Interval Probability and Measurement for Fuzzy Quantum Theories

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1 Introduction

Fuzzy quantum mechanics:

- http://cds.cern.ch/record/518511/files/0107054.pdf
- http://link.springer.com/chapter/10.1007%2F978-3-642-35644-5_18#page-1
- http://link.springer.com/chapter/10.1007%2F978-3-540-93802-6_20#page-1
- http://www.du.edu/nsm/departments/mathematics/media/documents/preprints/m0412.pdf
- http://www.space-lab.ru/files/pages/PIRT_VII-XII/pages/text/PIRT_X/Bobola.pdf
- http://www.vub.ac.be/CLEA/aerts/publications/1993LiptovskyJan.pdf

Pseudo-randomness:

- https://people.csail.mit.edu/silvio/Selected%20Scientific%20Papers/Pseudo%20Randomness/How_To_Generate_Cryptographically_Strong_Sequences_Of_Pseudo-Random_Bits.pdf: "the randomness of an event is relative to a specific model of computation with a specified amount of computing resources."
- Another version https://pdfs.semanticscholar.org/3e9c/5f6f48d9ef426655dc799e9b287d754e86c1.pdf

1.1 Plan

In the remainder of the paper, we consider variations of quantum probability spaces motivated by computation of numerical quantities in a world with limited resources:

- Instead of the Hilbert space $\mathcal{H}$ (constructed over the uncountable and uncomputable complex numbers $\mathbb{C}$), we will consider variants constructed over finite fields [1, 2].
- Instead of real-valued probability measures producing results in the uncountable and uncomputable interval $[0, 1]$, we will consider finite set-valued probability measures [3].

We will then ask if it is possible to construct variants of quantum probability spaces under these conditions. The main question is related to the definition of probability measures: is it possible to still define a probability measure as a function that depends on a single state? Specifically,

- given a state $|\psi\rangle$, is there a probability measure mapping events to probabilities that only depends on $|\psi\rangle$? In the conventional quantum probability space, the answer is yes by the Born rule [4, 5] and the map is given by: $P \mapsto \langle\psi| P\psi\rangle$.

- given a probability measure μ mapping each event P to a probability, is there a *unique* state ψ such that $\mu(P) = \langle\psi| P|\psi\rangle$? In the conventional case, the answer is yes by Gleason's theorem [6, 7, 8].

Andy says: Quantum meeting: the basics of classical and quantum standard and interval probabilities are becoming clearer (except need clearer exposition). The key step is the replacement of "sum" and "=" in the rule $\mu\left(\bigcirc\!\!\!\!\sum_{i=1}^{N} P_i\right) = \mathbb{R}\sum_{i=1}^{N} \mu(P_i)$ by various logical operations on sets instead of actually summing unit norm partitions of unity in $\mathbb{R}$. What remains is to determine how the Meyer/Mermin debate plays out for finite precision or uncertain measurements of events, and its implication for the validity of the Kochen-Specker theorem, and hence for the validity of Gleason's theorem. Do we have anything ANALOGOUS to a Gleason theorem for interval probability quantum mechanics? If so, what are the implications for Kochen-Specker and Bell analogs for interval probabilities, and what are the implications for the Meyer/Mermin debate? And if not, what are the consequences? Finally, given whatever remains of Gleason/Kochen-Specker for uncertain event measurements, what happens when we redefine "sum" yet again for $\mathbb{F}_{p^2}$ valued quantum theories (and degenerate states, and density matrices) to create a non-wrapping extension of interval probability calculus to Galois fields? (The objective is to put possible/impossible and its extensions on a solid footing for DQC, and also to set up a seamless transition back to a continuous CQC limit that is consistent, while incorporating such issues as the cost of measurement precision.)

Yu-Tsung says: A mathematical reason why Gleason's theorem should not be valid in the interval-valued probability quantum mechanics. Every classical probability measure has a one-to-one correspondence to its Radon-Nikodym derivative. The Radon-Nikodym theorem extends to Gleason's theorem in quantum, while the role of Radon-Nikodym derivative is replaced by the density matrix. If there is no easy Radon-Nikodym theorem for classical interval-valued probability measure, it is natural that there is no easy Gleason's theorem for quantum interval-valued probability measure...
If Gleason's theorem is not valid, a non-Born quantum probability measure might exist in reality... If there exists a non-Born quantum probability measure which cannot correspond to any state vectors, then what's the post-measurement quantum probability measure of it after measurement? If the updated post-measurement postulate is strange enough, then commuting observables may not interchange their measurement order... And commuting observables is compatible is the fundamental assumption for the Kochen-Specker theorem, the Bell theorem, and everything... If commuting observables becomes non-compatible, everything might need to be rewritten...

2 Classical Probability Spaces

A *probability space* specifies the necessary conditions for reasoning coherently about collections of uncertain events [9, 10, 11, 12]. We review the conventional presentation of probability spaces and then discuss the computational resources needed to estimate probabilities.

2.1 Classical Real-Valued Probability Spaces

The conventional definition of a probability space builds upon the field of real numbers. In more detail, a probability space consists of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ mapping events in $\mathcal{E}$ to the real interval $[0, 1]$. In this paper, we will only consider *finite* sets of events: we therefore restrict our attention to non-empty finite sets Ω as the sample space. The space of events $\mathcal{E}$ includes every possible subset of Ω: it is the powerset $2^{\Omega} = \{E | E \subseteq \Omega\}$.

Definition 1 (Probability Measure). Given the set of events $\mathcal{E}$, a *probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\emptyset) = 0$.

- $\mu(\Omega) = 1$.
- for any event E,
$$\mu(\Omega \backslash E) = 1 \ {}_{\mathbb{R}}{-}\ \mu(E) \ , \tag{1}$$
where $\Omega \backslash E$ is the complement event, and $1 \ {}_{\mathbb{R}}{-}\ \mu(E)$ explicitly specifies 1 and $\mu(E) \in \mathbb{R}$. Besides $\mathbb{R}$, we will prepose other symbols later to specify the type of operations, and they may be dropped when there is no ambiguity.
- for a collection $\{E_i\}_{i=1}^N$ of pairwise disjoint events, $\mu(\bigcup_{i=1}^N E_i) = {}_{\mathbb{R}}\sum_{i=1}^N \mu(E_i)$. Similar to the above condition, ${}_{\mathbb{R}}\sum_{i=1}^N \mu(E_i)$ explicitly specifies $\mu(E_i) \in \mathbb{R}$.

