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5 Randomness in Random Dynamics

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Abstract. Unlike the conventional schemes where the fundamental level is assumed to be simple, the Random Dynamics approach is based on the assumption that there is an irreducibly complex, random, bottom layer.

The purpose of this paper is to discuss the possible meanings of this assumption.

5.1 Random dynamics

Symmetry is an essential guiding principle in the development of physical theories. The highly successful Standard Model is an outstanding example [1] of this. According to the Standard Model philosophy, as we go up in energy, we expect a larger symmetry group. In the search of the ultimate fundamental theory, Grand Unification [1] is a scenario in this spirit, advocating that at higher energies there are forces corresponding to larger symmetry groups which break down at lower energies, ultimately giving rise to the symmetry group corresponding to the forces we see here at our low energy level.

The striving to reach an ultimate, fundamental force or forces that we find in the conventional Theory of Everything schemes, is based on the assumption that all physical phenomena eventually can be brought back onto a finite set of laws of nature, where a law is characterized by a finite complexity. There are however some stumbling blocks on this path, such as the inherent randomness of quantum mechanics. That is hard to reconcile with a conventional Theory of Everything scheme.

If symmetry is one corner stone in the search for physical laws, another guiding principle is simplicity. This is Occam's principle - simplicity is "sigillum veri". The idea is that fundamental principles are simple, which however does not necessarily imply that Nature itself is "simple" at a fundamental scale, on the contrary: as we climb up the energy scale there are more and more degrees of freedom, meaning a growing complexity. What goes on at a fundamental scale, like the Planck scale, is probably enormously complicated and most simply described in terms of randomness. This is the punch line of Random Dynamics, the theory developed by Holger Bech Nielsen [2] and his collaborators. Unlike the conventional schemes where fundamentality is assumed to be characterized by simplicity, the Random Dynamics approach is based on the contrary assumption that there is an irreducibly complex bottom layer.

The idea is that a sufficiently complex and general model for the fundamental physics at or above the Planck scale, will in the low energy limit where we operate, yield the physics we know. The reason is that as we slide down the energy scale, the structure and complexity characteristic for the high energy level is shaved away. The features that survive are those that are common for the long wavelength limit of any generic model of fundamental supra-Planck scale physics. The ambition of Random Dynamics is to "derive" [3] all the known physical laws as an almost unavoidable consequence of a random fundamental "world machinery".

According to this scheme, the fundamental "world machinery" is a very general, random mathematical structure $\mathcal{M}$, which contains non-identical elements and some set-theoretical notions. There are also strong exchange forces present. There is as yet no physics. At some stage $\mathcal{M}$ comes about, and then physics follows.

That the fundamental structure $\mathcal{M}$ comes without differentiability and with no concept of distance, that is, no geometry, implies an apriori lack of locality in the model. We cannot put in locality by hand, since the lack of geometry forbids locality to be properly stated. Thus the principle of locality, taken say as a path way integration $\int \mathcal{D}e^{\int \mathcal{L} d^4 x}$ with a Lagrangian density $\mathcal{L}$ only locally depending on the fields, cannot be put in before we have space and time.

We are so used to the urge for simplicity in science, that the idea that something fundamental could be non-simple seems impossible to imagine, the Random Dynamics assumption of non-simple fundamental laws thus appears as an oxymoron. How do we furthermore recognize a law as fundamental if it is non-simple?

Even more alarming is the Random Dynamics assumption that the fundamental level is characterized by a high degree of randomness. The purpose of this paper is to discuss the possible meaning of this assumption.

5.2 A comprehensible Universe

What does it mean that the world is comprehensible? According to Gottfried Wilhelm Leibniz [4] it means that the Universe is rationally comprehensible: that God used but a few principles to create the whole Universe, with all its complex beauty. And that we can backtrack it all by tracing the whole world back to the laws of nature.

The act of understanding thus amounts to a reduction of complexity, a kind of mapping from a set of ideas onto a smaller set of already accepted, well-defined notions. That is, we look for a cause, which is a phenomenon which is "smaller" and more general than the phenomenon we want to explain; the ultimate causes being the laws of nature.

