

Union-additive multimeasures and self-similarity

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Abstract

In this paper, we study some IFS Markov operators on set-valued measures (multimeasures). Unlike the usual multimeasures in the literature, the multimeasures considered are additive with respect to the union operation on sets. This provides an (somewhat unusual) alternative class of multimeasures and it can be considered as an extension of non-additive measures to set-valued measures. After defining appropriate metrics and proving completeness, we define several variations on IFS Markov operators on multimeasures and give many examples.

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Dedicated to the memory of Professor Bruno Forte

1 Introduction

Iterated Function Systems are one very nice way to formalize the notion of self-similarity or scale invariance of some mathematical object. Hutchinson [10] and Barnsley and Demko [2] showed how systems of contractive maps with associated probabilities – called “iterated function systems” by the latter – acting in a parallel manner either deterministically or probabilistically, can be used to construct fractal sets and measures. In the IFS literature, these are called IFS with probabilities (IFSP) and are based on the action of a contractive Markov operator on the complete metric space of all probability measures endowed with the Monge-Kantorovich metric. Applications of these methods can be found in image compression, approximation theory, signal analysis, denoising, and density estimation (see [4, 5, 6, 12]).

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IFS Markov operators on probability measures yield self-similar measures (in a generalized sense) which are invariant with respect to the action of the functions in the IFS. One way of thinking about usual IFS Markov operators for self-similar measures is that they provide a way of recursively partitioning the total probability over the space.

To explain this, we give a brief review of the construction of classical IFS invariant measures. Let Ω be a complete metric space and \mathcal{B} be the Borel σ -algebra for Ω . In addition, let $w_i : \Omega \rightarrow \Omega$, for $i = 1, 2, \dots, N$, be a collection of contractive mappings and p_i be an associated collection of probabilities (so that $p_i \geq 0$ and $\sum_i p_i = 1$). Then the Markov operator associated to this IFS is defined as

$$M\mu(B) = \sum_i p_i \mu(w_i^{-1}(B))$$

for any probability measure μ on (Ω, \mathcal{B}) and any Borel $B \in \mathcal{B}$. The result of this operator is to assign probability p_i to the set $w_i(\Omega)$. Upon a second application of M , we now have assigned probability $p_i p_j$ to the set $w_i(w_j(\Omega))$, and a third application assigns probability $p_i p_j p_k$ to the set $w_i(w_j(w_k(\Omega)))$, and so on. This clearly is recursively partitioning the total probability over Ω (in the limit, the measure is concentrated on the fractal defined by the w_i 's). Our wish is to generalize this process to set-valued measures. That is, to recursively partition some set over some other set. We give many examples of this process.

In [11], the authors construct a similar IFS framework for the more standard type of multimeasure – those set-valued functions which are additive with respect to Minkowski addition of sets. That is, where

$$\mu\left(\bigcup_i A_i\right) = \left\{\sum_i y_i : y_i \in \mu(A_i)\right\}$$

for disjoint A_i . Using Minkowski addition is a more direct generalization of the notion of the usual scalar-valued (or vector-valued) measure. Changing to using the union as the addition substantially changes the properties of the multimeasure. For instance, it is possible for $\mu(A \cup B) = \mu(A)$, even if $\mu(B)$ is not trivial. The analogy with positive measures would be to take $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ instead of the usual $\mu(A \cup B) = \mu(A) + \mu(B)$. The current paper is complementary to [11].

2 Union-additive multimeasures

In this section we define multimeasures which are additive with respect to the union operation. This notion can be considered as an extension of non-additive measures to the case of set-valued measures. Non-additive measures appeared in classical measure theory concerning countably additive set functions and, in the literature, it is well recognized that the pioneer of this theory was G.Choquet in his theory of capacities (see [3]). As pointed out by Pap in [14], one mathematical property which illustrates the difference between additive and non-additive

measures is the following: for a fixed A and an additive measure μ , the difference $\mu(A \cup B) - \mu(B)$ is constant for all sets B such that $A \cap B = \emptyset$. This is in general not true for non-additive measures, for which the difference depends on B ; this can be interpreted as the effect of B joining A .

