Measurement and Probability in Fuzzy Quantum Theories

1 Overview, Objectives, and Expected Significance

This is a *theoretical* investigation of *experimental* physics using *computational* methods. All experiments and computations are processes bounded in space, time, energy, and other resources. Yet, for centuries, the mathematical formalization of such processes has been founded on the infinitely precise real or complex numbers. Indeed, almost every description of quantum mechanics, quantum computation, or quantum experiments refers to entities such as e, π, $\sqrt{2}$, etc. From a computational perspective, such numbers do not exist in their entirety "for free." For example, the state of the art algorithms for computing the nth digit of π require on the order of n^2 operations [3]. In other words, simply referring to the nth digit of π requires more and more resources as n gets larger. Taking such resource bounds in consideration is what founded computer science as a discipline and is crucial for understanding the very nature of computation and, following Feynman [10], Landauer [24], and others, for understanding the very nature of physical processes.

We propose to revisit quantum mechanics, quantum information, and quantum computation from this resource-aware perspective. Our initial result in that domain showed how subtle the issues can be [43]: a straightforward replacement of the complex numbers by a finite field yields a variant of quantum mechanics in which computationally hard problems like UNIQUE-SAT (which decides whether a given Boolean formula is unsatisfiable or has exactly one satisfying assignment) can be deterministically solved in constant time. To eliminate such unrealistic theories requires delicate analysis of the structure of the Hilbert space, the process of observation, and the notion of probability teasing apart their reliance on the infinitely precise real numbers [18, 19]. In this proposal, our aim is to shift focus from the infinitely-specified but not directly observable quantum states, to observable measurable properties of quantum systems and their probabilities. Furthermore, we insist that our theories of measurement and probability only refer to finitely communicable evidence within feasible computational bounds. It follows that states, observations, and probabilities all become "fuzzy", i.e., specified by intervals of confidence that can only increase in precision if the available resources increase proportionally. Our notion of "fuzzy quantum mechanics" is related to existing work [14, 33, 35, 16, 2] but, as will be explained in more detail, is distinguished by its computational character.

We will begin by reviewing existing work that recasts classical probability spaces in a resource-aware setting and move to our proposal which, briefly speaking, aims at recasting quantum probability and hence quantum measurement to a corresponding resource-aware setting. In particular, we plan on addressing the Meyer-Mermin debate [29, 27, 5] on the impact of finite precision measurements on the relevance of the Kochen-Specker theorem. We argue that no matter what the results are their impact will be strong. Positive results that formulate a computable, with clear resource bounds, theory of quantum measurement and quantum probability would be essential for understanding and realizing quantum computation. Any negative results would redirect research that aims at realizing quantum computation to other approaches. At an intellectual level, the results will solve or clarify several paradoxes in quantum mechanics, quantum information, computability, and complexity theory.

2 Background I: Classical Probability

A *probability space* specifies the necessary conditions for reasoning coherently about collections of uncertain events [23, 36, 15, 38]. We review the conventional presentation of probability spaces and then discuss the computational resources needed to estimate probabilities.

2.1 Classical Probability Spaces

The conventional definition of a probability space builds upon the field of real numbers. In more detail, a probability space consists of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ mapping events in $\mathcal{E}$ to the real interval $[0, 1]$. We will only consider *finite* sets of events and restrict our attention to non-empty finite sets Ω as the sample space. The space of events $\mathcal{E}$ includes every possible subset of Ω: it is the powerset $2^{\Omega} = \{E \mid E \subseteq \Omega\}$. For future reference, we emphasize that events are the primary notion of interest and that the sample space is a convenient artifact that allows us to treat events as sets obeying the laws of Boolean algebra.

Definition 1 (Probability Measure). Given the set of events $\mathcal{E}$, a *probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(\Omega) = 1$,
- for every event E, $\mu(\Omega \backslash E) = 1 - \mu(E)$ where $\Omega \backslash E$ is the complement event of E, and
- for every collection $\{E_i\}_{i=1}^{N}$ of pairwise disjoint events, $\mu\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} \mu(E_i)$.

There is some redundancy in the definition that will be useful when moving to quantum probability spaces.

Example 1 (Two-coins experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. There are 16 total events including the event $\{HH, HT\}$ that the first coin lands heads up, the event $\{HT, TH\}$ that the two coins land on opposite sides, and the event $\{HT, TH, TT\}$ that at least one coin lands tails up. Here is a possible probability measure for these events:

$$
\begin{array}{rclrcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

It is useful to think that this probability measure is completely determined by the "reality" of the two coins in question and their characteristics, in the sense that each pair of coins induces a measure, and each measure must correspond to some pair of coins. The measure above would be induced by two particular coins such that the first coin is twice as likely to land tails up than heads up and the second coin is double-headed. In a strict computational or experimental setting, one should question the reliance of the definition of probability space on the uncountable and uncomputable real interval $[0, 1]$. This interval includes numbers like $0.h_1h_2h_3\ldots$ where h_i is 1 or 0 depending on whether Turing machine M_i halts or not. Such numbers cannot be computed. This interval also includes numbers like $\frac{\pi}{4}$ which can only be computed with increasingly large resources as the precision increases. Therefore, in a resource-aware computational or experimental setting, it is more appropriate to consider probability measures that map events to a set of elements computable with a fixed set of resources. We expand on this observation in the next section and then recall its formalization using interval-valued probability measures [42, 22].

2.2 Measuring Probabilities: Buffon's Needle Problem

In the previous example, we assumed the probability $\mu(E)$ of each event E is known a priori. In reality, although each event is assumed to have a probability, the exact value of $\mu(E)$ may not be known. According to the *frequency interpretation of probability* (which we will revisit when moving to the quantum case), to

determine the probability of an event, we run N independent trials which gives us an approximation of the (assumed) "true" or "real" probability. Let x_i be 1 or 0 depending on whether the event E occurs in the i-th trial or not, then $\mu(E)$ could be approximated to given accuracy $\epsilon > 0$ by the relative frequency $\frac{1}{N}\sum_{i=1}^{N} x_i$ with the probability converging to one as N goes to infinity, i.e.,

$$\forall \epsilon > 0, \lim_{N\to\infty} \mu\left(\left|\mu(E) - \frac{1}{N}\sum\nolimits_{i=1}^{N} x_i\right| < \epsilon\right) = 1 \ .$$

This fact is called the law of large numbers [6, 23, 40, 36, 31].

Let's look at a concrete example. Suppose we drop a needle of length ℓ onto a floor made of equally spaced parallel lines a distance h apart, where $\ell < h$. It is a known fact that the probability of the needle crossing a line is $\frac{2\ell}{\pi h}$ [12, 9, 17, 40]. Consider an experimental setup consisting of a collection of N identical needles of length ℓ. We throw the N needles one needle at a time, and observe the number M of needles that cross a line, thus estimating the probability of a needle crossing a line to be $\frac{M}{N}$. In an actual experiment with 500 needles and the ratio $\frac{\ell}{h} = 0.75$ [17], it was found that 236 crossed a line so the relative frequency is 0.472 whereas the idealized mathematical probability is $0.4774\ldots$. In a larger experiment with 5000 needles and the ratio $\frac{\ell}{h} = 0.8$ [40], the relative frequency was calculated to be 0.5064 whereas the idealized mathematical probability is $0.5092\ldots$. We see that the observed probability approaches $\frac{2\ell}{\pi h}$ but only if *larger and larger resources* are expended. These resource considerations suggest that it is possible to replace the real interval $[0, 1]$ with rational numbers up to a certain precision related to the particular experiment in question. There is clearly a connection between the number of needles and the achievable precision: in the hypothetical experiment with 3 needles, it is not sensible to retain 100 digits in the expansion of $\frac{2\ell}{\pi h}$.

