

Probability

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1 Classical Probability Spaces

A *probability space* [5, 7, 16] collects necessary conditions for reasoning coherently about collections of uncertain events. It consists of a *sample space* Ω , a space of *events* \mathcal{E} , and a *probability measure* μ . In this paper, we will only consider *finite* sets of events: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events \mathcal{E} as 2^Ω , the powerset of Ω . Given the set of events \mathcal{E} , the *probability measure* is a function $\mu : \mathcal{E} \rightarrow [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection E_i , of pairwise disjoint events, $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two-coin probability space). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is “heads” is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$\begin{array}{ll}
 \mu(\emptyset) &= 0 \\
 \mu(\{HH\}) &= 1/3 \\
 \mu(\{HT\}) &= 0 \\
 \mu(\{TH\}) &= 2/3 \\
 \mu(\{TT\}) &= 0 \\
 \mu(\{HH, HT\}) &= 1/3 \\
 \mu(\{HH, TH\}) &= 1 \\
 \mu(\{HH, TT\}) &= 1/3 \\
 \mu(\{HT, TH\}) &= 2/3 \\
 \mu(\{HT, TT\}) &= 0 \\
 \mu(\{TH, TT\}) &= 2/3 \\
 \mu(\{HH, HT, TH\}) &= 1 \\
 \mu(\{HH, HT, TT\}) &= 1/3 \\
 \mu(\{HH, TH, TT\}) &= 1 \\
 \mu(\{HT, TH, TT\}) &= 2/3 \\
 \mu(\{HH, HT, TH, TT\}) &= 1
 \end{array}$$

The assignment satisfies the two constraints for probability measures: the probability of the entire sample space is 1, and the probability of every collection of disjoint events (e.g., $\{HT\} \cup \{TH\}$) is the sum of the individual probabilities.

In a strict computational or experimental setting, one may question the reliance of the definition of probability space on the uncountable and uncomputable real interval $[0, 1]$. This interval includes numbers like $0.h_1h_2h_3\dots$ where h_i is 1 or 0 depending on whether Turing machine M_i halts or not. Such numbers cannot be computed. This interval also includes numbers like $\frac{\pi}{4}$ which can be computed but with increasingly large resources as the precision increases. Therefore it is more appropriate in such situations to consider probability measures mapping events to a finite set of elements computable with a fixed set of resources [2, 14]. The simplest such set, and the one we will consider exclusively in this paper, is the set $\mathcal{L}_2 = \{\text{impossible}, \text{possible}\}$ together with the operation \vee where $x \vee y = \text{impossible}$ if and only if $x = y = \text{impossible}$. In relation to the first definition, one can interpret impossible as the closed interval $[0, 0]$, possible as the open interval $(0, 1]$, and \vee as the addition of intervals. The definition of a probability measure in this case is modified as being a function $\mu : \mathcal{E} \rightarrow \mathcal{L}_2$ such that:

- $\mu(\Omega) = \text{possible}$, and

- for a collection E_i , of pairwise disjoint events, $\mu(\bigcup E_i) = \bigvee \mu(E_i)$.

Example 2 (Two-coin probability space with finite set-valued probability measure.). Under the new set-valued requirement, the probability measure in the first example becomes:

$\mu(\emptyset)$	= impossible	$\mu(\{HT, TH\})$	= possible
$\mu(\{HH\})$	= possible	$\mu(\{HT, TT\})$	= impossible
$\mu(\{HT\})$	= impossible	$\mu(\{TH, TT\})$	= possible
$\mu(\{TH\})$	= possible	$\mu(\{HH, HT, TH\})$	= possible
$\mu(\{TT\})$	= impossible	$\mu(\{HH, HT, TT\})$	= possible
$\mu(\{HH, HT\})$	= possible	$\mu(\{HH, TH, TT\})$	= possible
$\mu(\{HH, TH\})$	= possible	$\mu(\{HT, TH, TT\})$	= possible
$\mu(\{HH, TT\})$	= possible	$\mu(\{HH, HT, TH, TT\})$	= possible

2 Quantum Probability Spaces

The mathematical framework above assumes that one has complete knowledge of the events and their relationships. But even in many classical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot be determined with certainty. In the quantum case, this partial knowledge is compounded by the fact that not all quantum events can be observed simultaneously. Indeed, in the quantum world, there are non-commuting events which cannot even happen simultaneously. To accommodate these more complex situations, we abandon the sample space Ω and define and reason directly about events. A quantum probability space consist of just two components: a set of events \mathcal{E} and a probability measure $\mu : \mathcal{E} \rightarrow [0, 1]$. We give an example before giving the formal definition.

