Quantum Interval-Valued Probability: Measurement and Probability

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1. Introduction

Textbook quantum mechanics asserts that, given a quantum spin system described by the wave function $|\psi\rangle = \frac{\sqrt{\pi}}{2}|{+}\frac{1}{2}\rangle + \frac{\sqrt{4-\pi}}{2}|{-}\frac{1}{2}\rangle$ and given the observable $S_z = \frac{\hbar}{2}(|{+}\frac{1}{2}\rangle\langle{+}\frac{1}{2}| - |{-}\frac{1}{2}\rangle\langle{-}\frac{1}{2}|)$, then the probability of observing the eigenvalue ${+}\frac{\hbar}{2}$ (with post-measurement state $|{+}\frac{1}{2}\rangle$) is $\frac{\pi}{4}$ and the probability of observing the eigenvalue ${-}\frac{\hbar}{2}$ (with post-measurement state $|{-}\frac{1}{2}\rangle$) is $1 - \frac{\pi}{4}$. According to the frequency interpretation of probability, the above assertion implies that when performing a very large number of experiments on quantum systems described by wave function $|\psi\rangle$, a fraction $\frac{\pi}{4}$ of the experiments will result in the observation ${+}\frac{\hbar}{2}$ and the remaining fraction $1 - \frac{\pi}{4}$ will result in the observation ${-}\frac{\hbar}{2}$.

In reality, i.e., in an actual experiment, simulation, or computation, bounded by space, time, and energy resources, the situation is much fuzzier: (i) we cannot *exactly* prepare the state $|\psi\rangle$; (ii) we cannot build a measurement apparatus corresponding *exactly* to observable S_z; and (iii) we cannot prepare *exact* replicas of any state. In other words, we can only perform a limited number of experiments on quantum systems closely related to wave function $|\psi\rangle$ and using an apparatus that closely approximates observable S_z. Given any fixed resources, these experiments will *not* result in the *exact* fraction $\frac{\pi}{4}$ of observations ${+}\frac{\hbar}{2}$ and $1 - \frac{\pi}{4}$ of observations ${-}\frac{\hbar}{2}$. In fact, these experiments can be viewed as a method (algorithm) to calculate the value π and it is known that the state of the art algorithms for computing the nth binary digit of π require on the order of $O\left(n \log^{O(1)}(n)\right)$ operations [1]. In other words, calculating the nth digit of π appears to require more and more physical resources as n gets larger.

All of this is well-known and holds in both the classical and quantum worlds. In the classical world, these observations are believed to be well-understood approximations to an idealized independent reality that can continuously approached with more and more resources. The question in the quantum world is more subtle; it is not evident at all that this resource-aware perspective does not alter the very foundations of quantum mechanics. Indeed taking such resource bounds into consideration is what founded computer science as a discipline and has become crucial for understanding the very nature of computation. Following Feynman [2], Landauer [3], and others, one might argue that this resource-bounded perspective is also crucial for understanding the very nature of physical processes.

In previous work, we analyzed the wave function description of quantum systems from a resource-bounded perspective. Technically, instead of postulating that quantum states are rays in a Hilbert space, i.e., we instead used special finite fields whose size reflected the available resources. ...

Yu-Tsung says: If the core is non-empty, could the core characterize $\bar{\mu}$ comprehensively?

This is a *theoretical* investigation of *experimental* physics using *computational* methods. All experiments and computations are processes bounded in space, time, energy, and other resources [4, 5]. Yet, for centuries, the mathematical formalization of such processes has been founded on the infinitely precise real or complex numbers [6, 7, 8]. Our purpose here is to exploit the consequences of replacing infinitely-specified quantum probabilities by finite number of intervals used in interval-valued probability measures (IVPM); in particular, we show that a quantum IVPM may not be induced by a infinitely-precise state, but it might be more and more likely to be identified to a particular state when the measurement resources increase.

Indeed, almost every description of quantum mechanics, quantum computation, or quantum experiments refers to entities such as e, π, $\sqrt{2}$, etc (see, e.g., [9, 10, 11]). From a computational perspective, such numbers do not exist in their entirety "for free [12, 13]." For example, the state of the art algorithms for computing the nth binary digit of π require on the order of $O\left(n \log^{O(1)}(n)\right)$ operations [1]. In other words, simply referring to the nth digit of π requires more and more resources as n gets larger. Taking such resource bounds into consideration is what founded computer science as a discipline and is crucial for understanding the very nature of computation and, following Feynman [2], Landauer [3], and others, for understanding the very nature of physical processes.

We have been revisiting quantum mechanics, quantum information, and quantum computation from this resource-aware perspective. Our initial results in that domain showed how subtle the issues can be [14, 15, 16]: a straightforward replacement of the complex numbers by a finite field yields a variant of quantum mechanics in which computationally hard problems like UNIQUE-SAT (which decides whether a given Boolean formula is unsatisfiable or has exactly one satisfying assignment [17, 18, 19]) can be deterministically solved in constant time. To eliminate such unrealistic theories requires delicate analysis of the structure of the Hilbert space, the process of observation, and the notion of probability teasing apart their reliance on the infinitely precise real numbers [15, 16].

In this paper, we shift focus from the infinitely-specified but not directly observable quantum states, to observable measurable properties of quantum systems and their probabilities. Furthermore, we insist that our theories of measurement and probability only refer to finitely communicable evidence within feasible computational bounds. It follows that states, observations, and probabilities all become "fuzzy", i.e., specified by intervals of confidence that can only increase in precision if the available resources increase proportionally. Our notion of "fuzzy quantum mechanics" is related to existing work [20, 21, 22, 23, 24] but, as will be explained in more detail, is distinguished by its unique computational character.

We will begin by reviewing existing work that recasts classical probability spaces in a resource-aware setting and move to aim at recasting quantum probability and quantum measurement. In particular, we develop a measurement framework based on quantum IVPM.

Surprisingly, we found a quantum IVPM which can not be induced by a state, while Shapley proved a classical convex IVPM can always be induced by a classical "state" [25, 26], and Gleason proved a infinitely-specified quantum probabilities measure can always be induced by a quantum state [27, 9, 28]. Nevertheless, a quantum IVPM could still correspond to many possible states if it is broken into pieces. Our examples also suggest that a quantum IVPM could be more and more likely induced by a state while the intervals become sharper, i.e., the measurement resources increase.

Since Gleason's theorem supports the idea of von Neumann to define a state as a density matrix [29], a density matrix might not be a adequate imprecise quantum state. Instead, a imprecise quantum state might be more like an interactive system suggested by Quantum Bayesianism or QBism [30, 31, 32], which would interact differently with different clients. Furthermore, the validity the fundamental theorems of quantum mechanics followed up by Gleason's theorem, such as Bell [33, 9, 28, 4] and Kochen-Specker [34, 9, 28, 4], might need to be reassessed based on our imprecise measurement. Our research then might provide a new insight to the old debate among Meyer, Mermin, and others about whether finite precision measurement would nullify the Kochen-Specker theorem or not [35, 36, 37].

2. Classical Probability

A *probability space* specifies the necessary conditions for reasoning coherently about collections of uncertain events [38, 39, 40, 41]. We review the conventional presentation of probability spaces and then discuss the computational resources needed to estimate probabilities.

2.1. Classical Probability Spaces

The conventional definition of a probability space builds upon the field of real numbers. In more detail, a probability space consists of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ mapping events in $\mathcal{E}$ to the real interval $[0, 1]$. We will only consider *finite* sets of events and restrict our attention to non-empty finite sets Ω as the sample space. The space of events $\mathcal{E}$ includes every possible subset of Ω: it is the powerset $2^\Omega = \{E \mid E \subseteq \Omega\}$. For future reference, we emphasize that events are the primary notion of interest and that the sample space is a convenient artifact that allows us to treat events as sets obeying the laws of Boolean algebra [42, 9, 40].

Definition 1 (Probability Measure)**.** Given the set of events $\mathcal{E}$, a *probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(\Omega) = 1$,
- for every event E, $\mu(\Omega \backslash E) = 1 - \mu(E)$ where $\Omega \backslash E$ is the complement event of E, and

- for every collection $\{E_i\}_{i=1}^N$ of pairwise disjoint events, $\mu\left(\bigcup_{i=1}^N E_i\right) = \sum_{i=1}^N \mu(E_i)$.

There is some redundancy in the definition that will be useful when moving to quantum probability spaces.

Example 2 (Two-coins experiment)**.** Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. There are 16 total events including the event $\{HH, HT\}$ that the first coin lands heads up, the event $\{HT, TH\}$ that the two coins land on opposite sides, and the event $\{HT, TH, TT\}$ that at least one coin lands tails up. Here is a possible probability measure for these events:

$$\begin{array}{rclrcl} \mu(\emptyset) &=& 0 & \mu(\{HT,TH\}) &=& \frac{2}{3} \\ \mu(\{HH\}) &=& \frac{1}{3} & \mu(\{HT,TT\}) &=& 0 \\ \mu(\{HT\}) &=& 0 & \mu(\{TH,TT\}) &=& \frac{2}{3} \\ \mu(\{TH\}) &=& \frac{2}{3} & \mu(\{HH,HT,TH\}) &=& 1 \\ \mu(\{TT\}) &=& 0 & \mu(\{HH,HT,TT\}) &=& \frac{1}{3} \\ \mu(\{HH,HT\}) &=& \frac{1}{3} & \mu(\{HH,TH,TT\}) &=& 1 \\ \mu(\{HH,TH\}) &=& 1 & \mu(\{HT,TH,TT\}) &=& \frac{2}{3} \\ \mu(\{HH,TT\}) &=& \frac{1}{3} & \mu(\{HH,HT,TH,TT\}) &=& 1 \end{array} \tag{1}$$

It is useful to think that this probability measure is completely determined by the "reality" of the two coins in question and their characteristics, in the sense that each pair of coins induces a measure, and each measure must correspond to some pair of coins. The measure above would be induced by two particular coins such that the first coin is twice as likely to land tails up than heads up and the second coin is double-headed.

2.2. Measuring Probabilities: Buffon's Needle Problem

In a strict computational or experimental setting, one should question the reliance of the definition of probability space on the uncountable and uncomputable real interval $[0, 1]$ [43, 6, 7]. This interval includes numbers like $0.h_1h_2h_3\ldots$ where h_i is 1 or 0 depending on whether Turing machine TM_i halts or not. Such numbers cannot be computed. This interval also includes numbers like $\pi/4$ which can only be computed with increasingly large resources as the precision increases. Therefore, in a resource-aware computational or experimental setting, it is more appropriate to consider probability measures that map events to a set of elements computable with a fixed set of resources. We expand on this observation in the next section and then recall its formalization using interval-valued probability measures [44, 45].

In the previous example, we assumed the probability $\mu(E)$ of each event E is known a priori. In reality, although each event is assumed to have a probability, the exact value of $\mu(E)$ may not be known. According to the *frequency interpretation of probability* (which we will revisit when moving to the quantum case) [46, 47], to determine the probability of an event, we run M independent trials which gives us an approximation of the (assumed) "true" or "real" probability. Let x_i be 1 or 0 depending on whether the event E occurs in the i-th trial or not, then $\mu(E)$ could be approximated to given accuracy $\epsilon > 0$ by the relative frequency $\sum_{i=1}^{M} x_i/M$ with the probability converging to one as M goes to infinity, i.e.,

$$\forall \epsilon > 0, \lim_{M\to\infty} \mu\left(\left|\mu(E) - \frac{1}{M}\sum_{i=1}^{M} x_i\right| < \epsilon\right) = 1\,. \tag{2}$$

This fact is called the law of large numbers [48, 38, 49, 39, 10].

Let's look at a concrete example. Suppose we drop a needle of length ℓ onto a floor made of equally spaced parallel lines a distance h apart, where $\ell < h$. It is a known fact that the probability of the needle crossing a line is $2\ell/(\pi h)$ [50, 51, 52, 49]. Consider an experimental setup consisting of a collection of M identical needles of length ℓ. We throw the M needles one needle at a time, and observe the number M_c of needles that cross a line, thus estimating the probability of a needle crossing a line to be M_c/M. In an actual experiment with 500 needles and the ratio $\ell/h = 0.75$ [52], it was found that 236 crossed a line so the relative frequency is 0.472 whereas the idealized mathematical probability is $0.4774\ldots$. In a larger experiment with 5000 needles and the ratio $\ell/h = 0.8$ [49], the relative frequency was calculated to be 0.5064 whereas the idealized mathematical probability is $0.5092\ldots$. We see that the observed probability approaches $2\ell/(\pi h)$ but only if *larger and larger resources* are expended. These resource considerations suggest that it is possible to replace the real interval $[0, 1]$ with rational numbers up to a certain precision related to the particular experiment in question. There is clearly a connection between the number of needles and the achievable precision: in the hypothetical experiment with 3 needles, it is not sensible to retain 100 digits in the expansion of $2\ell/(\pi h)$.

There is another more subtle assumption of unbounded computational power in the experiment. We are assuming that we can always determine with certainty whether a needle is crossing a line. But "lines" on the the floor have thickness, their distance apart is not exactly h, and the needles' lengths are not all absolutely equal to ℓ. These perturbations make the events "fuzzy." Thus, in an experiment with limited resources, it is not possible to talk about the idealized event that exactly M_c needles cross lines as this would require the most expensive needles built to the most precise accuracy, laser precision for drawing lines on the floor, and the most powerful microscopes to determine if a needle does cross a line. Instead we might talk about the event that $M_c - \delta$ needles evidently cross lines and $M_c + \delta'$ needles plausibly cross lines where δ and δ' are resource-dependent approximations. This fuzzy notion of events leads to probabilities being only calculable within intervals of confidence reflecting the certainty of events and their plausibility. This is indeed consistent with published experiments: in an experiment with 3204 needles and the ratio $\ell/h = 0.6$ [51], 1213 needles clearly crossed a line and 11 needles were close enough to plausibly be considered as crossing the line: we would express the probability in this case as the interval $\left[\frac{1213}{3204}, \frac{1224}{3204}\right]$

expressing that we are certain that the event has probability at least $\frac{1213}{3204}$ but it is possible that it would have probability $\frac{1224}{3204}$.

Another way to re-express the above observations is that the calculus of probabilities requires all probabilities to be *coherent* (e.g., the probability of the union of two disjoint events is the sum of their probabilities). But given limited resources, the calculation of a probability measure can only be approximate and hence the probabilities cannot be absolutely guaranteed to be coherent in the ideal sense.

2.3. Classical Interval-Valued Probability Measures

As motivated above, an event E_1 may have an interval of probability $[l_1, r_1]$. Assume that another disjoint event E_2 has an interval of probability $[l_2, r_2]$, what is the interval of probability for the event $E_1 \cup E_2$? The answer is somewhat subtle: although it is possible to use the sum of the intervals $[l_1 + l_2, r_1 + r_2]$ as the combined probability, one can find a much tighter interval if information *against* the event (i.e., information about the complement event) is also taken into consideration. Formally, for a general event E with probability $[l, r]$, the evidence that contradicts E is evidence supporting the complement of E. The complement of E must therefore have probability $[1 - r, 1 - l]$ which we abbreviate $[1, 1] - [l, r]$. In general, the scalar multiplication of an interval is defined by

$$x[l, r] = \begin{cases} [xl, xr] & \text{for } x \geq 0 \ ; \\ [xr, xl] & \text{for } x \leq 0 \ . \end{cases} \tag{3}$$

Given a sample space Ω, its set of events $\mathcal{E}$, and a collection of intervals $\mathscr{I}$, a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ is a classical interval-valued probability measure if and only if $\bar{\mu}$ satisfies the following conditions [45] where the last line uses $\subseteq$ to allow for tighter intervals that exploit the complement event:

- $\bar{\mu}(\emptyset) = [0, 0]$.
- $\bar{\mu}(\Omega) = [1, 1]$.
- For any event E,
$$\bar{\mu}\left(\Omega \backslash E\right) = [1, 1] - \bar{\mu}\left(E\right) \ . \tag{4}$$
- For a collection $\{E_i\}_{i=1}^{N}$ of pairwise disjoint events, we have
$$\bar{\mu}\left(\bigcup_{i=1}^{N} E_i\right) \subseteq \sum_{i=1}^{N} \bar{\mu}\left(E_i\right) \ . \tag{5}$$

Example 3 (Two-coin experiment with interval probability)**.** We split the unit interval $[0, 1]$ in the following four closed sub-intervals: $[0, 0]$ which we call *impossible*, $[0, \frac{1}{2}]$ which we call *unlikely*, $[\frac{1}{2}, 1]$ which we call *likely*, and $[1, 1]$ which we call *certain*. Using these new values, we can modify the probability measure of example 2 by mapping each numeric value to the smallest sub-interval containing it to get the following:

$$\begin{aligned}
\bar{\mu}(\emptyset) &= \mathit{impossible} & \bar{\mu}(\{HT,TH\}) &= \mathit{likely} \\
\bar{\mu}(\{HH\}) &= \mathit{unlikely} & \bar{\mu}(\{HT,TT\}) &= \mathit{impossible} \\
\bar{\mu}(\{HT\}) &= \mathit{impossible} & \bar{\mu}(\{TH,TT\}) &= \mathit{likely} \\
\bar{\mu}(\{TH\}) &= \mathit{likely} & \bar{\mu}(\{HH,HT,TH\}) &= \mathit{certain} \\
\bar{\mu}(\{TT\}) &= \mathit{impossible} & \bar{\mu}(\{HH,HT,TT\}) &= \mathit{unlikely} \\
\bar{\mu}(\{HH,HT\}) &= \mathit{unlikely} & \bar{\mu}(\{HH,TH,TT\}) &= \mathit{certain} \\
\bar{\mu}(\{HH,TH\}) &= \mathit{certain} & \bar{\mu}(\{HT,TH,TT\}) &= \mathit{likely} \\
\bar{\mu}(\{HH,TT\}) &= \mathit{unlikely} & \bar{\mu}(\{HH,HT,TH,TT\}) &= \mathit{certain}
\end{aligned} \tag{6}$$

Despite the absence of infinitely precise numeric information, the probability measure is quite informative: it reveals that the second coin is double-headed and that the first coin is biased. To understand the $\subseteq$-condition, consider the following calculation:

$$\begin{aligned}
&\bar{\mu}(\{HH\}) + \bar{\mu}(\{HT\}) + \bar{\mu}(\{TH\}) + \bar{\mu}(\{TT\}) \\
=&\mathit{impossible} + \mathit{unlikely} + \mathit{impossible} + \mathit{likely} \\
=&[0,0] + \left[0, \frac{1}{2}\right] + [0,0] + \left[\frac{1}{2}, 1\right] = \left[\frac{1}{2}, \frac{3}{2}\right] .
\end{aligned} \tag{7}$$

If we were to equate $\bar{\mu}(\Omega)$ with the sum of the individual probabilities, we would get that $\bar{\mu}(\Omega) = \left[\frac{1}{2}, \frac{3}{2}\right]$. However, using the fact that $\bar{\mu}(\emptyset) = \mathit{impossible}$, we have $\bar{\mu}(\Omega) = 1 - \bar{\mu}(\emptyset) = \mathit{certain} = [1,1]$. This interval is tighter and a better estimate for the probability of the event Ω, and of course it is contained in $[\frac{1}{2}, \frac{3}{2}]$. However it is only possible to exploit the information about the complement when all four events are combined. Thus the $\subseteq$-condition allows us to get an estimate for the combined event from each of its constituents and then gather more evidence knowing the aggregate event.

3. Quantum Probability

The mathematical framework of classical probability above assumes that there exists a predetermined set of events that are independent of the particular experiment — classical physics is non-contextual [34, 9, 28, 4]. However, even in classical situations, the structure of the event space is often only partially known and the precise dependence of two events on each other cannot, a priori, be determined with certainty. In the quantum framework, this partial knowledge is compounded by the fact that there exist non-commuting events which cannot happen simultaneously. To accommodate these more complex situations, conventional approaches to quantum probability abandon the sample space Ω and reason directly about events which are generalized from plain sets to projection operators. A quantum probability space therefore consists of just two components: a set of events $\mathcal{E}$

often formalized as projection operators and a probability measure $\mu : \mathcal{E} \rightarrow [0, 1]$ formalized using the Born rule [53, 11, 4].

3.1. Quantum Events

Definition 4 (Projection Operators; Orthogonality [54, 9, 28, 40, 41])**.** Given a Hilbert space $\mathcal{H}$, an event (an experimental proposition [55], a question [54, 56], or an elementary quantum test [28]) is represented as a (self-adjoint or orthogonal [40, 57]) projection operator $P : \mathcal{H} \rightarrow \mathcal{H}$ onto a linear subspace of $\mathcal{H}$. The following define projections and list some of their properties:

- $\mathbb{0}$ is a projection.
- For any pure state $|\psi\rangle$, $|\psi\rangle\langle\psi|$ is a projection operator.
- Projection operators P_0 and P_1 are *orthogonal* if $P_0 P_1 = P_1 P_0 = \mathbb{0}$. The sum of two projection operators $P_0 + P_1$ is also a projection operator if and only if they are orthogonal.
- Conversely, every projection P can be expressed as $\sum_{j=1}^{N} |\psi_j\rangle\langle\psi_j|$, where P actually projects onto the linear subspace with orthonormal basis $\{|\psi_j\rangle\}_{j=1}^{N}$.
- A set of projections $\{P_i\}_{i=1}^{N}$ is called an *ideal measurement* if it is a partition of the identity, i.e., $\sum_{i=1}^{N} P_i = \mathbb{1}$ [41]. In this case, projections $\{P_i\}_{i=1}^{N}$ must be mutually orthogonal [40, 58], and N must be less or equal to the dimension of the Hilbert space.
- If P is a projection operator, then $\mathbb{1} - P$ is also a projection operator, called its *complement*. It is orthogonal to P, and corresponds to the complement event $\Omega \backslash E$ in classical probability [40].
- Projection operators P_0 and P_1 *commute* if $P_0 P_1 = P_1 P_0$. The product of two projection operators $P_0 P_1$ is also a projection operator if and only if they commute. This corresponds to the classical intersection between events [28, 40].
- For two commuting projection operators P_0 and P_1, their *disjunction* $P_0 \vee P_1$ is defined to be $P_0 + P_1 - P_0 P_1$ [40].

Example 5 (One-qubit quantum probability space)**.** Consider a one-qubit Hilbert space with each event interpreted as a possible post-measurement state [28, 11, 4]. For example, the event $|0\rangle\langle 0|$ indicates that the post-measurement state will be $|0\rangle$; the probability of such an event depends on the current state; the event $|1\rangle\langle 1|$ indicates that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ indicates that the post-measurement state will be $|+\rangle$; the event $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ indicates that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$ and is clearly certain; finally the empty event $\mathbb{0}$ states that the post-measurement state will be the empty state and is impossible. As in the classical case, a probability measure is a function that maps events to $[0, 1]$. Here is a partial specification of a possible probability measure that would be induced by a system whose current state is $|0\rangle$, $\mu(\mathbb{0}) = 0$, $\mu(\mathbb{1}) = 1$, $\mu(|0\rangle\langle 0|) = 1$, $\mu(|1\rangle\langle 1|) = 0$, $\mu(|+\rangle\langle +|) = \frac{1}{2}$, ... Note that, similarly to the classical case, the probability of $\mathbb{1}$ is 1 and the probability of collections of orthogonal events (e.g., $|0\rangle\langle 0| + |1\rangle\langle 1|$) is the sum of the individual probabilities. A collection of non-orthogonal events (e.g., $|0\rangle\langle 0|$ and $|+\rangle\langle +|$) is however not even a valid event. In the classical example, we argued that each probability measure is uniquely determined by two actual coins. A similar (but much more subtle) argument is valid also in the quantum case. By postulates of quantum mechanics and Gleason's theorem, it turns out that for large enough quantum systems, each probability measure is uniquely determined by an actual quantum state as discussed next.

3.2. Quantum Probability Measures

Given our setup, the definition of a quantum probability measure is a small variation on the classical definition.

Definition 6 (Quantum Probability Measure [54, 27, 9, 57])**.** Given a Hilbert space $\mathcal{H}$ with its set of events $\mathcal{E}$, a *quantum probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\mathbb{0}) = 0$.
- $\mu(\mathbb{1}) = 1$.
- For any projection P, $\mu(\mathbb{1} - P) = 1 - \mu(P)$.
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have $\mu\left(\sum_{i=1}^N P_i\right) = \sum_{i=1}^N \mu(P_i)$.