There is another more subtle assumption of unbounded computational power in the experiment. We are assuming that we can always determine with certainty whether a needle is crossing a line. But "lines" on the the floor have thickness, their distance apart is not exactly h, and the needles' lengths are not all absolutely equal to ℓ. These perturbations make the events "fuzzy." Thus, in an experiment with limited resources, it is not possible to talk about the idealized event that exactly M needles cross lines as this would require the most expensive needles built to the most precise accuracy, laser precision for drawing lines on the floor, and the most powerful microscopes to determine if a needle does cross a line. Instead we might talk about the event that $M - \delta$ needles evidently cross lines and $M + \delta'$ needles plausibly cross lines where δ and δ' are resource-dependent approximations. This fuzzy notion of events leads to probabilities being only calculable within intervals of confidence reflecting the certainty of events and their plausibility. This is indeed consistent with published experiments: in an experiment with 3204 needles and the ratio $\frac{\ell}{h} = 0.6$ [9], 1213 needles clearly crossed a line and 11 needles were close enough to plausibly be considered as crossing the line: we would express the probability in this case as the interval $\left[\frac{1213}{3204}, \frac{1224}{3204}\right]$ expressing that we are certain that the event has probability at least $\frac{1213}{3204}$ but it is possible that it would have probability $\frac{1224}{3204}$.

2.3 Classical Interval-Valued Probability Measures

As motivated above, an event E_1 may have an interval of probability $[l_1, r_1]$. Assume that another disjoint event E_2 has an interval of probability $[l_2, r_2]$, what is the interval of probability for the event $E_1 \cup E_2$? The answer is somewhat subtle: although it is possible to use the sum of the intervals $[l_1 + l_2, r_1 + r_2]$ as the combined probability, one can find a much tighter interval if information *against* the event (i.e., information about the complement event) is also taken into consideration. Formally, for a general event E with probability $[l, r]$, the evidence that contradicts E is evidence supporting the complement of E. The complement of E must therefore have probability $[1 - r, 1 - l]$ which we abbreviate $[1, 1] - [l, r]$. Given a sample space Ω and its set of events $\mathcal{E}$, a function $\bar{\mu} : \mathcal{E} \to [0, 1]$ is a classical interval-valued probability measure if and only if $\bar{\mu}$ satisfies the following conditions [22] where the last line uses $\subseteq$ to allow for tighter intervals that exploit the complement event:

- $\bar{\mu}(\emptyset) = [0, 0]$.
- $\bar{\mu}(\Omega) = [1, 1]$.

- For any event E, $\bar{\mu}(\Omega \backslash E) = [1,1] - \bar{\mu}(E)$
- For a collection $\{E_i\}_{i=0}^{N}$ of pairwise disjoint events, we have $\bar{\mu}\left(\bigcup_{i=0}^{N} E_i\right) \subseteq \sum_{i=0}^{N} \bar{\mu}(E_i)$.

Example 2 (Two-coin experiment with interval probability). We split the unit interval $[0,1]$ in the following four closed sub-intervals: $[0,0]$ which we call *impossible*, $[0,\frac{1}{2}]$ which we call *unlikely*, $[\frac{1}{2},1]$ which we call *likely*, and $[1,1]$ which we call *certain*. Using these new values, we can modify the probability measure of Ex. 1 by mapping each numeric value to the smallest sub-interval containing it to get the following:

$$\begin{array}{rcl@{\qquad}rcl}
\bar{\mu}(\emptyset) & = & \textit{impossible} & \bar{\mu}(\{HT,TH\}) & = & \textit{likely} \\
\bar{\mu}(\{HH\}) & = & \textit{unlikely} & \bar{\mu}(\{HT,TT\}) & = & \textit{impossible} \\
\bar{\mu}(\{HT\}) & = & \textit{impossible} & \bar{\mu}(\{TH,TT\}) & = & \textit{likely} \\
\bar{\mu}(\{TH\}) & = & \textit{likely} & \bar{\mu}(\{HH,HT,TH\}) & = & \textit{certain} \\
\bar{\mu}(\{TT\}) & = & \textit{impossible} & \bar{\mu}(\{HH,HT,TT\}) & = & \textit{unlikely} \\
\bar{\mu}(\{HH,HT\}) & = & \textit{unlikely} & \bar{\mu}(\{HH,TH,TT\}) & = & \textit{certain} \\
\bar{\mu}(\{HH,TH\}) & = & \textit{certain} & \bar{\mu}(\{HT,TH,TT\}) & = & \textit{likely} \\
\bar{\mu}(\{HH,TT\}) & = & \textit{unlikely} & \bar{\mu}(\{HH,HT,TH,TT\}) & = & \textit{certain}
\end{array}$$

Despite the absence of infinitely precise numeric information, the probability measure is quite informative: it reveals that the second coin is double-headed and that the first coin is biased. To understand the $\subseteq$-condition, consider the following calculation:

$$\begin{aligned}
\bar{\mu}(\{HH\}) + \bar{\mu}(\{HT\}) + \bar{\mu}(\{TH\}) + \bar{\mu}(\{TT\}) &= \textit{impossible} + \textit{unlikely} + \textit{impossible} + \textit{likely} \\
&= [0,0] + \left[0,\frac{1}{2}\right] + [0,0] + \left[\frac{1}{2},1\right] \\
&= \left[\frac{1}{2},\frac{3}{2}\right]
\end{aligned}$$

If we were to equate $\bar{\mu}(\Omega)$ with the sum of the individual probabilities, we would get that $\bar{\mu}(\Omega) = [\frac{1}{2},\frac{3}{2}]$. However, using the fact that $\bar{\mu}(\emptyset) = \textit{impossible}$, we have $\bar{\mu}(\Omega) = 1 - \bar{\mu}(\emptyset) = \textit{certain} = [1,1]$. This interval is tighter and a better estimate for the probability of the event Ω, and of course it is contained in $[\frac{1}{2},\frac{3}{2}]$. However it is only possible to exploit the information about the complement when all four events are combined. Thus the $\subseteq$-condition allows us to get an estimate for the combined event from each of its constituents and then gather more evidence knowing the aggregate event.

3 Background II: Quantum Probability

The mathematical framework of classical probability above assumes that there exists a predetermined set of events that are independent of the particular experiment — classical physics is non-contextual. However, even in classical situations, the structure of the event space is often only partially known and the precise dependence of two events on each other cannot, a priori, be determined with certainty. In the quantum framework, this partial knowledge is compounded by the fact that there exist non-commuting events which cannot happen simultaneously. To accommodate these more complex situations, conventional approaches to quantum probability abandon the sample space Ω and reason directly about events which are generalized from plain sets to projection operators. A quantum probability space therefore consists of just two components: a set of events $\mathcal{E}$ often formalized as projection operators and a probability measure $\mu : \mathcal{E} \to [0,1]$ formalized using the Born rule.

3.1 Quantum Events

Definition 2 (Projection Operators; Orthogonality [26, 34, 32, 15, 38]). Given a Hilbert space $\mathcal{H}$, an event (an experimental proposition [7], a question [26, 1], or an elementary quantum test [32]) is represented as

a (self-adjoint or orthogonal [15, 25]) projection operator $P : \mathcal{H} \to \mathcal{H}$ onto a linear subspace of $\mathcal{H}$. The following define projections and list some of their properties:

- $\mathbb{0}$ is a projection.
- For any pure state $|\psi\rangle$, $|\psi\rangle\langle\psi|$ is a projection operator.
- Projection operators P_0 and P_1 are *orthogonal* if $P_0P_1 = P_1P_0 = \mathbb{0}$. The sum of two projection operators $P_0 + P_1$ is also a projection operator if and only if they are orthogonal.
- Conversely, every projection P can be expressed as $\sum_{j=1}^{N} |\psi_j\rangle\langle\psi_j|$, where P actually projects onto the linear subspace with orthonormal basis $\{|\psi_j\rangle\}_{j=1}^{N}$.
- A set of projections $\{P_i\}_{i=1}^{N}$ is called an *ideal measurement* if it is a partition of the identity, i.e., $\sum_{i=1}^{N} P_i = \mathbb{1}$ [38]. In this case, projections $\{P_i\}_{i=1}^{N}$ must be mutually orthogonal [15] and N must correspond to the dimension of the Hilbert space.
- If P is a projection operator, then $\mathbb{1} - P$ is also a projection operator, called its *complement*. It is orthogonal to P, and corresponds to the complement event $\Omega \backslash E$ in classical probability [15].
- Projection operators P_0 and P_1 *commute* if $P_0P_1 = P_1P_0$. The product of two projection operators P_0P_1 is also a projection operator if and only if they commute. This corresponds to the classical intersection between events [32, 15].
- For two commuting projection operators P_0 and P_1, their *disjunction* $P_0 \vee P_1$ is defined to be $P_0 + P_1 - P_0P_1$ [15].