There are many applications of set functions to different areas such as mathematical economics, decision theory and social sciences; for this reason, many variations on and extensions of measures have been provided in the literature including, for instance, subadditive and superadditive set functions, submeasures, null-additive set functions and so on. The notion of union-additive multimeasure, provided in this paper, is one of these possible generalizations which consider set-valued set functions instead of set functions. It is also a generalization of fuzzy measures and fuzzy set-valued measures [9, 15]. Some practical motivations behind this kind of mathematical structures can be found, for instance, in mathematical economics, when coalitions are considered as primitive economic units (see [16]).

We provide a metric for such multimeasures, a definition of an IFS Markov operator on these multimeasures and show contractivity for this Markov operator. We use the Hausdorff metric to induce a metric on these multimeasures.

Let Ω be a complete metric space with \mathcal{B} its Borel σ -algebra. In addition, let X be another complete metric space and $\mathcal{H}(X)$ be the collection of all non-empty compact subsets of X under the Hausdorff metric d_H .

For a set $B \subset X$, we notate the ϵ -dilation of B as $B_\epsilon = \{x \in X : d(x, B) < \epsilon\}$. Recall that one way of characterizing the Hausdorff distance is

$$d_H(A, B) < \epsilon \text{ iff } A \subset B_\epsilon \text{ and } B \subset A_\epsilon.$$

We begin with a simple proposition.

Proposition 1. *Let $A_n \subset X$ be compact and suppose that $S = \overline{\bigcup_{n=1}^{\infty} A_n}$ is compact. Let $S_n = \bigcup_{i=1}^n A_i$. Then $S_n \rightarrow S$ in the Hausdorff metric.*

Proof. Let $\epsilon > 0$ be fixed. Obviously for all n

$$S_n \subset S \subset S_\epsilon.$$

On the other hand, the collection $\{(S_n)_\epsilon\}_{n \in \mathbb{N}}$ is an open cover for the compact set S , so there is some n_0 so that

$$S \subset (S_{n_0})_\epsilon$$

for all $n \geq n_0$. □

For us, a multimeasure is a set-valued function $\phi : (\Omega, \mathcal{B}) \rightarrow \mathcal{H}(X)$ with the one exception that we require that $\phi(\emptyset) = \emptyset$. Thus, $\phi(B) \subset X$ is compact and non-empty for all non-empty B .

We must specify the sense in which these multimeasures are additive. So, let $A_i \in \mathcal{B}$ be pairwise disjoint. We require that any multimeasure ϕ satisfy

$$\phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_n \bigcup_{i=1}^n \phi(A_i) = \overline{\bigcup_{i=1}^{\infty} \phi(A_i)} \quad (1)$$

where we take the limit in the Hausdorff metric. This is necessary, since in general a countable union of compact sets need not be compact. Notice that since $\phi(\Omega) \in \mathcal{H}(X)$ and $\phi(A_i) \subset \phi(\Omega)$ we have $\bigcup_i \phi(A_i) \subset \phi(\Omega)$ and thus the closure above will result in a compact set. This and Proposition 1 explains why the closure is the Hausdorff limit.

We comment that it is equivalent to ask that countable union-additivity, equation (1), holds for any sets A_i , not necessarily only pairwise disjoint. This is also quite different from the case of usual measures where in general additivity holds only for disjoint sets.

Definition 2.1. Let $UA(\Omega, X)$ denote the collection of countably union-additive multimeasures with $\phi(\emptyset) = \emptyset$ and $\phi(A) \in \mathcal{H}(X)$ for all nonempty $A \in \mathcal{B}$.