A quantum probability measure can be easily constructed if one knows the current state of the quantum system by using the Born rule. Specifically, for each pure normalized quantum state $|\phi\rangle$, the Born rule induces a probability measure μ_ϕ^{B} defined as $\mu_\phi^{\mathrm{B}}(P) = \langle \phi | P \phi \rangle$. The situation generalizes to mixed states $\rho = \sum_{j=1}^N q_j |\phi_j\rangle\langle\phi_j|$, where $\sum_{j=1}^N q_j = 1$ in which case the generalized Born rule induces a probability measure μ_ρ^{B} defined as $\mu_\rho^{\mathrm{B}}(P) = \mathrm{Tr}(\rho P) = \sum_{j=1}^N q_j \mu_{\phi_j}^{\mathrm{B}}(P)$ [28, 10, 4]. Conversely every probability measure must be of this form.

Theorem 7 (Gleason's theorem [27, 9, 28])**.** *In a Hilbert space $\mathcal{H}$ of dimension $d \geq 3$, given a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$, there exists a unique mixed state ρ such that $\mu = \mu_\rho^{\mathrm{B}}$.*

3.3. Measuring Quantum Probabilities

Similarly to the classical case, it is possible to estimate quantum probabilities by utilizing the frequentist approach of the previous section, assuming identical measurements conditions in each repeated experiment [28]. For instance, if one wants to determine the probability that the spin of a given silver atom is $+\hbar/2$, a Stern-Gerlach apparatus is built where ideally an inhomogeneous magnetic field is generated along, let's say, the quantization axis z. One then produces a collimated beam of identically prepared (neutral) silver atoms that is directed between the poles of the magnet where a predetermined field-gradient along the z direction has been established. Under appropriate experimental conditions we will observe that the beam, after traversing the magnetic-field region, will be deflected towards two regions identified by distinguished spots on a detector situated behind the apparatus [59, 28, 10, 40]. Each of the two discrete values is associated to either $+\hbar/2$ or $-\hbar/2$, commonly called "spin up" and "spin down", respectively. By "counting" the number of atoms that are deflected in the "spin up" region one can, in principle, estimate the probability that the prepared state of the silver atom state has spin $+\hbar/2$. Notice that a real experiment does not necessarily represent an "ideal measurement". For example, not all silver atoms will be identically prepared, or the field-gradient could not be large enough to distinguish between the spin up and down situations simply producing a large single blot. In other words, the closer we get to an ideal measurement the better we determine those probabilities at the cost of significantly increasing the number of resources. It is not very well appreciated in the literature that Bohr attempted to argue against the measurability of the spin of a free electron [60, 61, 62]. Essentially, Bohr argued (and Mott later on justified his assertion by an elegant use of uncertainty relations [63]) that a Stern-Gerlach experiment could not succeed in establishing the spin of an unbound electron because the Lorentz force would blur the detected pattern. This example illustrates the case of a fundamental physical limitation that not even infinite resources could mitigate.

4. Cryptodeterminism

Although a real or interval-valued probability measure could model whether we believe an event will happen or not, the classical world could be modeled in a simpler way. If the complete initial conditions for tossing a coin or throwing a needle is known, we can determine whether an event will happen or not for sure [64]. In another words, Newtonian physics could be modeled by a cryptodeterministic‡ measure defined as follows.

Definition 8 (Classical Cryptodeterministic Measure)**.** Given the set of events $\mathcal{E}$, a *classical cryptodeterministic measure* is a function $\mu^{\mathrm{D}} : \mathcal{E} \to \{0, 1\}$ such that:

‡ The word "cryptodeterminism" is used by Peres which means

> We now introduce the cryptodeterministic hypothesis: It is possible to specify all the details of a preparation, so that the result of any measurement becomes completely deterministic.

in [65], and

> ... any purported cryptodeterministic theory which would attribute a definite result to each quantum measurement, and still reproduce the statistical properties of quantum theory ...

in [66, 28].

- $\mu^{\mathrm{D}}(\emptyset)=0$,
- $\mu^{\mathrm{D}}(\Omega)=1$,
- for every event E, $\mu^{\mathrm{D}}\left(\Omega\backslash E\right)=1-\mu^{\mathrm{D}}\left(E\right)$ where $\Omega\backslash E$ is the complement event of E, and
- for every collection $\{E_i\}_{i=1}^{N}$ of pairwise disjoint events, $\mu^{\mathrm{D}}\left(\bigcup_{i=1}^{N}E_i\right)=\sum_{i=1}^{N}\mu^{\mathrm{D}}(E_i)$.

Example 9 (Two-coin experiment with cryptodeterministic measure)**.** Except the prefect prediction, a cryptodeterministic measure can model the belief of events which has already happened. For example, if we tossed the coin in example 2, and saw HH, our belief of which event has happened is represented by the following example.

$$\begin{array}{rclrcl}
\mu^{\mathrm{D}}(\emptyset) & = & 0 & \mu^{\mathrm{D}}(\{HT,TH\}) & = & 0\\
\mu^{\mathrm{D}}(\{HH\}) & = & 1 & \mu^{\mathrm{D}}(\{HT,TT\}) & = & 0\\
\mu^{\mathrm{D}}(\{HT\}) & = & 0 & \mu^{\mathrm{D}}(\{TH,TT\}) & = & 0\\
\mu^{\mathrm{D}}(\{TH\}) & = & 0 & \mu^{\mathrm{D}}(\{HH,HT,TH\}) & = & 1\\
\mu^{\mathrm{D}}(\{TT\}) & = & 0 & \mu^{\mathrm{D}}(\{HH,HT,TT\}) & = & 1\\
\mu^{\mathrm{D}}(\{HH,HT\}) & = & 1 & \mu^{\mathrm{D}}(\{HH,TH,TT\}) & = & 1\\
\mu^{\mathrm{D}}(\{HH,TH\}) & = & 1 & \mu^{\mathrm{D}}(\{HT,TH,TT\}) & = & 0\\
\mu^{\mathrm{D}}(\{HH,TT\}) & = & 1 & \mu^{\mathrm{D}}(\{HH,HT,TH,TT\}) & = & 1
\end{array} \tag{8}$$

The ability to predict classical events unambiguously rather than probabilistically inspired Einstein, Podolsky, and Rosen to ask whether quantum mechanics has any chance to be improved and predict in the same way [67, 28, 10, 11]. This question can be formulated by whether quantum mechanics could be modeled by quantum cryptodeterministic measures.

Definition 10 (Quantum Cryptodeterministic Measure)**.** Given a Hilbert space $\mathcal{H}$ with its set of events $\mathcal{E}$, a *quantum cryptodeterministic measure* is a function $\mu^{\mathrm{D}}:\mathcal{E}\to\{0,1\}$ such that:

- $\mu^{\mathrm{D}}(\mathbb{0})=0$,
- $\mu^{\mathrm{D}}(\mathbb{1})=1$,
- for every event P, $\mu^{\mathrm{D}}\left(\Omega\backslash P\right)=1-\mu^{\mathrm{D}}\left(P\right)$ where $\Omega\backslash P$ is the complement event of P, and
- for every collection $\{P_i\}_{i=1}^{N}$ of pairwise disjoint events, $\mu^{\mathrm{D}}\left(\bigcup_{i=1}^{N}P_i\right)=\sum_{i=1}^{N}\mu^{\mathrm{D}}(P_i)$.

The answer is "no" due to the Kochen-Specker theorem whose proof can also be formulated in the language of cryptodeterministic measure as in section Appendix G.5.

Theorem 11 (Kochen-Specker [34, 28, 9])**.** *Given a Hilbert space of dimension $d\geq 3$, there is no cryptodeterministic measure $\mu^{\mathrm{D}}:\mathcal{E}\to\{0,1\}$.*

Although sharing a lot of similarity, the Kochen-Specker theorem highlight the difference between quantum and classical measures. When we think a cryptodeterministic measure as a model of pasted events, the Kochen-Specker theorem also consistent with the usual interpretation of the quantum theory: we should not ask whether an event happened or not if an event is incompatible with the event really happened; events compatible with the event really happened are the only askable events.

5. Quantum Interval-valued Probability

5.1. Quantum Interval-valued Probability Measures

As argued in the previous sections, given fixed finite resources, it is only possible to estimate the quantum probabilities within an interval of confidence. It is therefore natural to propose the notion of a "quantum interval-valued probability measure" that combines the definitions of conventional quantum probability measures with classical interval-probabilities.

Definition 12 (Quantum Interval-valued Probability Measure)**.** Given a Hilbert space $\mathcal{H}$ with quantum events (projections) $\mathcal{E}$, and a collection of intervals $\mathscr{I}$, a *quantum $\mathscr{I}$-interval-valued probability measure* is a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that:

- $\bar{\mu}(\mathbb{0}) = [0, 0]$.
- $\bar{\mu}(\mathbb{1}) = [1, 1]$.
- For any projection P,

$$\bar{\mu}\left(\mathbb{1} - P\right) = [1, 1] - \bar{\mu}\left(P\right) . \tag{9}$$

- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have

$$\bar{\mu}\left(\sum_{i=1}^{N} P_i\right) \subseteq \sum_{i=1}^{N} \bar{\mu}\left(P_i\right) . \tag{10}$$

It is easy to establish that quantum interval-valued probability measures generalize conventional quantum probability measures. In particular, any quantum probability measure can be recast as an interval-valued measure using arbitrary small intervals as discussed in Appendix D. However, to simplify our discussion, we will focus on only three intervals $[0, 0]$, $[1, 1]$ and $[0, 1]$, where $[0, 0]$ and $[1, 1]$ are called $\mathscr{impossible}$ and $\mathscr{certain}$ as before, and $[0, 1]$ is called $\mathscr{unknown}$ because it provides no information. Given a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$, we define a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ by $\bar{\mu}(P) = \iota\left(\mu(P)\right)$, where $\iota : [0, 1] \to \mathscr{I}$ is defined by

$$\iota(x) = \begin{cases} \mathscr{certain} & \text{if } x = 1 \text{ ;} \\ \mathscr{impossible} & \text{if } x = 0 \text{ ;} \\ \mathscr{unknown} & \text{otherwise.} \end{cases} \tag{11}$$

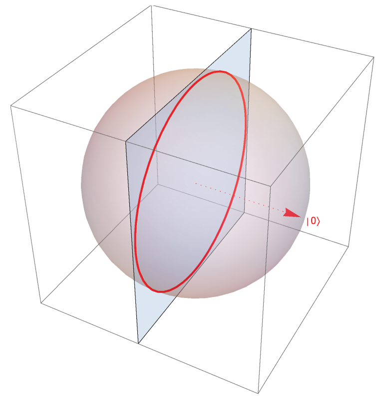

Figure 1. This figure illustrates the 1-dimensional projectors mapped by $\bar{\mu}^{\mathrm{B}}_{|0\rangle}(P) = \iota\left(\mu^{\mathrm{B}}_{|0\rangle}(P)\right)$ plotted in $\mathbb{R}^3$. The red dotted vector is $|0\rangle$, and $\bar{\mu}^{\mathrm{B}}_{|0\rangle}(|0\rangle\langle 0|) = \textit{certain}$. $|0\rangle$ is the normal vector of the plane so that we have $\bar{\mu}^{\mathrm{B}}_{|0\rangle}(|\psi\rangle\langle\psi|) = \textit{impossible}$ for any vector $|\psi\rangle$ in the plane. For any other vector $|\psi\rangle$, we have $\bar{\mu}^{\mathrm{B}}_{|0\rangle}(|\psi\rangle\langle\psi|) = \textit{unknown}$.

For example, figure 1 plots the 1-dimensional projectors mapped by $\iota\left(\mu^{\mathrm{B}}_{|0\rangle}(P)\right)$. This measure represents the beliefs of an experimenter with no prior knowledge about the particular quantum system in question.§

Similarly, we can recast the result of cryptodeterminism to quantum interval-valued probability measures as well. In another word, for any classically event space $\mathcal{E}$,there is always a classical interval-valued probability measure mapping to $\mathscr{I}_{\mathrm{D}} = \left\{\textit{impossible}, \textit{certain}\right\}$. However, given a Hilbert space of dimension $d \geq 3$, there is no quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}_{\mathrm{D}}$ by corollary 57 of the Kochen-Specker theorem. The Kochen-Specker theorem provides an example that quantum interval-valued probability measures behave differently from the classical one, and the convexity and core in the next section provides another example that quantum interval-valued probability measures behave unexpectedly.

5.2. Convexity and Core

Formally, we can ask: what can we deduce about the state of a quantum system given a quantum interval-valued probability measure, i.e., given observations done with finite resources. In the case that the intervals are infinitely precise, the question reduces to Gleason's theorem which states that the state of the quantum system is uniquely determined

§ Yu-Tsung says: Actually, $\iota\left(\mu^{\mathrm{B}}_{|0\rangle}(P)\right)$ is quite informative. Since $\iota\left(\mu^{\mathrm{B}}_{|0\rangle}(P)\right) = \textit{certain}$ if and only if $P = |0\rangle\langle 0|$, we know the core of $\iota\left(\mu^{\mathrm{B}}_{|0\rangle}(P)\right)$ is exactly $\left\{\mu^{\mathrm{B}}_{|0\rangle}\right\}$. The quantum interval-valued probability measure for an experimenter with no prior knowledge should be $\bar{\mu}(P) = \textit{unknown}$ for every projection P.

by the probability measure. But surely the less resources are available, the less precise the intervals, and the less we expect to know about the state of the system. To formally state and answer this question we begin with defining the *core* and *convexity* of a probability measure as follows.

Definition 13. Given a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$:

- A *core* of $\bar{\mu}$ is the set $\mathrm{core}\,(\bar{\mu}) = \{\mu : \mathcal{E} \to [0,1] \mid \forall P \in \mathcal{E}.\ \mu\,(P) \in \bar{\mu}\,(P)\}$.
- $\bar{\mu}$ is called convex if
$$\bar{\mu}\,(P_0 \vee P_1) + \bar{\mu}\,(P_0 P_1) \subseteq \bar{\mu}\,(P_0) + \bar{\mu}\,(P_1) \tag{12}$$
for all commuting $P_0, P_1 \in \mathcal{E}$.

The core of an interval-valued probability measure is the set of real-valued infinitely-precise probability measures it approximates. An interval-valued probability measure is convex if whenever certain intervals exist then combinations of these intervals must also exist, guaranteeing we can add and manipulate probabilities coherently. In the classical world, every convex measure has a non-empty core, which means that the interval-valued probability measure must approximate at least one infinitely-precise measure.

Theorem 14 (Shapley [25, 26])**.** *Every (classical) convex interval-valued probability measure has a non-empty core.*

In informal terms, this result states that although measurements done with limited resources and poor precision may not uniquely identify the true state of the system, there is always at least one system that is consistent with the measurements. Surprisingly, we discovered a counterexample to this statement in the quantum case, and have:

Theorem 15 (Empty core for general QIVPMs)**.** *In a Hilbert space of dimension $d \geq 3$, there exists a convex QIVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that $\overline{\mathcal{H}}\,(\bar{\mu}, \mathcal{E}) = \emptyset$.*

Proof. To prove this theorem, we need to construct a QIVPM on the Hilbert space, and verify that there are no states that are consistent with it on all possible events. Assume a Hilbert space of dimension $d \geq 3$ with orthonormal basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ and $\mathscr{I}_0 = \{\mathit{certain}, \mathit{impossible}, \mathit{unknown}\}$. For $i = 1, \ldots, d-1$, let $|i'\rangle = (|0\rangle + |i\rangle)/\sqrt{2}$. Given two distinguish states $|i'\rangle \neq |j'\rangle$, since the inner products $\langle i'|j'\rangle$ and $\langle i'|0\rangle$ are both non-zero, all projectors in $\mathcal{S} = \{|0\rangle\langle 0|\} \cup \{|i'\rangle\langle i'| \mid i = 1, \ldots, d-1\}$ are not commuting with each other. Consider the function $\bar{\mu} : \mathcal{E} \to \mathscr{I}_0$ defined by

$$\bar{\mu}\,(P) = \begin{cases} \mathit{impossible}, & \text{if } P = \mathbb{0} \text{ or } P \in \mathcal{S}\ ; \\ \mathit{certain}, & \text{if } P = \mathbb{1} \text{ or } \mathbb{1} - P \in \mathcal{S}\ ; \\ \mathit{unknown}, & \text{otherwise.} \end{cases} \tag{13}$$

The situation when $d = 3$ is illustrated in figure 2. To verify $\bar{\mu}$ be a convex QIVPM, it is sufficient to verify the convexity condition, equation (12). There are several cases:

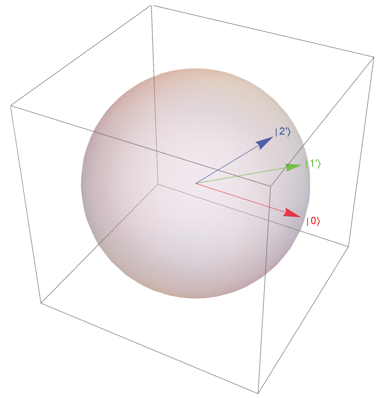

Figure 2. This figure illustrates theorem 15 plotted in $\mathbb{R}^3$. The red, green, and blue vectors are $|0\rangle$, $|1'\rangle$, and $|2'\rangle$ respectively.

- When $P_0 = \mathbb{0}$, equation (12) holds by lemma 38. This lemma also implies equation (12) when $P_0 = \mathbb{1}$.
- When $P_0 = |\psi\rangle\langle\psi| \in \mathcal{S}$, if $P_1 = |\psi\rangle\langle\psi| + P_2$, then equation (12) holds by lemma 38. Otherwise, we have $P_0 P_1 = \mathbb{0}$, and $\bar{\mu}\left(P_0 P_1\right) = \bar{\mu}\left(P_0\right) = \mathit{impossible}$. There are two cases:
  - When $P_1 = \mathbb{1} - |\psi\rangle\langle\psi|$, we have
$$\begin{aligned} &\bar{\mu}\left(P_0 \vee P_1\right) + \bar{\mu}\left(P_0 P_1\right) = \bar{\mu}\left(\mathbb{1}\right) + \bar{\mu}\left(\mathbb{0}\right) \\ &= \mathit{certain} + \mathit{impossible} = \bar{\mu}\left(P_0\right) + \bar{\mu}\left(P_1\right) . \end{aligned} \tag{14}$$
  - When $P_1 \neq \mathbb{1} - |\psi\rangle\langle\psi|$, neither $P_1 \in \mathcal{S}$ nor $\mathbb{1} - P_1 \in \mathcal{S}$ because all projectors in $\mathcal{S}$ are not commuting. Hence,$\bar{\mu}\left(P_0 \vee P_1\right) + \bar{\mu}\left(P_0 P_1\right) \subseteq \mathit{unknown} + \mathit{impossible} = \bar{\mu}\left(P_0\right) + \bar{\mu}\left(P_1\right)$.

  Therefore, equation (12) holds when $P_0 = |\psi\rangle\langle\psi| \in \mathcal{S}$.
- When $P_0 = \mathbb{1} - |\psi\rangle\langle\psi|$ for $|\psi\rangle\langle\psi| \in \mathcal{S}$, equation (12) follows from the previous case with lemma 38.
- If neither the above cases holds, we have $\bar{\mu}\left(P_0 \vee P_1\right) + \bar{\mu}\left(P_0 P_1\right) \subseteq \mathit{unknown} + \mathit{unknown} = \bar{\mu}\left(P_0\right) + \bar{\mu}\left(P_1\right)$.

Next we will prove by contradiction that $\overline{\mathcal{H}}\left(\bar{\mu}, \mathcal{E}\right)$ is the empty set. Suppose there is a state $\rho = \sum_{j=1}^{N} q_j \left|\phi_j\right\rangle\left\langle\phi_j\right| \in \overline{\mathcal{H}}\left(\bar{\mu}, \mathcal{E}\right)$, where $\sum_{j=1}^{N} q_j = 1$ and $q_j > 0$. Since $\mu_\rho^{\mathrm{B}}(P) \in \bar{\mu}(P)$ for any P, the state ρ must satisfy $\mu_\rho^{\mathrm{B}}(P) = 0$ for all $P \in \mathcal{S}$. This implies

$$\left\langle\phi_j|0\right\rangle = \left\langle\phi_j|i'\right\rangle = 0 \tag{15}$$

for all $i = 1, \ldots d-1$ and for all $j = 1, \ldots N$. Eq. (15) never holds proving the theorem. □

6. Why Empty Core?

The possibility that a convex quantum interval-valued probability measure has an empty core departs from the classical case. To understand the meaning of it, we will first study how to recover the non-empty core with extra conditions. We will then consider having an empty core as an legitimate result, and understand its consequence.

6.1. Convexity

One possibility for having an empty core is that the definition of quantum convexity is too weak. On one hand, strengthening equation (12) to non-commuting projectors P_0 and P_1 is unlikely to succeed because the product of non-commuting projectors is not a projector. On the other hand, strengthening equation (12) among commuting projectors cannot guarantee a non-empty core as well if we follow what people strengthen convexity classically. Equation (12) classically is a special case of

$$\bar{\mu}'\left(\bigcup_{i=1}^{N} E_i\right) + \sum_{\substack{I\subseteq\{1,\ldots,N\}\\ |I| \text{ is odd}}} \bar{\mu}'\left(\bigcap_{i\in I} E_i\right) \subseteq \sum_{\substack{I\subseteq\{1,\ldots,N\}\\ |I| \text{ is even}}} \bar{\mu}'\left(\bigcap_{i\in I} E_i\right) \tag{16}$$

when $N = 2$, where $\bar{\mu}'$ is a classical interval-valued probability measure. If $\bar{\mu}'$ satisfies equation (16) for all set of event $\{E_i\}_{i=1}^{N} \subseteq 2^{\Omega}$, then the left-end point of $\bar{\mu}'$ is called a Dempster-Shafer belief function. It is reasonable to call a quantum interval-valued probability $\bar{\mu}$ Dempster-Shafer if $\bar{\mu}$ satisfies

$$\bar{\mu}\left(\bigvee_{i=1}^{N} P_i\right) + \sum_{\substack{I\subseteq\{1,\ldots,N\}\\ |I| \text{ is odd}}} \bar{\mu}\left(\prod_{i\in I} P_i\right) \subseteq \sum_{\substack{I\subseteq\{1,\ldots,N\}\\ |I| \text{ is even}}} \bar{\mu}\left(\prod_{i\in I} P_i\right) . \tag{17}$$

for all set of mutually orthogonal projections $\{P_i\}_{i=1}^{N} \subseteq \mathcal{E}$. However, the quantum interval-valued probability measure in theorem 15 is Dempster-Shafer and has an empty core. In summary, it is unlikely to guarantee a non-empty core by strengthening convexity.

6.2. Commuting Sub-Event Space

Another reason for having an empty core is the existence of non-commuting observables. Non-commutativity is one of the difference between quantum and classical since Heisenberg's uncertainty principle states there is an intrinsic uncertainty measuring non-commuting observables in quantum mechanics [68, 28, 10, 40, 4]. Similarly, we argue that the quantum phenomenon disappears when we restrict the domain of a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ on a mutually commuting subspace $\mathcal{E}'$. As proved in theorem 32, the restricted $\bar{\mu}|_{\mathcal{E}'}$ is essentially a classical interval-valued probability measure. Furthermore, if $\bar{\mu}$ is convex, then $\bar{\mu}|_{\mathcal{E}'}$ will be convex, and have a non-empty core classically.

When $\bar{\mu}$ is restricted to different commuting $\mathcal{E}'$, the core elements of each $\bar{\mu}|_{\mathcal{E}'}$ may not be glued together, and gives a core element of $\bar{\mu}$. The inability of gluing core elements among commuting $\mathcal{E}'$ to non-commuting $\mathcal{E}$ is consistent with the proof of the Kochen-Specker theorem, where each triple of commuting observables could be colored well, but we cannot color all observables which are not all commuting. This is also consistent with the assertion that mutually commuting observables could be measured simultaneously [69]. In another word, if Alice and Bob in Bell's theorem can only pick from a set of mutually commuting observables, they could never observe any "quantum," and prove the quantum theory cannot be simulated by a local hidden variable theory.