Example 3 (One-qubit quantum probability space). Consider a one-qubit Hilbert space with each event interpreted as a possible post-measurement state. For example, the event $|0\rangle\langle 0|$ indicates that the post-measurement state will be $|0\rangle$; the probability of such an event depends on the current state; the event $|1\rangle\langle 1|$ indicates that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ indicates that the post-measurement state will be $|+\rangle$; the event $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ indicates that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$ and is clearly certain; finally the empty event $\mathbb{0}$ states that the post-measurement state will be the empty state and is impossible. As in the classical case, a probability measure is a function that maps events to $[0, 1]$. Here is a partial specification of a possible probability measure that would be induced by a system whose current state is $|0\rangle$:

$$\mu(\mathbb{0}) = 0, \quad \mu(\mathbb{1}) = 1, \quad \mu(|0\rangle\langle 0|) = 1, \quad \mu(|1\rangle\langle 1|) = 0, \quad \mu(|+\rangle\langle +|) = 1/2, \quad \ldots$$

Note that, similarly to the classical case, the probability of $\mathbb{1}$ is 1 and the probability of collections of orthogonal events (e.g., $|0\rangle\langle 0| + |1\rangle\langle 1|$) is the sum of the individual probabilities. A collection of non-orthogonal events (e.g., $|0\rangle\langle 0|$ and $|+\rangle\langle +|$) is however not even a valid event. In the classical example, we argued that each probability measure is uniquely determined by two actual coins. A similar (but much more subtle) argument is valid also in the quantum case. By postulates of quantum mechanics and Gleason's theorem, it turns out that for large enough quantum systems, each probability measure is uniquely determined by an actual quantum state as discussed next.

3.2 Quantum Probability Measures

Given our setup, the definition of a quantum probability measure is a small variation on the classical definition.

Definition 3 (Quantum Probability Measure [26, 13, 34, 25]). Given a Hilbert space $\mathcal{H}$ with its set of events $\mathcal{E}$, a *quantum probability measure* is a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\mathbb{0}) = 0$.

- $\mu(\mathbb{1}) = 1$.
- For any projection P, $\mu(\mathbb{1} - P) = 1 - \mu(P)$.
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^N$, we have $\mu\left(\sum_{i=1}^N P_i\right) = \sum_{i=1}^N \mu(P_i)$.

A quantum probability measure can be easily constructed if one knows the current state of the quantum system by using the Born rule [8, 28, 21]. Specifically, for each pure normalized quantum state $|\phi\rangle$, the Born rule induces a probability measure μ_ϕ^B defined as $\mu_\phi^B(P) = \langle\phi|P\phi\rangle$. The situation generalizes to mixed states $\rho = \sum_{j=1}^N q_j |\phi_j\rangle\langle\phi_j|$, where $\sum_{j=1}^N q_j = 1$ in which case the generalized Born rule induces a probability measure μ_ρ^B defined as follows [32, 31, 21]:

$$\mu_\rho^B(P) = \mathrm{Tr}(\rho P) = \sum\nolimits_{j=1}^N q_j \mu_{\phi_j}^B(P)$$

Conversely every probability measure must be of this form.

Theorem 1 (Gleason's theorem [13, 34, 32]). In a Hilbert space $\mathcal{H}$ of dimension $d \geq 3$, given a quantum probability measure $\mu : \mathcal{E} \to [0, 1]$, there exists a unique mixed state ρ such that $\mu = \mu_\rho^B$.

3.3 Measuring Quantum Probabilities

By using the law of large numbers, quantum probabilities can be also estimated by relative frequencies. For example, if we want to know the probability of "spin up" in a Stern-Gerlach experiment [37, 32, 31, 15], we can put a beam of silver atoms in a highly inhomogeneous magnetic field, and count the number of atoms deflects up. Ideally, if the local field of strength directs to the z-axis, and all particles have the same velocity, the Stern-Gerlach experiment only produces two spots corresponding to $|0\rangle$ and $|1\rangle$. Notice that whenever the magnetic field of the Stern-Gerlach is fixed, i.e., an ideal measurement is picked, measuring the spin is exactly the same as tossing a coin. In other words, if someone claims to be tossing a coin behind a veil, and only shows the resulting heads or tails, we cannot distinguish whether the results are really deriving from tossing a coin, or by running a Stern-Gerlach experiment, and showing the head, the tail, or the side of coin if the silver atoms is spin up, spin down, or just hit the middle, respectively. Following our theme, a Stern-Gerlach experiment requires significant resources to keep the local field of strength directing to the z-axis precisely, to keep the atoms having almost the same velocity, and to point out the exact position each particle landed on. If the field of strength does not perfectly direct to the z-axis, we do not really test the quantum event we want to test. Even if the field of strength perfectly directs to the z-axis, the variant velocity makes spots broader and more washed out. Together with the precision limit of the detector, it may sometimes be hard to decide a particle corresponding to which state.

4 Proposal: General Plan

A major aim of this proposal is to recast quantum probability and measurement in a computational resource-aware setting. Our vision is to unify the strands above in a novel way as explained in this section. We start with some immediate observations that justify the rest of the development.

- If we insist that probabilities cannot be infinitely precise and are bound to be approximations represented by intervals of confidence, then by Gleason's theorem, quantum states themselves cannot be infinitely precise. This raises an important philosophical question with immense practical relevance in the context of quantum computing. The question, which we could largely bypass in the classical case, is whether there is, in the background of all the measurements and probabilities, a "real" infinitely-precise quantum state that is being approximated and that, in principle, can be more and more accurately described given more and more resources. The answer will have to be *no* if we are to respect major theorems of quantum mechanics and quantum information. Thus our first radical departure from the

classical case will be that the quantum state itself must be fuzzy in some sense to be formalized. In other words, does a Gleason, Bell, or Kochen-Specker theorem still hold in a world with finite resources?

- Going further we actually have to concede not just the infinite accuracy of the quantum state, but its very existence as a "real" entity that exists independently of measurements and probabilities. The reasons for this radical position are as follows. First arguments embedded in the proof of the Kochen-Specker theorem essentially rule out the existence of a sample space as the probabilities of events cannot be consistently expressed using core assignments to elements of the sample space. Second, the frequency interpretation of probability which is already problematic in the classical case, becomes untenable in the quantum case. In both cases, but much more in the quantum case, it is much more sensible to interpret probability in the Bayesian sense as "lack of knowledge". This, and similar arguments elegantly expressed by recent work on Quantum Bayesianism [11, 41], suggest that the quantum state is more like an interactive system in computer science parlance. In other words, the state is only determined by its interactions with the environment or observers. A rough but useful analogy in the context of computer science would be a web server: the state of such a system cannot be said to exist independently of the interactions with clients; each interaction with a client determines a future state of the system and may affect future interactions. Crucially, each client may have a *different* view of the web server that is consistent with their own history of interactions. This computational perspective agrees with the philosophical position of QBism [41] that asserts that the quantum state is subjective: each observer has a different view of the quantum system that is consistent with their previous observations and that allows that observer, independently of other observers, to assign beliefs (i.e., probabilities) to future interactions.

4.1 Proposed Activity I: Quantum States as Interactive Processes

The early formalization of the process of computing using Turing machines [39] or the λ-calculus [4] viewed a computation as realizing a mathematical function. From our perspective, the most important consequence of this view is that the properties of the computed function existed independently of any observation, calculation, or interaction. An agent can only interact with the computation by asking questions regarding its input-output behavior and although such interaction would reveal fragments of the computed function it could not change it or affect it in any way.