There is one easy general way to construct a measure $\phi \in UA(\Omega, X)$. Let $f : \Omega \rightarrow X$ be any function with $\overline{f(\Omega)} \subset X$ compact and define $\phi(A) = \overline{f(A)}$. Then $\phi(\emptyset) = \emptyset$ and for any set $A \subset \Omega$, we have $\phi(A) \in \mathcal{H}(X)$. Furthermore, $\overline{f(A \cup B)} = \overline{f(A)} \cup \overline{f(B)}$ and so by Proposition 1,

$$\phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \overline{f\left(\bigcup_{i=1}^{\infty} A_i\right)} = \lim_N \bigcup_{i=1}^N \overline{f(A_i)} = \lim_N \bigcup_{i=1}^N \phi(A_i) = \overline{\bigcup_{i=1}^{\infty} \phi(A_i)},$$

and thus ϕ is countably union-additive. One can even use a multifunction $f : \Omega \rightrightarrows X$ and define

$$\phi(A) = \overline{\bigcup_{a \in A} f(a)}.$$

Not every $\phi \in UA(\Omega, X)$ is of this form, however. As an example, consider $X = \Omega = \mathbb{R}$ with $\phi(\emptyset) = \emptyset$, and $\phi(F) = \{1\}$ for any countable set F , and $\phi(A) = [0, 1]$ for any uncountable Borel set A . In Section 2.1 we provide another general construction method using the theory of IFS.

$UA(\Omega, X)$ as a Complete Metric Space

We now define a metric on $UA(\Omega, X)$ to turn it into a complete metric space. We define

$$\hat{d}_H(\phi_1, \phi_2) = \sup_{\emptyset \neq A \in \mathcal{B}} d_H(\phi_1(A), \phi_2(A)) \quad (2)$$

(thus, \hat{d}_H is a metric of uniform convergence for measures in $UA(\Omega, X)$).

Theorem 1. The metric \hat{d}_H makes $UA(\Omega, X)$ a complete metric space.

Proof. Let ϕ_n be a Cauchy sequence of measures in $(UA(\Omega, X), \hat{d}_H)$. We define the set-valued function ϕ by taking $\phi(\emptyset) = \emptyset$ and for each non-empty $A \in \mathcal{B}$, $\phi(A) = \lim_n \phi_n(A)$ in the Hausdorff metric on X . Since $\mathcal{H}(X)$ is complete under the Hausdorff metric, this limit exists.

We must show that ϕ is an element of $UA(\Omega, X)$. Clearly for each non-empty A we have that $\phi(A)$ is non-empty and compact, since each $\phi_n(A)$ is non-empty and compact and the Hausdorff limit preserves these properties.

Let $A_i \in \mathcal{B}$ be pairwise disjoint and non-empty.

From the hypotheses, we have that $\forall \epsilon > 0$ there exists $m_0(\epsilon)$ such that $d_h(\phi_m(A), \phi(A)) < \epsilon/2$ for all $m \geq m_0$. But this means that $\phi_m(A) \subset (\phi(A))_{\epsilon/2}$ and $\phi(A) \subset (\phi_m(A))_{\epsilon/2}$. So now these inclusions are true for all A_i with the same m_0 . Then

$$\overline{\bigcup_{i=1}^{\infty} \phi_m(A_i)} \subset \overline{\bigcup_{i=1}^{\infty} (\phi(A_i))_{\epsilon/2}} \subset \left(\overline{\bigcup_{i=1}^{\infty} (\phi(A_i))} \right)_{\epsilon}$$

and

$$\overline{\bigcup_{i=1}^{\infty} \phi(A_i)} \subset \overline{\bigcup_{i=1}^{\infty} (\phi_m(A_i))_{\epsilon/2}} \subset \left(\overline{\bigcup_{i=1}^{\infty} (\phi_m(A_i))} \right)_{\epsilon}.$$

This means that $d_H(\overline{\bigcup_{i=1}^{\infty} \phi(A_i)}, \overline{\bigcup_{i=1}^{\infty} \phi_m(A_i)}) < \epsilon$ and thus

$$\lim_n \lim_m \bigcup_{i=1}^n \phi_m(A_i) = \lim_m \lim_n \bigcup_{i=1}^n \phi_m(A_i).$$

