Probability

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1 Classical Conventional Probability Spaces

Textbook probability theory is defined using the notions of *sample spaces*, *events*, and *measures* [3, 2, 5].

1.1 Sample Space Ω

In this paper, we will only consider **finite** sample spaces. We therefore define a sample space Ω as a non-empty finite set.

Example 1 (A Classical Sample Space.). Consider an experiment that tosses three coins. A possible outcome of the experiment is HHT which means that the first and second coins landed with "heads" as the face-up side and that the third coin landed with "tails" as the face-up side. There are clearly a total of eight possible outcomes, and this collection constitutes the sample space:

$$\Omega_C = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Example 2 (A Quantum Sample Space.). Consider a quantum system composed of three electrons. By the postulates of quantum mechanics, an experiment designed to measure whether the spin of each electron along the x axis is left (L) or right (R) can only result in one of eight outcomes:

$$\Omega_H = \{LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR\}$$

1.2 Events $\mathcal{F}$

The space of events $\mathcal{F}$ associated with a sample space Ω is 2^Ω, the powerset of Ω. In other words, every subset of Ω is a possible event.

Example 3 (Some classical events.). The following are events associated with Ω_C:

- E_0, exactly zero coins are H, is the set $\{TTT\}$.
- E_1, exactly one coin is H, is the set $\{HTT, THT, TTH\}$.
- E_2, exactly two coins are H, is the set $\{HHT, HTH, THH\}$.
- E_3, exactly three coins are H, is the set $\{HHH\}$.
- $E_{>0}$, at least one coin is H, is the set $\{HHH, HHT, HTH, HTT, THH, THT, TTH\}$.

As the examples illustrate, events are *indirect* questions built from elementary elements of the sample space using logical connectives. Also note that some events may be disjoint and that some events may be expressed as combinations of other events. For example, we have $E_{>0} = E_1 \cup E_2 \cup E_3$ and each of these four events is disjoint from event E_0.

Example 4 (Some quantum events.). The following are events associated with Ω_H:

- F_0, exactly zero electrons are spinning L, is the set $\{RRR\}$.
- F_1, exactly one electron is spinning L, is the set $\{LRR, RLR, RRL\}$.
- F_2, exactly two electrons are spinning L, is the set $\{LLR, LRL, RLL\}$.
- F_3, exactly three electrons are spinning L, is the set $\{LLL\}$.
- $F_{>0}$, at least one electron is spinning L, is the set $\{LLL, LLR, LRL, LRR, RLL, RLR, RRL\}$.

As the examples illustrate, quantum events are, at first glance, similar to classical events. There are however some subtle differences that we point out in the next section.

1.3 Measures $\mathbb{P}$

The last ingredient of a probability space is a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns to each event a real number in the closed interval $[0, 1]$ subject to the following conditions:

- $\mathbb{P}(\Omega) = 1$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \Sigma_i\ \mathbb{P}(A_i)$.

Example 5 (Classical probability measure). There are 2^8 events associated with Ω_C. A possible probability measure for these events is:

$$\mathbb{P}(E) = \begin{cases} 1 & \text{if } E = \Omega \\ 0 & \text{otherwise} \end{cases}$$

Yu-Tsung says: Actually, the above $\mathbb{P}$ is not a probability measure because

$$\mathbb{P}\left(\Omega\right) = 1 \neq 0 + 0 = \mathbb{P}\left(E_0\right) + \mathbb{P}\left(E_{>0}\right)$$

A more interesting measure is defined recursively as follows:

$$\begin{aligned}
\mathbb{P}(\emptyset) &= 0 \\
\mathbb{P}(\{HHH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E) \\
\mathbb{P}(\{HHT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{HTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E) \\
\mathbb{P}(\{HTT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{THH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E) \\
\mathbb{P}(\{THT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{TTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E) \\
\mathbb{P}(\{TTT\} \cup E) &= \mathbb{P}(E)
\end{aligned}$$

Yu-Tsung says: Because $\mathbb{P}(\bigcup_i A_i) = \Sigma_i \, \mathbb{P}(A_i)$ requires disjoint events, the above formula should write like:

$$\begin{aligned}
\mathbb{P}(\emptyset) &= 0 \\
\mathbb{P}(\{HHH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E), \text{ if } HHH \notin E \\
\mathbb{P}(\{HHT\} \cup E) &= \mathbb{P}(E), \text{ if } HHT \notin E \\
\mathbb{P}(\{HTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E), \text{ if } HTH \notin E \\
\mathbb{P}(\{HTT\} \cup E) &= \mathbb{P}(E), \text{ if } HTT \notin E \\
\mathbb{P}(\{THH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E), \text{ if } THH \notin E \\
\mathbb{P}(\{THT\} \cup E) &= \mathbb{P}(E), \text{ if } THT \notin E \\
\mathbb{P}(\{TTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E), \text{ if } TTH \notin E \\
\mathbb{P}(\{TTT\} \cup E) &= \mathbb{P}(E), \text{ if } TTT \notin E
\end{aligned}$$

