DISCRETE QUANTUM THEORIES AND COMPUTING

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DEDICATION

I would like to dedicate my thesis to my parents, Cheng-Tien Tai (戴振泏) and Feng-Ming Chang (張鳳鳴). Thanks for encouraging me to study abroad and all your support during my Ph.D. study.

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TODO. Type a table for acknowledgment so that I would have to write “I thank” so many times here?Many people in both department of Mathematics and Computer Science need to do something more to make my double-major works smoothly. I need to thank all of them, especially directors of graduate study in Mathematics, especially Prof. Christopher Judge and Prof. Matthias Weber; directors of Ph.D. study in Computer Science, especially Prof. Yuqing Melanie Wu (吴愈青) and Prof. Funda Ergun.

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PREFACE

- The latest version of this thesis is in GitHub:

https://github.com/yuttai/Contextuality-and-Nonlocality-in-Discrete-Quantum-Theory/blob/master/dissertation.pdf

- Its source code is typeset using LyX [7] in:

https://github.com/yuttai/Contextuality-and-Nonlocality-in-Discrete-Quantum-Theory/blob/master/dissertation.lyx

- Any comments can be left in the Issue part of the GitHub Repository:

https://github.com/yuttai/Contextuality-and-Nonlocality-in-Discrete-Quantum-Theory/issues

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DISCRETE QUANTUM THEORIES AND COMPUTING

Our primary research interest is to build a quantum computing model characterizing realistic quantum computers. While most of the quantum computing models based on uncomputable numbers, that is, the continuum of real numbers, most of the classical computers in our daily life are digital instead of analog computers. This highlight the necessity to investigate discrete models for quantum theory and computing. Specifically, we start from replacing the continuum of complex numbers by the discrete finite fields. Although we have fruitful results on their geometric implications and computing powers, their probability models are still not completely satisfactory. To address this issue, we further exploited quantum interval-valued probability, and proved an imprecise version of foundational results such as the Gleason and Kochen-Specker theorems.

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CONTENTS

Chapter 1

INTRODUCTION

The marriage of quantum mechanics and computer science first envisioned and popularized by Feynman has created an awkward, but opportune, moment. This embarrassing dilemma was concisely summarized by Prof. Aaronson that one of the the following three statements is false [8, 9]:

(i) Textbook quantum mechanics is correct.

(ii) There does not exist an efficient classical factoring algorithm.

(iii) The extended Church-Turing thesis —that probabilistic Turing machines can efficiently simulate any physically realizable model of computation —is correct [10, 11].

There is overwhelming evidence to support each of these statements. The theoretical framework of quantum mechanics (i) has withstood decades of experimental confirmation. Entire industries are founded on the assumption (ii) that algorithms like RSA are secure and they also have withstood years of attempted attacks [12, 13]. Finally the entire field of complexity theory in computer science which has also withstood years of field testing rests, in essence, on assumption (iii) [14, 15]. And yet at least one of these three statements *must be false*! Indeed if there is a corresponding efficient classical factoring algorithm, then we concede (ii). If it is correct that Shor's efficient factoring algorithm is realizable, and we can prove there is no efficient classical factoring algorithm, then we concede (iii). Otherwise, if we cannot implement Shor's algorithm no matter how hard we try,

textbook quantum mechanics, i.e., (i) may need to be improved by a better theory. It is unlikely that there will be a simple resolution to this awkward situation. It is more likely that the resolution will emerge from deep and careful analyses of the foundations of each field.

When we check the compatibility between quantum mechanics and computer science, we found their fundamental assumptions are different. On one hand, the quantum theory is based on infinitely precise real and complex numbers. Its prediction could fit the physical reality by utilizing error analysis technique although these prediction cannot be completely faithful because of the measurement precision. On the other hand, current computers mostly perform digital computation, except some analog chips communicating with physical world [16]. While the computability of digital parts is faithfully characterized by its theoretical model, the Turing machine [17], the computability of analog chips in reality is far weaker than the theoretical prediction of real computation [18–21]. One of the reason behind this gap is also due to measurement precision, and some of their computability difference cannot be compensated by error analysis technique. All these problems for classical computation could potentially apply to quantum computation. Because we are agnostic about whether the physical reality is ultimately discrete or continuous, we tried to develop two different types of quantum theories and computing models to address these issues. Since the discrete classical computing model faithfully represents digital computers, we first tried to build discrete quantum theories and computing by considering quantum theories and computing over finite fields in Chapter 3. Since error analysis technique cannot always compensate the inevitable problem of measurement precision, we then incorporate the idea of finite precision measurement into the quantum theory in Chapter 4, and hope it could describe the physical reality and computability more faithful.

Chapter 2

CONVENTIONAL QUANTUM THEORY AND COMPUTING

The part of conventional quantum theory (CQT) used by quantum circuit model is described by the following:

(i) D orthonormal basis vectors for a Hilbert space of dimension D,

(ii) the normalized D complex probability amplitude coefficients describing the contribution of each basis vector,

(iii) a set of probability-conserving unitary matrix operators that suffice to describe all required state transformations of a quantum circuit,

(iv) and a measurement framework.

In Sec. 2.1, we focus on the discrete geometric issues raised by the properties (i) and (ii) given above for CQT. In Sec. 2.2, we introduce the important issues of (iv) and the foundations of quantum probability space. In Sec. **??**, we address the property (iii) by describing product and entangled n-qubit states and unitary matrices in quantum circuit model.

2.1 GEOMETRICAL STRUCTURE OF STATES

There are many things that are assumed in CQT, such as the absence of zero norm states for non-zero vectors, and the decomposition of complex amplitudes into a pair of ordinary real numbers. One also typically assumes the existence of a D-dimensional Hilbert space with an orthonormal basis, allowing us to write *pure* states in general as Hilbert space vectors with an Hermitian inner product:

$$|\Psi\rangle \quad = \quad \sum_{i=0}^{D-1} \alpha_i \, |i\rangle \ . \tag{2.1}$$

Here $\alpha_i \in \mathbb{C}$ are complex probability amplitudes, $\vec{\boldsymbol{\alpha}} \in \mathbb{C}^D$, and the $\{|i\rangle\}$ is an orthonormal basis of states obeying $\langle i|k\rangle = \delta_{ik}$.

The meaning of this is that any state $|\Phi\rangle = \sum_{i=0}^{D-1} \beta_i \, |i\rangle$ can be projected onto another state $|\Psi\rangle$ by writing

$$\langle\Phi|\Psi\rangle \quad = \quad \sum_{i=0}^{D-1} \beta_i^* \alpha_i \ , \tag{2.2}$$

thus quantifying the proximity of the two states. (Here * denotes complex conjugation.) This is one of many properties we take for granted in continuum quantum mechanics that challenge us in defining a discrete quantum geometry. To facilitate the transition to DQT carried out in later sections, we concern ourselves first with the properties of the simplest possible abstract state object in CQT, the single qubit state.

2.1.1 TWO-DIMENSIONAL HILBERT SPACE

A state in a two-dimensional Hilbert space, known as a qubit, already provides access to a wealth of geometric information and context. When we write the single qubit state as $|\psi_1\rangle = \alpha_0 \, |0\rangle + \alpha_1 \, |1\rangle$,

a convenience for computing probability and relative state properties is the normalization condition

$$\|\psi_1\|^2 = |\alpha_0|^2 + |\alpha_1|^2 = \alpha_0^*\alpha_0 + \alpha_1^*\alpha_1 = 1\,, \tag{2.3}$$

which identifies α_0 and $\alpha_1 \in \mathbb{C}$ as probability amplitudes and implies the conservation of probability in the closed world spanned by $\{|0\rangle\,, |1\rangle\}$. Note that we distinguish for future use the *norm* $\|\cdot\|$ of a vector from the *modulus* $|\cdot|$ of a complex number. Continuing, we see that if we want only the irreducible state descriptions, we must supplement the process of computing Eq. (2.3) by finding a way to remove the distinction between states that differ only by an overall phase transformation $\mathrm{e}^{\mathrm{i}\theta}$, that is, $\alpha_0\,|0\rangle + \alpha_1\,|1\rangle$ and $\mathrm{e}^{\mathrm{i}\theta}\alpha_0\,|0\rangle + \mathrm{e}^{\mathrm{i}\theta}\alpha_1\,|1\rangle$ are representing the same physical state. This can be accomplished by the Hopf fibration [22–27], which can be written down as follows: let $\alpha_0 = x_0 + \mathrm{i}y_0$ and $\alpha_1 = x_1 + \mathrm{i}y_1$. Then Eq. (2.3) becomes the condition that the four real variables describing a qubit denote a point on the three-sphere $\mathbf{S}^3$ (a 3-manifold) embedded in $\mathbb{R}^4$:

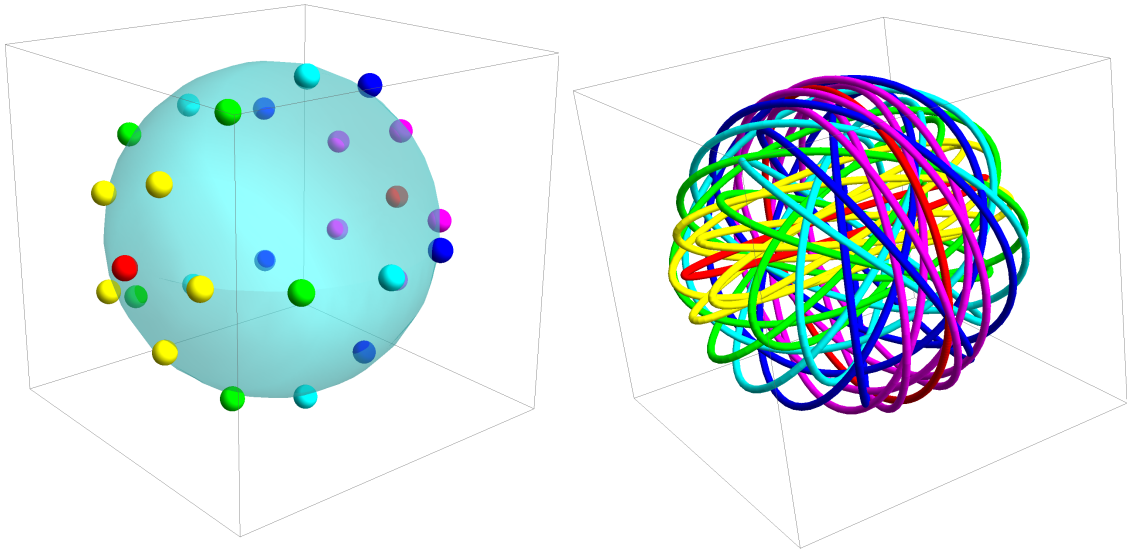
$$x_0{}^2 + y_0{}^2 + x_1{}^2 + y_1{}^2 = 1\,. \tag{2.4}$$

We can reduce 3 degrees of freedom in Eq. (2.4) to 2 degrees of freedom by effectively removing $\mathrm{e}^{\mathrm{i}\theta}$ ("fibering out by the circle $\mathbf{S}^1$"). The standard form of this maps ("the Hopf fibration") is

$$\begin{aligned} X &= 2\,\mathrm{Re}\;\alpha_0\alpha_1^* = 2x_0x_1 + 2y_0y_1\,, \\ Y &= 2\,\mathrm{Im}\;\alpha_0\alpha_1^* = 2x_1y_0 - 2x_0y_1\,, \\ Z &= |\alpha_0|^2 - |\alpha_1|^2 = x_0{}^2 + y_0{}^2 - x_1{}^2 - y_1{}^2\,. \end{aligned} \tag{2.5}$$

By denoting the three-dimensional vector (X, Y, Z) as $\hat{a}$, Eq. (2.4) implies these transformed coordinates obeying

$$\|\hat{a}\|^2 = X^2 + Y^2 + Z^2 = \left(|\alpha_0|^2 + |\alpha_1|^2\right)^2 = 1 \tag{2.6}$$



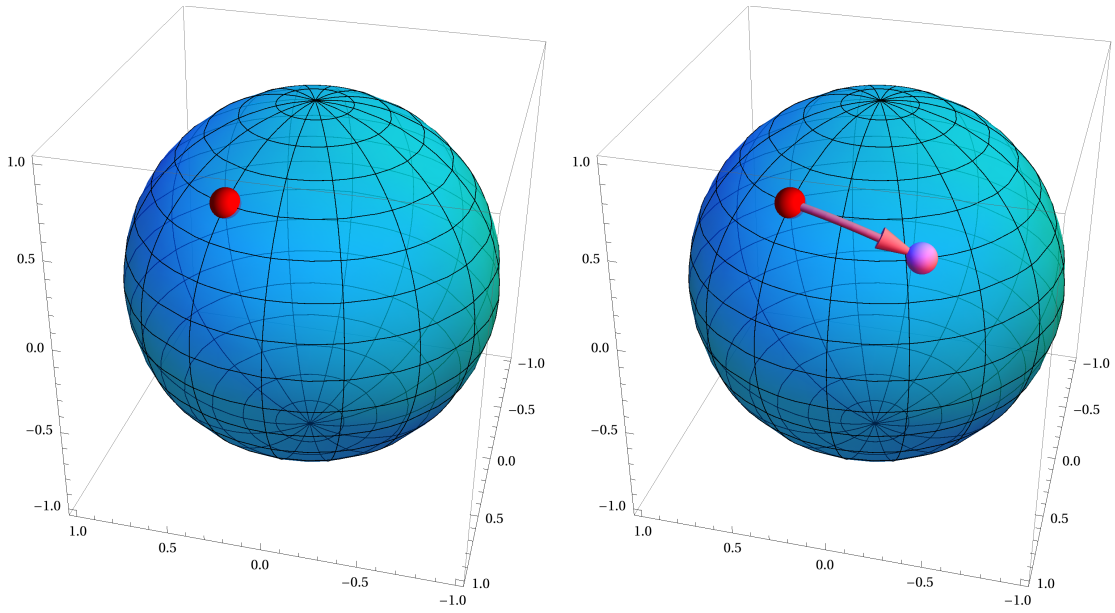
(a) (b)

Figure 2.1: (a) The two-sphere $\mathbf{S}^2$ represented by Eq. (2.6), which is the irreducible space of one-qubit states, along with a representative set of points on the sphere. Each single point on the sphere in (a) corresponding to a circle in (b), and a whole family of circles (the paths of $\mathrm{e}^{\mathrm{i}\theta}$) on the three-sphere $\mathbf{S}^3$ represents the Hopf fibration, Eq. (2.5). Although $\mathbf{S}^3$ cannot be directly embedded in $\mathbb{R}^3$, three-sphere $\mathbf{S}^3$ can be regarded as attaching two three-dimensional ball on two sides of two-sphere $\mathbf{S}^2$. In this way, each circle in $\mathbf{S}^3$ can be represented as a circle in the three-dimensional ball as shown in (b). Moreover, points in (a) are color coded corresponding to circles in (b), e.g., one pole contains the red elliptical circle that would become an infinite-radius circle by a slightly different way to represent $\mathbf{S}^3$ in $\mathbb{R}^3$, and the opposite pole corresponds to the large perfectly round red circle at the equator.

and therefore have only two remaining degrees of freedom describing all possible distinct one-qubit quantum states. In Fig. 2.1, we illustrate schematically the family of circles *each one of which is collapsed to a point* (ϕ, ψ) on the surface $X^2 + Y^2 + Z^2 = 1$ by the Hopf map.

The resulting manifold is the two-sphere $\mathbf{S}^2$ (a 2-manifold) embedded in $\mathbb{R}^3$. If we choose one of many possible coordinate systems describing $\mathbf{S}^3$ via Eq. (2.4) such as

$$(x_0, y_0, x_1, y_1) = (\cos(\theta+\phi)\cos\psi,\ \sin(\theta+\phi)\cos\psi,\ \cos(\theta-\phi)\sin\psi,\ \sin(\theta-\phi)\sin\psi)\ , \tag{2.7}$$



(a) (b)

Figure 2.2: (a) The conventional Bloch sphere with a unique state represented by the point at the red sphere. (b) The geodesic shortest-distance arc connecting two one-qubit quantum states.

where $0 \leq \psi \leq \frac{\pi}{2}$, with $0 \leq \theta + \phi < 2\pi$ and $0 \leq \theta - \phi < 2\pi$, we see that

$$(X, Y, Z) = (\cos(2\phi)\sin(2\psi)\,,\ \sin(2\phi)\sin(2\psi)\,,\ \cos(2\psi))\ . \tag{2.8}$$

Thus the one-qubit state is independent of θ, and we can choose $\theta = \phi$ without loss of generality, reducing the form of the unique one-qubit states to $|\psi_1\rangle = \mathrm{e}^{2\mathrm{i}\phi}\cos\psi\,|0\rangle + \sin\psi\,|1\rangle$, and an irreducible state can be represented as a point on a sphere called the Bloch sphere, as shown in Fig. 2.2(a).

Thus the geometry of a single qubit reduces to transformations among points on $\mathbf{S}^2$, which can be parametrized in an infinite one-parameter family of transformations, one of which is the geodesic or minimal-length transformation. Explicitly, given two one-qubit states denoted by points $\hat{a}$ and $\hat{b}$ on $\mathbf{S}^2$, the shortest rotation carrying $\hat{a}$ to $\hat{b}$ is the SLERP (spherical linear interpolation) [28, 29]

$$S\left(\hat{a}, \hat{b}, t\right) = \hat{a}\,\frac{\sin((1-t)\omega)}{\sin\omega} + \hat{b}\,\frac{\sin(t\omega)}{\sin\omega}\,, \tag{2.9}$$

where $\hat{a}\cdot\hat{b} = \cos\omega$. Figure 2.2(b) illustrates the path traced by a SLERP between two irreducible one-qubit states on the Bloch sphere. Because states in CQC are defined by infinite precision real numbers, it is not possible, even in principle, to make an exact state transition as implied by Fig. 2.2(b). In practice, one has to be content with approximate, typically exponentially expensive, transitions from state to state.

2.1.2 D-DIMENSIONAL HILBERT SPACE

The irreducible states in a D-dimensional Hilbert space are encoded in a similar family of geometric structures known technically as the complex projective space $\mathbb{C}\mathbf{P}^{D-1}$. We obtain these structures starting with the D initially unnormalized complex coefficients of the D-dimensional basis $|\Psi\rangle = \sum_{i=0}^{D-1} \alpha_i |i\rangle$. We then follow the analog of the two-dimensional procedure: Conservation of probability requires that the norm of the vector $\vec{\boldsymbol{\alpha}}$ be normalized to unity:

$$\langle\Psi|\Psi\rangle = \|\vec{\boldsymbol{\alpha}}\|^2 = \sum_{i=0}^{D-1} |\alpha_i|^2 = 1\,. \tag{2.10}$$

Thus the initial equation for the geometry of a quantum state describes a *topological sphere* $\mathbf{S}^{2D-1}$ embedded in $\mathbb{R}^{2D}$. To see this, remember that we can write the real and imaginary parts of α_i as $\alpha_i = x_i + \mathrm{i}y_i$, so

$$\sum_{i=0}^{D-1} |\alpha_i|^2 = \sum_{i=0}^{D-1} {x_i}^2 + {y_i}^2 = 1 \tag{2.11}$$

describes the locus of a $2D$-dimensional real unit vector in $\mathbb{R}^{2D}$, which is by definition $\mathbf{S}^{2D-1}$, the $(2D-1)$-sphere.

This $\mathbf{S}^{2D-1}$ in turn is ambiguous up to the usual overall phase, inducing an $\mathbf{S}^1$ symmetry action, and identifying $\mathbf{S}^{2D-1}$ as an $\mathbf{S}^1$ bundle, whose base space is the $(D-1)$-complex-dimensional projective space $\mathbb{C}\mathbf{P}^{D-1}$. There are thus $2D-2$ irreducible real degrees of freedom ($D-1$ complex degrees of freedom) for a quantum state with a D-dimensional basis, $\{|i\rangle \mid i = 0, \ldots, D-1\}$.

In summary, the full space of a D-dimensional quantum state, including its overall phase defining its relationship to other quantum states, is the topological space $\mathbf{S}^{2D-1}$. For an isolated system, the overall phase is not measurable, and eliminating the phase dependence in turn corresponds to identifying $\mathbf{S}^{2D-1}$ as a circle bundle over the base space $\mathbb{C}\mathbf{P}^{D-1}$, and therefore $\mathbb{C}\mathbf{P}^{D-1}$ defines the $2D-2$ intrinsic, irreducible, degrees of freedom of the isolated D-dimensional state's dynamics. In mathematical notation, this would be written $\mathbf{S}^1 \hookrightarrow \mathbf{S}^{2D-1} \rightarrow \mathbb{C}\mathbf{P}^{D-1}$**TODO. Citation?**. For $D = 2$, the single qubit, we have $2 - 1 = 1$, and the base space of the circle bundle is $\mathbb{C}\mathbf{P}^1 = \mathbf{S}^2$, the usual Bloch sphere. Note that only for $D = 2$ is this actually a sphere-like geometry due to an accident of low-dimensional topology.

2.1.3 EXPLICIT GENERALIZATION OF THE HOPF FIBRATION CONSTRUCTION

For a two-dimensional system, we could easily solve the problem of reducing the full unit-norm space to its irreducible components $\hat{a} = (X, Y, Z)$ characterizing the Bloch sphere. We have just argued that essentially the same process is possible for D-dimensional system: in the abstract argument, we simply identify the family of coefficients $\{\alpha_i\}$ as being the same if they differ only by an overall phase $\mathrm{e}^{\mathrm{i}\theta}$. However, in practice this is not a construction that is easy to realize in a practical computation. We now outline an explicit algorithm for accomplishing the reduction to the irreducible D-dimensional state space $\mathbb{C}\mathbf{P}^{D-1}$; this construction will turn out to be useful for the validation of our discrete results to follow below.

Given a normalized pure state $|\Psi\rangle = \sum_{i=0}^{D-1} \alpha_i |i\rangle$, a natural quantity characterizing an D-

dimensional system is its *density matrix*, $\rho = |\Psi\rangle\langle\Psi|$, or

$$\rho = \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* & \cdots & \alpha_0\alpha_{D-1}^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 & \cdots & \alpha_1\alpha_{D-1}^* \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{D-1}\alpha_0^* & \cdots & \alpha_{D-1}\alpha_{D-2}^* & |\alpha_{D-1}|^2 \end{pmatrix} . \tag{2.12}$$

We can now use the complex generalization of the classical Veronese coordinate system for projective geometry to remove the overall phase ambiguity $\mathrm{e}^{\mathrm{i}\theta}$ from the D-dimensional states. If we take a particular weighting of the elements of the density matrix ρ, we can construct a *unit vector* of real dimension D^2 with the form:

$$\hat{a} = \left(|\alpha_i|^2, \ldots, \sqrt{2}\,\mathrm{Re}\;\alpha_i\alpha_j^*, \ldots, \sqrt{2}\,\mathrm{Im}\;\alpha_i\alpha_j^*, \ldots \right) , \tag{2.13}$$

where

$$\hat{a}\cdot\hat{a} = \sum_{i=0}^{D-1} \left(|\alpha_i|^2 \right)^2 + \sum_{i=0}^{D-1} \sum_{\substack{j=0 \\ j\neq i}}^{D-1} \left(\mathrm{Re}\;\alpha_i\alpha_j^* \right)^2 + \left(\mathrm{Im}\;\alpha_i\alpha_j^* \right)^2 = \left(\sum_{i=0}^{D-1} |\alpha_i|^2 \right) \left(\sum_{j=0}^{D-1} |\alpha_j|^2 \right) = 1 .$$

This construction gives an explicit embedding of the $(D-1)$-dimensional complex, or $(2D-2)$-dimensional real, object in a real space of dimension D^2. However, this is somewhat subtle because the vector is of unit length, so technically the embedding space is a sphere of dimension $D^2 - 1$ embedded in $\mathbb{R}^{D^2}$. For example, the two-dimensional irreducible states could be represented in a four-dimensional embedding, but the magnitude of every coordinate would be one; furthermore, the object embedded in the resulting $\mathbf{S}^3$ is indeed $\mathbf{S}^2$ because we can fix one complex coordinate to be unity, and let one vary, giving a total of two irreducible dimensions. In fact one must choose *two* coordinate patches, one covering one pole of $\mathbf{S}^2$ with coordinates $\alpha_0 = 1 + \mathrm{i}0$ and $\alpha_1 = x_1 + \mathrm{i}y_1$, and the other patch covering the other pole of $\mathbf{S}^2$ with coordinates $\alpha_0 = x_0 + \mathrm{i}y_0$ and $\alpha_1 = 1 + \mathrm{i}0$.

