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RM-4571-PR
OCTOBER 1965

NOTES ON N-PERSON GAMES VII: CORES OF CONVEX GAMES

Lloyd Shapley

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The RAND Corporation
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PREFACE

This Memorandum is one of a series of technical notes dealing with topics in the mathematical theory of n-person games. Convex games are competitive situations in which there are strong incentives towards formation of large coalitions; examples can be found in both commercial and diplomatic contexts.

The results in this Memorandum were presented at the Fifth Informal Conference on Game Theory at Princeton University, April 5-7, 1965.

ABSTRACT

The core of an n-person game is the set of outcomes that cannot be blocked by any coalition of players. It is shown in this Memorandum that the core of a convex game has an especially regular structure, and that it is closely related to two other solution concepts. Specifically, (1) the value solution is the center of gravity of the extreme points of the core; (2) the von Neumann-Morgenstern stable set solution is unique and coincides with the core.

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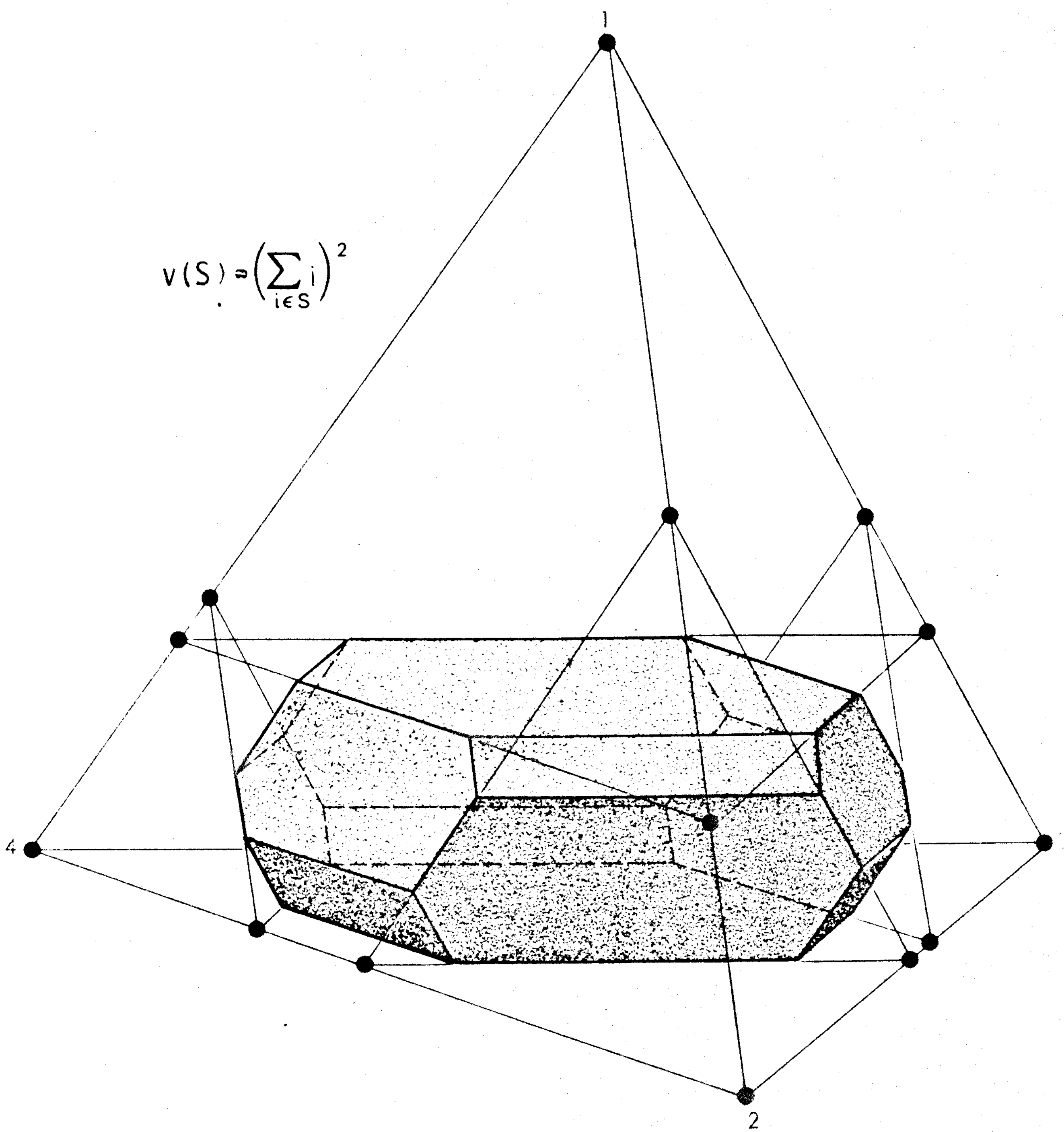

