A KOCHEN-SPECKER THEOREM FOR INTEGER MATRICES AND NONCOMMUTATIVE SPECTRUM FUNCTORS

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WITH APPENDIX BY ALEXANDRU CHIRVASITU

Abstract. We investigate the possibility of constructing Kochen-Specker uncolorable sets of idempotent matrices whose entries lie in various rings, including the rational numbers, the integers, and finite fields. Most notably, we show that there is no Kochen-Specker coloring of the $n \times n$ idempotent integer matrices for $n \geq 3$, thereby illustrating that Kochen-Specker contextuality is an inherent feature of pure matrix algebra. We apply this to generalize recent no-go results on noncommutative spectrum functors, showing that any contravariant functor from rings to sets (respectively, topological spaces or locales) that restricts to the Zariski prime spectrum functor for commutative rings must assign the empty set (respectively, empty space or locale) to the matrix ring $\mathbb{M}_n(R)$ for any integer $n \geq 3$ and any ring R. An appendix by Alexandru Chirvasitu shows that Kochen-Specker colorings of idempotents in partial subalgebras of $\mathbb{M}_3(F)$ for a perfect field F can be extended to partial algebra morphisms into the algebraic closure of F.

1. Introduction

The Bell-Kochen-Specker Theorem [6, 31] is a no-go theorem that shows the impossibility of certain hidden variable theories for quantum mechanics. The usual formulation of the Heisenberg Uncertainty Principle in terms of matrix mechanics shows that we can only expect to have precise knowledge of the values of two quantum-mechanical observables P and Q if these observables (represented as operators on some Hilbert space) commute: $PQ = QP$. Thus commuting observables are also called *commeasurable*. The nature of the algebra of operators on a Hilbert space H (of dimension $\dim(H) \geq 2$) is such that one may have an observable P commeasurable with two observables Q and Q', but such that Q and Q' are not commeasurable with one another. Thus one may expect to have precise simultaneous knowledge of the values of P and Q, or of P and Q', but not of the triple $\{P, Q, Q'\}$. A hidden variable theory is called *non-contextual* if the value $v(P)$ assigned to an observable P is independent of the choice of pairwise commeasurable set of observables $\{P, Q_1, Q_2, \dots\}$ that also happen to be measured by an experimental setup. This property was emphasized by Bell in [6, Section V].

Date: September 21, 2015.

2010 *Mathematics Subject Classification.* Primary: 81P13, 16B50; Secondary: 03G05, 15B33, 15B36, 06D22.

Key words and phrases. Kochen-Specker Theorem, contextuality, idempotent integer matrix, prime spectrum, noncommutative spectrum, prime partial ideal, partial Boolean algebra.

This material is based upon work supported by the National Science Foundation under grant no. DMS-1407152.

Now suppose that one restricts to observables that are projections (i.e., self-adjoint idempotent operators) on H. The value of each projection, being an eigenvalue of the operator, is either 0 or 1, so that such observables represent "yes-no questions" that may be asked about the underlying quantum system. Further, if one has an orthogonal set $\{P_i\}$ of projections whose sum is the identity (such as the projections onto an orthonormal basis of H), these classically correspond to mutually exclusive, collectively exhaustive propositions about the system. If one measures the values of the P_i simultaneously, then compatibility with the classical logic of Boolean algebras would require that one of these projections is assigned the value 1 and the rest are assigned value 0. Thus, a non-contextual hidden variable theory is expected to "color" every projection on H with a value 0 or 1 in such a way that, for each basis $\{v_i\}$ of H, the projection onto exactly one of the v_i is assigned the value 1. But Kochen and Specker proved such an assignment to be impossible whenever $\dim(H) \geq 3$, by providing an explicit set of orthogonal projections (represented by vectors in their ranges) for which such a $\{0, 1\}$-valued function does not exist. Bell [6] provided an alternative proof using Gleason's Theorem. (Our own methods follow closely those of Kochen and Specker, especially considering colorings of finite sets of vectors or idempotents. For this reason, we regularly refer to the no-hidden-variables theorem as the *Kochen-Specker Theorem*.)

By now there are many fine discussions of the role of the Bell-Kochen-Specker theorem in the logical foundations of quantum mechanics, so we have limited our discussion of this background to only a brief explanation of the physical intuition behind the mathematical result. Aside from the original papers of Bell and Kochen-Specker, we refer readers to the textbooks [5, 33] for introductions to the theorem in the broader context of hidden variable theories, to the article [3] for a discussion of various claimed "loopholes" to the theorem, and to the detailed survey [24] for further discussion and references to the literature.

There is much recent interest in examining the Kochen-Specker Theorem from new perspectives. One of the most notable such programs is the formulation of the theorem in the context of topos theory [28]. There are also approaches to the general theory of contextuality through sheaf theory [1] and through graphs and hypergraphs [11, 2]. There has been recent progress [42] on the problem of determining lower bounds for the size of a Kochen-Specker uncolorable set of three-dimensional vectors. The theorem has also found recent application in the well-known "Free Will Theorems" of Conway and Kochen [13, 14].

Our present work seeks to push the study of the Kochen-Specker Theorem in a new direction by allowing the study of contextuality for vectors and matrices whose entries lie in more general coefficient rings than the real or complex numbers, and it is motivated by applications in the setting of noncommutative algebraic geometry. The original analysis of Kochen and Specker framed the discussion of hidden variables in algebraic terms as an assignment of values to all observables on a quantum system whose restriction to any commesaurable set of observables forms a *homomorphism*. Such an assignment of values will be called a *morphism of partial rings*, as discussed in more detail in Section 2 below. From this perspective, one may view any noncommutative ring R as a purely algebraic analogue of the observables of a quantum system, with its commutative subrings as "commeasurable" subsets of observables, and investigate the possibility of "noncontextual hidden variable theories" on R as morphisms of partial rings from R to a commutative ring.

Table 1. Idempotent colorings and partial spectra of partial rings of matrices

partial ring R	prime p	$\mathrm{Idpt}(R)$	p-$\mathrm{Spec}(R)$
$\mathbb{M}_3(\mathbb{F}_p)_{\mathrm{sym}}$	$p = 2, 3$	colorable	nonempty
$\mathbb{M}_3(\mathbb{Z})_{\mathrm{sym}}$		colorable	nonempty
$\mathbb{M}_3(\mathbb{F}_p)_{\mathrm{sym}}$	$p \geq 5$	uncolorable	empty
$\mathbb{M}_3(\mathbb{Z}[1/30])_{\mathrm{sym}}$		uncolorable	empty
$\mathbb{M}_3(\mathbb{Q})_{\mathrm{sym}}$		uncolorable	empty
$\mathbb{M}_3(\mathbb{Z})$		uncolorable	empty

In this paper we establish that contextuality—in the purely algebraic sense of inadmissibility of such morphisms of partial rings—is a property inherent to any matrix ring of the form $\mathbb{M}_n(R)$ for $n \geq 3$, independent of the choice of the ring of scalars R. We consider this problem from the intimately related perspectives of Kochen-Specker colorings of idempotents and of morphisms of partial rings. Section 2 contains background and fundamental results on partial rings and partial Boolean algebras in the sense of Kochen and Specker, showing the precise relationships between colorability of idempotents, morphisms of partial rings, and the spectrum p-$\mathrm{Spec}(R)$ of *prime partial ideals* of a partial ring R. Most of these relationships are expressed in the basic language of categories [34], in terms of various functors and natural transformations.

Then in Section 3 we prove that algebraic analogues of the Kochen-Specker theorem do or do not hold in various (partial) rings of matrices. These results are summarized in Table 1. Given a ring S, we let $\mathrm{Idpt}(S)$ denote the set of idempotents of S, which carries the structure of a partial Boolean algebra. A formal definition of a Kochen-Specker coloring is given in Definition 2.6. For a commutative ring R, the set of matrices in $\mathbb{M}_3(R)$ that are symmetric (equal to their own transpose) is denoted $\mathbb{M}_3(R)_{\mathrm{sym}}$. We remark that the partial rings S in Table 1 for which p-$\mathrm{Spec}(S) = \varnothing$ admit no morphism of partial rings $S \to C$ for any (total) commutative ring C, yielding a direct analogue of the type of obstruction that Kochen and Specker originally sought.

Our motivation for this study stems from the recent application of the Kochen-Specker theorem to noncommutative geometry in [38]. There it was shown that any contravariant functor F from rings to sets (or to topological spaces) whose restriction to commutative rings is the prime spectrum functor Spec must satisfy $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$. In Section 4 we strengthen this result to conclude that such functors F in fact satisfy $F(\mathbb{M}_n(R)) = \varnothing$ for every ring R and integer $n \geq 3$.

Our results may be of particular interest in the study of models of quantum mechanics defined over fields other than the real or complex numbers. Quantum physics over p-adic fields has been a subject of interest for some time; for versions of p-adic quantum theory in which the amplitudes of wavefunctions are p-adic (as surveyed, for instance, in [16, Section 9]), the "matrix entries" of observable operators will also have p-adic values. Quantum physics over finite fields has become a topic of recent interest, including investigations involving modal quantum theory [41], quantum computing [21, 22], and even quantum field theory [40, 4].

Our results show that some form of contextuality persists in either these settings, where observables on finite-dimensional systems form matrices with entries over exotic commutative rings.

We wish to thank Alexandru Chirvasitu for several discussions and suggestions throughout the writing of this paper, as well as Chris Heunen for useful comments on a draft of the paper.

2. Partial algebraic structures and Kochen-Specker colorings

We will largely follow the basic definitions of [31], as adapted to the ring-theoretic setting in [38]. We follow the convention that every ring is associative and contains a multiplicative identity, and every ring homomorphisms preserves multiplicative identity elements.

The physical intuition for the terminology below is that a partial k-algebra A consists of "observables of a quantum system," with $x, y \in A$ commeasurable if and only if the values of x and y can be simultaneously measured with arbitrarily high precision.

Definitions 2.1. Let k be a commutative ring. A *partial algebra* A over k is a set equipped with:

- a reflexive and symmetric binary relation $\odot \subseteq A \times A$, called *commeasurability*,
- "partial" addition and multiplication operations $+$ and $\cdot$ that are functions $\odot \to A$,
- a scalar multiplication operation $k \times A \to A$, and
- zero and unity elements $0, 1 \in A$,

satisfying the following axioms:

(1) 0 and 1 are commeasurable with all elements of A,
(2) the parital binary operations preserve commeasurability,
(3) for every pairwise commeasurable subset $S \subseteq A$, there exists a pairwise commeasurable subset $T \subseteq A$ containing S such that the restriction of the partial operations of A make T into a (unital, associative) commutative k-algebra.

If A and B are partial k-algebras, then a function $f : A \to B$ is called a *morphism of partial k-algebras* if $f(0) = 0$, $f(1) = 1$, and whenever $x, y \in A$ are such that $x \odot y$, it follows that

- $f(x) \odot f(y)$ in B,
- $f(x + y) = f(x) + f(y)$, and
- $f(xy) = f(x)f(y)$.