TODO. Explain the last part more clear or add a picture.

We finally see that the irreducible D-dimensional state space $\mathbb{C}\mathbf{P}^{D-1}$ is described by D projectively equivalent coordinates, one of which can always be scaled out to leave $(D-1)$ actual (complex) degrees of freedom. We must choose, in turn, D different local sets of complex variables defined by taking the value $\alpha_k = 1$, with $k = 0, \ldots, D-1$, and allowing the remaining $D-1$ complex (or $2D-2$ real) variables to run free. No single set of coordinates will work, since the submanifold including $\alpha_k = 0$ is undefined and another coordinate system must be chosen to cover that coordinate patch. This is a standard feature of the topology of non-trivial manifolds such as $\mathbb{C}\mathbf{P}^{D-1}$ (see any textbook on geometry [30]).

2.2 QUANTUM PROBABILITY

A *probability space* is a mathematical abstraction specifying the necessary conditions for reasoning coherently about collections of uncertain events [31, 32, 33, 34]. In the quantum case, the events of interest are specified by *projection operators* P satisfying the condition $P^2 = P$. These include the empty projector $\mathbb{0}$, the identity projector $\mathbb{1}$, projectors of the form $|\Phi\rangle\langle\Phi|$ where $|\Phi\rangle$ is a pure quantum state (an element of a Hilbert space $\mathcal{H}$), sums of *orthogonal* projectors P_0 and P_1 with $P_0 P_1 = \mathbb{0}$, and products of *commuting* projectors P_0 and P_1 with $P_0 P_1 = P_1 P_0$. In a quantum probability space [4, 35–38], each event P_i is mapped to a probability $\mu(P_i)$ using a probability measure $\mu : \mathcal{E} \to [0, 1]$, where $\mathcal{E}$ is the set of all events, (i.e., projectors on a given Hilbert space), subject to the following constraints: $\mu(\mathbb{0}) = 0$, $\mu(\mathbb{1}) = 1$, $\mu(\mathbb{1} - P) = 1 - \mu(P)$, and for each pair of *orthogonal* projectors P_0 and P_1:

$$\mu(P_0 + P_1) = \mu(P_0) + \mu(P_1) \ . \tag{2.14}$$

Given a Hilbert space $\mathcal{H}$ of dimension D and a probability assignment for every projector P, we can define the expectation value of an observable $\mathbf{O}$ having spectral decomposition $\mathbf{O} = \sum_{i=1}^{D} \lambda_i P_i$, with eigenvalues $\lambda_i \in \mathbb{R}$, as [32, 39]:

$$\langle \mathbf{O} \rangle_{\mu} = \sum_{i=1}^{D} \lambda_i \mu\left(P_i\right) , \tag{2.15}$$

where the subscript μ might be omitted if it is clear according to the context.

A conventional quantum probability measure can easily be constructed using the Born rule if one knows the current pure unnormalized quantum state $|\Phi\rangle \in \mathcal{H}$; then the Born rule induces a probability measure μ_{Φ}^{B} defined as

$$\mu_{\Phi}^{\mathrm{B}}(P) = \frac{\langle \Phi | P | \Phi \rangle}{\langle \Phi | \Phi \rangle} . \tag{2.16}$$

If $|\Phi\rangle$ is normalized, Eq. (2.16) could be simplified as $\mu_{\Phi}^{\mathrm{B}}(P) = \langle \Phi | P | \Phi \rangle$. For mixed states $\rho = \sum_{j=1}^{N} q_j \left|\Phi_j\right\rangle\left\langle\Phi_j\right|$, where $\left|\Phi_j\right\rangle \in \mathcal{H}$ are normalized, $q_j > 0$, and $\sum_{j=1}^{N} q_j = 1$, the generalized Born rule induces a probability measure μ_{ρ}^{B} defined as $\mu_{\rho}^{\mathrm{B}}\left(P\right) = \mathrm{Tr}\left(\rho P\right) = \sum_{j=1}^{N} q_j \mu_{\Phi_j}^{\mathrm{B}}\left(P\right)$ [40, 41, 39].

As an example, consider a three-dimensional Hilbert space with orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$ and an observable $\mathbf{O}$ with spectral decomposition $\mathbf{O} = |0\rangle\langle 0| + 2\,|1\rangle\langle 1| + 3\,|2\rangle\langle 2|$, i.e., the ith eigenvalue $\lambda_i = i$ and the ith eigenprojector $P_i = |i-1\rangle\langle i-1|$. Two fragments of valid probability measures μ_1 and μ_2 that can be associated with this space are defined in Table 2.1. By the Born rule, the first probability measure corresponds to the quantum system being in the pure state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and the second corresponds to the quantum system being in the state $\frac{|0\rangle\langle 0|+|2\rangle\langle 2|}{2}$. The expectation values of the observable $\mathbf{O}$, $\langle \mathbf{O} \rangle_{\mu_{1,2}}$, are 1.5 in the first case and 2 in the second.

Table 2.1: Two fragments of valid probability measures μ_1 and μ_2.

$\|\Psi\rangle$	$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\mathrm{i}\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|2\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\mathrm{i}\|2\rangle}{\sqrt{2}}$	$\frac{\|1\rangle+\|2\rangle}{\sqrt{2}}$	$\frac{\|1\rangle+\mathrm{i}\|2\rangle}{\sqrt{2}}$	$\ldots$
$\mu_1\left(\|\Psi\rangle\langle\Psi\|\right)$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\ldots$
$\mu_2\left(\|\Psi\rangle\langle\Psi\|\right)$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\ldots$

2.2.1 GLEASON'S THEOREM AND THE PROPERTIES OF THE BORN RULE

A conventional quantum probability measure can be easily constructed from a state ρ according to the Born rule; conversely, this is the only way to obtain a conventional quantum probability measure according to Gleason's theorem [36, 37, 42].

Theorem 2.1 (Gleason's theorem)**.** *In a Hilbert space $\mathcal{H}$ of dimension $D \geq 3$, given a quantum probability measure $\mu : \mathcal{E} \to [0,1]$, there exists a unique mixed state ρ such that $\mu = \mu_\rho^{\mathrm{B}}$.*

When $D = 2$, there are some non-Born quantum probability measures. In particular, some non-Born quantum probability measures mapping every event to either 0 or 1. This kind of probability measure is called a non-contextual hidden variable model, and will be discussed in Sec. 2.4.

Although Gleason's theorem associates every quantum probability measure μ to a mixed state ρ, it is not obvious how to find ρ for a given μ. While hard in general, this task can be done easily if μ satisfies $\mu(P) = 1$ for some projection P when the dimension $D \geq 3$. According to Gleason's theorem, there exists a unique mixed state $\rho = \sum_{j=1}^{N} q_j \left|\Phi_j\right\rangle\left\langle\Phi_j\right|$ such that

$$1 = \mu(P) = \mu_\rho^{\mathrm{B}}(P) = \sum_{j=1}^{N} q_j \mu_{\Phi_j}^{\mathrm{B}}(P) = \sum_{j=1}^{N} q_j \left\langle\Phi_j\right|P\left|\Phi_j\right\rangle . \tag{2.17}$$

Together with the fact that $0 \leq \left\langle\Phi_j\right|P\left|\Phi_j\right\rangle \leq 1$, $q_j > 0$, and $\sum_{j=1}^{N} q_j = 1$, we must conclude that

$\langle\Phi_j|P|\Phi_j\rangle = 1$ for all j, and thus $P|\Phi_j\rangle = |\Phi_j\rangle$. Hence,

$$P\rho = \sum_{j=1}^{N} q_j P|\Phi_j\rangle\langle\Phi_j| = \sum_{j=1}^{N} q_j |\Phi_j\rangle\langle\Phi_j| = \rho\,. \tag{2.18}$$

Conversely, $P\rho = \rho$ also implies $\mu_\rho^{\mathrm{B}}(P) = 1$ because the trace of any mixed state is 1. The whole idea can then be summarized as the following proposition.

Proposition 2.1. $\mu_\rho^{\mathrm{B}}(P) = 1$ *if and only if* $P\rho = \rho$.

The final property is that the Born rule is basis independent [32, 42]: **TODO. Update the basis independent part with more details...**

Proposition 2.2. $\mu_{U\rho U^\dagger}^{\mathrm{B}}(UPU^\dagger) = \mu_\rho^{\mathrm{B}}(P)$, *where* U *is any unitary map.*

In Sec. 3.9, we will use an analog of Gleason's theorem and these two properties to deduce that there is no "sensible" real-valued Born rule over finite fields.

2.2.2 FUZZY MEASUREMENTS

The quantum probability postulates assume a mathematical idealization in which quantum states and measurements are both infinitely precise, i.e., *sharp*. In an actual experimental setup with an ensemble of quantum states that would ideally be identical, but are not actually identically prepared, with imperfections and inaccuracies in measuring devices, an experimenter might not be able to determine that the probability of an event P is precisely 0.5. To address this issue, we investigate the cardinal probability in Sec. 3.7 and the interval-valued probability in Chapter 4.

The quantum expectation value can also used to decide whether a state is entangled or not for multipartite systems as we describe in the following section.

2.3 THE GEOMETRY OF ENTANGLEMENT

Entanglement may be regarded as one of the main characteristics distinguishing quantum from classical mechanics. Entanglement involves quantum correlations such that the measurement outcomes in one subsystem are related to the measurement outcomes in another one. To discuss entanglement, we consider a D-dimensional quantum system composed of n-qubit subsystems, i.e., $D = 2^n$. A pure state of the total system $|\Psi\rangle$ is said to be entangled if it cannot be written as a product of states of each subsystem [42, 32, 39]. That is, a state $|\Psi\rangle$ is entangled if $|\Psi\rangle \neq |\psi_1\rangle \otimes \cdots \otimes \left|\psi_j\right\rangle \otimes \cdots \otimes |\psi_n\rangle$, where $\left|\psi_j\right\rangle$ refers to an arbitrary state of the j-th qubit, and $\otimes$ represents the tensor product. This is equivalent to saying that if one calculates the reduced density operator ρ_j of the j-th subsystem by tracing out all the other subsystems,

$$\rho_j = \mathrm{Tr}_{\{1,\cdots,j-1,j+1,\cdots,n\}}\left(\rho\right) , \tag{2.19}$$

with $j = 1, \cdots, n$ and $\rho = |\Psi\rangle\langle\Psi|$, the normalized state $|\Psi\rangle$ is entangled if and only if at least one subsystem state is *mixed*, i.e., $\mathrm{Tr}_j\left(\rho_j^2\right) < 1$ [32, 39].

The reduced density operator could be expressed explicitly by the expectation value of the Pauli operators. Therefore, we can decide whether a system is entangled or not by examining these expectation values. Let σ_η^j be the Pauli operators acting on the j-th spin [32],

$$\sigma_\eta^j = \overbrace{\sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \underbrace{\sigma_\eta}_{j^{\text{th}}\text{ factors}} \otimes \sigma_0 \otimes \cdots \otimes \sigma_0}^{n\text{ factors}} , \tag{2.20}$$

and $\left\langle\sigma_\eta^j\right\rangle$ be the corresponding expectation value, $\left\langle\sigma_\eta^j\right\rangle = \left\langle\Psi\middle|\sigma_\eta^j\middle|\Psi\right\rangle$, where $\eta = x$, y, and z, and

$$\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle 1| , \qquad \sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1| , \tag{2.21a}$$

$$\sigma_y = \mathrm{i}\,|1\rangle\langle 0| - \mathrm{i}\,|0\rangle\langle 1| , \qquad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| . \tag{2.21b}$$

For example, given a normalized two-qubit system $|\Psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, some of its expectation values are

$$\begin{aligned}
\langle\sigma_0^1\rangle &= |\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1\,,\\
\langle\sigma_x^1\rangle &= \alpha_{00}\alpha_{10}^* + \alpha_{01}\alpha_{11}^* + \alpha_{10}\alpha_{00}^* + \alpha_{11}\alpha_{01}^*\,,\\
\langle\sigma_y^1\rangle &= -\alpha_{00}\alpha_{10}^*\mathrm{i} - \alpha_{01}\alpha_{11}^*\mathrm{i} + \alpha_{10}\alpha_{00}^*\mathrm{i} + \alpha_{11}\alpha_{01}^*\mathrm{i}\,,\\
\langle\sigma_z^1\rangle &= |\alpha_{00}|^2 + |\alpha_{01}|^2 - |\alpha_{10}|^2 - |\alpha_{11}|^2\,.
\end{aligned} \tag{2.22}$$

Then, the reduced density operator ρ_1 can be expressed by these expectation values as following:

$$\begin{aligned}
\rho_1 =& \mathrm{Tr}_{\{2\}}\left(|\Psi\rangle\langle\Psi|\right)\\
=& \left(|\alpha_{00}|^2 + |\alpha_{01}|^2\right)|0\rangle\langle 0| + \left(\alpha_{00}\alpha_{10}^* + \alpha_{01}\alpha_{11}^*\right)|0\rangle\langle 1|\\
&+ \left(\alpha_{10}\alpha_{00}^* + \alpha_{11}\alpha_{01}^*\right)|1\rangle\langle 0| + \left(|\alpha_{10}|^2 + |\alpha_{11}|^2\right)|1\rangle\langle 1|\\
=& \frac{\langle\sigma_0^1\rangle\,\sigma_0 + \langle\sigma_x^1\rangle\,\sigma_x + \langle\sigma_y^1\rangle\,\sigma_y + \langle\sigma_z^1\rangle\,\sigma_z}{2}\,.
\end{aligned} \tag{2.23}$$

In general**TODO. Ask Gerardo why?**, the reduced density operator ρ_j of the j-th subsystem can always be expressed as

$$\rho_j = \frac{1}{2}\sum_{\eta=0,x,y,z}\left\langle\sigma_\eta^j\right\rangle\sigma_\eta\,, \tag{2.24}$$

and its coefficients can be summarized as the vector

$$\mathbf{X}_j = \left(\left\langle\sigma_x^j\right\rangle, \left\langle\sigma_y^j\right\rangle, \left\langle\sigma_z^j\right\rangle\right) \in \mathbb{R}^3 \tag{2.25}$$

that allows a geometric representation of each reduced state in $\mathbb{R}^3$, satisfying $0 \leq \left\|\mathbf{X}_j\right\| \leq 1$. Since $\mathrm{Tr}_j\left(\rho_j^2\right) = \frac{1}{2}\left(1 + \left\|\mathbf{X}_j\right\|^2\right)$, the state $|\Psi\rangle$ is entangled if $\left\|\mathbf{X}_j\right\| < 1$ for at least one j, represented by a point *inside* the corresponding local Bloch sphere embedded in $\mathbb{R}^3$. One may therefore consider $|\Psi\rangle$ to be maximally entangled if $\left\|\mathbf{X}_j\right\| = 0$ for all j. On the other hand, the state $|\Psi\rangle$ is unentangled

(i.e., a product state) if $\|\mathbf{X}_j\| = 1$ for all j, corresponding to points lying on the surface of the Bloch spheres.

A natural geometric measure of multipartite entanglement is obtained by defining the *purity of a state relative to a set of observables* [43, 44]. If the set is chosen to be the set of *all local observables*, i.e., corresponding to each of the subsystems that compose the actual system, one recovers the standard notion of entanglement for multipartite systems. For example, if the system consists of n qubits, we obtain a measure of conventional entanglement by calculating the purity relative to the semi-simple Lie algebra $\mathfrak{h}$ spanned by $\left\{\sigma_x^1, \sigma_y^1, \sigma_z^1, \ldots, \sigma_x^n, \sigma_y^n, \sigma_z^n\right\}$,

$$P_{\mathfrak{h}} = \frac{1}{n}\sum_{j=1}^{n}\sum_{\eta=x,y,z}\left\langle\sigma_\eta^j\right\rangle^2 = \frac{1}{n}\sum_{j=1}^{n}\left\|\mathbf{X}_j\right\|^2 . \tag{2.26}$$

Since the norm of the geometric representation state $\|\mathbf{X}_j\|$ defined in Eq. (2.25) is between 0 and 1, we have $0 \leq P_{\mathfrak{h}} \leq 1$, where $\frac{1}{n}$ in Eq. (2.26) is just a normalization factor. All the product states of the form $|\Psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$, have maximum purity (i.e., $P_{\mathfrak{h}} = 1$). Other states such as the Greenberger-Horne-Zeilinger state $|\Psi\rangle = |\mathsf{GHZ}_n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes \cdots \otimes |0\rangle + |1\rangle \otimes \cdots \otimes |1\rangle\right)$ are (maximally) entangled relative to the set of local observables (i.e., $P_{\mathfrak{h}} = 0$).

Different entanglement measures are obtained when an algebra $\mathfrak{h}$ different from the local observables is chosen. An obvious example, in particular, is given by the set of all observables. In this case, the purity takes its maximum value independently of the pure quantum state [43, 44], expressing the fact that any state is a generalized coherent state of the Lie algebra of all observables.

2.4 HIDDEN VARIABLE MODEL AND QUANTUM CONTEXTUALITY

TODO. Explain enough background knowledge to support the discussion about contextuality in the end of Sec. 3.2. I have typed some literature review in writing course, but I need to

decide whether it could be used or not before finish Sec. 2.2.

Chapter 3

QUANTUM THEORIES AND COMPUTING OVER FINITE FIELDS

3.1 FUNDAMENTALS OF FINITE FIELDS

3.1.1 BACKGROUND

A field $\mathbb{F}$ is an algebraic structure consisting of a set of elements equipped with the operations of addition, subtraction, multiplication, and division [45, 46, 47]. Fields may contain an infinite or a finite number of elements. The rational $\mathbb{Q}$, real $\mathbb{R}$, and complex numbers $\mathbb{C}$ are examples of infinite fields, while the set $\mathbb{F}_3 = \{0, 1, 2\}$, under multiplication and addition modulo 3, is an example of a finite field.

There are two distinguished elements in a field, the addition identity 0, and the multiplication identity 1. Given the field $\mathbb{F}$, the closed operations of addition, "$+$," and multiplication, "$*$," satisfy the following set of axioms:

1. $\mathbb{F}$ is an Abelian group under the addition operation $+$ (additive group);

2. The multiplication operation $*$ is associative and commutative. The field has a multiplicative identity and the property that every nonzero element has a multiplicative inverse;

3. Distributive laws: For all $a, b, c \in \mathbb{F}$

$$a * (b + c) = a * b + a * c\,, \qquad (b + c) * a = b * a + c * a\,. \tag{3.1}$$

From now on, unless specified, we will omit the symbol $*$ whenever we multiply two elements of a field.

Finite fields of q elements, $\mathbb{F}_q = \{0, \ldots, q-1\}$, will play a special role in this work. A simple explicit example is $\mathbb{F}_3$ with the following addition and multiplication tables:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

3.1.2 CYCLIC PROPERTIES OF FINITE FIELDS

The characteristic of a field is the least positive integer m such that $m = 1 + 1 + 1 + \cdots + 1 = 0$, and if no such m exists we say that the field has characteristic zero (which is the case for $\mathbb{R}$ for example). It turns out that if the characteristic is non-zero, it must be a prime p. For every prime p and positive integer r there is a finite field $\mathbb{F}_{p^r}$ of size $q = p^r$ and characteristic p, which is unique up to field isomorphism [22, 45]. The exponent r is known as the *degree* of the field over its prime subfield[1] [48]. If the characteristic p is an arbitrary prime number, we call the field *unrestricted*.

For every $a \in \mathbb{F}_q$, $a \neq 0$, then $a^{q-1} = 1$, implying the Frobenius endomorphism (also a consequence of Fermat's little theorem) $a^q = a$, which in turn permits us to write the multiplicative inverse of any non-zero element in the field as $a^{-1} = a^{q-2}$, since $a^{q-2}a = a^{q-1} = 1$. Every subfield of the field $\mathbb{F}_q$, of size $q = p^r$, has $p^{r'}$ elements with some r' dividing r, and for a given r' it is unique.

[1] Fields $\mathbb{F}_q$ where q is a power of a prime p, i.e., $q = p^r$, are known as Galois fields.

3.2 MODAL QUANTUM THEORY

Recently, Schumacher and Westmoreland [49, 50] and Chang et al. [51, 52] defined versions of quantum theory over *unrestricted* finite fields, which they call modal quantum theories (MQT) or Galois field quantum theories. Such theories retain several key quantum characteristics including notions of superposition, interference, entanglement, and mixed states, along with time evolution using invertible linear operators, complementarity of incompatible observables, exclusion of local hidden variable theories, impossibility of cloning quantum states, and the presence of natural counterparts of quantum information protocols such as superdense coding and teleportation. These modal theories are obtained by collapsing the Hilbert space structure over the field of complex numbers to that of a vector space over an *unrestricted* finite field. In the resulting structure, all non-zero vectors represent valid quantum states, and the evolution of a closed quantum system is described by *arbitrary* invertible linear maps.

Specifically, consider a one-qubit system with basis vectors $|0\rangle$ and $|1\rangle$. In conventional quantum theory, there exists an infinite number of states for a qubit of the form $\alpha_0 |0\rangle + \alpha_1 |1\rangle$, with α_0 and α_1 elements of the underlying field of complex numbers subject to the normalization condition $|\alpha_0|^2 + |\alpha_1|^2 = 1$. Moving to a finite field immediately limits the set of possible states as the coefficients α_0 and α_1 are now drawn from a finite set. In particular, in the field $\mathbb{F}_2 = \{0, 1\}$ of booleans, there are exactly four possible vectors: the zero vector, the vector $|0\rangle$, the vector $|1\rangle$, and the vector $|0\rangle + |1\rangle = |+\rangle$. Since the zero vector is considered non-physical, a one-qubit system can be in one of only three states. The dynamics of these one-qubit states is realized by any invertible linear map, i.e., by any linear map that is guaranteed never to produce the zero vector from a valid state. There are exactly 6 such maps, and their matrix representation with respect to the standard

basis are:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \tag{3.2a}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S^\dagger = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2b}$$

For example,

$$S\left|0\right\rangle = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left|+\right\rangle, \quad S\left|+\right\rangle = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|0\right\rangle. \tag{3.3}$$

This set of maps is clearly quite impoverished compared to the full set of one-qubit unitary maps in conventional quantum theory. In particular, it does not include the Hadamard transformation. However, this set also includes non-unitary maps such as S and $S^\dagger$ that are not allowed in conventional quantum computation.

Measurement in the standard basis is fairly straightforward: measuring $\left|0\right\rangle$ or $\left|1\right\rangle$ deterministically produces the same state while measuring $\left|+\right\rangle$ nondeterministically produces $\left|0\right\rangle$ or $\left|1\right\rangle$ with no assigned probability distribution. When measuring an arbitrary state $\left|\phi\right\rangle \in \{\left|0\right\rangle, \left|1\right\rangle, \left|+\right\rangle\}$ in other bases $\{\left|\psi_0\right\rangle, \left|\psi_1\right\rangle\}$, we first represents $\left|\phi\right\rangle$ as the linear combination of the basis vectors $\beta_0\left|\psi_0\right\rangle + \beta_1\left|\psi_1\right\rangle$, where β_0 and β_1 are elements in the field $\mathbb{F}_2$. If β_i is zero, measuring $\left|\phi\right\rangle$ is impossible to produce $\left|\psi_i\right\rangle$; otherwise, measuring $\left|\phi\right\rangle$ is possible to produce $\left|\psi_i\right\rangle$. Since only possibility and impossibility is predicted by the theory, modal quantum theories are named after these "modal" concepts.