In a series of breakthrough papers that led to his Turing Award [30], Milner argued that this "closed system" view of computation is too limiting as it cannot adequately describe *interactive computations*. Milner explained that "concurrency requires a new conceptual framework" and "not merely an extension of the repertoire of entities and constructions which explain sequential computing." He gives the following simple but compelling example. Consider the following two programs:

```
Program P1: x := 1 ; x := x + 1
Program P2: x := 2
```

As closed systems, these two programs define the same relation, namely the one that maps an input memory to an updated memory in which the location named `x` is assigned the value `2`. But place the programs in an environment which may involve interactions and the equivalence is lost. Concretely, place each of the two programs concurrently with the program:

```
Program Q: x := 3
```

and we can observe the following. In the combined system of `P1` and `Q` running concurrently, the location `x` may end up containing `2`, `3`, or `4`. But in the combined system of `P2` and `Q` running concurrently, the location `x` can only end up containing `2` or `3`. The lesson, phrased in physics terms, is that it is not sensible to think of the properties of a system (like `P1` or `P2`) independently of the observer (`Q` in this case). Indeed the observation `x=4` cannot said to have an independent existence independently from the interaction between `P1` and `Q`: it really emerges or is created as part of the observation process and is not a pre-existing "element of reality".

This phenomenon is hardly a surprise to computer scientists or physicists but embracing it fully does have far reaching consequences. In computer science, the big shift was to abandon the idea that a computation has inherent internal meaning, e.g., representing a pure mathematical function. Instead *the meaning of a computational process is its observable behavior.* It is not too difficult to imagine that, in the presence of multiple interacting observers, the behavior observed by just one agent would appear to involve hidden variables or action at a distance. The well-established theory of observation of concurrent processes and their equivalences [20] relies on neither.

do quantum version; application to CS and quantum information

4.2 Proposed Activity II: Quantum Interval-valued Probability Measures

Just as it is the case for Buffon's needles, estimating quantum probabilities (e.g., in the case of a Stern-Gerlach experiment) with increasing accuracy requires increasingly large resources. Therefore, given some finite resources, it is only possible to estimate the quantum probabilities within an interval of confidence. It is therefore natural to introduce the notion of an "interval-valued quantum probability measure" that combines the definitions of conventional quantum probability measures with classical interval-probabilities.

Definition 4 (Quantum Interval-valued Probability Measure). Given a Hilbert space $\mathcal{H}$ with quantum events (projections) $\mathcal{E}$, and a collection of intervals $\mathscr{I}$, a *quantum $\mathscr{I}$-interval-valued probability measure* is a function $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that:

- $\bar{\mu}(\mathbb{0}) = [0, 0]$.
- $\bar{\mu}(\mathbb{1}) = [1, 1]$.
- For any projection P,
$$\bar{\mu}\left(\mathbb{1} - P\right) = [1, 1] - \bar{\mu}\left(P\right) \ . \tag{1}$$
- For a set of mutually orthogonal projections $\{P_i\}_{i=1}^{N}$, we have:
$$\bar{\mu}\left(\sum\nolimits_{i=1}^{N} P_i\right) \subseteq \sum\nolimits_{i=1}^{N} \bar{\mu}\left(P_i\right) \tag{2}$$

It is easy to establish that quantum interval-valued probability measures generalize conventional quantum probability measures. In particular, any quantum probability measure can be recast as an interval-valued measure using the three intervals $[0, 0]$, $[1, 1]$ and $[0, 1]$, where $[0, 0]$ and $[1, 1]$ are called *impossible* and *certain* as before, and $[0, 1]$ is called *unknown* because it provides no information as follows. Given a probability measure $\mu : \mathcal{E} \to [0, 1]$, we define a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ by $\bar{\mu}(P) = \iota\left(\mu(P)\right)$, where $\iota : [0, 1] \to \mathscr{I}$ is defined by

$$\iota(x) = \begin{cases} \textit{certain} & \text{if } x = 1 \ ; \\ \textit{impossible} & \text{if } x = 0 \ ; \\ \textit{unknown} & \text{otherwise.} \end{cases}$$

The interesting question is the converse one corresponding to Gleason's theorem. Given an interval-valued probability measure, i.e., given observations done with finite resources, do these observations uniquely determine a quantum state? Our initial investigations indicate that the answer is *no.*

Definition 5. Given a quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$:

- A *core* of $\bar{\mu}$ is the set $\mathrm{core}\left(\bar{\mu}\right) = \{\text{quantum probability measure } \mu : \mathcal{E} \to [0, 1] | \forall E \in \mathcal{E}. \mu\left(E\right) \in \bar{\mu}\left(E\right)\}$.
- $\bar{\mu}$ is called convex if
$$\bar{\mu}\left(P_0 \vee P_1\right) + \bar{\mu}\left(P_0 P_1\right) \subseteq \bar{\mu}\left(P_0\right) + \bar{\mu}\left(P_1\right) \tag{3}$$
for all commuting $P_0, P_1 \in \mathcal{E}$.

Theorem 2. There is a convex quantum interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ such that core $(\bar{\mu}) = \emptyset$.

This theorem can be proved by the following example.

Example 4 (Three-dimensional quantum three-value interval-valued probability measure). Given a three dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Consider $\mathscr{I} = \{\textit{certain}, \textit{impossible}, \textit{unknown}\}$ and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $|+'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$. Let

$$\begin{aligned} \bar{\mu}(\mathbb{0}) = \bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) = \bar{\mu}(|+'\rangle\langle +'|) &= \textit{impossible}\,, \\ \bar{\mu}(\mathbb{1}) = \bar{\mu}(\mathbb{1} - |0\rangle\langle 0|) = \bar{\mu}(\mathbb{1} - |+\rangle\langle +|) = \bar{\mu}(\mathbb{1} - |+'\rangle\langle +'|) &= \textit{certain}\,, \\ \bar{\mu}(P) &= \textit{unknown}, \text{ otherwise.} \end{aligned}$$

To verify $\bar{\mu}$ is a quantum interval-valued probability measure, first notice that $\bar{\mu}(P_0 + P_1) \subseteq \bar{\mu}(P_0) + \bar{\mu}(P_1)$ implies equation (2) by induction. Second, equation (1) implies $\bar{\mu}(\mathbb{1}) \subseteq \bar{\mu}(P) + \bar{\mu}(\mathbb{1} - P)$. Therefore, to prove equation (2), it is sufficient to check

$$\bar{\mu}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|) \subseteq \bar{\mu}(|\psi_0\rangle\langle\psi_0|) + \bar{\mu}(|\psi_1\rangle\langle\psi_1|)$$

for orthogonal $|\psi_0\rangle$ and $|\psi_1\rangle$. This equation always holds because we always have $\textit{unknown} \subseteq \bar{\mu}(|\psi_0\rangle\langle\psi_0|) + \bar{\mu}(|\psi_1\rangle\langle\psi_1|)$.

Next, we need to verify that $\bar{\mu}$ is convex. First, for orthogonal projections P_0 and P_1, equation (2) implies equation (3). Second, when $P_0 = P_1 + P_2$, equation (3) holds trivially. Hence, we only need to verify the case which $P_0 = |\psi_0\rangle\langle\psi_0| + |\psi_2\rangle\langle\psi_2|$ and $P_1 = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$, where $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ is an orthonormal basis. Because we have

$$\begin{aligned} \bar{\mu}(P_0 \vee P_1) &= \bar{\mu}(\mathbb{1}) = \textit{certain} \\ \bar{\mu}(P_0 P_1) &= \bar{\mu}(|\psi_2\rangle\langle\psi_2|) \subseteq \textit{unknown} \end{aligned}$$

and $\bar{\mu}(P_0) + \bar{\mu}(P_1)$ is either $\textit{certain} + \textit{unknown}$ or $\textit{unknown} + \textit{unknown}$. Therefore, we have $\bar{\mu}(P_0 \vee P_1) + \bar{\mu}(P_0 P_1) \subseteq \bar{\mu}(P_0) + \bar{\mu}(P_1)$.

Finally, we will prove that $\bar{\mu}$ cannot correspond to any quantum probability measure by contradiction. Suppose $\mu(P) \in \bar{\mu}(P)$ for some quantum probability measure $\mu : \mathcal{E} \to [0, 1]$. We must have

$$\mu(|0\rangle\langle 0|) = \mu(|+\rangle\langle +|) = \mu(|+'\rangle\langle +'|) = 0\,. \tag{4}$$

By Gleason's theorem, there is a mixed state $\rho = \sum_{j=1}^{N} q_j\, |\phi_j\rangle\langle\phi_j|$ such that $\mu(P) = \sum_{j=1}^{N} q_j\, \langle\phi_j|P\phi_j\rangle$, where $\sum_{j=1}^{N} q_j = 1$ and $q_j > 0$. However, no pure state $|\phi\rangle$ satisfies $\langle\phi|0\rangle = \langle\phi|+\rangle = \langle\phi|+'\rangle = 0$ so that we have $\mu(P) \notin \bar{\mu}(P)$ for all quantum probability measure μ.