When phenomena that we judge to have a resemblance with each other occur in a way that however cannot be assigned any regularity or pattern - i.e. no cause can be defined - we perceive them as *random*. Something random cannot be ascribed to some *simpler* underlying mechanism or algorithm. Albeit the concept of simplicity according to Herman Weyl [5], " appears to be inaccessible to objective formulation...it has been attempted to reduce it to that of probability", we think of something simple as opposed to something complex, as something with few elements related in a transparent way. A simple dress is unadorned, with a straightforward cut, a simple language usually means short sentences with common words, and a simple model comprehends uncomplicted rules and few elements.

There certainly is simplicity out there, in the sense of comprehensible regularity, otherwise we would not have the insight that we actually have, otherwise our cars would not be running, our lamps would not be shining.. The question is how deep the simplicity sticks, i.e. whether the Universe is simple also at the most fundamental level - regular, ruled by some few and formulable principles - or complex, maybe even random, at heart. Is the idea that we should look for a simple ultimate law or principle, nothing but a prejudice? Maybe nature is not fundamentally simple, maybe nature is fundamentally quite complex. It may even be fundamentally random, and yet subjected to simple rules. Or does the randomness we find in the world only exist on the interface between us and Nature? Whatever the answer is, it is clear that we cannot get rid of the point where language touches upon Nature, like the nerve of sight touches the eye in the blind spot.

According to the Random Dynamics approach, the question is not whether the Universe is complex or not, the question is rather *how* complex it is, finitely or infinitely, i.e random.

5.2.1 Randomness

There is a story by Honoré de Balzac [6] called "The Unknown Masterpiece". It is about the young Nicolas Poussin who in the early 17th century came to Paris in the hope of becoming the apprentice of one of the great painters of his time.

As the Master's apprentice, the young man every day saw the Master produce the most exquisite paintings. But the painting that most of all preoccupied the apprentice's imagination, was a big canvas standing in a corner, covered by a piece of cloth. He knew the Master considered it his chef d'oeuvre, and that he also considered it as unfinished. The Master now and then worked on the canvas, but never in the presence of the apprentice.

Then one day when nobody else was around, the apprentice gave in to his curiosity, tiptoed to the chef d'oeuvre canvas, and lifted the cloth. He fixed the canvas in astonishment, but could not see a thing. The surface was covered with paint, layer on layer, a meningless chaos of colour. In one corner a first layer of paint was still visible. There an absolutely perfect human foot stuck out, as from under a blanket.

In perfecting the painting, the Master had painted and painted on top of the first layer, hiding every structure under another structure, until all structure was muddled away. The canvas had become patternless. The young man was staring at a *random* pattern.

When we use the word random we think of something disorganized, without a plan. We speak of making a random choice, and mean picking a choice from a set of possible choices, without a plan or reason for that particular choice. Something organized, patterened, on the other hand, is *ordered*, non-random.

Order and randomness can be assigned to a process, to a number, to a furnished room... This is something we know from everyday life - when we tidy up, the room becomes tidy, ordered. But how do we produce something random?

Tossing a coin is a classical procedure to obtain a *random number*. Each toss is independent and random, in the sense that each toss is a fact that is what it is, head or tails, for no reason. When you toss a coin N times, and write down the result by letting 0 and 1 represent heads and tails, you get one of 2^N binary series which all have the same probability. In the traditional understanding of a random sequence of tokens is a sequence where all the tokens appear with the same probablitity, so the series obtained by tossing the coin should be reliably random [7]. Some of the 2^N series may however display some recognizable inner structure, i.e. non-randomness. So merely to emerge from a probabilistic event does not guarantee randomness.

Something random is by definition not specific, in some sense it is so average that there is no way of tagging it. To single out a number as being random is thus an oxymoron - since there is no way to pin down something random, there can be no consensus about the definition of randomness.

5.3 The axioms at the bottom

A random event is an event that cannot be ascribed a cause, it just happens "for no reason". The scheme of explaining a phenomenon by relating it to a cause, which ultimately is a principle or a law of nature, originates from Euclid's idea of mathematical proof: a mathematical truth is established by reducing it to simpler truths until self-evident truths, i.e. axioms or postulates, are obtained. At the bottom of the world there is a firm layer of true axioms.