A partial algebra over the ring $k = \mathbb{Z}$ is called a *partial ring.* A morphism of partial $\mathbb{Z}$-algebras is also called a *morphism of partial rings.* The category of partial rings with morphisms of partial rings is denoted pRing.

We will use the terms *total k-algebra* and *total ring* to distinguish the usual notions of k-algebra and ring from their partial counterparts.

We define a *partial k-subalgebra* of a partial k-algebra R to be a subset $S \subseteq R$ that is a partial k-algebra under the restricted commeasurability relation and partial operations inherited from R, or equivalently, such that such that S is closed under k-scalar multiplication as well as sums and products of commeasurable elements. If all elements of S are pairwise commeasurable, we say that S is a *commeasurable* subalgebra; it is clear that S becomes a total subalgebra under the induced operations. In case $k = \mathbb{Z}$, we use the term *partial subring* for a partial $\mathbb{Z}$-subalgebra. (We note without further discussion that there is a subtlety in this terminology: it is possible to have a partial ring S, whose underlying set is a subset of a second partial ring R, such that the commeasurability relation on S is strictly coarser than that of R. Then the inclusion function $\imath\colon S \to R$ is a morphism of partial rings, but S is not a partial subring of R in the sense above.)

In the algebraic formulation of quantum mechanics, one views commutative algebras as corresponding to classical systems. Then commeasurable subalgebras of a partial algebra can be seen as "measurements" that may be performed on the corresponding quantum system. For more detail on this perspective, we refer readers to the recent survey [25].

The *(Zariski) spectrum* of a commutative ring R, denoted $\mathrm{Spec}(R)$, is the set of prime ideals of R. As discussed in Section 4 below, the spectrum is a spatial invariant of a commutative ring; but form the time being, we will view it merely as a set. We recall the extension of the spetcrum to an invariant of partial rings as in [38].

Definition 2.2. A subset $\mathfrak{p}$ of a partial ring R is a *prime partial ideal* if, for every commeasurable subring $C \subseteq R$, the intersection $C \cap \mathfrak{p}$ is a prime ideal of C; this is equivalent to the conditions that $1 \notin \mathfrak{p}$ and if $a, b \in R$ are commeasurable with $ab \in \mathfrak{p}$, then either $a \in \mathfrak{p}$ or $b \in P$. The set of all prime partial ideals of R is denoted p-$\mathrm{Spec}(R)$.

Given a morphism of partial rings $f\colon R \to S$ and $\mathfrak{p} \in p$-$\mathrm{Spec}(S)$, one may verify that $f^{-1}(\mathfrak{p}) \in p$-$\mathrm{Spec}(R)$; see [38, Lemma 2.10]. Using this assignment on morphisms, way we consider p-$\mathrm{Spec}\colon \mathsf{pRing}^{\mathrm{op}} \to \mathsf{Set}$ as a functor to the category of sets. In particular, if we consider the category of rings as a subcategory of pRing (via the obvious "forgetful" functor $\mathsf{Ring} \to \mathsf{pRing}$), the partial spectrum restricts to the usual prime spectrum functor $\mathrm{Spec}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Set}$.

The remark and lemma below illustrate that the partial spectrum of a partial ring may be viewed as an invariant to detect obstructions of morphisms in pRing to total commutative rings, reminiscent of Kochen and Specker's treatment of hidden variable theories.

Remark 2.3. It is well-known that every nonzero commutative ring has a (maximal, hence) prime ideal and consequently has a nonempty spectrum; for instance, see [35, Theorem 1.1]. On the other hand, the results of [38] (to be generalized in Corollary 3.11) show that sufficiently large matrix rings have empty partial spectrum. A useful technique to show that a particular partial ring R has empty partial spectrum is to produce a ring morphism of partial rings $R_0 \to R$ such that p-$\mathrm{Spec}(R_0) = \varnothing$. For then functoriality of the partial spectrum yields a function p-$\mathrm{Spec}(R) \to p$-$\mathrm{Spec}(R_0) = \varnothing$, and because the only set with a function to the empty set is the empty set itself, we deduce p-$\mathrm{Spec}(R) = \varnothing$.

Lemma 2.4. *Given a partial ring R, if p-$\mathrm{Spec}(R) = \varnothing$ then:*

(1) *There is no morphism of partial rings $R \to C$ for any nonzero (total) commutative ring C;*
(2) *The colimit in cRing of the diagram of commutative subrings of R is zero.*

Proof. Let $f\colon R \to C$ be a morphism as in (1) where $C \neq 0$. There exists a prime ideal $\mathfrak{p} \in \mathrm{Spec}(C)$ as in Remark 2.3. Thus $f^{-1}(\mathfrak{p}) \in p$-$\mathrm{Spec}(R)$, contradicting the assumption that p-$\mathrm{Spec}(R) = \varnothing$.

For (2), let $L = \varinjlim C$ be the colimit in cRing of all commeasurable subrings $C \subseteq R$, equipped with canonical morphisms $f_C \colon C \to L$. Each $x \in R$ is contained in a commeasurable subring $C \subseteq R$, and the construction of the colimit is such that the value $f(x) = f_C(x) \in L$ is independent of the choice of C. In this way we obtain a well-defined function $f \colon R \to L$ that is readily verified to be a morphism of partial rings (since f restricts on each commeasurable subring to a ring homomorphism). It now follows from (1) that if $p\text{-Spec}(R) = \varnothing$, then $L = 0$. □

Thanks to the above, an important special case for us is the integer matrix ring $\mathbb{M}_n(\mathbb{Z})$. This ring plays a *universal* role in no-go theorems, due to the fact that each ring R admits a unique ring homomorphism $\mathbb{Z} \to R$, which induces a ring homomorphism $\mathbb{M}_n(\mathbb{Z}) \to \mathbb{M}_n(R)$. Thus a Kochen-Specker type of theorem proved for $\mathbb{M}_n(\mathbb{Z})$ typically extends to $\mathbb{M}_n(R)$ for *any* ring R.

Kochen and Specker also considered "partial logical structures" in the following way. (We follow the terse, but efficient, alternative definition given in [7].) Recall that a *Boolean algebra* $(B, \vee, \wedge, \neg, 0, 1)$ is a structure such that $(B, \vee, \wedge, 0, 1)$ is a distributive lattice with bottom element 0 and top element 1, and a unary orthocomplement operation $\neg \colon B \to B$ (i.e., $\neg$ is an order-reversing involution that maps each element to a lattice-theoretic complement). The category Bool of Boolean algebras has as its morphisms the lattice homomorphisms that preserve top and bottom elements along with the orthocomplement operation. We refer readers to [20, 18] for the basic theory of Boolean algebras.

Definition 2.5. A *partial Boolean algebra* is a set B equipped with:

- a reflexive and symmetric commeasurability relation $\odot \subseteq B \times B$,
- a unary operation of negation $\neg \colon A \to A$,
- partially defined binary operations of meet and join $\wedge, \vee \colon \odot \to A$,
- elements $0, 1 \in B$,

such that every set $S \subseteq B$ of pairwise commeasurable elements is contained in a pairwise commesaurable set $T \subseteq B$ containing 0 and 1 for which the restriction of the operations makes T into a Boolean algebra.

There is a classical correspondence [20, §2] between Boolean algebras and *Boolean rings,* which are rings in which every element is idempotent. We may easily extend this correspondence to the setting of partial structures as follows. Define a *partial Boolean ring* to be a partial ring in which every element is idempotent. Every partial Boolean algebra B inherits the structure of a partial Boolean ring by keeping the same commeasurability relation and zero and unity elements, and defining addition via symmetric difference and multiplication as meet: if $x \odot y$ then

$$\begin{aligned} x + y &= (x \vee y) \wedge \neg(x \wedge y), \\ x \cdot y &= x \wedge y. \end{aligned}$$

Conversely, if R is a partial Boolean ring, then it inherits the structure of a partial Boolean algebra using the same commeasurability relation, zero, and unity, while using the following meet and join operations: if $r \odot s$ then

$$r \vee s = r + s - r \cdot s,$$
$$r \wedge s = r \cdot s.$$

As in the case of partial rings, we say that a subset of a partial Boolean algebra B is a *partial Boolean subalgebra* if it forms a partial Boolean algebra under the restricted commeasurability relation and partial operations from B. We also use the terms *total Boolean algebra* and *commeasurable Boolean subalgebra* to refer to the obvious Boolean analogues of the corresponding ring-theoretic notions.

In fact, this correspondence is functorial. Let pBRing denote the full subcategory of pRing whose objects are the partial Boolean rings, and let pBool denote the category whose objects are the partial Boolean algebras and whose morphisms are functions which preserve commeasurability, zero, unity, and the operations (when defined). Given (total) Boolean algebras A and B, the set of Boolean algebra homomorphisms from A to B is equal to the set of ring homomorphisms from A to B when both are considered as Boolean rings [20, §9]. Applying this observation piecewise to each commeasurable Boolean subalgebra (resp., subring) of two partial Boolean algebras A and B, we find that $\mathsf{pBool}(A, B) = \mathsf{pBRing}(A, B)$ as subsets of the set of functions B^A. These "identity" correspondences determine functors $\mathsf{pBool} \to \mathsf{pBRing}$ and $\mathsf{pBRing} \to \mathsf{pBool}$, which are mutually inverse and provide an equivalence (even isomorphism!) of categories $\mathsf{pBool} \cong \mathsf{pBRing}$.

We say that two elements p and q of a partial Boolean algebra B are *orthogonal* if they are commeasurable and $p \wedge q = 0$. Similarly, we define a partial ordering on B by declaring $p \leq q$ if p and q are commeasurable and $p \vee q = q$.

Definition 2.6. Let B be a partial Boolean algebra with a subset $\mathcal{S} \subseteq B$. A black-and-white coloring of $\mathcal{S}$ is called a *Kochen-Specker coloring* if, for every list of pairwise orthogonal elements $p_1, \dots, p_n \in B$,

(1) there is at most one index i such that p_i is colored white, and
(2) if furthermore $p_1 \vee \cdots \vee p_n = 1$, then there is exactly one index i such that p_i is colored white.

While the definition above is suited to an arbitrary subset of a partial Boolean algebra B, the algebraic theory of such colorings is best behaved in the case when one takes $\mathcal{S} = B$ to be the entire algebra, as illustrated in the theory outlined in the remainder of this section. On the other hand, to prove that the (possibly infinite) partial Boolean algebra B has no Kochen-Specker colorings, it clearly suffices to exhibit a smaller (finite) subset $\mathcal{S}$ of B that has no such coloring.

Notice immediately that for any Kochen-Specker coloring of a nontrivial ($0 \neq 1$) partial Boolean algebra, 0 is black and 1 is white by applying the condition above to the orthogonal decomposition $1 = 1 \vee 0 \vee 0$. On the other hand, the trivial Boolean algebra has no Kochen-Specker coloring, as evidenced by condition (2) applied to $0 = 0 \vee 1 = 1$.