Recall that when approximating the probability classically, the quality of the experimental equipment cannot be compensated by the number of independent trials. If the precision of the intervals $\mathscr{I}$ represents the quality of the quantum experimental equipment, example 4 tells us that we cannot identify a particular state if the experimental equipment reveals too little information. For example, $\bar{\mu}$ might correspond to $|2\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) = \textit{impossible}$, $|1\rangle$ because $\bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+'\rangle\langle +'|) = \textit{impossible}$, or the density matrix $\frac{\mathbb{1}}{3}$ because $\bar{\mu}(|\phi\rangle\langle\phi|) \neq \textit{certain}$ for all $|\phi\rangle$. Also, example 4 is no longer valid if our measurement equipment is a little more precise, say $\mathscr{I} = \{\textit{impossible}, \textit{unlikely}, \textit{likely}, \textit{certain}\}$, because example 4 really use the property of $\textit{unknown}$.

If we consider $\mathscr{I} = \{\textit{impossible}, \textit{unlikely}, \textit{likely}, \textit{certain}\}$, example 5 provides another example with an empty core. Comparing to example 4, example 5 is closer to the Born rule in a sense that it might only correspond to $|2\rangle$ or $\frac{\mathbb{1}}{3}$, but not $|1\rangle$. Example 5 is also no longer valid for more precise measurement equipment as well. In general, we believe if the measurement equipment is more and more precise, its interval-valued probability measure will be closer and closer to the the Born rule. In the limit case, $\mathscr{I} = \{\{a\} | a \in [0, 1]\}$ recovers conventional Gleason's theorem and the Born rule.

Example 5 (Three-dimensional quantum 4-value interval-valued probability measure). [1]Given a three dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Consider $\mathscr{I} = \{\mathit{impossible}, \mathit{unlikely}, \mathit{likely}, \mathit{certain}\}$, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\text{-}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and a quantum $\mathscr{I}$-interval-valued probability measure $\bar{\mu} : \mathcal{E} \to \mathscr{I}$ defined as follow, where the state vectors corresponding to 1-dimensional projector are also illustrated in figure 1.

1. Let
$$\begin{aligned} \bar{\mu}(\mathbb{0}) = \bar{\mu}(|0\rangle\langle 0|) = \bar{\mu}(|+\rangle\langle +|) &= \mathit{impossible} \\ \bar{\mu}(\mathbb{1}) = \bar{\mu}(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |0\rangle\langle 0|) = \bar{\mu}(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |+\rangle\langle +|) &= \mathit{certain}\ , \end{aligned}$$
where $|0\rangle$ and $|+\rangle$ is illustrated as the red and green dotted vector, respectively.

2. Consider the red and green circles in figure 1. They are the states orthogonal to $|0\rangle$ and $|+\rangle$, respectively, and can be parametrized as $\left|0^{\perp}_{\theta,\gamma}\right\rangle = \mathrm{e}^{\mathrm{i}\gamma}\sin\theta|1\rangle + \cos\theta|2\rangle$ and $\left|+^{\perp}_{\theta,\gamma}\right\rangle = -\mathrm{e}^{\mathrm{i}\gamma}\sin\theta|\text{-}\rangle + \cos\theta|2\rangle$, where $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \gamma < 2\pi$. The dotted half of those states need special treatment, i.e., whenever $0 \le \theta < \frac{\pi}{2}$ and $0 \le \gamma < \pi$, we define
$$\begin{aligned} \bar{\mu}\left(\left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) = \bar{\mu}\left(\left|+^{\perp}_{\theta,\gamma}\right\rangle\left\langle +^{\perp}_{\theta,\gamma}\right|\right) &= \mathit{likely} \\ \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} \left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) = \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} \left|+^{\perp}_{\theta,\gamma}\right\rangle\left\langle +^{\perp}_{\theta,\gamma}\right|\right) &= \mathit{unlikely}\ . \end{aligned}$$

3. Otherwise, $\bar{\mu}(|\psi\rangle\langle\psi|) = \mathit{unlikely}$ and $\bar{\mu}(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |\psi\rangle\langle\psi|) = \mathit{likely}$.

That $\bar{\mu}$ has an empty core can be proved by contradiction. Assume there is a real-valued probability measure satisfying $\mu^B_\rho(P) \in \bar{\mu}(P)$ for all $P \in \mathcal{E}$. Because $\mu^B_\rho(|0\rangle\langle 0|) \in \bar{\mu}(|0\rangle\langle 0|) = \mathit{impossible}$ and $\mu^B_\rho(|+\rangle\langle +|) \in \bar{\mu}(|+\rangle\langle +|) = \mathit{impossible}$, we must have $\mu^B_\rho(|0\rangle\langle 0|) = \mu^B_\rho(|+\rangle\langle +|) = 0$ so that $\mu^B_\rho = \mu^B_{|2\rangle}$. However,
$$\mu^B_{|2\rangle}(|2\rangle\langle 2|) = 1 \notin \mathit{unlikely} = \bar{\mu}(|2\rangle\langle 2|)\ .$$

Similar to example 4, to verify $\bar{\mu}$ is a quantum interval-valued probability measure, we only need to verify equation (2), and it is sufficient to check
$$\begin{aligned} \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |\psi_0\rangle\langle\psi_0|\right) &\subseteq \bar{\mu}\left(|\psi_1\rangle\langle\psi_1|\right) \mathbin{_{\mathscr{I}}\!+} \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) \\ \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |\psi_1\rangle\langle\psi_1|\right) &\subseteq \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) \mathbin{_{\mathscr{I}}\!+} \bar{\mu}\left(|\psi_0\rangle\langle\psi_0|\right) \\ \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |\psi_2\rangle\langle\psi_2|\right) &\subseteq \bar{\mu}\left(|\psi_0\rangle\langle\psi_0|\right) \mathbin{_{\mathscr{I}}\!+} \bar{\mu}\left(|\psi_1\rangle\langle\psi_1|\right) \end{aligned} \tag{5}$$
for every orthonormal basis $\mathcal{B} = \{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$. We are going to enumerate all possible orthonormal bases to verify equation (5), and case 1 and 2 are illustrated in figure 1.

1. When $|\psi_0\rangle$ is $|0\rangle$, then either $|\psi_1\rangle$ or $|\psi_2\rangle$ must be $\left|0^{\perp}_{\theta,\gamma}\right\rangle$ for some θ and γ. Because of
$$|0\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\ \mathrm{e}^{\mathrm{i}\gamma}\sin\theta\\ \cos\theta\end{pmatrix} = \begin{pmatrix}0\\ -\cos\theta\\ \mathrm{e}^{-\mathrm{i}\gamma}\sin\theta\end{pmatrix} = \mathrm{e}^{-\mathrm{i}\gamma}\begin{pmatrix}0\\ \mathrm{e}^{\mathrm{i}(\pi+\gamma)}\sin\left(\frac{\pi}{2}-\theta\right)\\ \cos\left(\frac{\pi}{2}-\theta\right)\end{pmatrix} = \mathrm{e}^{-\mathrm{i}\gamma}\left|0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right\rangle$$
the pair $|\psi_1\rangle = \left|0^{\perp}_{\theta,\gamma}\right\rangle$ and $|\psi_2\rangle = \left|0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right\rangle$ for $(0 < \theta < \frac{\pi}{2}$ and $0 \le \gamma < \pi)$ or $\theta = 0$ enumerate all possible orthonormal bases. Equation (5) can then be verified as follow.
$$\begin{aligned} \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} |0\rangle\langle 0|\right) = \mathit{certain} \subseteq \mathit{likely} \mathbin{_{\mathscr{I}}\!+} \mathit{unlikely} = \bar{\mu}\left(\left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) \mathbin{_{\mathscr{I}}\!+} \bar{\mu}\left(\left|0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right\rangle\left\langle 0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right|\right) \\ \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} \left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) = \mathit{unlikely} = \mathit{unlikely} \mathbin{_{\mathscr{I}}\!+} \mathit{impossible} = \bar{\mu}\left(\left|0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right\rangle\left\langle 0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right|\right) \mathbin{_{\mathscr{I}}\!+} \bar{\mu}\left(|0\rangle\langle 0|\right) \\ \bar{\mu}\left(\mathbb{1} \mathbin{_{\mathscr{O}}\!-} \left|0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right\rangle\left\langle 0^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right|\right) = \mathit{likely} = \mathit{impossible} \mathbin{_{\mathscr{I}}\!+} \mathit{likely} = \bar{\mu}\left(|0\rangle\langle 0|\right) \mathbin{_{\mathscr{I}}\!+} \bar{\mu}\left(\left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) \end{aligned} \tag{6}$$

[1]Need to verify whether this example is convex or not.