6.3. Increase the Measurement Resource

This section extend the idea of restricting the event space, but instead of commuting sub-event space, we will try to find some subset $\mathcal{E}' \subseteq \mathcal{E}$ such that $\bar{\mu}|_{\mathcal{E}'}$ has an non-empty core. This could help us understand the core because of the following identity

$$\operatorname{core}(\bar{\mu}) = \bigcap_{\mathcal{E}' \subseteq \mathcal{E}} \operatorname{core}(\bar{\mu}|_{\mathcal{E}'}) \tag{18}$$

if we know more about the right-hand side of the equation, we can have more idea about the left-hand side. This idea can be applied to theorem 15, and the following are three of all possibilities:

- When $\mathcal{E}_1 = \mathcal{E} \setminus \{|{+}'\rangle\langle{+}'|\}$, because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|{+}\rangle\langle{+}|) =$ *impossible*, we have $\operatorname{core}(\bar{\mu}|_{\mathcal{E}_1}) = \left\{\mu^{\mathrm{B}}_{|2\rangle}\right\}$.
- When $\mathcal{E}_2 = \mathcal{E} \setminus \{|{+}\rangle\langle{+}|\}$, because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|{+}'\rangle\langle{+}'|) =$ *impossible*, we have $\operatorname{core}(\bar{\mu}|_{\mathcal{E}_2}) = \left\{\mu^{\mathrm{B}}_{|1\rangle}\right\}$.
- When $\mathcal{E}_3 = \mathcal{E} \setminus \{|0\rangle\langle 0|, |{+}\rangle\langle{+}|, |{+}'\rangle\langle{+}'|\}$, because $\bar{\mu}(|\phi\rangle\langle\phi|) \neq$ *certain* for all $|\phi\rangle$, we have $\mu^{\mathrm{B}}_{\frac{\mathbb{1}}{3}} \in \operatorname{core}(\bar{\mu}|_{\mathcal{E}_3})$ as well as $\mu^{\mathrm{B}}_{|2\rangle}$ and $\mu^{\mathrm{B}}_{|1\rangle} \in \operatorname{core}(\bar{\mu}|_{\mathcal{E}_3})$.

Each of these possibilities is compatible with some but not all of the observations. In another word,

$$\operatorname{core}(\bar{\mu}) = \bigcap_{\mathcal{E}' \subseteq \mathcal{E}} \operatorname{core}(\bar{\mu}|_{\mathcal{E}'}) \subseteq \operatorname{core}(\bar{\mu}|_{\mathcal{E}_1}) \cap \operatorname{core}(\bar{\mu}|_{\mathcal{E}_2}) = \left\{\mu^{\mathrm{B}}_{|2\rangle}\right\} \cap \left\{\mu^{\mathrm{B}}_{|1\rangle}\right\} = \emptyset\,. \tag{19}$$

The first two possibilities are particular interesting since each two 1-dimensional projectors $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$ mapped to *impossible* suggests there is a subset $\mathcal{E}' \subseteq \mathcal{E}$ such that $\operatorname{core}(\bar{\mu}|_{\mathcal{E}'}) = \left\{\mu^{\mathrm{B}}_{|\psi_0\rangle \times |\psi_1\rangle}\right\}$ in the Hilbert space of dimension 3, where the cross product of complex vectors is defined in definition 21.

As we illustrated in figure 1, if $\bar{\mu}$ has a non-empty core, the states mapping to *impossible* is contained in a plane, which contains no area in the sphere. On the other hand, if the states mapping to *impossible* contain some area in the sphere as in figure B1, and this area intersects a plane $\mathcal{E}'$, then $\operatorname{core}(\bar{\mu}|_{\mathcal{E}'}) = \left\{\mu^{\mathrm{B}}_{|\psi\rangle}\right\}$, where $|\psi\rangle$ is the normal vector of $\mathcal{E}'$. Since different planes have different normal vectors, $\bar{\mu}$ has an empty core. Furthermore, the larger the area, the more state vectors which are inconsistent with any particular Born rule probability measure. In another word, there are more different singleton sets in the right-hand side of equation (18). Therefore, the area of the 1-dimensional projectors mapping to $\mathscr{impossible}$ could indicate how much $\bar{\mu}$ deviate from the Born rule probability measures.

In this sense, we found that the area of the 1-dimensional projectors mapping to $\mathscr{impossible}$ depends on the maximal length of the intervals, which represents the ability of measurement equipment. In particular, we have the following results whose details are described in Appendix B.

- The radius of any disk inside the area of the 1-dimensional projectors mapping to $\mathscr{impossible}$ must smaller than $\pi/4$ (theorem 26).
- When the radius of any disk inside the area of the 1-dimensional projectors mapping to $\mathscr{impossible}$ between $\arcsin\left(1/\sqrt{3}\right)$ and $\pi/4$, we have $\mathscr{unknown} \in \mathscr{I}$, i.e., $\max\limits_{[l,r]\in\mathscr{I}} r - l = 1$ (theorem 27).
- When the radius of any disk inside the area of the 1-dimensional projectors mapping to $\mathscr{impossible}$ between $\pi/6$ and $\arcsin\left(1/\sqrt{3}\right)$, we have $\max\limits_{[l,r]\in\mathscr{I}} r - l = \frac{1}{3}$ (to be typed...).

In general, we conjecture that if the measurement equipment is made more and more precise, the corresponding interval-valued probability measure will be closer and closer to the Born rule. In the limit case, $\mathscr{I} = \{\{a\} \mid a \in [0,1]\}$ we do indeed recover the conventional Gleason's theorem and the Born rule.

6.4. Wigner's Friends

The situation of empty core might not be that strange if we think in terms of Wigner's "friends." In the original story of Wigner and his friend [70, 31], the friend makes a measurement in a closed laboratory and experiences an outcome. Wigner, outside the laboratory, doesn't experience an outcome. If he believes what his friend has told him about her plans in her laboratory he will assign an entangled state to her, her apparatus, and the system on which she is making her intervention. If Wigner goes on to ask his friend about her experience, then the entangled state to her, her apparatus, and the system will change into one in which they even have separate wave functions, and Wigner and his friend will use the same wave function to describe the system.

The question is what happens if Wigner has more than one friend in the laboratory. Suppose Wigner has three friends: E, P, and R in the laboratory. E, P, and R measure the spin-1 system described in theorem 15. After the experiment, E tells Wigner $|0\rangle\langle 0|$ never happened, P tells Wigner $|{+}\rangle\langle{+}|$ never happened, and R tells Wigner $|{+'}\rangle\langle{+'}|$ never happened. Wigner may then need to use the quantum Dempster-Shafer rule(?) or other

Table 1. Relation between measurement resources and interval-valued probability measures.

Measurement resources	Lowest	$\longleftrightarrow$		Highest
Count of $\mathscr{I}$	3	4	...	∞
$\sup_{[l,r]\in\mathscr{I}} \lvert r-l \rvert$	1	$\frac{1}{2}$	...	0
Count of core $(\bar{\mu})$	0	0		1
Count of states correspond to pieces of $\bar{\mu}$	3	2	...	1
How precise we could identify a state?	Coarse	$\longleftrightarrow$		Precise

rules(?) to combine the belief told by his friends, and the combined belief might be the quantum interval-valued probability measure in theorem 15 which has an empty core.

7. Conclusion and Discussion

If we insist that probabilities cannot be computed to infinite precision and are bound to be approximations represented by intervals of confidence, then quantum states themselves can only be discussed within intervals of confidence. Despite the fact that classically a IVPM could always correspond to a real-valued probability measure, we found that there may be no "real" infinitely-precise quantum state approximated by a quantum IVPM as a whole. However, if a quantum IVPM is decomposed into pieces, each piece might still be induced by a quantum state. Moreover, our examples and Gleason's theorem summarized in table 1 suggests that as the measurement resources increase, an entire quantum IVPM could more and more likely be identified to a particular state. Additional work is in progress on the following research questions:

Research Question. Confirm that there is always a quantum IVPM with an empty-core if the length of the intervals is bigger than zero.

Research Question. Find a general method to determine how a quantum IVPM is close to the Born rule.

Research Question. Confirm that as the number and precision of the intervals increase the quantum IVPM converges to the measure induced by the Born rule.

Research Question. Investigate the status of the theorems of Bell and Kochen-Specker for quantum IVPM.

Although we discussed how a quantum IVPM could be induced from a real quantum state, our investigation leaves an open question of whether there exists a "real" entity that exists independently of measurements and probabilities. The possibility of no "real" underlying state is consistent with the elegantly recent work on Quantum Bayesianism or QBism [30, 31, 32], which suggests that the quantum state is more like an interactive system in computer science parlance. In another word, the quantum state is subjective: each observer has a different view of the quantum system that is consistent with their previous observations and that allows that observer, independently of other observers, to assign beliefs (i.e., probabilities) to possible future interactions.

Acknowledgments

Anyone who has discussed this with us?

Appendix A. Interval-valued Frame Functions

Although quantum IVPMs in theorem 15 and 60 have empty cores, we might still think that the probability measure $\bar{\mu}$ is induced by some states. In theorem 15, the probability measure $\bar{\mu}$ might be induced by the state $|2\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|{+}\rangle\langle{+}|) = \mathit{impossible}$, or induced by the state $|1\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|{+'}\rangle\langle{+'}|) = \mathit{impossible}$. In example 60, the probability measure $\bar{\mu}$ might be induced by the state $|2\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|{+}\rangle\langle{+}|) = \mathit{impossible}$. In general, each two real 1-dimensional projectors $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$ mapping to impossible suggests $\bar{\mu}$ might be induced by the pure state $|\psi_0\rangle \times |\psi_1\rangle$ in the Hilbert space of dimension 3.

Since a quantum IVPM is completely determined by its value on 1-dimensional projectors in the Hilbert space of dimension 3, we could define the interval-valued frame function to simplify the later discussion.

Definition 16. [71, 72] The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1} = \left\{|\psi\rangle \in \mathbb{R}^d \;\middle|\; \langle\psi|\psi\rangle = 1\right\}$, while the unit sphere in $\mathbb{C}^d$ is denoted by $\mathbb{C}S^{d-1} = \left\{|\psi\rangle \in \mathbb{C}^d \;\middle|\; \langle\psi|\psi\rangle = 1\right\}$. Since $\mathbb{C}$ is one-to-one correspondence to $\mathbb{R}^2$ [73], $\mathbb{C}S^{d-1}$ has the same structure as S^{2d-1}. However, different symbols emphasis that they are embedded in different spaces.

Definition 17. [27, 28, 74] For any quantum probability measure μ on the Hilbert space of dimension d, we can define its *frame function* $f : \mathbb{C}S^{d-1} \to [0,1]$ by $f(|\psi\rangle) = \mu(|\psi\rangle\langle\psi|)$. For a quantum probability measure μ^{B}_{ϕ} induced by the Born rule, its frame function is denoted by f^{B}_{ϕ}.

Theorem 18. *[27, 28, 74] A frame function $f : \mathbb{C}S^{d-1} \to [0,1]$ satisfies the following properties:*

- *$f(\lambda|\psi\rangle) = f(|\psi\rangle)$ for any $|\psi\rangle \in \mathbb{C}S^{d-1}$ and $\lambda \in \mathbb{C}S^0$.*
- *For any orthonormal basis $\{|\psi_i\rangle\}_{i=0}^{2}$, we have $1 = \sum_{i=0}^{2} f(|\psi_i\rangle)$.*

Conversely, if a function $f : \mathbb{C}S^{d-1} \to [0,1]$ satisfies the above properties, f must be a frame function.

Definition 19. For any quantum IVPM $\bar{\mu}$ on the Hilbert space of dimension 3, we can define an *interval-valued frame function* $\bar{f}:\mathbb{C}S^2\to\mathscr{I}$ by $\bar{f}(|\psi\rangle)=\bar{\mu}(|\psi\rangle\langle\psi|)=\left[f^l(|\psi\rangle),f^r(|\psi\rangle)\right]$, where $f^l:\mathbb{C}S^2\to[0,1]$ and $f^r:\mathbb{C}S^2\to[0,1]$ are the left-end and the right-end of $\bar{f}$, respectively. Also, if f is the frame function of μ, $\bar{f}$ is the interval-valued frame function of $\bar{\mu}$, and μ is in the core of $\bar{\mu}$, then we say f is in the core of $\bar{f}$, and denoted by $f\in\text{core}\left(\bar{f}\right)$.

Theorem 20. *An interval-valued frame function* $\bar{f}:\mathbb{C}S^2\to\mathscr{I}$ *satisfies the following properties:*

- $\bar{f}(\lambda|\psi\rangle)=\bar{f}(|\psi\rangle)$ *for any* $|\psi\rangle\in\mathbb{C}S^2$ *and* $\lambda\in\mathbb{C}S^0$.
- *For any orthonormal basis* $\{|\psi_i\rangle\}_{i=0}^2$, *we have* $[1,1]-\bar{f}(|\psi_i\rangle)\subseteq\bar{f}(|\psi_j\rangle)+\bar{f}(|\psi_k\rangle)$, *where* i, j, *and* k *are any permutations among* 0, 1, *and* 2.

Conversely, if a function $\bar{f}:\mathbb{C}S^2\to\mathscr{I}$ *satisfies the above properties,* $\bar{f}$ *must be an interval-valued frame function.*

Proof. Given a quantum IVPM $\bar{\mu}$, it is straightforward to verify that its interval-valued frame function $\bar{f}:\mathbb{C}S^2\to\mathscr{I}$ satisfies the above properties.

Conversely, given a function $\bar{f}:\mathbb{C}S^2\to\mathscr{I}$ satisfying the above properties, consider $\bar{\mu}:\mathcal{E}\to\mathscr{I}$ defined by

- $\bar{\mu}(\mathbb{0})=[0,0]$ and $\bar{\mu}(\mathbb{1})=[1,1]$.
- For any $|\psi\rangle\in\mathbb{C}S^2$, we have $\bar{\mu}(|\psi\rangle\langle\psi|)=\bar{f}(|\psi\rangle)$ and $\bar{\mu}(\mathbb{1}-|\psi\rangle\langle\psi|)=[1,1]-\bar{f}(|\psi\rangle)$.

Since other conditions are trivial, we only need to verify equation (10). First notice that $\bar{\mu}(P_0+P_1)\subseteq\bar{\mu}(P_0)+\bar{\mu}(P_1)$ implies equation (10) by induction. Second, equation (9) implies $\bar{\mu}(\mathbb{1})\subseteq\bar{\mu}(P)+\bar{\mu}(\mathbb{1}-P)$. Therefore, to prove equation (10), it is sufficient to check

$$\bar{\mu}(|\psi_0\rangle\langle\psi_0|+|\psi_1\rangle\langle\psi_1|)\subseteq\bar{\mu}(|\psi_0\rangle\langle\psi_0|)+\bar{\mu}(|\psi_1\rangle\langle\psi_1|) \tag{A.1}$$

for orthogonal $|\psi_0\rangle$ and $|\psi_1\rangle$,which always holds because the properties of $\bar{f}$. □

Definition 21. For $|\psi\rangle=\alpha_0|0\rangle+\alpha_1|1\rangle+\alpha_2|2\rangle=\begin{pmatrix}\alpha_0\\\alpha_1\\\alpha_2\end{pmatrix}$ and $|\phi\rangle=\beta_0|0\rangle+\beta_1|1\rangle+\beta_2|2\rangle=\begin{pmatrix}\beta_0\\\beta_1\\\beta_2\end{pmatrix}$, their cross product is defined as follow:

$$|\psi\times\phi\rangle=|\psi\rangle\times|\phi\rangle=\begin{vmatrix}\alpha_1^* & \beta_1^*\\\alpha_2^* & \beta_2^*\end{vmatrix}|0\rangle+\begin{vmatrix}\alpha_2^* & \beta_2^*\\\alpha_0^* & \beta_0^*\end{vmatrix}|1\rangle+\begin{vmatrix}\alpha_0^* & \beta_0^*\\\alpha_1^* & \beta_1^*\end{vmatrix}|2\rangle\ , \tag{A.2}$$

where $\begin{vmatrix}\alpha & \beta\\\alpha' & \beta'\end{vmatrix}=\alpha\beta'-\alpha'\beta$. Also, even if $|\psi\rangle$ and $|\phi\rangle$ are normalized, their cross product $|\psi\rangle\times|\phi\rangle$ need not be normalized as usual. We will use $|\psi\rangle\times|\phi\rangle$ and $|\psi\rangle\times|\phi\rangle/\||\psi\rangle\times|\phi\rangle\|$ interchanged when there is no confusion.

Definition 22. For $|\psi\rangle$ and $|\phi\rangle \in \mathbb{C}^3$, if $|\psi\rangle = \lambda |\phi\rangle$ for some $\lambda \neq 0$, we call $|\psi\rangle$ is parallel to $|\phi\rangle$ denoted by $|\psi\rangle \parallel |\phi\rangle$. In this case, $|\psi\rangle$ and $|\phi\rangle$ represent the same physical state. "$|\psi\rangle$ is not parallel to $|\phi\rangle$" is denoted by $|\psi\rangle \nparallel |\phi\rangle$.

Lemma 23. *Given $|\psi\rangle$, $|\phi\rangle$ and $|\varphi\rangle \in \mathbb{C}^3$ such that $|\psi\rangle \nparallel |\phi\rangle$. Then, $|\varphi\rangle \parallel (|\psi\rangle \times |\phi\rangle)$ if and only if $\langle\psi|\varphi\rangle = \langle\phi|\varphi\rangle = 0$.*

Proof. To prove that $|\varphi\rangle \parallel (|\psi\rangle \times |\phi\rangle) \Rightarrow \langle\psi|\varphi\rangle = \langle\phi|\varphi\rangle = 0$, we can just verify

$$\begin{aligned} &\langle\psi|\psi \times \phi\rangle = \langle\psi \times \phi|\psi\rangle^* \\ =& \left[(\alpha_1\beta_2 - \alpha_2\beta_1)\,\alpha_0 + (\alpha_2\beta_0 - \alpha_0\beta_2)\,\alpha_1 + (\alpha_0\beta_1 - \alpha_1\beta_0)\,\alpha_2\right]^* = 0\,, \\ &\langle\phi|\psi \times \phi\rangle = \langle\psi \times \phi|\phi\rangle^* \\ =& \left[(\alpha_1\beta_2 - \alpha_2\beta_1)\,\beta_0 + (\alpha_2\beta_0 - \alpha_0\beta_2)\,\beta_1 + (\alpha_0\beta_1 - \alpha_1\beta_0)\,\beta_2\right]^* = 0\,. \end{aligned} \tag{A.3}$$

To prove the other direction, consider the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^2$ defined by $T(|\varphi\rangle) = \begin{pmatrix} \langle\psi|\varphi\rangle \\ \langle\phi|\varphi\rangle \end{pmatrix}$. Since $|\psi\rangle \nparallel |\phi\rangle$, we have $\operatorname{rank}(T) = 2$. Together with the rank equation [75], we have

$$\dim(\ker T) = \operatorname{nullity}(T) = 3 - \operatorname{rank}(T) = 3 - 2 = 1\,. \tag{A.4}$$

By the other direction of the proof, we have already known $|\psi\rangle \times |\phi\rangle \in \ker T$. Therefore, $|\varphi\rangle \parallel (|\psi\rangle \times |\phi\rangle)$. $\square$

Since interval-valued probability measures $\bar{\mu}$ and interval-valued frame functions $\bar{f}$ are one-to-one correspondence, studying 1-dimensional projectors mapping to impossible by $\bar{\mu}$ is the same as studying the points on $\mathbb{C}S^2$ mapping to impossible by $\bar{f}$. These points, denoted by $\bar{f}^{-1}\left(\mathit{impossible}\right) = \left\{|\psi\rangle \in \mathbb{C}S^2 \;\middle|\; \bar{f}(|\psi\rangle) = \mathit{impossible}\right\}$, play an important role to understand $\bar{f}$. There are 4 different cases when analyzing $\bar{f}^{-1}\left(\mathit{impossible}\right)$:

(i) $\bar{f}^{-1}\left(\mathit{impossible}\right) = \emptyset$. If $f^{\mathrm{B}}_{\phi} \in \operatorname{core}\left(\bar{f}\right)$ for some pure state $|\phi\rangle$, we could say nothing about $|\phi\rangle$.

(ii) $\bar{f}^{-1}\left(\mathit{impossible}\right) = \{|\psi_0\rangle\}$. If $f^{\mathrm{B}}_{\phi} \in \operatorname{core}\left(\bar{f}\right)$ for some pure state $|\phi\rangle$, $|\phi\rangle$ could be any state orthogonal to $|\psi_0\rangle$.

(iii) Consider $\{|\psi_0\rangle, |\psi_1\rangle\} \subseteq \bar{f}^{-1}\left(\mathit{impossible}\right)$. If $(|\psi_0\rangle \times |\psi_1\rangle) \parallel (|\psi'_0\rangle \times |\psi'_1\rangle)$ for any $\{|\psi'_0\rangle, |\psi'_1\rangle\} \subseteq \bar{f}^{-1}\left(\mathit{impossible}\right)$, then $|\psi_0\rangle \times |\psi_1\rangle$ is the only pure state which might induce $\bar{f}$. In another word, if $f^{\mathrm{B}}_{\phi} \in \operatorname{core}\left(\bar{f}\right)$ for some pure state $|\phi\rangle$, then $|\phi\rangle \parallel (|\psi_0\rangle \times |\psi_1\rangle)$. For example, example 60 is in this case when the core is empty; an interval-valued frame function defined by $\bar{f}(|\psi\rangle) = \iota\left(f^{\mathrm{B}}_{\phi}(|\psi\rangle)\right)$ is also in this case when the core is the singleton set $\left\{f^{\mathrm{B}}_{\phi}\right\}$, where ι is defined in equation (11).

(iv) If there exist two pair of states and $\{|\psi_0\rangle, |\psi_1\rangle\}$ and $\{|\psi'_0\rangle, |\psi'_1\rangle\} \subseteq \bar{f}^{-1}\left(\mathit{impossible}\right)$ such that $(|\psi_0\rangle \times |\psi_1\rangle) \nparallel (|\psi'_0\rangle \times |\psi'_1\rangle)$, then both $|\psi_0\rangle \times |\psi_1\rangle$ and $|\psi'_0\rangle \times |\psi'_1\rangle$ might induce $\bar{f}$. However, neither $f^{\mathrm{B}}_{\psi_0 \times \psi_1}$ nor $f^{\mathrm{B}}_{\psi'_0 \times \psi'_1}$ is in the core of $\bar{f}$ because $f^{\mathrm{B}}_{\psi_0 \times \psi_1}$

is inconsistent with $\bar{f}(|\psi_0'\rangle) = \bar{f}(|\psi_1'\rangle) = \mathscr{impossible}$ and $f^{\mathrm{B}}_{\psi_0' \times \psi_1'}$ is inconsistent with $\bar{f}(|\psi_0\rangle) = \bar{f}(|\psi_1\rangle) = \mathscr{impossible}$. Actually, $\bar{f}$ has an empty core, and this is the case for theorem 15.

In summary, if we have an idea of $\bar{f}^{-1}(\mathscr{impossible})$, we could have some idea of the pure states in core.

Appendix B. Real Interval-valued Frame Functions

In order to simplify the discussion, we call a function $\bar{f} : S^2 \to \mathscr{I}$ satisfying conditions in theorem 20 a real interval-valued frame function. Obviously, any results in appendix Appendix A still hold if we replace every "complex" by "real," in particular, replace every $\mathbb{C}S^2$ by S^2. Furthermore, since restricting the domain of a complex interval-valued frame function to S^2 gives a real interval-valued frame function, if we prove there is no real interval-valued frame function satisfying a condition, then there must be no complex interval-valued frame function satisfying the corresponding condition. From now on, when we write an interval-valued frame function $\bar{f}$ without other specification, we would always means a real interval-valued frame function $\bar{f} : S^2 \to \mathscr{I}$.

If $\bar{f}$ has a non-empty core, $\bar{f}$ must be in the case 1 - 3 in the end of appendix Appendix A so that $\bar{f}^{-1}(\mathscr{impossible})$ contains no area in S^2. In contrast, if $\bar{f}^{-1}(\mathscr{impossible})$ contains some area in S^2, it must be the situation of case 4, and $\bar{f}$ has an empty core. Furthermore, the larger the area, the more state vectors which are inconsistent with any particular Born rule frame function. Therefore, the area of $\bar{f}^{-1}(\mathscr{impossible})$ could indicate how much $\bar{f}$ deviate from the Born rule frame functions. In the following paragraphs, we will establish that if the area of $\bar{f}^{-1}(\mathscr{impossible})$ is big, then the interval $\mathscr{I}$ must be coarse. In another word, if the interval $\mathscr{I}$ is finer, then the area of $\bar{f}^{-1}(\mathscr{impossible})$ would be smaller so that $\bar{f}$ is closer to a Born rule frame function.

In order to simplify the discussion, consider the center of $\bar{f}^{-1}(\mathscr{impossible})$ as the north pole, and assume $\bar{f}$ maps every state whose latitude is larger than θ_0 to impossible.

Definition 24. Given a vector $|\psi\rangle \in S^2$ with a specified north pole $|0\rangle \in S^2$, then $\angle(|\psi\rangle) = \arccos\langle 0|\psi\rangle$ denotes the latitude of $|\psi\rangle$. Conversely, $\angle^{-1}(I) = \{|\psi\rangle \in S^2 \mid \angle(|\psi\rangle) \in I\}$. For example, the north pole has the latitude 0, the equator has the the latitude $\pi/2$, and $\angle^{-1}([0,\theta_0]) = \{|\psi\rangle \in S^2 \mid \theta_0 \leq \angle(|\psi\rangle) \leq \pi/2\}$ contains all states whose latitude is larger than θ_0.