Remark 2.7. In the physics literature, Kochen-Specker colorings are usually considered on sets of vectors in real or complex Hilbert spaces H. Such colorings can be considered as colorings subsets of the orthomodular lattice of orthogonal projections on H (viewed as a partial Boolean algebra as in [26, Lemma 3.3], for instance) by identifying a vector v with the rank-1 orthogonal projection of H onto the line spanned by v.

Next we aim to show that Kochen-Specker colorings on a piecewise Boolean B algebra are intimately related to the appropriate notion of prime ideals and ultrafilters of B. The suitable generalizations of these objects are as follows.

Definition 2.8. A subset I of a partial Boolean algebra B is called a *partial ideal* if it satisfies the following conditions for commeasurable elements $p, q \in B$:

(i) $0 \in I$;
(ii) If $q \in I$ and $p \leq q$, then $p \in I$;
(iii) If $p, q \in I$, then $p \vee q \in I$.

A partial ideal I of B is called a *prime* partial ideal if it additionally satisfies the following condition for all commeasurable $p, q \in B$:

(iv) $1 \notin I$, and if $p \wedge q \in I$, then $p \in B$ or $q \in B$ (or equivalently, for all $p \in B$, either p or $\neg p$ is in B but not both).

The set of prime partial ideals of B will be denoted p-Spec(B).

Definition 2.9. A subset F of a partial Boolean algebra B is called a *partial filter* if it satisfies the following conditions for commeasurable elements $p, q \in F$:

(i) $1 \in F$;
(ii) $p \in F$ and $p \leq q$ imply $q \in F$;
(iii) $p, q \in F$ implies $p \wedge q \in F$.

A partial filter F is called a *partial ultrafilter* if it additionally satisfies the following condition for all commeasurable $p, q \in B$:

(iv) $0 \notin B$ and if $p \vee q \in F$, then $p \in B$ or $q \in B$ (or equivalently, for all $p \in B$, either p or $\neg p$ is in F but not both).

The definitions above coincide with the usual definitions of prime ideals and ultrafilters in case the partial Boolean algebra is in fact a total Boolean algebra. Furthermore, it is clear that a subset X of a partial Boolean algebra B is a partial ideal (respectively, prime partial ideal, partial filter, or partial ultrafilter) if and only if $X \cap C$ is an ideal (respetively, prime ideal, filter, or ultrafilter) of C for every commeasurable Boolean subalgebra C of B. As in the classical case of total Boolean algebras, one may readily verify that a subset I of a partial Boolean algebra B is a partial ideal if and only if $\neg I = \{\neg x \mid x \in I\}$ is a partial filter of B, and that I is a prime partial ideal if and only if $\neg I$ is an ultrafilter, if and only if $B \setminus I$ is an ultrafilter (equal to $\neg I$).

We also note that if B is a partial Boolean algebra and I is a subset of B, then I is a prime partial ideal of B considered as a partial Boolean algebra if and only if I is a prime partial ideal of B when considered as a partial Boolean ring. (This is perhaps most easily verified considering the intersection $I \cap C$ for all commeasurable Boolean subalgebras $C \subseteq B$, and recalling that the two notions coincide in the classical case of total Boolean algebras and rings.) Thus there is no danger in our use of the notation p-Spec(B) for the spectrum of prime partial ideals in both senses, as these two spectra in fact coincide. This assignment forms a functor p-Spec: $\mathsf{pBool}^{\mathrm{op}} \to \mathsf{Set}$, which acts on a morphism $f\colon A \to B$ by sending $\mathfrak{p} \in p\text{-Spec}(B)$ to $\mathrm{Spec}(f)(\mathfrak{p}) = f^{-1}(\mathfrak{p}) \in p\text{-Spec}(A)$.

Proposition 2.10. *Let B be a partial Boolean algebra, and fix a black-and-white coloring of B. The coloring is a Kochen-Specker coloring if and only if the set of white elements forms a partial ultrafilter of B, if and only if the set of black elements forms a prime partial ideal of B.*

Proof. First suppose that the coloring is Kochen-Specker; we verify the four conditions of Definition 2.9 for the set of white elements. Condition (i) follows by applying the Kochen-Specker condition to the orthogonal decomposition $1 = 1 \vee 0 \vee 0$. Condition (iv) follows easily by applying the Kochen-Specker condition to the orthogonal decomposition $1 = p \vee (\neg p)$. For condition (ii), suppose that $p \leq q$ in B with p white. In the orthogonal decomposition $1 = p \vee (q \wedge \neg p) \vee \neg q$, because p is white the third term $\neg q$ must be black, so that it follows from (iv) that q is white. For (iii), suppose that $p, q \in B$ are commeasurable and white. In the decomposition

$$1 = (p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee \neg(p \vee q),$$

exactly one of the four joined terms on the right is white. If any of the second, third, or fourth terms is white, then its complement is black by (iv); as each of these elements x has either $p \leq x$ or $q \leq x$, we would deduce from condition (ii) the contradiction that either p or q is black. Thus we must have $p \wedge q$ white as desired.

Conversely, suppose that the set of white elements of the coloring satisfies conditions (i)–(iv) of Definition 2.9. To verify the Kochen-Specker condition, suppose

$$1 = p_1 \vee \cdots \vee p_n \tag{2.11}$$

for pairwise orthogonal elements $p_i \in B$; we prove inductively that exactly one of the p_i is white. The trivial case $n = 1$ follows from (i). In case $n = 2$, the fact that exactly one of p_1 or $p_2 = \neg p_1$ is white follows from (iv). Proceeding inductively, suppose the Kochen-Specker condition holds for all orthogonal decompositions of the unit into $n - 1 \geq 2$ elements. We may rewrite (2.11) as

$$1 = (p_1 \vee p_2) \vee p_3 \vee \cdots \vee p_n$$

and deduce by the inductive hypothesis that exactly one of $q = (p_1 \vee p_2), p_3, \ldots, p_n$ is white. If one of $p_3, \ldots, p_n$ is white then q is black. Applying (ii) to $p_1, p_2 \leq q$ we obtain that p_1 and p_2 are both black, as desired. Now in case $p_3, \ldots, p_n$ are black and q is white, we only need to verify that exactly one of p_1 or p_2 is white. In the orthogonal decomposition $1 = q \vee \neg q = p_1 \vee p_2 \vee (\neg q)$, condition (iv) implies that $\neg q$ is black. If p_1 and p_2 are both black, then $\neg(p_1 \vee p_2) = (\neg p_1) \wedge (\neg p_2)$ is a join of white elements and therefore is white by (iii), implying the contradiction that $p_1 \vee p_2 = q$ is black. Thus at least one of p_1 or p_2 is white. Because $p_1 \wedge p_2 = 0 = \neg 1$ is black, condition (iii) now shows that only one of p_1 or p_2 can be white, as desired.

The set of white elements of the coloring is an ultrafilter if and only if its complement, the set of black elements, is a prime partial ideal. This completes the proof. $\square$

Let $\mathrm{KS}(B)$ denote the set of Kochen-Specker colorings of a partial Boolean algebra B. Given a morphism $f\colon B_1 \to B_2$ in pBool and a Kochen-Specker coloring of B_2, one may readily verify using Proposition 2.10 that the coloring of B_1 given by declaring $b \in B_1$ white if and only if $f(b) \in B_2$ is white yields a Kochen-Specker coloring of B_1. In this way, we may consider $\mathrm{KS}\colon \mathsf{pBool}^{\mathrm{op}} \to \mathsf{Set}$ to be a functor.

In the following, we let $\mathbf{2} = \{0, 1\}$ denote the two-element Boolean algebra, which is the initial object of both the category of Boolean algebras and pBool.

Theorem 2.12. *Let B be a partial Boolean algebra. Then the following three sets are in bijection:*

1. *The set $\mathsf{pBool}(B, \mathbf{2})$ of morphisms of partial Boolean algebras $B \to \mathbf{2}$;*
2. *The set $\mathrm{KS}(B)$ of Kochen-Specker colorings of B;*
3. *The set $p\text{-}\mathrm{Spec}(B)$ of prime partial ideals of B.*

These bijections are natural in B and thus form natural isomorphisms $\mathsf{pBool}(-, \mathbf{2}) \cong \mathrm{KS} \cong p\text{-}\mathrm{Spec}$ as functors $\mathsf{pBool}^{\mathrm{op}} \to \mathsf{Set}$.

Proof. We will define functions

$$\begin{tikzcd} & p\text{-}\mathrm{Spec}(B) \arrow[dr] & \\ \mathsf{pBool}(B, \mathbf{2}) \arrow[ur] & & \mathrm{KS}(B) \arrow[ll] \end{tikzcd} \tag{2.13}$$

as follows. Given $\phi \in \mathsf{pBool}(B, \mathbf{2})$, set $\mathfrak{p}_\phi = \phi^{-1}(0) \subseteq B$. Because ϕ can equivalently be viewed as a morphism of partial Boolean rings $B \to \mathbf{2}$, we find that $\mathfrak{p}_\phi = p\text{-}\mathrm{Spec}(\phi)(0) \in p\text{-}\mathrm{Spec}(B)$.

Next, given $\mathfrak{p} \in p\text{-}\mathrm{Spec}(B)$, it follows from Proposition 2.10 that the coloring of B assigning black to all elements of $\mathfrak{p}$ and white to all elements of $B \setminus \mathfrak{p}$ is a Kochen-Specker coloring. This yields our function $p\text{-}\mathrm{Spec}(B) \to \mathrm{KS}(B)$.

Finally, given a Kochen-Specker coloring of B, define a function $\phi\colon B \to \mathbf{2}$ by $\phi(b) = 0$ if $b \in B$ is colored black and $\phi(b) = 1$ if b is colored white. Then $\phi^{-1}(0)$ is a prime partial ideal of B by Proposition 2.10. Now the restriction of ϕ to any commeasurable subalgebra C of B is such that $\phi|_C^{-1}(0) = \phi^{-1}(0) \cap C$ is a prime ideal, and it is well-known [18, Lemma 22.1] that this implies that $\phi|_C$ is a homomorphism of Boolean algebras. So ϕ restricts to a Boolean algebra homomorphism on all commeasurable subalgebras, from which we conclude that it is a morphism in $\mathsf{pBool}(B, \mathbf{2})$.