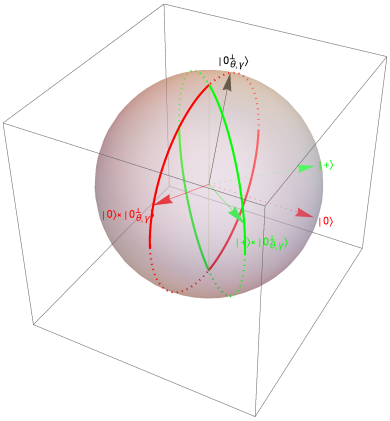

Figure 1: This figure illustrates case 1 and 2 in $\mathbb{R}^3$ in example 5. The red and green dotted vector are $|0\rangle$ and $|+\rangle$ respectively. All possible vectors of $\left|0^{\perp}_{\theta,\gamma}\right\rangle$ and $\left|+^{\perp}_{\theta,\gamma}\right\rangle$ are drawn in the red and green circles, respectively. Within the circles, a vector $|\psi\rangle$ in dotted part means $\bar{\mu}(|\psi\rangle\langle\psi|) = \textit{likely}$; otherwise, $\bar{\mu}(|\psi\rangle\langle\psi|) = \textit{unlikely}$. The gray vector is a generic vector $\left|0^{\perp}_{\theta,\gamma}\right\rangle$, and the red and green solid vectors are normalized $|0\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle$ and $|+\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle$, respectively.

Similarly, when $|\psi_0\rangle$ is $|+\rangle$, equation (5) holds.

2. When $|0\rangle \notin \mathcal{B}$ and $|\psi_0\rangle = \left|0^{\perp}_{\theta,\gamma}\right\rangle$ for $0 \le \theta < \frac{\pi}{2}$ and $0 \le \gamma < \pi$, we want to prove $\bar{\mu}\left(|\psi_1\rangle\langle\psi_1|\right) = \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) = \textit{unlikely}$. Then, we have

$$\begin{aligned} \bar{\mu}\left(\mathbb{1} \mathbin{\mathscr{O}-} \left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) = \textit{unlikely} \subseteq \textit{unlikely} \mathbin{\mathscr{I}+} \textit{unlikely} = \bar{\mu}\left(|\psi_1\rangle\langle\psi_1|\right) \mathbin{\mathscr{I}+} \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) \\ \bar{\mu}\left(\mathbb{1} \mathbin{\mathscr{O}-} |\psi_1\rangle\langle\psi_1|\right) = \textit{likely} \subseteq \textit{unlikely} \mathbin{\mathscr{I}+} \textit{likely} = \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) \mathbin{\mathscr{I}+} \bar{\mu}\left(\left|0^{\perp}_{\theta,\gamma}\right\rangle\left\langle 0^{\perp}_{\theta,\gamma}\right|\right) \end{aligned} \tag{7}$$

In order to verify $\bar{\mu}\left(|\psi_1\rangle\langle\psi_1|\right) = \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) = \textit{unlikely}$, it is sufficient to prove that if $\left|+^{\perp}_{\theta',\gamma'}\right\rangle \in \mathcal{B}$, then $(0 < \theta' < \frac{\pi}{2}$ and $\pi \le \gamma' < 2\pi)$ or $\theta' = \frac{\pi}{2}$. Recall $\left\langle + \middle| +^{\perp}_{\theta',\gamma'} \right\rangle = 0$. Hence, if $\left|+^{\perp}_{\theta',\gamma'}\right\rangle \in \mathcal{B}$, we have $\left|+^{\perp}_{\theta',\gamma'}\right\rangle$ parallel to $|+\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle$.

$$|+\rangle \times \left|0^{\perp}_{\theta,\gamma}\right\rangle = \begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\ e^{i\gamma}\sin\theta\\ \cos\theta\end{pmatrix} = \begin{pmatrix}\cos\theta\\ -\cos\theta\\ e^{-i\gamma}\sin\theta\end{pmatrix} = e^{-i\gamma}\begin{pmatrix}-e^{i(\pi+\gamma)}\sin\left(\frac{\pi}{2}-\theta\right)\\ e^{i(\pi+\gamma)}\sin\left(\frac{\pi}{2}-\theta\right)\\ \cos\left(\frac{\pi}{2}-\theta\right)\end{pmatrix} = \sqrt{2}e^{-i\gamma}\left|+^{\perp}_{\frac{\pi}{2}-\theta,\pi+\gamma}\right\rangle \tag{8}$$

Because of $0 \le \theta < \frac{\pi}{2}$ and $0 \le \gamma < \pi$, we have $(0 < \theta' = \frac{\pi}{2} - \theta < \frac{\pi}{2}$ and $\pi \le \gamma' = \pi + \gamma < 2\pi)$ or $\theta' = \frac{\pi}{2} - \theta = \frac{\pi}{2}$. Similarly, when $|\psi_0\rangle$ is $\left|+^{\perp}_{\theta,\gamma}\right\rangle$ and $|+\rangle \notin \mathcal{B}$, equation (5) holds.

3. Finally, when $|0\rangle \notin \mathcal{B}$, $|+\rangle \notin \mathcal{B}$, $\left|0^{\perp}_{\theta,\gamma}\right\rangle \notin \mathcal{B}$, and $\left|+^{\perp}_{\theta,\gamma}\right\rangle \notin \mathcal{B}$ for $0 \le \theta < \frac{\pi}{2}$ and $0 \le \gamma < \pi$, i.e., the "otherwise" case. Then, equation (5) can easily be verified.

$$\bar{\mu}\left(\mathbb{1} \mathbin{\mathscr{O}-} |\psi_0\rangle\langle\psi_0|\right) = \textit{likely} \subseteq \textit{unlikely} \mathbin{\mathscr{I}+} \textit{unlikely} = \bar{\mu}\left(|\psi_1\rangle\langle\psi_1|\right) \mathbin{\mathscr{I}+} \bar{\mu}\left(|\psi_2\rangle\langle\psi_2|\right) \tag{9}$$

2

[2]This example is a little bit too long to human-verify, and we need more examples. Maybe we need to write a program for further verification.

The following example verifies example 5 cannot hold with more precise intervals.

Example 6 (Three-dimensional quantum 4-value interval-valued probability measure (Continue)). Consider $\mathscr{I} = \{\mathit{impossible}, [l_1, r_1], [1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1], \mathit{certain}\}$, where $[l_1, r_1]$ and $[1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1]$ will be used to replace every *unlikely* and *likely* in example 5, respectively. The bound of l_1 and r_1 can then be estimated by the equations corresponding to equation (6), (7), and (9).

When we replace *unlikely* and *likely* by $[l_1, r_1]$ and $[1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1]$, equation (6) will become

$$\begin{aligned} \mathit{certain} &\subseteq [1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1] \ _{\mathscr{I}}+ [l_1, r_1] \\ [l_1, r_1] &\subseteq [l_1, r_1] \ _{\mathscr{I}}+ \mathit{impossible} \\ [1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1] &\subseteq \mathit{impossible} \ _{\mathscr{I}}+ [1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1] \end{aligned} \tag{10}$$

which are all tautologies. Equation (7) will become

$$\begin{aligned} [l_1, r_1] &\subseteq [l_1, r_1] \ _{\mathscr{I}}+ [l_1, r_1] \\ [1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1] &\subseteq [l_1, r_1] \ _{\mathscr{I}}+ [1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1] \end{aligned} \tag{11}$$

which implies $l_1 = 0$. Finally, equation (9) will become

$$[1 \ _{\mathbb{R}}- r_1, 1 \ _{\mathbb{R}}- l_1] \subseteq [l_1, r_1] \ _{\mathscr{I}}+ [l_1, r_1] \tag{12}$$

which implies $\frac{1}{2} \leq r_1 \leq 1$. Therefore, $l_1 = 0$ and $r_1 = \frac{1}{2}$ provides the most precise intervals which make example 5 work, and in this case, the intervals are $\{\mathit{impossible}, \mathit{unlikely}, \mathit{likely}, \mathit{certain}\}$.