Lemma 25. *Given $\theta_0 \in [0, \pi/2]$ and an interval-valued frame function $\bar{f} : S^2 \to \mathscr{I}$, if $\bar{f}(|\psi\rangle) = \mathscr{impossible}$ for every $|\psi\rangle \in \angle^{-1}([0,\theta_0])$, then there is $l_0 \in \left[0, \frac{1}{2}\right]$ such that $\bar{f}(|\psi_0\rangle) = [l_0, 1 - l_0] \in \mathscr{I}$ for any $|\psi_0\rangle \in \angle^{-1}([\pi/2 - \theta_0, \pi/2])$.*

Proof. We could illustrate the situation in figure B1. Let the yellow circle is the states whose latitude is θ_0. Then, every states above the yellow circle maps to $\mathscr{impossible}$ by $\bar{f}$. In

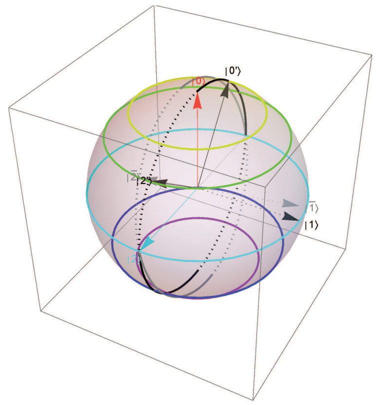

Figure B1. This figure is used in the proof of lemma 25, where every vector above the light green circle maps to $\mathscr{impossible}$, and we want to prove that every vector between dark green and light blue maps to the same interval.

particular, $\bar{f}(|0\rangle) = \bar{f}(|0'\rangle) = \mathscr{impossible}$. Since $\{|0'\rangle, |1\rangle, |2'\rangle\}$ is an orthonormal basis, by theorem 20, we have

$$\begin{aligned} [1,1] - \bar{f}(|1\rangle) &\subseteq \bar{f}(|0'\rangle) + \bar{f}(|2\rangle) = \bar{f}(|2'\rangle) \\ [1,1] - \bar{f}(|2'\rangle) &\subseteq \bar{f}(|0'\rangle) + \bar{f}(|1\rangle) = \bar{f}(|1\rangle) \end{aligned} \tag{B.1}$$

so that $\bar{f}(|2'\rangle) = [1,1] - \bar{f}(|1\rangle)$. By the same reason, for any state $|\psi\rangle$ in the dotted black circle between $|2'\rangle$ and $|2\rangle$, we have $\bar{f}(|\psi\rangle) = [1,1] - \bar{f}(|1\rangle) = \bar{f}(|2'\rangle)$. Since the angle between $|0\rangle$ and $|0'\rangle$ is the same as the angle between $|2\rangle$ and $|2'\rangle$, for any $|\psi_0\rangle$ and $|\psi_1\rangle \in \angle^{-1}([0,\theta_0])$, if $|\psi_0\rangle$ and $|\psi_1\rangle$ are in the same longitude, then $\bar{f}(|\psi_0\rangle) = \bar{f}(|\psi_1\rangle)$.

Moreover, the same argument can apply on the great circle which is not longitude. Since $|\bar{1}\rangle$ is the normal vector of the gray circle which includes $|2\rangle$ and $|\bar{2}\rangle$, we have $\bar{f}(|2\rangle) = [1,1] - \bar{f}(|\bar{1}\rangle) = \bar{f}(|\bar{2}\rangle)$. In another words, for any $|\psi_0\rangle$ and $|\psi_1\rangle \in \angle^{-1}([\pi/2-\theta_0, \pi/2])$, even if $|\psi_0\rangle$ and $|\psi_1\rangle$ are not in the same longitude, then $\bar{f}(|\psi_0\rangle)$ and $\bar{f}(|\psi_1\rangle)$ are still the same.

Thus, we could plug $\bar{f}(|1\rangle) = \bar{f}(|2'\rangle) = [l_0, r_0]$ into equation (B.1), and get

$$[1-r_0, 1-l_0] = [1,1] - [l_0, r_0] = [l_0, r_0] \Rightarrow l_0 + r_0 = 1\,. \tag{B.2}$$

Also, we have

$$1 \in \bar{f}(|0'\rangle) + \bar{f}(|1\rangle) + \bar{f}(|2'\rangle) = [2l_0, 2r_0] \Rightarrow l_0 \le \frac{1}{2} \le r_0\,. \tag{B.3}$$

Thus the lemma is proved. □

Theorem 26. *Given $\theta_0 \in [0,\pi/2]$ and an interval-valued frame function $\bar{f} : S^2 \to \mathscr{I}$, if $\bar{f}(|\psi\rangle) = \mathscr{impossible}$ for every $|\psi\rangle \in \angle^{-1}([0,\theta_0])$, then $\theta_0 < \pi/4$.*

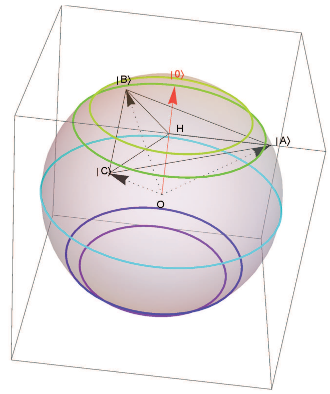

Figure B2. This figure is used in the proof of theorem 27. The latitude of the yellow circle is $\theta_0 \in \left[\arcsin\left(1/\sqrt{3}\right), \pi/4\right)$, and the latitude of the green circle is $\pi/2 - \theta_0$.

Proof. We are going to prove by contradiction. Suppose $\theta_0 \geq \pi/4$, we have $[0, \theta_0] \cap [\pi/2 - \theta_0, \pi/2] = [\pi/2 - \theta_0, \theta_0] \neq \emptyset$. Pick any $|\psi_0\rangle \in \angle^{-1}([\pi/2 - \theta_0, \theta_0])$, by lemma 25, we have $\mathscr{impossible} = \bar{f}(|\psi_0\rangle) = [l_0, 1 - l_0]$, where $l_0 \in [0, \pi/2]$. A contradiction! $\square$

Theorem 27. *Given* $\theta_0 \in \left[\arcsin\left(1/\sqrt{3}\right), \pi/4\right)$ *and an interval-valued frame function* $\bar{f} : S^2 \to \mathscr{I}$, *if* $\bar{f}(|\psi\rangle) = \mathscr{impossible}$ *for every* $|\psi\rangle \in \angle^{-1}([0, \theta_0])$, *then* $\mathscr{unknown} \in \mathscr{I}$.

Proof. We could illustrate the situation in figure B2, where the latitude of of the yellow circle is θ_0 and the latitude of of the green circle is $\pi/2 - \theta_0$. Therefore, $\bar{f}$ should map any state above the yellow circle to $\mathscr{impossible}$ and any states between the green circle and the equator to the same interval, say, $[l_0, 1 - l_0]$, where $0 \leq l_0 \leq \frac{1}{2}$ by lemma 25.

When $\theta_0 \geq \arcsin\left(1/\sqrt{3}\right)$, the latitude of the green circle is $\theta_0 \leq \pi/2 - \arcsin\left(1/\sqrt{3}\right)$. Now consider three states $|A\rangle$, $|B\rangle$, and $|C\rangle$ whose latitude is $\pi/2 - \arcsin\left(1/\sqrt{3}\right)$ so that they are below the green circle, and these three states form an equilateral triangle whose center is H. Since the latitude of this triangle is $\pi/2 - \arcsin\left(1/\sqrt{3}\right)$, we know $\overline{OH} = 1/\sqrt{3}$, where O is the center of the sphere. By the Pythagorean theorem, $\overline{HA} = \overline{HB} = \sqrt{\frac{2}{3}}$. Notice that $\triangle HAB$ is an isosceles triangle with $\angle AHB = 2\pi/3$, we have $\overline{AB} = \sqrt{2}$. Since $\overline{OA} = \overline{OB} = 1$ and $\overline{AB} = \sqrt{2}$, $\triangle OAB$ is an isosceles right triangle with $\angle AHB = \pi/2$, i.e., $|A\rangle$ and $|B\rangle$ are orthogonal. Applying the same reason on all $|A\rangle$, $|B\rangle$, and $|C\rangle$, $\{|A\rangle, |B\rangle, |C\rangle\}$ is an orthonormal basis.

Since $\{|A\rangle, |B\rangle, |C\rangle\}$ is an orthonormal basis and $\bar{f}(|A\rangle) = \bar{f}(|B\rangle) = \bar{f}(|C\rangle) = [l_0, 1 - l_0]$, we have the following inequalities by theorem 20,

$$[1, 1] - [l_0, 1 - l_0] \subseteq [2l_0, 2 - 2l_0] \Rightarrow l_0 \leq 0\,. \tag{B.4}$$

Recall that $0 \leq l_0 \leq \frac{1}{2}$. Therefore, $l_0 = 0$, that is, $\mathscr{unknown} = [l_0, 1 - l_0] \in \mathscr{I}$. $\square$

Appendix C. Between Quantum and Classical

This section discusses the relation between quantum and classical. We first prove that any quantum phenomenon would not be observed if we could only access commuting projectors since commuting projectors could be measured simultaneously [69]. Based on this fact, we show a quantum IVPM $\bar{\mu}$ is convex if its restriction on any orthonormal basis Ω, $\bar{\mu}^{\Omega}$, is a convex classical IVPM. Together with the fact that every classical IVPM $\bar{\mu}^{\Omega}$ is convex when the number of elements in $\Omega \leq 3$, every quantum IVPM must be convex on a three-dimensional Hilbert space.

Appendix C.1. Classical and Quantum Probability Measures

Before diving into the lemmas and proofs, we start from two definitions.

Definition 28. Given a Hilbert space $\mathcal{H}$, and a set of projections $\mathcal{S} \subseteq \mathcal{E}$, we define $\sum_{P\in\mathcal{S}} P = \mathbb{0}$ when $\mathcal{S} = \emptyset$ since $\mathbb{0}$ is the additive identity in operators.

Definition 29. Given a Hilbert space $\mathcal{H}$ with an orthonormal basis $\Omega = \{|j\rangle\}_{j=0}^{d-1}$.

- For any quantum probability measure $\mu : \mathcal{E} \to [0,1]$, we define $\mu^{\Omega} : 2^{\Omega} \to [0,1]$ by $\mu^{\Omega}(E) = \mu\left(\sum_{|j\rangle\in E} |j\rangle\langle j|\right)$.
- For any quantum IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$, we define $\bar{\mu}^{\Omega} : 2^{\Omega} \to \mathscr{I}$ by $\bar{\mu}^{\Omega}(E) = \bar{\mu}\left(\sum_{|j\rangle\in E} |j\rangle\langle j|\right)$.

Lemma 30. *μ^{Ω} and $\bar{\mu}^{\Omega}$ in the previous definition are classical real-valued and interval-valued probability measures, respectively.*

Proof. When $\mathscr{I} = \{\{a\} \mid a \in [0,1]\}$, $\bar{\mu}$ and $\bar{\mu}^{\Omega}$ are essentially quantum and classical real-valued probability measures, respectively. Hence, it is sufficient to prove the interval-valued case.

- $\bar{\mu}^{\Omega}(\emptyset) = \bar{\mu}\left(\sum_{|j\rangle\in\emptyset} |j\rangle\langle j|\right) = \bar{\mu}(\mathbb{0}) = [0,0]$.
- $\bar{\mu}^{\Omega}(\Omega) = \bar{\mu}\left(\sum_{|j\rangle\in\Omega} |j\rangle\langle j|\right) = \bar{\mu}(\mathbb{1}) = [1,1]$.
- For any event $E \in 2^{\Omega}$,

$$\bar{\mu}^{\Omega}(\Omega \backslash E) = \bar{\mu}\left(\mathbb{1} - \sum_{|j\rangle\in E} |j\rangle\langle j|\right) = [1,1] - \bar{\mu}^{\Omega}(E)\ . \tag{C.1}$$

- For a collection $\{E_i\}_{i=1}^{N} \subseteq 2^{\Omega}$ of pairwise disjoint events, we have

$$\bar{\mu}^{\Omega}\left(\bigcup_{i=1}^{N} E_i\right) = \bar{\mu}\left(\sum_{i=1}^{N}\sum_{|j\rangle\in E_i} |j\rangle\langle j|\right) \subseteq \sum_{i=1}^{N} \bar{\mu}^{\Omega}(E_i)\ . \tag{C.2}$$

$\square$

Definition 31. Given a Hilbert space $\mathcal{H}$, a set of event $\mathcal{E}' \subseteq \mathcal{E}$ is called a subspace if it satisfies the follow conditions.

- $\mathbb{0} \in \mathcal{E}'$.
- $\mathbb{1} \in \mathcal{E}'$.
- For any projection $P \in \mathcal{E}'$, we have $\mathbb{1} - P \in \mathcal{E}'$.
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N \subseteq \mathcal{E}'$, we have $\sum_{i=1}^N P_i \in \mathcal{E}'$.

By replacing every occurrence of $\mathcal{E}$ by $\mathcal{E}'$ in definition 10, 6, and 12, it is straightforward to define a quantum cryptodeterministic, real-valued probability, and interval-valued probability measure on $\mathcal{E}'$. Also, $\mathcal{E}'$ is called convex if we have $P_1P_2 \in \mathcal{E}'$ for a pair of mutually commuting projections P_1 and $P_2 \in \mathcal{E}'$.

Theorem 32. *Given a Hilbert space $\mathcal{H}$ and a mutually commuting sub-event space $\mathcal{E}' \subseteq \mathcal{E}$, there is a set Ω, another commuting sub-event space $\widehat{\mathcal{E}}$ with $\mathcal{E}' \subseteq \widehat{\mathcal{E}} \subseteq \mathcal{E}$, and a bijection $\tau : \widehat{\mathcal{E}} \to 2^\Omega$ such that:*

(i) If $\mu' : \mathcal{E}' \to [0,1]$ is a quantum real-valued probability measure, then there is a classical real-valued probability measure $\mu : 2^\Omega \to [0,1]$ such that $\mu'(P) = \mu(\tau(P))$ for all $P \in \mathcal{E}'$.

(ii) If $\bar{\mu}' : \mathcal{E}' \to \mathscr{I}$ is a quantum interval-valued probability measure, then there is a classical interval-valued probability measure $\bar{\mu} : 2^\Omega \to \mathscr{I}$ such that $\bar{\mu}'(P) = \bar{\mu}(\tau(P))$ for all $P \in \mathcal{E}'$.

(iii) If $\mu : 2^\Omega \to [0,1]$ is a classical real-valued probability measure, then $\mu \circ \tau : \widehat{\mathcal{E}} \to [0,1]$ is a QRVPM.

(iv) Given a QRVPM $\widehat{\mu} : \widehat{\mathcal{E}} \to [0,1]$, there is a density matrix ρ such that $\mu_\rho^{\mathrm{B}}(P) = \widehat{\mu}(P)$ for all $P \in \widehat{\mathcal{E}}$.

(v) ρ in the previous case may not be unique.

(vi) The bijection $\tau : \widehat{\mathcal{E}} \to 2^\Omega$ preserves the following operations.

- $\tau(\mathbb{0}) = \emptyset$.
- $\tau(\mathbb{1}) = \Omega$.
- *For any projection $P \in \widehat{\mathcal{E}}$, we have $\tau(\mathbb{1} - P) = \Omega \backslash \tau(P)$.*
- *For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N \subseteq \widehat{\mathcal{E}}$, we have $\tau\left(\sum_{i=1}^N P_i\right) = \bigcup_{i=1}^N \tau(P_i)$.*
- *For a pair of mutually commuting projections P_1 and $P_2 \in \mathcal{E}'$, we have $\tau(P_1P_2) = \tau(P_1) \cap \tau(P_2)$.*

Proof. (i) Since this case is a special case of the next case, it is sufficient to prove the next case.

(ii) Since $\mathcal{E}'$ is a set of mutually commuting projections, they can be diagonalized by a common orthonormal basis $\Omega = \{|j\rangle\}_{j=0}^{d-1}$. Pick arbitrary projector $P \in \mathcal{E}'$, its eigenvalues are 0 or 1. Select its eigenvectors whose eigenvalues are 1 into a set $\tau(P) \subseteq \Omega$. Then, we have $P = \sum_{|j\rangle\in\tau(P)} |j\rangle\langle j|$, $\widehat{\mathcal{E}} = \left\{\sum_{|j\rangle\in E} |j\rangle\langle j| \;\middle|\; E \subseteq \Omega\right\}$, and τ is a bijection between $\widehat{\mathcal{E}}$ and 2^{Ω}. By definition 29, $\bar{\mu}'(P) = \bar{\mu}^{\Omega}(\tau(P))$, and $\bar{\mu}^{\Omega}$ is a classical interval-valued probability measure by lemma 30.

(iii) Trivial.

(iv) If we pick $\rho = \sum_{j=1}^{d} \widehat{\mu}(|j\rangle\langle j|)\, |j\rangle\langle j|$, then ρ is a density matrix because [10]

$$\mathrm{Tr}(\rho) = \sum_{j=1}^{d} \widehat{\mu}(|j\rangle\langle j|)\, \mathrm{Tr}(|j\rangle\langle j|) = 1\,. \tag{C.3}$$

Also,

$$\mu_{\rho}^{\mathrm{B}}(|i\rangle\langle i|) = \sum_{j=1}^{N} \widehat{\mu}(|j\rangle\langle j|)\, \mu_{|j\rangle}^{\mathrm{B}}(|i\rangle\langle i|) = \widehat{\mu}(|i\rangle\langle i|) \text{ for all } i\,, \tag{C.4}$$

$$\mu_{\rho}^{\mathrm{B}}(P_0) + \mu_{\rho}^{\mathrm{B}}(P_1) \overset{\mathrm{IH}}{=} \widehat{\mu}(P_0) + \widehat{\mu}(P_1) = \widehat{\mu}(P_0 + P_1) \tag{C.5}$$

for orthogonal $P_0, P_1 \in \mathcal{E}$. Therefore, we have $\mu_{\rho}^{\mathrm{B}}(P) = \widehat{\mu}(P)$ for all $P \in \widehat{\mathcal{E}}$ by induction.

(v) Consider a Hilbert space $\mathcal{H}$ of dimension 2, a mutually commuting sub-event space $\widehat{\mathcal{E}} = \{\mathbb{0}, |0\rangle\langle 0|, |1\rangle\langle 1|, \mathbb{1}\}$, and a QRVPM $\widehat{\mu} : \widehat{\mathcal{E}} \to [0,1]$ such that $\widehat{\mu}(|0\rangle\langle 0|) = \widehat{\mu}(|1\rangle\langle 1|) = \frac{1}{2}$. Then, $|{+}\rangle\langle{+}|$ and $|{-}\rangle\langle{-}|$ are both density matrices, and

$$\mu_{|+\rangle}^{\mathrm{B}}(|0\rangle\langle 0|) = \mu_{|-\rangle}^{\mathrm{B}}(|0\rangle\langle 0|) = \frac{1}{2} = \widehat{\mu}(|0\rangle\langle 0|)\ ,$$
$$\mu_{|+\rangle}^{\mathrm{B}}(|1\rangle\langle 1|) = \mu_{|-\rangle}^{\mathrm{B}}(|1\rangle\langle 1|) = \frac{1}{2} = \widehat{\mu}(|1\rangle\langle 1|)\ .$$

(vi) Trivial.

□

Given an orthonormal basis $\Omega = \{|j\rangle\}_{j=0}^{d-1}$, if we consider sub-event space generated by Ω, $\mathcal{E}^{\Omega} = \left\{\sum_{|j\rangle\in E} |j\rangle\langle j| \;\middle|\; E \subseteq \Omega\right\}$, then any quantum IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ restricting on $\mathcal{E}^{\Omega}$, denoted by $\bar{\mu}|_{\mathcal{E}^{\Omega}}$, behaves classically. In particular, if $\bar{\mu}$ is convex, each $\bar{\mu}|_{\mathcal{E}^{\Omega}}$ is convex and has a non-empty core. However, the core elements of each $\bar{\mu}|_{\mathcal{E}^{\Omega}}$ may not be glued together, and gives a core element of $\bar{\mu}$. The inability of gluing core elements among commuting $\mathcal{E}^{\Omega}$ to non-commuting $\mathcal{E}$ is consistent with the idea of the Kochen-Specker theorem, that each triple of commuting observables could be colored well, but we cannot color all observables which are not all commuting. This is also consistent with the assertion that mutually commuting observables could be measured simultaneously [69]. In another word, if Alice and Bob in Bell's theorem can only pick from a set of mutually commuting observables, they could never observe any "quantum," and we cannot eliminate the local hidden variable theory in this setting.

Appendix C.2. Describe Classical World by Quantum Probability Measures with a Commuting Sub-event Space

Since there is a correspondence between classical probability measures and quantum ones on a commuting sub-event space, we can use the quantum language to describe the classical world given an orthonormal basis.

Definition 33. Given a Hilbert space $\mathcal{H}$, an orthonormal basis $\Omega = \{|j\rangle\}_{j=0}^{d-1}$, and a classical real-valued probability measures $\mu : 2^\Omega \to [0,1]$ on the sample space Ω, the density matrix $\sum_{j=0}^{d-1} \mu(\{j\}) |j\rangle\langle j|$ is called classical with respect to Ω.

Definition 34. A state ρ is consistent with $\bar{\mu}$ on a projector P if $\mu_\rho^{\mathrm{B}}(P) \in \bar{\mu}(P)$, and denoted by $\rho \in \overline{\mathcal{H}}(\bar{\mu}, \{P\})$. Moreover, $\rho \in \overline{\mathcal{H}}(\bar{\mu}, S)$ means ρ is consistent with $\bar{\mu}$ on every projector in S. Notice that $\rho \in \overline{\mathcal{H}}(\bar{\mu}, S)$ if and only if $\mu_\rho^{\mathrm{B}} \in \mathrm{core}(\bar{\mu}, S)$.

Then, we can rephrase theorem 14, in the language of quantum IVPM with an orthonormal basis.

Corollary 35. *For every convex quantum IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$, if a sub-event space $\mathcal{E}' \subseteq \mathcal{E}$ commutes, then $\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}') \neq \emptyset$.*

Proof. Since $\mathcal{E}'$ is a set of mutually commuting projections, they can be diagonalized by a common orthonormal basis $\Omega = \{|j\rangle\}_{j=0}^{d-1}$. By theorem 32, we can find a bijection $\tau : \widehat{\mathcal{E}} \to 2^\Omega$ and a classical IVPM $\bar{\mu}' : 2^\Omega \to \mathscr{I}$ such that $\bar{\mu}(P) = \bar{\mu}'(\tau(P))$ for all $P \in \mathcal{E}'$. Since τ is a bijection, $\bar{\mu}'$ is convex. By theorem 14, there is a classical probability measure $\mu : 2^\Omega \to [0,1]$ such that

$$
\begin{aligned}
&\mu \in \mathrm{core}(\bar{\mu}') \\
\Rightarrow\ & \mu(E) \in \bar{\mu}'(E) \text{ for all } E \in 2^\Omega \\
\Rightarrow\ & \mu(\tau(P)) \in \bar{\mu}'(\tau(P)) = \bar{\mu}(P) \text{ for all } P \in \widehat{\mathcal{E}} \\
\Rightarrow\ & \mu(\tau(P)) \in \bar{\mu}(P) \text{ for all } P \in \widehat{\mathcal{E}}
\end{aligned}
$$

By theorem 32, there is a density matrix ρ such that $\mu_\rho^{\mathrm{B}}(P) = \mu'(\tau(P))$ for all $P \in \mathcal{E}'$. Therefore, $\mu_\rho^{\mathrm{B}} \in \mathrm{core}(\bar{\mu}, \mathcal{E}')$, i.e., $\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')$. □

Appendix C.3. Application on Convexity

The rest of the section focus on convexity.

Theorem 36. *Given a classical IVPM $\bar{\mu}^\Omega : 2^\Omega \to \mathscr{I}$, the following statements are true:*

(i) If $E_0 \subseteq E_1$, then $\bar{\mu}^\Omega(E_0 \cup E_1) + \bar{\mu}^\Omega(E_0 \cap E_1) \subseteq \bar{\mu}^\Omega(E_0) + \bar{\mu}^\Omega(E_1)$.

(ii) If $\bar{\mu}^\Omega(E_0 \cup E_1) + \bar{\mu}^\Omega(E_0 \cap E_1) \subseteq \bar{\mu}^\Omega(E_0) + \bar{\mu}^\Omega(E_1)$, then $\bar{\mu}^\Omega\left(\overline{E_0} \cup \overline{E_1}\right) + \bar{\mu}^\Omega\left(\overline{E_0} \cap \overline{E_1}\right) \subseteq \bar{\mu}^\Omega\left(\overline{E_0}\right) + \bar{\mu}^\Omega\left(\overline{E_1}\right)$, where $\overline{E_0} = \Omega \backslash E_0$ and $\overline{E_1} = \Omega \backslash E_1$.