□

Notice that $\mu(\emptyset) = 0$ and equation (1) can be induced by other conditions. They are listed in here so that we can compare definition 1, lemma 1, definition 5, and definition 6 easily. We will compare these lemma and definitions when we try to formulate quantum interval-valued probability measures later.

Example 1 (Two-coins experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. There are 16 total events including for example the event $\{HH, HT\}$ that the first coin lands heads up, the event $\{HT, TH\}$ that the two coins land on opposite sides, and the event $\{HT, TH, TT\}$ that at least one coin lands tails up. Here is a possible probability measure for these events:

$$\begin{array}{rclrcl}
\mu(\emptyset) &=& 0 & \mu(\{HT, TH\}) &=& 2/3 \\
\mu(\{HH\}) &=& 1/3 & \mu(\{HT, TT\}) &=& 0 \\
\mu(\{HT\}) &=& 0 & \mu(\{TH, TT\}) &=& 2/3 \\
\mu(\{TH\}) &=& 2/3 & \mu(\{HH, HT, TH\}) &=& 1 \\
\mu(\{TT\}) &=& 0 & \mu(\{HH, HT, TT\}) &=& 1/3 \\
\mu(\{HH, HT\}) &=& 1/3 & \mu(\{HH, TH, TT\}) &=& 1 \\
\mu(\{HH, TH\}) &=& 1 & \mu(\{HT, TH, TT\}) &=& 2/3 \\
\mu(\{HH, TT\}) &=& 1/3 & \mu(\{HH, HT, TH, TT\}) &=& 1
\end{array}$$

The assignment satisfies the two constraints for probability measures: the probability of the entire sample space is 1, and the probability of every collection of disjoint events (e.g., $\{HT\} \cup \{TH\} = \{HT, TH\}$) is the sum of the individual probabilities. The probability of collections of non-disjoint events (e.g., $\{HT, TH\} \cup \{TH, TT\} = \{HT, TH, TT\}$) may add to something different than the probabilities of the individual events. It is useful to think that this probability measure is completely determined by the two coins in question and their characteristics, in the sense that each pair of coins induces a measure, and each measure must correspond to some pair of coins. The measure above is induced by two coins such that the first coin is twice as likely to land tails up than heads up and the second coin is double-headed. □

Although specifying a probability measure for every event looks complex, it can be simply constructed by

$$\mu(E) = {}_{\mathbb{R}}\sum_{\omega \in E} f(\omega) \ , \tag{2}$$

where $f : \Omega \to [0, 1]$, ${}_{\mathbb{R}}\sum_{\omega \in \Omega} f(\omega) = 1$, and f is called the Radon-Nikodym derivative of μ (with respect to the counting measure) [13, 14]. The converse is also valid and called the Radon-Nikodym theorem.

Theorem 1 (Radon-Nikodym theorem for finite probability space [9, 12]). For every probability measure μ, there exists a unique Radon-Nikodym derivative f such that equation (2) holds. □

For example, the Radon-Nikodym derivative of μ in example 1 is:

$$f(HH) = 1/3, \quad f(HT) = 0, \quad f(TH) = 2/3, \quad f(TT) = 0 \ .$$

In a strict computational or experimental setting, one may question the reliance of the definition of probability space on the uncountable and uncomputable real interval $[0, 1]$. This interval includes numbers like

$0.h_1h_2h_3\ldots$ where h_i is 1 or 0 depending on whether Turing machine M_i halts or not. Such numbers cannot be computed. This interval also includes numbers like $\frac{\pi}{4}$ which can only be computed with increasingly large resources as the precision increases. Therefore, in a resource-aware computational or experimental setting, it is more appropriate to consider probability measures that map events to a set of elements computable with a fixed set of resources. We expand on this observation and then consider interval-valued probability measures [15, 16] in detail.

2.2 Measuring Probabilities: Buffon's Needle Problem

In the previous section, the probability $\mu(E)$ of each event E is known a priori. In reality, although each event is assumed to have a probability, the exact value of $\mu(E)$ may not be known. If we want to know its value, we can run N independent trials. Let $x_i = 1$ or 0 denote whether the event E occurs in the i-th trial or not, respectively, then $\mu(E)$ could be approximated to given accuracy $\epsilon > 0$ by the relative frequency $\frac{1}{N}\sum_{i=1}^{N} x_i$ with the probability converging to one as N goes to infinity, i.e.,

$$\forall \epsilon > 0, \lim_{N\to\infty} \mu\left(\left|\mu(E) - \frac{1}{N}\sum_{i=1}^{N} x_i\right| < \epsilon\right) = 1 \ .$$

This fact is called the law of large numbers [17, 9, 18, 10, 19].

Suppose we drop a needle of length ℓ onto a floor made of equally spaced parallel lines a distance h apart, where $\ell < h$. It is a known fact that the probability of the needle crossing a line is $\frac{2\ell}{\pi h}$ [20, 21, 22, 18]. We analyze this situation in the mathematical framework of probability spaces paying special attention to the resources needed to estimate the probability computationally or experimentally.

To formalize the experiment, we consider an experimental setup consisting of a collection of N identical needles of length ℓ. We throw the N needles one needle at a time, and observe the number M of needles that cross a line. The sample space can be expressed as the set $\{X, -\}^N$ of sequences of characters of length N where each character is either X to indicate a needle crossing a line or $-$ to indicate a needle not crossing a line. If $N = 3$, the probability of the event that exactly 2 needles cross lines $\{-XX, X-X, XX-\}$ can be estimated by the relative frequency $\frac{2}{3}$. Generally, the probability of the event that exactly M needles out of the N total needles cross lines can be estimated by $\frac{M}{N}$.

