CRYPTODETERMINISM AND QUANTUM THEORY

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ABSTRACT

The Kochen-Specker proof of nonexistence of hidden variables is reassessed. We argue that a physical situation where hidden variables would have definite values might violate not only quantum statistics, but even the quantum predictions for possible results of *individual* measurements. It is also shown in an Appendix that the Kochen-Specker theorem can be proved by using only 109 rays, rather than 117 in their original proof.

Quantum theory allows us to compute *probabilities* for the observation of events of a specified kind, following a specified preparation.[1] The preparation and the observation are *macroscopic* procedures, which are described in classical language[2] and are not subject to further analysis.[3] The algorithm allowing us to compute these transition probabilities involves abstract concepts, such as vectors and linear operators in Hilbert space, which are purported to describe real microscopic objects (electrons, photons, etc.). As this algorithm supplies only statistical information, it is natural to ask whether there could be a more detailed description of nature, such that all our predictions would become unambiguous. For example, we would be able to predict whether any specified silver atom passing through a Stern-Gerlach magnet will be deflected up or down. This more detailed description would presumably involve additional data on the silver atom and perhaps also on the Stern-Gerlach magnet and the oven from which the atom originated. These additional data are commonly called "hidden variables" (HV).[4] The tentative goal of HV theories is the following: In the absence of detailed knowledge (or control) of the HV, calculations could still be based on an *ensemble* average over their purported statistical distribution and would then yield the statistical predictions of quantum theory.

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There have been over the years many claims that HV theories are inconsistent or at least violate some cherished principles of physics, such as locality and causality. The early impossibility proofs are discussed –and demolished –in lucid reviews reviews by Albertson[5] and Bell.[6] In fact, Bell[6] was able to construct an explicit HV algorithm specifying a definite result for the measurement of any spin component of a spin-$\frac{1}{2}$ particle and yielding the correct quantum expectation value upon averaging the HV. A similar construction was independently proposed by Kochen and Specker[7] (hereafter KS).

Here, a word of caution is in order: The term "measuring" a spin component does not mean "acquiring knowledge about the value" of that component. In general, that value does no exist in any sense, prior to the interaction of the quantum system with the measuring apparatus.[8,9] The virtue of the HV algorithms of Bell and KS is only to yield some definite value for the macroscopically observable outcome of that interaction.

However, the paper of KS[7] also claimed that the results of measurements involving spin-1 (rather than spin-$\frac{1}{2}$) particles could not be simulated by *any* HV theory. The remarkable feature of their proof is that it does not involve quantum statistics at all, but only the fact that the results of *individual* measurements are *discrete* and may be *correlated.* This radical simplification due to KS thus neatly disposes of any theoretical arguments on how to take averages over the HV distribution and also of any experimental arguments about detector efficiencies. Unfortunately, the KS article is couched in a rather esoteric mathematical language, and it had much less impact on the physics community than Bell's celebrated theorem,[10] which is actually weaker than the KS result.

The purpose of this essay is to explain the KS theorem in simple language, and then to analyze its physical significance. We start from the well-known fact that, for a spin-1 particle, orthogonal spin components can be represented by matrices

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \qquad J_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1)$$

in units such as $\hbar = 1$. These matrices indeed satisfy $[J_x, J_y] = iJ_z$ etc. In that representation,

$$J_x{}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_y{}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_z{}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

These matrices commute, so that, by the tenets of quantum theory, all the corresponding observables can be measured simultaneously, and these measurements "do not disturb one another." In fact, all these $J_m{}^2$ can be considered as functions of a single nondegenerated dynamical variable, such as

$$K = J_x{}^2 - J_y{}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Indeed, we have

$$J_x{}^2 = 2 + (K - K^2)/2, \tag{4}$$

$$J_y{}^2 = 2 - (K + K^2)/2, \tag{5}$$

$$J_z{}^2 = K^2. \tag{6}$$

A variable such as K can in principle be measured in a single operation.[11] It is tacitly assumed that the result of measuring $J_x{}^2$, say, is the same whether we measure directly $J_x{}^2$, or J_x (and then take the square of the result) or K (and compute $J_x{}^2$ as $2 + (K - K^2)/2$). This tacit assumption will be discussed below.

It follows from the above that the three projection operators $P_x = J_x{}^2$, $P_y = J_y{}^2$, and $P_z = J_z{}^2$ have the same eigenvalues (1,1,0), and their sum is

$$P_x + P_y + P_z = 1. \tag{7}$$

Their physical meaning is very simple. The question "Is $P_m = 1$?" (more precisely: "Does a measurement of P_m yield the result 1?") is the same as "Is $J_m = 0$?". We can simultaneously test the values of P_x, P_y, and P_z. One result must be 1, the two others must be 0, by virtue of Eq. (7). The same naturally applies to any three spin components along three arbitrary orthogonal directions.