Proof. (i) $\bar{\mu}^\Omega(E_0 \cup E_1) + \bar{\mu}^\Omega(E_0 \cap E_1) = \bar{\mu}^\Omega(E_0) + \bar{\mu}^\Omega(E_1)$.

(ii)

$$\begin{aligned}
&\bar{\mu}^{\Omega}\left(E_0 \cup E_1\right)+\bar{\mu}^{\Omega}\left(E_0 \cap E_1\right) \subseteq \bar{\mu}^{\Omega}\left(E_0\right)+\bar{\mu}^{\Omega}\left(E_1\right) \\
\Rightarrow &\bar{\mu}^{\Omega}\left(\Omega \backslash\left(\overline{E_0} \cap \overline{E_1}\right)\right)+\bar{\mu}^{\Omega}\left(\Omega \backslash\left(\overline{E_0} \cup \overline{E_1}\right)\right) \subseteq \bar{\mu}^{\Omega}\left(\Omega \backslash\left(\overline{E_0}\right)\right)+\bar{\mu}^{\Omega}\left(\Omega \backslash\left(\overline{E_1}\right)\right) \\
\Rightarrow &2-\left[\bar{\mu}^{\Omega}\left(\overline{E_0} \cap \overline{E_1}\right)+\bar{\mu}^{\Omega}\left(\overline{E_0} \cup \overline{E_1}\right)\right] \subseteq 2-\left[\bar{\mu}^{\Omega}\left(\overline{E_0}\right)+\bar{\mu}^{\Omega}\left(\overline{E_1}\right)\right] \\
\Rightarrow &\bar{\mu}^{\Omega}\left(\overline{E_0} \cap \overline{E_1}\right)+\bar{\mu}^{\Omega}\left(\overline{E_0} \cup \overline{E_1}\right) \subseteq \bar{\mu}^{\Omega}\left(\overline{E_0}\right)+\bar{\mu}^{\Omega}\left(\overline{E_1}\right) .
\end{aligned}$$

□

Lemma 37. *When the number of elements in Ω is less than or equal to 3, every classical IVPM $\bar{\mu}^{\Omega}: 2^{\Omega} \rightarrow \mathscr{I}$ is convex.*

Proof. Given two events $E_0, E_1 \subseteq \Omega$, we want to verify

$$\bar{\mu}^{\Omega}\left(E_0 \cup E_1\right)+\bar{\mu}^{\Omega}\left(E_0 \cap E_1\right) \subseteq \bar{\mu}^{\Omega}\left(E_0\right)+\bar{\mu}^{\Omega}\left(E_1\right) . \tag{C.6}$$

When the number of elements in $\Omega \leq 3$, one of the following must happen

(i) When $E_0 \cap E_1=\emptyset$, equation (C.6) is a special case of equation (5).

(ii) When $E_0 \cup E_1=\Omega$, we have $\overline{E_0} \cap \overline{E_1}=\emptyset$, where $\overline{E_0}=\Omega \backslash E_0$ and $\overline{E_1}=\Omega \backslash E_1$. By equation (5), we have $\bar{\mu}^{\Omega}\left(\overline{E_0} \cup \overline{E_1}\right)+\bar{\mu}^{\Omega}\left(\overline{E_0} \cap \overline{E_1}\right) \subseteq \bar{\mu}^{\Omega}\left(\overline{E_0}\right)+\bar{\mu}^{\Omega}\left(\overline{E_1}\right)$. Then, theorem 36 proves equation (C.6).

(iii) When $E_0 \subseteq E_1$ or $E_1 \subseteq E_0$, equation (C.6) is also followed by theorem 36.

so that $\bar{\mu}^{\Omega}$ is convex. □

Lemma 38. *Given a QIVPM $\bar{\mu}: \mathcal{E} \rightarrow \mathscr{I}$, the following statements are true:*

(i) If $P_1=P_0+P_2$, then $\bar{\mu}\left(P_0 \vee P_1\right)+\bar{\mu}\left(P_0 P_1\right) \subseteq \bar{\mu}\left(P_0\right)+\bar{\mu}\left(P_1\right)$.

(ii) Given commuting P_0 and $P_1 \in \mathcal{E}$, if $\bar{\mu}\left(P_0 \vee P_1\right)+\bar{\mu}\left(P_0 P_1\right) \subseteq \bar{\mu}\left(P_0\right)+\bar{\mu}\left(P_1\right)$, then $\bar{\mu}\left(\overline{P_0} \vee \overline{P_1}\right)+\bar{\mu}\left(\overline{P_0 P_1}\right) \subseteq \bar{\mu}\left(\overline{P_0}\right)+\bar{\mu}\left(\overline{P_1}\right)$, where $\overline{P_0}=\Omega \backslash P_0$ and $\overline{P_1}=\Omega \backslash P_1$.

Proof. The direct consequence of theorem 36 and theorem 32. □

Theorem 39. *Given a Hilbert space $\mathcal{H}$ of dimension 3, every quantum IVPM $\bar{\mu}: \mathcal{E} \rightarrow \mathscr{I}$ is convex.*

Proof. The direct consequence of lemma 37 and theorem 32. □

Appendix D. From QRVPM to QIVPM

Recall that given a set of intervals $\mathscr{I}$, we want to know whether there is an QIVPM mapping to $\mathscr{I}$. If there is a QIVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$, we want to further verify whether $\bar{\mu}$ has a non-empty core. Usually, the easiest way to prove the existence is to construct an example. As in example 60, constructing a QIVPM directly is long and tedious. In contrast, a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$ can be easily constructed by the Born rule. Therefore, it would be desirable if we can construct a QIVPM $\bar{\mu}$ from μ, and expect μ is in the core of $\bar{\mu}$.

Notice that the range of μ and $\bar{\mu}$ are different. If there is a "homomorphism" $\iota : [0, 1] \to \mathscr{I}$ preserving the desiring properties of the ranges, then we could easily inducing a map from quantum real-valued probability measures to QIVPMs by function composition $\mathcal{E} \xrightarrow{\mu} [0, 1] \xrightarrow{\iota} \mathscr{I}$. This idea gives the following definition.

Definition 40. Given $\mathscr{I}$ is a collection of intervals, we say a interval-valued function $\iota : [0, 1] \to \mathscr{I}$ is additive if ι satisfies the following properties.

- $\iota\left(0\right) = [0, 0]$.
- $\iota\left(1\right) = [1, 1]$.
- For any $x \in [0, 1]$, $\iota\left(1 - x\right) = [1, 1] - \iota\left(x\right)$.
- For a set $\{x_i\}_{i=1}^{N} \subseteq [0, 1]$, if $\sum_{i=1}^{N} x_i \in [0, 1]$, then $\iota\left(\sum_{i=1}^{N} x_i\right) \subseteq \sum_{i=1}^{N} \iota\left(x_i\right)$.

Moreover, ι is called convex if for any three numbers x_0, x_1, and $x_2 \in [0, 1]$, $x_0+x_1+x_2 \in [0, 1]$ implies $\iota\left(x_0 + x_1 + x_2\right) + \iota\left(x_2\right) \subseteq \iota\left(x_0 + x_2\right) + \iota\left(x_1 + x_2\right)$.

Although a interval-valued function $\iota : [0, 1] \to \mathscr{I}$ may look exotic, its left-end and right-end $\left[\iota^{\mathrm{L}}\left(x\right), \iota^{\mathrm{R}}\left(x\right)\right] = \iota\left(x\right)$ are just usual functions from a unit interval to another unit interval. For an additive interval-valued function, ι^{L} is superadditive, i.e., $\iota^{\mathrm{L}}\left(x_0 + x_1\right) \geq \iota^{\mathrm{L}}\left(x_0\right) + \iota^{\mathrm{L}}\left(x_1\right)$, and ι^{R} is subadditive, i.e., $\iota^{\mathrm{R}}\left(x_0 + x_1\right) \leq \iota^{\mathrm{R}}\left(x_0\right) + \iota^{\mathrm{R}}\left(x_1\right)$ [76]. For a convex interval-valued function, ι^{L} is Wright-convex, i.e., $\iota^{\mathrm{L}}\left(x_0 + x_1 + x_2\right) + \iota^{\mathrm{L}}\left(x_2\right) \geq \iota^{\mathrm{L}}\left(x_0 + x_2\right) + \iota^{\mathrm{L}}\left(x_1 + x_2\right)$, and ι^{R} is Wright-concave, i.e., $\iota^{\mathrm{R}}\left(x_0 + x_1 + x_2\right) + \iota^{\mathrm{R}}\left(x_2\right) \leq \iota^{\mathrm{R}}\left(x_0 + x_2\right) + \iota^{\mathrm{R}}\left(x_1 + x_2\right)$ [77, 78, 76, 79]. Also, if $\mathscr{I}$ contains only finite number of intervals, ι^{L} and ι^{R} must be step functions. After we define interval-valued function, we are ready to construct a QIVPM from a quantum probability measure.

Lemma 41. *Consider an additive interval-valued function $\iota : [0, 1] \to \mathscr{I}$. Then, given a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$, a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ defined by $\bar{\mu}\left(P\right) = \iota\left(\mu\left(P\right)\right)$ is a quantum IVPM. Moreover, if ι is convex, then $\bar{\mu}$ is convex.*

Proof.

- $\bar{\mu}(\mathbb{0}) = \iota\left(\mu(\mathbb{0})\right) = \iota\left(0\right) = [0, 0]$.
- $\bar{\mu}(\mathbb{1}) = \iota\left(\mu(\mathbb{1})\right) = \iota\left(1\right) = [1, 1]$.
- For any projection P, $\bar{\mu}\left(\mathbb{1} - P\right) = \iota\left(1 - \mu\left(P\right)\right) = [1, 1] - \bar{\mu}\left(P\right)$.

- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have $\sum_{i=1}^N \mu(P_i) \in [0,1]$ so that $\bar{\mu}\left(\sum_{i=1}^N P_i\right) = \iota\left(\sum_{i=1}^N \mu(P_i)\right) \subseteq \sum_{i=1}^N \bar{\mu}(P_i)$.
- For convexity part, given a pair of commuting operators P_0 and P_1, they and $P_0 \vee P_1$ can be expressed by the summation among $P_0(\mathbb{1} - P_1)$, $(\mathbb{1} - P_0)P_1$, and P_0P_1. Hence,

$$\begin{aligned} &\bar{\mu}(P_0 \vee P_1) + \bar{\mu}(P_0P_1) \\ =&\iota(\mu(P_0(\mathbb{1} - P_1)) + \mu((\mathbb{1} - P_0)P_1) + \mu(P_0P_1)) + \iota(\mu(P_0P_1)) \\ \subseteq&\iota(\mu(P_0(\mathbb{1} - P_1)) + \mu(P_0P_1)) + \iota(\mu((\mathbb{1} - P_0)P_1) + \mu(P_0P_1)) \\ =&\bar{\mu}(P_0) + \bar{\mu}(P_1) \end{aligned} \tag{D.1}$$

□

Since we want to consider QIVPMs induced by the Born rule, it is easier to work with the states instead of the probability measures, which gives us the following definition.

Definition 42. A state ρ is consistent with $\bar{\mu}$ on a projector P if $\mu_\rho^{\mathrm{B}}(P) \in \bar{\mu}(P)$, i.e., $\mu_\rho^{\mathrm{B}} \in \operatorname{core}\left(\bar{\mu}|_{\{P\}}\right)$, and denoted by $\rho \in \overline{\mathcal{H}}(\bar{\mu}, \{P\})$. Moreover, $\rho \in \overline{\mathcal{H}}(\bar{\mu}, S)$ means ρ is consistent with $\bar{\mu}$ on every projector in S, i.e., $\mu_\rho^{\mathrm{B}} \in \operatorname{core}(\bar{\mu}|_S)$.

Lemma 43. *Consider a set of intervals $\mathscr{I}$ with an additive interval-valued function $\iota : [0,1] \to \mathscr{I}$. For any normalized pure state $|\phi\rangle$, $\iota \circ \mu_\phi^{\mathrm{B}}$ is a quantum IVPM such that $\overline{\mathcal{H}}\left(\iota \circ \mu_\phi^{\mathrm{B}}\right)$ has an unique element. Obviously, if ι is convex, so does $\iota \circ \mu_\phi^{\mathrm{B}}$.*

Proof. $\left(\iota \circ \mu_\phi^{\mathrm{B}}\right)(|\phi\rangle\langle\phi|) = \iota\left(\mu_\phi^{\mathrm{B}}(|\phi\rangle\langle\phi|)\right) = \iota(1) = [1,1]$, i.e., for any quantum probability measure $\mu \in \overline{\mathcal{H}}\left(\iota \circ \mu_\phi^{\mathrm{B}}\right)$, we have $\mu(|\phi\rangle\langle\phi|) = 1$. Therefore, $\mu = \mu_\phi^{\mathrm{B}}$. □

Example 44. For any positive integer n and $\delta \in [0,1]$, there is an additive interval-valued function $\iota_{n,\delta} : [0,1] \to \mathscr{I}_{n,\delta}$ defined by

$$\begin{aligned} \mathscr{I}_{n,\delta} &= \left\{ \left[\frac{k-\delta}{n}, \frac{k+\delta}{n}\right] \,\middle|\, k = 1, \ldots, n-1 \right\} \\ &\quad \cup \left\{ \left[\frac{k}{n}, \frac{k+1}{n}\right] \,\middle|\, k = 0, \ldots, n-1 \right\} \cup \{[0,0], [1,1]\} \\ \iota_{n,\delta}(x) &= \begin{cases} [x,x] \ , & \text{if } x = 0 \text{ or } x = 1 \ ; \\ \left[\frac{k-\delta}{n}, \frac{k+\delta}{n}\right] , & \text{if } x = \frac{k}{n} \in (0,1) \ ; \\ \left[\frac{k}{n}, \frac{k+1}{n}\right] , & \text{if } \frac{k}{n} < x < \frac{k+1}{n} \ . \end{cases} \end{aligned} \tag{D.2}$$

When we use the same notation in other place in this paper, we may write $\iota_n : [0,1] \to \mathscr{I}_n$ to represent $\iota_{n,0} : [0,1] \to \mathscr{I}_{n,0}$. Notice that the usual $\{\mathit{certain}, \mathit{impossible}, \mathit{unknown}\}$ is $\mathscr{I}_1$. Also, let $\mathscr{I}_\infty = \{\{x\} \mid x \in [0,1]\}$ and $\iota_\infty(x) = \{x\}$. Of course, $\iota_\infty : [0,1] \to \mathscr{I}_\infty$ is a convex interval-valued function.

Proof.

- Since $\delta \in [0,1]$ and $\iota_{n,\delta}(x) = \left[\frac{k-\delta}{n}, \frac{k+\delta}{n}\right]$ only when $0 < x = \frac{k}{n} < 1$, we have $\iota_{n,\delta}(x) \subseteq [0,1]$.
- $\iota_{n,\delta}(0) = [0,0]$.
- $\iota_{n,\delta}(1) = [1,1]$.
- When $x = 0$ or $x = 1$, we have $\iota_{n,\delta}(1-x) = [1-x, 1-x] = [1,1] - \iota_{n,\delta}(x)$.
  When $x = k/n \in (0,1)$, we have
  $$\iota_{n,\delta}(1-x) = \left[\frac{n-k-\delta}{n}, \frac{n-k+\delta}{n}\right] = [1,1] - \iota_{n,\delta}(x) \ . \tag{D.3}$$
  When $k/n < x < (k+1)/n$, we have $1 - k/n > 1 - x > 1 - (k+1)/n$ so that
  $$\iota_{n,\delta}(1-x) = \left[1 - \frac{k+1}{n}, 1 - \frac{k}{n}\right] = [1,1] - \iota_{n,\delta}(x) \ . \tag{D.4}$$
- It is sufficient to prove the case when $N = 2$, i.e., for a set $\{x_1, x_2\} \subseteq [0,1]$, if $x_1 + x_2 \in [0,1]$, then $\iota_{n,\delta}(x_1 + x_2) \subseteq \iota_{n,\delta}(x_1) + \iota_{n,\delta}(x_2)$, and the rest can be done by the induction.
  - When $x_1 = 0$, we have $\iota_{n,\delta}(x_1 + x_2) = \iota_{n,\delta}(x_2) = \iota_{n,\delta}(x_1) + \iota_{n,\delta}(x_2)$.
  - When $x_1 = k_1/n \in (0,1)$, we have
    $$\iota_{n,\delta}(x_1 + x_2) \subseteq \left[\frac{k_1}{n}, \frac{k_1}{n}\right] + \iota_{n,\delta}(x_2) \subseteq \iota_{n,\delta}(x_1) + \iota_{n,\delta}(x_2) \ , \tag{D.5}$$
    where $\iota_{n,\delta}(x_1 + x_2)$ is a proper subset of $\left[\frac{k_1}{n}, \frac{k_1}{n}\right] + \iota_{n,\delta}(x_2)$ only when $x_1 + x_2 = 1$.
  - When $k_1/n < x_1 < (k_1+1)/n$ and $k_2/n < x_2 < (k_2+1)/n$, we have
    $$\frac{k_1 + k_2}{n} < x_1 + x_2 < \frac{k_1 + k_2 + 2}{n} \tag{D.6}$$
    so that
    $$\begin{aligned} \iota_{n,\delta}(x_1 + x_2) &\subseteq \left[\frac{k_1 + k_2}{n}, \frac{k_1 + k_2 + 2}{n}\right] \\ &= \left[\frac{k_1}{n}, \frac{k_1 + 1}{n}\right] + \left[\frac{k_2}{n}, \frac{k_2 + 1}{n}\right] = \iota_{n,\delta}(x_1) + \iota_{n,\delta}(x_2) \ . \end{aligned} \tag{D.7}$$

□

Example 45. Consider the following intervals
$$\begin{aligned} \widehat{\mathscr{I}_n} = &\left\{ \left[\tfrac{k-1}{2n}, \tfrac{k+1}{2n}\right] \cap [0,1] \;\middle|\; k = 0, \ldots, 2n \right\} \\ &\cup \{\mathscr{impossible}, \mathscr{certain}\} \end{aligned} \tag{D.8}$$
which are overlapping intervals with length $1/n$ except for the intervals with length $(2n)^{-1}$ at each end. As shown in Table D1, for each interval $[\ell, r]$, we define $\mathrm{Int}([\ell, r])$, the non-overlapping interior part of $[\ell, r]$. Since their interior parts are non-overlapping, we can define a function $\widehat{\iota_n} : [0,1] \to \widehat{\mathscr{I}_n}$ by $\widehat{\iota_n}(x) = [\ell, r]$ if $x \in \mathrm{Int}([\ell, r])$ to map a real-valued probability

Table D1. Intervals in $\widehat{\mathscr{I}_n}$, their lengths, and their interiors, where $2 \mid k$ means k is even, and $2 \nmid k$ means k is odd. Although intervals $[\ell, r]$ are overlapping, their interior $\operatorname{Int}([\ell, r])$ are non-overlapping.

k	$[\ell, r]$	$r - \ell$	$\operatorname{Int}([\ell, r])$
	impossible	0	*impossible*
0	$\left[0, \frac{1}{2n}\right]$	$\frac{1}{2n}$	$\left(0, \frac{0.5}{2n}\right]$
$0 < k < 2n$ and $2 \nmid k$	$\left[\frac{k-1}{2n}, \frac{k+1}{2n}\right]$	$\frac{1}{n}$	$\left(\frac{k-0.5}{2n}, \frac{k+0.5}{2n}\right)$
$0 < k < 2n$ and $2 \mid k$	$\left[\frac{k-1}{2n}, \frac{k+1}{2n}\right]$	$\frac{1}{n}$	$\left[\frac{k-0.5}{2n}, \frac{k+0.5}{2n}\right]$
$2n$	$\left[\frac{2n-1}{2n}, 1\right]$	$\frac{1}{2n}$	$\left[\frac{2n-0.5}{2n}, 1\right)$
	certain	0	*certain*

Table D2. Intervals in $\widehat{\mathscr{I}_2}$, , their lengths, and their interiors. Although intervals $[\ell, r]$ are overlapping, their interior $\operatorname{Int}([\ell, r])$ are non-overlapping.

k	$[\ell, r]$	$r - \ell$	$\operatorname{Int}([\ell, r])$
	impossible	0	*impossible*
0	$\left[0, \frac{1}{4}\right]$	$\frac{1}{4}$	$\left(0, \frac{1}{8}\right]$
1	$\left[0, \frac{1}{2}\right]$	$\frac{1}{2}$	$\left(\frac{1}{8}, \frac{3}{8}\right)$
2	$\left[\frac{1}{4}, \frac{3}{4}\right]$	$\frac{1}{2}$	$\left[\frac{3}{8}, \frac{5}{8}\right]$
3	$\left[\frac{1}{2}, 1\right]$	$\frac{1}{2}$	$\left(\frac{5}{8}, \frac{7}{8}\right)$
4	$\left[\frac{3}{4}, 1\right]$	$\frac{1}{4}$	$\left[\frac{7}{8}, 1\right)$
	certain	0	*certain*

to an interval-valued one. Table D2 lists a special case $\widehat{\mathscr{I}_2}$ to illustrate these intervals and their interior parts more clearly.

Proof.

- $\widehat{\iota}_n(0) = [0, 0]$.
- $\widehat{\iota}_n(1) = [1, 1]$.
- When $x = 0$ or $x = 1$, we have $\widehat{\iota}_n(1 - x) = [1 - x, 1 - x] = [1, 1] - \widehat{\iota}_n(x)$. Otherwise, since $2 \mid k \Leftrightarrow 2 \mid (2n - k)$, we always have
$$\widehat{\iota}_n(1 - x) = \left[1 - \frac{k+1}{2n}, 1 - \frac{k-1}{2n}\right] \cap [0, 1] = [1, 1] - \widehat{\iota}_n(x) \ . \tag{D.9}$$
- It is sufficient to prove the case when $N = 2$, i.e., for a set $\{x_1, x_2\} \subseteq [0, 1]$, if $x_1 + x_2 \in [0, 1]$, then $\widehat{\iota}_n(x_1 + x_2) \subseteq \widehat{\iota}_n(x_1) + \widehat{\iota}_n(x_2)$, and the rest can be done by the induction.
  - When $x_1 = 0$, we have $\widehat{\iota}_n(x_1 + x_2) = \widehat{\iota}_n(x_2) = \widehat{\iota}_n(x_1) + \widehat{\iota}_n(x_2)$.
  - When $x_1 = 1$, we must have $x_2 = 0$ which reduces to the previous case.
  - When $\widehat{\iota}_n(x_1) = \left[\frac{k_1-1}{2n}, \frac{k_1+1}{2n}\right] \cap [0, 1]$ and $\widehat{\iota}_n(x_2) = \left[\frac{k_2-1}{2n}, \frac{k_2+1}{2n}\right] \cap [0, 1]$, we have $k_1 - 0.5 \le 2nx_1 \le k_1 + 0.5$ and $k_2 - 0.5 \le 2nx_2 \le k_2 + 0.5$ so that $k_1 + k_2 - 1 \le 2n(x_1 + x_2) \le k_1 + k_2 + 1$. Hence, no matter whether $x_1 + x_2 = 1$ or not, we always have
$$\begin{aligned} &\widehat{\iota}_n(x_1 + x_2) \subseteq \left[\frac{k_1 + k_2 - 2}{2n}, \frac{k_1 + k_2 + 2}{2n}\right] \cap [0, 1] \\ =&\left[\frac{k_1 - 1}{2n}, \frac{k_1 + 1}{2n}\right] \cap [0, 1] + \left[\frac{k_2 - 1}{2n}, \frac{k_2 + 1}{2n}\right] \cap [0, 1] \\ =&\widehat{\iota}_n(x_1) + \widehat{\iota}_n(x_2) \ . \end{aligned} \tag{D.10}$$

□

Although a convex QIVPM can be constructed by a convex interval-valued function easily, its range either is $\{\mathscr{certain}, \mathscr{impossible}, \mathscr{unknown}\}$ or contains infinitely many intervals. The reason is that a Wright-convex function is midpoint-convex or Jensen-convex, i.e.,

$$\frac{\iota^{\mathrm{L}}(x_0) + \iota^{\mathrm{L}}(x_1)}{2} \ge \iota^{\mathrm{L}}\left(\frac{x_0 + x_1}{2}\right) , \tag{D.11}$$

and a midpoint-convex function is continuous on an open interval if we cannot use the Axiom of Choices [80, 81, 78].

Even if an additive interval-valued function $\iota : [0, 1] \to \mathscr{I}$ is not convex, it may still induce a convex QIVPM. In particular, every QIVPM on a three-dimensional Hilbert space is convex by theorem 39. The next natural question is whether a non-convex interval-valued function always induces some non-convex QIVPM, and the answer is “yes” by the following lemma.