But this gives

$$\begin{aligned} \phi\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_m \phi_m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_m \lim_n \phi_m\left(\bigcup_{i=1}^n A_i\right) \\ &= \lim_m \lim_n \bigcup_{i=1}^n \phi_m(A_i) = \lim_n \lim_m \bigcup_{i=1}^n \phi_m(A_i) \\ &= \lim_n \bigcup_{i=1}^n \lim_m \phi_m(A_i) = \lim_n \bigcup_{i=1}^n \phi(A_i) \end{aligned}$$

which is what we wished to prove. \square

There is another useful way to look at union-additive multimeasures. The collection \mathcal{B} of Borel subsets of Ω ordered under inclusion is a lattice, as is $\mathcal{H}(X)$. A *join homomorphism* from \mathcal{B} to $\mathcal{H}(X)$ is an order-preserving map $\phi : \mathcal{B} \rightarrow \mathcal{H}(X)$ for which $\phi(A \cup B) = \phi(A) \cup \phi(B)$. This is the same condition as that for a finitely union-additive multimeasure. Thus, a countably union-additive multimeasure is simply a join homomorphism which preserves countable joins.

2.1 IFS Markov Operator

We now define an IFS Markov operator on $UA(\Omega, X)$. Let $w_i : \Omega \rightarrow \Omega$ for $i = 1, 2, \dots, N$ map Borel sets to Borel sets (so continuous is more than enough) and $T_i : \mathcal{H}(X) \rightarrow \mathcal{H}(X)$ be contractions with contractivity factor k_i and such that

$$T_i\left(\overline{\bigcup_{n=1}^{\infty} A_n}\right) = \overline{\bigcup_{n=1}^{\infty} T_i(A_n)}$$

for disjoint A_n . This condition on T_i is easily met, for example if $T_i(A) = t_i(A)$ for some continuous $t_i : X \rightarrow X$. We are not requiring any of the w_i to be contractive.

Using this, we define two different types of operators $M : UA(\Omega, X) \rightarrow UA(\Omega, X)$. For the first one we assume that $\bigcup_i w_i(\Omega) = \Omega$, so that Ω is the attractor of the IFS $\{w_i\}$. In particular, for any non-empty $S \in \mathcal{B}$, we have $w_i^{-1}(S) \neq \emptyset$ for at least one i . In this case we define

$$M_1\phi(B) = \bigcup_{w_i^{-1}(B) \neq \emptyset} T_i(\phi(w_i^{-1}(B))), \quad \emptyset \neq B \in \mathcal{B}. \quad (3)$$

For the second type we make no additional assumption on the maps w_i but in addition we take some fixed $\psi \in \mathcal{H}(X)$ and define

$$M_2\phi(B) = \psi(B) \cup \bigcup_{w_i^{-1}(B) \neq \emptyset} T_i(\phi(w_i^{-1}(B))), \quad \emptyset \neq B \in \mathcal{B}. \quad (4)$$

In both cases we see that since this is a finite union we have no need to take a Hausdorff limit of the union.

Clearly $M_j\phi(\emptyset) = \emptyset$ for either Markov operator. Take $\emptyset \neq S \in \mathcal{B}$. Then in (3), we have $w_i^{-1}(S) \neq \emptyset$ for at least one i and thus $M_1\phi(S) \neq \emptyset$. For the M_2 as defined in (4), we have $\psi(S) \neq \emptyset$ so again $M_2\phi(S) \neq \emptyset$. The fact that $M_j\phi$ is countably union-additive follows from the assumptions on T_i . Thus in either case $M_j : UA(\Omega, X) \rightarrow UA(\Omega, X)$.

Theorem 2. *Let $k = \max_i k_i$ be the maximum of the contractivities of the operators T_i . Then*

$$\hat{d}_H(M_1\phi_1, M_1\phi_2) \leq k\hat{d}_H(\phi_1, \phi_2).$$

Proof. Let $A \in \mathcal{B}$. Then starting with M_1 we have

$$\begin{aligned} d_H(M_1\phi_1(A), M_1\phi_2(A)) &= d_H\left(\bigcup_i T_i(\phi_1(w_i^{-1}(A))), \bigcup_i T_i(\phi_2(w_i^{-1}(A)))\right) \\ &\leq \max_i d_H(T_i(\phi_1(w_i^{-1}(A))), T_i(\phi_2(w_i^{-1}(A)))) \\ &\leq k \max_i d_H(\phi_1(w_i^{-1}(A)), \phi_2(w_i^{-1}(A))) \\ &\leq k \hat{d}_H(\phi_1, \phi_2) \end{aligned}$$

and thus

$$\hat{d}_H(M_1\phi_1, M_1\phi_2) \leq k\hat{d}_H(\phi_1, \phi_2),$$

as desired.