Frontispiece — Core of a 4-person convex game

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1. CONVEX GAMES

A game on $\mathcal{N}$ is a function v from a Boolean ring $\mathcal{N}$ to the reals satisfying

$$(1) \qquad v(0) = 0.$$

It is superadditive if

$$(2) \qquad v(S) + v(T) \leq v(S \cup T), \quad \text{all } S, T \in \mathcal{N} \text{ with } S \cap T = 0.$$

It is convex if

$$(3) \qquad v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \qquad \text{all } S, T \in \mathcal{N}.$$

It is strictly convex if inequality holds in (3) whenever the sets S, T, $S \cup T$, $S \cap T$ are all distinct. Note that (1) and (3) together imply (2).

To appreciate the term "convex", define a differencing operator Δ_R by:

$$[\Delta_R v](S) = v(S \cup R) - v(S - R), \qquad \text{all } S \in \mathcal{N},$$

for arbitrary $R \in \mathcal{N}$. Then (3) is equivalent to the assertion that the second differences $\Delta_Q[\Delta_R v]$ are nonnegative functions, for all $Q, R \in \mathcal{N}$.

Throughout this paper $\mathcal{N}$ will be the ring of subsets of a finite set N: $\mathcal{N} = 2^N$. In the standard game-theory application, the elements of N are "players"; the elements of $\mathcal{N}$ are "coalitions"; and $v(S)$, the so-called <u>characteristic function</u>, gives for each coalition S the maximum expected payoff it can achieve without help from players outside S.

Superadditivity is a necessary consequence of this interpretation, but not convexity.* To see what the latter entails, let us think of the quantity $e(S, T) = v(S \cup T) - v(S) - v(T)$ as measuring the "merger incentive" for the disjoint pair of coalitions S, T. Then we verify at once that (3) is equivalent to the assertion that $e(S, T)$ is nondecreasing in both variables. Another condition equivalent to (3) (in the finite case) is the following:

$$v(S) - v(S - \{i\}) \leq v(T) - v(T - \{i\}), \qquad \text{all} \quad i \in S \subseteq T \subseteq N.$$

This expresses a sort of increasing marginal utility for coalition membership.

1.1 Measure Games

A trivial example of convexity is provided by the <u>additive</u> functions, defined by equality in (2) (or

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* For example, in a voting game let S and T, but not $S \cap T$, be winning coalitions. Then (3) fails.

equivalently by equality in (3)). For something less trivial, let v be expressed in the form*

$$(4) \qquad v(S) = f(m(S)), \qquad \text{all } S \in \mathcal{N},$$

where m is a measure on $\mathcal{N}$ — i.e., a nonnegative additive function, and $f(0) = 0$. Then it is easy to see that such a <u>measure game</u> is convex if f is convex, and strictly convex if f is strictly convex.

Sums of convex measure games are convex games, but are not necessarily convex measure games. An additive game can always be written as the sum of two convex measure games, since every additive function is the difference of two measures and $f(x) \equiv x$ and $f(x) \equiv -x$ are convex functions. But there are convex games that are not sums of any number of convex measure games. An example, due to S. A. Cook, is the following: Take $|N| = 4$, $P \subset N$, $|P| = 2$, and define v_P by

$$\begin{cases} v_P(S) = 0,\ 0,\ 1/2,\ 1 \quad \text{for} \quad |S| = 0,\ 1,\ 3,\ 4 \quad \text{respectively,} \\ v_P(S) = 1/4 \quad \text{for} \quad |S| = 2,\ S \neq P, \quad \text{and} \\ v_P(P) = 0. \end{cases}$$

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* Actually, any game on $\mathcal{N}$ can be put into this form, if $\mathcal{N}$ is finite and no restriction is placed on f; we merely choose m so that all the numbers $m(S)$ are distinct, and define f accordingly. In economic applications, m often represents the initial distribution of some resource, while f represents a production function.