Notice that the measurement process is complicated by the fact that the possibility to produce a basis vector $\left|\psi_i\right\rangle$ depending on the measurement basis. For example, measuring $\left|+\right\rangle$ is possible to produce $\left|0\right\rangle$ in the standard basis $\{\left|0\right\rangle, \left|1\right\rangle\}$ but is impossible to produce $\left|0\right\rangle$ in another basis $\{\left|+\right\rangle, \left|0\right\rangle\}$. In contrast, when measuring a state $\left|\phi\right\rangle$ in CQT, the probability to produce a basis vec-

tor $|\psi_i\rangle$ is completely determined by $|\psi_i\rangle$ and $|\phi\rangle$ no matter $|\psi_i\rangle$ is in which measurement basis. This phenomena of the measurement basis dependence in CQT only exists when discussing quantum contextuality. Despite this kind of "supercontextuality" of MQT, its computational model, modal quantum computing (MQC), having "supernatural" computational power is also far from conventional quantum computing as we will describe next.

3.3 MODAL QUANTUM COMPUTING

To understand the computational implications of the modal quantum theory defined over the field $\mathbb{F}_2$ of booleans, we developed a quantum computing model and established its correspondence to a classical model of logical programming with a feature that has quantum-like behavior [53]. In a conventional logic program, answers produced by different execution paths are collected in a sequence with *no* interference. However, in this modal quantum computing model over $\mathbb{F}_2$, these answers may interfere destructively with one another.

Our computations with this "toy" modal quantum theory showed that it possesses "supernatural" computational power. For example, one can solve a black box version of the UNIQUE-SAT problem [54] in a way that outperforms conventional quantum computing. The classical UNIQUE-SAT problem (also known as USAT or UNAMBIGUOUS-SAT**TODO. Add citation about where these two names come from.**) is the problem of deciding whether a given boolean formula has a satisfying assignment, assuming that it has at most one such assignment [55]. This problem is, in a precise sense [56], just as hard as the general satisfiability problem and hence all problems in the NP complexity class. Our black-box version of the UNIQUE-SAT problem replaces the boolean formula with an arbitrary black box. Solutions to this generalized problem can be used to solve an unstructured database search of size N using $O(\log N)$ black box evaluations by binary search on the database. This algorithm then outperforms the known asymptotic bound $O\left(\sqrt{N}\right)$ for unstructured database search in conventional quantum computing.

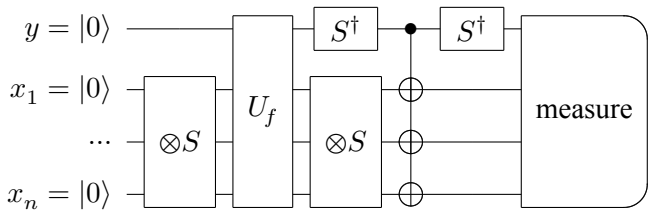


Figure 3.1: Circuit for black box UNIQUE-SAT in modal quantum theory over the field $\mathbb{F}_2$. For further notation see text.

We can prove the unreasonable power of the arbitrary-function UNIQUE-SAT starting with a classical function $f : \mathsf{Bool}^n \to \mathsf{Bool}$ that takes n bits and returns at most one `true` result. To build a quantum algorithm, f is first represented as the Deutsch quantum black box U_f with [32, 57]

$$U_f \left|y\right\rangle \left|\overline{x}\right\rangle = \left|y \oplus f\left(\overline{x}\right)\right\rangle \left|\overline{x}\right\rangle = \begin{cases} \left|y\right\rangle \left|\overline{x}\right\rangle & \text{if } f\left(\overline{x}\right) = \texttt{false}\,; \\ \left|\mathsf{not}\left(y\right)\right\rangle \left|\overline{x}\right\rangle & \text{if } f\left(\overline{x}\right) = \texttt{true}\,, \end{cases} \tag{3.4}$$

where $\overline{x}$ denotes a sequence $x_1, x_2, \ldots, x_n$ of n bits, $\oplus$ is exclusive disjunction, and 0 and 1 are identified as `false` and `true`, respectively. Then, we can give an algorithm (see Fig. 3.1) taking as input such a classical function that decides, deterministically and in a constant number of black box evaluations, whether f is satisfiable or not:

CASE I: f IS UNSATISFIABLE; THE MEASUREMENT DETERMINISTICALLY PRODUCES $|0\rangle\left|\overline{0}\right\rangle$. TODO. This part use $|\overline{a}\rangle = |a_1\rangle \ldots |a_n\rangle$ while previous parts use $|\Psi\rangle = |\psi_1\rangle \cdots \left|\psi_j\right\rangle \cdots |\psi_n\rangle$. Moreover, bar is heavily used in QIVPM discussion later, so maybe not using bar here????The state is initialized to $|0\rangle\left|\overline{0}\right\rangle$, with $\left|\overline{0}\right\rangle = |0\rangle|0\rangle\cdots|0\rangle$, i.e., the tensor product of n $|0\rangle$ states. As Eq. (3.3), applying the map S to each qubit in the second component of the state produces $|0\rangle|\overline{+}\rangle$ where $|\overline{+}\rangle$ denotes the sequence $|+\rangle \ldots |+\rangle$ of length n. Applying U_f to the entire state has no effect since U_f is the identity when f is unsatisfiable. Applying S to each qubit in the second component of the state produces $|0\rangle\left|\overline{0}\right\rangle$. Applying $S^\dagger$ to the first component leaves the state unchanged. As the first component of the state is 0, applying the map σ_0 (which

is the identity) leaves the state unchanged.**TODO. Control-not need to be defined and explained in Sec. ??** Applying $S^\dagger$ to the first component leaves the state unchanged. Measuring the state will deterministically produce $|0\rangle \left|\overline{0}\right\rangle$.

CASE II: f IS SATISFIABLE; THE MEASUREMENT PRODUCES SOME STATE OTHER THAN $|0\rangle \left|\overline{0}\right\rangle$. Assume the function f is satisfiable at some input $a_1, a_2, \ldots, a_n$ denoted $\overline{a}$, and where $|\overline{a}\rangle = |a_1\rangle \ldots |a_n\rangle$. In the second step, the state becomes $|0\rangle |\overline{+}\rangle$ as above. We can write this state as $|0\rangle |\overline{a}\rangle + \Sigma_{\overline{x}\neq\overline{a}} |0\rangle |\overline{x}\rangle$. Applying U_f produces $|1\rangle |\overline{a}\rangle + \Sigma_{\overline{x}\neq\overline{a}} |0\rangle |\overline{x}\rangle$. We can rewrite this state as $|+\rangle |\overline{a}\rangle + \Sigma_{\overline{x}} |0\rangle |\overline{x}\rangle = |+\rangle |\overline{a}\rangle + |0\rangle |\overline{+}\rangle$, where the summation is now over all vectors (notice that $|0\rangle |\overline{a}\rangle + |0\rangle |\overline{a}\rangle$ is the zero vector). Applying S to each qubit in the second component produces $|+\rangle \left|\overline{S(a)}\right\rangle + |0\rangle \left|\overline{0}\right\rangle$. Applying $S^\dagger$ to the first component produces: $|1\rangle \left|\overline{S(a)}\right\rangle + |0\rangle \left|\overline{0}\right\rangle$. Applying control-not gate, which applying σ_0 or σ_x on the second component depending on the first component of the state, and produces

$$|1\rangle \left(\sigma_x \left|\overline{S(a)}\right\rangle\right) + |0\rangle \left(\sigma_0 \left|\overline{0}\right\rangle\right) = |1\rangle \left|\overline{\mathsf{not}\,(S(a))}\right\rangle + |0\rangle \left|\overline{0}\right\rangle \ . \tag{3.5}$$

Applying $S^\dagger$ to the first component produces $|+\rangle \left|\overline{\mathsf{not}\,(S(a))}\right\rangle + |0\rangle \left|\overline{0}\right\rangle$. For the measurement of $|+\rangle \left|\overline{\mathsf{not}\,(S(a))}\right\rangle + |0\rangle \left|\overline{0}\right\rangle$ to be guaranteed to never be $|0\rangle \left|\overline{0}\right\rangle$, we need to verify that $|+\rangle \left|\overline{\mathsf{not}\,(S(a))}\right\rangle$ has one occurrence $|0\rangle \left|\overline{0}\right\rangle$. **TODO. The following need to be rewritten.**This can be easily proved as follows. Since each a_i is either 0 or 1, then each $S(a_i)$ is either $+$ or 1, and hence each $\mathsf{not}\,(S(a_i))$ is either $+$ or 0. The result follows since any state with a combination of $+$ and 0, when expressed in the standard basis, would consist of a superposition containing the state $|0 \ldots\rangle$.

3.4 DISCRETE QUANTUM THEORY (I)

3.4.1 COMPLEXIFIED FINITE FIELDS

Our next objective is to develop more realistic discrete quantum theory variants that exclude "supernatural" algorithms such as the one presented above. Our first such plausible framework [58] is based on complexifiable finite fields. To incorporate complex numbers for quantum amplitudes, we exploit the fact that the polynomial x^2+1 is *irreducible* ($x^2+1=0$ has no solution) over a prime field $\mathbb{F}_p$ with p odd if and only if p is of the form $4\ell+3$, with ℓ a non-negative integer [22, 45, 47]. For example, when $p=3$, x could be 0 or ± 1. Since $0^2+1\neq 0$ and $(\pm 1)^2+1\neq 0$, none of the element in $\mathbb{F}_3$ solves $x^2+1=0$, and x^2+1 is irreducible over $\mathbb{F}_3$. In contrast, $2^2+1=0$ over $\mathbb{F}_5$ so that x^2+1 is reducible.

Since $x^2+1=0$ has no solution in any field $\mathbb{F}_p$ with $p=4\ell+3$, we can extend $\mathbb{F}_p$ to a field $\mathbb{F}_{p^2}$ whose elements can be viewed as discrete complex numbers with the real and imaginary parts in $\mathbb{F}_p$. Therefore, every element in $\mathbb{F}_{p^2}$ can be expressed as $a+b\mathrm{i}$ with $a,b\in\mathbb{F}_p$, and a $\mathbb{F}_{p^2}$ is called a complexified finite field. Since the multiplicative group of any finite field is cyclic [22], there is a generator $g\in\mathbb{F}_{p^2}$ such that every non-zero element $a+b\mathrm{i}$ can also be represented as the power of a generator, i.e., $a+b\mathrm{i}=g^j$ for some j. For example, $1-\mathrm{i}$ is a generator in $\mathbb{F}_{3^2}$ means a particular element $1+\mathrm{i}\in\mathbb{F}_{3^2}$ can be expressed as $1+\mathrm{i}=1-3\mathrm{i}-3+\mathrm{i}=(1-\mathrm{i})^3$. All possible choices of generators in $\mathbb{F}_{3^2}$ is listed in Table 3.1.

In Table 3.1, one can notice that $(a+b\mathrm{i})^3=a-b\mathrm{i}$. In general, the p-th power $(a+b\mathrm{i})^p=a-b\mathrm{i}$ is called the *Frobenius automorphism* acts like complex conjugation $(a+b\mathrm{i})^*=a-b\mathrm{i}$ [22, 45, 59]. Then, we define the *field norm* $\mathsf{N}(\cdot):\mathbb{F}_{p^2}\to\mathbb{F}_p$ as an element $a+\mathrm{i}b$ multiplying its complex conjugation $(a+b\mathrm{i})^*$ [60],

$$\mathsf{N}(a+\mathrm{i}b)=(a+b\mathrm{i})(a+b\mathrm{i})^*=(a+b\mathrm{i})^{p+1}=a^2+b^2\,, \tag{3.6}$$

Table 3.1: Generators in $\mathbb{F}_{3^2}$

j	0	1	2	3	4	5	6	7
$(1+\mathrm{i})^j$	1	$1+\mathrm{i}$	$-\mathrm{i}$	$1-\mathrm{i}$	-1	$-1-\mathrm{i}$	i	$-1+\mathrm{i}$
$(1-\mathrm{i})^j$	1	$1-\mathrm{i}$	i	$1+\mathrm{i}$	-1	$-1+\mathrm{i}$	$-\mathrm{i}$	$-1-\mathrm{i}$
$(-1+\mathrm{i})^j$	1	$-1+\mathrm{i}$	i	$-1-\mathrm{i}$	-1	$1-\mathrm{i}$	$-\mathrm{i}$	$1+\mathrm{i}$
$(-1-\mathrm{i})^j$	1	$-1-\mathrm{i}$	$-\mathrm{i}$	$-1+\mathrm{i}$	-1	$1+\mathrm{i}$	i	$1-\mathrm{i}$

where the square root in the usual definition of norm is avoided because, unlike the continuous case, the square root does not always exist, and the field norm of an element $\mathsf{N}(\cdot)$ should be the direct counterpart of the norm-squared $|\cdot|^2$ in the conventional quantum theory. For example, the field norm of every generator g in Table 3.1 is the same number $\mathsf{N}(g) = g^{3+1} = -1$. In fact, these four generators are the only elements in $\mathbb{F}_{3^2}$ whose field norm is $-1 \in \mathbb{F}_3$. Generally, given any $c \in \mathbb{F}_p$, let $\mathsf{N}^{-1}(\{c\})$ denote the set of element whose field norm is c, i.e., $\mathsf{N}^{-1}(\{c\}) = \left\{\alpha \in \mathbb{F}_{p^2} \;\middle|\; \mathsf{N}(\alpha) = c\right\}$. The set $\mathsf{N}^{-1}(\{c\})$ is the discrete analog of phase-equivalence under the modulus-preserving transformation $z \to \mathrm{e}^{\mathrm{i}\phi} z$, and the number of its elements is characterized by the following proposition.

Proposition 3.1. *Given any $c \in \mathbb{F}_p$, the number of elements in $\mathsf{N}^{-1}(\{c\})$ is $p+1$, i.e., there are always $p+1$ elements in $\mathbb{F}_{p^2}$ whose field norm is c.*

Proof. To prove Proposition 3.1, we start by proving $\mathsf{N}^{-1}(\{c\})$ is non-empty. Consider a special case of the field norm $\mathsf{N}(.)$, namely the real quadratic map $Q(e) = e^2$ taking an arbitrary element $e \in \mathbb{F}_p$ to its square in the field. Since $(\pm 1)^2 = 1$, the image of $Q(e)$ has only $\frac{p+1}{2}$ elements in $\mathbb{F}_p$, including the zero element. We let A be the image of the map $Q(e)$ in $\mathbb{F}_p$, and note that the set A_c resulting from displacing an element $x = b^2$ of A to $c - x = c - b^2$ with $c \in \mathbb{F}_p$ also has $\frac{p+1}{2}$ elements because the result is simply a cyclic shift of element labels. We now observe that

for any non-zero $c \in \mathbb{F}_p$, the sum of the elements in two sets A and A_c is $\frac{p+1}{2} + \frac{p+1}{2} = p+1$, which is greater than the size p of $\mathbb{F}_p$, and so there must be at least one common element such that $a^2 = c - b^2$. Thus every element $c \in \mathbb{F}_p$ is the field norm of some element $\alpha = a + b\mathrm{i} \in \mathbb{F}_{p^2}$ such that $\mathsf{N}(\alpha) = a^2 + b^2 = c$, and $\mathsf{N}^{-1}(\{c\})$ is non-empty.

We then want to show for all non-zero $c \in \mathbb{F}_p$, the size of $\mathsf{N}^{-1}(\{c\})$ is always the same. Given a particular non-zero $c_0 \in \mathbb{F}_p$ and $\alpha_0 \in \mathbb{F}_{p^2}$ with $\mathsf{N}(\alpha_0) = c_0$, consider the map $f(\alpha) = \alpha_0\alpha$. When $\mathsf{N}(\alpha) = 1$, we have [45]

$$\mathsf{N}(f(\alpha)) = \mathsf{N}(\alpha_0\alpha) = \mathsf{N}(\alpha_0)\,\mathsf{N}(\alpha) = c_0 \tag{3.7}$$

so that $f(\alpha) \in \mathsf{N}^{-1}(\{c_0\})$. Since $\mathsf{N}(a + b\mathrm{i}) = 0$ only for $a = b = 0$, α_0 is non-zero, and f is actually a bijection between $\mathsf{N}^{-1}(\{1\})$ and $\mathsf{N}^{-1}(\{c_0\})$. This means the number of elements in $\mathsf{N}^{-1}(\{1\})$ and $\mathsf{N}^{-1}(\{c_0\})$ are the same. Because c_0 can be any non-zero element, the number of elements in the equivalence classes $\mathsf{N}^{-1}(\{c\})$ is always the same.

We can now compute the size of the equivalence class of complex unit-modulus phases corresponding to the Hopf fibration circle. **TODO. Explain the Hopf fibration circle here... ?** Since $\mathbb{F}_{p^2}$ has $p^2 - 1$ non-zero values, and the map $\mathsf{N}(\alpha)$ distributes these equally across the domain of $p-1$ non-zero elements $c \in \mathbb{F}_p$, there are $\frac{p^2-1}{p-1} = p+1$ (non-zero) domain elements in $\mathbb{F}_{p^2}$ for each (non-zero) image element in $\mathbb{F}_p$. We illustrate this graphically in Figure 3.2. Thus the Hopf circle always has size $p+1$, corresponding essentially to a discrete projective line, and that is the size of each equivalence class of the map $\mathsf{N}(\alpha)$ for non-vanishing α, including in particular the map to the unit norm value $c = 1 \in \mathbb{F}_p$. □

3.4.2 VECTOR SPACES

In this section we want to build a theory of discrete vector spaces that approximates as closely as possible the features of conventional quantum theory. Such a structure would ideally consist of the

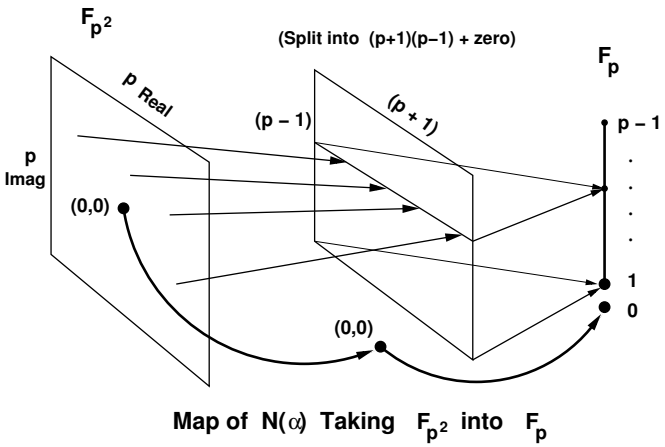


Figure 3.2: Sketch of the map from $\mathbb{F}_{p^2}$ to $\mathbb{F}_p$ using $\mathsf{N}(\alpha)$, showing the decomposition of $\mathbb{F}_{p^2}$ into the zero element $(0,0)$ and the $p^2-1=(p+1)(p-1)$ non-zero elements that map onto the $p-1$ non-zero elements of $\mathbb{F}_p$ with multiplicity $p+1$.

following: (i) a vector space over the field of complex numbers, and (ii) an inner product $\langle\Phi|\Psi\rangle$ associating to each pair of vectors a complex number, and satisfying the following properties:

(A) $\langle\Phi|\Psi\rangle$ is the complex conjugate of $\langle\Psi|\Phi\rangle$;

(B) $\langle\Phi|\Psi\rangle$ is conjugate linear in its first argument and linear in its second argument;

(C) $\langle\Psi|\Psi\rangle$ is always non-negative and is equal to 0 only if $|\Psi\rangle$ is the zero vector.

It turns out that a vector space defined over a finite field cannot have an inner product satisfying the properties above. However, we will introduce an Hermitian "dot product" satisfying some of those properties.

We are interested in the vector space $\mathcal{H}$ of dimension D defined over the complexified field $\mathbb{F}_{p^2}$. Let $|\Psi\rangle = \begin{pmatrix}\alpha_0 & \alpha_1 & \ldots & \alpha_{D-1}\end{pmatrix}^T$ and $|\Phi\rangle = \begin{pmatrix}\beta_0 & \beta_1 & \ldots & \beta_{D-1}\end{pmatrix}^T$ represent vectors in $\mathcal{H}$, with numbers α_i and β_i drawn from $\mathbb{F}_{p^2}$, and where $(\cdot)^T$ is the transpose.

Definition 3.1 (Hermitian dot product)**.** Given vectors $|\Phi\rangle$ and $|\Psi\rangle \in \mathcal{H}$, it can be shown [6, 59] the Hermitian dot product is always reducible to the form

$$\langle\Phi|\Psi\rangle = \sum_{i=0}^{D-1} \beta_i^p \alpha_i \,. \tag{3.8}$$

Two vectors $|\Phi\rangle$ and $|\Psi\rangle \in \mathcal{H}$ are said to be orthogonal if $\langle\Phi|\Psi\rangle = 0$. This product satisfies conditions (A) and (B) for inner products but violates condition (C) since in every finite field there always exists a non-zero vector $|\Psi\rangle$ such that $\langle\Psi|\Psi\rangle = 0$. The reason is that addition in finite fields eventually "wraps around" (because of their cyclic or modular structure), allowing the sum of non-zero elements to be zero. The fraction of non-zero vectors satisfying $\langle\Psi|\Psi\rangle = 0$ decreases with the order p.

For any vector $|\Psi\rangle = \begin{pmatrix} \alpha_0 & \alpha_1 & ... & \alpha_{D-1} \end{pmatrix}^T$, the Hermitian dot product $\langle\Psi|\Psi\rangle$ is equal to $\sum_{i=0}^{D-1} \mathsf{N}(\alpha_i)$, which is the sum of the field norms for the complex coefficients. For convenience, we now extend the field norm to include vector arguments by defining

$$\mathsf{N}(|\Psi\rangle) = \langle\Psi|\Psi\rangle = \sum_{i=0}^{D-1} \mathsf{N}(\alpha_i) \ . \tag{3.9}$$

Although the field norm of a vector can vanish for non-vanishing vectors, if a vector $|\Psi\rangle$ has a non-vanishing field norm c, then $|\Psi\rangle$ can be normalized by utilizing its field norm. Recalled in Sec. 3.4.1, we defined $\mathsf{N}^{-1}(\{c\})$ to be the set of element whose field norm is c. Given any $\alpha \in \mathsf{N}^{-1}(\{c\})$, the field norm of $\frac{|\Psi\rangle}{\alpha}$ is

$$\mathsf{N}\left(\frac{|\Psi\rangle}{\alpha}\right) = \frac{\mathsf{N}(|\Psi\rangle)}{\mathsf{N}(\alpha)} = \frac{c}{c} = 1 \, , \tag{3.10}$$

i.e., $\frac{|\Psi\rangle}{\alpha}$ is normalized. However, since the size of $\mathsf{N}^{-1}(\{c\})$ is $p+1$, we cannot identify a "unique" way to normalize any given vector.