Consider the one-qubit Hilbert space with computational basis $|0\rangle$ and $|1\rangle$ and states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The set of events associated with this Hilbert space consists of all projections including the empty projection \emptyset and the unit projection $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$:

$$\{\emptyset, |0\rangle\langle 0|, |1\rangle\langle 1|, |+\rangle\langle +|, |-\rangle\langle -|, \dots, \mathbb{1}\}$$

Each event is interpreted as a possible post-measurement state of a quantum system as follows: given some arbitrary current quantum state $|\psi\rangle$ to be measured, the event $|0\rangle\langle 0|$ states that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ states that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ states that the post-measurement state will be $|+\rangle$; the event $|-\rangle\langle -|$ states that the post-measurement state will be $|-\rangle$; the event $\mathbb{1}$ states that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the event \emptyset states that the post-measurement state will be the empty state.

In classical probability space, two particular coins can define a probability measure on every event. Similarly, a quantum system can define a quantum probability measure on its event. The difference is that only orthogonal events can test together. For example, asking whether event $\mathbb{1}$ occurs is the same as asking whether event $|0\rangle\langle 0|$ or event $|1\rangle\langle 1|$ occurs simultaneously. However, we can never ask whether event $|0\rangle\langle 0|$ or event $|-\rangle\langle -|$ occurs simultaneously. Similarly to the classical case, a *quantum probability space* [3, 6, 15, 1, 11] can be summarized into the following two rules¹:

- $\mu(\mathbb{1}) = 1$, and
- for a collection of pairwise orthogonal E_i , we have $\mu(\sum_i E_i) = \sum_i \mu(E_i)$.

¹Recently, people extend the domain of μ to all operators \mathcal{A} on \mathcal{H} and consider $\mu : \mathcal{A} \rightarrow \mathbb{C}$ [11, 17]. When an operator $A \in \mathcal{A}$ is Hermitian, $\mu(A)$ is the expectation value of A . We does not take this approach because we want to focus only on probability.

Example 3. [One-qubit Hilbert space] Consider an experiment that with a qubit. The events are described previously. Depending on the systems, we can define different quantum probability measures. Here is a possible one:

$$\begin{array}{rcl} \mu_0(\mathbb{0}) & = & 0 \\ \mu_0(|0\rangle\langle 0|) & = & 1 \\ \mu_0(|1\rangle\langle 1|) & = & 0 \\ & \vdots & \\ \mu_0(\mathbb{1}) & = & 1 \\ \mu_0(|+\rangle\langle +|) & = & 1/2 \\ \mu_0(|-\rangle\langle -|) & = & 1/2 \\ & \vdots & \end{array}$$

The assignment satisfies the two constraints for quantum probability measures: the probability of $\mathbb{1}$ is 1, and the probability of every summation of orthogonal events (e.g., $|0\rangle\langle 0| + |1\rangle\langle 1|$) is the sum of the individual probabilities.

Although it seems that we need an infinite long table to specify the quantum probability measure μ_0 , our μ_0 is actually given by a simple formula $\langle 0|E|0\rangle$. In general, Born discovered each quantum state $|\psi\rangle \in \mathcal{H} \setminus \{0\}$ induces a probability measure $\tilde{\mu}_\psi : \mathcal{E} \rightarrow [0, 1]$ on the space of events defined for any event $E \in \mathcal{E}$ as follows [4, 12]:

$$\tilde{\mu}_\psi(E) = \frac{\langle \psi|E|\psi\rangle}{\langle \psi|\psi\rangle} \quad (1)$$

The Born rule satisfies the following properties:

- It can be extended to mixed states. Given a mixed state represented by a density matrix $\rho = \sum_{j=1}^N q_j \frac{|\psi_j\rangle\langle\psi_j|}{\langle\psi_j|\psi_j\rangle}$, where $\sum_{j=1}^N q_j = 1$, i.e., $\text{Tr}(\rho) = 1$, then the Born rule can be extended to ρ by

$$\tilde{\mu}_\rho(E) = \text{Tr}(\rho E) = \sum_{j=1}^N q_j \tilde{\mu}_{\psi_j}(E) . \quad (2)$$

Notice that $(\{1, \dots, N\}, 2^{\{1, \dots, N\}}, \mu(J) = \sum_{j \in J} q_j)$ is a classical probability space. Therefore, when we discretize the Hilbert space later, we may need to discretize this probability space as well. Yu-Tsung says: Check!!!