Or add a sentence "where the element in the singleton set is not belong to E for each equation." Or write like:

$$\begin{aligned}
\mathbb{P}(\{HHH\}) &= \tfrac{1}{5} \\
\mathbb{P}(\{HHT\}) &= 0 \\
\mathbb{P}(\{HTH\}) &= \tfrac{3}{10} \\
\mathbb{P}(\{HTT\}) &= 0 \\
\mathbb{P}(\{THH\}) &= \tfrac{1}{5} \\
\mathbb{P}(\{THT\}) &= 0 \\
\mathbb{P}(\{TTH\}) &= \tfrac{3}{10} \\
\mathbb{P}(\{TTT\}) &= 0 \\
\mathbb{P}(E) &= \textstyle\sum_{\omega \in E} \mathbb{P}(\{\omega\})
\end{aligned}$$

Because this is a *classical* situation, the probability assignments can be understood *locally* and *non-contextually*. In other words, we can reason about each coin separately and perform experiments on it ignoring the rest of the context. If we were to perform such experiments we may find that for the first coin, the probability of either outcome H or T is $\frac{1}{2}$; for coin two, the probabilities are skewed a little with the probability of outcome H being $\frac{2}{5}$ and the probability of outcome T being $\frac{3}{5}$; and that coin 3 is a fake double-headed coin where the probability of outcome H is 1 and the probability of outcome T is 0. The reader may check that these local observations are consistent with the probability measure above.

Example 6. [Quantum probability measure] Like in the classical case, there are 2^8 events. But as Mermin explains in a simple example [4], here is a possible probability measure:

$$\begin{aligned}
\mathbb{P}_{xxx}(\{LLL\}) &= \tfrac{1}{4} \\
\mathbb{P}_{xxx}(\{LLR\}) &= 0 \\
\mathbb{P}_{xxx}(\{LRL\}) &= 0 \\
\mathbb{P}_{xxx}(\{LRR\}) &= \tfrac{1}{4} \\
\mathbb{P}_{xxx}(\{RLL\}) &= 0 \\
\mathbb{P}_{xxx}(\{RLR\}) &= \tfrac{1}{4} \\
\mathbb{P}_{xxx}(\{RRL\}) &= \tfrac{1}{4} \\
\mathbb{P}_{xxx}(\{RRR\}) &= 0 \\
\mathbb{P}_{xxx}(E) &= \textstyle\sum_{\omega \in E} \mathbb{P}_{xxx}(\{\omega\})
\end{aligned}$$

In contrast with the classical example previously, the event of different electrons are not independent. More precisely, consider the event for each electron separately:

$$\begin{aligned} F_{1,L} &= \{LLL, LLR, LRL, LRR\} \\ F_{2,L} &= \{LLL, LLR, RLL, RLR\} \\ F_{3,L} &= \{LLL, LRL, RLL, RRL\} \end{aligned}$$

They are not independent means

$$\mathbb{P}_{xxx}(F_{1,L} \cap F_{2,L} \cap F_{3,L}) = \mathbb{P}_{xxx}(\{LLL\}) = \frac{1}{4} \neq \frac{1}{8} = \mathbb{P}_{xxx}(F_{1,L})\mathbb{P}_{xxx}(F_{2,L})\mathbb{P}_{xxx}(F_{3,L}) \ .$$

Classical events may also not be independent even if they seems unrelated. For example, events defined by the temperature is usually not independent to ones defined by how much Coca Cola is sold. Another example can be formulated by tossing three coins as we discussed previously. However, this time the coins are tossed behind a veil where someone tosses the coins for you. Because we cannot see how she tosses the coins, she might actually roll an tetrahedron die with $\{HHH, HTT, THT, TTH\}$ in its four faces. If HTT is face-down, she faces H, T, and T up by hand, respectively. Then, she uncovers the veil, and claims she has tossed the coins. If the coins are tossed in this way, the result of coin-tossing is correlated, and we will never see TTT no matter how many times we toss these coins.

Because we do not know how nature decides the spin of an electron, Einstein, Podolsky, and Rosen (EPR) [1] suggested the nature might give us the probability measure $\mathbb{P}_{xxx}$ because she rolled an tetrahedron die or performed other classical and deterministic process behind the veil. This claim may be convincing if $\mathbb{P}_{xxx}$ is the only probability measure we have, but will lead to a contradiction if we consider other probability measures as well. Notice that after the coins are placed by hand and before uncovering the veil, which side up has already been decided although we do not know. This would be also true for the quantum probability measure. Because the three electrons can be spacially separated, and each electron can be measured along the x axis separately, if the nature rolled a tetrahedron die, this die should be rolled before the electrons are separated and measured, and she should know the result of measurement. Let the result of the j-th electron measured along the x axis be $w\left(\sigma_x^j\right)$. Because

$$\mathbb{P}_{xxx}(\{LLR\}) = \mathbb{P}_{xxx}(\{LRL\}) = \mathbb{P}_{xxx}(\{RLL\}) = \mathbb{P}_{xxx}(\{RRR\}) = 0 \ ,$$

we have $w\left(\sigma_x^1\right) w\left(\sigma_x^2\right) w\left(\sigma_x^3\right) \in \{LLL, LRR, RLR, RRL\}$, i.e., the number of L in $w\left(\sigma_x^1\right)$, $w\left(\sigma_x^2\right)$, and $w\left(\sigma_x^3\right)$ should be odd.