In an actual experiment with 500 needles and the ratio $\frac{\ell}{h} = 0.75$ [22], it was found that 236 crossed a line so the relative frequency is 0.472 whereas the idealized mathematical probability is $0.4774\ldots$. In a larger experiment with 5000 needles and the ratio $\frac{\ell}{h} = 0.8$ [18], the relative frequency was calculated to be 0.5064 whereas the idealized mathematical probability is $0.5092\ldots$. We see that the observed probability approaches $\frac{2\ell}{\pi h}$ but only if *larger and larger resources* are expended. These resource considerations suggest that it is possible to replace the real interval $[0, 1]$ with rational numbers up to a certain precision related to the particular experiment in question. There is clearly a connection between the number of needles and the achievable precision: in the hypothetical experiment with 3 needles, it is not sensible to retain 100 digits in the expansion of $\frac{2\ell}{\pi h}$.

There is however another more subtle assumption of unbounded computational power in the experiment. We are assuming that we can always determine with certainty whether a needle is crossing a line. But "lines" on the the floor have thickness, their distance apart is not exactly h, and the needles lengths are not all absolutely equal to ℓ. These perturbations make the events "fuzzy." Thus, in an experiment with limited resources, it is not possible to talk about the idealized event that exactly M needles cross lines as this would require the most expensive needles built to the most precise accuracy, laser precision for drawing lines on the floor, and the most powerful microscopes to determine if a needle does cross a line. Instead we might talk about the event that $M - \delta$ needles evidently cross lines and $M + \delta'$ needles plausibly cross lines where δ and δ' are resource-dependent approximations. This fuzzy notion of events leads to probabilities being only calculable within intervals of confidence reflecting the certainty of events and their plausibility. This is indeed consistent with published experiments: in an experiment with 3204 needles and the ratio $\frac{\ell}{h} = 0.6$ [21], 1213 needles clearly crossed a line and 11 needles were close enough to plausibly be considered as crossing

the line: we would express the probability in this case as the interval $\left[\frac{1213}{3204}, \frac{1224}{3204}\right]$ expressing that we are certain that the event has probability at least $\frac{1213}{3204}$ but it is possible that it would have probability $\frac{1224}{3204}$.

Recall that the relative frequency will approximate the probability of an event if the event has a probability. What if the event doesn't have infinity precise probability because of the experimental limit? In this case, two sequences of independent copies of experimental results can have their relative frequencies converge almost surely to different limits [23, 24]. In other words, to get a better approximation of the probability, the quality of the experimental equipment cannot be compensated by the number of independent trials.

2.3 Classical Interval-valued probability measures

As motivated above, an event E_1 may have an interval of probability $[l_1, r_1]$. Assume that another disjoint event E_2 has interval probability $[l_2, r_2]$, what is the interval probability of the event $E_1 \cup E_2$? The answer is somewhat subtle: although it is possible to use the sum of the intervals $[l_1 + l_2, r_1 + r_2]$ as the combined probability, one can do find a much tighter interval if information *against* the event (i.e., information about the complement event) is also taken into consideration. Formally, for a general event E with probability $[l, r]$, the evidence that contradicts E is an evidence supporting the complement of E. The complement of E must therefore have probability $[1 - r, 1 - l]$ which we abbreviate $[1, 1] \ {}_{\mathscr{I}}- [l, r]$, where the preposing $\mathscr{I}$ specifies we subtract intervals. Given a collection of intervals $\mathscr{I}$, an $\mathscr{I}$–interval-valued probability measure is a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that [16]:[1]

- $\bar{\mu}(\emptyset) = [0, 0]$,
- $\bar{\mu}(\Omega) = [1, 1]$,
- for every collection of pairwise disjoint events $\{E_i\}_{i=1}^{N}, \{E'_i\}_{i=1}^{N'} \subseteq \mathcal{E}$ with $\Omega = \left(\bigcup_{i=1}^{N} E_i\right) \bigcup \left(\bigcup_{i=1}^{N'} E'_i\right)$, we have

$$\bar{\mu}\left(\bigcup_{i=1}^{N} E_i\right) \subseteq \left[\max\left\{1 - {}_{\mathbb{R}}\sum_{i=1}^{N'} r'_i,\ {}_{\mathbb{R}}\sum_{i=1}^{N} l_i\right\}, \min\left\{1 - {}_{\mathbb{R}}\sum_{i=1}^{N'} l'_i,\ {}_{\mathbb{R}}\sum_{i=1}^{N} r_i\right\}\right],$$

  where $\bar{\mu}(E_i) = [l_i, r_i]$ and $\bar{\mu}(E'_i) = [l'_i, r'_i]$ for all i.

In order to understand this definition, we tear the last condition apart and provide an equivalent definition of interval-valued probability measures.

Lemma 1. Given a sample space Ω and its event $\mathcal{E}$, a function $\mu : \mathcal{E} \to [0, 1]$ is a classical interval-valued probability measure if and only if μ satisfies the following conditions:

- $\bar{\mu}(\emptyset) = [0, 0]$,
- $\bar{\mu}(\Omega) = [1, 1]$,
- for any event E, $\bar{\mu}(\Omega \backslash E) = [1, 1] \ {}_{\mathscr{I}}- \bar{\mu}(E)$, and
- for a collection $\{E_i\}_{i=1}^{N}$ of pairwise disjoint events, we have $\bar{\mu}\left(\bigcup_{i=1}^{N} E_i\right) \subseteq {}_{\mathscr{I}}\sum_{i=1}^{N} \bar{\mu}(E_i)$, where ${}_{\mathscr{I}}\sum_{i=1}^{N}[l_i, r_i] = \left[{}_{\mathbb{R}}\sum_{i=1}^{N} l_i,\ {}_{\mathbb{R}}\sum_{i=1}^{N} r_i\right]$. We may drop the preposing $\mathscr{I}$ when summands are clearly intervals.

□

We will explain why the last condition is expressed using $\subseteq$ by the following.

[1]The right-end of the interval-valued probability measure is a special case of (sub-additive) capacity [25, 26, 27]. The left-end of the interval-valued probability measure is a special case of game [28].