We now introduce the cryptodeterministic hypothesis: *It is possible to specify all the details of a preparation, so that the result of any measurement becomes completely deterministic.* For example, the result of measuring

$$P_n = 1 - (\mathbf{n} \cdot \mathbf{J})^2, \tag{8}$$

along the direction $\mathbf{n}$, is unique for every unit vector $\mathbf{n}$.

Assuming that this is true, we can now imagine that we paint a unit sphere in such a way that there is a black dot at the tip of $\mathbf{n}$ if a measurement of P_n yields 1, and a white dot if it yields 0. The whole unit sphere is thereby painted in black and white. Now, if we consider a triad of orthonormal vectors $\mathbf{m}$, $\mathbf{n}$, $\mathbf{s}$, we have

$$P_m + P_n + P_s = 1. \tag{9}$$

Therefore these three orthonormal vectors must hit the unit sphere at one black point and two white ones. It is a matter of elementary geometry to show that this is impossible: A sphere cannot be painted in black and white in such a way that *any* triad will hit it at exactly one black point and two white ones. It follows that the cryptodeterministic hypothesis, as stated above, is inconsistent with quantum theory.

Kochen and Specker prove more than that. They do not need the complete unit sphere (an infinite number of directions) but only 117 unit vectors, which include a large number of orthogonal triads such that some vectors belong to more than one triad. Stated in purely algebraic language, KS construct 117 Hermitean 3×3 matrices P_k, each one with eigenvalues 1, 0, 0, and show that there is no mapping $P_k \rightarrow p_k = 0, 1$ such that, if

$$P_m P_n = P_n P_s = P_s P_m = 1, \tag{10}$$

and

$$P_m + P_n + P_s = 1, \tag{11}$$

then

$$p_m + p_n + p_s = 1. \tag{12}$$

Thus, contrary to the cryptodeterministic hypothesis, it is impossible to assign a unique result to the measurement of each matrix, consistent with Eqs. (10-12). We show in the Appendix how it is possible to reduce the number of directions from 117 to only 109.

The KS proof is extremely elegant and powerful. It does not involve at all quantum statistics, but only the fact that individual outcomes of measurements are *discrete* and may be *correlated*, as in Eq. (12). Let us examine carefully the tacit assumption underlying that proof. It was already mentioned after Eq. (6): To prove the KS theorem, one must assume that to each $\mathbf{n}$ corresponds a unique result, 0 or 1, irrespective of whether we measure P_n alone or P_n together with P_k and P_m (k, m, n being an orthonormal triad) or P_n together with P_s and P_t (n, s, t being a *different* orthonormal triad, sharing only $\mathbf{n}$ with the first one). This kind of assumption was already mentioned by Bell[6] in his analysis of Gleason's proof[12] (a precursor of KS):

> "It was tacitly assumed that measurement of an observable must yield the same value independently of what other [compatible] measurements may be made simultaneously .... There is no a priori reason to believe that the results should be the same. The result of an observation may reasonably depend not only on the state of the system (including HV) but also on the complete disposition of the apparatus ...."

This line of thought can in fact be traced back to Bohr[2] who emphasized "the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

In brief, *the same dynamical variable may correspond to different observables.* (This sibyllic statement means that a given Hermitean matrix is purported to represent an "observable" P_n, but actually the symbol P_n has a different meaning if P_n is measured alone or measured with P_m or with P_s, etc.) The only exception is a nondegenerate variable, corresponding to a *complete* set of compatible observables: If A is nondegenerate, its measurement completely determines that of any B commuting with A, because one can always write B as a single-valued function $B = f(A)$. If, on the other hand, A is degenerate, it is possible that $[A, B] = 0$ and $[A, C] = 0$ but $[B, C] \neq 0$. In that case, "a measurement of A" is not a complete specification of the experimental setup, and the result of that measurement may depend on whether one measures A alone, or A and B, or A and C. This is essentially Bell's comment, quoted above.