For M_2 , we notice that $d_H(A \cup S, B \cup S) \leq \max\{d_H(A, B), d_H(S, S)\} = d_H(A, B)$ for any $A, B, S \in \mathcal{H}(X)$ and thus the contractivity factor for M_2 is similarly obtained. \square

2.2 Examples

Example 2.1.

Our first example is one where the values of the fractal multimeasure are subsets of some fractal. Let $\Omega = [0, 1]$ and $X = [0, 1] \times [0, 1]$. We let $w_i : [0, 1] \rightarrow [0, 1]$ be defined by $w_i(x) = x/3 + i/3$ for $i = 0, 1, 2$ and $T_i : X \rightarrow X$ be defined as

$$T_0(x, y) = (x/2, y/2), \quad T_1(x, y) = (x/2 + 1/2, y/2), \quad T_2(x, y) = (x/2, y/2 + 1/2).$$

Since $[0, 1] = w_0([0, 1]) \cup w_1([0, 1]) \cup w_2([0, 1])$, the maps w_i satisfy the conditions necessary to define M_1 using equation (3).

The attractor for the IFS $\{w_0, w_1, w_2\}$ is $[0, 1]$ while the attractor for the IFS $\{T_0, T_1, T_2\}$ is the classical Sierpinski Gasket, G .

Using these to define a Markov operator, we get the fixed point multimeasure ϕ . This multimeasure has the property that for all Borel $S \subset [0, 1]$,

$$\phi(S) = \{x \in G : \text{the address of } x \text{ (represented in ternary)} \in S\}.$$

Example 2.2.

For our next example, we take $\Omega = X = [0, 1]$ and $w_0(x) = x/3$ and $w_1(x) = x/3 + 2/3$ (so that the attractor of the IFS $\{w_0, w_1\}$ is the classical Cantor Set) and $T_0(x) = px$ and $T_1(x) = (1 - p)x + p$ (so that the attractor of the IFS $\{T_0, T_1\}$ is $[0, 1]$), for some $0 < p < 1$.

As the attractor of $\{w_0, w_1\}$ is the Cantor set, we must use equation (4) to define the Markov operator. We are free to choose any $\psi \in UA(\Omega, X)$ for this and for each different choice we are likely to get a different fixed point. A simple choice is to select some fixed $S_0 \in \mathcal{H}(X)$ and define $\psi(B) = S_0$ for all non-empty B . For a specific choice, let $S_0 = \{0\}$.

In this case, the fixed point multimeasure ϕ has the property that $\phi(S) \subset [0, 1]$ and the Lebesgue measure λ of $\phi(S)$ could be thought of as some sort of “probability” of the set S . For instance, $\lambda(\phi([0, 1/3])) = \lambda([0, p]) = p$ and $\lambda(\phi([1/3, 2/3])) = \lambda(\emptyset) = 0$ and $\lambda(\phi([2/3, 1])) = \lambda([p, 1]) = 1 - p$.

Further, $\{0\} \subset \phi(B)$ for any non-empty B and $\phi(B) = \{0\}$ if the intersection of B with the Cantor set is empty.

3 Further generalizations

In the most basic sense, an IFS with probabilities is recursively partitioning probability (total mass of one) over some recursively defined set. So, an ultimate generalization of an IFS with probability would be to define some set of operators which perform a recursive partition. We briefly outline this generalization and give some examples of general constructions along this line.

Let Ω and X be sets. Our multimeasures will take as input subsets in some algebra \mathcal{A} of subsets of Ω (an algebra, not necessarily a σ -algebra). The

output of the multimeasure will be subsets of X , which will represent parts of a recursively defined partition of X .