The composite of the three functions in the cycle (2.13) beginning at any of the three sets yields is readily seen to be the identity. Thus each of the functions is bijective. Finally, it is straightforward to see from the construction of these bijections that they are natural in B, yielding natural isomorphisms between the three functors $\mathsf{pBool}^{\mathrm{op}} \to \mathsf{Set}$ as claimed. □

Given a partial ring R, let $\mathrm{Idpt}(R) = \{e \in R \mid e = e^2\}$ denote the set of idempotents elements of R. Given commeasurable elements $e, f \in \mathrm{Idpt}(R)$, it is straightforward to verify that $e \vee f = e + f - ef$ and $e \wedge f = ef$ are both idempotents. Clearly $0, 1 \in \mathrm{Idpt}(R)$ as well. It is straightforward to verify that the above operations endow $\mathrm{Idpt}(R)$ with the structure of a partial Boolean algebra. Furthermore, any momorphism of partial rings $f\colon R \to S$ restricts to a morphism of partial Boolean algebras $\mathrm{Idpt}(R) \to \mathrm{Idpt}(S)$. In this way we may view this assignment as a functor

$$\mathrm{Idpt}\colon \mathsf{pRing} \to \mathsf{pBool}\,.$$

In particular, if we again consider the category of rings to be a subcategory of pRing, then this restricts to a functor $\mathrm{Idpt}\colon \mathsf{Ring} \to \mathsf{pBool}$.

It is straightforward to verify that with the above definitions in place, if $\mathfrak{p} \in p\text{-}\mathrm{Spec}(R)$ for a partial ring R, then $\mathfrak{p} \cap \mathrm{Idpt}(R)$ is a prime partial ideal of the partial Boolean algebra of idempotents. In this way one obtains a natural transformation of functors $\mathsf{pRing}^{\mathrm{op}} \to \mathsf{Set}$

$$p\text{-}\mathrm{Spec} \to p\text{-}\mathrm{Spec} \circ \mathrm{Idpt} \cong \mathrm{KS} \circ \mathrm{Idpt}\,. \tag{2.14}$$

This allows us to deduce information about the partial spectrum of a ring from the (un)colorability of its idempotents.

Corollary 2.15. *If R is a partial ring such that the partial Boolean algebra* $\mathrm{Idpt}(R)$ *has no Kochen-Specker colorings, then* $p\text{-}\mathrm{Spec}(R) = \varnothing$.

Proof. The natural transformation (2.14) provides a function $p\text{-}\mathrm{Spec}(R) \to \mathrm{KS}(\mathrm{Idpt}(R))$. Because the latter set is empty, so is the former. $\square$

A square matrix over a commutative ring is said to be *symmetric* if it is equal to its own transpose. For a commutative ring R and positive integer n, we let $\mathbb{M}_n(R)_{\mathrm{sym}}$ denote the subset of $\mathbb{M}_n(R)$ consisting of symmetric matrices, and we let $\mathrm{Proj}(R) = \mathrm{Idpt}(\mathbb{M}_n(R)_{\mathrm{sym}})$ denote the set of symmetric idempotents, which we call *projections*. It is clear that $\mathbb{M}_n(R)_{\mathrm{sym}}$ is a partial R-subalgebra of $\mathbb{M}_n(R)$, and that $\mathrm{Proj}(\mathbb{M}_n(R))$ is a partial Boolean subalgebra of $\mathrm{Idpt}(\mathbb{M}_n(R))$. Together, we obtain a diagram of sets

$$\begin{tikzcd}
p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)_{\mathrm{sym}}) \arrow[d] & p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)) \arrow[l] \arrow[d] \\
p\text{-}\mathrm{Spec}(\mathrm{Proj}(\mathbb{M}_n(R)) & p\text{-}\mathrm{Spec}(\mathrm{Idpt}(\mathbb{M}_n(R)) \arrow[l]
\end{tikzcd}$$

that is easily shown to commute. Thus, to show that $p\text{-}\mathrm{Spec}(\mathbb{M}_n(k)) = \varnothing$, it suffices to show that any one of the other three partial spectra is empty.

The next lemma shows that the nonexistence of either Kochen-Specker colorings or of prime partial ideals extends from matrix rings of a fixed order to all matrix rings of larger order. Throughout the following, for a ring R, we let $E_{ij} \in \mathbb{M}_n(R)$ denote the matrix unit whose (i,j)-entry is 1 and whose other entries are 0.

Lemma 2.16. *Let R be a ring and let $m \geq 1$ be an integer.*

(1) *If* $\mathrm{Idpt}(\mathbb{M}_m(R))$ *has no Kochen-Specker colorings, then also* $\mathrm{Idpt}(\mathbb{M}_n(R))$ *has no Kochen-Specker colorings for all integers $n \geq m$.*
(2) *If* $p\text{-}\mathrm{Spec}(\mathbb{M}_m(R)) = \varnothing$, *then also* $p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)) = \varnothing$ *for all integers $n \geq m$.*

Now assume furthermore that R is commutative.

(3) *If* $\mathrm{Proj}(\mathbb{M}_m(R))$ *has no Kochen-Specker colorings, then also* $\mathrm{Proj}(\mathbb{M}_n(R))$ *has no Kochen-Specker colorings for all integers $n \geq m$.*
(4) *If* $p\text{-}\mathrm{Spec}(\mathbb{M}_m(R)_{\mathrm{sym}}) = \varnothing$, *then also* $p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)_{\mathrm{sym}}) = \varnothing$ *for all integers $n \geq m$.*

Proof. To prove (1), it suffices by induction assume that $\mathrm{Idpt}(\mathbb{M}_n(R))$ has no Kochen-Specker coloring and deduce that $\mathrm{Idpt}(M)$ has no such coloring for $M = \mathbb{M}_{n+1}(R)$. Suppose toward a contradiction that there $\mathrm{Idpt}(M)$ does have a Kochen-Specker coloring. Consider the diagonal idempotents E_{ii} for $i = 1, \dots, n, n+1$. It follows that exactly one of the E_{ii} is white; assume without loss of generality that this is E_{11}.

For the idempotent $E = E_{11} + \cdots + E_{nn} = 1 - E_{n+1,n+1} \in M$, because $E_{11} \leq E$ in $\mathrm{Idpt}(M)$ we have that E is white according to Proposition 2.10. The corner ring EME has multiplicative identity E, so that the restriction of the coloring to $\mathrm{Idpt}(EME)$ satisfies condition (i) of Proposition 2.10. But conditions (ii)–(iv) of the same lemma are easily seen to pass to $\mathrm{Idpt}(EME)$, from which it follows that this restriction is a Kochen-Specker coloring. But it is clear from the choice of E that $EME \cong \mathbb{M}_n(R)$. Thus we obtain the contradiction that $\mathrm{Idpt}(\mathbb{M}_n(R)) \cong \mathrm{Idpt}(EME)$ has a Kochen-Specker coloring.

The proof of (2) also proceeds inductively, showing that any $\mathfrak{p} \in p\text{-}\mathrm{Spec}(M)$ for $M = \mathbb{M}_{n+1}(R)$ induces a prime partial ideal of $\mathbb{M}_n(R)$. This is done similarly to the proof of part (1) above, this time noting that without loss of generality we may assume $E_{11} \in \mathfrak{p}$ with the rest of the $E_{ii} \notin \mathfrak{p}$, so that for $E = E_{11} + \cdots + E_{nn}$ the restriction $\mathfrak{p} \cap EME$ is a prime partial ideal of $EME \cong \mathbb{M}_n(R)$.

The proofs of (3) and (4) follow the same arguments as those given above with only minor modifications: the idempotents E_{ii} and E above are symmetric, so that the transpose restricts to an involution of the corner ring EME, resulting in isomorphisms $EME_{\mathrm{sym}} \cong \mathbb{M}_n(R)_{\mathrm{sym}}$. ∎

Remark 2.17. If $\mathcal{S} \subseteq \mathrm{Idpt}(\mathbb{M}_n(R))$ is a subset that has no Kochen-Specker coloring, then one may adapt the proof above to explicitly construct a new set $\mathcal{S}^+ \subseteq \mathrm{Idpt}(\mathbb{M}_{n+1}(R))$ that also has no Kochen-Specker coloring, by taking the diagonal matrix units E_{ii} for $i = 1, \dots, n, n+1$ along with isomorphic copies of $\mathcal{S}$ in each of the partial Boolean rings $\mathrm{Idpt}((1 - E_{ii})\mathbb{M}_{n+1}(R)(1 - E_{ii})) \cong \mathrm{Idpt}(\mathbb{M}_n(R))$.

3. Colorability of idempotent matrices over various rings

In the following, for a field F, we consider the F-vector spaces F^n to consist of column vectors.

Given vectors $u, v \in F^n$, we denote their usual "dot product" by

$$u \cdot v = u^T v = \sum u_i v_i.$$

This defines a bilinear form on F^n, but this may be a degenerate pairing depending upon the ground field F.

Lemma 3.1. *Let F be a field, and let $v \in F^n \setminus \{0\}$. Then the following are equivalent:*

1. *There is a symmetric idempotent in $\mathbb{M}_n(F)$ with range* $\mathrm{Span}(v)$*;*
2. *The sum of squares $v^T v \in F$ is nonzero.*

In case $v^T v = \lambda \neq 0$, the symmetric idempotent with range $\mathrm{Span}(v)$ *is $P_v = \lambda^{-1} v v^T$. Finally, given $u, v \in F^n$ with $u^T u \neq 0 \neq v^T v$, the projections P_u and P_v are orthogonal if and only if $u \cdot v = 0$.*

Proof. Assume (1) holds, so that $P = P^2 = P^T \in \mathbb{M}_n(F)$ with $\mathrm{range}(P) = \mathrm{Span}(v)$. Because $v \neq 0$, we have $P \neq 0$. Thus some entry of P is nonzero, say the (i, j)-entry. Let v_i and v_j denote the ith and jth rows of P respectively. Then the (i, j)-entry of $P = P^2 = P^T P$ is equal to $v_i^T v_j \neq 0$. But as $v_i, v_j \in \mathrm{range}(P) = \mathrm{Span}(v)$, the product $v_i^T v_j$ is a scalar multiple of $v^T v$. Thus (2) must hold.

Conversely, suppose (2) holds, and set $\lambda = v^T v \neq 0$. Then $P = \lambda^{-1} v v^T$ satisfies $P = P^T$ and

$$P^2 = (\lambda^{-1} v v^T)(\lambda^{-1} v v^T) = \lambda^{-2} v(v^T v)v^T = \lambda^{-1} v v^T = P.$$

Given any $w \in F^n$, since $Pw = \lambda^{-1} v v^T w = (\lambda^{-1} v \cdot w)v \in \mathrm{Span}(v)$ and $Pv = v$, we see that $\mathrm{range}(P) = \mathrm{Span}(v)$. Thus (1) holds.

Finally, suppose $u, v \in F^n$ are as in the last sentence of the lemma. If $u \cdot v = 0$ then we have

$$P_u P_v = (u^T u)^{-1} u u^T \cdot (v^T v)^{-1} v v^T = (u^T u \cdot v^T v)^{-1} u(u^T v)v^T = 0,$$

and $P_v P_u = (P_u P_v)^T = 0$. Conversely, suppose that $P_u P_v = 0$. Then

$$u \cdot v = (P_u u) \cdot (P_v v) = (P_v^T P_u u) \cdot v = (P_v P_u u) \cdot v = 0$$

as desired. □

The lemma above allows us to equate Kochen-Specker colorings of rank-1 symmetric projections over a field F with Kochen-Specker colorings of vectors v satisfying $v^T v \neq 0$, in a manner analogous to Remark 2.7.