As a concrete example, consider an orthonormal triad $(\mathbf{m}, \mathbf{n}, \mathbf{p})$ and an operator similar to Eq. (3):

$$K(\mathbf{m}, \mathbf{n}) = (\mathbf{m} \cdot \mathbf{J})^2 - (\mathbf{n} \cdot \mathbf{J})^2, \tag{13}$$

which is nondegenerate. Then if the result of measuring $K(\mathbf{m}, \mathbf{n})$ for any $(\mathbf{m}, \mathbf{n}, \mathbf{p})$ is cryptodeterministic, it determines that of

$$(\mathbf{m} \cdot \mathbf{J})^2 = 2 + [K(\mathbf{m}, \mathbf{n}) - K(\mathbf{m}, \mathbf{n})^2]/2. \tag{14}$$

However, the latter could well be different from

$$(\mathbf{m} \cdot \mathbf{J})^2 = 2 + [K(\mathbf{m}, \mathbf{r}) - K(\mathbf{m}, \mathbf{r})^2]/2, \tag{15}$$

obtained by choosing a *different* triad $(\mathbf{m}, \mathbf{r}, \mathbf{s})$ containing the *same* unit vector $\mathbf{m}$. Up to now, there is nothing surprising: As $K(\mathbf{m}, \mathbf{n})$ does not commute with $K(\mathbf{m}, \mathbf{r})$, these two observables cannot be measured simultaneously; therefore the two values of $(\mathbf{m} \cdot \mathbf{J})^2$ in (14) and (15) result from incompatible setups and may well be different.

There is however a difficulty. Suppose that we measure $(\mathbf{m} \cdot \mathbf{J})^2$ first and only at a later time decide whether $\mathbf{m}$ belongs to the triad $(\mathbf{m}, \mathbf{n}, \mathbf{p})$ or the triad $(\mathbf{m}, \mathbf{r}, \mathbf{s})$. More generally, if $[A, B] = [A, C] = 0$ but $[B, C] \neq 0$, suppose that we measure A first and only at a later time decide whether to measure B or C or none of them. How can the outcome of the measurement of A depend on this *future* decision?

A solution to this puzzle was recently proposed by Kraus[13] in the context of Bell's theorem.[10] Kraus remarks that, if there is an experimental setup to measure A, no contradiction can result from the following assumptions:

(a) The result of a measurement of A is deterministic and is independent of the performance of any compatible simultaneous measurement or any future measurement whatsoever.

(b) The results of any compatible simultaneous measurements are also deterministic and independent of whether or not A is measured.

(c) Nothing however is assumed about the outcomes of future measurements (that is, after A will be measured) or of *impossible* measurements which do *not* commute with A.

In the context of the KS algorithm, this means the following: One should not attempt to *simultaneously* specify the colors of all the points on the unit sphere. Rather, one must specify them *successively*, one triad at a time; and the outcomes for each triad may well depend on the order in which the preceding triads were considered.

What is the physical meaning of this cryptodeterminism? Imagine that some more advanced technology than ours could prepare a setup such that a future HV theory (more detailed than quantum mechanics) would be capable of assigning a well-defined value to a given P_k (while quantum mechanics could only make statistical predictions). Then that supertheory could also assign values to all P_n, with $\mathbf{n}$ orthogonal to $\mathbf{k}$, with no contradiction. However, there would remain many *other* P_n which could not be predicted, and then the premises of the KS theorem would not be satisfied.

In summary, although there can be no *causal*, or *separable* cryptodeterministic theory predicting *all* variables, it might not be impossible to concoct one which would predict *more* than quantum theory. This raises however a serious question. Such a causal, cryptodeterministic supertheory would undoubtedly refer to experimental setups more sophisticated than those which we know how to prepare today and which are adequately described by quantum theory. If this is indeed the case, there is *no reason to demand that the results of individual measurements still obey quantum theory.* For example, if the HV can be driven away from their standard statistical distribution, the result of a measurement of $(\mathbf{n} \cdot \mathbf{J})^2$ may no longer be 0 or 1. More generally, in these hypothetical post-quantum experimental conditions, the notion of "measurement" would acquire a new meaning, about which unfortunately little can be said today.

APPENDIX: THE KOCHEN-SPECKER THEOREM

Imagine a finite number of rays in a 3-dimensional Euclidean space, each one labelled "black" or "white." The goal is to dispose these rays in such a way that it is impossible to fulfill the following requirement: If three rays are mutually orthogonal, exactly one must be black.

LEMMA. Two rays making an angle of 72° cannot both be black.

The proof, which requires the introduction of six auxiliary rays, is given by Fig. 1, in which two points are connected by a line if the corresponding rays are orthogonal. It is easily shown that this graph is geometrically possible if the angle α between rays A and B satisfies $\cos\alpha < 1/3$. Then, if A and B are black, C, D, E, F must be white, therefore G and H must be black—a contradiction.