- $\tilde{\mu}_\rho$ is a probability measure for all mixed state ρ .
- $\langle \psi|\phi\rangle = 0 \Leftrightarrow \tilde{\mu}_\psi(|\phi\rangle\langle\phi|) = 0$.
- $\tilde{\mu}_\psi(E) = \tilde{\mu}_{\mathbf{U}|\psi\rangle}(\mathbf{U}E\mathbf{U}^\dagger)$, where \mathbf{U} is any unitary map, i.e., $\mathbf{U}^\dagger\mathbf{U} = \mathbb{1}$.

Yu-Tsung says: ...I think I need to check these properties identify an unique Born rule in CQT or something...

Naturally, we may ask: is every probability measure induced from a state by the Born rule? The answer is yes by Gleason's theorem when the dimension ≥ 3 and extend the domain of the Born rule to mixed state [6, 13, 15].

Yu-Tsung says: There are three ways to characterize a quantum probability:

1. The above two conditions as Gleason did.
2. As a linear functional on a $*$ algebra.
3. As induced by a state vector by the Born rule (1).

According to equation (37) in section 4 Quantum Probability in [11], the equalivance between 1. and 2. only hold for $d \geq 3$ because of Gleason' theorem. Therefore, it seems hard to get insight for how to handle Gleason's theorem by $*$ -algebra formulism...

The equalivance between 2. and 3. can be found in Proposition 4.1.1 in [17]. Here, the Born rule $\mu_\psi(E)$ is replaced by a faithful pseudotrace τ and $\mu_\psi(E) = \tau(|\psi\rangle\langle\psi|E)$. The faithful pseudotrace is defined in section 5.2 Decomposition of algebra with a list of conditions. The problem is:

- Lemma 5.2.1 proved there exists a faith pseudotrace, but said nothing about uniqueness. Therefore, these conditions may not uniquely determine the Born rule?
- We don't formulate thing in algrebra... So although we may have an analog or something, we cannot quite sure our conditions will be exactly the same as theirs.

If we admit the above, then the condition (a) to (c) and $\tau(AB) = \tau(BA)$ implies

- Our map $\tilde{\mu} : \mathcal{H} \rightarrow \mathcal{E} \rightarrow [0, 1]$ should satisfy $\tilde{\mu}_\psi$ is a probability measure for all $|\psi\rangle \in \mathcal{H} \setminus \{0\}$.

The condition (a) and (d) implies

- Our map $\tilde{\mu}$ satisfy $\langle\psi|\phi\rangle = 0 \Leftrightarrow \tilde{\mu}_\psi(|\phi\rangle\langle\phi|) = \tilde{0}$, where $\tilde{0}$ is 0 for $[0, 1]$ and $\tilde{0}$ is impossible for $\mathcal{L}_2 = \{\text{impossible}, \text{possible}\}$.

$\tau(AB) = \tau(BA)$ also implies

- $\tilde{\mu}_\psi(|\phi\rangle\langle\phi|) = \tilde{\mu}_{\mathbf{U}|\psi\rangle}(\mathbf{U}|\phi\rangle\langle\phi|\mathbf{U}^\dagger)$, where $|\psi\rangle, |\phi\rangle$ are states and \mathbf{U} is any unitary map, i.e., $\mathbf{U}^\dagger\mathbf{U} = \mathbb{1}$.

Besides, the condition (a) also gives the probability of mixed states form pure states... OK... it seems that I know what I am going to type for DQT, but I think I need to sort them out before type them...

2.1 Plan

In the remainder of the paper, we consider variations of quantum probability spaces motivated by computation of numerical quantities in a world with limited resources:

- Instead of the Hilbert space \mathcal{H} (constructed over the uncountable and uncomputable complex numbers \mathbb{C}), we will consider variants constructed over finite fields [10, 9, 8].
- Instead of real-valued probability measures producing results in the uncountable and uncomputable interval $[0, 1]$, we will consider finite set-valued probablity measures [2, 14].