Example 2 (Two-coin experiment with interval probability). We split the unit interval $[0,1]$ in the following four closed sub-intervals: $[0,0]$ which we call *impossible*, $[0,\frac{1}{2}]$ which we call *unlikely*, $[\frac{1}{2},1]$ which we call *likely*, and $[1,1]$ which we call *certain*. Using these new values, we can modify the probability measure of Ex. 1 by mapping each numeric value to the smallest sub-interval containing it to get the following:

$$\begin{array}{rcl@{\qquad}rcl}
\bar{\mu}(\emptyset) & = & \textit{impossible} & \bar{\mu}(\{HT,TH\}) & = & \textit{likely} \\
\bar{\mu}(\{HH\}) & = & \textit{unlikely} & \bar{\mu}(\{HT,TT\}) & = & \textit{impossible} \\
\bar{\mu}(\{HT\}) & = & \textit{impossible} & \bar{\mu}(\{TH,TT\}) & = & \textit{likely} \\
\bar{\mu}(\{TH\}) & = & \textit{likely} & \bar{\mu}(\{HH,HT,TH\}) & = & \textit{certain} \\
\bar{\mu}(\{TT\}) & = & \textit{impossible} & \bar{\mu}(\{HH,HT,TT\}) & = & \textit{unlikely} \\
\bar{\mu}(\{HH,HT\}) & = & \textit{unlikely} & \bar{\mu}(\{HH,TH,TT\}) & = & \textit{certain} \\
\bar{\mu}(\{HH,TH\}) & = & \textit{certain} & \bar{\mu}(\{HT,TH,TT\}) & = & \textit{likely} \\
\bar{\mu}(\{HH,TT\}) & = & \textit{unlikely} & \bar{\mu}(\{HH,HT,TH,TT\}) & = & \textit{certain}
\end{array}$$

Despite the absence of any numeric information, the probability measure is quite informative: it reveals that the second coin is double-headed and that the first coin is biased. To understand the $\subseteq$-condition, consider the following calculation:

$$\begin{aligned}
& \bar{\mu}(\{HH\}) + \bar{\mu}(\{HT\}) + \bar{\mu}(\{TH\}) + \bar{\mu}(\{TT\}) \\
= \; & \textit{impossible} + \textit{unlikely} + \textit{impossible} + \textit{likely} \\
= \; & [0,0] + \left[0,\frac{1}{2}\right] + [0,0] + \left[\frac{1}{2},1\right] = \left[\frac{1}{2},\frac{3}{2}\right]
\end{aligned}$$

If we were to equate $\bar{\mu}(\Omega)$ with the sum of the individual probabilities, we would get that $\bar{\mu}(\Omega) = \left[\frac{1}{2},\frac{3}{2}\right]$. However, using the fact that $\bar{\mu}(\emptyset) = \textit{impossible}$, we have $\bar{\mu}(\Omega) = 1 - \bar{\mu}(\emptyset) = \textit{certain} = [1,1]$. This interval is tighter and a better estimate for the probability of the event Ω, and of course it is contained in $\left[\frac{1}{2},\frac{3}{2}\right]$. However it is only possible to exploit the information about the complement when all four events are combined. Thus the $\subseteq$-condition allows us to get an estimate for the combined event from each of its constituents and then gather more evidence knowing the aggregate event. □

In contrast to real-valued probability measures, there is no easy Radon-Nikodym theorem to simplify an interval-valued probability measure to its Radon-Nikodym derivative. So far, the interval-valued Radon-Nikodym Theorems are either too restricted to apply to all interval-valued probability measures [25, 26, 27] or corresponding an interval-valued probability measure $\bar{\mu}: \mathcal{E} \to \mathscr{I}$ to another set function $\bar{f}: \mathcal{E} \to \bar{\mathscr{I}}$ which is not as simple as the original Radon-Nikodym derivative [29, 27].

Although there is no easy Radon-Nikodym theorem, an interval-valued probability measure can still be understood by its relation with real-valued probability measures. For example, the real-valued probability measure in example 1 can construct the interval-valued probability measure in example 2. Conversely, we could understand the interval-valued probability measure in example 2 by the real-valued probability measure and its Radon-Nikodym derivative in example 1. The relation between example 1 and 2 can be abstracted into the definition of core. Besides of core, we also define another property, convex, which will be used later.

Definition 2. Given an interval-valued probability measure $\bar{\mu}: \mathcal{E} \to \mathscr{I}$:

- A *core*[2] of $\bar{\mu}$ is the set $\text{core}(\bar{\mu}) = \{\mu: \mathcal{E} \to [0,1] \text{ probability measure} | \mu(E) \in \bar{\mu}(E) \, \forall E \in \mathcal{E}\}$ [29, 23, 24].
- $\bar{\mu}$ is called *convex* or *2-monotone* if $\bar{\mu}(E_1 \cup E_2) \; {}_{\mathscr{I}}+ \; \bar{\mu}(E_1 \cap E_2) \subseteq \bar{\mu}(E_1) \; {}_{\mathscr{I}}+ \; \bar{\mu}(E_2)$ for all $E_1, E_2 \in \mathcal{E}$ [28, 29, 23, 24].[3]

□

[2] The core is also called a structure [15].

[3] Yu-Tsung says: If we want to use convex, we need to explain the physical meaning of convex interval-valued probability measures...? If there is a good explanation, we may only consider convex interval-valued probability measures, and replace the definition in the beginning...?

It is common to consider convex interval-valued probability measures. For example, a special case of convex interval-valued probability measures consists the Dempster-Shafer belief function as the left-end and the Dempster-Shafer plausibility as the right-end [30, 10, 29, 23, 27, 24]. Dempster-Shafer functions, also called *completely or totally monotone*, plays an important role in the theory of non-additive belief. Furthermore, convex interval-valued probability measures have a good property that they can always be associated with real-valued probability measures.

Theorem 2 (Shapley [28, 29]). Every convex interval-valued probability measure has a nonempty core. □

Yu-Tsung says: Because the physical meaning of convex is still not clear, it would be better if we can find or prove a theorem which removes "convex" in the theorem. However, I still have not found an example with an empty core or a real-valued probability measure in the core of each interval-valued probability measure. What we know:

1. The cloud in [16] cannot induce an interval-valued probability measure. Therefore, any results related to clouds cannot be used directly.
2. The definition of interval-valued probability measure in [16] is different from any later definition of interval-valued probability measure written by Lodwick, including [31].
3. About linear programming, the Mathematica program solving the case $|\Omega| = 1$ or 2 gives positive answer. The program ceases to respond when outputting $|\Omega| = 3$. Maybe I should simplify the program?
4. Check whether the Shapley value still in the core or not in our setting...?