Our first uncolorability result makes use of one of the few Kochen-Specker vector configurations in the literature that makes use of vectors with integer entries. An account is given in [9] (also [10, Chapter 3]) of a proof of the Kochen-Specker Theorem due to Kurt Schütte, making use of a certain classical tautology that does not remain a tautology when interpreted in the partial Boolean algebra $\mathrm{Proj}(\mathbb{M}_3(\mathbb{R}))$. The orthogonal projections used to realize represent this logical proposition happen to be projections onto lines spanned by vectors with integer entries. Though these vectors do not have unit length, and their normalizations have irrational entries, we observe in the proof below that the resulting projection matrices do in fact have rational entries. We recall below that for any rational $q \in \mathbb{Q}$, the ring $\mathbb{Z}[q]$ denotes the subring of $\mathbb{Q}$ generated by the integers and q, and consists of elements of the form $f(q)$ where f is any polynomial with integer entries.

Theorem 3.2. *The partial Boolean algebra* $\mathrm{Proj}(\mathbb{M}_3(\mathbb{Z}[1/30])$ *has no Kochen-Specker coloring. Consequently, the partial ring* $\mathbb{M}_3(\mathbb{Z}[1/30])_{\mathrm{sym}}$ *has no prime partial ideals.*

Proof. Consider the uncolorable set of vectors [9, Section 4] used in Schütte's proof of the Kochen-Specker theorem. These are vectors in $\mathbb{Z}^3$, and each of the vectors v is such that $\|v\|^2 = v^T v$ divides 30. Thus each of the corresponding orthogonal projections $p_v = (v^T v)^{-1} v v^T$ lies in $\mathrm{Proj}(\mathbb{M}_3(\mathbb{Z}[1/30])$ by Lemma 3.1. Thus the argument of Schütte and Bub in fact shows more generally that $\mathrm{Proj}(\mathbb{M}_3(\mathbb{Z}[1/30]))$ has no Kochen-Specker coloring.

Because $\mathrm{Proj}(\mathbb{M}_3(\mathbb{Z}[1/30])) = \mathrm{Idpt}(\mathbb{M}_3(\mathbb{Z}[1/30])_{\mathrm{sym}})$, the claim about prime partial ideals follows from Corollary 2.15. □

Let F be a field and consider the canonical ring homomorphism $\mathbb{Z} \to F$. If the characteristic of F does not divide 30, then this homomorphism factors uniquely as $\mathbb{Z} \to \mathbb{Z}[1/30] \to F$.

This induces a ring homomorphism $\mathbb{M}_3(\mathbb{Z}[1/30]) \to \mathbb{M}_3(F)$, along with a morphism of partial Boolean algebras $\operatorname{Proj}(\mathbb{M}_3(\mathbb{Z}[1/30])) \to \operatorname{Proj}(\mathbb{M}_3(F))$. By functoriality of Kochen-Specker colorings, the theorem above implies that $\operatorname{Proj}(\mathbb{M}_3(F))$ has no Kochen-Specker colorings and therefore that $\mathbb{M}_3(F)_{\mathrm{sym}}$ has no prime partial ideals. These remarks apply in particular for the field $F = \mathbb{Q}$ of rational numbers.

Remark 3.3. In the literature addressing finite-precision loopholes to the Kochen-Specker theorem (as in [36, 12] and many further references discussed in [3]) it is well-documented that the set $S = \mathbb{Q}^3 \cap S^2$ of vectors with rational coordinates on the unit sphere has a Kochen-Specker coloring [19]. The apparent conflict between this fact and the uncolorability of $\operatorname{Proj}(\mathbb{M}_3(\mathbb{Q}))$ may be resolved as follows. The mapping $\phi\colon S \to \operatorname{Proj}(\mathbb{M}_3(\mathbb{Q}))$ given by $\phi(v) = P_v = \|v\|^{-2}vv^T = vv^T$ preserves orthogonality by Lemma 3.1 and has image contained in the rank-1 rational projections. By the same lemma, every rank-1 projection in $\operatorname{Proj}(\mathbb{M}_3(\mathbb{Q}))$ is of the form $P = P_v = \|v\|^{-2}vv^T$ for any nonzero vector v in the range of P. The image of ϕ forms a proper subset of the rank-1 rational projections, as one readily verifies that for vectors such as $v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ such that $v/\|v\|$ has irrational entries, the projection P_v lies outside of the image of ϕ. So the rational unit vectors correspond to a Kochen-Specker colorable subset of the Kochen-Specker uncolorable set of *all* rank-1 rational projections.

On the other hand, if F has characteristic p dividing 30 (i.e., $p = 2, 3, 5$), then the ring homomorphism $\mathbb{Z} \to F$ *does not* factor through $\mathbb{Z}[1/30]$, so we cannot make the same conclusion about Kochen-Specker colorings of projections or prime partial ideals in $\mathbb{M}_3(F)_{\mathrm{sym}}$. In the following we use $\mathbb{F}_q$ to denote the finite field with q elements. In case $F = \mathbb{F}_p$ we will show below that such Kochen-Specker colorings and prime partial ideals do in fact exist in case $p = 2, 3$ but not in case $p = 5$.

Theorem 3.4. *There exist Kochen-Specker colorings of* $\operatorname{Proj}(\mathbb{M}_3(\mathbb{F}_p))$ *and prime partial ideals of the partial rings* $\mathbb{M}_3(\mathbb{F}_p)_{\mathrm{sym}}$ *for* $p = 2, 3$.

Proof. We establish the existence of Kochen-Specker colorings below. Then it will follow from Lemma A.1 that there exists a morphism of partial F-algebras $\phi\colon \mathbb{M}_3(\mathbb{F}_p)_{\mathrm{sym}} \to K$ for a field extension K of $\mathbb{F}_p$, making $\phi^{-1}(0)$ a prime partial ideal of $\mathbb{M}_3(\mathbb{F}_p)_{\mathrm{sym}}$ for $p = 2, 3$.

$p = 2$: There are four vectors $v \in \mathbb{F}_2^3$ satisfying $v^T v \neq 0$, yielding four rank-1 projections in $\mathbb{M}_3(\mathbb{F}_2)$; three of these projections are the diagonal matrix units $E_{ii} = e_i e_i^T$ from the standard basis vectors $\{e_i \mid i = 1, 2, 3\}$, and the fourth is

$$U = uu^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{for} \quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus $\operatorname{Proj}(\mathbb{M}_3(\mathbb{F}_2))$ has two maximal commeasurable Boolean subalgebras: one is generated by the E_{ii} (and therefore isomorphic to the power set algebra on a three-element set), and the other is given by $\{0, U, I - U, I\}$ (isomorphic to the power set of a two-element set). Now any independent choice of a Kochen-Specker coloring on each of these maximal commeasurable subalgebras (given by any homomorphism into $\mathbf{2}$) gives a Kochen-Specker coloring of $\operatorname{Proj}(\mathbb{M}_3(\mathbb{F}_2))$.

$p = 3$: In this case every projection in $\mathbb{M}_3(\mathbb{F}_3)$ is a sum of orthogonal rank-1 projections, of which there are nine. In fact, there are only four orthogonal triples of rank-1 projections that sum to the identity (the off-diagonal entries that are omitted below are zero):

$$\begin{aligned} I &= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 2 & 2 \\ & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 2 & 1 \\ & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 & \\ 2 & 2 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 & \\ 1 & 2 & \\ & & 0 \end{pmatrix} \end{aligned}$$

As each of the diagonal matrix units E_{ii} is contained in $T = \{E_{11}, E_{22}, E_{33}\}$ and exactly one other orthogonal triple, it is easy to verify that any Kochen-Specker coloring of the triple T can be extended to a Kochen-Specker coloring of the nine projections above and thereby to all of $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_3))$. □

Corollary 3.5. *There exist morphisms of partial rings* $\mathbb{M}_3(\mathbb{Z})_{\mathrm{sym}} \to \mathbb{F}_{p^6}$ *for* $p = 2, 3$. *Consequently,* p-$\mathrm{Spec}(\mathbb{M}_3(\mathbb{Z})_{\mathrm{sym}}) \neq \varnothing$.

Proof. For $p = 2, 3$ let $\mathbb{M}_3(\mathbb{Z})_{\mathrm{sym}} \to \mathbb{M}_3(\mathbb{F}_p)_{\mathrm{sym}}$ be the canonical homomorphism that acts "modulo p" in each matrix entry. By Theorem 3.4 and Remark A.2 following Lemma A.1, there is a morphism of partial $\mathbb{F}_p$-algebras $\mathbb{M}_3(\mathbb{F}_p) \to \mathbb{F}_{p^6}$, since $\mathbb{F}_{p^6}$ is up to isomorphism the unique degree six extension of $\mathbb{F}_p$. The composite of these morphisms yields the desired function, and the preimage of the zero ideal of $\mathbb{F}_{p^6}$ is a prime partial ideal of $\mathbb{M}_3(\mathbb{Z})_{\mathrm{sym}}$. □

Next we will show that the projections in $\mathbb{M}_3(\mathbb{F}_5)$ do not have a Kochen-Specker coloring. To this end, we note that each $v \in \mathbb{F}_5^3$ satisfying $v^T v \neq 0$ is a scalar multiple of a unique vector in the list below.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad v_7 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad v_8 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad v_9 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad v_{10} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_{11} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad v_{12} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad v_{13} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad v_{14} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad v_{15} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$v_{16} = \begin{pmatrix}1\\1\\2\end{pmatrix} \qquad v_{17} = \begin{pmatrix}-2\\1\\1\end{pmatrix} \qquad v_{18} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \qquad v_{19} = \begin{pmatrix}1\\1\\-2\end{pmatrix} \qquad v_{20} = \begin{pmatrix}-2\\2\\1\end{pmatrix}$$

$$v_{21} = \begin{pmatrix}-2\\1\\2\end{pmatrix} \qquad v_{22} = \begin{pmatrix}1\\-2\\2\end{pmatrix} \qquad v_{23} = \begin{pmatrix}2\\-2\\1\end{pmatrix} \qquad v_{24} = \begin{pmatrix}2\\1\\-2\end{pmatrix} \qquad v_{25} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$$

Thus the 25 rank-1 symmetric projections are exactly those of the form $P_i = P_{v_i}$ for the vectors v_i above.

Suppose that $Q \in \mathbb{M}_3(\mathbb{F}_5)$ is an invertible matrix with $Q^{-1} = Q^T$. If $A \in \mathbb{M}_3(\mathbb{F}_5)_{\mathrm{sym}}$ then $(QAQ^{-1})^T = (Q^{-1})^T A^T Q^T = QAQ^{-1}$, so that conjugation by Q restricts from an automorphism of the $\mathbb{F}_5$-algebra $\mathbb{M}_3(\mathbb{F}_5)$ to a partial $\mathbb{F}_5$-algebra automorphism of $\mathbb{M}_3(\mathbb{F}_5)^{\mathrm{sym}}$ and to an automorphism of the partial Boolean algebra $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_5)$. This applies in particular if Q is any permutation matrix.

