Free probability

Free probability is a mathematical theory that studies non-commutative random variables. The "freeness" or free independence property is the analogue of the classical notion of independence, and it is connected with free products. This theory was initiated by Dan Voiculescu around 1986 in order to attack the **free group factors isomorphism problem**, an important unsolved problem in the theory of operator algebras. Given a free group on some number of generators, we can consider the von Neumann algebra generated by the group algebra, which is a type II_1 factor. The isomorphism problem asks if these are isomorphic for different numbers of generators. It is not even known if any two free group factors are isomorphic. This is similar to Tarski's free group problem, which asks whether two different non-abelian finitely generated free groups have the same elementary theory.

Later connections to random matrix theory, combinatorics, representations of symmetric groups, large deviations, quantum information theory and other theories were established. Free probability is currently undergoing active research.

Typically the random variables lie in a unital algebra A such as a C-star algebra or a von Neumann algebra. The algebra comes equipped with a **noncommutative expectation**, a linear functional φ: $A \to \mathbf{C}$ such that $\varphi(1) = 1$. Unital subalgebras A_1, ..., Am are then said to be **freely independent** if the expectation of the product $a_1 ... an$ is zero whenever each aj has zero expectation, lies in an Ak, and no adjacent aj's come from the same subalgebra Ak. Random variables are freely independent if they generate freely independent unital subalgebras.

One of the goals of free probability (still unaccomplished) was to construct new invariants of von Neumann algebras and free dimension is regarded as a reasonable candidate for such an invariant. The main tool used for the construction of free dimension is free entropy.

The free cumulant functional (introduced by Roland Speicher)[1] plays a major role in the theory. It is related to the lattice of noncrossing partitions of the set $\{ 1, ..., n \}$ in the same way in which the classic cumulant functional is related to the lattice of *all* partitions of that set.[1]

1 See also

- Random matrix
- Wigner semicircle distribution
- Circular law
- Free deconvolution

2 Notes

[1] Barndorff-Nielsen, Ole E.; Thorbjørnsen, Steen (2006), "Classical and free infinite divisibility and Lévy processes", *Quantum independent increment processes. II*, Lecture Notes in Math. **1866**, Berlin: Springer, pp. 33–159, doi:10.1007/11376637_2, MR 2213448. Barndorff-Nielsen and Thorbjørnsen credit the introduction of free cumulants to Speicher, Roland (1994), "Multiplicative functions on the lattice of non-crossing partitions and free convolution", *Mathematische Annalen* **298** (4): 611–628, doi:10.1007/BF01459754, MR 1268597.

3 References

- A. Nica, R. Speicher: *Lectures on the Combinatorics of Free Probability.* Cambridge University Press, 2006, ISBN 0-521-85852-6
- Fumio Hiai and Denis Petz, *The Semicircle Law, Free Random Variables, and Entropy*, ISBN 0-8218-2081-8
- Mitchener, P.D. (2005) *Non-Commutative Probability Theory*, preprint
- Voiculescu, D. V.; Dykema, K. J.; Nica, A. *Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.* CRM Monograph Series, 1. American Mathematical Society, Providence, RI, 1992. ISBN 0-8218-6999-X
- Terence Tao, 254A, Notes 5: Free probability (10 February, 2010), course notes for graduate course on "Topics in random matrix theory"

4 External links

- Voiculescu receives NAS award in mathematics — contains a readable description of free probability.
- RMTool — A MATLAB-based free probability calculator.

- Alcatel-Lucent Chair on Flexible Radio Applications of Free Probability to Wireless Communications.
- survey articles of Roland Speicher on free probability.

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