Actually, the similar problem has already happened in conventional quantum theory. For example, assume we want to normalize $|\Psi\rangle = |0\rangle + |1\rangle$. Its inner product with itself is $\langle\Psi|\Psi\rangle = 2$. Since $\left(\pm\sqrt{2}\right)^2 = 2$, both $\frac{|\Psi\rangle}{\sqrt{2}}$ and $\frac{|\Psi\rangle}{-\sqrt{2}}$ are normalized and representing the same state as $|\Psi\rangle$. In this case, we systematically choose the state divided by the positive square root as "the" normalized vector of $|\Psi\rangle$ in conventional quantum theory, and the positive square root function is called the *principal*

Table 3.2: Inverse field norm over $\mathbb{F}_{3^2}$ with respect to the generator $1-\mathrm{i}$

$c = g^{(p+1)k}$	$(p+1)\,k$	k	$\mathsf{N}^{-1}\left(g^{(p+1)k}\right) = g^k$
-1	4	1	$1-\mathrm{i}$
1	0	0	1

branch of $\sqrt{w}$ [61]. In discrete case, we can also systematically choose the *principal inverse field norm* by utilizing a generator $g \in \mathbb{F}_{p^2}$ discussed in Sec. 3.4.1. Because g is a generator, any non-zero element $c \in \mathbb{F}_p \backslash \{0\}$ can be expressed as $g^{(p+1)k}$ where k is an integer and $0 \leq k < p-1$, so we can define the principal inverse field norm $\mathsf{N}^{-1}\left(g^{(p+1)k}\right)$ as g^k. For example, the inverse field norm over $\mathbb{F}_{3^2}$ with respect to the generator $1-\mathrm{i}$ is shown in Table 3.2. Given the non-normalized state $|\Psi\rangle = |0\rangle + |1\rangle$, since its field norm is $\mathsf{N}\left(|\Psi\rangle\right) = \mathsf{N}\left(1\right) + \mathsf{N}\left(1\right) = -1$, it can be normalized as

$$\frac{|\Psi\rangle}{\mathsf{N}^{-1}\left(-1\right)} = \frac{|0\rangle + |1\rangle}{1-\mathrm{i}} = (1+\mathrm{i})\,|0\rangle + (1+\mathrm{i})\,|1\rangle\ . \tag{3.11}$$

3.4.3 IRREDUCIBLE DISCRETE D-DIMENSIONAL STATES: GENERALIZED DISCRETE BLOCH SPHERE

In the one-qubit state with coefficients in $\mathbb{F}_{p^2}$, the discrete analog of the Bloch sphere is constructed by exact analogy to the continuous case: we first require that the coefficients of the single qubit basis obey

$$\mathsf{N}\left(|\psi_1\rangle\right) = \mathsf{N}\left(\alpha_0\right) + \mathsf{N}\left(\alpha_1\right) = 1 \tag{3.12}$$

in the discrete field. We show that there are $p\left(p^2-1\right)$ such values later in the general theorem, Proposition 3.2. Given this requirement, which is similar in form to the conservation of probability, but not as useful due to the lack of orderable probability values, we can immediately conclude that

the discrete analog of the Hopf fibration is again

$$\begin{aligned} X &= 2\,\mathrm{Re}\;\alpha_0\alpha_1^* = 2x_0x_1 + 2y_0y_1\,, \\ Y &= 2\,\mathrm{Im}\;\alpha_0\alpha_1^* = 2x_1y_0 - 2x_0y_1\,, \\ Z &= \mathsf{N}(\alpha_0) - \mathsf{N}(\alpha_1) = {x_0}^2 + {y_0}^2 - {x_1}^2 - {y_1}^2\,. \end{aligned} \tag{3.13}$$

but now with all computations in $(\mathrm{mod}\ p)$. At this point one simply writes down all possible discrete values for the complex numbers (α_0, α_1) satisfying Eq. (3.12) and enumerates those that project to the same value of (X, Y, Z). This equivalence class is the discrete analog of the circle in the complex plane that was eliminated in the continuous case. In Proposition 3.1, we show that $p+1$ discrete values of (α_0, α_1) with unit norm map to the same point under the Hopf map Eq. (3.13)**TODO. Does Proposition 3.1 really show this?**; we may think of these as discrete circles or projective lines of equivalent, physically indistinguishable, complex phase. The surviving $p\,(p-1)$ values of (α_0, α_1) correspond to irreducible physical states of the discrete single qubit system. Thus, for example, choosing the underlying field to be $\mathbb{F}_{3^2}$, there are exactly 6 single-qubit state vectors to populate the Bloch sphere; the four equivalent phase-multiples mapping to each of the six points on the $\mathbb{F}_{3^2}$ Bloch sphere are collapsed and regarded as physically indistinguishable. In Figure 3.3, we plot the irreducible states on the Bloch sphere for $p = 3$, 7, and 11. Note that the Cartesian lengths of the real vectors corresponding to the points on the Bloch sphere vary considerably due to the nature of discrete fields; we have artificially normalized them to a "continuous world" unit radius sphere for conceptual clarity.

3.4.3.1 COUNTING STATES ON THE DISCRETE BLOCH SPHERE

We have the unique opportunity in the finite-field approach to quantum computing to precisely identify and enumerate the physical states. In the conventional theory, as we have seen in Sec. 2.1.3, we employ a generalized Hopf fibration on the normalized states to project out a circle of phase-

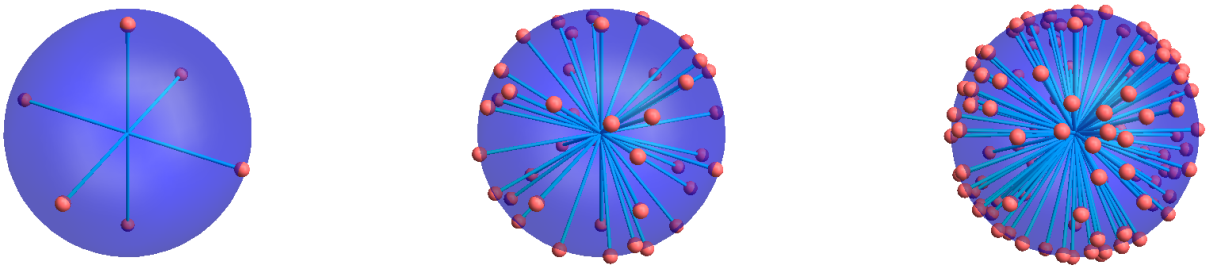


Figure 3.3: Schematically normalized plots of the elements of the discrete Bloch sphere, the irreducible single-qubit (two-dimensional) state vectors with unit norm over the field $\mathbb{F}_{p^2}$. We show the results for $p = 3$, 7, and 11. For example, in $\mathbb{F}_{3^2}$, there are 24 vectors of unit norm, but only the 6 inequivalent classes appear in the plot. The $p + 1 = 4$ equivalent vectors in each class differ only by a complex discrete phase.

equivalent states, yielding the generalized Bloch sphere.

In the introduction to this subsection, we sketched the counting of the irreducible single-qubit discrete states. To count the number of inequivalent discrete states for the general n-qubit case with coefficients in $\mathbb{F}_{p^2}$, we first must find the set of unit-norm states, and then determine the equivalence classes of unit-norm states under discrete phase transformations; we can then enumerate the list of states on the discrete generalized Bloch sphere. By executing computer searches of these spaces, we discovered an hypothesis for a closed-form solution for the counting of the states, and find a rigorous proof of the enumeration.

This process of describing the discrete D-dimensional irreducible states can again be understood geometrically by following the discrete analog of the Hopf fibration. First, we construct the discrete version of the quadratic unit-length form that automatically annihilates the distinction among states differing only by a discrete phase,

$$\hat{a} = \left(\mathsf{N}\left(\alpha_i\right), \ldots, \sqrt{2}\,\mathrm{Re}\;\alpha_i\alpha_j^*, \ldots, \sqrt{2}\,\mathrm{Im}\;\alpha_i\alpha_j^*, \ldots\right) , \tag{3.14}$$

where

$$\hat{a} \cdot \hat{a} = \left(\sum_{i=0}^{D-1} \mathsf{N}\left(\alpha_i\right) \right)^2 = 1 \,. \tag{3.15}$$

From Proposition 3.1, we know that $p+1$ elements of this discrete $\mathbf{S}^{2D-1}$ structure map to the *same point* in $\hat{a}$**TODO. Does Proposition 3.1 really show this?**. Each set of $p+1$ redundant points is, geometrically speaking, the *discrete Hopf fibration circle* living above each *irreducible* point of the D-dimensional state description. These $p+1$ points are interpretable as the p finite points plus the single point at infinity of the projective discrete line (see, e.g., [62]).

The next part of this argument is the determination of the unit-norm states, effectively the space of allowed discrete partitions of unity; we cannot exactly call these "probability-conserving" sectors of the state coefficients since we do not have a well defined notion of probability, but we do have a well-defined notion of partition of unity. Compared to the total number p^{2D} of possible complex integer state vectors that could be chosen, the number of unit-norm states is given by the following proposition. This unit-norm state structure is the discrete analog of $\mathbf{S}^{2D-1}$.

Proposition 3.2. *The number of unit-norm states described by a D-dimensional vector* $\left(\alpha_0, \ldots, \alpha_{D-1}\right)$ *with coefficients* $\alpha_i \in \mathbb{F}_{p^2}$ *is* $p^{D-1}\left(p^D - (-1)^D\right)$.

Proof. Proposition 11.27 in Grove [59] provides the count of the zero-norm states $\zeta\left(D,p\right) = p^{D-1}\left(p^D + (-1)^D\left(p-1\right)\right)$. Since there are p^2 elements $\alpha \in \mathbb{F}_{p^2}$, we must have $\left(p^2\right)^D = p^{2D}$ possible values of a D-dimensional vector $\left(\alpha_0, \ldots, \alpha_{D-1}\right)$. There are $p^2 - 1$ non-zero values of $\alpha \in \mathbb{F}_{p^2}$, and we showed in Proposition 3.1 that $\mathsf{N}\left(\alpha\right)$ maps exactly $p+1$ values in that set to each of the $p-1$ non-zero values in $\mathbb{F}_p$. Therefore, the *unit-norm case* has a count of domain elements that is $\frac{1}{p-1}$ of the total number of non-zero-norm cases,

$$\frac{p^{2D} - \zeta\left(D,p\right)}{p-1} = \frac{p^{2D} - p^{2D-1} - (-1)^D p^{D-1}\left(p-1\right)}{p-1} = p^{D-1}\left(p^D - (-1)^D\right) \,. \tag{3.16}$$

□

Finally, we repeat the last step of the D-dimensional continuous Hopf fibration process for discrete D-dimensional states, eliminating the discrete set of $p+1$ equivalent points that map to the same point $\hat{a}$ on the generalized Bloch sphere. Dividing the tally $p^{D-1}\left(p^D-(-1)^D\right)$ of unit norm states by the $p+1$ elements of each phase-equivalent discrete circle, we find

$$\frac{p^{D-1}\left(p^D-(-1)^D\right)}{p+1} \tag{3.17}$$

as the total count of unique irreducible states in a discrete D-dimensional configuration. The resulting object is precisely the discrete version of $\mathbb{C}\mathbf{P}^{D-1}$, which we might call a *discrete complex projective space* or $\mathbf{D}\mathbb{C}\mathbf{P}^{D-1}$.

3.4.4 GEOMETRY OF ENTANGLED STATES

To discuss entanglement, we consider a D-dimensional quantum system composed of n-qubit subsystems, i.e., $D=2^n$ as usual. Without regard to uniqueness, an n-qubit state with discrete complex coefficients in $\mathbb{F}_{p^2}$ will have the total possible space of coefficients with dimension $p^{2\times 2^n}$ (including the null state). Imposing the condition of a length-one norm in $\mathbb{F}_p$, this number is reduced to $p^{2^n-1}\left(p^{2^n}-1\right)$. The ratio of all the states to the unit-norm states is asymptotically p:

$$\frac{p^{2^n+1}}{p^{2^n}-1}\to p\,, \tag{3.18}$$

so there are roughly p sets of coefficients, for any number of qubits n, that are discarded for each retained unit-length state vector. A factor of $p+1$ more states are discarded in forming the discrete Bloch sphere of irreducible states. Selected plots of the full space compared to both the unit-norm space and the irreducible space for a selection of complexified finite fields are shown in Figure 3.4 for 1, 2, 3, and 4 qubits.

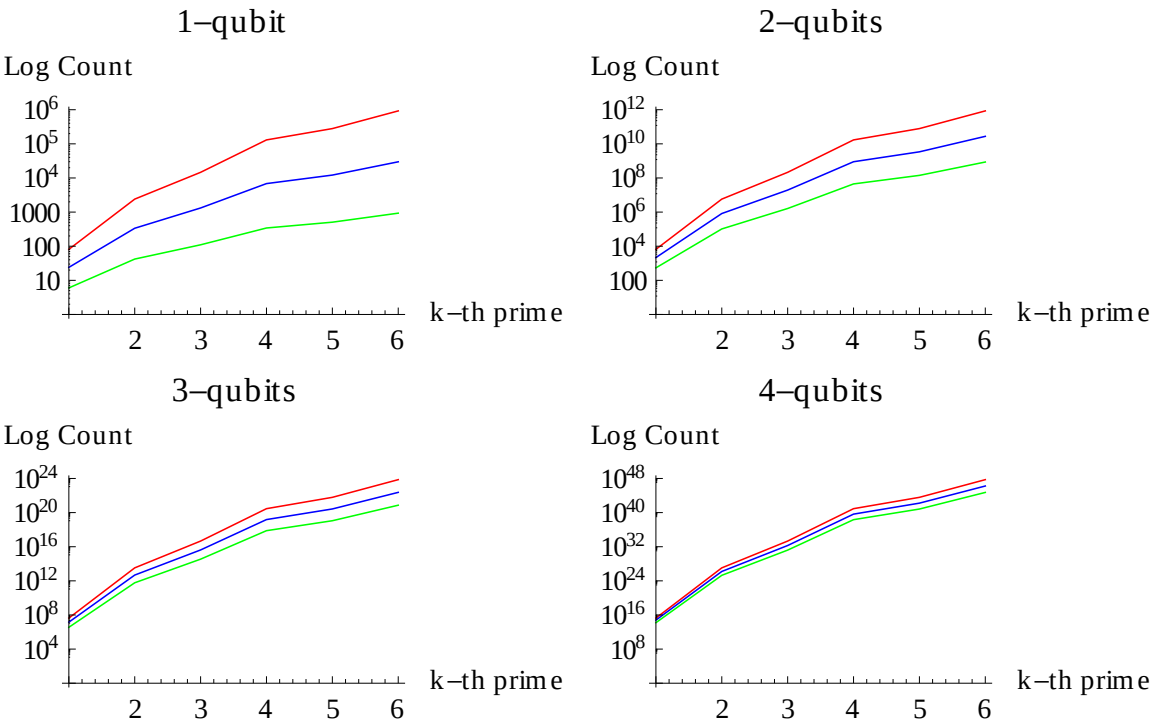


Figure 3.4: Logarithmic plot of the number of discrete unnormalized states (top, in red), vs the number of normalized discrete states (middle, in blue), vs the irreducible states (bottom, in green) for the first 6 $\mathbb{F}_{p^2}$-compatible primes, (3, 7, 11, 19, 23, 31), for the number of qubits 1, 2, 3, and 4.

3.4.4.1 UNENTANGLED VS ENTANGLED DISCRETE STATES

For a given p and the corresponding complexified field $\mathbb{F}_{p^2}$, the n-qubit discrete quantum states with coefficients in $\mathbb{F}_{p^2}$ can be classified by their degree of entanglement to a level of precision that is unavailable in the continuous theory. We look first at the unentangled n-qubit states, which are direct product states of the form

$$|\Psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_j\rangle \otimes \cdots \otimes |\psi_n\rangle \ . \tag{3.19}$$

Without regard to normalization, there are $\left(p^4\right)^n$ possible unentangled states out of the total of $p^{2\times 2^n}$ states noted above. When we normalize the individual product states to unit norm, the norm of the entire n-qubit state becomes the product of those unit norms, and is automatically normalized to one. We have already seen that each single-qubit normalized state in the tensor product Eq. (3.19)

has precisely $p\,(p-1)$ irreducible components due to $D=2$ case in Eq. (3.17).

3.4.4.2 COMPLETELY UNENTANGLED STATES AND THE DISCRETE BLOCH SPHERE

In effect, the irreducible states for unentangled n-qubit configurations reduce to a single Bloch sphere for each one-qubit component $\left|\psi_j\right\rangle$, and thus the whole set of states is defined by an n-tuple of discrete Bloch sphere coordinates. Since each Bloch sphere in $\mathbb{F}_{p^2}$ has $p\,(p-1)$ distinct irreducible components, we have

$$\textbf{Count of Unentangled States} = p^n\,(p-1)^n\ . \tag{3.20}$$

According to Eq. (3.17), we know that the total number of irreducible states (points in the generalized $\mathbf{D}\mathbb{C}\mathbf{P}^{2^n-1}$ Bloch sphere) for an n-qubit state is $\frac{p^{2^n-1}\left(p^{2^n}-1\right)}{p+1}$, and so the number of states containing some measure of entanglement is

$$\textbf{Count of Entangled States} = \frac{p^{2^n-1}\,(p^{2^n}-1)}{p+1} - p^n\,(p-1)^n\ . \tag{3.21}$$

Therefore a very small fraction of the unit norm states are unentangled.

3.4.4.3 MAXIMAL ENTANGLEMENT

Equation (2.26) for $P_{\mathfrak{h}}$ includes a normalization factor $\frac{1}{n}$. In the discrete case, this normalization factor is undefined when $p \mid n$. Equation (2.26) also includes a summation of n terms. In the discrete case, certainly when $p \mid n$ but also in other cases, this summation may vanish in the field even if the individual summands are non-zero. These anomalies are irrelevant for the classification of unentangled states as this computation is performed by directly checking the possibility of direct decomposition into product states, disregarding equation (2.26).

For maximally entangled states, the purity calculation in conventional quantum mechanics using

equation (2.26) produces 0. Given the above observations, in a discrete field, equation (2.26) may be undefined or may report a purity of 0 even for partially entangled states. For example, the normalized 5-qubit state $|\Psi\rangle = (1-\mathrm{i})\left(|00\rangle + |11\rangle\right) \otimes |000\rangle$ has $P_{\mathfrak{h}} = 0$ for $p = 3$, and is not maximally entangled because only the first two qubits are entangled. In the discrete case, we therefore check for maximally entangled states using the following equations [5],

$$\forall j, \forall \eta \in \{x, y, z\}\,, \left\langle \sigma_{\eta}^{j} \right\rangle^{2} = 0\,, \tag{3.22}$$

which avoids the normalization factor and simply checks that each summand is 0.

We now implement these procedures to enumerate the maximally entangled states for the specific cases for $n = 2, 3$ and compare these to the counts for product states. We have verified explicitly in Eq. (3.20) that the numbers of unit-norm product states for $n = 2$, $p = \{3, 7, 11, 19, ...\}$ are

$$(p+1)p^2(p-1)^2 = \{144, 14\,112, 145\,200, 2339\,280, ...\}\,, \tag{3.23}$$

and for general n, $(p+1)p^n(p-1)^n$. The irreducible state counts are reduced by $(p+1)$, giving

$$p^2(p-1)^2 = \{36, 1764, 12\,100, 116\,964, ...\}\,, \tag{3.24}$$

and in general for n-qubits, there are $p^n\,(p-1)^n$ instances of pure product states.

Performing the computation using equation (3.22), we find the numbers of maximally entangled states for two qubits to be

$$p\,(p^2-1)\,(p+1) = \{96, 2688, 15\,840, 136\,800, ...\}\ . \tag{3.25}$$

The irreducible state counts for maximal entanglement are reduced by $(p+1)$, giving, for $n=2$,

$$p\left(p^2-1\right)=\{24,336,1320,6840,...\}\ . \tag{3.26}$$

For three qubits, there are $p^3\left(p^4-1\right)(p+1)$ (total) and $p^3\left(p^4-1\right)$ (irreducible) instances of pure maximally entangled states, while the general formula for 4-qubit states remains unclear.

Therefore, the ratio of maximally entangled to product states is

$$\frac{\textbf{Max entangled}}{\textbf{Product}}=\frac{p+1}{p\left(p-1\right)}\text{ and }\frac{\left(p^2+1\right)(p+1)}{\left(p-1\right)^2} \tag{3.27}$$

for $n=2$ and 3, respectively.

3.5 DISCRETE QUANTUM COMPUTING (I)

Given a complexified finite field $\mathbb{F}_{p^2}$ and its Hermitian dot product (Eq. (3.8)) much of the structure of conventional quantum computing can be recovered. For example, the smallest field $\mathbb{F}_{3^2}$ is already rich enough to express the standard Deutsch-Jozsa algorithm [32, 39, 63], which requires only normalized versions of vectors or matrices with the scalars 0, 1, and -1. Similarly, other deterministic quantum algorithms (algorithms for which we may determine the outcome with certainty), such as Simon's [39, 41, 64] and Bernstein-Vazirani [10, 41], perform as desired. In the following subsection, we will present the discrete Deutsch algorithm as an example. However, this quantum computing model is still different from the conventional one. On one hand, algorithms such as Grover's search [39, 41, 65] will not work in the usual way because we lack (the notion of) ordered angles and probability in general. On the other hand, this computational model still leads to excessive computational power for the unstructured database search problem for certain database sizes.**TODO. Gil Kalai thought it would be interested if we can so Simon's algorithm in DQC(I). Since I need to re-type everything, let's postpone this idea for a while...**

Table 3.3: Possible f for Deutsch black box U_f

Input	f_1	f_2	f_3	f_4
`false`	`false`	`false`	`true`	`true`
`true`	`false`	`true`	`false`	`true`
constant or balanced?	constant	balanced	balanced	constant

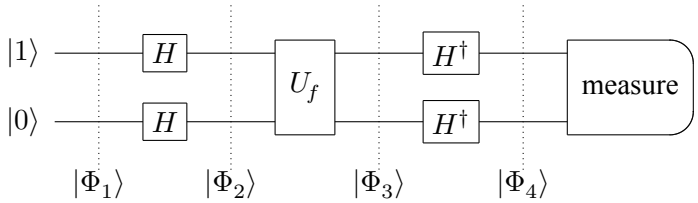


Figure 3.5: Quantum Circuit for Deutsch Algorithm.

3.5.1 DISCRETE DEUTSCH ALGORITHM

Although having no realistic application, the Deutsch algorithm is the first quantum algorithm which outperforms any possible classical algorithm for the Deutsch problem [32, 41, 57]. The Deutsch problem is to decide whether a function $f : \mathsf{Bool} \to \mathsf{Bool}$ is constant or balanced. As listed in Table 3.3, we have only 4 different f: 2 of them are constant while another 2 are balanced. Similar to our UNIQUE-SAT algorithm in Sec. 3.3, we start by representing f as a Deutsch black box U_f in the middle of the quantum circuit, Figure 3.5. To explain why this circuit solves the Deutsch problem, we then compute the state in each step explicitly, and express the Dirac bracket notation with its matrix representation in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ parallelly. **TODO. We typeset U_f either directly or use macro \uf, but the definition of \uf has a negative space between U and f which is different from U_{f}... I need to understand whether the negative space is necessary or not, or whether the negative space has any semantic meaning...**

First, a 2-qubit pure state is initialized to $|\Phi_1\rangle = |1\rangle\,|0\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

Second, on both initialized qubits, we apply the Hadamard matrix $H = \frac{1}{\alpha}\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$ over $\mathbb{F}_{p^2}$,

where $\alpha = \mathsf{N}^{-1}(2)$ is the principal inverse field norm which is used to replace the square root $\sqrt{2}$ in the conventional Hadamard matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$ as discussed in Sec. 3.4.2. The second step produces

$$|\Phi_2\rangle = (H \otimes H)|\Phi_1\rangle = \left[\frac{1}{\alpha}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\right] \otimes \left[\frac{1}{\alpha}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\right] = |-\rangle|+\rangle\ ,$$

where

$$|+\rangle = \frac{1}{\alpha}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{|0\rangle + |1\rangle}{\alpha}\,, \qquad\qquad |-\rangle = \frac{1}{\alpha}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{|0\rangle - |1\rangle}{\alpha}\,. \tag{3.28}$$

Third, the Deutsch black box U_f is applied to the state $|\Phi_2\rangle$. According to Eq. (3.4), these Deutsch black box will be used to apply exclusive disjunction on the first qubit, where we respectively identify `false` and `true` as 0 and 1 as usual. Since our first qubit is $|-\rangle$, the value $f(x)$ could be moved outside as a phase no matter $f(x)$ is `false` or `true`:

$$\begin{aligned} U_f|-\rangle|x\rangle &= \tfrac{1}{\alpha}\left[|0 \oplus f(x)\rangle|x\rangle - |1 \oplus f(x)\rangle|x\rangle\right] \\ &= \begin{cases} \frac{1}{\alpha}\left[|0\rangle|x\rangle - |1\rangle|x\rangle\right] & \text{if } f(x) = \texttt{false} = 0\,; \\ \frac{1}{\alpha}\left[|1\rangle|x\rangle - |0\rangle|x\rangle\right] & \text{if } f(x) = \texttt{true} = 1 \end{cases} \\ &= (-1)^{f(x)}|-\rangle|x\rangle\ . \end{aligned} \tag{3.29}$$

Then, $|\Phi_3\rangle$ can be evaluated as follow:

$$\begin{aligned} |\Phi_3\rangle = U_f|-\rangle|+\rangle &= \tfrac{1}{\alpha}\left[U_f|-\rangle|0\rangle + U_f|-\rangle|1\rangle\right] \\ &= \tfrac{1}{\alpha}\left[(-1)^{f(0)}|-\rangle|0\rangle + (-1)^{f(1)}|-\rangle|1\rangle\right] \\ &= \begin{cases} (-1)^{f(0)}|-\rangle|+\rangle & \text{if } f(0) = f(1)\ ; \\ (-1)^{f(0)}|-\rangle|-\rangle & \text{if } f(0) \neq f(1)\ . \end{cases} \end{aligned} \tag{3.30}$$

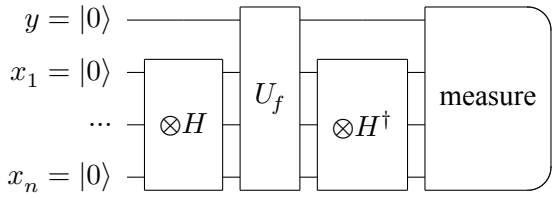


Figure 3.6: Circuit for black box UNIQUE-SAT in discrete quantum computing.