Theorem 3.6. *There is no Kochen-Specker coloring of* $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_5))$. *Thus* $\mathbb{M}_3(\mathbb{F}_5)_{\mathrm{sym}}$ *has no prime partial ideals.*

Proof. Assume for contradiction that there is a Kochen-Specker coloring of the rank-1 projections in $\mathbb{M}_3(\mathbb{F}_5)$; this induces a coloring on the vectors $\{v_i \mid i = 1, \ldots, 25\}$ above.

As remarked above, conjugation by any 3×3-permutation matrix restricts to an automorphism of the partial Boolean algebra $\mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_5))$. This automorphism clearly preserves the rank of a projection. So it restricts to a bijection on the set of rank-1 projections $\{P_i \mid i = 1, \ldots, 25\}$ defined by the vectors above. These automorphisms permute the diagonal matrix units $E_{ii} = P_i$ for $i = 1, 2, 3$. Thus after conjugating by an appropriate permutation matrix, we may assume that the coloring of the vectors is such that v_1, v_2 are black and v_3 is white. By orthogonality of the triple $\{v_3, v_4, v_7\}$, we must have v_4 and v_7 colored black.

Similarly, conjugation by the symmetric matrix $Q = \left(\begin{smallmatrix}1&0&0\\0&-1&0\\0&0&1\end{smallmatrix}\right)$ restricts to an automorphism of $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_5))$ that preserves rank. This automorphism fixes P_1 (as $Qv_1 = v_1$) and permutes P_6 with P_9 (as $Qv_6 = v_9$ and $Qv_9 = v_6$). Thus, after conjugating by Q if necessary, we assume without loss of generality that v_6 is colored black and v_9 is white. From the following orthogonality relations , we deduce the colorings below:

$$\begin{aligned}
\{v_9, v_{10}, v_{17}\} \text{ orthogonal } &\implies v_{10} \text{ black},\\
\{v_9, v_{11}, v_{14}\} \text{ orthogonal } &\implies v_{11} \text{ black},\\
\{v_4, v_{11}, v_{20}\} \text{ orthogonal } &\implies v_{20} \text{ white},\\
\{v_7, v_{10}, v_{19}\} \text{ orthogonal } &\implies v_{19} \text{ white},\\
\{v_{18}, v_{20}, v_{25}\} \text{ orthogonal } &\implies v_{25} \text{ black},\\
\{v_{19}, v_{21}, v_{22}\} \text{ orthogonal } &\implies v_{21} \text{ black},\\
\{v_5, v_{11}, v_{21}\} \text{ orthogonal } &\implies v_5 \text{ white},\\
\{v_6, v_{13}, v_{25}\} \text{ orthogonal } &\implies v_{13} \text{ white}.
\end{aligned}$$

But now orthogonality of the triple $\{v_5, v_{13}, v_{24}\}$ contradicts the coloring of the vectors v_5 and v_{13} above, establishing uncolorabililty of the set of projections $\{P_i\}_{i=1}^{25}$ and consequently the uncolorability of $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_5))$.

Because $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_5)) = \mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_5)_{\mathrm{sym}})$, the second claim follows from the first by Corollary 2.15. $\square$

Our results on Kochen-Specker colorings on symmetric idempotents over finite fields has the following consequence for Kochen-Specker colorings of vectors in $\mathbb{R}^3$ whose coordinate entries happen to be integers (such as those considered in [9]).

Corollary 3.7. *Suppose that $\{v_i\}$ is a set of vectors in $\mathbb{R}^3$ for which there is no Kochen-Specker coloring. If all v_i have integer coordinates, then the least common multiple of the integers $v_i \cdot v_i = \|v_i\|^2$ is divisible by* 6.

Proof. Let $N = \mathrm{lcm}\{\|v_i\|^2\}$ be the least common multiple described in the statement. Given any prime p, entrywise application of the canonical map $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$, denoted $x \mapsto \overline{x}$, induces a linear mapping $\mathbb{Z}^3 \to \mathbb{F}_p^3$, which we similarly denote by $v \mapsto \overline{v}$. Suppose that $p \nmid N$; then each of the images $\overline{v_i} \cdot \overline{v_i} = \overline{v_i \cdot v_i} \in \mathbb{F}_p$ are nonzero. Thus we obtain projections $Q_i = (\overline{v_i \cdot v_i})^{-1}\overline{v_i}\overline{v_i}^T$ in $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_p))$ for all i, with Q_i and Q_j orthogonal whenever v_i is orthogonal to v_j. So if the set $\{v_i\}$ has no Kochen-Specker coloring, the same must be true of the set $\{Q_i\}$, making $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_p))$ uncolorable.

But now it follows from Theorem 3.4 that N must be divisible by both $p = 2$ and $p = 3$, yielding the desired result. $\square$

Related to the results presented above, we ask two related questions:

Question 3.8. *(A) Can the conclusion of the corollary above be strengthened to state that* $30 = 2 \cdot 3 \cdot 5$ *must divide the least common multiple of the* $\|v_i\|^2$*?*

(B) Does $\mathrm{Proj}(\mathbb{M}_3(\mathbb{Z}[1/6]))$ *have a Kochen-Specker coloring?*

The method of proof used above does not extend to the prime $p = 5$ because of the uncolorability of $\mathrm{Proj}(\mathbb{M}_3(\mathbb{F}_5))$, which leads to question (A). Note that every rank-1 projection in $\mathbb{M}_3(\mathbb{Z}[1/6])$ is of the form $\|v\|^{-2}vv^T$ for some $v \in \mathbb{Z}^3$. Choosing v to have the least common multiple of its entries equal to 1, then we may conclude that $\|v\|^2 = v \cdot v$ divides a power of 6. So a negative answer to question (B) would imply a negative answer to question (A).

We now turn our attention to partial Boolean algebras of (non-symmetric) idempotents over various rings, beginning with finite fields. While Theorem 3.9 will be generalized in Corollary 3.11 below, the motivation for the uncolorable set of idempotents to be constructed in the proof of the more general result is actually motivated by this preliminary result regarding finite fields.

The following result was communicated to us by Alexandru Chirvasitu, whom we thank for kindly allowing us to include it here. Its proof is combinatorial in nature and uses some well-known methods of counting subspaces in vector spaces over finite fields via Gaussian binomial coefficients; see [37], for instance. Let q be a prime power, let $F = \mathbb{F}_q$ be the field of q elements, and let V be an F-vector space of dimension n. The number of k-dimensional subspaces of V is given by the number of ordered linearly independent lists of k vectors in V (each of which spans a k-dimensional subspace) divided by the number of bases for a k-dimensional vector space:

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q^2)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

The dual vector space $V^* = \mathrm{Hom}_F(V,F)$ is also an n-dimensional vector space. Given a subspace $W \subseteq V$ of dimension k, there is a corresponding subspace $W^\perp = \{f \in V* \mid f(W)=0\}$ of dimension $n-k$ in V^*. Note that $W_1 \subseteq W_2$ implies $W_2^\perp \subseteq W_1$ for subspaces W_i of V. Because $V \cong V^*$ as vector spaces, this provides a bijection between the subspaces of V having dimension k and the subspaces of V having dimension $n-k$, which reverses inclusion of subspaces.

For instance, when $V = F^3$ so that $n=3$, the number of 1-dimensional subspaces in V is $\binom{3}{1}_q = (q^3-1)/(q-1) = q^2+q+1$ and the number of 2-dimensional subspaces is also $\binom{3}{2}_q = q^2+q+1$. Also, the number of 2-dimensional subspaces of V that contain a given 1-dimensional subspace is equal to the number of 1-dimensional subspaces contained in a given 2-dimensional subspace, and is therefore equal to $\binom{2}{1}_q = (q^2-1)/(q-1) = q+1$.

Theorem 3.9. *Let p be a prime such that $p \equiv 2 \pmod 3$. Then $\mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_p))$ has no Kochen-Specker coloring.*

Proof. We prove the contrapositive. Suppose that there exists a Kochen-Specker coloring of $\mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_p))$. Consider the set $\mathcal{S}$ of unordered orthogonal triples $\{E_1,E_2,E_3\}$ of rank-1 idempotents in $\mathbb{M}_3(\mathbb{F}_p)$ such that $E_1+E_2+E_3 = I$. Given a rank-1 idempotent E, let $\mathcal{S}_E = \{T \in \mathcal{S} \mid E \in T\}$; we claim that these sets have the same cardinality for all E. Indeed, the general linear group $G = \mathrm{GL}_3(\mathbb{F}_p)$ acts transitively on the rank-1 idempotents in $\mathbb{M}_3(\mathbb{F}_p)$ via conjugation, and this induces an action on $\mathcal{S}$ by

$${}^uT = \{UP_iU^{-1} \mid i=1,2,3\} \quad \text{for } U \in G \text{ and } T = \{P_1,P_2,P_3\} \in \mathcal{S}.$$

Now given rank-1 idempotents E and F, if we fix $U \in G$ with $UEU^{-1} = F$, the action of U on $\mathcal{S}$ carries the elements of $\mathcal{S}_E$ to the elements of $\mathcal{S}_F$. Thus these two sets are in bijection, establishing the claim. We let $N = |\mathcal{S}_E|$ denote the number of triples in $\mathcal{S}$ that contain any given rank-1 idempotent E, independent of E.

By the Kochen-Specker property of the coloring, each $T \in \mathcal{S}$ contains exactly one white idempotent and two black idempotents. Thus the number of white rank-1 idempotents is equal to $|\mathcal{S}|/N$ and the number of black rank-1 idempotents is equal to $2(|\mathcal{S}|/N)$. In particular, the number of rank-1 idempotents is $3(|\mathcal{S}|/N)$, a multiple of 3.

On the other hand, each rank-1 idempotent is uniquely determined by the choice of its range, which can be any line in the vector space $V = \mathbb{F}_p^3$, along with its kernel, which can be any plane in V not containing that line. The number of lines in V is equal to $\binom{3}{1}_p = p^2+p+1$. The number of planes not containing a given line in V is equal (by duality) to the number of lines not contained in a given plane, and is therefore equal to $\binom{3}{1}_p - \binom{2}{1}_p = (p^2+p+1)-(p+1) = p^2$. Thus we may alternatively calculate the number of rank-1 idempotents in $\mathbb{M}_3(\mathbb{F}_p)$ to be equal to $(p^2+p+1)p^2$.