That the language for describing the Universe is maths implies that the Universe is perceived as made out of eternal mathematical truth: like in Plato's Timaeus where the building blocks of the Universe are given, as simple, symmetrical geometrical forms. And mathematical truth is most certainly based on a set of simple axioms. This was at least David Hilbert's [8] conviction when he addressed his colleagues at the International Congress of Mathematicians in Paris in 1900, where he outlined 23 major mathematical problems to be studied in the coming century. One of the problems was the the axiomatization of mathematics: to formulate the axioms onto which all mathematical truth can ultimately be brought back. This led to the opening of the first Pandora box.

At the end of the 19th century, Georg Cantor [9] had investigated the properties of infinite sets. His work lead to many worrisome results, so worrysome in fact, that many mathematicians - including the great Poincaré - turned against Cantor. What upset people were the insights brought about by Cantor's work, like when he proved that "almost all" numbers are transcendental by proving that the real numbers were not countable, or the status of reality that he bestowed on the concept of infinity (Gauss had stated that infinity should only be used as "a way of speaking"). Cantor furthermore discovered the set theoretical paradoxes that constituted the basis for Bertrand Russell's [10] work on paradoxes in logic itself (like in "is the set of all set a member of itself?").

David Hilbert was an follower of Leibniz, who thought that one could avoid not only logical paradoxes, but all conflict, by formulating all statements in an algebraic form (we may laugh at this, but it is noteworthy that the majority of dissidents in former Soviet-Union were natural scientists and mathematicians, people for whom retouching of facts is not that easy to swallow). Hilbert thus had the idea that one could escape paradoxes like Russell's paradox, by creating a completely formal axiomatic system. Once all statements were formulated within this system there would not be room for paradoxes, Hilbert thought.

He expected every formal system to be consistent and complete, and any well-posed mathematical problem to be decideable, in the sense that there is a mechanical procedure (for example a computer program) for deciding whether a statement is true or not. If you can prove both the statement A and the statement $\neg A$ within a given formalism, the formalism is inconsistent. A formal axiomatic system is moreover complete if any statement A formulated within the system can be settled by proving or disproving it. That is, from the set of axioms of the system, you should be able to prove the whole truth and nothing but the truth implied by these axioms.

Hilbert's ambition was thus to project all of mathematics onto a formal complete and consistent system. In 1931 Kurt Gödel [11] however showed that Hilbert's idea that all paradoxes would evaporate when we decide to exclusively use formal language, was wrong. Gödel's point of departure was (a variant of) the liar's paradox, which is classically formulated as "this is a lie", or "I lie now"; an equivalent statement is "I am unprovable". Gödel showed that there is no formal axiomatic system that can make it clear whether a sentence like this is true or not - the implication being that any formal system, any language, is either incomplete or inconsistent. Hilbert's first two demands were thus shown to be mutually exclusive. Moreover, in 1936, Hilbert's third demand was abolished as Alan Turing [12] discovered *uncomputability*.

A number is *computable* if there is an algorithm for computing its digits one by one, approximating it to arbitrary precision. π is a computable number, even though its decimals may look totally patternless, without redundancy, as its digits all seem to have the same probability. π however has but finite complexity:, since there are algorithms [13] for calculating it, like the Bailey-Borwein-Plouffe algorithm

$$\pi = \sum_{n=0}^{\infty} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \left(\frac{1}{16} \right)^{n} \tag{5.1}$$

Most real numbers are however not computable. In physics there is a large occurrence of measurable albeit non-computable numbers, and likewise in maths itself. "A rather startling result" writes Roger Penrose [14] about the fact that the wave equation with computable initial conditons can have non-computable solutions.

Turing addressed the halting problem, which concerns whether a computer program can tell in advance if another program will eventually halt or not. He concluded that no such program exists, since if you can find a mechanical procedure for deciding whether a computer program will halt, you end up being able to compute a real number which is not computable, i.e. in self-contradiction.