The proof consists in using sets S, T that give equality in (3) to obtain restriction on the functions in any supposed representation $v_P = \Sigma f_i(m_i)$.

Call a game normalized (in the finite case) if it vanishes on all one-element sets, and call two games equivalent if their difference is an additive function. Then every (finite) game is equivalent to one and only one normalized game. The set of all normalized convex games on 2^N constitutes a closed convex polyhedral cone, of dimension $2^{|N|} - |N| - 1$. It would be interesting to know the extremal elements of this cone.*

Acknowledgment. The theorems in the sequel were obtained first for convex measure games (except for the "if" part of Theorem 3), and then extended to their sums. The idea of trying to extend them to all convex games was stimulated by a relayed request from Jack Edmonds, last December, for references concerning functions (3), and I should like to acknowledge with thanks the benefit of this stimulus. It now appears that some of these results have been obtained independently by Edmonds, in a more general setting; but we are as yet unable to give specific attribution.

Edmonds favors the term "supermodular" in place of "convex".

1.2 Notation

We shall systematically use the letters $n, s, t, \ldots$

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* S. A. Cook (private communication) has solved this problem for $|N| = 4$.

to denote the number of elements in $N, S, T, \ldots$. Let E^N denote the n-dimensional cartesian space of <u>payoff vectors</u>, with coordinates indexed by the elements of N. If $a \in E^N$ and $S \in \mathcal{N}$, we shall often write $a(S)$ for $\Sigma_S a_i$, treating a as an additive set-function on $\mathcal{N}$. Let H_S denote the hyperplane in E^N defined by the equation $a(S) = v(S)$, for all $0 \subset S \subseteq N$.

2. CORES

A point $a \in E^N$ is said to be feasible (for v) if $a(N) \leq v(N)$.* The core of the game is defined as the set C of all feasible $a \in E^N$ such that

$$(5) \qquad a(S) \geq v(S), \qquad \text{all} \quad S \subseteq N.$$

The core is a compact convex polyhedron of dimension at most $n - 1$. It is contained in H_N and is bounded by the intersection of that hyperplane with other hyperplanes H_S, $0 \subset S \subset N$. We define $C_S = H_S \cap C$ for $0 \subset S \subseteq N$, and, formally, $C_0 = C$. Note that it is quite possible for the core to be empty.

A core in which none of the sets C_S are empty will be called complete. A core in which each of the C_S has the highest possible dimension will be called strictly complete. Thus, a strictly complete core is an $(n-1)$-dimensional polyhedron in E^N having exactly $2^n - 2$ faces of dimension $n - 2$.

Figure 1 shows a complete core for $n = 3$ that is not strictly complete. The Frontispiece depicts a strictly complete core for $n = 4$.

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* In application, there is likely to be a lower bound β such that only payoff vectors with $\beta \leq a(N) \leq v(N)$ are truly "feasible". It will be apparent, however, that only the upper bound is of any significance in the theory.

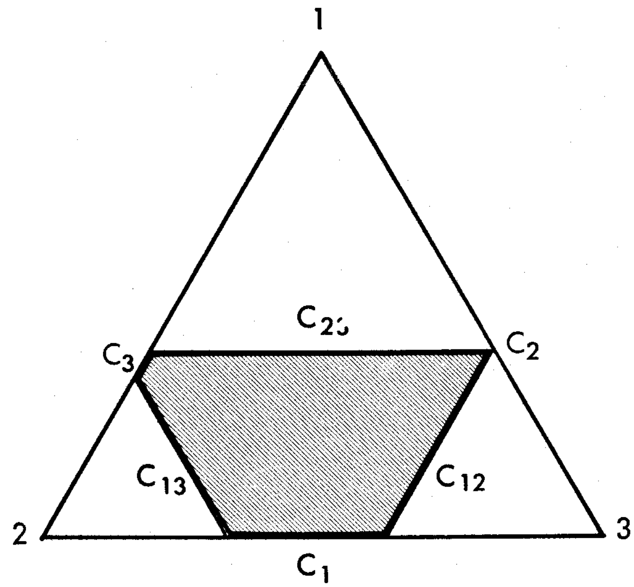

Fig. 1—Core of a 3-person game

2.1 Regular cores

Although the direction cosines of the hyperplanes H_S are independent of v, there is still freedom in the possible incidence relations among the faces of C, even if we know the core to be strictly complete. One "natural" arrangement of faces can be singled out, however, which occurs, for example, when all of the faces are tangent to an inscribed (n—2)-sphere.* In this arrangement two faces touch if and only if their associated player sets are comparable:

(6) $$C_S \cap C_T \neq 0 \quad \text{if and only if} \quad S \subseteq T \quad \text{or} \quad T \subseteq S.$$

We shall call this kind of core strictly regular.

For our immediate purpose, however, the important concept is a slight generalization: let us call a core regular if it is not empty, and satisfies the condition

(7) $$C_S \cap C_T \subseteq C_{S \cap T} \cap C_{S \cup T}, \qquad \text{all} \quad S, T \subseteq N.$$

This condition is tantamount to saying that each of the collections $\mathcal{S}_a = \{S \mid a \in C_S\}$, $a \in E^N$, is closed under union and intersection. We see at once that a strictly regular core is regular.

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* This condition is fulfilled by the strictly convex symmetric measure game $v(S) = s - \sqrt{s(n-s)/(n-1)}$.

The formal similarity between (7) and (3) is not accidental; in due course we shall show that regularity of the core is necessary and sufficient for convexity of the game. First, we shall investigate the structure of regular cores.

2.2 Faces

THEOREM 1. In a regular core we have

$$C_{S_1} \cap C_{S_2} \cap \ldots \cap C_{S_m} \neq 0 \tag{8}$$

for any increasing sequence $S_1 \subset S_2 \subset \ldots \subset S_m \subseteq N$. In particular, a regular core is complete.

The proof proceeds with the aid of two lemmas concerning regular cores. Let us write $S \subset\subset T$ to mean that $S \subset T$ and $|T - S| \geq 2$.

LEMMA 1. If $S \subset\subset T$ and $a \in C_S \cap C_T$, then there exists Q and b such that $S \subset Q \subset T$ and $b \in C_S \cap C_Q \cap C_T$. Moreover, we can require that b differ from a only on $T - S$, and that $j \in Q$, $k \notin Q$ for any two given elements j, k of $T - S$.

Proof. The core is compact, being a closed subset of the simplex bounded by the $n + 1$ hyperplanes $H_{\{i\}}$, $i \in N$, and H_N. Choose $j, k \in T - S$ and define $b_j = a_j - \rho$,

$b_k = a_k + \rho$, and $b_i = a_i$ for all other $i \in N$, with ρ set at the largest possible value compatible with $b \in C$. Clearly $b \in C_S \cap C_T$. Since a larger ρ would have taken b out of the core entirely, there must be a set R, containing j and <u>not</u> containing k, such that $b \in C_R$. If we now set $Q = (S \cup R) \cap T$, the result follows by two applications of (7).

LEMMA 2. Let $0 \subseteq S \subset\subset N$ and $a \in C_S$. Then, for any $j \in N - S$ there exists $b \in C_S \cap C_{S\cup\{j\}}$ such that $b_i = a_i$, all $i \in S$.

<u>Proof</u>. Set $T = N$ and use Lemma 1 to find Q, b such that $S \cup \{j\} \subseteq Q \subset N$ and $b \in C_S \cap C_Q$, the latter agreeing with a on S. If $Q = S \cup \{j\}$ we are through; if not, set $T = Q$ and repeat.

<u>Proof of Theorem 1</u>. We may assume that the sequence $\{S_k\}$ is maximal, i.e., that $m = n + 1$. Then we have $C_{S_1} = C_0$, which is nonempty by definition. Take $a_{(1)} \in C_{S_1}$, and for $k = 1, \ldots, n-1$, apply Lemma 2 to get $a_{(k+1)} \in C_{S_k} \cap C_{S_{k+1}}$, agreeing with $a_{(k)}$ on S_k. Then the $a_{(n)}$ obtained by this construction will be in all of the C_{S_k}, $1 \leq k \leq n$, and it will also be in $C_{S_{n+1}} = C_N = C$. Hence (8) follows.