It follows that 3 divides $(p^2+p+1)p^2$. This is only possible if $p \not\equiv 2 \pmod 3$, completing the proof. $\square$

At this point, we are able to deduce that *for any prime $p \neq 3$, there is no Kochen-Specker coloring of* $\mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_p))$. Indeed, if $p \notin \{2,3,5\}$, then the existence of a (unique) ring homomorphism $\mathbb{Z}[1/30] \to \mathbb{F}_p$ induces morphisms of partial Boolean algebras $\mathrm{Proj}(M_3(\mathbb{Z}[1/30])) \subseteq \mathrm{Idpt}(\mathbb{M}_3(\mathbb{Z}[1/30]) \to \mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_p))$. It follows from Theorem 3.2 and functoriality of Kochen-Specker colorings that $\mathrm{Idpt}(\mathbb{M}_3(\mathbb{F}_p))$ has no Kochen-Specker colorings. On the other hand, for $p \in \{2,5\}$ the result follows directly from Theorem 3.9. But as we have already mentioned, a far stronger conclusion will be made in Corollary 3.11 below.

We will now define a set $\mathcal{S} \subseteq \mathrm{Idpt}(\mathbb{M}_3(\mathbb{Z}))$ that will be shown to have no Kochen-Specker coloring. For a commutative ring R, given a basis $\{u,v,w\}$ of the free R-module R^3, we use the notation $\left[u \,\middle|\, v \quad w\right]$ to denote the the idempotent in $\mathbb{M}_3(R)$ with range spanned by u and with kernel spanned by v and w. (This may be explicitly constructed via the invertible matrix $U = \begin{pmatrix} u & v & w \end{pmatrix}$ as $U\,\mathrm{diag}(1,0,0)U^{-1}$.) The argument given in the proof of Theorem 3.9 shows that number of rank-1 idempotents in $\mathbb{M}_3(\mathbb{F}_2)$ is equal to $2^2(2^2+2+1) = 28$. Below we list all 28 idempotent matrices in $\mathbb{M}_3(\mathbb{F}_2)$ in terms of the above notation.

$$P_1 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad P_2 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad P_3 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \qquad P_4 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$P_5 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \qquad P_6 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}\right] \qquad P_7 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{array}\right] \qquad P_8 = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$P_9 = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \qquad P_{10} = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \qquad P_{11} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \qquad P_{12} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

$$P_{13} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad P_{14} = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \qquad P_{15} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \qquad P_{16} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$P_{17} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \qquad P_{18} = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \qquad P_{19} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad P_{20} = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$P_{21} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \qquad P_{22} = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right] \qquad P_{23} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right] \qquad P_{24} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$P_{25}=\left[\begin{array}{c|cc}0&1&0\\0&1&1\\1&0&1\end{array}\right]\qquad P_{26}=\left[\begin{array}{c|cc}1&1&0\\1&0&1\\0&1&0\end{array}\right]\qquad P_{27}=\left[\begin{array}{c|cc}1&0&1\\0&0&1\\1&1&0\end{array}\right]\qquad P_{28}=\left[\begin{array}{c|cc}1&1&0\\1&1&1\\1&0&1\end{array}\right]$$

Considering the column vectors above as elements of $\mathbb{Z}^3$, note that each triple of vectors given above forms a basis of the free $\mathbb{Z}$-module $\mathbb{Z}^3$. (This can be easily verified, for instance, by noting that the matrix $U=\begin{pmatrix}u & v & w\end{pmatrix}$ formed by placing the three vectors from a triple has determinant in the group of units $\{\pm 1\}\subseteq \mathbb{Z}$.) Thus if we interpret the above notation in $\mathbb{M}_3(\mathbb{Z})$, the P_i for $1\le i\le 28$ define distinct rank-1 idempotents in $\mathbb{M}_3(\mathbb{Z})$. We let $\mathcal{S}=\{P_i \mid 1\le i\le 28\}$ denote the set of 28 idempotents in $\mathbb{M}_3(\mathbb{Z})$ given above.

Note that two idempotents $P=\left[u\,\middle|\,v\quad w\right]$ and $Q=\left[x\,\middle|\,y\quad z\right]$ presented as above are orthogonal if and only if the range vector u is contained in the $\mathbb{Z}$-span of the kernel vectors y and z, and also x is contained in the $\mathbb{Z}$-span of v and w.

Under the canonical ring homomorphism $\phi\colon \mathbb{M}_3(\mathbb{Z})\to\mathbb{M}_3(\mathbb{F}_2)$ induced entrywise by $\mathbb{Z}\twoheadrightarrow\mathbb{F}_2$, the set $\mathcal{S}$ is constructed in such a way that ϕ restricts to a bijection $\mathcal{S}\xrightarrow{\sim}\operatorname{Idpt}(\mathbb{M}_3(\mathbb{F}_2))$. In this sense, we may think of $\mathcal{S}$ as a set of "lifts" of the idempotents of $\mathbb{M}_3(\mathbb{F}_2)$ to the integer 3×3 matrices.

If the idempotents in $\mathcal{S}$ satisfied the same orthogonality relations as their images in $\mathbb{M}_3(\mathbb{F}_2)$, then it would follow directly from Theorem 3.9 that $\mathcal{S}$ has no Kochen-Specker coloring. But as it happens, there are triples of idempotents in $\mathcal{S}$ that are not pairwise orthogonal over $\mathbb{Z}$, but whose image under ϕ become pairwise orthogonal over $\mathbb{F}_2$.

To be precise, the following list displays all triples $\{i,j,k\}$ such that $\{P_i,P_j,P_k\}\subseteq\mathbb{M}_3(\mathbb{Z})$ is orthogonal and also $\{\phi(P_i),\phi(P_j),\phi(P_k)\}\subseteq\mathbb{M}_3(\mathbb{F}_2)$.

$$\begin{array}{llll}
\mathcal{O}_1=\{1,2,3\} & \mathcal{O}_2=\{1,4,5\} & \mathcal{O}_3=\{1,6,7\} & \mathcal{O}_4=\{2,8,9\}\\
\mathcal{O}_5=\{2,10,11\} & \mathcal{O}_6=\{3,12,13\} & \mathcal{O}_7=\{3,14,15\} & \mathcal{O}_8=\{4,8,16\}\\
\mathcal{O}_9=\{4,17,18\} & \mathcal{O}_{10}=\{5,19,20\} & \mathcal{O}_{11}=\{5,15,21\} & \mathcal{O}_{12}=\{6,17,22\}\\
\mathcal{O}_{13}=\{6,12,23\} & \mathcal{O}_{14}=\{8,23,24\} & \mathcal{O}_{15}=\{10,14,17\} & \mathcal{O}_{16}=\{10,23,27\}\\
\mathcal{O}_{17}=\{12,16,19\} & \mathcal{O}_{18}=\{15,22,25\} & \mathcal{O}_{19}=\{14,16,26\} & \mathcal{O}_{20}=\{19,22,28\}
\end{array}$$

On the other hand, the list below displays the pairs $\mathcal{O}_m^a=\{i,j\}$ and $\mathcal{O}_m^b=\{i,k\}$ of indices such that $\{P_i,P_j\}$ and $\{P_j,P_k\}$ are orthogonal pairs with P_i and P_k not orthogonal over $\mathbb{Z}$, but $\{\phi(P_i),\phi(P_j),\phi(P_k)\}$ forms an orthogonal triple over $\mathbb{F}_2$.

$$\begin{array}{llll}
\mathcal{O}_{21}^a=\{7,24\} & \mathcal{O}_{21}^b=\{7,20\} & \mathcal{O}_{22}^a=\{7,11\} & \mathcal{O}_{22}^b=\{7,25\}\\
\mathcal{O}_{23}^a=\{9,26\} & \mathcal{O}_{23}^b=\{21,26\} & \mathcal{O}_{24}^a=\{9,13\} & \mathcal{O}_{24}=\{13,20\}\\
\mathcal{O}_{25}^a=\{11,18\} & \mathcal{O}_{25}^b=\{18,21\} & \mathcal{O}_{26}^a=\{13,27\} & \mathcal{O}_{26}^b=\{13,25\}\\
\mathcal{O}_{27}^a=\{18,28\} & \mathcal{O}_{27}^b=\{18,24\} & \mathcal{O}_{28}^a=\{26,28\} & \mathcal{O}_{28}^b=\{26,27\}
\end{array}$$

Thus the proof that $\mathcal{S}$ is uncolorable requires a different argument. Note that the following proof only gives a complete argument that a larger set $\mathcal{S}'\supseteq\mathcal{S}$ is uncolorable. But as we indicate below, with extra work it is possible to prove that $\mathcal{S}$ itself is uncolorable.

Theorem 3.10. *There is no Kochen-Specker coloring of* $\operatorname{Idpt}(\mathbb{M}_3(\mathbb{Z}))$.

Proof. It suffices to show that the set $\mathcal{S}$ defined above is uncolorable. This is achieved through a case-splitting argument which shows that various colorings of the triples $\mathcal{O}_i$ for $i = 1, 2, 3$ lead to contradictions. Thus, we assume toward a contradiction that $\mathcal{S}$ has a Kochen-Specker coloring.

Case I: *E_1 black, E_2 black, and E_3 white in $\mathcal{O}_1$.* Then in the orthogonal triple $\mathcal{O}_6$ we have that E_{12} is black. We examine two possible subcases.

Case I.1: *E_4 is black and E_5 is white in $\mathcal{O}_2$.* We obtain the following sequence of deductions:

$$\begin{aligned} E_{19} \text{ black in triple } \mathcal{O}_{10} &\Rightarrow E_{16} \text{ white in triple } \mathcal{O}_{17} \\ &\Rightarrow E_8 \text{ white in triple } \mathcal{O}_8 \end{aligned}$$

We deduce contradictions in two further subcases of Case I.1 as follows.

Case I.1.a: *E_6 black and E_7 white in $\mathcal{O}_3$.* In this case we deduce:

$$\begin{aligned} E_{24} \text{ black in pair } \mathcal{O}_{21}^a &\Rightarrow E_{11} \text{ black in pair } \mathcal{O}_{22}^a \\ &\Rightarrow E_{10} \text{ black in triple } \mathcal{O}_5 \\ &\Rightarrow E_{23} \text{ black in triple } \mathcal{O}_{14}, \end{aligned}$$

arriving at the contradiction that two idempotents in triple $\mathcal{O}_{16}$ are white.

Case I.1.b: *E_6 black and E_7 white in $\mathcal{O}_3$.* Now we deduce that E_{17} and E_{22} are black in triple $\mathcal{O}_{12}$, which implies that E_{18} is white in triple $\mathcal{O}_9$ and E_{28} is white in triple $\mathcal{O}_{20}$. We obtain a contradictory coloring of the pair $\mathcal{O}_{28}^a$. This completes the proof that case I.1 leads to a contradiction.

Case I.2: *E_4 is white and E_5 is black in $\mathcal{O}_2$.* In this setting we have E_{14} black in triple $\mathcal{O}_7$, while E_{16} and E_8 are black in triple $\mathcal{O}_8$. It follows that E_9 is white in triple $\mathcal{O}_4$ and E_{26} is white in triple $\mathcal{O}_{19}$. We obtain a contradictory coloring of the pair $\mathcal{O}_{23}^a$.