Do the logical limitations in the formal system constitute limitations to scientific knowledge? Is there moreover any physical Gödel's theorem, implying unanswerable questions and limits to scientific knowledge [15]?

One answer is that the limitations discovered by Gödel and Turing imply is that certain mathematical "observables", i.e. deduced results that scientists can understand, cannot be obtained. According to this view, since this is information that is excluded from the tentative mathematical model, it cannot contribute to the scientific method and should therefore be discarded. To pay a more serious attention to the limitations within the formal system would namely be to consecrate mathematical equations with a physical reality that they do not have. (Einstein: "as far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality").

Physical observation consists of finite detectable information corresponding to a finite set of domains of real numbers (precision determined by the precision of the instruments), while a mathematical model is associated with infinitely precise real numbers, and computer algorithms correspond to a finite set of integers. These forms of information are thus inherently distinct, to the point that there is no one-to-one mapping between them. The correlations emerging from the physical observations are not necessarily the same as those emerging from the mathematical models or from the computer algorithms, the scientific method is however expected to supply a relationship between the correlations from physical observation, and mathematical or computer modelling.

The Gödelian trap is that we are able to intuit the truth of a sentence without being able to formally prove it. If you can prove the sentence "this is an unprovable sentence", it is a false sentence. Thus you have formally proven something false - a terrible oxymoron. On the other hand, if you `cannot` prove it, albeit you perceive its truth, you sit in the trap. The crux is that to possess information is not the same as having knowledge or understanding. Insight and understanding come with the correlations between observables. Since understanding per se cannot be logically formalized, it cannot be logically proved or disproved either.

5.3.1 Information Theory

In information theory, one studies how to measure the rate at which a message source generates information. It tells us how to represent or encode messages from a particular source over a given channel, avoiding errors of transmission. The question is basically how messages are conveyed from a message source, such as a writer or speaker, to a recipient. The amount of information in a message is measured in `bits`, one bit being the answer to a yes/no-question. Bits are thought of as abstract zeros and ones, but information is always encoded in real physical objects. A string of bits can thus be regarded as a physical resource. The essential elements of information science, classical or quantum, can be summarized in a three step procedure.

1. Identify a physical resource - e.g. A string of bits.
2. Identify an information-compressing task, like gunzipping.
3. Identify a criterion for successful completion of 2. (like controling that the output from the compression stage perfectly matches the input of the compression stage).

The fundamental question of information science is then "what is the minimal quantity of physical resource 1. needed to perform the information processing task 2. in compliance with the success criterion 3".

In classical information science it can be stated as "what is the minimum number of bits needed to store the information produced by some source"? This was solved by Shannon in 1948 [16]. Shannon quantified the information content produced by an information source, defining it to be the minimum number of bits needed to reliably store the output of the source.

5.3.2 Algorithmic Information Theory

While classical information theory (Shannon[16], Wiener[17]) uses concepts like ensembles and probability distributions, Algorithmic Information Theory [18] instead focuses on individual objects, by posing questions like: what is the size of the smallest program for calculating a given object, how many bits are needed to compute it? In algorithmic information theory the complexity of an object is measured by the size (say in bits) of the smallest algorithm generating it, i.e. the amount of information needed to give to a computer in order to have it perform a given task. The size of a computer program is analogous to the degree of disorder of a physical system, algorithmic information theory in this way supplies a definition of what it means for a string to be unstructured, i.e. random. According to algorithmic information theory the simplest theory is defined as one corresponding to the smallest algorithm, that is, the scheme using the fewest bits.

As algorithmic complexity of a sequence is measured as the length of the smallest algorithm that reproduces the sequence in question, randomness refers to something that informationwise cannot be compressed at all. A completely random sequence requires an algorithm as long as the sequence itself, i.e. something random is algorithmically incompressible or irreducible. Most strings are moreover algorithmically irreducible and therefore random.

Consider a string of the length of N bits. The algorithmic information content of such a string is (generically) $H(S) \leq N + H(N)$, where $H(N)$ is the algorithmic information content for N in the base-2, $H(N)$ given as $H(N) \sim N + \log N$.