The interesting part is that three electrons cannot be measured the spin only along the x axis, but also along the y axis with the result down (D) or up (U). We only consider to measure even number of electrons along the y axis, and the probability measure of non-singleton set could be defined by

$$\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
\mathbb{P}_{xyy}(\{LDD\}) &=& 0 & \mathbb{P}_{yxy}(\{DLD\}) &=& 0 & \mathbb{P}_{yyx}(\{DDL\}) &=& 0 \\
\mathbb{P}_{xyy}(\{LDU\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{DLU\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{DDR\}) &=& \frac{1}{4} \\
\mathbb{P}_{xyy}(\{LUD\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{DRD\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{DUL\}) &=& \frac{1}{4} \\
\mathbb{P}_{xyy}(\{LUU\}) &=& 0 & \mathbb{P}_{yxy}(\{DRU\}) &=& 0 & \mathbb{P}_{yyx}(\{DUR\}) &=& 0 \\
\mathbb{P}_{xyy}(\{RDD\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{ULD\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{UDL\}) &=& \frac{1}{4} \\
\mathbb{P}_{xyy}(\{RDU\}) &=& 0 & \mathbb{P}_{yxy}(\{ULU\}) &=& 0 & \mathbb{P}_{yyx}(\{UDR\}) &=& 0 \\
\mathbb{P}_{xyy}(\{RUD\}) &=& 0 & \mathbb{P}_{yxy}(\{URD\}) &=& 0 & \mathbb{P}_{yyx}(\{UUL\}) &=& 0 \\
\mathbb{P}_{xyy}(\{RUU\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{URU\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{UUR\}) &=& \frac{1}{4}
\end{array}$$

with $\mathbb{P}_{ijk}(E) = \sum_{\omega \in E} \mathbb{P}_{ijk}(\{\omega\})$. Similarly, the nature should predetermine $w\left(\sigma_x^j\right)$ and $w\left(\sigma_y^j\right)$ for these probabilities. Furthermore, because she do not know along which axis we are going to measure, she should predetermine the same $w\left(\sigma_x^j\right)$ and $w\left(\sigma_y^j\right)$ for all different probability measures. Notice that the number of L or D in $\left\{w\left(\sigma_x^1\right), w\left(\sigma_y^2\right), w\left(\sigma_y^3\right)\right\}$, $\left\{w\left(\sigma_y^1\right), w\left(\sigma_x^2\right), w\left(\sigma_y^3\right)\right\}$, and $\left\{w\left(\sigma_y^1\right), w\left(\sigma_y^2\right), w\left(\sigma_x^3\right)\right\}$ should be even. If we look these 9 letters carefully, we can find every $w\left(\sigma_x^j\right)$ appear once and every $w\left(\sigma_y^j\right)$ appear twice. Hence, the number of L in $w\left(\sigma_x^1\right)$, $w\left(\sigma_x^2\right)$, and $w\left(\sigma_x^3\right)$ should be even. This contradict to the conclusion in our last paragraph. Therefore, EPR's assumption is wrong, and it is not always true that the nature can predetermine the measurement result before we perform the measurement.

1.4 Finite Precision of Measurements

In a laboratory setting or a computational setting, there are neither uncountable entities nor uncomputable entities. We are thus looking at alternative probability spaces which do not depend on the real numbers and revisit the mysteries of quantum mechanics in that setting. In other words, is it possible that at least part of the quantum mysteries related to probability and measurement are due to the reliance on uncomputable probability values?

Following previous work on probability, we will replace the closed interval $[0, 1]$ by the *finite set* $S = \{\mathbf{possible}, \mathbf{impossible}\}$ and adapt the definition of probability measure as follows.

A set-valued probability measure $\mathbb{P} : \mathcal{F} \to S$ assigns to each event either the tag **possible** or the tag **impossible** subject to the following conditions:

- $\mathbb{P}(\Omega) = \mathbf{possible}$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \mathbf{possible}$ if any event A_i is **possible** and **impossible** otherwise.

2 Conventional Quantum Mechanics

Attempting to modify the probability measure to be set-valued, while keeping the rest of the mathematical framework of quantum mechanics intact leads to a contradiction. More precisely, it is not possible to maintain infinite precision probability amplitudes in the presence of set-valued probabilities without violating essential aspects of quantum theory.

. . . explain and give theorem

3 Discrete Quantum Theory

The next question to ask is therefore whether the infinite precision of probability amplitudes is itself justified. If all measurements are finite and all probabilities are computable, then it is plausible that the internal mathematical representation of quantum states should also be based on countable computable entities.

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References

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