So, let $w_i : \Omega \rightarrow \Omega$ be (somewhat arbitrary) functions and $T_i : Pow(X) \rightarrow Pow(X)$ be functions which satisfy

1. $\bigcup_i T_i(S) = S$, for all $S \in \mathcal{A}$.
2. $T_i(S) \cap T_j(S) = \emptyset$ for all $S \in \mathcal{A}$, $i \neq j$.
3. $T_i(A \cup B) = T_i(A) \cup T_i(B)$ for all $A, B \in \mathcal{A}$ and i .

Clearly properties 1 and 2 ensure that $\{T_i(S)\}$ is a partition of S for any appropriate S . Property 3 will ensure that if ϕ is union-additive then so is $M\phi$.

Define our “operator” M by

$$M\phi(B) = \bigcup_{i=1}^N T_i(\phi(w_i^{-1}(B))). \quad (5)$$

Then any fixed point of M is a self-similar recursively defined partition-valued multimeasure. In particular, once we have specified the value of $\phi(\Omega)$, equation (5) will recursively define the values of the fixed point ϕ on sets of the form

$$w_{i_1} \circ w_{i_2} \circ \dots \circ w_{i_k}(\Omega), \text{ some } i_1, i_2, \dots, i_k \in \{1, 2, \dots, N\} \quad (6)$$

and thus any fixed point of M will be uniquely defined on the algebra of sets generated by sets of this form. However, in general there is no reason to believe that a fixed point ϕ is countably union-additive.

We will say that an algebra \mathcal{A} of subsets of Ω is $\{w_i\}$ -compatible if for all $A \in \mathcal{A}$ and all sequences $i_1, i_2, \dots, i_k \in \{1, 2, \dots, N\}$ we have

$$w_{i_1} \circ w_{i_2} \circ \dots \circ w_{i_k}(A) \in \mathcal{A}.$$

Choosing some $\{w_i\}$ -compatible algebra \mathcal{A} , define $UFA(\Omega, X)$ as the collection of all finitely union-additive multimeasures from (Ω, \mathcal{A}) to $Pow(X)$. With this notation, it is clear that M as defined in equation (5) defines a linear operator $M : UFA(\Omega, X) \rightarrow UFA(\Omega, X)$.

We end with two examples which are variations on this type of generalization.

Example 3.1.

Let $\Omega = X = [0, 1]$, $w_i(x) = x/2 + i/2$ and $T_i(x) = x/2 + i/2$ for $i = 0, 1$. Then the fixed point multimeasure ϕ satisfies $\phi(S) = S$ for any $S \subset [0, 1]$. This is clearly a rather silly example. It can be made slightly more interesting by defining $T_0(x) = x/2 + 1/2$ and $T_1(x) = x/2$. To describe the invariant multimeasure ϕ we need to define an auxiliary function. Let $\tau : [0, 1] \rightarrow [0, 1]$ be defined by the i th binary digit of $\tau(x)$ is 1 iff the i th binary digit of x is 0 (and similarly for 0 and 1). That is, τ “flips” all the binary digits of x . Using τ , we see that $\phi(S) = \tau(S)$. Actually, these situations fall into the framework

of Section 2 with each T_i contractive with contractivity factor $1/2$ and thus the fixed point is unique and is a countably union-additive multimeasure.

Another slightly different example is $\Omega = [0, 1]$ with the same w_i but $X = \mathbb{N} \cup \{0\}$, the set of whole numbers. Then we define $T_0(x) = 2x$ and $T_1(x) = 2x+1$ (so that $T_0(X) = 2X$ and $T_1(X) = 2X + 1$, the set of even and odd numbers respectively). The operator M in this case first partitions X into even and odd, then into the modular four classes, then modular eight, modular sixteen, etc.

Interestingly, this example can also be put into the framework from Section 2. To do this, we recall some facts about *2-adic* numbers (see [8]). We place the metric d_2 on X by setting

$$d_2(n, m) = 2^{-k}, \text{ whenever } n - m = a2^k \text{ with } \gcd(a, 2) = 1.$$

Using this metric, $d_2(T_i(n), T_i(m)) = (1/2)d_2(n, m)$, so the two maps T_i are contractive. However, X is not complete under this metric. The completion of X is contained in the set of *2-adic integers*, call this \mathbb{Z}_2 , which can be viewed as formal infinite sums of the form $\sum_{n=1}^{\infty} a_n 2^n$. It turns out that \mathbb{Z}_2 is not only complete but also compact.