Although it is easier to deduce an interval-valued probability measure has an empty core by considering the state mapped to *impossible* as in example 4 and 5, it is not the only way to prove an interval-valued probability measure has an empty core. The following family of interval-valued probability measures satisfy that $\bar{\mu}(P) = \mathit{impossible}$ if and only if $P = \mathbb{0}$, but we can still prove them have empty cores.

Although it is easier to deduce an interval-valued probability measure has an empty core by considering the state mapped to *impossible* as in example 4 and 5, it is not the only way to prove an interval-valued probability measure has an empty core. The following family of interval-valued probability measures satisfy that $\bar{\mu}(P) = \mathit{impossible}$ if and only if $P = \mathbb{0}$, but we can still prove them have empty cores.

4.3 Proposed Activity III: Quantum Information Theorems

Born KS: Mermin-Meyer

4.4 Proposed Activity IV: Computational Aspects

may find surprises like we found with UNIQUE-SAT; not sure where that belongs

clear goals: KS with interval probability or with quantum states as processes

talk about finite fields; previous work; unique-sat surprise; etc

... and quantum measurement expressed using game semantics

This is really about unifying classical and quantum theories of measurement. One of the most well-developed and formalized theories of classical measurement is the one for programming language observational contextual equivalence.

This worldview can be maintained as long as the impact of the measurements is really negligible on the execution time of the functions of interest.

measurement affects runtime and hence affects other measurements; could even conceive of programs whose input-output behavior is affected by measurement

This intended level of abstraction is reflected in the first line of the function declaration in Haskell. This *interface* (or *type* or *signature* or *contract*) line:

```
square :: Int -> Int
```

summarizes the publicly stated, computationally relevant, aspects of the function. In this case, it is announced that the function is a computer realization of a pure mathematical map from one `Int` to another. The fact that executing the computer realization of the map on a particular input might consume 10% of the battery power on your laptop is something that—although potentially critical knowledge in some circumstances—is not even expressible in the notation above.

Partial Functions (Time)

Computations take time. At some level of abstraction, we can ignore the time taken by a computation. For example, we can pretend that a calculation that takes one or two microseconds is similar to the "instantaneous" lookup in a mathematical mapping. But we can hardly keep pretending this as the time used by the calculation tends towards infinity.

Consider the following Java method:

```
int collatz (int n) {
  if (n == 1) return 0;
  else if (n % 2 == 0) return 1 + collatz (n / 2);
  else return 1 + collatz (3 * n + 1);
}
```

As evident from the use of `collatz` in the body of the method, this method is *recursive*. It takes an input number `n` and:

- if the input number is equal to one, the method immediately stops with the value zero, which represents in this case the total number of recursive calls performed before stopping;
- if the input number is even, the method divides it by two and recursively invokes itself, adding one to the total number of recursive calls;
- if the input number is odd, the method multiplies it by three, adds one, and then recursively invokes itself, after again adding one to the total number of recursive calls.

It is a famous mathematical conjecture, *the Collatz conjecture*, that this calculation always terminates for all positive integers. One can quickly check a few cases:

- invoking `collatz(3)` iterates through 10, 5, 16, 8, 4, 2, 1, and then stops returning the result 7.
- invoking `collatz(17)` iterates through 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, and then stops returning the result 12.

There is however no known proof that this method always stops. Thus, even though the method has the same interface as the method `square` above, one may not be able to interpret the two interfaces in the same way. To account for the possibility that the calculation never stops, the interface:

```
int collatz (int n)
```

should in general be read as saying:

> Given a particular `int` the method *may or may not* return another `int` in a finite amount of time.

Another way to look at the situation is that we have introduced "time" as an implicit parameter of computations. Some computations as the method `square` update—as a side effect—this time parameter by consuming a few ticks. By general agreement, this consumption is deemed negligible and not "observable" and does not generally constitute a computational effect. But when the consumption of time is unbounded, it arguably becomes an observable computational effect.

Assignments (Memory)

Computations require memory. Again, at some level of abstraction we can pretend that memory does not exist. But what happens if a computation starts changing external memory locations?

Consider the following variant of the method `square`:

```
int square (int n) {
  count++;
  return n * n;
}
```

As before, the method takes an integer `n` from the domain of Java integers `int` and returns another integer whose value is the square of the input. But, as a side effect the method reads and updates an external reference `count`, which can be used to find out how many times the method was called. Thus if during the execution of a particular program the method `square` is called ten times, then the value of `count` will be ten.

Thus the interface:

```
int square (int n)
```

should now be read as saying:

> Given a particular `int` the method *may possibly* read and update external memory references and then it *may or may not* return another `int` in a finite amount of time.

Another way to look at the situation is that we have introduced "memory" as an implicit parameter of computations. Some computations as the original method `square` might—as a side effect—allocate, update, and read several memory locations (registers, cache entries, stack frames, etc.). By general agreement, these actions on memory are not considered observable and do not generally constitute a computational effect. But when a computation updates an external memory location that other program fragments can inspect, the update of the memory location becomes an observable computational effect.

Runtime Exceptions (Context)

Computations occur in a context. When a computation occurs in the context of a larger computation, it is typically expected to return its result to that outer computation. But what if a computation cannot return a sensible value to its context and decides instead to abort it?

Consider again the method `square` whose interface is:

```
int square (int n)
```

We initially argued that it is possible to interpret this interface to say that given a particular `int` the method's execution promises to return another `int`. This is clearly an idealized view: in real life, any of an unlimited number of failures may occur in the hardware or underlying software components including the operating system or the Java virtual machine. The Java language collectively refers to these conditions as `Error`s or `RuntimeException`s.

So to be more accurate, and putting together our observations so far, one should interpret the interface:

```
int square (int n)
```

as saying:

> Given a particular `int` the method *may possibly* read and update external references, and then it *may or may not* return another `int` in a finite time or it *may fail to return to its context* due to any of a number of errors and runtime exceptions.

Another way to look at the situation is that we have introduced the "context" as an implicit parameter of computations. It is expected that every inner computation would resume its context with its result. But when an inner computation encounters a fatal error condition, it can ignore the context parameter, which causes the error condition to immediately terminate the entire computation. This escape clearly constitutes an observable computational effect.

Other Computational Effects

The unifying theme in all the above effects is that computations do not happen in isolation in some abstract mathematical world: rather computations are realized in some physical computing platform. The physical interactions of a computation with the computing platform and with other computations may be ignored in some idealized situation corresponding to simple computations, but they become crucial if one is interested in modeling the actual behavior of a sophisticated realistic computation. If these interactions happen "behind the scene" they are considered side effects of the computation.

Without further constraints on computations and their contexts, it is impossible to enumerate all the possible side effects that a computation may have. The reader may find it instructive to consider the following list of effects that could be performed by a method with the interface:

```
int square (int n)
```

Given a particular `int`, the method's behavior may include various combinations of the following actions:

- Reading and updating external memory locations, environment parameters, local or remote file systems or databases, etc.
- Communicating with a user or another local or remote process.
- Suspending itself for a certain period of time or until a certain outside condition happens.
- Resuming other suspended computations.
- Loading, compiling, and executing some other computation.
- Using reflection to examine and perhaps modify its own machine representation.
- Producing multiple answers by arranging for one `int` to be returned now and subsequent answers to be returned when the method is called again (with the same parameters).
- Checkpointing its entire state to a log, which can be examined and resumed later.
- Failing due to any of a number of errors and runtime exceptions.
- Consuming half of the battery life of a laptop and generating enough heat to keep you warm on a cold winter night.

And yes by the way, the method may actually return another `int` in a finite amount of time.

It is clear that the method's interface is nowhere close to giving us a reasonable approximation of the possible behaviors of the method: it sweeps too many things under the rug. Pragmatically, if a programmer invokes a method with the interface:

```
int square (int n)
```

the implicit computational effects of the method might come as a surprise.