Finally, $|\Phi_4\rangle$ can then be obtained by applying the Hermitian conjugate of Hadamard matrix $H^\dagger = \frac{1}{\alpha^*}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ on both qubits. If $f(0) = f(1)$, i.e., f is constant, we have

$$|\Phi_4\rangle = (-1)^{f(0)} \left[\frac{1}{\alpha^*}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\alpha}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \otimes \left[\frac{1}{\alpha^*}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = (-1)^{f(0)} |1\rangle |0\rangle \ ;$$

if $f(0) \neq f(1)$, i.e., f is balanced, we have

$$|\Phi_4\rangle = (-1)^{f(0)} \left[\frac{1}{\alpha^*}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\alpha}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \otimes \left[\frac{1}{\alpha^*}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\alpha}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = (-1)^{f(0)} |1\rangle |1\rangle \ .$$

Hence, we can decide whether f is constant or balanced by measuring $|\Phi_4\rangle$ in the computational basis.

3.5.2 PARTIAL UNIQUE-SAT ALGORITHM

It is possible, in some situations, to exploit the cyclic behavior of the field to creatively cancel probability amplitudes and solve problems with what again appears to be "supernatural" efficiency. We illustrate this behavior with the algorithm in Fig. 3.6, which is a variant of the one in Fig. 3.1. Unlike the modal quantum algorithm, the new algorithm does not always succeed deterministically using a constant number of black box evaluations. We can, however, show that supernatural behavior occurs if the characteristic p of the field divides $2^N - 1$. For a database of fixed size N, matching the conditions becomes less likely as the size of the field increases. Nevertheless, for a *given* field,

it is always possible to expand any database with dummy records to satisfy the divisibility property. Physically, we are taking advantage of additional interference processes that happen because of the possibility of "wrapping around" due to modular arithmetic. We do not know, in general, whether this version of discrete quantum computing actually enables the rapid solution of NP-complete problems.

3.6 DISCRETE QUANTUM THEORY (II): INNER PRODUCT SPACE

We next discuss an approach using finite complexifiable fields that conditionally resolves the inner product condition (C) discussed in Sec. 3.4.2, which is violated by the theory just presented. A possible path is suggested by the work of Reisler and Smith [66]. The general idea is that while the cyclic properties of arithmetic in finite fields make it impossible to *globally* obtain the desired properties of the conventional Hilbert space inner product, it *is* possible to recover them *locally*, thereby restoring, with some restrictions, all the usual properties of the inner product needed for conventional quantum mechanics and conventional quantum computing. As the size of the discrete field becomes large, the size of the locally valid computational framework grows as well, leading to the *effective emergence of conventional quantum theory*. We next briefly outline such a context for local orderable subspaces of a finite field, and introduce an improvement on the original method [66] suggested by recent number theory resources [67].

Let us first note that the range of the quadratic map, $\left\{x^2 \bmod p \mid x \in \mathbb{F}_p\right\}$, is always one-half of the non-zero elements of $\mathbb{F}_p$, and is the set of elements with square roots in the field. This is the set of *quadratic residues*, and the complementary set (the other half of the non-zero field elements) is the set of *quadratic non-residues*. For example, in $\mathbb{F}_7$, the elements $\{1, 2, 4\}$ are considered positive as they have the square roots $\{1, 3, 2\}$ respectively; the remaining elements $\{3, 5, 6\}$ do not have square roots in the field. What is interesting is that if we have an uninterrupted sequence of numbers that are all quadratic residues, then we can define a *transitive order*, with $a > c$ if $a > b$ and $b > c$, provided $a - b$, $b - c$, and $a - c$ are all quadratic residues.**TODO. Should we remove the quadratic residue**

p	3	7	23	71	311	479	1559	5711	10559	18191	…
k	**2**	**3**	**5**	**7**	**11**	**13**	**17**	**19**	**23**	**29**	…
$\pi(k)$	1	2	3	4	5	6	7	8	9	10	…

Table 3.4: Number k of transitively ordered elements for a given field $\mathbb{F}_p$.

part? Since we never use take the square roots of numbers in order range later, whether they are the quadratic residues or not doesn't really matter...

As a concrete example, consider a finite field in which the sequential elements 0, 1, 2, 3, …, and $k-1$ are all quadratic residues (including 0). Then any sequence of odd length k and centered around an arbitrary $x \in \mathbb{F}_p$, i.e., $S_x(k) = x - \frac{k-1}{2}, \ldots, x-2, x-1, x, x+1, x+2, \ldots, x + \frac{k-1}{2}$, is *transitively ordered*. Indeed, we have $(x+1) - x = 1$ which is a quadratic residue and hence $x+1 > x$. Similarly, $x-(x-1) = 1$ and hence $x > x-1$. Also $(x+1)-(x-1) = 2$ which is a quadratic residue and hence $x+1 > x-1$. Clearly this process may be continued to show that the sequence $S_x(k)$ is transitively ordered. We can construct examples using the sequence A000229 in the encyclopedia of integer sequences [67][2]. The nth element of that sequence (which must be prime) is the least number such that the nth prime is the *least* quadratic non-residue for the given element. The first few elements of this sequence are listed in the top row of Table 3.4. The next row lists the number k of transitively ordered consecutive elements in that field, and $\pi(k)$ in the bottom row is the prime counting function (the number of primes up to k).

As an example, consider the field $\mathbb{F}_{23}$. Looking at the squares of the numbers $\mathbb{F}_{23} = \{0, \ldots, 22\}$ modulo 23, we find the 2-centered uninterrupted sequence $S_2(5) = \{0, 1, 2, 3, 4\}$, followed by

[2]For computational purposes, this sequence is preferable to the one proposed by Reisler and Smith [66] because it produces smaller primes. Their work showed that a sufficient condition on finite fields to produce sequences of quadratic residues is to further constrain the underlying prime numbers to be of the form $8\prod_{i=1}^{m} q_i - 1$, where q_i is the ith odd prime. While all such primes are of the form $4\ell + 3$, the set is severely restricted to astronomical numbers because the first few such primes are 7, 23, 839, 9239, 2042039, …

	allowed probability amplitudes $F^D(k)$
$D=1$	$F^1(11)=\{0,\pm1,\pm2,\pm\mathrm{i},\pm2\mathrm{i},(\pm1\pm\mathrm{i}),(\pm1\pm2\mathrm{i}),(\pm2\pm\mathrm{i})\}$
$D=2$	$F^2(11)=\{0,\pm1,\pm\mathrm{i},(\pm1\pm\mathrm{i})\}$
$D=3$	$F^3(11)=\{0,\pm1,\pm\mathrm{i}\}$
$D=4$	$F^4(11)=\{0,\pm1,\pm\mathrm{i}\}$
$D=5$	$F^5(11)=\{0,\pm1,\pm\mathrm{i}\}$
$D\geq6$	$F^D(11)=\{0\}$

Table 3.5: Allowed probability amplitudes for different vector space dimensions D and $k=11$.

5, which is both the smallest quadratic non-residue and the size of the uninterrupted sequence of quadratic residues (including 0) of interest. In particular, it is possible to construct a total order for the elements $S_0(5)=\{-2,-1,0,1,2\}$ in the fields $\mathbb{F}_{23}$, $\mathbb{F}_{71}$, $\mathbb{F}_{311}$, etc., but not in the smaller fields $\mathbb{F}_3$ and $\mathbb{F}_7$.

Given a D-dimensional vector space over $\mathbb{F}_{p^2}$ where p is one of the primes above, it is possible to define a *region* over which an inner product and norm can be identified. Let the length of the sequence of quadratic residues be k. The region of interest includes all vectors $|\Psi\rangle=\sum_{i=0}^{D-1}\alpha_i|i\rangle=\begin{pmatrix}\alpha_0 & \alpha_1 & \ldots & \alpha_{D-1}\end{pmatrix}^T$, for which $D<p-\frac{k-1}{2}$ and each α_i satisfies

$$D\,\mathsf{N}(\alpha_i)=D\,(a_i^2+b_i^2)\leq\frac{k-1}{2}\,, \tag{3.31}$$

with a_i and b_i drawn from the set $S_0(k)$. Consider, for example, $\mathbb{F}_{311^2}$ ($p=311$, $k=11$). We find that we can trade off the dimension D of the vector space against the range of probability amplitudes available for each α_i in Table 3.5.

We can now verify, by using Table 3.5, that for any vector $|\Psi\rangle$ in the selected region the value of $\langle\Psi|\Psi\rangle$ is ≥0 and vanishes precisely when $|\Psi\rangle$ is the zero vector. Thus, in the selected region,

condition (C) is established. Although the set of vectors defined over that region is not closed under addition, and hence the set is not a vector subspace, we can still have a theory by restricting our computations. In other words, *as long as our computation remains within the selected region*, we may pretend to have an inner product space. The salient properties of conventional quantum mechanics emerge, but the price to be paid is that the state space is no longer a vector space. This is basically a rigorous formulation of Schwinger's intuition (See, in particular, Chapter 1, Section 1.16. in [68]).

Readers with backgrounds in computer science or numerical analysis will notice, significantly, that this model for discrete quantum computing is reminiscent of practical computing with a classic microprocessor having only integer arithmetic and a limited word length. We cannot perform a division having a fractional result at all, since there are no fractional representations; we do have the basic constants zero and one, as well as positive and negative numbers, but multiplications or additions producing results outside the integer range wrap around modulo the word length and typically yield nonsense. This implies that, for the local discrete model, we must accept an operational world view that *has no awareness of the value of* p, and depends on having set up in advance an environment with a field size, analogous to the word size of a microprocessor, that happily processes *any* calculation we are prepared to perform. This is the key step, though it may seem strange because we are accustomed to arithmetic with real numbers: we list the calculations that must be performed in our theory, discover an *adequate size of the processor word* —implying a possibly ridiculously large value of p chosen as described above —and from that point on, we calculate necessarily valid values within that processor, never referring in any way to p itself in the sequel.

3.7 DISCRETE QUANTUM THEORY (II): CARDINAL PROBABILITY

The final issue that must be addressed in the discrete theory put forward in Section 3.6 concerns measurement. To recap, within the theory, states are D-dimensional vectors with complex discrete-

valued amplitudes drawn from a totally-ordered range, $F^D(k)$, in the underlying finite field. These states possess, by construction, having field norms in the non-negative integers, all in the ordered range of Eq. (3.31), and hence potentially produce probabilities that can be ordered. Our point is that, although the mathematical framework of conventional quantum mechanics relies on infinite precision probabilities, it is impossible in practice to measure exact equality of real numbers —we can only achieve an approximation within measurement accuracy. Significantly, when we use finite fields, this measurement accuracy will be encoded in the size of the finite field used for measurements.

Given a D-dimensional Hilbert space in conventional quantum theory, although we can measure the probability for every eigenprojector of an observable as discussed in Sec. 2.2, our previous quantum circuits in Figures 3.1, 3.5, and 3.6 always measure in the computational basis $\{|0\rangle, ..., |i\rangle, ..., |D-1\rangle\}$. Indeed, it is sufficient to only consider measuring in the computational basis because measuring in another basis is the same as applying a quantum gate and measuring in the computational basis. In this situation, the Born rule for pure states, Eq. (2.16), can be simplified as

$$\mu_{\Psi}^{\mathrm{B}}\left(|i\rangle\langle i|\right) \equiv \mu_{\Psi}^{\mathrm{B}}\left(i\right) = \frac{\langle\Psi|i\rangle\,\langle i|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\left|\langle i|\Psi\rangle\right|^2}{\langle\Psi|\Psi\rangle} = \frac{\left|\alpha_i\right|^2}{\langle\Psi|\Psi\rangle}\,, \tag{3.32}$$

where $|\Psi\rangle = \begin{pmatrix}\alpha_0 & \alpha_1 & ... & \alpha_{D-1}\end{pmatrix}^T$ is an unnormalized state. Hereafter, we will simplify by calling $\mu_{\Psi}^{\mathrm{B}}\left(i\right)$ the probability of measuring $|i\rangle$.

Although division is not an allowed operation for the elements in the an ordered region, following the standard procedure to define a conventional fraction as a pair of integers [22, 45, 69, 70], we could define a *cardinal probability* as a pair of order-region elements as well:

$$\mu_{\Psi}^{\mathrm{C}}\left(i\right) = \mathsf{N}\left(\langle i|\Psi\rangle\right) \mathbin{/\!\!/} \mathsf{N}\left(|\Psi\rangle\right) = \mathsf{N}\left(\alpha_i\right) \mathbin{/\!\!/} \mathsf{N}\left(|\Psi\rangle\right)\,, \tag{3.33}$$

where every probability amplitude $\alpha_i \in F^D\left(k\right)$ so that both $\mathsf{N}\left(\alpha_i\right)$ and $\mathsf{N}\left(|\Psi\rangle\right)$ are within the order

$\|\Psi\rangle$	$\mathsf{N}(\langle 0\|\Psi\rangle)$	$\mathsf{N}(\langle 1\|\Psi\rangle)$	$\mathsf{N}(\|\Psi\rangle)$	$\mu_{\Psi}^{\mathrm{C}}(0)$	$\mu_{\Psi}^{\mathrm{C}}(1)$
$1\|0\rangle$	1	0	1	$1 /\!\!/ 1$	$0 /\!\!/ 1$
$1\|0\rangle + 1\|1\rangle$	1	1	2	$1 /\!\!/ 2$	$1 /\!\!/ 2$
$1\|0\rangle + (1+\mathrm{i})\|1\rangle$	1	2	3	$1 /\!\!/ 3$	$2 /\!\!/ 3$
$(1-\mathrm{i})\|0\rangle + (1+\mathrm{i})\|1\rangle$	2	2	4	$2 /\!\!/ 4$	$2 /\!\!/ 4$

Table 3.6: Field norms and probabilities for one-qubit states $|\Psi\rangle$ in $F^2(11)$.

range $S_0(k)$, and $\mu_{\Psi}^{\mathrm{C}}(i)$ is called the cardinal probability of measuring $|i\rangle$. For example, let $p = 311$, $k = 11$, and $D = 2$. The permitted range is $S_0(11) = \{-5, \ldots, -1, 0, 1, \ldots, 5\}$, given the dimension $D = 2$, the allowed probability amplitude coefficients are $F^2(11) = \{0, \pm 1, \pm \mathrm{i}, (\pm 1 \pm \mathrm{i})\}$ (see Table 3.5). Now the cardinal probabilities of several representative one-qubit states are listed in Table 3.6.

When measuring cardinal probabilities, *inequalities* can be preserved with appropriate resources (in the form of a sufficiently large choice of the field), while *equalities* cannot be guaranteed in the theory, and in fact can be represented as *inequalities of any order*. That is, given two cardinal probabilities $\mathsf{N}(\alpha_i) /\!\!/ \mathsf{N}(|\Psi\rangle)$ and $\mathsf{N}(\alpha_j) /\!\!/ \mathsf{N}(|\Psi\rangle)$ with the same "denominator", $\mathsf{N}(\alpha_i) > \mathsf{N}(\alpha_j)$ physically means it is more likely to measure $|i\rangle$ than $|j\rangle$ if we have enough resource; $\mathsf{N}(\alpha_i) = \mathsf{N}(\alpha_j)$ means the experimental results might not always favor $|i\rangle$ or $|j\rangle$ no matter how much resource we use. The same principle can also apply to two cardinal probabilities with the different "denominator". The details of the comparison can easily formulated by following the standard procedure [22, 45, 69, 70] and will left as an exercise for the readers.

3.8 DISCRETE QUANTUM COMPUTING (II)

We now examine two particularly important types of examples within the discrete theory of the previous section: the first is the deterministic Deutsch-Jozsa algorithm [32, 39, 63], which determines

the balanced or unbalanced nature of an unknown function with a single measurement step ($O(1)$), and the second is the (normally) probabilistic Grover algorithm [39, 41, 65], determining the result of an unstructured search in $O\left(\sqrt{N}\right)$ time. In the following, we use k to denote the upper bound of the ordered range of integers needed to perform a given calculation; this in turn is assumed to be implemented using a choice of a finite prime number p that supports calculation in the range of k.

3.8.1 DISCRETE DEUTSCH-JOZSA ALGORITHM: DETERMINISTIC

To examine the Deutsch-Jozsa algorithm in the discrete theory of the previous section, we assume we are given a classical function $f : \mathsf{Bool}^n \to \mathsf{Bool}$, and are told that f is either constant or balanced [32, 39, 63]. The algorithm is expressed in a space of dimension $D = 2^{n+1}$: it begins with the $n+1$ qubit state $|1\rangle\left|\overline{0}\right\rangle$ where the overline denotes a sequence of length n. A straightforward calculation [32] shows that the final state is[3]**TODO. A little bit more explanation?**

$$\sum_{\overline{z}\in\{0,1\}^n}\sum_{\overline{x}\in\{0,1\}^n}(-1)^{f(\overline{x})+\overline{x}\cdot\overline{z}}\left(|0\rangle\left|\overline{z}\right\rangle-|1\rangle\left|\overline{z}\right\rangle\right), \tag{3.34}$$

and that its field norm is 2^{n+1}.**TODO. Verify?** To make sure that the algorithm works properly, we note that all the probability amplitudes involved in the calculation are in the range -2^n, …, 2^n and therefore, by Eq. (3.31), we get the following constraint on the size of the ordered region in the finite field:

$$2^{n+1}\left(2^n\right)^2 \leq \frac{k-1}{2} \quad \Leftrightarrow \quad k \geq 2^{3n+2}+1\,. \tag{3.35}$$

Now we need to choose a prime number p that supports calculation in the range of k. Assume that k is the least prime satisfying $k \geq 2^{3n+2}+1$, and let p be the $\pi(k)$th element of the sequence

[3]Note that the algorithm in reference [32] makes use of the Hadamard matrix. We have eliminated the factor $\frac{1}{\sqrt{2}}$ to ensure that all quantities are expressed in terms of integers. Also notice that the positioning of the initial qubit state $|1\rangle$ is reversed from [32].

p	…	422231	…	196265095009	…		…		…
k	…	**37**	…	**131**	…	**257**	…	**32771**	…
$\pi(k)$	…	12	…	32	…	55	…	3513	…

Table 3.7: Extension of transitively ordered elements.

A000229 [67]. We argue that no prime less than this value of p can support calculation in the ordered range of k, and that this p is sufficient to support such calculation. In particular, since k is the least quadratic non-residue of p, every number less than k is a quadratic residue, and thus 0, 1, 2, 3, …, 2^{3n+2} are all quadratic residues. Hence the numbers -2^n, …, 2^n are all inside the ordered range $S_0(k)$. On the other hand, if we choose any prime smaller than p, there is a quadratic non-residue smaller than k, and we also know that the least quadratic non-residue is a prime [47]. Thus, there is a quadratic non-residue in 0, 1, 2, 3, …, 2^{3n+2}, and therefore, for this smaller p, there would be a number in -2^n, …, 2^n that is not in the ordered range $S_0(k)$.

When f is constant, the cardinal probability of measuring $|0\rangle|\overline{0}\rangle$ or $|1\rangle|\overline{0}\rangle$ is $(2^n)^2 + (2^n)^2 = 2^{2n+1} /\!\!/ 2^{2n+1}$; i.e., the cardinal probability of measuring any other state is $0 /\!\!/ 2^{2n+1}$. When f is balanced, the cardinal probability of measuring $|0\rangle|\overline{0}\rangle$ or $|1\rangle|\overline{0}\rangle$ is $0 /\!\!/ 2^{2n+1}$. Therefore, if we find that the post-measurement state is either $|0\rangle|\overline{0}\rangle$ or $|1\rangle|\overline{0}\rangle$, we know f is constant; otherwise, f is balanced.

For a single qubit Deutsch problem, the probability amplitudes are between -2 and 2, and the dimension $D = 2^{1+1} = 4$, so we want to have

$$k \geq 2^{3\cdot 1+2} + 1 = 2^5 + 1 = 33\,. \tag{3.36}$$

The least prime satisfying the above condition is $k = 37$, and thus $\pi(37) = 12$ and $p = 422231$, as shown in the extended elements Table 3.7.

For the 2-qubit Deutsch-Jozsa, the computation is already quite challenging. Now the probability

amplitudes are between -4 and 4, and the dimension $D = 2^{2+1} = 8$, so we need

$$k \geq 2^{3 \cdot 2+2} + 1 = 2^8 + 1 = 257\,. \tag{3.37}$$

Because 257 is a prime, we can pick $k = 257$ and $\pi\,(257) = 55$. The actual value of p is already outside the range of the published sequence A000229.

These examples illustrate that the value of p plays an essential role: its size grows with the numerical range of the intermediate and final results of the algorithms being implemented. Therefore, we naturally recover a deterministic measure of the intrinsic resources required for a given level of complexity; this measure is normally completely hidden in computations with real numbers, and explicitly exposing it is one of the significant achievements of our discrete field analysis of quantum computation. This solves the conundrum that the conventional Deutsch-Jozsa algorithm mysteriously continues to work for larger and larger input functions without any apparent increase in resources. Our analysis of this problem reveals that as the size of the input increases, it is necessary to increase the size of p and hence the size of the underlying available numeric coefficients. This observation does not fully explain the power of quantum computing over classical computing, but at least it explains that some of the power of quantum computing depends on increasingly larger precision in the underlying field of numbers.

3.8.2 DISCRETE GROVER SEARCH: NONDETERMINISTIC

As an example of how to apply our cardinal probability framework to a nondeterministic algorithm, we consider discrete Grover's algorithm searching an unstructured database of size $N = 2^n$ [39, 41, 65]. Let $f : \mathsf{Bool}^n \to \mathsf{Bool}$ be the function we want to search. To simplify our discussion, we will

only consider

$$f(\overline{x}) = \begin{cases} 1\,, & \text{if } \overline{x} = \overline{0}\,; \\ \\ 0\,, & \text{if } \overline{x} \neq \overline{0}\,, \end{cases} \tag{3.38}$$

where we identify `false` and `true` as 0 and 1 as usual, and $\overline{0} = (0, \ldots, 0)$.

To solve Grover's problem, we represent the search states as n-qubit states as usual. However, instead of the Deutsch black box U_f, f is represented as the $N \times N$ "phase rotation" matrix

$$R = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \tag{3.39}$$

where the "marked" element is in the first position. Beside R, we also need the $N \times N$ "diffusion" matrix

$$\Delta = \begin{pmatrix} 1 - \frac{N}{2} & 1 & 1 & \ldots & 1 \\ 1 & 1 - \frac{N}{2} & 1 & \ldots & 1 \\ 1 & 1 & 1 - \frac{N}{2} & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 - \frac{N}{2} \end{pmatrix}, \tag{3.40}$$

where we have eliminated, in matrix Δ, the scaling factor $\frac{2}{N}$ to enforce the requirement that all matrix coefficients in our framework are integer-valued. By applying the transformation ΔR repeatedly

$$j = \text{round}\left(\frac{\pi}{4 \arccos \sqrt{1 - \frac{1}{N}}} - \frac{1}{2} \right) \approx \text{round}\left(\frac{\pi}{4} \sqrt{N} \right) \tag{3.41}$$

times, we can find the target element $\overline{0}$. In our context, we also need to choose a prime number that is large enough to ensure that all the numbers that occur during the calculation and after measurement are within the transitively-ordered subrange.