Lemma 46. *Given an additive but non-convex interval-valued function $\iota : [0,1] \to \mathscr{I}$, there is a non-convex quantum IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$.*

Proof. Since $(\mathscr{I}, \iota)$ is additive but non-convex, there are three numbers x_0, x_1, and $x_2 \in [0,1]$ such that $x_0+x_1+x_2 \in [0,1]$ but $\iota(x_0+x_1+x_2)+\iota(x_2) \nsubseteq \iota(x_0+x_2)+\iota(x_1+x_2)$. Consider a 4-dimensional Hilbert space and a density matrix

$$\rho = x_0 |0\rangle\langle 0| + x_1 |1\rangle\langle 1| + x_2 |2\rangle\langle 2| + (1-x_0-x_1-x_2) |3\rangle\langle 3| \ . \tag{D.12}$$

Let $P_0 = |0\rangle\langle 0| + |2\rangle\langle 2|$, $P_1 = |1\rangle\langle 1| + |2\rangle\langle 2|$, and $\bar{\mu}(P) = \iota\left(\mu_\rho^{\mathrm{B}}(P)\right)$. Then,

$$\begin{aligned} &\bar{\mu}(P_0 \vee P_1) + \bar{\mu}(P_0 P_1) \\ =&\iota\left(\mu_\rho^{\mathrm{B}}(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)\right) + \iota\left(\mu_\rho^{\mathrm{B}}(|2\rangle\langle 2|)\right) \\ =&\iota(x_0+x_1+x_2) + \iota(x_2) \nsubseteq \iota(x_0+x_2) + \iota(x_1+x_2) \\ =&\iota\left(\mu_\rho^{\mathrm{B}}(|0\rangle\langle 0| + |2\rangle\langle 2|)\right) + \iota\left(\mu_\rho^{\mathrm{B}}(|1\rangle\langle 1| + |2\rangle\langle 2|)\right) = \bar{\mu}(P_0) + \bar{\mu}(P_1) \ . \end{aligned} \tag{D.13}$$

□

In conclusion, we may still use an additive but non-convex interval-valued function to construct QIVPMs, but we need to verify the convexity of the constructed QIVPM one-by-one which might be long and tedious.

Appendix D.1. Convex but Infinite Number of Intervals

Since we cannot restrict our convex interval-valued function mapping to only finite number of intervals, let's consider another family of natural interval-valued functions.

Lemma 47. *Given a function $c : [0,1] \to [0,1]$ satisfying the following conditions:*

- *For all $x \in [0,1]$, $c(x) \leq x$.*
- *For all $x \in [0,1]$, $c(1-x) = c(x)$.*
- *c is Wright-concave [79], i.e., for any three numbers x_0, x_1, and $x_2 \in [0,1]$, $x_0+x_1+x_2 \in [0,1]$ implies $c(x_0+x_1+x_2) + c(x_2) \leq c(x_0+x_2) + c(x_1+x_2)$.*

Then, there is a convex interval-valued function $\iota'_c : [0,1] \to \mathscr{I}'_c$ defined by $\iota'_c(x) = [x-c(x), x+c(x)]$, where $\mathscr{I}'_c = \{\iota'_c(x) \mid x \in [0,1]\}$.

Proof.

- Since $c(x) \leq x$, $0 \leq x - c(x)$. Also, $c(x) = c(1-x) \leq 1-x$ implies $x + c(x) \leq 1$. Therefore, $\iota'_c(x) = [x-c(x), x+c(x)] \subseteq [0,1]$.
- For any $x \in [0,1]$, $\iota'_c(1-x) = [1,1] - [x-c(1-x), x+c(1-x)] = [1,1] - \iota'_c(x)$.
- Since $c(0) \in [0,1]$ and $c(0) \leq 0$, we have $c(0) = 0$. Therefore, $\iota'_c(0) = [0,0]$, and $\iota'_c(1) = [1,1] - \iota'_c(0) = [1,1]$.

- Given any three numbers x_0, x_1, and $x_2 \in [0,1]$, if $x_0 + x_1 + x_2 \in [0,1]$, we have

$$\begin{aligned} & c\left(x_0+x_1+x_2\right)+c\left(x_2\right) \leq c\left(x_0+x_2\right)+c\left(x_1+x_2\right) \\ \Rightarrow & \left[x_0+x_1+x_2+c\left(x_0+x_1+x_2\right)\right]+\left[x_2+c\left(x_2\right)\right] \\ & \leq\left[x_0+x_2+c\left(x_0+x_2\right)\right]+\left[x_1+x_2+c\left(x_1+x_2\right)\right] . \end{aligned} \tag{D.14}$$

  Also,

$$\begin{aligned} & c\left(x_0+x_1+x_2\right)+c\left(x_2\right) \leq c\left(x_0+x_2\right)+c\left(x_1+x_2\right) \\ \Rightarrow & -c\left(x_0+x_2\right)-c\left(x_1+x_2\right) \leq -c\left(x_0+x_1+x_2\right)-c\left(x_2\right) \\ \Rightarrow & \left[x_0+x_2-c\left(x_0+x_2\right)\right]+\left[x_1+x_2-c\left(x_1+x_2\right)\right] \\ & \leq\left[x_0+x_1+x_2-c\left(x_0+x_1+x_2\right)\right]+\left[x_2-c\left(x_2\right)\right] . \end{aligned} \tag{D.15}$$

  The above two equations implies $\iota_c'\left(x_0+x_1+x_2\right)+\iota_c'\left(x_2\right) \subseteq \iota_c'\left(x_0+x_2\right)+\iota_c'\left(x_1+x_2\right)$.
- For a set $\{x_i\}_{i=1}^N \subseteq [0,1]$, if $\sum_{i=1}^N x_i \in [0,1]$, then $\iota_c'\left(\sum_{i=1}^N x_i\right) \subseteq \sum_{i=1}^N \iota_c'\left(x_i\right)$ by inducting on the convex condition listed previously with $x_2 = 0$.

□

Appendix D.2. Sates Consistent with a QIVPM Induced by the Born Rule

When we work with mixed states or density matrices, we also need a norm to compute the distance among them.

Definition 48. Given a operator or matrix A, the norm of A is defined by [10]

$$\|A\| = \max_{\langle\psi|\psi\rangle=1} \left|\langle\psi|A|\psi\rangle\right| . \tag{D.16}$$

Notice that the norm in definition 48 is not one of the common matrix norms.

Example 49. Consider $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. It is easy to verify $\|A\| = 3/2$. On the other hand, its Frobenius norm or Hilbert-Schmidt norm is [82, 4]

$$\|A\|_{\mathrm{F}} = \operatorname{Tr}\left(A^\dagger A\right) = \left(\sum_{i,j} \left|a_{ij}\right|^2\right)^{1/2} = \sqrt{3}\,, \tag{D.17}$$

where a_{ij} are the (i,j)-th entry of A. If we consider the operator p-norms [78, 28, 82]

$$\|A\|_p = \sup_{\||\psi\rangle\|_p=1} \|A\,|\psi\rangle\|_p = \sup_{|\psi\rangle\neq 0} \frac{\|A\,|\psi\rangle\|_p}{\||\psi\rangle\|_p}\,, \tag{D.18}$$

we have $\|A\|_1 = \|A\|_\infty = 2$ and $\|A\|_2 = \left[\left(3+\sqrt{5}\right)/2\right]^{1/2}$, where $|\psi\rangle = \sum_{i=1}^d \alpha_i\,|i\rangle$, $\||\psi\rangle\|_\infty = \max_i |\alpha_i|$, and $\||\psi\rangle\|_p = \left(\sum_{i=1}^d |\alpha_i|^p\right)^{1/p}$ for $1 \leq p < \infty$. The common matrix norms of A are all different from $3/2$ so that our norm defined in definition 48 is different from all of them.

Lemma 50. *Given a Hilbert space with the set of all projectors $\mathcal{E}$, and an additive interval-valued function $\iota : [0,1] \to \mathscr{I}$ with the maximum length of intervals in $\mathscr{I}$ is δ. For two mixed states ρ and ρ', if $\rho' \in \overline{\mathcal{H}}\left(\iota \circ \mu^{\mathrm{B}}_{\rho}, \mathcal{E}\right)$, then $\|\rho - \rho'\| \leq \delta$.*

Proof. By definition, we have $\mu^{\mathrm{B}}_{\rho'}(P) \in \iota\left(\mu^{\mathrm{B}}_{\rho}(P)\right)$ for all $P \in \mathcal{E}$. By substitute $P = |\psi\rangle\langle\psi|$, we have

$$\mu^{\mathrm{B}}_{\rho'}(|\psi\rangle\langle\psi|) \in \iota\left(\mu^{\mathrm{B}}_{\rho}(|\psi\rangle\langle\psi|)\right) . \tag{D.19}$$

Since the maximum length of intervals in $\mathscr{I}$ is δ, we can move ρ and ρ' to the same side and rewrite equation (D.19) as

$$\delta \geq \left|\mu^{\mathrm{B}}_{\rho'}(|\psi\rangle\langle\psi|) - \mu^{\mathrm{B}}_{\rho}(|\psi\rangle\langle\psi|)\right| = |\langle\psi|\rho' - \rho|\psi\rangle| . \tag{D.20}$$

Applying maximum on both side, we have

$$\delta \geq \max_{\langle\psi|\psi\rangle=1} |\langle\psi|\rho' - \rho|\psi\rangle| = \|\rho' - \rho\| . \tag{D.21}$$

□

By applying the previous lemma on example 45, we can estimate the states consistent with a QIVPM more tightly.

Example 51. Given the additive interval-valued function $\widehat{\iota_n} : [0,1] \to \widehat{\mathscr{I}_n}$, the maximum length of intervals in $\widehat{\mathscr{I}_n}$ is $1/n$. Therefore, for any totally mixed state ρ on a d-dimensional Hilbert space, if $\rho' \in \overline{\mathcal{H}}\left(\widehat{\iota_n} \circ \mu^{\mathrm{B}}_{\rho}, \mathcal{E}\right)$, then $\|\rho - \rho'\| \leq 1/n$.

For example, when $n = 2$, two states $\rho_0 = \frac{1}{3} \cdot |0\rangle\langle 0| + \frac{1}{3} \cdot |1\rangle\langle 1| + \frac{1}{3} \cdot |2\rangle\langle 2|$ and $\rho_1 = \frac{7}{24} \cdot |0\rangle\langle 0| + \frac{1}{3} \cdot |1\rangle\langle 1| + \frac{3}{8} \cdot |2\rangle\langle 2|$ can induce two QIVPMs $\widehat{\iota_2} \circ \mu^{\mathrm{B}}_{\rho_0}$ and $\widehat{\iota_2} \circ \mu^{\mathrm{B}}_{\rho_1}$. Given any projector P in the minimal subspace of events containing $|0\rangle\langle 0|$, $|1\rangle\langle 1|$, and $|2\rangle\langle 2|$, $\mathcal{E}_C$, the interval-valued probability of the induced QIVPMs, $\widehat{\iota_2}\left(\mu^{\mathrm{B}}_{\rho_0}(P)\right)$ and $\widehat{\iota_2}\left(\mu^{\mathrm{B}}_{\rho_1}(P)\right)$, can be computed straightforwardly in Table D3. Table D3 shows that ρ_0 is consistent with $\widehat{\iota_2} \circ \mu^{\mathrm{B}}_{\rho_1}$ and ρ_1 is consistent with $\widehat{\iota_2} \circ \mu^{\mathrm{B}}_{\rho_0}$ both on every projectors in $\mathcal{E}_C$. Indeed, they are consistent with each other on every projectors in $\mathcal{E}$ that implies $\|\rho_0 - \rho_1\| \leq \frac{1}{2}$ by lemma 50. This is true because the exact value of $\|\rho_0 - \rho_1\|$ is $\frac{1}{24}$ [10].

Previously, we proved that if ρ' is consistent with $\iota \circ \mu^{\mathrm{B}}_{\rho}$, then ρ' is close to ρ. However, if ρ' is close to ρ, there is no way to guarantee ρ' is consistent with $\iota \circ \mu^{\mathrm{B}}_{\rho}$ because no matter how close between two real numbers x and y, we can never guarantee $y \in \iota(x)$ as in the following example.

Example 52. Given an additive interval-valued function $\iota : [0,1] \to \mathscr{I}$, consider let $x_0 = \sup\{x \in [0,1] \mid \iota(x) = [0,0]\}$. Since $\iota(1) = [1,1]$, $x_0 < 1$. For any $\varepsilon > 0$, $\left|\left(x + \frac{\varepsilon}{2}\right) - x\right| = \frac{\varepsilon}{2} < \varepsilon$, and $x + \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2} > 0$ implies $x + \frac{\varepsilon}{2} \notin [0,0] = \iota(x)$.

Notice that even if we explicitly exclude 0 and 1, the previous argument applies on any boundary points among intervals. Therefore, there is no way to avoid this problem if we have more than three intervals.

Table D3. Compute $\widehat{\iota}_2 \circ \mu^{\mathrm{B}}_{\rho_0}$ and $\widehat{\iota}_2 \circ \mu^{\mathrm{B}}_{\rho_1}$ based on $\mu^{\mathrm{B}}_{\rho_0}$ and $\mu^{\mathrm{B}}_{\rho_1}$ on $\mathcal{E}_C$.

P	$\mu^{\mathrm{B}}_{\rho_0}(P)$	$\widehat{\iota}_2\left(\mu^{\mathrm{B}}_{\rho_0}(P)\right)$	$\mu^{\mathrm{B}}_{\rho_1}(P)$	$\widehat{\iota}_2\left(\mu^{\mathrm{B}}_{\rho_1}(P)\right)$
$\mathbb{0}$	0	*impossible*	0	*impossible*
$\lvert 0\rangle\langle 0\rvert$	$\frac{1}{3}$	$\left[0, \frac{1}{2}\right]$	$\frac{7}{24}$	$\left[0, \frac{1}{2}\right]$
$\lvert 1\rangle\langle 1\rvert$	$\frac{1}{3}$	$\left[0, \frac{1}{2}\right]$	$\frac{1}{3}$	$\left[0, \frac{1}{2}\right]$
$\lvert 2\rangle\langle 2\rvert$	$\frac{1}{3}$	$\left[0, \frac{1}{2}\right]$	$\frac{3}{8}$	$\left[\frac{1}{4}, \frac{3}{4}\right]$
$\mathbb{1} - \lvert 2\rangle\langle 2\rvert$	$\frac{2}{3}$	$\left[\frac{1}{2}, 1\right]$	$\frac{5}{8}$	$\left[\frac{1}{4}, \frac{3}{4}\right]$
$\mathbb{1} - \lvert 1\rangle\langle 1\rvert$	$\frac{2}{3}$	$\left[\frac{1}{2}, 1\right]$	$\frac{2}{3}$	$\left[\frac{1}{2}, 1\right]$
$\mathbb{1} - \lvert 0\rangle\langle 0\rvert$	$\frac{2}{3}$	$\left[\frac{1}{2}, 1\right]$	$\frac{17}{24}$	$\left[\frac{1}{2}, 1\right]$
$\mathbb{1}$	1	*certain*	1	*certain*

Appendix D.3. Category Theory

It looks like ι might be a morphism in a category. Let's try to check if it is true here.

Definition 53. Given $\mathscr{I}$ is a collection of intervals, we say a interval-valued function $\iota : \mathscr{I}' \to \mathscr{I}''$ is additive if ι satisfies the following properties.

- $\iota\left(\textit{impossible}\right) = \textit{impossible}$.
- $\iota\left(\textit{certain}\right) = \textit{certain}$.
- For any $[\ell, r] \in \mathscr{I}'$, $\iota\left(\textit{certain} - [\ell, r]\right) = \textit{certain} - \iota\left([\ell, r]\right)$.
- For a pair $[\ell_0, r_0]$ and $[\ell_1, r_1] \in \mathscr{I}'$, if $[\ell_0, r_0] \subseteq [\ell_1, r_1]$, then $\iota\left([\ell_0, r_0]\right) \subseteq \iota\left([\ell_1, r_1]\right)$.
- For a pair $[\ell_0, r_0]$ and $[\ell_1, r_1] \in \mathscr{I}'$, if $[\ell_0, r_0] + [\ell_1, r_1] \cap [0, 1] \neq \emptyset$, then $[\ell_0, r_0] + [\ell_1, r_1] \in \mathscr{I}'$ and $\iota\left([\ell_0, r_0] + [\ell_1, r_1]\right) \subseteq \iota\left([\ell_0, r_0]\right) + \iota\left([\ell_1, r_1]\right)$.

Moreover, ι is called convex if for any three intervals $[\ell_0, r_0]$, $[\ell_1, r_1]$, and $[\ell_2, r_2] \in \mathscr{I}'$, their sum $[\ell_0, r_0] + [\ell_1, r_1] + [\ell_2, r_2] \in \mathscr{I}'$ implies $\iota\left([\ell_0, r_0] + [\ell_1, r_1] + [\ell_2, r_2]\right) + \iota\left([\ell_2, r_2]\right) \subseteq \iota\left([\ell_0, r_0] + [\ell_2, r_2]\right) + \iota\left([\ell_1, r_1] + [\ell_2, r_2]\right)$.

Lemma 54. *Consider an additive interval-valued function $\iota : \mathscr{I}' \to \mathscr{I}''$. Then, given a QIVPM $\bar{\mu}' : \mathcal{E} \to \mathscr{I}'$, a function $\bar{\mu}'' : \mathcal{E} \to \mathscr{I}''$ defined by $\bar{\mu}''(P) = \iota\left(\bar{\mu}'(P)\right)$ is a QIVPM.*

Proof.

- $\bar{\mu}''(\mathbb{0}) = \iota\left(\bar{\mu}'(\mathbb{0})\right) = \iota\left(\textit{impossible}\right) = \textit{impossible}$.

- $\bar{\mu}''(\mathbb{1}) = \iota(\bar{\mu}'(\mathbb{1})) = \iota(\mathit{certain}) = \mathit{certain}$.
- For any projection P, $\bar{\mu}''(\mathbb{1} - P) = \iota(\mathit{certain} - \bar{\mu}'(P)) = \mathit{certain} - \bar{\mu}''(P)$.
- For a pair of mutually orthogonal projections P_0 and $P_1 \in \mathcal{E}$, since $\emptyset \neq \bar{\mu}'(P_0 + P_1) \cap [0,1] \subseteq \bar{\mu}'(P_0) + \bar{\mu}'(P_1) \cap [0,1]$, we have $\bar{\mu}'(P_0) + \bar{\mu}'(P_1) \in \mathscr{I}'$ and

$$\bar{\mu}''(P_0 + P_1) = \iota(\bar{\mu}'(P_0 + P_1)) \subseteq \iota(\bar{\mu}'(P_0) + \bar{\mu}'(P_1)) \subseteq \iota(\bar{\mu}'(P_0)) + \iota(\bar{\mu}'(P_1)) = \bar{\mu}''(P_0)$$

□

Appendix E. From Interval-valued to Real-valued

Previously we discussed how to construct a quantum IVPM from a quantum real-valued probability measure; we then switch our focus to how to construct a quantum real-valued probability measure from a special kind of quantum IVPMs. Since the only properties we use in this section is whether its interval-valued or real-valued, the results in this section applies on every subspace of events $\mathcal{E}'$. In particular, when $\mathcal{E}'$ is commuting, any quantum results will be classical as shown in theorem 32.

Definition 55. An interval-valued probability measure $\bar{\mu} : \mathcal{E}' \to \mathscr{I}$ is called $(1/n, \delta)$-deterministic if for every $[l, r] \in \mathscr{I}$, there is a $k \in \mathbb{Z}$ such that $[l, r] \subseteq [k/n - \delta, k/n + \delta]$.

Recall $\mathscr{I}_{\mathrm{D}} = \{\mathit{impossible}, \mathit{certain}\}$ so that a $\mathscr{I}_{\mathrm{D}}$-valued QIVPM is $(1, 0)$-deterministic. For another example, given a three-dimensional Hilbert space,

$$\iota_\infty \circ \mu^{\mathrm{B}}_{\frac{\mathbb{1}}{3}} : \mathcal{E}' \to \left\{\mathit{impossible}, \left[\frac{1}{3}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{2}{3}\right], \mathit{certain}\right\} \tag{E.1}$$

is $\left(1, \frac{1}{3}\right)$-deterministic and $\left(\frac{1}{3}, 0\right)$-deterministic. However, we will prove there is generally no $(1/n, \delta)$-deterministic probability measures when δ is much smaller than $1/n$.

Lemma 56. *Given a Hilbert space of dimension d, and a quantum $(1/n, \delta)$-deterministic probability measure $\bar{\mu} : \mathcal{E}' \to \mathscr{I}$ with $\delta < 1/(3n)$, if $\bar{\mu}(P) \subseteq [k/n - \delta, k/n + \delta]$, let $\mu(P) = k/n$, then $\mu : \mathcal{E}' \to [0, 1]$ is a quantum probability measure. In another word, $\iota_\infty \circ \mu$ is $(1/n, 0)$-deterministic by lemma 41.*

Proof.

- Since $\bar{\mu}(\mathbb{0}) = \mathit{impossible}$, we have $\mu(\mathbb{0}) = 0$.
- Since $\bar{\mu}(\mathbb{1}) = \mathit{certain}$, we have $\mu(\mathbb{1}) = 1$.
- For every event P, we have

$$\begin{aligned} \mu(P) = \frac{k}{n} &\Rightarrow \bar{\mu}(P) \subseteq \left[\frac{k}{n} - \delta, \frac{k}{n} + \delta\right] \\ &\Rightarrow \bar{\mu}(\mathbb{1} - P) = [1, 1] - \bar{\mu}(P) \subseteq \left[1 - \frac{k}{n} - \delta, 1 - \frac{k}{n} + \delta\right] \\ &\Rightarrow \mu(\mathbb{1} - P) = 1 - \frac{k}{n} = 1 - \mu(P) \ . \end{aligned} \tag{E.2}$$

- Given a pair of orthogonal operators P_0 and P_1, let $\mu(P_0) = k_0/n$ and $\mu(P_1) = k_1/n$. Then,

$$\bar{\mu}(P_0 + P_1) \subseteq \bar{\mu}(P_0) + \bar{\mu}(P_1) \subseteq \left[\frac{k_0 + k_1}{n} - 2\delta, \frac{k_0 + k_1}{n} + 2\delta\right]. \tag{E.3}$$

Since $\delta < 1/(3n)$, we have

$$\frac{k_0 + k_1 - 1}{n} + \delta < \frac{k_0 + k_1}{n} - 2\delta \leq \frac{k_0 + k_1}{n} + 2\delta < \frac{k_0 + k_1 + 1}{n} - \delta. \tag{E.4}$$

The previous inequality implies that if $\bar{\mu}(P_0 + P_1)$ overlap with any interval of the form $[k/n - \delta, k/n + \delta]$ for some k, k must be $k_0 + k_1$, i.e.,

$$\bar{\mu}(P_0 + P_1) \subseteq \left[\frac{k_0 + k_1}{n} - \delta, \frac{k_0 + k_1}{n} + \delta\right]. \tag{E.5}$$

Therefore,

$$\mu(P_0 + P_1) = \frac{k_0 + k_1}{n} = \mu(P_0) + \mu(P_1). \tag{E.6}$$

By induction, we have $\mu\left(\bigcup_{i=1}^{N} P_i\right) = \sum_{i=1}^{N} \mu(P_i)$ for any mutually orthogonal events $\{P_i\}_{i=1}^{N}$.

□

Together with the Kochen-Specker theorem, we will have the following corollary.

Corollary 57. *Given a Hilbert space of dimension $d \geq 3$, there is no quantum $(1, \delta)$-deterministic probability measure for $\delta < \frac{1}{3}$.*

In general [9], Gleason's theorem asserts that a quantum probability measure must be continuous. Therefore, a $(1/n, 0)$-deterministic QIVPM exists if and only if $d \mid n$, and so does a $(1/n, \delta)$-deterministic QIVPM for $\delta < 1/(3n)$.

Lemma 56 may be extended in another trivial way.

Lemma 58. *If every interval in $\mathscr{I}$ is a singleton set, given a QIVPM $\bar{\mu} : \mathcal{E}' \to \mathscr{I}$, then $\mu : \mathcal{E}' \to [0, 1]$ defined by $\bar{\mu}(P) = \{\mu(P)\}$ is a quantum probability measure.*

Proof.