The situation is not unique to Java. Almost every other programming language allows similar computational effects to occur implicitly. Even hardware circuits and assembly language instructions often perform implicit effects. For example, an assembly instruction might change the contents of the flag register as a side effect. In fact natural languages also include various "non-compositional phenomena," which are essentially

computational effects. For this reason, programmers are universally taught to avoid side effects if possible and to be careful when analyzing code that might contain side effects. But when side effects are unavoidable, the solution is to *expose them* as the rest of the article explains.

Common wisdom is that classical physics has no randomness. Only ignorance of the fine details or lack of control over them causes *statistical randomness*. In principle, but not in practice, randomness is absent from classical physics.

In sharp contrast, quantum randomness is unavoidable even in principle.

But but but ... Nature is one. The distinction between "classical" and "quantum" can only be in our minds, or more precisely in our mathematical models of the universe. We have two mathematical models, one which is, in principle deterministic, and one which is probabilistic. Nature must really be one or the other.

Consider this unsettling quote from QBism:

> Instead, the wavefunction obeys two fundamentally different laws:
>
> 1. As long as the electron is left to its own devices unobserved its wavefunction unfolds smoothly, continuously, and predictably. It obeys fixed mathematical laws, just like a bullet in flight and a wave on a lake.
> 2. When the electron reveals its whereabouts by leaving a dot on a screen, the wavefunction "collapses" suddenly into a new much more compact form concentrated on the point of impact.

Can we develop a mathematical model that unifies both the classical deterministic perspective and the quantum probabilistic perspective? Quantum information and quantum computation opens the door to using computational insights. The main one is that, in a computation, everything produced during a computation and everything observed about a computation is finite. The computational perspective imposes a principle we might call the principle of "finitely communicable evidence."

So imprecision in classical physics becomes, not some second-class principle that is considered to stem from practical limitations but rather a foundational principle. If everything is "fuzzy" so to speak and if our mathematical models embrace this fuzziness then a different worldview emerges. That worldview agrees with classical physics but embraces imprecision within the mathematical model not as an external non-mathematical nuisance. This shift in perspective enables a smooth transition to the world of quantum physics since our mathematical model has already embraced probabilistic effects.

Three insights:

- Interactive systems: response of quantum system determined *after* the observer initiates the experiment used to observe the system
- Bayesian probability, personal
- Finite precision

Goal: build computational model of QM that encompasses classical (reversible), computation and observation with finite resources, and hopefully revolutionize "quantum thinking"

5 Broader Impact and Education

6 PI Team and Prior NSF Support

References

[1] Abramsky, S. Big toy models: Representing physical systems as Chu spaces. *CoRR abs/0910.2393* (2009).

[2] Aerts, D., Durt, T., and Van Bogaert, B. A physical example of quantum fuzzy sets and the classical limit. 5–14.

[3] Bailey, D., Borwein, P., and Plouffe, S. On the rapid computation of various polylogarithmic constants. *Math. Comput. 66*, 218 (1997), 903–913.

[4] Barendregt, H. P. *The lambda calculus : its syntax and semantics*, revised ed. North-Holland ; Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co., Nov. 1984.

[5] Barrett, J., and Kent, A. Non-contextuality, finite precision measurement and the kochen–specker theorem. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 35*, 2 (2004), 151–176.

[6] Bernoulli, J. *The Art of Conjecturing*. Johns Hopkins University Press, 2006.

[7] Birkhoff, G., and Von Neumann, J. The logic of quantum mechanics. *Annals of mathematics* (1936), 823–843.

[8] Born, M. On the quantum mechanics of collisions. In *Quantum Theory and Measurement*. Princeton University Press, 1983, pp. 52–55.

[9] De Morgan, A. *A Budget of Paradoxes*. Longmans, Green, and Co., 1872.

[10] Feynman, R. Simulating physics with computers. *International Journal of Theoretical Physics 21*, 6-7 (June 1982), 467–488.

[11] Fuchs, C. A. Qbism, the perimeter of quantum bayesianism.

[12] Georges Louis Leclerc, C. d. B. Essai d'arithmétique morale. In *Histoire naturelle, générale et particulière, Supplément*, vol. 4. De l'Imprimerie royale, an 12, 1777, pp. 46–109.

[13] Gleason, A. Measures on the closed subspaces of a Hilbert space. *Indiana Univ. Math. J. 6* (1957), 885–893.

[14] Granik, A., and Caulfield, H. J. Fuzziness in quantum mechanics. 496.

[15] Griffiths, R. B. *Consistent quantum theory*. Cambridge University Press, 2003.

[16] Gudder, S. Fuzzy quantum probability theory. *AIP Conference Proceedings 750*, 1 (2005), 114–121.

[17] Hall, A. On an experimental determination of π. *Messeng. Math 2* (1873), 113–114.

[18] Hanson, A. J., Ortiz, G., Sabry, A., and Tai, Y.-T. Geometry of discrete quantum computing. *Journal of Physics A: Mathematical and Theoretical 46*, 18 (2013), 185301.

[19] Hanson, A. J., Ortiz, G., Sabry, A., and Tai, Y.-T. Discrete quantum theories. *Journal of Physics A: Mathematical and Theoretical 47*, 11 (2014), 115305.

[20] Hennessy, M. *Observing processes*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989, pp. 173–200.

[21] Jaeger, G. *Quantum Information*. Springer New York.

[22] Jamison, K. D., and Lodwick, W. A. Interval-valued probability measures, 2004.

[23] Kolmogorov, A. N. *Foundations of the Theory of Probability*. Chelsea Publishing Company, 1950.

[24] Landauer, R. The physical nature of information. *Physics Letters A 217* (1996), 188 – 193.

[25] Maassen, H. Quantum probability and quantum information theory. In *Quantum information, computation and cryptography*. Springer, 2010, pp. 65–108.

[26] Mackey, G. W. Quantum mechanics and Hilbert space. *The American Mathematical Monthly 64*, 8 (1957), 45–57.

[27] Mermin, N. D. A kochen-specker theorem for imprecisely specified measurement. *arXiv preprint quant-ph/9912081* (1999).

[28] Mermin, N. D. *Quantum Computer Science*. Cambridge University Press, 2007.

[29] Meyer, D. Finite precision measurement nullifies the Kochen-Specker theorem. *Phys. Rev. Lett. 83* (11 1999), 3751–3754.

[30] Milner, R. Elements of interaction: Turing award lecture. *Commun. ACM 36*, 1 (Jan. 1993), 78–89.

[31] Nielsen, M. A., and Chuang, I. L. *Quantum computation and quantum information*. Cambridge University Press, New York, NY, USA, 2000.

[32] Peres, A. *Quantum Theory: Concepts and Methods*. Fundamental Theories of Physics. Springer, 1995.

[33] Pykacz, J. *Fuzzy Sets in Foundations of Quantum Mechanics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013, pp. 553–557.

[34] Redhead, M. *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics*. Oxford University Press, 1987.

[35] Schmitt, I., Nürnberger, A., and Lehrack, S. *On the Relation between Fuzzy and Quantum Logic*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009, pp. 417–438.

[36] Shafer, G. *A mathematical theory of evidence*. Princeton University Press, 1976.

[37] Stern, O. A way towards the experimental examination of spatial quantisation in a magnetic field. *Zeitschrift für Physik D Atoms, Molecules and Clusters 10*, 2 (1988), 114–116.

[38] Swart, J. Introduction to quantum probability. Lecture Notes for the course NSTP187 at Charles University, Prague, Summer 2013.

[39] Turing, A. On computable numbers with an application to the "Entscheidungsproblem". *Proceeding of the London Mathematical Society* (1936).

[40] Uspensky, J. V. *Introduction to mathematical probability*. McGraw-Hill Book Co., Inc., 1937.

[41] Von Baeyer, H. C., and von Baeyer, L. *QBism*. Harvard University Press.

[42] Weichselberger, K. The theory of interval-probability as a unifying concept for uncertainty. *International Journal of Approximate Reasoning 24*, 2 (2000), 149–170.

[43] Willcock, J., and Sabry, A. Solving UNIQUE-SAT in a modal quantum theory. `arXiv:1102.3587` [quant-ph], 2011.