2.3 Vertices

Let ω represent a simple ordering of the players. Specifically, let ω be one of the $n!$ functions that map N onto $\{1, 2, \ldots, n\}$. Define

(9) $$S_{\omega,k} = \{i \in N | \omega(i) \leq k\}, \qquad k = 0, 1, \ldots, n;$$

these are the "initial segments" of the ordering. The equations of the hyperplanes $H_{S_{\omega,k}}$ for $k = 1, \ldots, n$ are linearly independent; solving them we obtain:

(10) $$a_i^\omega = v(S_{\omega,\omega(i)}) - v(S_{\omega,\omega(i)-1}), \qquad \text{all } i \in N.$$

There is, of course, no assurance that the points a^ω are all distinct.

THEOREM 2. The vertices of a regular core are precisely the points a^ω.

<u>Proof</u>. We have, for each ω,

$$C_{S_{\omega,1}} \cap \ldots \cap C_{S_{\omega,n}} \subseteq H_{S_{\omega,1}} \cap \ldots \cap H_{S_{\omega,n}} = \{a^\omega\}.$$

The left-hand side is not empty, by Theorem 1; it therefore consists of the single point a^ω. Hence a^ω is in the core. If it were not a vertex (i.e., an extreme point), then for some nonzero d we would have both $a^\omega \pm d \in C$.

But this is impossible, since at least one of the hyperplanes $H_{S_{\omega,k}}$ that meet at a^{ω} must pass between the two points $a^{\omega} \pm d$, excluding one from the core.

It remains to show that C has no other vertices. Let a be a vertex and let $\mathscr{S}_a = \{S | a \in C_S\}$. Let $0 = S_1 \subset S_2 \subset \cdots \subset S_m = N$ be a maximal increasing sequence contained in $\mathscr{S}_a$. If there are no "gaps" in this sequence, i.e., if $m = n + 1$, then a is of the form a^{ω}, and we are through. Suppose therefore that $S_k \subset\subset S_{k+1}$, for some k, and take $i, j \in S_{k+1} - S_k$, $i \neq j$. Since a is a vertex, it is the unique solution of the equations $a(S) = v(S)$, all $S \in \mathscr{S}_a$. But in order for us to be able to solve for a_i and a_j, and not just their sum $a_i + a_j$, there must be some $Q \in \mathscr{S}_a$ that contains one of i, j and not the other. Since $\mathscr{S}_a$ is closed under union and intersection, we have $T \in \mathscr{S}_a$, where $T = (S_k \cup Q) \cap S_{k+1}$. But $S_k \subset T \subset S_{k+1}$. This contradicts the assumed maximality of the sequence $\{S_k\}$, and the proof is complete.

2.4 Strictly Regular Cores

This section is mainly descriptive, and proofs are omitted. A strictly regular core can be defined as at (6) above, but it would have sufficed to require that $C \neq 0$ and

(11) $\quad C_S \cap C_T \neq 0$ implies $S \subseteq T$ or $T \subseteq S$, all $S, T \subseteq N$.

To see this, note that (11) implies regularity, which in turn implies the converse of (11) by way of Theorem 1 with $m = 2$.

A strictly regular core is strictly complete, and possesses $n!$ distinct extreme points. A regular core that is not strictly regular, on the other hand, always has fewer than $n!$ extreme points, but it may still be strictly complete, i.e., have the full number of $(n-2)$-dimensional faces.

The combinatorial structure of a strictly regular core is uniquely determined, given the value of n. For example, each $(n-2)$-face C_S is bounded by $2^s + 2^{n-s} - 4$ $(n-3)$-faces of the form $C_S \cap C_T$ with $0 \subset S \subset T$ or $S \subset T \subset N$. The symmetries of this structure are generated by the permutations of N, together with complementation: $S \leftrightarrow N - S$. The latter automorphism maps each vertex a^ω into its antipode $a^{\omega'}$, where ω' reverses ω: $\omega'(i) \equiv n + 1 - \omega(i)$.

We can now distinguish several classes of games, according to the kinds of cores they possess:

games with nonempty cores	games with $(n-1)$-dimensional cores
games with complete cores	games with strictly complete cores
games with regular cores	games with strictly regular cores

Each class includes those listed below it and to the right. For $n > 3$, the inclusions are all strict. If a game is regarded as a point in E^{2^n-1}, then each of the six classes is convex, and those on the left [right] are the closures [interiors] of those on the right [left]. In particular, every regular core is a limit of strictly regular cores — indeed it can be approximated by both increasing and decreasing sequences of strictly regular cores.

