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MATHEMATICAL NOTES

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SOLUTION OF A PROBLEM OF E. M. WRIGHT ON CONVEX FUNCTIONS*

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With R denoting the real number field and f a function on R to R, consider the following two statements about f:

(A) $f(x+\delta)-f(x) \geqq f(y+\delta)-f(y)$ for all $\delta>0$ and $x>y$.

(B) $f(\frac{1}{2}x+\frac{1}{2}y) \leqq \frac{1}{2}f(x)+\frac{1}{2}f(y)$ for all x, y.

It is well known that (A) implies (B), and that for continuous f the two are equivalent. (See [2], for example.) In a recent note [3], Professor E. M. Wright showed that (A) is equivalent to certain other interesting inequalities, and raised a question as to the existence of a function f for which (B) is true but (A) is false. The purpose of this note is to describe such a function†. Needless to say, the argument given leans heavily on the Axiom of Choice.

THEOREM. *There is a function f on R to R such that for all x, $y \in R$ it is true that*

(i) $f(\lambda x+(1-\lambda)y) \leqq \lambda f(x)+(1-\lambda)f(y)$ *for all rational* λ *with* $0 \leqq \lambda \leqq 1$;

(ii) *if* $x \neq y$, *there is a* $\delta>0$ *such that* $f(x+\mu\delta)-f(x)<f(y+\mu\delta)-f(y)$ *for all rational* $\mu>0$.

Proof. Recall first that the line R and the plane R^2 are both of dimension $2^{\aleph_0}$ as vector spaces over the rational field.‡ (This follows, for example, from the lemma on p. 20 of [1].) Thus there is an (additive, rationally homogeneous) isomorphism τ of R onto R^2. For each $x \in R$ let $f(x)=|\tau(x)|^2$, where $|\,|$ is the Euclidean norm in the plane R^2. That (i) is true follows readily from the corresponding property of $|\,|^2$ and the fact that τ is an isomorphism. It remains to establish (ii).

Consider an arbitrary pair x and y of distinct points of R and let $x'=\tau(x)$, $y'=\tau(y)$. Let U be the non-empty open set of all $p \in R^2$ such that $(y', p)>(x', p)$, where $(\ ,\ )$ denotes the inner-product in R^2. Since for all z, $q \in R^2$ it is true that $d|z+tq|^2/dt=2(z, q)+2t|q|^2$, it then follows that $|x'+tp|^2-|x'|^2<|y'+tp|^2-|y'|^2$ for all $p \in U$ and $t>0$. Thus with $\delta=\tau^{-1}(p)$ and μ a positive rational number, it is true that $f(x+\mu\delta)-f(x)<f(y+\mu\delta)-f(y)$. It remains only to show that $\tau^{-1}U$ includes at least one positive number. Suppose not. Then τ^{-1} is an additive real function on R^2 which is bounded above on the non-empty open set

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† Since Wright works with functions on $[0, \infty[$ to R, his question is answered by the restriction to $[0, \infty[$ of the function described in the Theorem.

‡ The Axiom of Choice enters in the proof of this fact.

U, and hence is bounded above on a translate V of U which includes the origin. But then τ^{-1} is bounded on the set $V\cap(-V)$, which is a neighborhood of the origin. Since τ^{-1} is rationally homogeneous, it follows that τ^{-1} is continuous at the origin, and hence by additivity at each point of R^2. Since, however, τ^{-1} maps R^2 biuniquely onto R, it cannot be continuous, and the contradiction completes the proof.

References

1. Reinhold Baer, Linear Algebra and Projective Geometry, New York, 1952.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934.
3. E. M. Wright, An inequality for convex functions, this MONTHLY, vol. 61, 1954, pp. 620–622.

NOTE ON CONVEX FUNCTIONS

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E. M. Wright in an interesting note [1] on a convex inequality states that it is unknown whether functions exist which satisfy condition (i) below and not condition (ii). It is not difficult to construct such a function, making use of a Hamel basis. (This makes use of the axiom of choice. See [2] or [3].) The conditions are as follows:

(i) $$f\left(\frac{a+b}{2}\right) \leqq \frac{f(a)+f(b)}{2} \qquad \text{for real } a \text{ and } b.$$

(ii) If $a \leqq b$ and $\delta > 0$ then $f(a+\delta) - f(a) \leqq f(b+\delta) - f(b)$.

Let H be a Hamel basis for the real numbers over the rationals. Suppose without loss of generality that 1 and π belong to H. Then each real number x has the unique representation $x = \sum_{h\in H} r_{x,h}\cdot h$, where the $r_{x,h}$ are rational numbers, only a finite number of which are not zero. Let

$$f(x) = \sum_{h\in H} r_{x,h}^2 \qquad \text{for each real } x.$$

It is easy to check that (i) holds. To see that (ii) does not hold, let $a=1$, $b=\pi$, and $\delta=1$. Then $a<b$, $\delta>0$, and $f(a+\delta)-f(a)=4-1>2-1=f(b+\delta)-f(b)$.

The function f may be modified so that (i) is still satisfied; and so that (ii) is satisfied for any preassigned set of values of $\delta>0$ of power less than the continuum, but not for all $\delta>0$.

References

1. E. M. Wright, An inequality for convex functions, this MONTHLY, vol. 61, 1954, pp. 620–622.
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3. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1934, p. 96.