Cross product in complex vector spaces

When inner product is defined in complex vector space, conjugation is performed on one of the vectors. What about is the cross product of two complex 3D vectors? I suppose that one possible generalization is $A \otimes B \rightarrow (A \times B)^*$ where $\times$ denotes the normal cross product. The conjugation here is to ensure that the result of the cross product is orthogonal to both vectors A and B. Is that correct ?

(linear-algebra) (complex-numbers) (complex-geometry) (cross-product)

edited Apr 8 '12 at 6:37 asked Apr 8 '12 at 6:18
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103 1 12

2 Maybe it would be natural to generalize the cross product after viewing it in a sufficiently abstract setting, for example as the Hodge dual of the wedge product? I don't know enough to say whether this would apply directly to $\mathbb{C}^3$ though. – Rahul Apr 8 '12 at 7:33

2 Answers

For finding the correct definition to apply, one needs to know whether the scalar product is taken to be anti-linear in its first or its second argument. Assuming the first convention, the relation one would want to preserve for $\vec{x} = (x_1, x_2, x_3)$ and similarly for $\vec{y}, \vec{z}$ is that one still has

$$(\vec{x} \times \vec{y}) \cdot \vec{z} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Note that the determinant is linear in all of its columns, so the left hand side needs to be an expression that is linear in the vector that appears directly as a column, which explains that one cannot use $\vec{x} \cdot (\vec{y} \times \vec{z})$ instead, which is anti-linear in $\vec{x}$. Now it is easy to see that the coordinates of $\vec{x} \times \vec{y}$ should be taken to be the complex conjugates of the expressions in their usual defintion, for instance $\overline{x_2 y_3 - x_3 y_2}$ for the first coordinate.

One actually arrives at the same conclusion for a scalar product that is defined to be anti-linear in its second argument. However the identity that leads to this definition is different, namely the one which equates $\vec{x} \cdot (\vec{y} \times \vec{z})$ to the above determinant.

edited Nov 2 '14 at 8:17 answered Apr 8 '12 at 8:20
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The meaning of triple product $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ of Euclidean 3-vectors is the volume form (SL(3, $\mathbb{R}$) invariant), that gets an expression through dot product (O(3) invariant) and cross product (SO(3) invariant, a subgroup of SL(3, $\mathbb{R}$)). We can complexify all the stuff (resulting in SO(3, $\mathbb{C}$)-invariant vector calculus), although we will not obtain an inner product space. But if we generalize "·" as a sesquilinear form, then its underlying symmetry becomes U(3), whereas proposed generalization of the triple product is still ruled by SL(3, $\mathbb{C}$). This seemingly leads to an SU(3) vector calculus. – Incnis Mrsi Nov 1 '14 at 18:33

Yes, this is correct definition. If v, w are perpendicular vectors in $\mathbb{C}^3$ (according to hermitian product) then $v, w, v \times w$ form matrix in SU_3.

We can define complex cross product using octonion multiplication (and vice versa). Let's use Cayley-Dickson formula twice:

$$(a + b^\iota)(c + d^\iota) = ac - \bar{d}\, b + (b\bar{c} + da)^\iota$$

for quaternions a, b, c, d. Next set $a = u\mathbf{j}, b = v + w\mathbf{j}, c = x\mathbf{j}, d = y + z\mathbf{j}$ for complex numbers u, v, w, x, y, z . Then we obtain from above formula

$$-u\bar{x} - v\bar{y} - \bar{w}z + (\bar{v}z - w\bar{y})\mathbf{j} + [w\bar{x} - \bar{u}z + (-vx + uy)\mathbf{j}]\iota$$

Applying complex conjugation to third complex coordinate we obtain formula for cross product. The first term is hermitian product of the vectors (u, v, w), (x, y, z).

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \overline{vz} - \overline{wy} \\ \overline{wx} - \overline{uz} \\ \overline{uy} - \overline{vx} \end{bmatrix}$$

SU_3 is subgroup of octonion automorphism group G_2. Any automorphism of octonions can be obtained by fixing unit vector $\mathbf{i}$ on imaginary sphere S^6. It defines complex structure on perpendicular space R^6 via multiplication. Now in this complex structure any SU_3 element is octonion automorphism. So G_2 is fiber bundle $S^6 \times \times SU_3$.

Now going to "vice versa". Let's define octonions as pairs $(a, \mathbf{v})$ where a is complex number and $\mathbf{v}$ vector in $\mathbb{C}^3$. Then octonion multiplication can be defined as

$$(a, \mathbf{v})(b, \mathbf{w}) = (ab - \mathbf{v} \cdot \mathbf{w}, a\mathbf{w} + b\mathbf{v} + \mathbf{v} \times \mathbf{w})$$

I hope that above argument with double Cayley-Dickson formula can be used to prove it although I have not done myself this calculation. The reader is urged to do it as an exercise :)

We can extend the definition of cross product to quaternions the same way. Extending it to octonions we need to be more careful. Freudenthal has done this using 3x3 matrices over octonions - so called Jordan algebra. Some kind of "cross product" is present in all exceptional Lie groups F_4, E_6, E_7, E_8 as these groups are called by Rosenfeld as automorphism groups of 2-dimensional projective planes over $\mathbb{O}, \mathbb{C} \otimes \mathbb{O}, \mathbb{H} \otimes \mathbb{O}, \mathbb{O} \otimes \mathbb{O}$. Have I flied away too far from original question ?

edited Mar 13 at 9:15

answered Mar 13 at 8:44
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