3. SOLUTIONS

Many different "solution concepts" have been proposed for n-person games in characteristic-function form. We shall be concerned here with three of them: the core, already defined; the (Shapley) value (Sec. 3.2); and the (von Neumann-Morgenstern) solutions, which we call stable sets (Sec. 3.3). We shall see that the three concepts are closely related in the case of convex games.

3.1 The Core

The core of a game may be interpreted as the set of sociologically stable outcomes, in that no coalition can upset any one of them. In a game with an empty core, at least one set of players must fail to realize its full potential, no matter how the winnings are divided.

THEOREM 3. A game is convex if and only if its core is regular.

LEMMA 3. The core of a convex game is not empty.

Proof. Let v be convex, and let a^{ω} be defined by (9), (10), where ω is a fixed ordering of N (see Sec. 2.3). We shall show that $a^{\omega} \in C$. Let $T \subset N$ and let j be the ω-first element of $N - T$, so that all the ω-predecessors of j are in T, but not j itself. Then we have at once

$$T \cup S_{\omega,\omega(j)} = T \cup \{j\},$$

$$T \cap S_{\omega,\omega(j)} = S_{\omega,\omega(j)-1},$$

using the notation of (9). Hence, by convexity,

$$v(T) + v(S_{\omega,\omega(j)}) \leq v(T\cup\{j\}) + v(S_{\omega,\omega(j)-1})$$

or, by (10),

$$a_j^{\omega} \leq v(T\cup\{j\}) - v(T).$$

Hence,

$$a^{\omega}(T) - v(T) \geq a^{\omega}(T\cup\{j\}) - v(T\cup\{j\}).$$

By induction,

$$a^{\omega}(T) - v(T) \geq a^{\omega}(N) - v(N) = 0.$$

Hence a^{ω} is in the core.

<u>Proof of Theorem 3</u>. Suppose C is regular. Let $S, T \subseteq N$. Since $S \cup T \supseteq S \cap T$, we can find $a \in C_{S\cap T} \cap C_{S\cap T}$, by Theorem 1. Then we have

$$v(S\cup T) + v(S\cap T) = a(S\cup T) + a(S\cap T) = a(S) + a(T) \geq v(S) + v(T).$$

Hence v is convex.

Conversely, suppose that v is convex. Then $C \neq 0$ by Lemma 3, and it remains to establish that $C_S \cap C_T \subseteq C_{S\cup T} \cap C_{S\cap T}$ for all $S, T \subseteq N$. Take any $a \in C_S \cap C_T$. Then we have

$$v(S\cup T) + v(S\cap T) \geq v(S) + v(T) = a(S) + a(T) = a(S\cup T) + a(S\cap T).$$

But we also have $a(S\cup T) \geq v(S\cup T)$ and $a(S\cap T) \geq v(S\cap T)$, because a is in the core. Hence equality prevails, and we have $a \in C_{S\cup T} \cap C_{S\cap T}$, as required.

COROLLARY. A game is strictly convex if and only if its core is strictly regular.

We omit the simple proof.

3.2 The Value

The value of the game v is the payoff vector $\varphi \in E^N$ defined by

$$\varphi_i = \sum_{\substack{S\subseteq N \\ S\ni i}} \frac{(s-1)!\,(n-s)!}{n!} [v(S) - v(S-\{i\})], \quad \text{all } i \in N.$$

We have $\varphi(N) = v(N)$ and, for superadditive games, $\varphi_i \geq v(\{i\})$ for all $i \in N$. The quantity φ_i may be interpreted as the "equity value" associated with being player i in the game v.

THEOREM 4. The value of a convex game is in the core.

Proof. It is well known (see [4]) that

(12) $$\varphi = \frac{1}{n!} \sum_{\omega\in\Omega} a^{\omega},$$

where Ω is the set of all orderings of N. Hence, by Theorems 2 and 3, the value is in the core, since it is a center of gravity of extreme points of the core.