- Since $\bar{\mu}(\mathbb{0}) = \mathscr{impossible}$, we have $\mu(\mathbb{0}) = 0$.
- Since $\bar{\mu}(\mathbb{1}) = \mathscr{certain}$, we have $\mu(\mathbb{1}) = 1$.
- For every event P, we have

$$\begin{aligned} &\bar{\mu}(\mathbb{1} - P) = [1, 1] - \bar{\mu}(P) \\ \Rightarrow\ &\{\mu(\mathbb{1} - P)\} = [1, 1] - \{\mu(P)\} = \{1 - \mu(P)\} \\ \Rightarrow\ &\mu(\mathbb{1} - P) = 1 - \mu(P). \end{aligned}$$

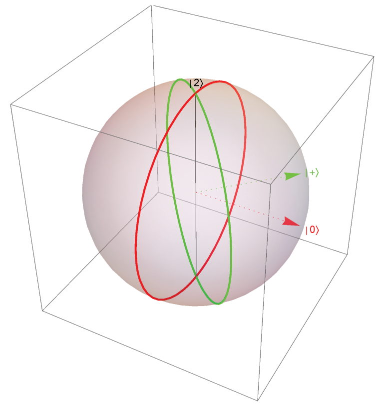

Figure F1. This figure illustrates example 60 plotted in $\mathbb{R}^3$. The red and green dotted vectors are $|0\rangle$ and $|+\rangle$ respectively. All possible real vectors of the subspaces orthogonal to $|0\rangle$ or $|+\rangle$ are drawn in the red and green circles, respectively.

- Given a pair of orthogonal operators P_0 and P_1, we have

$$\begin{aligned}
&\bar{\mu}\left(P_0+P_1\right) \subseteq \bar{\mu}\left(P_0\right)+\bar{\mu}\left(P_1\right) \\
\Rightarrow\ &\left\{\mu\left(P_0+P_1\right)\right\} \subseteq\left\{\mu\left(P_0\right)\right\}+\left\{\mu\left(P_1\right)\right\}=\left\{\mu\left(P_0\right)+\mu\left(P_1\right)\right\} \\
\Rightarrow\ &\mu\left(P_0+P_1\right)=\mu\left(P_0\right)+\mu\left(P_1\right) .
\end{aligned}$$

□

Together with example 44, we know QIVPMs with $\mathscr{I}_\infty$ is one-to-one correspondence to quantum real-valued probability measures. Moreover, since $\iota_\infty : [0,1] \to \mathscr{I}_\infty$ is convex, we have the following corollary.

Corollary 59. *If every interval in $\mathscr{I}$ is a singleton set, then every QIVPM $\bar{\mu} : \mathcal{E}' \to \mathscr{I}$ is convex.*

Appendix F. Examples for Quantum Interval-valued Probability Measures

Example 60 (Three-dimensional quantum 5-interval-valued probability measure)**.** Given a three dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Let $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and $\mathscr{I} = \left\{\mathit{impossible}, \mathit{unlikely}, \mathit{likely}, \mathit{certain}, \mathit{middle}\right\}$, where middle represents $\left[\frac{1}{4}, \frac{3}{4}\right]$. The definition of $\bar{\mu}$ below refers to figure F1 which plots the 1-dimensional projectors:

(i) Let:

$$\begin{aligned}
\bar{\mu}(\mathbb{0}) &= \bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) = \mathit{impossible}, \\
\bar{\mu}(\mathbb{1}) &= \bar{\mu}(\mathbb{1} - |0\rangle\langle 0|) = \bar{\mu}(\mathbb{1} - |+\rangle\langle +|) = \mathit{certain},
\end{aligned} \tag{F.1}$$

where $|0\rangle$ and $|+\rangle$ are plotted as the red and green dotted vectors, respectively.

(ii) The red and green circles are the states orthogonal to $|0\rangle$ and $|+\rangle$, respectively. We define $\bar{\mu}\left(|\psi\rangle\langle\psi|\right) = \bar{\mu}\left(\mathbb{1} - |\psi\rangle\langle\psi|\right) = \mathit{middle}$, where $\langle\psi|0\rangle = 0$ or $\langle\psi|+\rangle = 0$.

(iii) Otherwise, $\bar{\mu}(|\psi\rangle\langle\psi|) = \mathit{unlikely}$ and $\bar{\mu}(\mathbb{1} - |\psi\rangle\langle\psi|) = \mathit{likely}$.

By theorem 20, to check $\bar{\mu}$ a quantum interval-valued probability measure is equivalent to check $\bar{f}$ satisfying

$$\begin{aligned}
[1,1] - \bar{f}\left(|\psi_0\rangle\right) &\subseteq \bar{f}\left(|\psi_1\rangle\right) + \bar{f}\left(|\psi_2\rangle\right) , \\
[1,1] - \bar{f}\left(|\psi_1\rangle\right) &\subseteq \bar{f}\left(|\psi_2\rangle\right) + \bar{f}\left(|\psi_0\rangle\right) , \\
[1,1] - \bar{f}\left(|\psi_2\rangle\right) &\subseteq \bar{f}\left(|\psi_0\rangle\right) + \bar{f}\left(|\psi_1\rangle\right)
\end{aligned} \tag{F.2}$$

for every orthonormal basis $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$, where $\bar{f}\left(|\psi\rangle\right) = \bar{\mu}\left(|\psi\rangle\langle\psi|\right)$. We are going to enumerate all possible orthonormal bases to verify equation (F.2). Also, to simplify the notation, we denote $\{|\psi\rangle \mid \langle\psi|0\rangle \neq 0 \text{ and } \langle\psi|+\rangle \neq 0\}$ by $\mathcal{T}$.

(i) When $|\psi_0\rangle$ is $|0\rangle$, then $\langle\psi_1|0\rangle = \langle\psi_2|0\rangle = 0$. Equation (F.2) can then be verified as follow.

$$\begin{aligned}
[1,1] - \bar{f}\left(|0\rangle\right) = \mathit{certain} \subseteq \mathit{middle} + \mathit{middle} &= \bar{f}\left(|\psi_1\rangle\right) + \bar{f}\left(|\psi_2\rangle\right) , \\
[1,1] - \bar{f}\left(|\psi_1\rangle\right) = \mathit{middle} = \mathit{middle} + \mathit{impossible} &= \bar{f}\left(|\psi_2\rangle\right) + \bar{f}\left(|0\rangle\right) ,
\end{aligned} \tag{F.3}$$

Similarly, when $|\psi_0\rangle$ is $|+\rangle$, equation (F.2) holds.

(ii) When $\langle\psi_0|0\rangle = \langle\psi_1|+\rangle = 0$ and $|\psi_2\rangle \in \mathcal{T}$, we have

$$\begin{aligned}
[1,1] - \bar{f}\left(|\psi_0\rangle\right) = \mathit{middle} \subseteq \mathit{middle} + \mathit{unlikely} &= \bar{f}\left(|\psi_1\rangle\right) + \bar{f}\left(|\psi_2\rangle\right) , \\
[1,1] - \bar{f}\left(|\psi_2\rangle\right) = \mathit{likely} \subseteq \mathit{middle} + \mathit{middle} &= \bar{f}\left(|\psi_0\rangle\right) + \bar{f}\left(|\psi_1\rangle\right) .
\end{aligned} \tag{F.4}$$

(iii) When $\langle\psi_0|0\rangle = 0$, and $|\psi_1\rangle$, $|\psi_2\rangle \in \mathcal{T}$, we have

$$\begin{aligned}
[1,1] - \bar{f}\left(|\psi_0\rangle\right) = \mathit{middle} \subseteq \mathit{unlikely} + \mathit{unlikely} &= \bar{f}\left(|\psi_1\rangle\right) + \bar{f}\left(|\psi_2\rangle\right) , \\
[1,1] - \bar{f}\left(|\psi_1\rangle\right) = \mathit{likely} \subseteq \mathit{unlikely} + \mathit{middle} &= \bar{f}\left(|\psi_2\rangle\right) + \bar{f}\left(|\psi_0\rangle\right) .
\end{aligned} \tag{F.5}$$

Similarly, when $\langle\psi_0|+\rangle = 0$, $|\psi_1\rangle \in \mathcal{T}$, and $|\psi_2\rangle \in \mathcal{T}$, equation (F.2) holds.

(iv) When $|\psi_0\rangle$, $|\psi_1\rangle$, and $|\psi_2\rangle \in \mathcal{T}$, i.e., the "otherwise" case. Then, equation (F.2) can easily be verified.

$$[1,1] - \bar{f}\left(|\psi_0\rangle\right) = \mathit{likely} \subseteq \mathit{unlikely} + \mathit{unlikely} = \bar{f}\left(|\psi_1\rangle\right) + \bar{f}\left(|\psi_2\rangle\right) . \tag{F.6}$$

Since we proved that $\bar{\mu}$ is a quantum interval-valued probability measure, $\bar{\mu}$ is convex by theorem 39.

The following argument establishes that $\bar{\mu}$ has an empty core. Assume there is a real-valued probability measure satisfying $\mu_\rho^{\mathrm{B}}(P) \in \bar{\mu}(P)$ for all $P \in \mathcal{E}$. Because $\mu_\rho^{\mathrm{B}}(|0\rangle\langle 0|) \in \bar{\mu}(|0\rangle\langle 0|) = \mathit{impossible}$ and $\mu_\rho^{\mathrm{B}}(|+\rangle\langle +|) \in \bar{\mu}(|+\rangle\langle +|) = \mathit{impossible}$, we must have $\mu_\rho^{\mathrm{B}}(|0\rangle\langle 0|) = \mu_\rho^{\mathrm{B}}(|+\rangle\langle +|) = 0$ so that $\mu_\rho^{\mathrm{B}} = \mu_{|2\rangle}^{\mathrm{B}}$. However,

$$\mu_{|2\rangle}^{\mathrm{B}}(|2\rangle\langle 2|) = 1 \notin \mathit{unlikely} = \bar{\mu}(|2\rangle\langle 2|) . \tag{F.7}$$

Appendix G. Expectation Values for IVPMs

Appendix G.1. Expectation Values for Classical IVPMs

We may need to understand how to define expectation values of an observable for quantum IVPMs. Recall an observable is the quantum version of a random variable. Hence, in order to define expectation values of an observable, we need to understand the definition of expectation values of a random variable for an classical IVPM. Since the expectation value of random variable X is the integral $\int X \mathrm{d}\mu$ [38], we need to understand the meaning of integration on IVPMs. In the rest of the section, we will consider the real-valued and interval-valued probability measure defined in example 2 and 3, and see how to compute the expectation value of different random variables. To simplify the discussion later, we will only consider a random variable $X : \Omega \to \mathbb{R}$ which is a step function, i.e., $X = \sum_{i=1}^{N} x_i \mathbf{1}_{E_i}$, where $E_i \in \mathcal{E}$ and $\mathbf{1}_E$ is the indicator function defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \text{ ;} \\ 0 & \text{if } x \notin A. \end{cases} \tag{G.1}$$

Given a random variable X and a probability measure μ, the expectation value of X is

$$\int X \mathrm{d}\mu = \sum_{i=1}^{N} x_i \mu\left(E_i\right) \ . \tag{G.2}$$

Intuitively, the expectation value is weighted average of the possible outcomes of X with the weight equal to the probability of each outcome could happen. For example, consider the random variable X_0 representing the number of tails. For the probability measure μ in example 2, the expectation value of X_0 is

$$\begin{aligned} \int X_0 \mathrm{d}\mu &= 0 \cdot \mu(\{HH\}) + 1 \cdot \mu(\{HT, TH\}) + 2 \cdot \mu(\{TT\}) \\ &= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} + 2 \cdot 0 = \frac{2}{3} \,. \end{aligned} \tag{G.3}$$

There is a subtlety happens when we want to define the expectation value of a random variable on IVPMs, and will be demonstrated by the following example.

Example 61. Consider an CIVPM $\bar{\mu}$ and an event E such that $\bar{\mu}(E) = [0.2, 0.3]$ and $\bar{\mu}(\overline{E}) = [0.7, 0.8]$. Given a random variable

$$X\left(\omega\right) = \begin{cases} 1 & \text{, if } \omega \in E \text{ ;} \\ 2 & \text{, if } \omega \notin E \text{ ,} \end{cases} \tag{G.4}$$

its expectation value with respect to $\bar{\mu}$ might be naïvely defined as

$$\begin{aligned} & 1 \cdot \bar{\mu}(E) + 2 \cdot \bar{\mu}(\overline{E}) \\ =& 1 \cdot [0.2, 0.3] + 2 \cdot [0.7, 0.8] \\ =& [1 \cdot 0.2 + 2 \cdot 0.7, 1 \cdot 0.3 + 2 \cdot 0.8] \\ =& [1.6, 1.9] \,. \end{aligned} \tag{G.5}$$

As we discussed when introducing the core, the true probability of $\bar{\mu}(E)$, $\mu(E)$, can be anywhere in the range $[0.2, 0.3]$ and the true probability of $\bar{\mu}(\overline{E})$ is $\mu(\overline{E}) = 1 - \mu(E)$, that is, $\mu \in \operatorname{core}(\bar{\mu})$. If $\mu(E) = 0.2$, the expectation value of X with respect to μ is

$$1 \cdot \mu(E) + 2 \cdot \mu(\overline{E}) = 1 \cdot 0.2 + 2 \cdot 0.8 = 1.8\,; \tag{G.6}$$

if $\mu(E) = 0.3$, the expectation value of X with respect to μ is

$$1 \cdot \mu(E) + 2 \cdot \mu(\overline{E}) = 1 \cdot 0.3 + 2 \cdot 0.7 = 1.7\,. \tag{G.7}$$

In another word, no matter how we pick $\mu(E)$, the expectation value of X is in $[1.7, 1.8]$ which is a more realistic expectation value of X with respect to $\bar{\mu}$. The interval $[1.7, 1.8]$ is narrower than the naïve expectation value $[1.6, 1.9]$ because the naïve one combines contradicting situations in the computation.

In general, the computation of the realistic expectation value, called the Choquet integral, is more complex than what we discussed because a random variable may have more than two values. Despite its complexity, it is still a weight average among the values of a random variable as given in the following definition.

Definition 62 (Choquet integral [83, 84, 85, 26]).

- Consider a classical IVPM $\bar{\mu}$ such that $\mu^{\mathrm{L}} : \mathcal{E} \to [0,1]$ and $\mu^{\mathrm{R}} : \mathcal{E} \to [0,1]$ are the left-end and the right-end of $\bar{\mu}$, respectively, i.e., $\bar{\mu}(E) = \left[\mu^{\mathrm{L}}(E), \mu^{\mathrm{R}}(E)\right]$.
- We still decompose a random variable X into step functions, but we separate the positive and negative terms and write

  $$X = \sum_{i=1}^{N^-} x_i^- \mathbf{1}_{E_i^-} + \sum_{i=1}^{N^+} x_i^+ \mathbf{1}_{E_i^+}\,, \tag{G.8}$$

  where

  $$x_{N^-}^- < \ldots < x_1^- < 0 \le x_1^+ < \ldots < x_{N^+}^+ \tag{G.9}$$

  and $\left\{E_i^-\right\}_{i=1}^{N^-}$ and $\left\{E_i^+\right\}_{i=1}^{N^+}$ are all disjoint.
- Let $E_i^{s\prime} = \bigcup_{j \ge i} E_j^s$ and $\Delta\mu^*\left(E_i^{s\prime}\right) = \mu^*\left(E_i^{s\prime}\right) - \mu^*\left(E_{i+1}^{s\prime}\right)$, where $s \in \{+,-\}$ and $\mu^* \in \left\{\mu^{\mathrm{L}}, \mu^{\mathrm{R}}\right\}$.

Then, the Choquet integral of X with respect to $\bar{\mu}$ is

$$\begin{aligned}\int X \mathrm{d}\bar{\mu} = \Bigg[& \sum_{i=1}^{N^+} x_i^+ \Delta\mu^{\mathrm{L}}\left(E_i^{+\prime}\right) + \sum_{i=1}^{N^-} x_i^- \Delta\mu^{\mathrm{R}}\left(E_i^{-\prime}\right), \\ & \sum_{i=1}^{N^+} x_i^+ \Delta\mu^{\mathrm{R}}\left(E_i^{+\prime}\right) + \sum_{i=1}^{N^-} x_i^- \Delta\mu^{\mathrm{L}}\left(E_i^{-\prime}\right) \Bigg]\,.\end{aligned} \tag{G.10}$$

Table G1.

$$\int X \mathrm{d}\bar{\mu} = [1 \cdot 0.3 + 2 \cdot 0.7, 1 \cdot 0.2 + 2 \cdot 0.8] = [1.7, 1.8] \ .$$

i	x_i^+	E_i^+	$E_i^{+\prime}$	$\mu^{\mathrm{L}}(E_i^{+\prime})$	$\Delta\mu^{\mathrm{L}}(E_i^{+\prime})$	$\mu^{\mathrm{R}}(E_i^{+\prime})$	$\Delta\mu^{\mathrm{R}}(E_i^{+\prime})$
1	1	E	Ω	1	0.3	1	0.2
2	2	$\overline{E}$	$\overline{E}$	0.7	0.7	0.8	0.8
3			$\emptyset$	0		0	

Table G1 follows the definition to compute $\int X \mathrm{d}\bar{\mu}$. Notice that equation (G.10) looks complex. On the one hand, it would be a little bit easier to think its right-hand side as

$$\sum_{s\in\{+,-\}} \sum_{i=1}^{N^s} x_i^s \left[\Delta\mu^{\mathrm{L}}(E_i^{s\prime}), \Delta\mu^{\mathrm{R}}(E_i^{s\prime})\right] \ , \tag{G.11}$$

$\Delta\mu^{\mathrm{L}}(E_i^{s\prime})$ can be larger than $\Delta\mu^{\mathrm{R}}(E_i^{s\prime})$ as in the second row in table G1. Therefore, if we want to use this notation, we need to stick to this convention that not reverse the two ends of intervals, even if the left-end is larger than the right-end. On the other hand, similar to equation (G.2), equation (G.11) can be considered as weighted average of the possible outcomes of X as well, despite the "weight"

$$\left[\Delta\mu^{\mathrm{L}}(E_i^{s\prime}), \Delta\mu^{\mathrm{R}}(E_i^{s\prime})\right] \subseteq \left[\mu^{\mathrm{L}}(E_i^{s}), \mu^{\mathrm{R}}(E_i^{s})\right] = \bar{\mu}(E_i^s) \text{ for } s \in \{+,-\} \tag{G.12}$$

is tighter than the interval-valued probability of the corresponding event. This makes it easier to remember.

Also, $[1.7, 1.8]$ in example 61 is exactly the same as the expectation value computed in table G1. This nature property can be generated to the following theorem.

Theorem 63. *[86, 85, 26]For every classical IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ and any random variable $X : \Omega \to \mathbb{R}$, $\bar{\mu}$ is convex if and only if*

$$\int X \mathrm{d}\bar{\mu} = \left[\min_{\mu\in\mathrm{core}(\bar{\mu})} \int X \mathrm{d}\mu, \max_{\mu\in\mathrm{core}(\bar{\mu})} \int X \mathrm{d}\mu\right] \ . \tag{G.13}$$

Yu-Tsung says: Example for when $\bar{\mu}$ is not convex, Eq. (G.13) doesn't hold. This can justify why we need convex...

The next lemmas shows our definition is exactly the same as in the literature, and is consistent with the CRVPM.

Lemma 64. *Follow the same notation in the definition, we also define $X^+ = \sum_{i=1}^{N^+} x_i^+ \mathbf{1}_{E_i^+}$ and $X^- = \sum_{i=1}^{N^-} x_i^- \mathbf{1}_{E_i^-}$. Then, $\int X \mathrm{d}\bar{\mu} = \int X^+ \mathrm{d}\bar{\mu} - \int (-X^-) \mathrm{d}\bar{\mu}$.*

Proof.

$$
\begin{aligned}
\int X \mathrm{d}\bar{\mu} &= \sum_{i=1}^{N^+} x_i^+ \left[\Delta\mu^{\mathrm{L}}\left(E_i^{+\prime}\right), \Delta\mu^{\mathrm{R}}\left(E_i^{+\prime}\right)\right] + \sum_{i=1}^{N^-} x_i^- \left[\Delta\mu^{\mathrm{L}}\left(E_i^{-\prime}\right), \Delta\mu^{\mathrm{R}}\left(E_i^{-\prime}\right)\right] \\
&= \int X^+ \mathrm{d}\bar{\mu} + \left[\sum_{i=1}^{N^-} x_i^- \Delta\mu^{\mathrm{L}}\left(E_i^{-\prime}\right), \sum_{i=1}^{N^-} x_i^- \Delta\mu^{\mathrm{R}}\left(E_i^{-\prime}\right)\right] \\
&= \int X^+ \mathrm{d}\bar{\mu} - \left[-\sum_{i=1}^{N^-} x_i^- \Delta\mu^{\mathrm{R}}\left(E_i^{-\prime}\right), -\sum_{i=1}^{N^-} x_i^- \Delta\mu^{\mathrm{L}}\left(E_i^{-\prime}\right)\right] \\
&= \int X^+ \mathrm{d}\bar{\mu} - \left[\sum_{i=1}^{N^-} \left(-x_i^-\right) \Delta\mu^{\mathrm{R}}\left(E_i^{-\prime}\right), \sum_{i=1}^{N^-} \left(-x_i^-\right) \Delta\mu^{\mathrm{L}}\left(E_i^{-\prime}\right)\right] \\
&= \int X^+ \mathrm{d}\bar{\mu} - \sum_{i=1}^{N^-} \left(-x_i^-\right) \left[\Delta\mu^{\mathrm{R}}\left(E_i^{-\prime}\right), \Delta\mu^{\mathrm{L}}\left(E_i^{-\prime}\right)\right] \\
&= \int X^+ \mathrm{d}\bar{\mu} - \int \left(-X^-\right) \mathrm{d}\bar{\mu}\,.
\end{aligned}
$$

□

Lemma 65. *[26]When $\mu^{\mathrm{L}} : \mathcal{E} \to [0,1]$ and $\mu^{\mathrm{R}} : \mathcal{E} \to [0,1]$ are real-valued probability measures, we have $\mu^{\mathrm{L}} = \mu^{\mathrm{R}}$ and $\int X \mathrm{d}\bar{\mu} = \left\{\int X \mathrm{d}\mu^{\mathrm{L}}\right\} = \left\{\int X \mathrm{d}\mu^{\mathrm{R}}\right\}$.*

Appendix G.2. Expectation Values for δ-deterministic CIVPMs

Finally, we are going to focus on δ-deterministic CIVPMs.

Lemma 66. *Consider a sample space Ω with d elements, and $\delta < 1/d$. Given a δ-deterministic CIVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$, the following statements are true:*

(i) If $\bar{\mu}$ is 0-deterministic, its left-end μ^{L} and right-end μ^{R} are the same CRVPM, and will be simply denoted by $\mu : \mathcal{E} \to \{1, 0\}$.

(ii) There is a unique element $\omega_0 \in \Omega$ such that $\bar{\mu}(\{\omega_0\}) \subseteq [1-\delta, 1]$, denoted by $\bar{\mu}^{-1}([1-\delta,1])$. In particular, if $\bar{\mu}$ is 0-deterministic, $\bar{\mu}(\{\omega_0\}) = \{\mu(\{\omega_0\})\} = \mathscr{certain}$, ω_0 could be denoted by $\mu^{-1}(1)$ or $\bar{\mu}^{-1}(\mathscr{certain})$.

Proof.

(i) The direct consequence of lemma 56.

(ii) We have either $\bar{\mu}(\{\omega\}) \subseteq [0,\delta]$ or $\bar{\mu}(\{\omega\}) \subseteq [1-\delta, 1]$.

Case 1. If $\bar{\mu}(\{\omega\}) \subseteq [0,\delta]$ for all $\omega \in \Omega$, then $\mathscr{certain} = \bar{\mu}(\Omega) \subseteq \sum_{\omega\in\Omega} \bar{\mu}(\{\omega\}) \subseteq [0, d\delta] \Rightarrow d\delta \geq 1$. A contradiction!

Case 2. If there are more than one $\omega \in \Omega$ such that $\bar{\mu}(\{\omega\}) \subseteq [1-\delta, 1]$, then $\mathscr{certain} = \bar{\mu}(\Omega) \subseteq \sum_{\omega\in\Omega} \bar{\mu}(\{\omega\}) \subseteq [2-2\delta, d] \Rightarrow 2\delta \geq 1$. A contradiction!

Therefore, there is a unique element $\omega_0 \in \Omega$ such that $\bar{\mu}(\{\omega_0\}) \subseteq [1-\delta, 1]$.

□

Lemma 67. *For any CIVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}_{\mathrm{D}}$ and a random variable $X = \sum_{i=1}^{N} x_i \mathbf{1}_{E_i}$, then $\int X \mathrm{d}\bar{\mu} = \{X(\bar{\mu}^{-1}(\mathscr{certain}))\}$.*

Proof. By adopting the notation in lemma 66, we have

$$\int X \mathrm{d}\bar{\mu} \overset{\text{Lemma 65}}{=} \left\{\int X \mathrm{d}\mu\right\}$$
$$\overset{\text{Equation } (G.2)}{=} \left\{X\left(\mu^{-1}(1)\right) \cdot 1\right\} = \left\{X\left(\bar{\mu}^{-1}(\mathscr{certain})\right)\right\} .$$

□

Given two singleton sets $\{x_1\}$ and $\{x_2\}$, it is intuitive to define their multiplication of to be $\{x_1 x_2\}$ which gives the following lemma.