For strings that have some pattern or regularity, $H(S) < N + H(N)$, while for strings that have no regularities that could diminish their information content, $H(S) = N + H(N)$. Such strings are random or algorithmically incompressible, the border between random and non-random occurring at $H(S) \approx N$. The probability that an infinite sequence obtained by tossing a (fair) coin is (algorithmically) random is 1, while the calculation of the first N bits of π takes $H(N) \approx \log N$ bits, implying that π is definitely a non-random number.

5.3.3 Attempts to define randomness

Order and randomness are both characteristic of the set of relations between parts of a system, not of the parts themselves. It is a whole system of entities that is random or ordered.

While "order" is a very special way of organizing the parts of a system (think of a tidy room), a random state is "typical", non-specific, and thus has no distinguishing features (think of all the different ways a room can be untidy). According to the traditional understanding of randomness, the parts of a random system are all on the same footing - complete democracy, complete symmetry rule. In that sense it is patternless and very hard to define, indeed impossible. To see this, assume that you actually can single out a random number. The property of being random is then a feature that makes random numbers stick out. But since randomness is by generic, non-specific, this is a self-contradiction, implying that there cannot be any definitive definition of randomness. The more one tries to pin it down, the more evasive it becomes. Every definition of randomness therefore has a taint of something preliminary.

In algorithmic information theory, the degree of randomness of a given sequence is however established in terms of the smallest algorithm that reproduces the sequence in question. If this algorithm is as long as the sequence itself, the sequence is considered as random. This still does not ensure that something is random in any absolute sense, it merely establishes a randomness hierarchy. It is thus impossible to prove randomness, but non-randomness can be stated exactly. We may establish that a number is not random by finding an algorithm that generates it, but we cannot by the same means establish that a number is random: that I have not succeeded to find an algorithm generating it does not prove that the number is random. How can I know that I will not find such an algorithm tomorrow - I cannot prove that such an algorithm will never be found. In this sense the property of randomness is unprovable. The definition of randomness is by necessity heuristic and preliminary - we can never say if there is some algorithm that is yet to be discovered. According to the algorithmic information theory scheme, the Random Dynamics philosophy thus means that fundamental algorithms and laws are not necessarily short.

The program-size randomness also implies that something random is undistinguished and typical. Incompressibility is characteristic of something typical or generic - i.e. there is no structure that singles out this object, that makes it non-random [20]. The only way to describe a random thing is by stipulative definition, to point at it: here it is!

So, something is random if we cannot find an algorithm, i.e. a shorter description, from which we can derive the structure in question: no redundancies remain to be removed. This clearly is true for the elementary parts of a system, like the axioms in an axiomatic system. In this sense axioms, and indeed any elementary entity is also - trivially - random. (A Gödelian statement, impossible to prove or disprove, is thus also a sort of axiom). When we speak of randomness we however usually do not mean the trivial randomness of axioms. When we speak of randomness and order, we mean properties of systems above the elementary, axiomatic level.

In the causal explanatory scheme, it is assumed that `if` it were possible to control all the influences over a physical experiment, the outcome would always be the same. Therefore, since a random event cannot be ascribed a well-defined cause, randomness is often perceived as a lack of knowledge, an ignorance on the part of the observer, In quantum mechanics it assumed to be possible to set up an experiment with perfect control of all relevant parameters. Even in such an experiment the outcome can still be totally random. There have been unsuccessful attempts to save the situation by means of "hidden variables"; the randomness of the outcome however remains, indicating that the world might be irreducibly random.

A large body of non-random physics and maths has been formulated, but quantum physics places randomness at the fundamental level of physics, and even in classical mechanics we find unpredictability and randomness. But this does not prove that there is an innermost random core - maybe what we perceive as random today, will one day be punctuated, maybe it is our ignorance that speaks, not nature's fundamental properties!