3 Quantum Probability Spaces

The mathematical framework above assumes that there exists a predetermined set of events that are independent of the particular experiment. However, in many practical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot, a priori, be determined with certainty. In the quantum framework, this partial knowledge is compounded by the fact that there exist non-commuting events which cannot happen simultaneously. To accommodate these more complex situations, we abandon the sample space Ω and reason directly about events. A quantum probability space therefore consists of just two components: a set of events $\mathcal{E}$ and a probability measure $\mu : \mathcal{E} \to [0, 1]$. To properly explain the quantum probability, we will first discuss projection operators as quantum events, provide an easy example, define quantum probability measures, and extend to quantum interval-valued probability.

3.1 Quantum Events

Definition 3 (Projection Operators; Orthogonality [32, 8, 7, 11, 12]). Given a Hilbert space $\mathcal{H}$, an event[4] mathematically is represented as a projection operator $P : \mathcal{H} \to \mathcal{H}$ onto a linear subspace S of $\mathcal{H}$. The set of all events $\mathcal{E}$ can be defined recursively as follow: [5]

- $\mathbb{0}$ is a projection.
- For any pure state $|\psi\rangle$, $|\psi\rangle\langle\psi|$ is a projection operator.
- Projection operators P_1 and P_2 are *orthogonal* if $P_1 P_2 = P_2 P_1 = \mathbb{0}$. The sum of two projection operators $P_1 \,{}_{\mathscr{O}}{+}\, P_2$ is also a projection operator if and only if they are orthogonal, where the preposing subscript $\mathscr{O}$ means ${}_{\mathscr{O}}{+}$ is an operation between operators.

[4] An event is formally called an experimental proposition [33], a question [32, 34], or an elementary quantum test [7].
[5] "Projection" is sometimes called "orthogonal projection" or "self-adjoint projection" to emphasize $P^\dagger = P$ [11, 35].

- Conversely, every projection P can be expressed as $\bigoplus_{j=1}^{N} |\psi_j\rangle\langle\psi_j|$, where P actually projects onto the linear subspace S which has an orthonormal basis $\{|\psi_j\rangle\}_{j=1}^{N}$.

□

Because quantum events are projection operators, operations and properties of quantum events can be written in terms of those of operators.

Definition 4 (Ideal Measurement; Complement; Commutativity).

- A set of projections $\{P_i\}_{i=1}^{N}$ is called an *ideal measurement* if it is a partition of the identity, i.e., $\bigoplus_{i=1}^{N} P_i = \mathbb{1}$ [12]. In this case, projections $\{P_i\}_{i=1}^{N}$ must be mutually orthogonal [11].
- If P is a projection operator, then $\mathbb{1} \ominus P$ is also a projection operator, called *complement*. It is orthogonal to P, and corresponds to the complement event $\Omega \backslash E$ in classical probability [11].
- Projection operators P_1 and P_2 *commute* if $P_1 P_2 = P_2 P_1$. The product of two projection operators $P_1 P_2$ is also a projection operator if and only if they commute. This corresponds to the classical intersection between events [7, 11].
- For two commuting projection operators P_1 and P_2, their *disjunction* $P_1 \vee P_2$ is defined to be $P_1 \oplus P_2 \ominus P_1 P_2$ [11].

□

Based on the above definition and properties of quantum events, we can present an example before giving the formal definition of quantum probability measures.

Example 3 (One-qubit quantum probability space). Consider a one-qubit Hilbert space with states $|\phi\rangle$, which can be expressed as a linear combination of $|0\rangle$ and $|1\rangle$, i.e., $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$, $\alpha, \beta \in \mathbb{C}$. The set of events associated with this Hilbert space consists of all projection operators. Each event is interpreted as a possible post-measurement state of a quantum system in current state $|\phi\rangle$. For example, the event $|0\rangle\langle 0|$ indicates that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ indicates that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ indicates that the post-measurement state will be $|+\rangle$; the event $\mathbb{1} = |0\rangle\langle 0| \oplus |1\rangle\langle 1|$ indicates that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the empty event $\mathbb{0}$ states that the post-measurement state will be the empty state. As in the classical case, a probability measure is a function that maps events to $[0, 1]$: here is a partial specification of a possible probability measure:

$$\mu(\mathbb{0}) = 0, \quad \mu(\mathbb{1}) = 1, \quad \mu(|0\rangle\langle 0|) = 1, \quad \mu(|1\rangle\langle 1|) = 0, \quad \mu(|+\rangle\langle +|) = 1/2, \quad \ldots$$

Note that, similarly to the classical case, the probability of $\mathbb{1}$ is 1 and the probability of collections of orthogonal events (e.g., $|0\rangle\langle 0| + |1\rangle\langle 1|$) is the sum of the individual probabilities. A collection of non-orthogonal events (e.g., $|0\rangle\langle 0|$ and $|+\rangle\langle +|$) is however not even a valid event. In the classical example, we argued that each probability measure is uniquely determined by two actual coins. A similar (but much more subtle) argument is valid also in the quantum case. By postulates of quantum mechanics and Gleason's theorem, it turns out that for large enough quantum systems, each probability measure is uniquely determined by an actual quantum state. □

3.2 Quantum Real-valued Probability Measures

Definition 5 (Quantum Probability Measure [32, 6, 8, 35]). Given a Hilbert space $\mathcal{H}$ with its set of events $\mathcal{E}$, a *quantum probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:[6]

[6]It is possible to define a more general space of events consisting of all operators $\mathcal{A}$ on $\mathcal{H}$ and to extend μ linearly to $\mathcal{A}$ and consider $\mu : \mathcal{A} \to \mathbb{C}$ [35, 12]. When an operator $A \in \mathcal{A}$ is Hermitian, $\mu(A)$ is the expectation value of A which could bigger than one because μ is linear. We does not take this approach because we want to focus only on probability.