Let's walk through the state in each iteration to make sure the algorithm works. Because the probability amplitudes of $|\overline{x}\rangle$ are all the same for $\overline{x} \neq \overline{0}$, we can let a_l be the probability amplitude of $|\overline{0}\rangle$, with b_l the probability amplitude of each of the other possibilities, which are all the same, after the operators ΔR is applied l times. Beginning at $l = 0$ with the information-less state, the normalization scaled to integer values as usual, the state after lth iteration can written as**TODO. Verify the computation??**

$$\begin{pmatrix} a_l \\ b_l \\ \vdots \\ b_l \end{pmatrix} = (\Delta R)^l \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \Delta R \begin{pmatrix} a_{l-1} \\ b_{l-1} \\ \vdots \\ b_{l-1} \end{pmatrix} . \tag{3.42}$$

We can solve the above iteration by the following recurrence relation for the successive coefficients:

$$a_0 = 1\,, \qquad a_{l+1} = \left(\frac{N}{2} - 1\right) a_l + (N-1)\, b_l\,, \tag{3.43a}$$

$$b_0 = 1\,, \qquad b_{l+1} = (-1)\, a_l + \left(\frac{N}{2} - 1\right) b_l\,. \tag{3.43b}$$

We also know $|a_j| > |b_j|$, so we can estimate an upper bound for the maximum cardinal probability as $\max \mathsf{N}(a_j) \leq 2\left(\frac{N}{2}\right)^{2j+1}$. By applying Eq. (3.31) with $D = N = 2^n$, we can estimate k using $k \geq 8\left(\frac{N}{2}\right)^{2j+2} + 1$. We can then pick a prime k, and choose the $\pi(k)$th prime in the sequence represented by Table 3.4 guaranteeing that every number we need for the computation is within the transitively ordered range $F^D(k)$.

For the 2-qubit Grover search, we have $N = D = 4$ and $j = 1$, with the maximum cardinal probability

$$\max \mathsf{N}(a_j) \leq 2\left(\frac{4}{2}\right)^{2+1} = 16\,, \tag{3.44}$$

so we need

$$k \geq 8\left(\frac{4}{2}\right)^{2\cdot 1+2} + 1 = 8 \cdot 2^4 + 1 = 129\,. \tag{3.45}$$

The least prime k satisfying the above condition is $k = 131$, and so $\pi(131) = 32$ and $p = 196265095009$.

When $p = 196265095009$, we assume that $f(\overline{x}) = 1$ if and only if $|\overline{x}\rangle = |0\rangle|0\rangle$, and so the final state is $\begin{pmatrix} 4 & 0 & \dots & 0 \end{pmatrix}^T$ with the field norm of 16. Then, the cardinal probability of obtaining $|0\rangle|0\rangle$ as the post-measurement state is $16 /\!\!/ 16$, and it is $0 /\!\!/ 16$ for the rest of the states.

For the 3-qubit Grover search, we have $N = D = 8$ and $j = 2$, with an upper bound $\max \mathsf{N}(a_j) \leq 2\left(\frac{8}{2}\right)^{4+1} = 2048$ on the cardinal probability. Thus

$$k \geq 8\left(\frac{8}{2}\right)^6 + 1 = 32769\,. \tag{3.46}$$

The nearest prime greater than this number is 32771, so we can pick $k = 32771$ and $\pi(32771) = 3513$, and so if we use the 3513th prime, we can implement Grover's algorithm for a database of size 8.

Continuing with the 3-qubit Grover example, we show how the cardinal probabilities evolve to single out the target state. First, assume that $f(\overline{x}) = 1$ if and only if $|\overline{x}\rangle = |0\rangle|0\rangle|0\rangle$. The initial information-less 8-dimensional state vector evolves under the application of ΔR as follows:**TODO. Change the whole paragraph to a table?**

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 10 \\ 2 \\ \vdots \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 44 \\ -4 \\ \vdots \\ -4 \end{pmatrix}. \tag{3.47}$$

These states have differing field norm so we cannot compare their cardinal probability directly. Since states differ by only scalar multiplication representing the same state, if we multiply the first and second states by 16 and 4, respectively, their field norms become the same value of 2048. The

now-consistently-normalized states become

$$\begin{pmatrix} 16 \\ 16 \\ \vdots \\ 16 \end{pmatrix} \rightarrow \begin{pmatrix} 40 \\ 8 \\ \vdots \\ 8 \end{pmatrix} \rightarrow \begin{pmatrix} 44 \\ -4 \\ \vdots \\ -4 \end{pmatrix}. \tag{3.48}$$

Therefore, the cardinal probabilities of measuring $|0\rangle|0\rangle|0\rangle$ in each state are

$$256 /\!\!/ 2048 \qquad 1600 /\!\!/ 2048 \qquad 1936 /\!\!/ 2048, \tag{3.49}$$

while the cardinal probabilities of measuring the other states become

$$256 /\!\!/ 2048 \qquad 64 /\!\!/ 2048 \qquad 16 /\!\!/ 2048. \tag{3.50}$$

We may thus conclude that the cardinal probability of measuring the satisfying assignment of f increases as we apply the diffusion Δ and phase rotation R matrices repeatedly.

Clearly, the required size of k increases systematically with the problem size, and the corresponding size of the required prime number p defining the discrete field increases in the fashion illustrated in Tables 3.4 and 3.7.

3.9 QUANTUM PROBABILITY MEASURES OVER FINITE FIELDS

The cardinal probability over finite fields defined in Eq. (3.33) is based on the conventional Born rule, Eq. (3.32). Although they are valid inside an ordered region, some quantum algorithms, like Shor's [39, 41, 71], don't localize their numbers in a small region. Moreover, we still haven't found how to manipulate the resulting cardinal probabilities, and hence have little idea how to define the mixed states and expectation values. To overcome these issues, we try a different approach to define

a "conventional" probability over finite fields axiomatically.

Recall Gleason's theorem in Sec. 2.2.1 states that every quantum probability measure is induced by the Born rule in a Hilbert space $\mathcal{H}$ of dimension $D \geq 3$. To find a Born rule over finite fields, we could follow the steps in Sec. 2.2 to define events and probability measures, and see whether we could have a Gleason-like theorem which induces a correspondence between states and probability measures qualified as a discrete Born rule.

Definition 3.2 (Quantum Probability Measures over Finite Fields)**.** Given a vector space $\mathcal{H}$ of dimension D over the complexified field $\mathbb{F}_{p^2}$, the set of events $\mathcal{E}_{p^2}$ are recursively defined as follows:

- $\mathbb{0}$ and $\mathbb{1} \in \mathcal{E}_{p^2}$.
- If $|\Psi\rangle$ is a unit-norm state in $\mathcal{H}$, i.e., $|\Psi\rangle \in \mathcal{H}$ with $\langle\Psi|\Psi\rangle = \mathsf{N}(|\Psi\rangle) = 1$, then the projector of the form $|\Psi\rangle\langle\Psi| \in \mathcal{E}_{p^2}$.
- For each pair of *orthogonal* events $P_0 \in \mathcal{E}_{p^2}$ and $P_1 \in \mathcal{E}_{p^2}$, i.e., $P_0 P_1 = \mathbb{0}$, their sum is $P_0 + P_1$ is an event, i.e., $P_0 + P_1 \in \mathcal{E}_{p^2}$.

Then we take a quantum probability measure over finite field (QPMFF) $\mu : \mathcal{E}_{p^2} \to [0, 1]$ to be an assignment of a probability to each event (projection operator P) subject to $\mu(\mathbb{0}) = 0$, $\mu(\mathbb{1}) = 1$, and satisfying for each pair of *orthogonal* events P_0 and P_1, $\mu(P_0 + P_1) = \mu(P_0) + \mu(P_1)$.

There are some correspondence qualified as "the discrete Born rule" when $p = D = 3$. However, when $D = 3$ and $p = 7$, we have numerical verified that the unique QPMFF $\mu : \mathcal{E}_{7^2} \to [0, 1]$ is

$$\mu(P) = \begin{cases} 0 & \text{if } P = \mathbb{0}\,; \\ \frac{1}{3} & \text{if } P \text{ is a one-dimensional projector;} \\ \frac{2}{3} & \text{if } P \text{ is a two-dimensional projector;} \\ 1 & \text{if } P = \mathbb{1}\,. \end{cases} \tag{3.51}$$

Since we don't have enough QPMFF to correspond even pure states in the vector space $\mathbb{F}_{7^2}^3$, there is no discrete Born rule in this case. In general, we conjecture that QPMFF is always unique for $D \geq 3$ except $D = p = 3$.

Although we couldn't prove there is no discrete Born rule by investigating QPMFF alone, we could prove there is no "sensible" Born rule which also satisfies an analog of Proposition 2.1 and 2.2.**TODO. Notice that the notation here and previous is a little bit different because we dealt with mixed states previously, but pure states here. Besides, should we explain the physical meaning of these propositions?**

Theorem 3.1. *For $D \geq 3$ except $p = D = 3$, there is no "sensible" Born rule μ^{F} parametrized by unit-norm states $|\Phi\rangle$ in $\mathcal{H}$ such that μ_Φ^{F} is a QPMFF;*

$$\mu_\Phi^{\mathrm{F}}(P) = 1 \tag{3.52}$$

if and only if $P|\Phi\rangle = |\Phi\rangle$; and

$$\mu_{U|\Phi\rangle}^{\mathrm{F}}(UPU^\dagger) = \mu_\Phi^{\mathrm{F}}(P)\ , \tag{3.53}$$

where U is any unitary map.

Proof. [4]Assume such a map μ^{F} satisfying all properties exists, we will use the listed properties to build a contradiction. Consider the computational basis $\{|0\rangle, |1\rangle, |2\rangle, ...\}$, and the projectors formed by these vectors, $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$, and $P_2 = |2\rangle\langle 2|$. Since $p \geq 7$, $1+1+1$ is not 0 in $\mathbb{F}_{p^2}$, and has the principal inverse field norm $\gamma = \mathsf{N}^{-1}(3)$ as defined in Sec. 3.4.2. The unit-norm state $|\oplus\rangle = \frac{1}{\gamma}[|0\rangle + |1\rangle + |2\rangle]$ can then be used as a parameter of μ^{F}, and induces a QPMFF $\mu_\oplus^{\mathrm{F}}$. To compute the probability values of $\mu_\oplus^{\mathrm{F}}$, we want to utilize Eq. (3.53) by letting U_i be the unitary map that permutes the basis vectors $|0\rangle$ and $|i\rangle$ and acts as the identity for the rest for $i = 1$ and 2. Note

[4]This proof assume $p \geq 7$, and there is a simpler proof for $D \geq 4$ applying to $p = 3$ case [5].

that $|\oplus\rangle$ is invariant under U_i so we have $\mu^{\mathrm{F}}_{\oplus}(P_0) = \mu^{\mathrm{F}}_{U_i|\oplus\rangle}\left(U_i P_0 U_i^{\dagger}\right) = \mu^{\mathrm{F}}_{\oplus}(P_i)$ by Eq. (3.53). Let $P' = P_0 + P_1 + P_2$. Since $P'|\oplus\rangle = |\oplus\rangle$, Eq. (3.52) implies

$$1 = \mu^{\mathrm{F}}_{\oplus}(P') = \mu^{\mathrm{F}}_{\oplus}(P_0) + \mu^{\mathrm{F}}_{\oplus}(P_1) + \mu^{\mathrm{F}}_{\oplus}(P_2) = 3\mu^{\mathrm{F}}_{\oplus}(P_0)\ , \tag{3.54}$$

and thus for $i = 0$, 1, and 2, we always have $\mu^{\mathrm{F}}_{\oplus}(P_i) = \frac{1}{3}$.

Similar to the previous paragraph, we now compute the QPMFF induced by $|\Phi'\rangle = U'|\oplus\rangle = \frac{2\alpha}{\gamma}|0\rangle + \frac{1}{\gamma}|2\rangle$, where $\alpha = \mathsf{N}^{-1}\left(\frac{1}{2}\right) \in \mathbb{F}_{p^2}$ and

$$U' = \begin{pmatrix} \alpha & \alpha & 0 \\ -\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\ . \tag{3.55}$$

Since $|2\rangle$ is invariant under U', Eq. (3.53) implies $\mu^{\mathrm{F}}_{\oplus}(P_2) = Hilbert\mu^{\mathrm{F}}_{U'|\oplus\rangle}(U'P_2U'^{\dagger}) = \mu^{\mathrm{F}}_{\Phi'}(P_2)$. Also, Eq. (3.52) implies $1 = \mu^{\mathrm{F}}_{\Phi'}(P_0 + P_2) = \mu^{\mathrm{F}}_{\Phi'}(P_0) + \mu^{\mathrm{F}}_{\Phi'}(P_2)$. By moving $\mu^{\mathrm{F}}_{\Phi'}(P_0)$ to the left-hand side of the equation, we have

$$\mu^{\mathrm{F}}_{\Phi'}(P_0) = 1 - \mu^{\mathrm{F}}_{\Phi'}(P_2) = 1 - \mu^{\mathrm{F}}_{\oplus}(P_2) = 1 - \tfrac{1}{3} = \tfrac{2}{3}\,. \tag{3.56}$$

Iterating the similar process with $|\Phi''\rangle = U''|\Phi'\rangle = \frac{2\alpha}{\gamma}(|0\rangle + |1\rangle) + \frac{\beta}{\gamma}|2\rangle$, where $\beta = \mathsf{N}^{-1}(-1) \in \mathbb{F}_{p^2}$ and

$$U'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^* & 2\alpha \\ 0 & 2\alpha^* & \beta \end{pmatrix}\ . \tag{3.57}$$

Because $|\Phi''\rangle$ is invariant under U_1, invoking Eq. (3.53) in $\mu^{\mathrm{F}}_{\Phi''}(P_1) = \mu^{\mathrm{F}}_{U_1|\Phi''\rangle}\left(U_1 P_1 U_1^{\dagger}\right) = \mu^{\mathrm{F}}_{\Phi''}(P_0)$, we know $\mu^{\mathrm{F}}_{\Phi''}(P_1)$ and $\mu^{\mathrm{F}}_{\Phi''}(P_0)$ are same. Actually, they are both equal to $\frac{2}{3}$ because $\mu^{\mathrm{F}}_{\Phi''}(P_0) = \mu^{\mathrm{F}}_{U''|\Phi'\rangle}(U''P_0U''^{\dagger}) = \mu^{\mathrm{F}}_{\Phi'}(P_0) = \frac{2}{3}$ by Eqs. (3.53) and (3.56). Since $P'|\Phi''\rangle = |\Phi''\rangle$,

Eq. (3.52) implies

$$1 = \mu^{\mathrm{F}}_{\Phi''}(P') = \mu^{\mathrm{F}}_{\Phi''}(P_0) + \mu^{\mathrm{F}}_{\Phi''}(P_1) + \mu^{\mathrm{F}}_{\Phi''}(P_2) = \tfrac{2}{3} + \tfrac{2}{3} + \mu^{\mathrm{F}}_{\Phi''}(P_2) > \tfrac{4}{3}\,. \tag{3.58}$$

This is inconsistent with the requirement that the probabilities for orthogonal outcomes add up to 1, and build a contradiction. □

The reason why there is no "sensible" Born rule might be that the state spaces are now discrete and finite, but we still consider mapping probability assignments to infinitely precise values in the unit interval $[0, 1]$. To overcome this issue, we want to consider a discrete Born rule mapping to finite number of intervals called interval-valued probability [3, 72]. To adopting the idea of interval-valued probability step-by-step, before attempting to study quantum interval-valued probability over finite fields, we will first review the classical interval-valued probability, and extend it with the conventional quantum theory in the next chapter.

Chapter 4

TOWARD A QUANTUM MEASUREMENT THEORY WITH ERROR: QUANTUM INTERVAL-VALUED PROBABILITY

4.1 INTERVALS OF UNCERTAINTY

4.1.1 DEFINITIONS OF CLASSICAL AND QUANTUM IVPMS

We will start by reviewing classical IVPMs and then propose our quantum generalization. In the classical setting, there are several proposals for "imprecise probabilities" [34, 72–78]. Although these proposals differ in some details, they all share the fact that the probability $\mu(E)$ of an event E is generalized from a single *real number* to an *interval* $[\ell, r]$, where ℓ intuitively corresponds to the strength of evidence for the event E and $1 - r$ corresponds to the strength of evidence against the same event. Under some additional assumptions, this interval could be interpreted as the Gaussian width of a probability distribution.

We next introduce probability axioms for IVPMs. First, for each interval $[\ell, r]$ we have the natural constraint $0 \leq \ell \leq r \leq 1$ that guarantees that every element of the interval can be interpreted as a conventional probability. We also include $\mathsf{F} = [0, 0]$ and $\mathsf{T} = [1, 1]$ as limiting intervals that

refer, respectively, to the probability interval for impossible events and for events that are certain. We can write the latter as $\mu(\emptyset) = \mathbf{F}$ and $\mu(\Omega) = \mathbf{T}$, where $\emptyset$ is the empty set and Ω is the event covering the entire sample space. For each interval $[\ell, r]$, we also need the dual interval $[1-r, 1-\ell]$ so that if one interval refers to the probability of an event E, the dual refers to the probability of the event's complement $\overline{E}$. For example, if we discover as a result of an experiment that $\mu(E) = [0.2, 0.3]$ for some event E, we may conclude that $\mu\left(\overline{E}\right) = [0.7, 0.8]$ for the complementary event $\overline{E}$. In addition to these simple conditions, there are some subtle conditions on how intervals are combined, which we discuss next.

Let E_0 and E_1 be two disjoint events with probabilities $\mu(E_0) = [\ell_0, r_0]$ and $\mu(E_1) = [\ell_1, r_1]$. A first attempt at calculating the probability of the combined event that *either* E_0 or E_1 occurs might be $\mu(E_0 \cup E_1) = [\ell_0 + \ell_1, r_0 + r_1]$. In some cases, this is indeed a sensible definition. For example, if $\mu(E_0) = [0.1, 0.2]$ and $\mu(E_1) = [0.3, 0.4]$ we get $\mu(E_0 \cup E_1) = [0.4, 0.6]$. But consider an event E such that $\mu(E) = [0.2, 0.3]$ and hence $\mu\left(\overline{E}\right) = [0.7, 0.8]$. The two events E and $\overline{E}$ are disjoint; the naïve addition of intervals would give $\mu\left(E \cup \overline{E}\right) = [0.9, 1.1]$, which is not a valid probability interval. Moreover the event $E \cup \overline{E}$ is the entire space; its probability interval should be $\mathbf{T}$ which is sharper than $[0.9, 1.1]$. The problem is that the two intervals are correlated: there is more information in the combined event than in each event separately so the combined event should be mapped to a sharper interval. In our example, even though the "true" probability of E can be anywhere in the range $[0.2, 0.3]$ and the "true" probability of $\overline{E}$ can be anywhere in the range $[0.7, 0.8]$, the values are not independent. Any value of $\mu(E) \leq 0.25$ will force $\mu\left(\overline{E}\right) \geq 0.75$. To account for such subtleties, the axioms of interval-valued probability do not use a strict equality for the combination of disjoint events. The correct constraint enforcing coherence of the probability assignment for $E_0 \cup E_1$ when E_0 and E_1 are disjoint is taken to be:

$$\mu(E_0 \cup E_1) \subseteq [\ell_0 + \ell_1, r_0 + r_1]\,. \tag{4.1}$$

Note that for any event E with $\mu(E) = [\ell, r]$, we always have $\mu(\Omega) = \mathbf{T} \subseteq [\ell, r] + [1-r, 1-\ell] = \mu(E) \cup \mu\left(\overline{E}\right)$.

When combining non-disjoint events, there is a further subtlety whose resolution will give us the final general condition for IVPMs. For events E_0 and E_1, not necessarily disjoint, we have:

$$\mu(E_0 \cup E_1) + \mu(E_0 \cap E_1) \subseteq \mu(E_0) + \mu(E_1)\,, \tag{4.2}$$

which is a generalization of the classical inclusion-exclusion principle that uses $\subseteq$ instead of $=$ for the same reason as before. The new condition, known as *convexity* [79, 75, 80, 76, 81, 34], reduces to the previously motivated Eq. (4.1) when the events are disjoint, i.e., when $\mu(E_0 \cap E_1) = 0$.

Previous discussions can be summarized as the following definition [72].

Definition 4.1 (IVPM)**.** Assume a collection of intervals $\mathcal{I}$ including $\mathbf{F}$ and $\mathbf{T}$ with addition and scalar multiplication defined as follows:

$$[\ell_0, r_0] + [\ell_1, r_1] = [\ell_0 + \ell_1, r_0 + r_1] \text{ and} \tag{4.3a}$$

$$x[\ell, r] = \begin{cases} [x\ell, xr] & \text{for } x \geq 0\,; \\ [xr, x\ell] & \text{for } x \leq 0\,. \end{cases} \tag{4.3b}$$

Given a finite sample space Ω, and its power set 2^Ω as the classical event space, a classical IVPM $\bar{\mu} : 2^\Omega \to \mathcal{I}$ is a function subject to the following constraints:

$$\bar{\mu}(\emptyset) = \mathbf{F}\,, \tag{4.4a}$$

$$\bar{\mu}(\Omega) = \mathbf{T}\,, \tag{4.4b}$$

$$\bar{\mu}\left(\overline{E}\right) = \mathbf{T} - \bar{\mu}\left(E\right)\,, \tag{4.4c}$$

and satisfying the convexity condition, Eq. (4.2), for each pair of events E_0 and E_1.

Note that the minus sign appearing in Eq. (4.4c) is accommodated by the $x \leq 0$ case in Eq. (4.3b).

We now have the necessary ingredients to define the quantum extension, QIVPMs, as a generalization of both classical IVPMs and conventional quantum probability measures in Sec. 2.2. We will show that QIVPMs reduce to classical IVPMs when the space of quantum events $\mathcal{E}$ is restricted to mutually commuting events $\mathcal{E}_C$, i.e., to compatible events that can be measured simultaneously. In Sec. 4.3 we will discuss the connection between QIVPMs and conventional quantum probability measures in detail.

Definition 4.2 (QIVPM)**.** we take a QIVPM $\bar{\mu}$ to be an assignment of an interval to each event (projection operator P) subject to the following constraints:

$$\bar{\mu}(\mathbb{0}) = \mathbf{F}\,, \qquad \bar{\mu}(\mathbb{1}) = \mathbf{T}\,, \qquad \bar{\mu}\left(\mathbb{1} - P\right) = \mathbf{T} - \bar{\mu}\left(P\right)\,, \tag{4.5}$$

and satisfying for each pair of *commuting* projectors P_0 and P_1 with $P_0 P_1 = P_1 P_0$,

$$\bar{\mu}\left(P_0 + P_1 - P_0 P_1\right) + \bar{\mu}\left(P_0 P_1\right) \subseteq \bar{\mu}\left(P_0\right) + \bar{\mu}\left(P_1\right)\,. \tag{4.6}$$

The first three constraints, Eqs. (4.5), are the direct counterpart of the corresponding ones for classical IVPMs. With the understanding that the union of classical sets $E_0 \cup E_1$ is replaced by $P_0 + P_1 - P_0 P_1$ in the case of quantum projection operators [33], the last condition, Eq. (4.6), is a direct counterpart of the convexity condition of Eq. (4.2). Thus our definition of QIVPMs merges aspects of both classical IVPMs and quantum probability measures.

4.1.2 RELATIONS BETWEEN CLASSICAL AND QUANTUM IVPMS

Our definition of QIVPMs is consistent with classical IVPMs in the sense that a restriction of QIVPMs to mutually commuting events, $\mathcal{E}_C$, recovers the definition of classical IVPMs. To see this, note that the commuting $\mathcal{E}_C$ can be diagonalized by a common orthonormal basis $\Omega = \{|\psi_0\rangle\,, |\psi_1\rangle\,, \ldots\,, |\psi_{D-1}\rangle\}$.