We will then ask if it is possible to construct variants of quantum probability spaces under these conditions. The main question is related to the definition of probability measures: is it possible to still define a probability measure as a function that depends on a single state? Specifically,

- given a state $|\psi\rangle$, is there a probability measure mapping events to probabilities that only depends on $|\psi\rangle$? In the conventional quantum probability space, the answer is yes by the Born rule [4, 12] and the map is given by: $E \mapsto \langle\psi|E\psi\rangle$.

- given a probability measure μ mapping each event E to a probability, is there a *unique* state ψ such that $\mu(E) = \langle \psi | E \psi \rangle$? In the conventional case, the answer is yes by Gleason's theorem [6, 13, 15].

3 All Continuous or All Discrete

Before we turn to the main part of the paper, we quickly dismiss the possibility of having one but not the other of the discrete variations. Specifically, it is impossible to maintain the Hilbert space and have a finite set-valued probability measure and it is also impossible to have a vector space constructed over a finite field with a real-valued probability measure.

3.1 Hilbert Space with Finite Set-Valued Probability Measure

However, there is a \mathcal{L}_2 -valued probability measure

$$\hat{\mu}_1(E) = \begin{cases} \text{impossible} & , \text{ if } E = |+\rangle\langle +|; \\ \bar{\mu}_0(E) & , \text{ otherwise.} \end{cases}$$

such that $\hat{\mu}_1 \neq \bar{\mu}_\psi$ for all mixed state $|\psi\rangle$.

3.2 Discrete Vector Space with Real-Valued Probability Measure

References

- [1] Samson Abramsky. Big toy models: Representing physical systems as Chu spaces. *CoRR*, abs/0910.2393, 2009.
- [2] Zvi Artstein. Set-valued measures. *Transactions of the American Mathematical Society*, 165:103–125, 1972.
- [3] Garrett Birkhoff and John Von Neumann. The logic of quantum mechanics. *Annals of mathematics*, pages 823–843, 1936.
- [4] Max Born. Zur quantenmechanik der stoßvorgänge (1926). In *Die Deutungen der Quantentheorie*, pages 48–52. Springer, 1984.
- [5] William G. Faris. Appendix: Probability in quantum mechanics. In *The infamous boundary : seven decades of controversy in quantum physics*. Boston : Birkhauser, 1995.
- [6] Andrew Gleason. Measures on the closed subspaces of a hilbert space. *Indiana Univ. Math. J.*, 6:885–893, 1957.
- [7] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, 1994.
- [8] Andrew J Hanson, Gerardo Ortiz, Amr Sabry, and Yu-Tsung Tai. Geometry of discrete quantum computing. *Journal of Physics A: Mathematical and Theoretical*, 46(18):185301, 2013.
- [9] Andrew J Hanson, Gerardo Ortiz, Amr Sabry, and Yu-Tsung Tai. Discrete quantum theories. *Journal of Physics A: Mathematical and Theoretical*, 47(11):115305, 2014.
- [10] Andrew J Hanson, Gerardo Ortiz, Amr Sabry, and Yu-Tsung Tai. Corrigendum: Geometry of discrete quantum computing. *Journal of Physics A: Mathematical and Theoretical*, 49(3):039501, 2015.
- [11] Hans Maassen. Quantum probability and quantum information theory. In *Quantum information, computation and cryptography*, pages 65–108. Springer, 2010.

- [12] N. D. Mermin. *Quantum Computer Science*. Cambridge University Press, 2007.
- [13] A. Peres. *Quantum Theory: Concepts and Methods*. Fundamental Theories of Physics. Springer, 1995.
- [14] Madan L Puri and Dan A Ralescu. Strong law of large numbers with respect to a set-valued probability measure. *The Annals of Probability*, pages 1051–1054, 1983.
- [15] Michael Redhead. *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics*. Oxford University Press, 1987.
- [16] V.K. Rohatgi and A.K.M.E. Saleh. *An Introduction to Probability and Statistics*. Wiley Series in Probability and Statistics. Wiley, 2011.
- [17] Jan Swart. Introduction to quantum probability. *Lecture Notes*, 2013.