For completeness, we sketch a proof of (12). Fix i. Given $S \ni i$, we ask: for how many orderings $\omega \in \Omega$ do we have $S_{\omega,\omega(i)} = S$? It is clearly both necessary and sufficient that ω be such that $\omega(j) < \omega(i)$ for all $j \in S - \{i\}$ and $\omega(j) > \omega(i)$ for all $j \in N - S$. Thus the number of orderings is $(s-1)!(n-s)!$. The result now follows easily.

3.3 Stable Sets

A feasible payoff vector b is said to be dominated by a payoff vector a (not necessarily feasible) if there is a set $S \in \mathcal{N}$ such that

(13) $$a(S) \leq v(S)$$

and

(14) $a_i > b_i$, all $i \in S$.

A set V of feasible* payoff vectors is said to be <u>stable</u> if every feasible* payoff vector is either a member of V or dominated by a member of V, but not both. A stable set may be interpreted as "a standard of behavior," or a system of "conventional" outcomes, against which any proposal that is put forward during the negotiation period is automatically tested for domination.

It is easily verified that the core is precisely the set of undominated payoff vectors, in the present sense, and hence that every stable set contains the core. Since no stable set can properly include another, it follows that if the core is stable then it is the only stable set.**

THEOREM 5. The core of a convex game is stable (i.e., is the unique Neumann-Morgenstern solution).

<u>Proof</u>. Let C be regular, and take any feasible $b \in E^N - C$. We shall show that b is dominated by an

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* The classical definition replaces "feasible" in these two places by "feasible and individually rational", where $a \in E^N$ is <u>individually rational</u> if and only if $a_i \geq v(\{i\})$, all $i \in N$. This distinction (discussed at length in [5]) makes no difference in the present case, since the core is stable in the one sense if and only if it is stable in the other. (See e.g., Theorems 12 and 13 of [3].)

** It has been conjectured that stable sets exist for all superadditive games, and that the core is the intersection of all stable sets.

element of C. Define $g(0) = 0$ and

$$g(S) = \frac{v(S) - b(S)}{s}, \qquad \text{all } 0 \subset S \subseteq N,$$

and let g attain its maximum, g_o, at $S = S_o$. Since $b \notin C$, we see that $g_o > 0$. Let c be any point in C_{S_o} (a regular core is complete), and define the point $a \in E^N$ by:

$$a_i = \begin{cases} b_i + g_o, & i \in S_o, \\ c_i, & i \in N - S_o. \end{cases}$$

Since $a(S_o) = b(S_o) + s_o g_o = v(S_o)$, conditions (13) and (14) are satisfied, and we see that a dominates b. Moreover, since $a(N) = v(S_o) + c(N - S_o) = v(N)$, we see that a is feasible. It remains to show that a satisfies the core inequalities (5).

Let $T \subseteq N$ be arbitrary, and break up T into $Q = T \cap S_o$ and $R = T - S_o$. Then we have, by easy steps,

$$\begin{aligned} a(T) &= a(Q) + a(R) = b(Q) + qg_o + c(R) \\ &\geq b(Q) + qg(Q) + c(R) \\ &= v(Q) + c(R) \\ &= v(Q) + c(T \cup S_o) - c(S_o) \\ &\geq v(Q) + v(T \cup S_o) - v(S_o) \\ &\geq v(T). \end{aligned}$$

Hence a is in the core. Q.E.D.

3.4 Remarks

An example of a stable core that is not regular, or even complete, is provided by the following nonconvex four-person game:

$$v(S) = 0,\ 0,\ 1,\ 1,\ 3 \quad \text{for} \quad |S| = 0,\ 1,\ 2,\ 3,\ 4 \text{ respectively.}$$

The core is a perfect cube, with vertices $(1, 1, 1, 0)$, $(1/2, 1/2, 1/2, 3/2)$, etc. (Figure 2). Stability is easily verified. The square faces are the sets C_S, $|S| = 2$. The sets C_S, $|S| = 3$ are empty, since each three-person coalition gets at least $3/2$ in the core.

We have obtained a complete theory of core stability in symmetric games — i.e., games that are invariant under all permutations of the players. This will be presented elsewhere.