Since Ω is a finite set, it can be the sample space of a classical probability space with the classical event space 2^Ω. Because a QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$ and a classical IVPM $\bar{\mu}^\Omega : 2^\Omega \to \mathcal{I}$ are only differed by their domains, if we can find a function $\upsilon : 2^\Omega \to \mathcal{E}$ preserving the structure, we can pullback [82] any QIVPM $\bar{\mu}$ to $\bar{\mu}^\Omega$ such that $\bar{\mu}(\upsilon(E)) = \bar{\mu}^\Omega(E)$ for any $E \subseteq \Omega$. **TODO. Although $\bar{\mu}^\Omega$ looks like the pullback of $\bar{\mu}$ by υ [82], $\mathcal{E}_C$ and 2^Ω are not categories, and υ is not preserving algebraic or analytical properties... So I don't know whether people call this "pullback" or not.**

We now explicitly define υ and $\bar{\mu}^\Omega$. First, $\mathcal{E}'$ is called a *subspace* of the set of events $\mathcal{E}$ if $\mathcal{E}'$ contains the projectors $\mathbb{0}$ and $\mathbb{1}$ and is closed under complements, sums, and products. In particular, for any projector $P \in \mathcal{E}'$, we have $\mathbb{1} - P \in \mathcal{E}'$ and for each pair of commuting projectors $P_0 \in \mathcal{E}'$ and $P_1 \in \mathcal{E}'$, we have $P_0 P_1$ and $P_0 + P_1 - P_0 P_1 \in \mathcal{E}'$. Given a mutually commuting subspace $\mathcal{E}_C$ diagonalized by a common orthonormal basis $\Omega = \{|\psi_0\rangle, |\psi_1\rangle, ..., |\psi_{D-1}\rangle\}$, we define the function $\upsilon : 2^\Omega \to \mathcal{E}$ maps any set E to the sum of the projectors formed by elements in E, i.e., $\upsilon(E) = \sum_{|j\rangle \in E} |j\rangle\langle j|$, where we follow the convention $\sum_{P \in \emptyset} P = \mathbb{0}$. According to its definition, the function υ naturally sends the set operations to the corresponding projector operations:

$$\upsilon(\emptyset) = \mathbb{0}, \qquad \upsilon(\Omega) = \mathbb{1}, \qquad \upsilon\left(\overline{E}\right) = \mathbb{1} - \upsilon(E), \tag{4.7}$$

and for each pair of set $E_0 \subseteq \Omega$ and $E_1 \subseteq \Omega$, we have

$$\upsilon(E_0 \cap E_1) = \upsilon(E_0)\,\upsilon(E_1), \qquad \upsilon(E_0 \cup E_1) = \upsilon(E_0) + \upsilon(E_1) - \upsilon(E_0)\,\upsilon(E_1). \tag{4.8}$$

Therefore, the function $\bar{\mu}^\Omega : 2^\Omega \to \mathcal{I}$ defined by $\bar{\mu}^\Omega(E) = \bar{\mu}(\upsilon(E))$ is the pullback of $\bar{\mu}$ by υ, and is a classical IVPM naturally.

Since we can pullback a QIVPM to a classical one, known properties of classical IVPMs directly hold for QIVPMs when one restricts to mutually commuting events, $\mathcal{E}_C$. In particular, in the classical world, it is *impossible* for experiments to result in probabilities that are inconsistent with *every* state

of the system under consideration, i.e., all IVPMs must have a non-empty "core" [1]. Interestingly, as we show in Sec. 4.3, it is possible in the quantum world for the probabilities associated with some events to be inconsistent with *any* quantum state, i.e., for the QIVPM to have an empty core; in that case, one cannot guarantee non-empty cores for finite-precision attempts at proving Gleason's theorem by extending the Born measure $\mu_\rho^{\mathrm{B}}(P)$ to QIVPMs $\bar{\mu}(P)$. However, if we restrict ourselves to the set $\mathcal{E}_C$ of mutually commuting events, the situation reverts to the classical case in which probabilities always determine at least one state.

We now give the necessary technical definitions to prove this non-empty core property.

Definition 4.3 (Consistency)**.** We say a QIVPM $\bar{\mu}$ is *consistent* with a state ρ and a projector P if the interval $\bar{\mu}(P)$ contains the exact probability calculated by the Born rule [40, 41, 39], i.e.,

$$\mu_\rho^{\mathrm{B}}(P) = \mathrm{Tr}(\rho P) \in \bar{\mu}(P) \ . \tag{4.9}$$

In contrast with classical probability spaces[1], there is *no guarantee* that there exists a state ρ that satisfies Eq. (4.9) and therefore is consistent with a QIVPM.

We next refine the concept of consistency by introducing the idea of a "core" set of states relative to subspaces.

Definition 4.4 (The core of a probability measure)**.** The *core* $\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')$ of a probability measure $\bar{\mu}$ relative to a subspace of events $\mathcal{E}'$ is the collection of all states ρ that are *consistent* with $\bar{\mu}$ on every projector in $\mathcal{E}'$, that is

$$\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}') = \left\{ \rho \;\middle|\; \forall P \in \mathcal{E}',\ \mu_\rho^{\mathrm{B}}(P) \in \bar{\mu}(P) \right\} \ . \tag{4.10}$$

We are now in position to state and prove that, for the special case of commuting events, a

[1] A result by Shapley [79, 75, 80, 34] proves that a classical IVPM always contains at least one state that is consistent with every event.

QIVPM will always have a non-empty core.

Theorem 4.1 (Non-empty Core for Compatible Measurements)**.** *For every QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$, if a subspace of events $\mathcal{E}_C \subseteq \mathcal{E}$ commutes, then $\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}_C) \neq \emptyset$.*

An outline of the proof, detailed in the forthcoming thesis [83], proceeds as follows. From a subspace $\mathcal{E}_C$ of mutually commuting events, one can construct a partial orthonormal basis by diagonalization, and complete this to a full orthonormal basis $\overline{\mathcal{E}_C}$. We can then build a bijection between the QIVPM on the set of projectors associated with this basis and the set of classical events corresponding to this basis. Using this correspondence together with the classical result by Shapley [79, 75, 80, 34], we can establish that for the special case of commuting events, a QIVPM will always have a non-empty core.

4.1.3 CLASSICAL CHOQUET INTEGRALS AND EXPECTATION VALUES OF OBSERVABLES

We conclude this section with a generalization of expectation values of observables in the context of QIVPMs. In conventional quantum mechanics the expectation value of an observable as defined in Eq. (2.15) is a unique real number. The generalization to QIVPMs implies that this expectation value will itself become bounded by an interval.

Definition 4.5 (Expectation Value of Observables over QIVPMs)**.** Let $\mathcal{I}$ be a set of intervals; $\mathcal{H}$ a Hilbert space of dimension D with event space $\mathcal{E}$; and $\mathbf{O}$ an observable with spectral decomposition $\sum_{i=1}^{D} \lambda_i P_i$. Let $\mathcal{E}'$ be the minimal subspace of events containing all the projectors P_i in the spectral decomposition of $\mathbf{O}$ and define:

$$\langle \mathbf{O} \rangle_{\bar{\mu}} = \left[\min_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}} , \max_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}} \right] . \tag{4.11}$$

Intuitively the expectation value of an observable relative to a QIVPM $\bar{\mu}$ lies between two possible

outcomes, which themselves lie between the minimum and maximum bounds of the probability intervals associated with each state ρ that is consistent with $\bar{\mu}$ on every projector in the spectral decomposition of the observable. If $\bar{\mu}$ is a conventional (Born) probability measure induced by a state ρ, then the Born rule probability induced by every state in $\overline{\mathcal{H}}\left(\bar{\mu}, \mathcal{E}'\right)$ will be μ_ρ^{B} and the interval collapses to a point, thus reducing the definition to that of Eq. (2.15) [83]. We also note that, when restricted to commuting projectors, Eq. (4.11) is consistent with the classical notion of the *Choquet integral* [34, 75, 84] which is used to calculate the expectation value of random variables as a weighted average [83].

4.2 THE KOCHEN-SPECKER THEOREM AND CONTEXTUALITY

Our generalization of quantum probability measures to QIVPMs allows us to strengthen the scope of one of the fundamental theorems of quantum physics: the Kochen-Specker theorem [37, 39, 42, 85–88]. Our finite-precision extension of that theorem will suggest a resolution to the debate initiated by Meyer and Mermin on the relevance of the Kochen-Specker to experimental, and hence finite-precision, quantum measurements [89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101]. Specifically, the original Kochen-Specker theorem is formulated using a model quantum mechanical system that has *definite values at all times* [88], i.e., its observables have infinitely precise values at all times. Our interval-valued probability framework will allow us to state, and prove, a stronger version of the theorem that holds even if the observables have values that are only definite up to some precision specified by a parameter δ. Our approach provides a quantitative realization of Mermin's intuition [91]:

> …although the outcomes deduced from such imperfect measurements will occasionally differ dramatically from those allowed in the ideal case, if the misalignment is very slight, the statistical distribution of outcomes will differ only slightly from the ideal case.

4.2.1 FINITE-PRECISION EXTENSION OF THE KOCHEN-SPECKER THEOREM

The first step in our formalization is to introduce a family of QIVPMs parameterized by an uncertainty δ, which we call δ-deterministic QIVPMs.

Definition 4.6 (δ-Determinism)**.** A QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$ is δ-deterministic if, for every event $P \in \mathcal{E}$, we have that either $\bar{\mu}(P) \subseteq [0, \delta]$ or $\bar{\mu}(P) \subseteq [1-\delta, 1]$.

This definition puts no restrictions on the set of intervals itself, only on which intervals are assigned to events. When $\delta = 0$, every event must be assigned a probability either in $\mathbf{F}$ or in $\mathbf{T}$, i.e., every event is completely determined with certainty. As δ gets larger, the QIVPM allows for more indeterminate behavior.

The expectation value of an observable $\mathbf{O}$ in a Hilbert space H of dimension D relative to a 0-deterministic QIVPM is fully determinate and is equal to one of the eigenvalues λ_i of that observable. To see this, note that given an orthonormal basis $\Omega = \{|\psi_0\rangle, |\psi_1\rangle, ..., |\psi_{D-1}\rangle\}$, a 0-deterministic QIVPM must map exactly one of the projectors $|\psi_i\rangle\langle\psi_i|$ to $\mathbf{T}$ and all others to $\mathbf{F}$. This is because, by Eq. (**??**), we have $\bar{\mu}\left(\sum_{j=0}^{D-1} |\psi_j\rangle\langle\psi_j|\right) = \mathbf{T}$ and by inductively applying Eq. (4.6), we must have one of the $\bar{\mu}(|\psi_i\rangle\langle\psi_i|) = \mathbf{T}$ and all others mapped to $\mathbf{F}$. Given any state ρ that is consistent with this QIVPM on all the projectors in Ω, we have by Eq. (4.10) that μ_ρ^{B} must also map exactly one of the projectors in Ω to 1 and all others to 0. If an observable has a spectral decomposition along Ω then, by Eq. (2.15), its expectation value relative to μ_ρ^{B} is the eigenvalue λ_i whose projector is mapped to 1. It therefore follows, by Eq. (4.11), that the expectation value relative to the 0-deterministic $\bar{\mu}$ is fully determinate and lies in the interval $[\lambda_i, \lambda_i]$.

We can now proceed with the main technical result of this section. We first observe that the original Kochen-Specker theorem is a statement regarding the non-existence of a 0-deterministic QIVPM, and generalize to a corresponding statement about δ-deterministic QIVPMs.

Theorem 4.2 (0-Deterministic Variant of the Kochen-Specker Theorem)**.** *Given a Hilbert space $\mathcal{H}$ of dimension $D \geq 3$, there is no 0-deterministic measure $\bar{\mu}$ mapping every event to either* **F** *or* **T**.

To explain why this result is equivalent to the original Kochen-Specker theorem and to prove it at the same time, we proceed by assuming a 0-deterministic QIVPM $\bar{\mu}$ and derive the same contradiction as the original Kochen-Specker theorem. Instead of adapting the more complicated proof for $D = 3$, the counterexample presented below uses the simpler proof for a Hilbert space of dimension $D = 4$ and is constructed as follows.

We consider a two spin-$\frac{1}{2}$ Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of dimension $D = 4$. We use the same nine observables $\mathbf{O}_{ij}$ with i and j ranging over $\{0, 1, 2\}$ from the Mermin-Peres "magic square" used to prove the Kochen-Specker theorem [33, 42, 87]:

$\mathbf{O}_{ij}$	$j = 0$	$j = 1$	$j = 2$
$i = 0$	$\mathbb{1} \otimes \sigma_z$	$\sigma_z \otimes \mathbb{1}$	$\sigma_z \otimes \sigma_z$
$i = 1$	$\sigma_x \otimes \mathbb{1}$	$\mathbb{1} \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$i = 2$	$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$
subspace			

The observables are constructed using the Pauli matrices $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ whose eigenvalues are all either 1 or -1 [32, 33, 37, 39, 41]. They are arranged such that in each row and column, *except the column* $j = 2$, every observable is the product of the other two. In the $j = 2$ column, we have instead that $(\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x) = -\sigma_y \otimes \sigma_y$. Now assume a 0-deterministic QIVPM $\bar{\mu}$; the expectation values of the observables in each row relative to this 0-deterministic QIVPM are fully determinate

and must lie in either the interval $[1, 1]$ or the interval $[-1, -1]$ depending on which eigenvalue is the one whose associated projector is certain. Since the product of any two observables in a row is equal to the third, there must be an even number of occurrences of the interval $[-1, -1]$ in each row and hence in the entire table [83]. However, looking at the expectation values of the observables in each column, there must be an even number of occurrences of the interval $[-1, -1]$ in the first two columns and an odd number in the $j = 2$ column and hence in the entire table [83]. The contradiction implies the non-existence of the assumed 0-deterministic QIVPM.

Our framework allows us to generalize the above theorem to state that for small enough δ, it is impossible to have δ-deterministic QIVPMs, which is a stronger statement of contextuality that includes the effects of finite-precision. Every QIVPM must map some events to truly uncertain intervals, not just "almost definite intervals." The proof requires two simple lemmas that we present first.

The first lemma shows a simpler way to prove the convexity condition. Recall that the convexity condition for a QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$ states that for each pair of *commuting* projectors P_0 and P_1 with $P_0 P_1 = P_1 P_0$, the following equation holds:

$$\bar{\mu}\left(P_0 + P_1 - P_0 P_1\right) + \bar{\mu}\left(P_0 P_1\right) \subseteq \bar{\mu}\left(P_0\right) + \bar{\mu}\left(P_1\right) . \tag{4.12}$$

Lemma 4.1. *To verify the convexity condition of a QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$, it is sufficient to check that:*

$$\bar{\mu}\left(P' + P''\right) = \bar{\mu}\left(P'\right) + \bar{\mu}\left(P''\right) \tag{4.13}$$

for all orthogonal projectors P' and P''.

The proof follows the outline of the proof of the classical inclusion-exclusion principle. From the commuting projectors P_0 and P_1, we construct the following three orthogonal projectors: $P_0 P_1$, $P_0\left(\mathbb{1} - P_1\right)$, and $\left(\mathbb{1} - P_0\right) P_1$. Then we proceed as follows:

$$
\begin{aligned}
&\bar{\mu}\left(P_0 + P_1 - P_0P_1\right) + \bar{\mu}\left(P_0P_1\right) \\
&\quad = \bar{\mu}\left(P_0P_1 + P_0(\mathbb{1} - P_1) + P_1 - P_0P_1\right) + \bar{\mu}\left(P_0P_1\right) && \text{(because } P_0 = P_0P_1 + P_0 - P_0P_1\text{)} \\
&\quad = \bar{\mu}\left(P_0(\mathbb{1} - P_1) + P_1\right) + \bar{\mu}\left(P_0P_1\right) \\
&\quad = \bar{\mu}\left(P_0(\mathbb{1} - P_1) + P_0P_1 + (\mathbb{1} - P_0)P_1\right) + \bar{\mu}\left(P_0P_1\right) && \text{(because } P_1 = P_0P_1 + P_1 - P_0P_1\text{)} \\
&\quad = \bar{\mu}(P_0(\mathbb{1} - P_1)) + \bar{\mu}(P_0P_1) + \bar{\mu}((\mathbb{1} - P_0)P_1) + \bar{\mu}\left(P_0P_1\right) && \text{(using Eq. (4.13) twice)} \\
&\quad = \bar{\mu}(P_0(\mathbb{1} - P_1) + P_0P_1) + \bar{\mu}((\mathbb{1} - P_0)P_1 + P_0P_1) && \text{(using Eq. (4.13) twice)} \\
&\quad = \bar{\mu}(P_0) + \bar{\mu}(P_1)
\end{aligned}
$$

The next lemma relates δ-deterministic QIVPMs with $\delta < \frac{1}{3}$ to 0-deterministic QIVPMs.

Lemma 4.2. *From any δ-deterministic QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$ with $\delta < \frac{1}{3}$, we can construct a* 0*-deterministic QIVPM $\bar{\mu}^{\mathrm{D}} : \mathcal{E} \to \{\mathsf{F}, \mathsf{T}\}$ defined as follows:*

$$
\bar{\mu}^{\mathrm{D}}\left(P\right) = \begin{cases} \mathsf{F} & \text{if } \bar{\mu}\left(P\right) \subseteq [0, \delta] \ ; \\ \mathsf{T} & \text{if } \bar{\mu}\left(P\right) \subseteq [1-\delta, 1] \ . \end{cases} \tag{4.14}
$$

The most important part of the proof is to verify the convexity condition for $\bar{\mu}^{\mathrm{D}}$. By Lemma 4.1, it is sufficient to verify the following equation for orthogonal projectors P' and P'',

$$
\bar{\mu}^{\mathrm{D}}\left(P' + P''\right) = \bar{\mu}^{\mathrm{D}}\left(P'\right) + \bar{\mu}^{\mathrm{D}}\left(P''\right) \ , \tag{4.15}
$$

for two cases, which we now examine in detail.

When one of $\bar{\mu}^{\mathrm{D}}\left(P'\right)$ and $\bar{\mu}^{\mathrm{D}}\left(P''\right)$ is T, say $\bar{\mu}^{\mathrm{D}}\left(P'\right) = \mathsf{F}$ and $\bar{\mu}^{\mathrm{D}}\left(P''\right) = \mathsf{T}$, we have $\bar{\mu}\left(P'\right) \subseteq [0, \delta]$ and $\bar{\mu}\left(P''\right) \subseteq [1-\delta, 1]$ which implies $\bar{\mu}\left(P' + P''\right) \subseteq [1-\delta, 1+\delta]$. Since $\bar{\mu}\left(P' + P''\right)$ is a subset of $[0, 1]$, $\bar{\mu}\left(P' + P''\right)$ must be a subset of $[1-\delta, 1]$, which implies $\bar{\mu}^{\mathrm{D}}\left(P' + P''\right)$ is also T, thus satisfying Eq. (4.15).

When both $\bar{\mu}^{\mathrm{D}}\left(P'\right)$ and $\bar{\mu}^{\mathrm{D}}\left(P''\right)$ are F, we have both $\bar{\mu}\left(P'\right)$ and $\bar{\mu}\left(P''\right) \subseteq [0, \delta]$ which implies

Table 4.1: Possible probability measures on a Hilbert space of dimension $D = 3$, where $\bar{\mu}'_2$ and $\bar{\mu}_3$ are QIVPMs while $\bar{\mu}_0$, $\bar{\mu}_1$, and $\bar{\mu}_2$ are not. Events are listed in the column labeled by P.

P	$\bar{\mu}_0(P)$	$\bar{\mu}_1(P)$	$\bar{\mu}_2(P)$	$\bar{\mu}'_2(P)$	$\bar{\mu}_3(P)$
$\mathbb{0}$	**F**	**F**	**F**	**F**	**F**
All one-dimensional projectors	$[0,0]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{3}]$	$[\frac{1}{3},\frac{1}{3}]$	$[0,\frac{1}{2}]$
All two-dimensional projectors	$[1,1]$	$[\frac{3}{4},1]$	$[\frac{2}{3},1]$	$[\frac{2}{3},\frac{2}{3}]$	$[\frac{1}{2},1]$
$\mathbb{1}$	**T**	**T**	**T**	**T**	**T**

$\bar{\mu}(P' + P'') \subseteq [0, 2\delta]$. Since we assume $\delta < \frac{1}{3}$, $[0, 2\delta]$ and $[1-\delta, 1]$ are disjoint, which implies $\bar{\mu}(P' + P'')$ and $[1-\delta, 1]$ are disjoint. Together with the fact that $\bar{\mu}(P' + P'')$ is a subset of either $[0, \delta]$ or $[1-\delta, 1]$, $\bar{\mu}(P' + P'')$ must be a subset of $[0, \delta]$, which implies $\bar{\mu}^{\mathrm{D}}(P' + P'') = \mathbf{F}$, and hence also Eq. (4.15) is again satisfied.

Theorem 4.3 (Finite-precision Extension of the Kochen-Specker Theorem)**.** *Given a Hilbert space $\mathcal{H}$ of dimension $D \geq 3$, there is no δ-deterministic QIVPM for $\delta < \frac{1}{3}$.*

The proof is by contradiction: Suppose there is a δ-deterministic QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$. By Lemma 4.2, we can construct a 0-deterministic QIVPM; however, by Thm. 4.2, such 0-deterministic QIVPMs do not exist.

The bound $\delta < \frac{1}{3}$ is tight as it is possible to construct a $\frac{1}{3}$-deterministic QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$. For example, $\bar{\mu}'_2$ defined in Table 4.1 is a valid $\frac{1}{3}$-deterministic QIVPM. When $\delta \geq \frac{1}{3}$, i.e, when the uncertainty in measurements becomes so large, it becomes possible to map every observable to some (quite inaccurate) probability interval, thus invalidating the Kochen-Specker theorem. We can summarize and illustrate the above arguments using Fig. 4.1.

As is the case for conventional, infinitely-precise, quantum probability measures, the theorem is only applicable to dimensions $D \geq 3$. Indeed when the Hilbert space has dimension 2, it is straightforward to construct a 0-deterministic QIVPM as follows. Consider a non-contextual hidden variable model for $D = 2$ (e.g., as proposed by Bell or Kochen-Specker [85, 86]). Such a two-dimensional

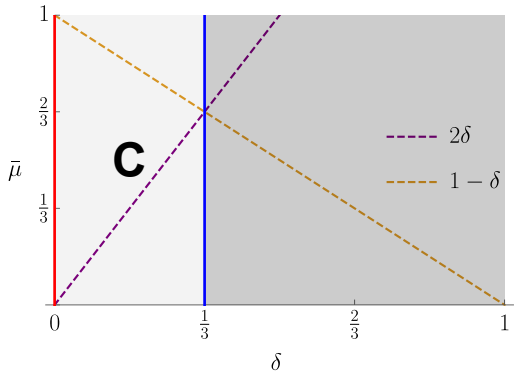


Figure 4.1: The region to the left of the vertical line at $\delta = \frac{1}{3}$ is where we assume small measurement degradation; in that region our extension of the KS theorem definitely demonstrates contextuality (C). In the region to the right, the degradation of the data is large and our extension of the KS theorem no longer refutes other explanations for the experimental data.

model assigns definite values to all observables at all times, and hence assigns a *determinate* probability (0 or 1) to each event. This probability measure directly induces a 0-deterministic QIVPM by changing 0 to **F** and 1 to **T**. It follows that every 0-deterministic QIVPM is δ-deterministic.

4.2.2 EXPERIMENTAL DATA AND δ-DETERMINISM

We have thus quantified one important aspect of uncertainty in quantum mechanics—the effect of the imprecise nature of devices—which is a novel addition to the theory of measurement. Indeed, as Heisenberg emphasized in his famous microscope example [102], the conventional theory of measurement states that it is impossible to precisely measure any property of a system without disturbing it somewhat. Thus, there are fundamental limits to what one can measure and these limits have traditionally been attributed to complementarity. Our imprecision represents an *additional* source of indeterminacy beyond the inherent probabilistic nature of quantum mechanics.

