

# Incompatible results of quantum measurements

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Quantum theory is incompatible with the following propositions. (1) The result of the measurement of an operator  $A$  depends solely on  $A$  and on the system being measured. (2) If operators  $A$  and  $B$  commute, the result of a measurement of their product  $AB$  is the product of the results of separate measurements of  $A$  and of  $B$ .

Various quantum “paradoxes” [1–5] are based on the assumption that the result of the measurement of an operator  $A$  depends only on  $A$  and on the state of the quantum system being measured (here, the word “state” includes not only the wave function  $\psi$ , but also any hidden variables that theorists may invent). In particular, that result does not depend on the choice of *other* measurements that may also be performed on distant physical systems. The purpose of this Letter is to give an elementary proof that these assumptions are incompatible with quantum theory.

Consider two spin  $\frac{1}{2}$  particles in a singlet state. The result of a measurement of  $\sigma_{1x}$  (which may be only  $\pm 1$ ) will be called  $x_1$ , and similar notations used for the other components. We may measure a product  $\sigma_{1x}\sigma_{2x}$  either by measuring  $\sigma_{1x}$  and  $\sigma_{2x}$  separately, with result  $x_1x_2$ , or in a single nonlocal procedure. The result of the latter must be  $-1$ , because  $\langle \sigma_{1x}\sigma_{2x} \rangle = -1$  for a singlet state. It follows that  $x_1x_2 = -1$ . Likewise  $y_1y_2 = -1$  and  $z_1z_2 = -1$ .

Consider now a measurement of  $\sigma_{1x}\sigma_{2y}$ . We can measure  $\sigma_{1x}$  and  $\sigma_{2y}$  separately, with results  $x_1$  and  $y_2$ . We assume that these are the same  $x_1$  and  $y_2$  as mentioned above – that is, the result of measuring  $\sigma_{1x}$  does not depend on whether the other measurement is (was, will be) a measurement of  $\sigma_{2x}$  or one of  $\sigma_{2y}$ . Therefore, if we perform a direct measurement of  $\sigma_{1x}\sigma_{2y}$ , the result is  $x_1y_2$ .

Note, however, that

$$[\sigma_{1x}\sigma_{2y}, \sigma_{1y}\sigma_{2x}] = 0, \quad (1)$$

so that we can measure  $\sigma_{1y}\sigma_{2x}$  together with  $\sigma_{1x}\sigma_{2y}$ , without mutual disturbance. (Of course, we cannot measure  $\sigma_{1y}\sigma_{2x}$  together with  $\sigma_{1x}$  and  $\sigma_{2y}$  separately.) The result of measuring  $\sigma_{1y}\sigma_{2x}$  is  $y_1x_2$ , by the same argument as above.

Now, we also have, for the product of these two commuting operators,

$$\sigma_{1x}\sigma_{2y}\sigma_{1y}\sigma_{2x} = \sigma_{1x}\sigma_{1y}\sigma_{2y}\sigma_{2x} = \sigma_{1z}\sigma_{2z}, \quad (2)$$

whose expectation value is  $\langle \sigma_{1z}\sigma_{2z} \rangle = -1$ . It follows that

$$x_1y_2y_1x_2 = \langle \sigma_{1x}\sigma_{2y}\sigma_{1y}\sigma_{2x} \rangle = -1, \quad (3)$$

in contradiction with  $x_1x_2 = y_1y_2 = -1$ .

This simple exercise shows that what we call “the result of a measurement of  $A$ ” cannot depend only on  $A$  and on the state of the system (unless the wave function is an eigenstate of  $A$ ). It also depends, in a way not yet understood, on the choice of other quantum measurements that may possibly be performed. This property could also be inferred from the references listed below. However, the proof given here is much simpler than that of Kochen and Specker [3] (who found a similar incompatibility among 117 measurements of a particle of spin 1) and the more recent proofs in refs. [4] and [5], which require four and three particles, respectively, instead of just two, as here.

*Note added.* It was shown by Mermin [6] that the use of a singlet state is not essential to the above argument: Consider the nine operators

$$\begin{array}{ccc} \sigma_{1x} & \sigma_{2x} & \sigma_{1x}\sigma_{2x} \\ \sigma_{2y} & \sigma_{1y} & \sigma_{1y}\sigma_{2y} \\ \sigma_{1x}\sigma_{2y} & \sigma_{1y}\sigma_{2x} & \sigma_{1z}\sigma_{2z} . \end{array}$$

The three operators in each row and each column commute, and their product is 1, except those of the last column, whose product is  $-1$ . There is obviously no way of assigning numerical values  $\pm 1$  to these nine operators, with this multiplicative property. I am grateful to N.D. Mermin for a stimulating correspondence.

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## References

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