5.4 The emergence of order

In the Random Dynamics scheme, the world is assumed to be fundamentally random. If one takes this view, the question is how order emerges, the order we see, in the form of symmetry, laws, all the organized forms - in short, the world we live in! Can there be an inherent randomness in a world with such a high degree of organization? To explain the emergence of the observed order of the world, is one of the challenges of the Random Dynamics scheme, while the ontological state of randomness remains a matter of discussion.

Some proponents claim that the world contains real randomness, others that what we believe is randomness is really pseudo-randomness.

A third approach is that randomness may exist in the realm of mathematics, while not in the physical world. An proponent of this view is physicist Karl Svozil [19], who claims that the randomness displayed in quantum mechanics is a matter of ignorance. He believes that a new, deeper hidden-variable quantum theory will eventually emerge, which amends the present quantum theory and gives us a randomness-free, deterministic theory. Svozil however accepts that mathematics contains real randomness.

5.4.1 Apparent randomness

According to the rationalist view of the physical world, everything happens for a reason, implying that the Universe is logical and comprehensible. Supporters of this view claim that there is no real randomness, only pseudo-randomness exixts. This is the kind of randomness produced by random-number generators, which are deterministic sequences of numbers generated by algorithms (not by quantum mechanical processes!). The Universe is accordingly deterministic, governed by deterministic physical laws: - the world has finite complexity. According to this view, albeit the world looks so complex, it is really quite simple, like in the case of π. We just don't know the underlying law, or algorithm, therefore this overwhelming impression of complexity and even randomness. This resembles the classical picture, where we formulate a story based on observations, and then deal with the information content of that story by using mathematics, with the intention of pinning down the substructure of laws and principles.

π really constitutes a very interesting example. The *number* π looks impossibly convoluted, while the *geometrical relationship* defining π is quite simple. And indeed, the *number* π turns out to be non-random. The simple, transparent geometrical description of π constitutes an algorithm with finite complexity, indicating that there should also exist numerical algorithms for generating the number π.

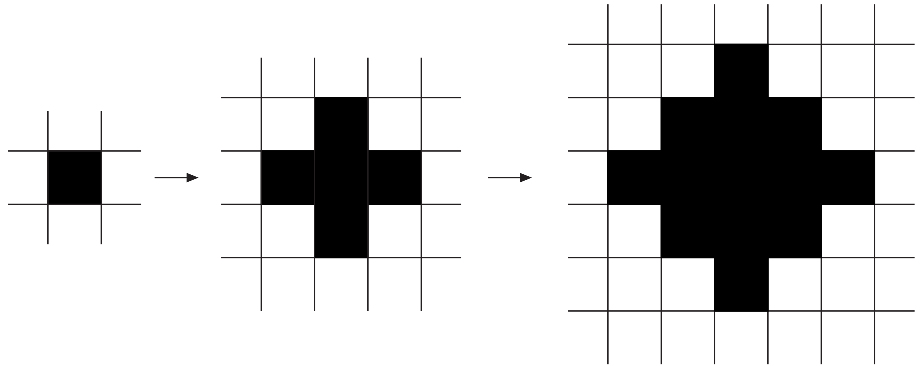
In his book "A New Kind of Science" [21], Stephen Wolfram reports on a systematic computer search for simple rules with very complicated consequences. In traditional physics one studies systems that satisfy certain constraints. Wolfram instead studies systems that develop according to given algorithms, and subsequently asks what pattern or algorithm corresponds to a given constraint, if any. He looks for fundamental algorithms rather than for fundamental laws, by investigating the emergence of organized patterns in a system of cellular automata, using simple rules à la the Game of Life. Challenging the Pythagorianian "Number rules the Universe", Wolfram asserts that it is not number that rules the Universe, but Algorithm. Namely discrete algorithm.

Wolfram's claim is moreover on the line with that of digital philosophy [22], which advocates that everything fundamentally comes in discrete bits. That implies a picture of the world as a giant digital information processor, a computer, in agreement with algorithmic information theory. This challenges traditional physics which is based on continuous mathematics. Digital philosophy is really a scheme for describing the world in terms of automata theory. The assumption is that everything fundamental is atomic or discrete, and the continuous flow of time is replaced by a sequence of time steps. So the basis of digital philosophy is:

- All information can ultimately be digitally represented.
- All change in information is due to digital information processes.

