\| f \|^2 + \| g \|^2} \leqq \frac{1}{2} \cdot \frac{2 \cdot 2 \cdot (\| f \|^2 + \| g \|^2)}{\| f \|^2 + \| g \|^2} = 2,$$

so $b \leqq 2$. 23 (Definition 1) for $\alpha = 2$ gives:

$$C_{f+g,f-g} = \frac{1}{2} \cdot \frac{\| 2f \|^2 + \| 2g \|^2}{\| f + g \|^2 + \| f - g \|^2} = 2 \cdot \frac{\| f \|^2 + \| g \|^2}{\| f + g \|^2 + \| f - g \|^2} = \frac{1}{C_{f,g}},$$

so $a = \frac{1}{b}$. Thus $b \leqq 2$ implies $a \geqq \frac{1}{2}$, and $a \leqq b$ implies $a \leqq 1 \leqq b$, together: $\frac{1}{2} \leqq a \leqq 1 \leqq b \leqq 2$.

That linear spaces with an inner product are characterized by $a = b = 1$ is the statement of Theorem I.

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