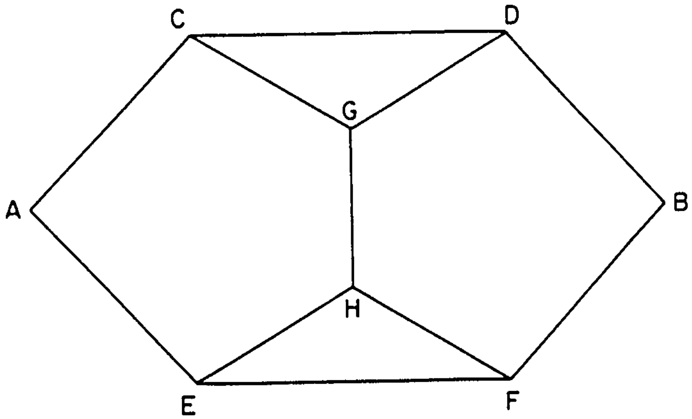

Figure 1: The Kochen-Specker graph used in the proof of the lemma.

Now consider any point O, the "pole," and ten rays separated by angles of 18° along the "equator," as in Fig. 2. Two cases must be distinguished. If O is assumed white, at least one of these equatorial points must be black. Call it F. Then B and J must be white (by virtue of the lemma). Therefore E and G are black (since they are orthogonal to O and also to J and B, respectively). Likewise C and I are white (lemma), and thus D and H are black (orthogonality), and this in turn violates the lemma. This construction requires 10 rays along the equator plus 60 auxiliary rays (6 for each pair along the equator, as shown in Fig. 2).

If, on the other hand, the polar ray is assumed black, then the entire equator is white, and one may consider nine rays along a meridian, as shown in Fig. 3, together with 5 sets of 6 auxiliary rays. The same argument as before then leads to a contradiction among the 39 new rays.

This construction thus requires 70+39=109 rays. Kochen and Specker needed 3×39=117—in a more symmetric but less economical choice of rays.

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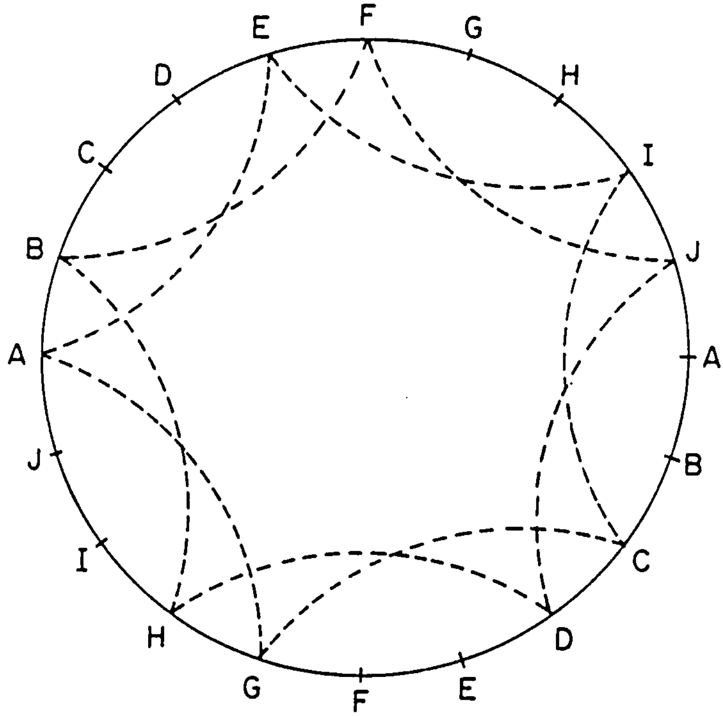

Figure 2: Ten rays separated by angles of 18° along the equator. Note that opposite rays are identified. Each dotted line represents six auxiliary rays, constructed as in Fig. 1.

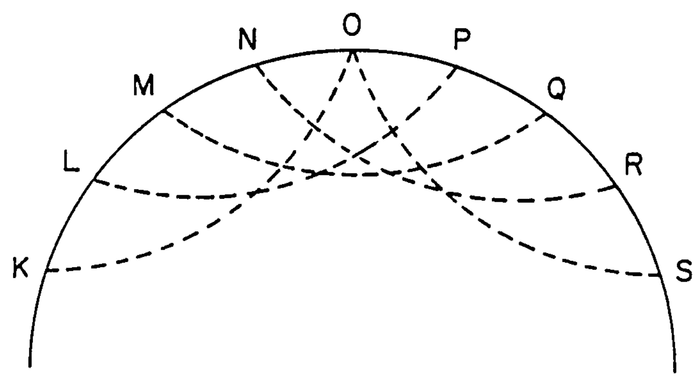

Figure 3: Nine rays separated by 18° along a meridian. Each dotted line represents six auxiliary rays, constructed as in Fig. 1.

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