The two maps T_i can be extended naturally to \mathbb{Z}_2 and $T_0(\mathbb{Z}_2)$ and $T_1(\mathbb{Z}_2)$ form a partition of \mathbb{Z}_2 just as $T_0(X)$ and $T_1(X)$ formed a partition of X . Thus, the unique fixed point of the operator M is a countably union-additive multimeasure with values in $\mathcal{H}(\mathbb{Z}_2)$.

Example 3.2.

For this example, we change our operator slightly. Let Ω be a complete metric space and \mathcal{H} be a Hilbert Space. We let $w_i : \Omega \rightarrow \Omega$ be contractive and $T_i : \mathcal{H} \rightarrow \mathcal{H}$ be continuous linear and injective and satisfy $\sum_i T_i = I$ (the identity). These conditions ensure that the T_i 's "partition" the space \mathcal{H} into an orthogonal sum of the subspaces $T_i(\mathcal{H})$.

Our multimeasures will be defined on the Borel σ -algebra of Ω and take values as closed linear subspaces of \mathcal{H} . We define the IFS "operator" M by

$$M\phi(B) = \sum_i T_i(\phi(w_i^{-1}(B))).$$

As a more concrete example of this situation, we let $w_i(x) = x/2 + i/2$ for $i = 0, 1$ and $\mathcal{H} = L^2[0, 1]$ and $T_i(f) = f \circ w_i^{-1}$ (so that $T_0(f)$ is supported on $[0, 1/2]$ and $T_1(f)$ is supported on $[1/2, 1]$). The fixed point of M then is the multimeasure

$$\phi(S) = \{f \in L^2[0, 1] : f \text{ is supported on } S\}.$$

Another slightly more interesting example of this construction is where we again have $w_i(x) = x/2 + i/2$, but we have $\mathcal{H} = L^2(\mathbb{R})$ and T_0 be the "high pass filter" and T_1 be the "low pass filter" from a two-scale MRA (Multi-Resolution Analysis) associated with a wavelet basis. These T_0, T_1 satisfy the recursive partitioning conditions, so the invariant multimeasure ϕ is a subspace-valued measure which is compatible with the wavelet basis.

3.1 Extension of finitely union-additive multimeasures

One might wish to use the standard procedures to extend a finitely union-additive multimeasure to a countably union-additive multimeasure. Unfortunately this doesn't work in general. Given $\phi \in UFA(\Omega, X)$, it is certainly possible to define an "outer measure" ϕ^* on $Pow(\Omega)$ by

$$\phi^*(S) = \bigcap \left\{ \bigcup_{i=1}^{\infty} \phi(A_i) : S \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{A} \right\}$$

where $S \subset \Omega$ (the intersection acts like taking the infimum). The following properties are easy to verify:

1. $\phi^*(\emptyset) = \emptyset$.
2. $A \in \mathcal{A}$ implies that $\phi^*(A) \subset \phi(A)$.
3. $A \subset B$ implies that $\phi^*(A) \subset \phi^*(B)$.
4. $\bigcup_{n=1}^{\infty} \phi^*(B_n) \subset \phi^*(\bigcup_{n=1}^{\infty} B_n)$.

Notice that the last property has the opposite inclusion of the usual countably additivity.

However, ϕ^* is not necessarily an extension of ϕ . To see this, let $\Omega = X = \mathbb{N}$, and let \mathcal{A} be the algebra generated by the finite subsets of Ω (so \mathcal{A} consists of finite and co-finite subsets of \mathbb{N}). We define $\phi(\emptyset) = \emptyset$ and $\phi(F) = \{1\}$ for any finite set F and $\phi(\mathbb{N}) = \phi(C) = \{0, 1\}$ for any co-finite set C . Then it is easy to see that ϕ^* will satisfy $\phi^*(\emptyset) = \emptyset$ and $\phi^*(F) = \{1\}$ for any finite set F and $\phi^*(I) = \{1\}$ for any infinite set I . Thus, ϕ^* does not agree with ϕ even on \mathcal{A} .

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