It follows from Theorem 5 that the set of solvable games is full dimensional. This was first pointed out by Gillies ([2], Corollary 30; see also [3], Theorem 25, and [1], Theorems 4.2 and 4.3), who proved core stability for another class of games: those whose normalizations (see Sec. 1.1) satisfy

$$\begin{cases} v(S) \leq v(N)/n, & \text{all} \quad S \subset N, \\ v(N) \geq 0. & \end{cases}$$

This includes some but not all convex games, as well as some that are not convex.

|S|	v(S)
0	0
1	0
2	1
3	1
4	3

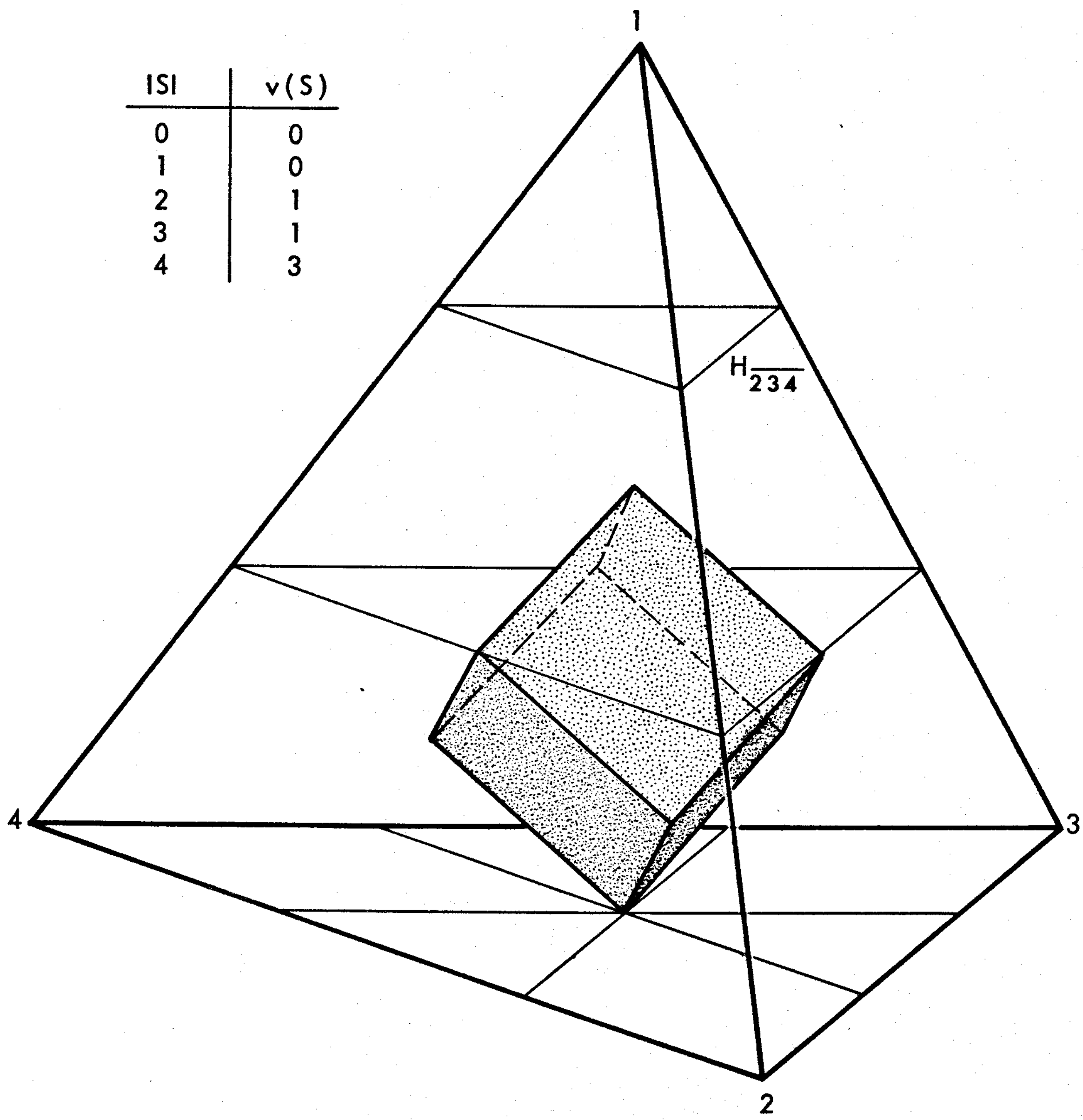

Fig. 2—Example of a stable, nonregular core

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