Probability

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1 Classical Conventional Probability Spaces

Textbook probability theory is defined using the notions of *sample spaces*, *events*, and *measures*.

1.1 Sample Space Ω

In this paper, we will only consider **finite** sample spaces. We therefore define a sample space Ω as a non-empty finite set.

Example 1 (A Classical Sample Space.). Consider an experiment that tosses three coins. A possible outcome of the experiment is HHT which means that the fist and second coins landed with "heads" as the face-up side and that the third coin landed with "tails" as the face-up side. There are clearly a total of eight possible outcomes, and this collection constitutes the sample space:

$$\Omega_C = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Example 2 (A Quantum Sample Space.). Consider a quantum system composed of three electrons. By the postulates of quantum mechanics, an experiment designed to measure whether the spin of each electron along the x axis is left (L) or right (R) can only result in one of eight outcomes:

$$\Omega_H = \{LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR\}$$

1.2 Events $\mathcal{F}$

The space of events $\mathcal{F}$ associated with a sample space Ω is 2^Ω, the power set of Ω. In other words, every subset of Ω is a possible event.

Example 3 (Some classical events.). The following are events associated with Ω_C:

- E_0, exactly zero coins are H, is the set $\{TTT\}$.
- E_1, exactly one coin is H, is the set $\{HTT, THT, TTH\}$.
- E_2, exactly two coins are H, is the set $\{HHT, HTH, THH\}$.
- E_3, exactly three coins are H, is the set $\{HHH\}$.
- $E_{>0}$, at least one coin is H, is the set $\{HHH, HHT, HTH, HTT, THH, THT, TTH\}$.

As the examples illustrate, events are *indirect* questions. Also note that some events may be disjoint and that some events may be expressed as combinations of other events. For example, we have $E_{>0} = E_1 \cup E_2 \cup E_3$ and each of these four events is disjoint from event E_0.

Example 4 (Some quantum events.). The following are events associated with Ω_H:

- F_0, exactly zero electrons are spinning L, is the set $\{RRR\}$.

- F_1, exactly one electron is spinning L, is the set $\{LRR, RLR, RRL\}$.
- F_2, exactly two electrons are spinning L, is the set $\{LLR, LRL, RLL\}$.
- F_3, exactly three electrons are spinning L, is the set $\{LLL\}$.
- $F_{>0}$, at least one electron is spinning L, is the set $\{LLL, LLR, LRL, LRR, RLL, RLR, RRL\}$.

As the examples illustrate, quantum events are, at first glance, similar to classical events. There are however some subtle differences that we point out in the next section.

1.3 Measures $\mathbb{P}$

The last ingredient of a probability space is a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns to each event a real number in the closed interval $[0, 1]$ subject to the following conditions:

- $\mathbb{P}(\Omega) = 1$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \Sigma_i\ \mathbb{P}(A_i)$.

Example 5 (Classical probability measure). There are 2^8 events associated with Ω_C. A possible probability measure for these events is:

$$\mathbb{P}(E) = \begin{cases} 1 & \text{if } E = \Omega \\ 0 & \text{otherwise} \end{cases}$$

A more interesting measure is defined recursively as follows:

$$\begin{array}{rcl} \mathbb{P}(\emptyset) & = & 0 \\ \mathbb{P}(\{HHH\} \cup E) & = & \frac{1}{5} + \mathbb{P}(E) \\ \mathbb{P}(\{HHT\} \cup E) & = & \mathbb{P}(E) \\ \mathbb{P}(\{HTH\} \cup E) & = & \frac{3}{10} + \mathbb{P}(E) \\ \mathbb{P}(\{HTT\} \cup E) & = & \mathbb{P}(E) \\ \mathbb{P}(\{THH\} \cup E) & = & \frac{1}{5} + \mathbb{P}(E) \\ \mathbb{P}(\{THT\} \cup E) & = & \mathbb{P}(E) \\ \mathbb{P}(\{TTH\} \cup E) & = & \frac{3}{10} + \mathbb{P}(E) \\ \mathbb{P}(\{TTT\} \cup E) & = & \mathbb{P}(E) \end{array}$$

Because this is a *classical* situation, the probability assignments can be understood *locally* and *non-contextually*. In other words, we can reason about each coin separately and perform experiments on it ignoring the rest of the context. If we were to perform such experiments we may find that for the first coin with sample space, the probability of either outcome H or T is $\frac{1}{7}2$; for coin two, the probabilities are skewed a little with the probability of outcome H being $\frac{2}{5}$ and the probability of outcome T being $\frac{3}{5}$; coin 3 is a fake double-headed coin and hence the probability of outcome H is 1 and the probability of outcome T is 0. The reader may check that these local observations are consistent with the probability measure above.

Example 6 (Quantum probability measure). Like in the classical case, there are 2^8 events. But as Mermin explains in a simple example (Quantum Mysteries Revisited), here is a possible probability measure:

$$\mathbb{P}(S) \quad = \quad \{ \begin{array}{ll} 1/8 & \text{if there is an odd number of } L \\ \ldots & \end{array}$$

Interestingly and paradoxically, it is *not* possible to analyze each electron independently and assign a consistent probability to each one locally and non-contextually.

1.4 Finite Precision of Measurements

In a laboratory setting or a computational setting, there are neither uncountable entities nor uncomputable entities. We are thus looking at alternative probability spaces which do not depend on the real numbers and revisit the mysteries of quantum mechanics in that setting. Following previous work on probability, we will replace the closed interval $[0, 1]$ by the *finite set* $S = \{\textbf{possible}, \textbf{impossible}$ and adapt the definition of probability measure as follows:

A set-valued probability mesure $\mathbb{P} : \mathcal{F} \to S$ assigns to each event either the tag **possible** or the tag **impossible** subject to the following conditions:

- $\mathbb{P}(\Omega) = \textbf{possible}$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \textbf{possible}$ if any event A_i is **possible** and **impossible** otherwise.