Lemma 68. *For any CIVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}_{\mathrm{D}}$ and a sequence of random variable $\{X_j\}$,*

$$\prod_j \int X_j \mathrm{d}\bar{\mu} = \int \prod_j X_j \mathrm{d}\bar{\mu} . \tag{G.14}$$

Proof.

$$\prod_j \int X_j \mathrm{d}\bar{\mu} \overset{\text{Lemma 67}}{=} \prod_j \left\{X_j\left(\bar{\mu}^{-1}(\mathscr{certain})\right)\right\}$$
$$= \left\{\prod_j X_j\left(\bar{\mu}^{-1}(\mathscr{certain})\right)\right\} \overset{\text{Lemma 67}}{=} \int \prod_j X_j \mathrm{d}\bar{\mu} .$$

□

Appendix G.3. Expectation Values for QIVPMs

Before discussing QIVPMs, we first review various notations brought forward from the conventional quantum theory. Assume that we wish to measure a physical quantity $\mathcal{O}$ of this system, with corresponding observable $\mathbf{O}$. Its spectral decomposition can either be represented in term of outer product or projector. Since we want to define the expectation value of $\mathbf{O}$ with respect to a QIVPM as the classical Choquet integral in definition 62, we are going to order the eigenvalues of $\mathbf{O}$ as we ordered the step functions for classical random variables, and to represent the spectral decomposition of $\mathbf{O}$ by [87, 75, 10, 88, 4]

$$\mathbf{O} = \sum_{s\in\{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s P_i^s , \tag{G.15}$$

Table G2. Compare the similarity between the classical and quantum

Classical	Quantum
CIVPM $\bar{\mu}$	QIVPM $\bar{\mu}$
Random variable X	Observable $\mathbf{O}$
Classical step function decomposition, equation (G.8)	Spectral decomposition, equation (G.15)
Distinct value x_i^s	Distinct eigenvalue λ_i^s
Non-overlapping sets E_i^s	Orthogonal projector E_i^s

where the positive and negative eigenvalues are ordered from the smallest to the largest

$$\lambda_{N^-}^- < \ldots < \lambda_1^- < 0 \leq \lambda_1^+ < \ldots < \lambda_{N^+}^+ , \tag{G.16}$$

and P_i^s is the projector onto the eigenspace of each distinct eigenvalue λ_i^s.

For the conventional quantum theory, a physical system can be represented by a density matrix ρ, the expectation value of the observable $\mathbf{O}$ is defined as [10, 4]

$$\langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}} = \mathrm{Tr}\left(\rho \mathbf{O}\right) = \sum_{s \in \{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \mu_\rho^{\mathrm{B}}\left(P_i^s\right) \ . \tag{G.17}$$

Equation (G.17) can be combined with definition 62 to define the expectation value of $\mathbf{O}$ with respect to a QIVPM $\bar{\mu}$. Similar to definition 62, we first separate the left-end and the right-end of $\bar{\mu}$ as $\mu^{\mathrm{L}} : \mathcal{E} \to [0,1]$ and $\mu^{\mathrm{R}} : \mathcal{E} \to [0,1]$. Then, let $P_i^{s\prime} = \sum_{j \geq i} P_j^s$ and $\Delta\mu^* \left(P_i^{s\prime}\right) = \mu^* \left(P_i^{s\prime}\right) - \mu^* \left(P_{i+1}^{s\prime}\right)$, where $s \in \{+,-\}$ and $\mu^* \in \left\{\mu^{\mathrm{L}}, \mu^{\mathrm{R}}\right\}$. Finally, the expectation value of $\mathbf{O}$ with respect to a QIVPM $\bar{\mu}$ is defined to be

$$\langle \mathbf{O} \rangle_{\bar{\mu}} = \sum_{s \in \{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \left[\Delta\mu^{\mathrm{L}}\left(P_i^{s\prime}\right), \Delta\mu^{\mathrm{R}}\left(P_i^{s\prime}\right)\right] \ . \tag{G.18}$$

Notice that the similarity between classical Choquet integral and quantum expectation values as in Table G2. ‖,

‖ Yu-Tsung says: functor? Galois connection?

we would prove some lemmas which can help us move the classical results to quantum.

Definition 69. Given an observable $\mathbf{O}$ with any its spectral decomposition (**??**) whose eigenvalues are not necessary distinct. A subspace of events $\mathcal{E}' \subseteq \mathcal{E}$ is called consistent with $\mathbf{O}$ if it is the minimal subspace of events satisfying $\{P_i\}_{i=1}^{N} \subseteq \mathcal{E}'$. Obviously, $\mathcal{E}'$ is commuting and convex.

By theorem 32, we can find a set Ω, a function $\tau : \mathcal{E}' \to 2^{\Omega}$ to pull quantum real-valued or interval-valued probability measures to classical ones. Although it is a powerful tool, it would be more powerful if we can pull an observable to a random variable as well.

Definition 70. Given an observable $\mathbf{O}$ with its spectral decomposition (G.15) and its inducing subspace of events $\mathcal{E}'$, set Ω, and function $\tau : \mathcal{E}' \to 2^{\Omega}$. Then, the random variable $X : \Omega \to \mathbb{R}$ defined by $X = \sum_{s\in\{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \mathbf{1}_{\tau\left(P_i^s\right)}$ is called the random variable induced by $\mathbf{O}$.

The following lemma shows that .

Lemma 71. *Given an observable $\mathbf{O}$ with an orthonormal basis $\Omega = \{|k\rangle\}_{k=0}^{d-1}$, $\widehat{\mathcal{E}} = \left\{\sum_{|k\rangle\in E} |k\rangle\langle k| \;\middle|\; E \subseteq \Omega\right\}$, and the bijection $\tau : \widehat{\mathcal{E}} \to 2^{\Omega}$ defined in theorem 32. By definition 70, $\mathbf{O}$ induces a random variable $X : \Omega \to \mathbb{R}$, if $\mathbf{O} = \sum_{k=0}^{d-1} \lambda_k |k\rangle\langle k|$, then $X = \sum_{k=0}^{d-1} \lambda_k \mathbf{1}_{\{|k\rangle\}}$.*

Proof. Since $\mathbf{O}$ can be diagonalized by Ω, each projector in the spectral decomposition P_i^s is the summation of some basis vectors $\sum_{k\in S_i^s} |k\rangle\langle k|$. Since

$$\mathbf{1}_{\tau\left(P_i^s\right)} = \mathbf{1}_{\tau\left(\sum_{k\in S_i^s}|k\rangle\langle k|\right)} = \mathbf{1}_{\bigcup_{k\in S_i^s}\tau(|k\rangle\langle k|)} = \sum_{k\in S_i^s} \mathbf{1}_{\{|k\rangle\}}\,, \tag{G.19}$$

we have $X = \sum_{k=0}^{d-1} \lambda_k \mathbf{1}_{|k\rangle}$. □

Lemma 72. *Adopting the definitions in this subsection. Given a QRVPM $\mu : \mathcal{E} \to [0,1]$, there is a classical real-valued probability measure $\mu' : 2^{\Omega} \to [0,1]$ such that $\mu|_{\mathcal{E}'} = \mu' \circ \tau$. Then, we have $\langle \mathbf{O} \rangle_\mu = \int X \mathrm{d}\mu'$.*

Proof. Since $\mu : \mathcal{E} \to [0,1]$ is a QRVPM, $\mu|_{\mathcal{E}'}$ is a QRVPM on the commuting sub-event space $\mathcal{E}'$. By case i in theorem 32, there is a CRVPM such that $\mu|_{\mathcal{E}'} = \mu' \circ \tau$.

$$\begin{aligned}
\langle \mathbf{O} \rangle_\mu &\overset{\text{Equation }(G.17)}{=} \sum_{s\in\{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \mu\left(P_i^s\right) \\
&= \sum_{s\in\{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \mu'\left(\tau\left(P_i^s\right)\right) \\
&\overset{\text{Equation }(G.2)}{=} \int X \mathrm{d}\mu'\,.
\end{aligned}$$

$\square$

Lemma 73. *Adopting the definitions in this subsection. Given a quantum IVPM $\bar{\mu}$, there is a classical IVPM $\bar{\mu}' : 2^{\Omega} \to \mathscr{I}$ such that $\bar{\mu} = \bar{\mu}' \circ \tau$. Then, we have $\langle \mathbf{O} \rangle_{\bar{\mu}} = \int X \mathrm{d}\bar{\mu}'$.*

Proof. Let $\mu^{\mathrm{L}\prime} : \mathcal{E}' \to [0,1]$ and $\mu^{\mathrm{R}\prime} : \mathcal{E}' \to [0,1]$ are the left-end and the right-end of $\bar{\mu}'$, respectively, i.e., $\bar{\mu}'(E) = \left[\mu^{\mathrm{L}\prime}(E), \mu^{\mathrm{R}\prime}(E)\right]$ for all $E \in 2^{\Omega}$. Then,

$$
\begin{aligned}
\langle \mathbf{O} \rangle_{\bar{\mu}} &\overset{\text{Equation }(G.18)}{=} \sum_{s \in \{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \left[\Delta\mu^{\mathrm{L}}\left(P_i^{s\prime}\right), \Delta\mu^{\mathrm{R}}\left(P_i^{s\prime}\right)\right] \\
&= \sum_{s \in \{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \left[\Delta\mu^{\mathrm{L}\prime}\left(\tau\left(P_i^{s\prime}\right)\right), \Delta\mu^{\mathrm{R}\prime}\left(\tau\left(P_i^{s\prime}\right)\right)\right] \\
&\overset{\text{Definition 62}}{=} \int X \mathrm{d}\bar{\mu}' .
\end{aligned}
$$

$\square$

Appendix G.4. Between Expectation Values for Quantum Real-valued and Interval-valued Probability Measures

Intuitively, the expectation value of an observable with respect to an IVPM should be the minimum and maximum expectation value of the same observable with respect to its consistent states. We found this is true when we require our IVPM to be convex, and on a commuting sub-event space.

Theorem 74. *Given a Hilbert space, for every quantum IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ and any observable $\mathbf{O}$ inducing a sub-event space $\mathcal{E}'$, if $\bar{\mu}$ is convex, then*

$$
\langle \mathbf{O} \rangle_{\bar{\mu}} = \left[\min_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}}, \max_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}} \right]. \tag{G.20}
$$

Conversely, if Eq. (G.20) holds for every observable, then $\bar{\mu}$ is convex.

Proof. Adopting the definitions in subsection **??**, it is easy to prove

$$
\begin{aligned}
\langle \mathbf{O} \rangle_{\bar{\mu}} &\overset{\text{Lemma 72}}{=} \int X \mathrm{d}\bar{\mu}' \\
&\overset{\text{Theorem 63}}{=} \left[\min_{\mu' \in \mathrm{core}(\bar{\mu}')} \int X \mathrm{d}\mu', \max_{\mu' \in \mathrm{core}(\bar{\mu}')} \int X \mathrm{d}\mu' \right] \\
&\overset{\text{Lemma 73}}{=} \left[\min_{\mu' \in \mathrm{core}(\bar{\mu}')} \langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}}, \min_{\mu' \in \mathrm{core}(\bar{\mu}')} \langle \mathbf{O} \rangle_{\mu_\rho^{\mathrm{B}}} \right].
\end{aligned}
$$

Therefore, equation (G.20) is equivalent to convexity if $\text{core}(\bar{\mu}')$ and $\overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')$ are one-to-one correspondence so that

$$\left[\min_{\mu' \in \text{core}(\bar{\mu}')} \langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}}, \min_{\mu' \in \text{core}(\bar{\mu}')} \langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}}\right] = \left[\min_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}}, \max_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}}\right]. \tag{G.21}$$

Notice that lemma 72 guarantees that for any $\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')$, there exists $\mu' \in \text{core}(\bar{\mu}')$ such that $\mu^{\mathrm{B}}_{\rho} = \mu' \circ \tau$. On the other hand, for any $\mu' \in \text{core}(\bar{\mu}')$, we have

$$\mu'(E) \in \bar{\mu}'(E) \text{ for all } E \in 2^{\Omega} \tag{G.22}$$

$$\Rightarrow \mu'(\tau(P)) \in \bar{\mu}'(\tau(P)) = \bar{\mu}(P) \text{ for all } P \in \mathcal{E}' \tag{G.23}$$

By theorem 32, there is a density matrix ρ such that $\mu^{\mathrm{B}}_{\rho}(P) = \mu'(\tau(P))$ for all $P \in \mathcal{E}'$. Therefore, $\mu^{\mathrm{B}}_{\rho} \in \text{core}(\bar{\mu}, \mathcal{E}')$, i.e., $\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')$. □

Our next lemma discusses the quantum version of lemma 65.

Lemma 75. *Given a Hilbert space of $d \geq 3$, when $\mu^{\mathrm{L}} : \mathcal{E} \to [0,1]$ and $\mu^{\mathrm{R}} : \mathcal{E} \to [0,1]$ are real-valued probability measures, we have $\mu^{\mathrm{L}} = \mu^{\mathrm{R}} = \mu$. Then, $\langle \mathbf{O} \rangle_{\bar{\mu}} = \left\{\langle \mathbf{O} \rangle_{\mu}\right\}$.*

Proof 1. Adopting the definitions in subsection **??**, we have

$$\langle \mathbf{O} \rangle_{\bar{\mu}} \overset{\text{Lemma 72}}{=} \int X \mathrm{d}\bar{\mu}' \overset{\text{Lemma 65}}{=} \left\{\int X \mathrm{d}\mu'\right\} \overset{\text{Lemma 73}}{=} \left\{\langle \mathbf{O} \rangle_{\mu}\right\}, \tag{G.24}$$

where $\mu' = \mu^{\mathrm{L}\prime} = \mu^{\mathrm{R}\prime}$. □

Proof 2. Let $\mathcal{E}'$ be the subspace of events induced by $\mathbf{O}$. For any state $\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')$, we have $\mu^{\mathrm{B}}_{\rho}(P) \in \bar{\mu}(P) = \{\mu(P)\}$ for all $P \in \mathcal{E}'$. Thus, $\langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}} = \langle \mathbf{O} \rangle_{\mu}$. By theorem 74, we have

$$\langle \mathbf{O} \rangle_{\bar{\mu}} = \left[\min_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}}, \max_{\rho \in \overline{\mathcal{H}}(\bar{\mu}, \mathcal{E}')} \langle \mathbf{O} \rangle_{\mu^{\mathrm{B}}_{\rho}}\right] = \left\{\langle \mathbf{O} \rangle_{\mu}\right\} \tag{G.25}$$

□

Appendix G.5. Expectation Values for δ-deterministic QIVPMs

After we defined the expectation value, we can provide another proof of corollary 57 which doesn't refer the Kochen-Specker theorem directly. Assume there is a δ-deterministic QIVPM $\bar{\mu}$, we will prove $\bar{\mu}$ has such bizarre properties that it cannot exist for small δ.

Lemma 76. *Given a Hilbert space of dimension d, $\delta < 1/d$, and a δ-deterministic QIVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}$, the following statements are true:*

(i) If $\bar{\mu}$ is 0-deterministic, its left-end μ^{L} and right-end μ^{R} are the same QRVPM, and will be simply denoted by $\mu : \mathcal{E} \to \{1, 0\}$.

(ii) *Given an orthonormal basis* Ω*, there is a unique element* $|\psi_{i_0}\rangle \in \Omega$ *such that* $\bar{\mu}\left(|\psi_{i_0}\rangle\langle\psi_{i_0}|\right) \subseteq [1-\delta, 1]$*, denoted by* $\bar{\mu}^{-1}\left([1-\delta,1]\right) \cap \Omega$*. In particular, if* $\bar{\mu}$ *is 0-deterministic,* $\bar{\mu}\left(|\psi_{i_0}\rangle\langle\psi_{i_0}|\right) = \{\mu\left(|\psi_{i_0}\rangle\langle\psi_{i_0}|\right)\} = \mathscr{certain}$*,* $|\psi_{i_0}\rangle\langle\psi_{i_0}|$ *could be denoted by* $\mu^{-1}(1) \cap \Omega$ *or* $\bar{\mu}^{-1}(\mathscr{certain}) \cap \Omega$*.*

Proof.

(i) The direct consequence of lemma 56.

□

Lemma 77. *For any quantum IVPM* $\bar{\mu} : \mathcal{E} \to \mathscr{I}_{\mathrm{D}}$*,* $\langle \mathbf{O} \rangle_{\bar{\mu}}$ *is a singleton set* $\{\lambda\}$*, where* λ *is an eigenvalue of* $\mathbf{O}$*.*

Proof 1. Adopt the definitions in subsection **??**. By lemma 56, μ^{L} and μ^{R} are the same real-valued probability measures, and is simply denoted by $\mu : \mathcal{E} \to \{1, 0\}$. By lemma 75, $\langle \mathbf{O} \rangle_{\bar{\mu}} = \left\{ \langle \mathbf{O} \rangle_{\mu} \right\}$. Since μ is a cryptodeterministic measure, there is only one pair of (s_0, i_0) such that $\mu\left(P_{i_0}^{s_0}\right) = 1$. Therefore,

$$\langle \mathbf{O} \rangle_{\mu} \overset{\text{Equation } (G.17)}{=} \sum_{s \in \{+,-\}} \sum_{i=1}^{N^s} \lambda_i^s \mu\left(P_i^s\right) = \lambda_{i_0}^{s_0} . \tag{G.26}$$

□

Proof 2. Adopt the definitions in subsection **??**, we have

$$\langle \mathbf{O} \rangle_{\bar{\mu}} = \int X \mathrm{d}\bar{\mu}' \overset{\text{Lemma 67}}{=} \left\{ X\left(\bar{\mu}^{-1}(\mathscr{certain})\right) \right\} , \tag{G.27}$$

where $X\left(\bar{\mu}^{-1}(\mathscr{certain})\right)$ is an eigenvalue of $\mathbf{O}$ by the definition of X. □

Given two singleton sets $\{\lambda_1\}$ and $\{\lambda_2\}$, it is intuitive to define their multiplication of to be $\{\lambda_1\lambda_2\}$ which gives the following lemma.

Lemma 78. *For a QIVPM* $\bar{\mu} : \mathcal{E} \to \mathscr{I}_{\mathrm{D}}$ *and a sequence of commuting observables* $\{\mathbf{O}_j\}$*,*

$$\prod_j \langle \mathbf{O}_j \rangle_{\bar{\mu}} = \left\langle \prod_j \mathbf{O}_j \right\rangle_{\bar{\mu}} . \tag{G.28}$$

Proof 1. Since $\{\mathbf{O}_j\}$ is a sequence of commuting observables, they can be diagonalized by a common orthonormal basis $\Omega = \{|k\rangle\}_{k=0}^{d-1}$. By theorem 32, there is a commuting sub-event space $\widehat{\mathcal{E}}$ with $\widehat{\mathcal{E}} \subseteq \mathcal{E}$ and a bijection $\tau : \widehat{\mathcal{E}} \to 2^{\Omega}$ such that there is a CIVPM $\bar{\mu} : 2^{\Omega} \to \mathscr{I}_{\mathrm{D}}$ such that $\bar{\mu}(P) = \bar{\mu}'(\tau(P))$ for all $P \in \widehat{\mathcal{E}}$.

Let $\mathcal{E}'_j$ be the subspace of events induced by $\mathbf{O}_j = \sum_{k=0}^{d-1} \lambda_{j,k} |k\rangle\langle k|$. Clearly, $\mathcal{E}'_j \subseteq \widehat{\mathcal{E}}$. Therefore, restricting τ on each $\mathcal{E}'_j$ gives a function $\tau|_{\mathcal{E}'_j} : \mathcal{E}'_j \to 2^{\Omega}$. Then, we can follow lemma 71 to induce random variables $X_j = \sum_{k=0}^{d-1} \lambda_{j,k} \mathbf{1}_{\{|k\rangle\}}$, and have $\langle \mathbf{O}_j \rangle_{\bar{\mu}} = \int X_j \mathrm{d}\bar{\mu}'$ by lemma 73. Therefore,

$$\begin{aligned}
&\prod_j \langle \mathbf{O}_j \rangle_{\bar{\mu}} = \prod_j \int X_j \mathrm{d}\bar{\mu}' \overset{\text{Lemma 68}}{=} \int \prod_j X_j \mathrm{d}\bar{\mu}' \\
&= \int \prod_j \sum_{k=0}^{d-1} \lambda_{j,k} \mathbf{1}_{\{|k\rangle\}} \mathrm{d}\bar{\mu}' = \int \sum_{k=0}^{d-1} \prod_j \lambda_{j,k} \mathbf{1}_{\{|k\rangle\}} \mathrm{d}\bar{\mu}' \\
&= \left\langle \sum_{k=0}^{d-1} \prod_j \lambda_{j,k} |k\rangle\langle k| \right\rangle_{\bar{\mu}} = \left\langle \prod_j \mathbf{O}_j \right\rangle_{\bar{\mu}} .
\end{aligned} \tag{G.29}$$

□

Proof 2. Since $\{\mathbf{O}_j\}$ is a sequence of commuting observables, they can be diagonalized by a common orthonormal basis $\Omega = \{|k\rangle\}_{k=0}^{d-1}$, i.e., $\mathbf{O}_j = \sum_{k=0}^{d-1} \lambda_{j,k} |k\rangle\langle k|$ and $\prod_j \mathbf{O}_j = \sum_{k=0}^{d-1} \prod_j \lambda_{j,k} |k\rangle\langle k|$. By lemma 56, μ^{L} and μ^{R} are the same real-valued probability measures, and is simply denoted by $\mu : \mathcal{E} \to \{1, 0\}$. Since μ is a cryptodeterministic measure, there is only one $|k_0\rangle\langle k_0|$ such that $\mu(|k_0\rangle\langle k_0|) = 1$.

$$\begin{aligned}
&\prod_j \langle \mathbf{O}_j \rangle_{\bar{\mu}} \overset{\text{Lemma 75}}{=} \prod_j \left\{ \langle \mathbf{O}_j \rangle_{\mu} \right\} = \prod_j \{\lambda_{j,k_0}\} \\
&= \left\{ \prod_j \lambda_{j,k_0} \right\} = \left\langle \prod_j \mathbf{O}_j \right\rangle_{\mu} \overset{\text{Lemma 75}}{=} \left\langle \prod_j \mathbf{O}_j \right\rangle_{\bar{\mu}} .
\end{aligned} \tag{G.30}$$

□

Based on the previous lemmas, we could follow Mermin's idea to prove the Kochen-Specker theorem again [89, 28].

Theorem 79. *Given a Hilbert space of dimension $d \geq 3$, there is no quantum IVPM $\bar{\mu} : \mathcal{E} \to \mathscr{I}_{\mathrm{D}}$, where $\mathscr{I}_{\mathrm{D}} = \left\{ \text{impossible}, \text{certain} \right\}$ as before.*

Proof. Since the expectation value $\langle \cdot \rangle_{\bar{\mu}}$ satisfying the previous lemmas, the following observables [89, 28]:

$$\begin{array}{ccc}
\mathbb{1} \otimes \sigma_z & \sigma_z \otimes \mathbb{1} & \sigma_z \otimes \sigma_z \\
\sigma_x \otimes \mathbb{1} & \mathbb{1} \otimes \sigma_x & \sigma_x \otimes \sigma_x \\
\sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_y
\end{array} \tag{G.31}$$

must have the the expectation value 1 or -1. In each row and column, operators commutes, and each operator is the product of the two others, except in the third column

$(\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x) = -\sigma_y \otimes \sigma_y$. Therefore, it is clear impossible to find a IVPM $\bar{\mu}$ such that

$$\begin{array}{ccc} \langle \mathbb{1} \otimes \sigma_z \rangle_{\bar{\mu}} & \langle \sigma_z \otimes \mathbb{1} \rangle_{\bar{\mu}} & \langle \sigma_z \otimes \sigma_z \rangle_{\bar{\mu}} \\ \langle \sigma_x \otimes \mathbb{1} \rangle_{\bar{\mu}} & \langle \mathbb{1} \otimes \sigma_x \rangle_{\bar{\mu}} & \langle \sigma_x \otimes \sigma_x \rangle_{\bar{\mu}} \\ \langle \sigma_x \otimes \sigma_z \rangle_{\bar{\mu}} & \langle \sigma_z \otimes \sigma_x \rangle_{\bar{\mu}} & \langle \sigma_y \otimes \sigma_y \rangle_{\bar{\mu}} \end{array} \tag{G.32}$$

satisfying equation (G.28). □

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