- $\mu(\mathbb{0}) = 0$.
- $\mu(\mathbb{1}) = 1$.
- For any projection P,
$$\mu\left(\mathbb{1} \mathbin{\mathscr{O}-} P\right) = 1 \mathbin{\mathbb{R}-} \mu\left(P\right) \ . \tag{3}$$
- for a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have $\mu\left(\mathop{\mathscr{O}\sum}_{i=1}^N P_i\right) = \mathop{\mathbb{R}\sum}_{i=1}^N \mu\left(P_i\right)$.

□

Similar to classical definition 1, $\mu(\mathbb{0}) = 0$ and equation (3) can be induced by other conditions. A set of events $\mathcal{E}$ together with a quantum probability measure is called a *quantum probability space*. Comparing to classical probability spaces, the empty set $\emptyset$ corresponds to the empty projection $\mathbb{0}$, and the event of whole space Ω corresponds to the identity projection $\mathbb{1}$. In contrast, the union $\cup$ of any two events always gives an event classically, but the operator addition $\mathscr{O}+$ of two projections may not be a projection. As the result, the classical condition $\mu(E_1 \cup E_2) = \mu(E_1) \mathbin{\mathbb{R}+} \mu(E_2)$ is always defined, and it is true when E_1 and E_2 are disjoint; however, $\mu\left(P_1 \mathbin{\mathscr{O}+} P_2\right) = \mu(P_1) \mathbin{\mathbb{R}+} \mu(P_2)$ is always true whenever the left-handed side is defined.
[7]

Recall that classical probability measure can be constructed from its Radon-Nikodym derivative. Similarly, a quantum probability measure can be easily constructed by states according to the Born rule [4, 5, 36]. For each pure normalized ($\langle\phi|\,\phi\rangle = 1$) quantum state $|\phi\rangle$, the Born rule induces a probability measure μ_ϕ^B as follows:
$$\mu_\phi^B(P) = \langle\phi|\,P\phi\rangle \ .$$

Moreover, the Born rule can be extended to a mixed state. When we prepare a state, suppose the probability of preparing state $|\phi_j\rangle$ is q_j, the state of the system can be expressed as a density matrix $\rho = \mathop{\mathscr{O}\sum}_{j=1}^N q_j|\phi_j\rangle\langle\phi_j|$, where $\mathop{\mathbb{R}\sum}_{j=1}^N q_j = 1$. Then, the quantum probability measure introduced by ρ is the combination of $\mu_{\phi_j}^B$ with respect to probability q_j [7, 19, 36]:
$$\mu_\rho^B\left(P\right) \quad = \quad \mathrm{Tr}\left(\rho P\right) = \mathop{\mathbb{R}\sum}_{j=1}^N q_j \mu_{\phi_j}^B\left(P\right) \ . \tag{4}$$

3.3 Measuring Quantum Probabilities Ideally: Radon-Nikodym and Gleason's Theorem

Similar to the classical case, by applying the law of large numbers, quantum probabilities can be estimated by relative frequencies. For example, if we want to know the probability of the spin up in the Stern-Gerlach experiment [37, 7, 19, 11], we can put a beam of silver atoms in a highly inhomogeneous magnetic field, and counting the number of atoms deflects up. Ideally, if the local field of strength directs to the z-axis, and all particles have the same velocity, the Stern-Gerlach experiment only produces two spots corresponding to $|0\rangle$ and $|1\rangle$. Notice that whenever the magnetic field of the Stern-Gerlach is fixed, i.e., an ideal measurement is picked, measuring the spin is exactly the same as tossing a coin. In another word, if somebody claims she is tossing a coin behind a veil, and only shows the resulting heads or tails, we cannot distinguish whether she has really tossed a coin, or she has run an Stern-Gerlach experiment, and shows us the head, the tail, or the side of coin if the silver atoms is spin up, spin down, or just hit the middle, respectively.

Mathematically, given a Hilbert space $\mathcal{H}$ of dimension d, we can pick a orthonormal basis $\Omega = \{|\psi_j\rangle\}_{j=1}^d$, and fix a density matrix ρ representing in the same basis, i.e., $\rho = \mathop{\mathscr{O}\sum}_{i=1}^d \mathop{\mathscr{O}\sum}_{i=1}^d r_{ij}\,|\psi_i\rangle\,\langle\psi_j|$. Applying the canonical inner product, we can define a Radon-Nikodym derivative $f_\rho : \Omega \to [0,1]$ by
$$f_\rho\left(|\psi_j\rangle\right) = \langle\psi_j|\,\rho\psi_j\rangle = r_{jj} \ . \tag{5}$$

[7] Yu-Tsung says: The definition of independence is interesting, and we definitely need to discuss it when we want to discuss Bell's theorem and the Kochen-Specker theorem. However, the definition diverges. So we just leave it so far, and we will go back if we really need it in this paper. (Maybe when discussing repeating experiments?)

Furthermore, we only consider the quantum events $\mathcal{E}'$ with the form $P = \mathscr{O}\sum_{|\psi_j\rangle\in E} |\psi_j\rangle\langle\psi_j|$, where $E \in 2^\Omega$. The classical probability measure $\mu' : 2^\Omega \to [0,1]$ induced by the Radon-Nikodym derivative f_ρ is exactly the Born rule of the mixed state ρ [35],

$$\mu'(E) \overset{(2)}{=} \mathbb{R}\sum_{|\psi_j\rangle\in E} f_\rho(|\psi_j\rangle) \overset{(5)}{=} \mathbb{R}\sum_{|\psi_j\rangle\in E} r_{jj} = \mathrm{Tr}(\rho P) \overset{(4)}{=} \mu_\rho^B(P) \ . \tag{6}$$

In this sense, the Radon-Nikodym derivative f_ρ is a special case of the mixed state ρ, and the Radon-Nikodym theorem is a special case of Gleason's theorem [38, 8, 39]: in Hilbert spaces of dimension $d \geq 3$, given a quantum probability measure μ, there exist a mixed state ρ that induces such a measure using the Born rule [6, 8, 7]. By applying Gleason's theorem, we can extend definition 5 to the following corollary.

Corollary 1. Given a Hilbert space $\mathcal{H}$ of dimension $d \geq 3$, for any function $\mu : \mathcal{E} \to [0,1]$ satisfying the conditions listed in definition 5, there exists an unique mixed state ρ such that $\mu = \mu_\rho^B$.