In an experimental setup, δ is calculated as follows. To determine the probability of any event, we typically repeat an experiment m times and count the number of times we witness the event. This assumes that for each run of the experiment we can determine, using our apparatus, whether the event occurred or not. Assume an event has an ideal mathematical probability of 0, and we repeat

the experiment 100 times. In a perfect world we should be able to refute the event 100 times and calculate that the probability is 0. We might also observe the event 2 times and refute it 98 times and therefore calculate the probability to be 0.02. Note that this situation assumes perfect measurement conditions and remains within the context of conventional (real-valued) probability theory. The question we focus on is what happens if we are only able to refute it 97 times and are *uncertain* 3 times? This is quite common in actual experiments. Mathematically we can model this idea by stating that the probability of the event is in the range $[0, 0.03]$ which says that the probability of the event could be 0, 0.01, 0.02, or 0.03 as each the three uncertain records could either be evidence for the event or against it. We just cannot nail it down given the current experimental results and therefore represent the evidence as a ($\delta =$)0.03-deterministic probability measure. The interesting observation is that the axioms of probability theory (like additivity and convexity) impose enough constraints on the structure of interval-valued quantum probability measures to make them robust in the face of small non-vanishing δ's.

To see this idea in the context of a quantum experiment, consider a three-dimensional Hilbert space with one-dimensional projectors P_ρ, two-dimensional projectors $P_\rho + P_\sigma$, and an experiment that is repeated 12 times. By the Kochen-Specker theorem, it is impossible to build a probability measure that maps every projection to either $0 = \frac{0}{12}$ or $1 = \frac{12}{12}$. That is, the assignment $\bar{\mu}_0$ defined in Table 4.1 is not a QIVPM.

Now consider what happens if $\frac{1}{4}$ of the data for *every* one-dimensional projector is uncertain. A potential account of this degradation is to assign to each event P the entire range of possibilities $\bar{\mu}_1(P)$ as defined in Table 4.1. This measure is not a valid QIVPM because it does not satisfy the convexity condition: for any two orthogonal one-dimensional events P_0 and P_1, the convexity condition requires $\bar{\mu}_1(P_0 + P_1) \subseteq \bar{\mu}_1(P_0) + \bar{\mu}_1(P_1)$, but $\bar{\mu}_1(P_0 + P_1) = [\frac{3}{4}, 1]$ which is not a subset of $[0, \frac{1}{2}] = \bar{\mu}_1(P_0) + \bar{\mu}_1(P_1)$. Interestingly, it is impossible to find any probability measure that would be consistent with these observations, as the interval $[\frac{3}{4}, 1]$ is completely disjoint from

the interval $[0, \frac{1}{2}]$ and no amount of shifting of assumptions regarding the precise outcome of the uncertain observations could change that disjointness. However, as shown next, a sharp transition occurs when $\delta = \frac{1}{3}$.

When the proportion of uncertain data reaches $\frac{1}{3}$, the probability measure that assigns to each event the entire range of possibilities is $\bar{\mu}_2$ defined in Table 4.1. This is also not a valid probability measure by the same argument as above. However, in this case $\bar{\mu}_2(P_0 + P_1) = [\frac{2}{3}, 1]$ and $[0, \frac{2}{3}] = \bar{\mu}_2(P_0) + \bar{\mu}_2(P_1)$ have a *common point*. Hence, by assuming that the uncertain data for one-dimensional projectors always support the associated event, while those for two-dimensional projectors always refute the event, we can find the probability measure $\bar{\mu}_2'$ that can verified to be a valid QIVPM and is consistent with the experimental data.

A similar situation happens when more than $\frac{1}{3}$ of data is uncertain. In particular, if half of the data is uncertain, the probability measure $\bar{\mu}_3$ that assigns to each event the entire range of possibilities is already a QIVPM.

4.3 THE BORN RULE AND GLEASON'S THEOREM

A conventional quantum probability measure can be easily constructed from a state ρ according to the Born rule [40, 41, 39]. According to Gleason's theorem [36, 37, 42], this state ρ is also the unique state consistent with any possible probability measure.

4.3.1 FINITE-PRECISION EXTENSION OF GLEASON'S THEOREM

In order to re-examine these results in our framework, we first reformulate Gleason's theorem in QIVPMs using infinitely precise uncountable intervals $\mathcal{I}_\infty = \{[x, x] \mid x \in [0, 1]\}$:

Theorem 4.4 ($\mathcal{I}_\infty$ Variant of the Gleason Theorem)**.** *In a Hilbert space $\mathcal{H}$ of dimension $D \geq 3$, given a QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}_\infty$, the state ρ consistent with $\bar{\mu}$ on every projector is unique, i.e., there exists a unique state ρ such that $\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}) = \{\rho\}$.*

Table 4.2: QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}_0$ on a Hilbert space of dimension $D = 3$. Events are listed in the column labeled by P.

P	$\bar{\mu}(P)$
$\mathbb{0}$, $\|0\rangle\langle 0\|$, $\|+\rangle\langle +\|$, $\|+'\rangle\langle +'\|$	**F**
$\mathbb{1}$, $\mathbb{1} - \|0\rangle\langle 0\|$, $\mathbb{1} - \|+\rangle\langle +\|$, $\mathbb{1} - \|+'\rangle\langle +'\|$	**T**
All other projectors	**U**

Now let us consider relaxing $\mathcal{I}$ to a countable set of finite-width intervals. As the intervals in the image of a QIVPM become less and less sharp, we expect more and more states to be consistent with it. In the limit of minimal sharpness, all states ρ are consistent with the QIVPM

$$\bar{\mu}(P) = \begin{cases} \mathbf{F} & \text{if } P = \mathbb{0}\,; \\ \mathbf{T} & \text{if } P = \mathbb{1}\,; \\ \mathbf{U} = [0,1] & \text{otherwise} \end{cases} \tag{4.16}$$

mapping nearly all projections to the *unknown* interval **U**. There is however a subtlety: as shown in the theorem below, it is possible for an arbitrary assignment of intervals to projectors to be globally inconsistent.

Theorem 4.5 (Empty Cores Exist for General QIVPMs)**.** *There exists a Hilbert space H and a QIVPM $\bar{\mu} : \mathcal{E} \to \mathcal{I}$ such that $\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}) = \emptyset$.*

To prove this theorem, we need to construct a QIVPM on some Hilbert space, and verify that there are no states that are consistent (see Defs. 4.3 and 4.4) with it on all possible events. Assume a Hilbert space of dimension $D = 3$ with orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$, let $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|+'\rangle = (|0\rangle + |2\rangle)/\sqrt{2}$, and assign

$$\mathcal{I}_0 = \{\mathbf{T}, \mathbf{F}, \mathbf{U}\}\ . \tag{4.17}$$

The map $\bar{\mu} : \mathcal{E} \to \mathcal{I}_0$ defined in Table 4.2 can be verified to be a QIVPM [83].

Next we will prove by contradiction that $\overline{\mathcal{H}}\,(\bar{\mu}, \mathcal{E})$ is the empty set. Suppose there is a state $\rho = \sum_{j=1}^{N} q_j \left|\phi_j\right\rangle\!\left\langle\phi_j\right| \in \overline{\mathcal{H}}\,(\bar{\mu}, \mathcal{E})$, where $\sum_{j=1}^{N} q_j = 1$ and $q_j > 0$. Since we assumed the core $\overline{\mathcal{H}}\,(\bar{\mu}, \mathcal{E})$ is non-empty, so $\mu_\rho^{\mathrm{B}}(P) \in \bar{\mu}(P)$, and Table 4.2 tells us that $\bar{\mu}(|0\rangle\langle 0|) = \mathbf{F} = [0, 0]$, we must conclude that $\mu_\rho^{\mathrm{B}}(|0\rangle\langle 0|) = 0 \in [0, 0]$, and similarly for $|+\rangle\langle +|$ and $|+'\rangle\langle +'|$. If this is true, then $\langle 0|\phi_j\rangle = \langle +|\phi_j\rangle = \langle +'|\phi_j\rangle = 0$ for all j, and thus

$$\langle 1|\phi_j\rangle = \sqrt{2}\,\langle +|\phi_j\rangle - \langle 0|\phi_j\rangle = 0\,, \tag{4.18a}$$

$$\langle 2|\phi_j\rangle = \sqrt{2}\,\langle +'|\phi_j\rangle - \langle 0|\phi_j\rangle = 0\,. \tag{4.18b}$$

The above equations imply $\left|\phi_j\right\rangle = |0\rangle\,\langle 0|\phi_j\rangle + |1\rangle\,\langle 1|\phi_j\rangle + |2\rangle\,\langle 2|\phi_j\rangle = 0$, violating the assumption that $\left|\phi_j\right\rangle$ is a normalized state, and thus the theorem is proved.

The fact that a collection of poor measurements on a quantum system cannot reveal the underlying state is not surprising. Under certain conditions, we can however guarantee that the uncertainty in measurements is consistent with *some* non-empty collection of quantum states. Furthermore, we can relate the uncertainty in measurements to the volume of quantum states such that, in the limit of infinitely precise measurements, the volume of states collapses to a single state.

To that end, we introduce the concept of *interval maps*, which we can use to construct a consistent family of QIVPMs. An interval map $f : [0, 1] \to \mathcal{I}$ maps every real-valued probability $x \in [0, 1]$ to a set of intervals $f\,(x) = [\ell, r]$ containing x, where $[0, 1]$ denotes the set of real-valued probabilities (this should not be confused with the interval-valued probability $\mathbf{U}$). We also need a notion of *norm* to quantify the distance between (pure or mixed) states. The norm of a pure state $\rho = |\psi\rangle\langle\psi|$ is defined as usual by $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. For any given Hermitian operator A, we choose the operator norm $\|A\| = \max_{\|\psi\|=1} \|A\,|\psi\rangle\|$, which is also known as the 2-norm or the spectral norm [42, 103–105]. In fact, for any such matrix, including the density matrix ρ, this norm is the maximum absolute value of its eigenvalues. Then, a finite-precision extension of Gleason's theorem can be stated as follows:

Theorem 4.6 (Finite-Precision Extension of the Gleason Theorem)**.** *Let $f : [0,1] \to \mathcal{I}$ be an interval map and let the composition $f \circ \mu_\rho^{\mathrm{B}}$ be a QIVPM, where μ_ρ^{B} is the probability measure induced by the Born rule for a given state ρ. Let α be the maximum length of intervals in $\mathcal{I}$. If a state ρ' is consistent with $f \circ \mu_\rho^{\mathrm{B}}$ on all events, i.e., $\rho' \in \overline{\mathcal{H}}\left(f \circ \mu_\rho^{\mathrm{B}}, \mathcal{E}\right)$, then the norm of their difference is bounded by α, i.e., $\|\rho - \rho'\| \leq \alpha$.*

The proof proceeds as follows. Given a state ρ' consistent with $f \circ \mu_\rho^{\mathrm{B}}$, we have $\mu_{\rho'}^{\mathrm{B}}\left(|\psi\rangle\langle\psi|\right) \in f\left(\mu_\rho^{\mathrm{B}}\left(|\psi\rangle\langle\psi|\right)\right)$ for any one-dimensional projector $P = |\psi\rangle\langle\psi|$. Since the maximum length of the intervals in $\mathcal{I}$ is α, it is also the upper bound of the difference:

$$\left|\mu_{\rho'}^{\mathrm{B}}\left(|\psi\rangle\langle\psi|\right) - \mu_\rho^{\mathrm{B}}\left(|\psi\rangle\langle\psi|\right)\right| = |\langle\psi|\rho - \rho'|\psi\rangle| \leq \alpha\,.$$

Since $\rho - \rho'$ is Hermitian, $\max_{\|\psi\|=1} |\langle\psi|\rho - \rho'|\psi\rangle|$ is the maximum absolute value of the eigenvalues of $\rho - \rho'$ [32], and equal to $\|\rho - \rho'\|$ [104, 105]. Hence, $\|\rho - \rho'\| \leq \alpha$.

4.3.2 ULTRAMODULAR FUNCTIONS

Theorem 4.6 generalizes Gleason's theorem in the sense that it accounts for a larger class of probability measures that includes the conventional one as a limit. The theorem is however "special" in the sense that it only applies to the particular class of QIVPMs constructed by composing an interval map with a conventional quantum probability measure. QIVPMs constructed in this manner have some peculiar properties that we examine next.

An interval map is called *ultramodular* if it satisfies the following properties:

Definition 4.7 (Ultramodular Functions)**.** Given a collection of intervals $\mathcal{I}$ including $\mathbf{F}$ and $\mathbf{T}$, an

interval map $\mathcal{M} : [0,1] \to \mathcal{I}$ is called ultramodular if:

$$\mathcal{M}(0) = \mathbf{F}\,, \tag{4.19a}$$

$$\mathcal{M}(1) = \mathbf{T}\,, \tag{4.19b}$$

$$\mathcal{M}\left(1-x\right) = \mathbf{T} - \mathcal{M}\left(x\right)\,, \tag{4.19c}$$

and for any three numbers x_0, x_1, and $x_2 \in [0,1]$ such that $y = x_0 + x_1 + x_2 \in [0,1]$, we have:

$$\mathcal{M}\left(y\right) + \mathcal{M}\left(x_2\right) \subseteq \mathcal{M}\left(x_0 + x_2\right) + \mathcal{M}\left(x_1 + x_2\right)\,. \tag{4.20}$$

The first three constraints, Eqs. (4.19), are the direct counterpart of the corresponding QIVPM constraints, Eqs. (4.5); the last condition, Eq. (4.20), is the direct counterpart of the convexity conditions, Eqs. (4.2) and (4.6) [84, 79, 80, 81]. Therefore, these conditions guarantee that, for any conventional quantum probability measure μ, the composition $\mathcal{M} \circ \mu$ defines a valid QIVPM. Conversely, if for every quantum probability measure μ, it is the case that $f \circ \mu$ is a QIVPM, then the interval map f is an ultramodular function. Formally, we have the following result:

Theorem 4.7 (Equivalence of Ultramodular Functions and IVPMs)**.** *The following three statements are equivalent:*

1. *A function $\mathcal{M} : [0,1] \to \mathcal{I}$ is ultramodular.*
2. *The composite function $\mathcal{M} \circ \mu : \mathcal{E}_C \to \mathcal{I}$ is a classical IVPM for all classical probability measures $\mu : \mathcal{E}_C \to [0,1]$.*
3. *The composite function $\mathcal{M} \circ \mu : \mathcal{E} \to \mathcal{I}$ is a QIVPM for all quantum probability measures $\mu : \mathcal{E} \to [0,1]$.*

Statement 1 implies 2 and 3 as we have outlined above. Conversely, for the quantum case, we want to show that if $\mathcal{M}$ is not ultramodular, then for some quantum probability measure μ, the

composite $\mathcal{M} \circ \mu$ might not be a QIVPM. Suppose there are three particular numbers x_0, x_1, and $x_2 \in [0,1]$ such that $y = x_0 + x_1 + x_2 \in [0,1]$, but they don't satisfy Eq. (4.20). Consider the state:

$$\rho = x_0 \left|0\right\rangle\!\left\langle 0\right| + x_1 \left|1\right\rangle\!\left\langle 1\right| + x_2 \left|2\right\rangle\!\left\langle 2\right| + (1-y) \left|3\right\rangle\!\left\langle 3\right| .$$

The induced map $\mathcal{M} \circ \mu_\rho^{\mathrm{B}}$ constructed using the Born rule and blurred by $\mathcal{M}$ fails to satisfy Eq. (4.6) when $P_0 = \left|0\right\rangle\!\left\langle 0\right| + \left|2\right\rangle\!\left\langle 2\right|$ and $P_1 = \left|1\right\rangle\!\left\langle 1\right| + \left|2\right\rangle\!\left\langle 2\right|$. In other words, this induced map fails to be a QIVPM. For the classical case, if $\mathcal{M}$ is not ultramodular, we also want to find a classical probability measure $\mu : \mathcal{E}_C \to [0,1]$ such that $\mathcal{M} \circ \mu$ is not a classical IVPM. This can be done by restricting our previous quantum probability measure μ_ρ^{B} to the space of events $\mathcal{E}_C$ generated by the mutually commuting projectors $\left|0\right\rangle\!\left\langle 0\right|$, $\left|1\right\rangle\!\left\langle 1\right|$, $\left|2\right\rangle\!\left\langle 2\right|$, and $\left|3\right\rangle\!\left\langle 3\right|$. The restricted function $\mu = \mu_\rho^{\mathrm{B}}|_{\mathcal{E}_C}$ is then a classical probability measure, and the induced map $\mathcal{M} \circ \mu$ fails to be a classical IVPM for the same reason as in the quantum case.

In other words, essential properties of QIVPMs constructed using interval maps can be gleaned from the properties of ultramodular functions. The following is a most interesting property in our setting:

Theorem 4.8 (Range of Ultramodular Functions)**.** *For any ultramodular function $\mathcal{M} : [0,1] \to \mathcal{I}$, either $\mathcal{I} = \mathcal{I}_0$ as defined in Eq. (4.17) or $\mathcal{I}$ contains uncountably many intervals.*

Since $\mathcal{M}$ maps to intervals, we can decompose it into two functions: its left-end and right-end, where $\left[\mathcal{M}^L(x), \mathcal{M}^R(x)\right] = \mathcal{M}(x)$. By Eq. (4.20), the left-end function $\mathcal{M}^L : [0,1] \to [0,1]$ is Wright-convex [103, 106, 107], i.e.,

$$\mathcal{M}^L(y) + \mathcal{M}^L(x_2) \geq \mathcal{M}^L(x_0 + x_2) + \mathcal{M}^L(x_1 + x_2)$$

for three numbers x_0, x_1, and $x_2 \in [0,1]$ with $y = x_0 + x_1 + x_2 \in [0,1]$. Together with the fact that $\mathcal{M}^L$ maps to a bounded interval $[0,1]$, the left-end function $\mathcal{M}^L$ must be continuous on the unit

open interval $(0, 1)$ [81]. Therefore, either $\mathcal{M}$ maps every number in $(0, 1)$ to the same interval, or the number of intervals to which $\mathcal{M}$ maps must be uncountable.

To summarize, a conventional quantum probability measure has an uncountable range $[0, 1]$. A QIVPM constructed by blurring such a conventional quantum probability measure must also have an uncountable range of intervals. Of course, any particular QIVPM, or any particular experiment, will use a fixed collection of intervals appropriate for the resources and precision of the particular experiment.

Chapter 5

FURTHER QUESTIONS

When people proved the original Gleason theorem, people usually exploited the geometrical structure of real 3-dimensional Hilbert space [36, 42, 108, 109]. Since our finite-precision extension of the Gleason theorem only applies on a class of QIVPMs, we might want to ask how to modify these geometrical arguments to have a Gleason-type theorem for general QIVPMs. We will further study the tensor product structure among QIVPMs which is essential for defining product and entangled states, and serves the basis to discuss quantum nonlocality [37, 39, 42, 110] and quantum computing with QIVPMs. Finally, we want to improve the discrete quantum theories to consider QIVPMs over finite fields in future research.

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EDUCATION

Indiana University Bloomington (IUB) (GPA: 3.802/4.0) **2010 – Present**

- Ph.D. double-major in Mathematics and Computer Science (expected) November 2018
- Master of Science in Computer Science May 2016
- Master of Arts in Mathematics December 2012

National Taiwan University (NTU) (GPA: 3.68/4.0) **2002 – 2006**

- Bachelor of Science in Mathematics (Rank: 4/48) June 2006

PUBLICATIONS

[1] Y.-T. Tai, A. J. Hanson, G. Ortiz and A. Sabry, "Quantum interval-valued probability: Contextuality and the Born rule," *Phys. Rev. A,* vol. 97, no. 5, p. 052121, May 2018.

[2] A. J. Hanson, G. Ortiz, A. Sabry and Y.-T. Tai, "Discrete Quantum Theories," *J. Phys. A: Math. Theor.,* vol. 47, p. 115305, 2014.

[3] A. J. Hanson, G. Ortiz, A. Sabry and Y.-T. Tai, "Geometry of Discrete Quantum Computing," *J. Phys. A: Math. Theor.,* vol. 46, p. 185301, 2013. Erratum "Corrigendum: Geometry of Discrete Quantum Computing," *J. Phys. A: Math. Theor.,* vol. 49, p. 039501, 12 2016.

CONFERENCES AND SEMINARS

Quantum Interval-Valued Probability: Contextuality and the Born Rule

- Talk in Interdisciplinary Logic Seminar, IUB August 2017
- Poster Session in Contextuality: Conceptual Issues, Operational Signatures, and Applications, Perimeter Institute for Theoretical Physics July 2017

Introduction to Discrete Quantum Theories and Computing

- Talk in Theory Seminar, Department of Computer Science, IUB March 2017

Real Computation

- Talk in Theory Reading Group, Department of Computer Science, IUB Feb 2016

TEACHING EXPERIENCE

Taught with Full Responsibility

- MATH-T101 Mathematics for Elementary Teachers I, IUB — Fall 2017
- MATH-M216 Calculus II (Online), Indiana University East — Summer 2012

Designed and Edited Online Courses, Data Science Program, IUB

- Basic Linear Algebra and Calculus with Python (Designer) — Summer 2017 – Spring 2018
- Machine Learning with Python (Editor) — Fall 2016 – Spring 2018
- Introduction to C++ (Designer) — Summer 2016 – Fall 2017

Taught Recitation Sessions, IUB

- MATH-M211 Calculus I — Fall 2016
- MATH-M212 Calculus II — Summer 2014, Fall 2014, Fall 2015
- CSCI-B501 Theory of Computing — Spring 2015

Assisted and Graded, IUB

- CSCI-B609 Topics in Algorithms and Computing Theory (AlphaGo) — Spring 2018
- INFO-I231 Introduction to the Mathematics of Cybersecurity — Spring 2017
- CSCI-B503 Algorithms Design and Analysis — Spring 2016
- MATH-M119 Brief Survey of Calculus I — Fall 2013, Spring 2014
- MATH-M303 Linear Algebra for Undergraduates — Spring 2013
- MATH-M118 Finite Mathematics — Fall 2010, Fall 2012
- MATH-M301 Linear Algebra and Applications — Spring 2012
- MATH-M365 Introduction to Probability and Statistics — Fall 2011
- MATH-M120 Brief Survey of Calculus II — Spring 2011
- MATH-S312 Honors Course in Calculus IV — Spring 2011

Taught Mini-Courses in NTU Math Camps

- There is No Formula for General Quintic Equations in Terms of Radicals — 2005
- Game Theory — 2004

RESEARCH APPOINTMENTS

- Research Assistant, Department of Computer Science, IUB — July 2018 – Present
- Research Assistant, Kelley School of Business, IUB — May 2016
- Research Assistant, Department of Computer Science, IUB — Summer 2015
- Research Associate, Department of Computer Science, IUB — Summer 2013
- Research Assistant, Department of Economics, NTU — January 2008 – July 2009

TECHNICAL SKILLS

- Programming Languages:
  Python (with NumPy, matplotlib, and SymPy), Mathematica, Visual Basic for Application, HTML, C/C++, LaTeX, MATLAB, Isabelle, Agda, Scheme, SQL
- Platforms:
  Microsoft Windows (7, 10, 8, XP, 98, 95, 3.1), Cygwin, Red Hat Linux, MS-DOS 6.22
- Office and Project Management Softwares:
  Microsoft Outlook, Microsoft PowerPoint, Microsoft Excel, Microsoft Word,
  Adobe Acrobat, Adobe Dreamweaver CC, Adobe Captivate 9,
  LyX, ShareLaTeX, Trello, Google Docs, Google Sheets, Slack, emacs
- Version Control Systems:
  Git, Apache Subversion
- Integrated Development Environments:
  Eclipse, PyCharm, Visual Studio 2013
- Fluency of Languages:
  Chinese (Native), English (Fluent), Japanese (Beginning), French Reading (Beginning)

AWARDS AND HONORS

- Studying Abroad Scholarship, Ministry of Education, Taiwan, R.O.C. 2010 – 2012
- Presidential Award, NTU Spring 2005, Fall 2005, Spring 2006
- Distinction Award, 1st Taiwan Mathematical Contest of Modeling for Undergraduate Students September 2003

CLUB ACTIVITIES

- Account Administrator of ptt2.cc 2006 – 2009
- NTU Go Club 2002 – 2006