According to digital philosophy a physical state is represented by a pattern of bits, like in a computer. The digital philosophy bits exist in a digital spacetime, and each point contains 1 bit of information. Digital spacetime thus consists of bits that all have integer coordinates. A digital philosophy model is always specific, unlike a mathematical model that is generic. Every digital mechanics model can be put into a computer where it runs and evolves. In physics, if space and time are discrete, all other physical quantities must also be discrete. In digital philosophy, the dynamics of a system is therefore described by difference equations, which can be transformed into computational algorithms.

A simple example of cellular automata is a fixed array of cells together with the rule that a cell becomes black if one of its neighbours is black:



One of the features that cellular automata and Turing machines have in common is that they consist of a fixed array of cells. The colours and arrangement of colours vary, but the number and organization of cells remains constant. With a substitutions system one may however also change the number of elements, if the substitution law e.g. implies to replace each element by a block of new elements. To complicate the matter further, one can combine the substitution of elements with some form of interaction between the elements. The changes then depend both on the "internal properties" of each element, as well as on its environment.

By studying cellular automata that develop according to given algorithms, Wolfram found that behaviour of great complexity can be produced by certain rules starting from certain initial states. Other systems than cellular automata can also develop complexity, Wolfram indeed concludes that complexity is a universal phenomenon that is quite independent of the details of the given system. If the rules governing a system are simple enough, the system will display very simple, repetetive behaviour. With rules somewhat less simple, nesting will appear, namely self-similar, fractal patterns. As the complexity in the underlying rules goes beyond a certain critical value, the system's overall behaviour will be complex. The criticality threshold is typically rather low, meaning that the deviation from simplicity is not very large. Quite simple rules may lead to a complex behaviour. Moreover, once the critical value has been passed, adding complexity to the underlying rules does not increase the complexity of the system's behaviour.

So the stages from simplicity to complexity can be described by repetition, nesting and complexity. The typical types of behaviour are quite universal and practically independent of the underlying rules. Behaviour with non-random features can develop even from completely random initial conditions, there are many instances of cellular automata that start from random initial conditions and quickly settle down in a stable state. The end result is not necessarily "simple" and transparent, it may be quite complicated, almost random. But also in a random final state, there is almost always some small non-random structures that emerge in the evolution of the system. The most complex results lie between the extremes where the end result is a completely trivial, uniform state; or a seemingly random state. In the complex final state the cellular automata are organized in a set of definite localized structures that do not remain fixed but move around and interact with each other.

The patterns that emerge, quite different among themselves, fall into four fundamental types of patterns or classes, numbered in growing order of complexity:

Class 1. displays very simple behaviour: almost all initial conditions lead to the same uniform final state. The information about the initial conditions is simply wiped out and the same final state is reached, redardless of the initial conditions.

Class 2. emcompasses many different final states, but they all display a certain set of simple structures that remain the same or repeat every few steps. Changes may persist, but are always localized in some small region of the system. That is, the information about the initial conditions always remains, but localized and not communicated from one part of the system to the others.

Class 3. more complicated, more random; but at some level some small-scale structures exist. Any change that is made typically spreads with a uniform rate throughout the system. That is, there is a long-range communication of information.

Class 4. mixture of order and randomness. Changes also spread here, but in a more sporadic way than in the class 3.-systems.

Class 1. And 2. rapidly settle down to states in which there is essentially no further activity, while class 3.-systems continue to change at each step; class 4. systems are somewhere between class 3 and the two first classes. The differences between the classes reflect that systems from different classes handle information in different ways. Also continuous cellular automata - where the underlying rules involve parameters that vary smoothly between 0 and 1 - can be classified in accordance with this scheme.

5.4.2 Randomness at the heart of mathematics

Unlike Stephen Wolfram, Gregory Chaitin [23] is convinced that true randomness exists. The crux is to pin it down.