It is instructive to study a counterexample when $d = 2$, i.e., the case of a one-qubit system.

Example 4 (One-qubit quantum probability measure). Consider a quantum probability measure $\mu : \mathcal{E} \to [0,1]$ defined as follow:

$$\mu(P) = \begin{cases} 1 & \text{, if } P = |+\rangle\langle+| \ ; \\ 0 & \text{, if } P = |-\rangle\langle-| \ ; \\ \mu_{|0\rangle}^B(P) & \text{, otherwise.} \end{cases}$$

On one hand, μ is a quantum probability measure. Because μ is almost the same as the quantum probability measure $\mu_{|0\rangle}^B$, we only need to check the orthogonal pair $|+\rangle\langle+|$ and $|-\rangle\langle-|$: $\mu(|+\rangle\langle+|)+\mu(|-\rangle\langle-|) = 1+0 = 1$. On the other hand, μ cannot be induced by any mixed state because $\mu(|+\rangle\langle+|) = \mu(|0\rangle\langle0|) = 1$, and $\mu_\rho^B(P) = 1$ if and only if ρ represents a pure state and $\rho = P$. □

3.4 Measuring Quantum Probabilities in Reality: Quantum Interval-valued Probability Measures

In reality, a Stern-Gerlach experiment requires a lot of resource to keep the local field of strength directing to the z-axis precisely, to keep the atoms having almost the same velocity, and to point out the exact position each particle landed on. If the field of strength does not perfectly direct to the z-axis, we do not really test the quantum event we want to test. This problem will be handled later when we introduce the discrete quantum theory. Even if the field of strength perfectly directs to the z-axis, the variant velocity makes spots broader and more washed out. Together with the precision limit of the detector, it may sometimes be hard to decide a particle corresponding to which state[8].

Similar to Buffon's needles, this kind of fuzziness can be taken into account by associating each quantum event with an interval-valued probability $[l, r]$. Therefore, we should plug in the definition of interval-valued probability measure into the definition of quantum probability space for each ideal measurement. More specifically, for classical probability measures, definition 1 and lemma 1 are almost the same except the last condition using the equal sign $=$ in real-valued definition 5 and the subset equal sign $\subseteq$ in interval-valued lemma 1. Therefore, for quantum probability measures, replacing the the equal sign $=$ in real-valued definition 5 by the subset equal sign $\subseteq$ gives the only reasonable definition for quantum interval-valued probability measures as follow.

Definition 6 (Quantum Interval-valued Probability Measure). Given a Hilbert space $\mathcal{H}$ with the set of quantum events $\mathcal{E}$, and a collection of intervals $\mathscr{I}$, a *quantum $\mathscr{I}$-interval-valued probability measure* is a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that:

- $\bar{\mu}(\mathbb{0}) = [0,0]$.

[8] Yu-Tsung says: Add citations to support the idea... Haven't found suitable ones...

Amr says: preparation fuzzy, device fuzzy, Meyer [40]

- $\bar{\mu}(\mathbb{1}) = [1,1]$.
- For any projection P,
$$\bar{\mu}\left(\mathbb{1} \ominus_{\mathscr{O}} P\right) = [1,1] \ominus_{\mathscr{I}} \bar{\mu}\left(P\right) \ . \tag{7}$$
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have
$$\bar{\mu}\left(\sum\nolimits_{\mathscr{O},i=1}^{N} P_i\right) \subseteq \sum\nolimits_{\mathscr{I},i=1}^{N} \bar{\mu}\left(P_i\right) \ . \tag{8}$$

□

Similar to the classical interval-valued probability measure, a quantum interval-valued probability measure can be constructed from a real-valued one.

Example 5 (Quantum three-value interval-valued probability measure). We consider three intervals $[0,0]$, $[1,1]$ and $[0,1]$, where $[0,0]$ and $[1,1]$ are called *impossible* and *certain* as before, and $[0,1]$ is called *unknown* because it provides no information. For any Hilbert space and any quantum probability measure $\mu : \mathcal{E} \to [0,1]$, we can define a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ by $\bar{\mu}(P) = \iota\left(\mu(P)\right)$, where $\iota : [0,1] \to \mathscr{I}$ is defined by
$$\iota(x) = \begin{cases} \textit{certain} & \text{, if } x = 1 \ ; \\ \textit{impossible} & \text{, if } x = 0 \ ; \\ \textit{unknown} & \text{, otherwise.} \end{cases}$$
ι has two interesting properties $\iota\left(\mathbb{1} \ominus_{\mathbb{R}} x\right) = [1,1] \ominus_{\mathscr{I}} \iota\left(x\right)$ and $\iota\left(\sum\nolimits_{\mathbb{R},i=1}^{N} x_i\right) \subseteq \sum\nolimits_{\mathscr{I},i=1}^{N} \iota\left(x_i\right)$, where x and $\sum\nolimits_{\mathbb{R},i=1}^{N} x_i \in [0,1]$. By applying these two properties, it is easy to verify $\bar{\mu}$ is a quantum $\mathscr{I}$-interval-valued probability measure. □

Similar to classical interval-valued probability measures which have no easy Radon-Nikodym theorem, quantum interval-valued probability measures have no easy Gleason's theorem. Although we could define core and convex for quantum interval-valued probability measures, in contrast to theorem 2, a convex quantum interval-valued probability measure may have a empty core, i.e., it cannot correspond to a quantum real-valued probability measure or a mixed state.

Definition 7. Given a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$:

- A *core* of $\bar{\mu}$ is the set $\mathrm{core}\left(\bar{\mu}\right) = \{\mu : \mathcal{E} \to [0,1] \text{ quantum probability measure} | \mu\left(P\right) \in \bar{\mu}\left(P\right) \forall P \in \mathcal{E}\}$.
- $\bar{\mu}$ is called convex if
$$\bar{\mu}\left(P_1 \oplus_{\mathscr{O}} P_2\right) \oplus_{\mathscr{I}} \bar{\mu}\left(P_1 P_2\right) \subseteq \bar{\mu}\left(P_1\right) \oplus_{\mathscr{I}} \bar{\mu}\left(P_2\right) \tag{9}$$
for all commuting $P_1, P_2 \in \mathcal{E}$.

□

Theorem 3. There is a convex quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that $\mathrm{core}\left(\bar{\mu}\right) = \emptyset$. □

This theorem can be proved by the following example.

Example 6 (Three-dimensional quantum three-value interval-valued probability measure). Given a three dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Consider $\mathscr{I} = \{\textit{certain}, \textit{impossible}, \textit{unknown}\}$ and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $|+'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$. Let
$$\begin{aligned} \bar{\mu}(\mathbb{0}) = \bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) = \bar{\mu}(|+'\rangle\langle +'|) &= \textit{impossible} \ , \\ \bar{\mu}(\mathbb{1}) = \bar{\mu}(\mathbb{1} - |0\rangle\langle 0|) = \bar{\mu}(\mathbb{1} - |+\rangle\langle +|) = \bar{\mu}(\mathbb{1} - |+'\rangle\langle +'|) &= \textit{certain} \ , \\ \bar{\mu}(P) &= \textit{unknown}, \text{ otherwise.} \end{aligned}$$

To verify $\bar{\mu}$ is a quantum interval-valued probability measure, first notice that $\bar{\mu}(P_1 \mathbin{\mathscr{O}+} P_2) \subseteq \bar{\mu}(P_1) \mathbin{\mathscr{I}+} \bar{\mu}(P_2)$ implies equation (8) by induction. Second, equation (7) implies $\bar{\mu}(\mathbb{1}) \subseteq \bar{\mu}(P) \mathbin{\mathscr{I}+} \bar{\mu}(\mathbb{1} \mathbin{\mathscr{O}-} P)$. Therefore, to prove equation (8), it is sufficient to check

$$\bar{\mu}(|\psi_1\rangle\langle\psi_1| \mathbin{\mathscr{O}+} |\psi_2\rangle\langle\psi_2|) \subseteq \bar{\mu}(|\psi_1\rangle\langle\psi_1|) \mathbin{\mathscr{I}+} \bar{\mu}(|\psi_2\rangle\langle\psi_2|)$$

for orthogonal $|\psi_1\rangle$ and $|\psi_2\rangle$. This equation always holds because we always have $\mathit{unknown} \subseteq \bar{\mu}(|\psi_1\rangle\langle\psi_1|) \mathbin{\mathscr{I}+} \bar{\mu}(|\psi_2\rangle\langle\psi_2|)$.

Next, we need to verify that $\bar{\mu}$ is convex. First, for orthogonal projections P_1 and P_2, equation (8) implies equation (9). Second, when $P_1 = P_2 \mathbin{\mathscr{O}+} P_3$, equation (9) holds trivially. Hence, we only need to verify the case which $P_1 = |\psi_1\rangle\langle\psi_1| \mathbin{\mathscr{O}+} |\psi_3\rangle\langle\psi_3|$ and $P_2 = |\psi_2\rangle\langle\psi_2| \mathbin{\mathscr{O}+} |\psi_3\rangle\langle\psi_3|$, where $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ is an orthonormal basis. Because we have

$$\begin{aligned} \bar{\mu}(P_1 \mathbin{\mathscr{O}\vee} P_2) &= \bar{\mu}(\mathbb{1}) = \mathit{certain} \\ \bar{\mu}(P_1 P_2) &= \bar{\mu}(|\psi_3\rangle\langle\psi_3|) \subseteq \mathit{unknown} \end{aligned}$$

and $\bar{\mu}(P_1) \mathbin{\mathscr{I}+} \bar{\mu}(P_2)$ is either $\mathit{certain} \mathbin{\mathscr{I}+} \mathit{unknown}$ or $\mathit{unknown} \mathbin{\mathscr{I}+} \mathit{unknown}$. Therefore, we have $\bar{\mu}(P_1 \mathbin{\mathscr{O}\vee} P_2) \mathbin{\mathscr{I}+} \bar{\mu}(P_1 P_2) \subseteq \bar{\mu}(P_1) \mathbin{\mathscr{I}+} \bar{\mu}(P_2)$.

$\bar{\mu}$ cannot correspond to any quantum probability measure. Suppose $\mu(P) \in \bar{\mu}(P)$ for some quantum probability measure $\mu : \mathcal{E} \to [0,1]$. We must have

$$\mu(|0\rangle\langle 0|) = \mu(|+\rangle\langle +|) = \mu(|+'\rangle\langle +'|) = 0 \ . \tag{10}$$

By Gleason's theorem, there is a mixed state $\rho = \sum_j q_j |\phi_j\rangle\langle\phi_j|$ such that $\mu(P) = \sum_{j=1}^{N} q_j \langle\phi_j| P |\phi_j\rangle$, where $\sum_j q_j = 1$ and $q_j > 0$. However, no pure state $|\phi\rangle$ satisfies $\langle\phi|0\rangle = \langle\phi|+\rangle = \langle\phi|+'\rangle = 0$ so that we have $\mu(P) \notin \bar{\mu}(P)$ for all quantum probability measure μ. □

4 Quantum Computing on Quantum Interval-valued Probability Measures

We give up the idea of representing a quantum state as a density matrix, and compute the quantum state as a quantum interval-valued probability measure directly.

4.1 Time Evolution and Tensor Product

In CQT, an time evolution operator U will map a state $|\phi\rangle$ to $U|\phi\rangle$. In this case, its quantum real-valued probability measure after evolution is

$$\mu^B_{U|\phi\rangle}(P) = \langle\phi| U^\dagger P U \phi\rangle = \mu^B_{|\phi\rangle}\left(U^\dagger P U\right) \ .$$

Hence, given a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$, the quantum interval-valued probability measure after time evolution operator U is defined as

$$\bar{\mu}_U(P) = \bar{\mu}\left(U^\dagger P U\right) \ .$$

4.2 Quantum Teleportation

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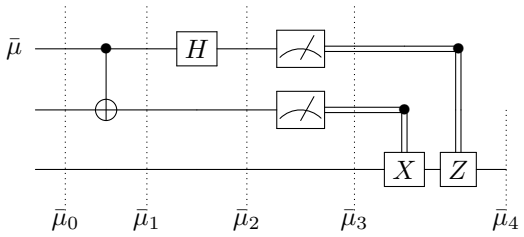

Figure 1: Circuit for Quantum Teleportation

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