According to Chaitin Gödel's and Turing's discoveries indicate that there is an inherent randomness in mathematics, Chaitin furthermore formulated the source of randomness at the heart of mathematics [24]. Chaitin's reasoning goes as follows: equations can be classified according to their numbers of solutions. For example, some equations have no solutions, like $x = x + 4$, or one solution, like $4x = 12$, or two solutions, like $x^2 = 3x - 5$, or an infinite number of solutions, like $(1 + x)^2 = x^2 + 2x + 1$. Now, Chaitin has found an equation for which it is mathematically undecideable whether it has a finite or an infinite number of solutions, leading to the definition of the number Ω = the probability that a probabilistically generated program will ever terminate. This number is so random that not one sole of its digits can be predicted!

The prescription for the random number Ω starts with running a program on a computer. Each time the computer requests the next bit of the program, insert the result obtained by flipping a fair coin. The computer must decide by itself when to stop reading the program, this turns the program into self-delimiting binary information. For each program p that halts, sum the probailities of getting precisely that program by chance:

$$\Omega = \Sigma_{(\text{program p halts})} \frac{1}{2^P} \tag{5.2}$$

where P is the size in bits of the program p. Each n-bit self-delimiting program p that halts contributes $1/2^n$ to the value of Ω.

Ω is an algorithmically random or irreducible number, thus a formal algebraic system can determine only finitely many bits of such a number; really only as many bits of Ω as its own complexity. The only way to determine the bits of Ω is thus to put the information directly into the axioms of the formal algebraic system. There are thus no short-cuts, no algorithms to compress the information needed to generate Ω. The information needed to generate Ω is Ω itself. The bits of Ω are logically irreducible, i.e. cannot be obtained from axioms simpler than they are.

Mathematics thus contains randomness - the bits of Ω. That doesn't mean that mathematics is random in the sense of being arbitrary. It means that mathematics contains irreducible information, like Ω. Ω is, both algorithmically or computationally and logically, i.e. by means of proofs, irreducible. No shift of perspective, no reparametrization can make Ω more transparent, like for example in the case of π. This means that Ω has many of the characteristics of the typical outcome of a random process.

Ω is a random real with lots of meaning, since it contains lots of information about the halting problem. This information is stored in Ω in an irreducible fashion, with no redundancy. Once redundancy is squeezed out of something meaningful, it may look meaninglesseven though it is in reality dense with meaning. Just like the Master's chef d'oeuvre. A random pattern may be meaningless or extremely meaningful, there is no way to distinguish. That is the crux of the matter.

This implies that the mathematical Universe is infinitely complex, and thus the whole of mathematical ideas cannot be comprehended in it entirety. This is Chaitin's radical refutation of Hilbert's project of summing all of mathematics up by formulating the set of mathematical axioms.

5.5 Conclusion

It is by definition impossible to formulate an exact definition of randomness, but degrees of randomness can nevertheless be established, and most precisely so in algorithmic information theory. There are divergeing opinions about the ontological state of randomness. While some claim that what we take for randomness is only an effective pseudo-randomness, others state that randomness exists at the heart of mathematics, and most probably also at the heart of physics.

Gödel's, Turing's and Chaitin's work certainly gives strong support to a fundamental randomness, as advocated by Random Dynamics, which on the one hand concerns the choice of a primary set $\mathcal{M}$ from the set $\mathbf{M}$ of all generic sets or proto-models. The choice is random in the sense that any set of such general nature will do. Nature picked $\mathcal{M}$, but could just as well have picked $\mathcal{M}'$. The randomness of choice of $\mathcal{M}$ can be interpreted as the type of randomness that occurs in coin tossing.

The randomness inherent in the structure of the set $\mathcal{M}$ corresponds to a lack of an organizing principle governing the elements of $\mathcal{M}$, as energy and information are evenly distributed over all the degrees of freedom. Only by going down in energy is redundancy added to the system, allowing the information that is inherent in $\mathcal{M}$ to be displayed. The elements of $\mathcal{M}$ are different, yet undistinguishible, because there is not enough structure at hand to enable categorization of the elements. This randomness inherent in the composition of $\mathcal{M}$ implies that "algorithms" or recipes for generating a set like $\mathcal{M}$ can by no means be short.

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