We deduce from these contradictions that there is no Kochen-Specker coloring of $\mathcal{S}$ satisfying the condition in case I. Let $G \subseteq \mathbb{M}_3(\mathbb{Z})$ be the group of permutation matrices, acting on $\operatorname{Idpt}(\mathbb{M}_3(\mathbb{Z}))$ by conjugation. While $\mathcal{S}$ is not fixed under the action of G, it is contained in a smallest set $\mathcal{S}' = G\mathcal{S}$ closed under this action. Now if $\mathcal{S}'$ had a Kochen-Specker coloring, conjugation by a suitable element of G would yield a coloring of $\mathcal{S}' \supseteq \mathcal{S}$ for which case I indeed holds. So we find that $\mathcal{S}'$ is uncolorable, proving the theorem.

(The interested reader may reason as above to verify similar contradictions in either **case II:** E_1 black, E_2 white, E_3 black, or **case III:** E_1 white, E_2 black, E_3 black. This proves that $\mathcal{S}$ itself has no Kochen-Specker coloring.) $\square$

Corollary 3.11. *Let R be a ring, and fix any integer $n \geq 3$.*

1. *There is no Kochen-Specker coloring of* $\operatorname{Idpt}(\mathbb{M}_n(R))$.
2. p-$\operatorname{Spec}(\mathbb{M}_n(R)) = \varnothing$.
3. *There is no morphism of partial rings from $\mathbb{M}_n(R)$ to any (total) commutative ring.*
4. *The colimit in* cRing *of the diagram of commutative subrings of $\mathbb{M}_n(R)$ is zero.*

Proof. (1) It follows from Lemma 2.16 and Theorem 3.10 that $\operatorname{Idpt}(\mathbb{M}_n(\mathbb{Z}))$ has no Kochen-Specker coloring. The existence of a ring homomorphism $\mathbb{M}_n(\mathbb{Z}) \to \mathbb{M}_n(R)$ and functoriality of Idpt yield a morphism of partial Boolean algebras $\mathrm{Idpt}(\mathbb{M}_n(\mathbb{Z})) \to \mathrm{Idpt}(\mathbb{M}_n(R))$. Now by functoriality of Kochen-Specker colorings, we obtain a function $\mathrm{KS}(\mathrm{Idpt}(\mathbb{M}_n(R))) \to \mathrm{KS}(\mathrm{Idpt}(\mathbb{M}_n(\mathbb{Z}))) = \varnothing$. Now we deduce that $\mathrm{Idpt}(\mathbb{M}_n(R))$ has no Kochen-Specker colorings.

Part (2) follows from (1) by Lemma 2.15. Then (3) and (4) are immediate from Lemma 2.4. □

4. Applications to spectrum functors in noncommutative algebraic geometry

In this final section, we apply the above results on the partial spectrum of integer matrix rings to strengthen the main result of [38].

Modern algebraic geometry provides a way to view *every* commutative ring as a ring of "'globally defined functions" (the global sections of a sheaf of rings) on a geometric object (a locally ringed space) called an *affine scheme*. The scheme associated to a commutative ring is called its *spectrum*, and the assignment of the spectrum to each ring forms an equivalence of categories $\mathsf{cRing}^{\mathrm{op}} \to \mathsf{AffSch}$. For a commutative ring R, the Zariski prime spectrum $\mathrm{Spec}(R)$ (the set of prime ideals of R) forms the underlying set of its affine scheme. We refer readers to [17, I.2] for an introduction to the spectrum of a ring in algebraic geometry.

In the spirit of noncommutative geometry, it is natural to wonder whether every noncommutative ring may be given a similar "spatial realization." The most obvious way to attempt to build a "noncommutative affine scheme" would be to use a ringed space for which the sheaf of rings is not necessarily commutative. Indeed, such constructions have been intensely pursued in past decades; an outstanding survey of these efforts may be found in [43]. In order to obtain a true correspondence between algebra and geometry, one would wish for such a construction to be a contravariant functor. Thus, at the very least, one would require a functor F from $\mathsf{Ring}^{\mathrm{op}}$ to the category Top of topological spaces, or even to Set if we forget about topology, that yields the underlying point set of each ringed space, such that the restriction of F to $\mathsf{cRing}^{\mathrm{op}}$ is (isomorphic to) the usual spectrum functor $\mathrm{Spec}\colon \mathsf{cRing}^{\mathrm{op}} \to \mathsf{Set}$. Furthermore, in order to obtain a nontrivial construction, one should require that if R is a nonzero ring then $F(R)$ is nonempty.

However, it was shown in [38] that any functor F as above necessarily assigns $F(\mathbb{M}_n(R)) = \varnothing$ for any ring R containing $\mathbb{C}$ as a subring and any integer $n \geq 3$. The proof of this result crucially relied upon the fact that the Kochen-Specker Theorem implies that $p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)) = \varnothing$ for any such R. Our algebraic analogues of Kochen-Specker will allow us to extend this result to any ring R, not only those that contain the complex numbers.

In hindsight, the connection between the connection between this algebro-geometric obstruction and the Kochen-Specker Theorem is arguably a natural one. For a ring R, let $\mathcal{C}(R)$ denote the diagram in cRing whose objects are the commutative subrings $C \subseteq R$ and whose morphisms $C_1 \to C_2$ are the inclusions $C_1 \subseteq C_2$. It is shown in [38, Proposition 2.14] that as sets, the partial spectrum of R is the limit in the category of sets of the spectra $\mathrm{Spec}(C)$ for $C \in \mathcal{C}(R)$:

$$p\text{-}\mathrm{Spec}(R) \cong \varprojlim_{C \in \mathcal{C}(R)} \mathrm{Spec}(C).$$

This bijection allows us to view a point in $p\text{-}\mathrm{Spec}(R)$ as a "noncontextual choice of points" in the spectra $\mathrm{Spec}(C)$ of all commutative subrings of R. The obstruction of [38] used the

Kochen-Specker Theorem to show that there does not exist any such "noncontextual choice of points" in the case when $R = \mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. (One may also see [15, Sec. 3–4] for a related discussion.)

Further, it is interesting to note that Kochen and Specker's motivating discussion in [31, Section 1] phrases the problem of hidden variables as the search for a probability space Ω of "hidden pure states," with a hidden variable theory being a morphism of partial algebras from the algebra of observables to the algebra of real-valued measurable functions on Ω. If one imagines Ω as a kind of spectrum associated to a quantum system, then it seems entirely natural that the Kochen-Specker theorem should have led to the results of [38].

We now prove the strengthened version of [38, Theorem 1.1], answering the question posed in [38, Question 4.2]. We use essentially the same argument, relying upon Theorem 3.10 in place of the Kochen-Specker Theorem.

Theorem 4.1. *Let* $F\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Set}$ *(or* $F\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Top}$*) be a functor whose restriction to the full subcategory* $\mathsf{cRing}^{\mathrm{op}}$ *is isomorphic to* Spec. *Then* $F(\mathbb{M}_n(R)) = \varnothing$ *for any ring* R *and any integer* $n \geq 3$*.*

Proof. By [38, Theorem 2.15], the hypothesis on F ensures that there exists a natural transformation $F \to p\text{-Spec}$. For a fixed ring R and integer $n \geq 3$, the unique ring homomorphism $\mathbb{Z} \to R$ induces a ring homomorphism $\mathbb{M}_n(\mathbb{Z}) \to \mathbb{M}_n(R)$. The natural transformation and functoriality of p-Spec yield a composite function

$$F(\mathbb{M}_n(R)) \to p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)) \to p\text{-}\mathrm{Spec}(\mathbb{M}_n(\mathbb{Z})) = \varnothing,$$

with the last equality following from Corollary 3.11. As the only set with a function to the empty set is $\varnothing$ itself, we conclude that $F(\mathbb{M}_n(R)) = \varnothing$. $\square$

This obstruction to "noncommutative spectrum functors" has the following immediate application, which strengthens [38, Corollary 4.3] regarding "abelianization functors" defined on the category of rings.

Corollary 4.2. *Let* R *be a ring and* $n \geq 3$ *be an integer. If* $\alpha\colon \mathsf{Ring} \to \mathsf{cRing}$ *is any functor whose restriction to* cRing *is isomorphic to the identity functor, then* $\alpha(\mathbb{M}_n(R)) = 0$*.*

Proof. The hypothesis on α ensures that the composite functor $\mathrm{Spec} \circ \alpha\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Set}$ has restriction to $\mathsf{cRing}^{\mathrm{op}}$ isomorphic to Spec. By Theorem 4.1 we have $\mathrm{Spec}(\alpha(\mathbb{M}_3(\mathbb{Z})) = \varnothing$. This implies that the commutative ring $\alpha(\mathbb{M}_3(\mathbb{Z}))$ is zero. $\square$

It was noted in [38, p. 689] that the statements of both Corollaries 3.11(4) and 4.2 fail in the case when $n = 2$.

The study of topological spaces and their sheaves, especially including ringed spaces, can be conducted without any reference to the underlying point set of the topological space via the use of *locales* and *toposes* [30]. One might therefore expect that obstructions such as the one in Theorem 4.1 could be avoided if one considers the Zariski spectrum as a functor taking values in the category of locales rather than in the category of topological spaces. (We will describe the localic Zariski spectrum below.) However, it was shown in [8] that the obstruction of [38] persists for functors taking values in such categories of "pointless spaces." We now show that our version of the Kochen-Specker theorem for integer matrices allows us to extend the obstruction of Theorem 4.1 in the same manner. From this point until the end of the paper, we now consider the Zariski spectrum $\mathrm{Spec}(R)$ of a commutative ring R as a topological space, with the usual Zariski topology whose open sets are those of the form $D(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \not\subseteq \mathfrak{p}\}$ for all ideals I of R.

While a complete review of the theory of locales would take us too far afield, we have attempted to supply sufficiently many basic definitions and appropriate references to assist interested readers who have limited experience with the theory of locales.

A *frame* $(F, \bigvee, \wedge, 0, 1)$ is a complete lattice which satisfies the "infinite distributive law" $a \wedge (\bigvee b_i) = \bigvee (a \wedge b_i)$ for any family $\{b_i\} \subseteq F$. The motivating example of a frame is the collection of open sets in a topological space X. Frames form a category Frm whose morphisms are the homomorphisms of posets that perserve finite meets and arbitrary joins; in particular, these maps preserve 0 and 1. The category $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}}$ of *locales* is defined to be the opposite of the category of frames. If L denotes a locale, we will use $\Omega(L)$ to denote its underlying frame (so that L is "opposite" to $\Omega(L)$ in $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}}$); we call the elements of $\Omega(L)$ the *opens* of L. Given a morphism $f\colon L \to S$ in Loc, we denote the corresponding morphism of frames as $f^*\colon \Omega(S) \to \Omega(L)$.

We recall one formulation of the localic Zariski spectrum from [29, V.3] (and especially Corollary V.3.2(i) of that reference). For a commutative ring R and an ideal I of R, recall that the *radical of* I is the ideal $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some integer } n \geq 1\}$, and that I is called a *radical ideal* if $I = \sqrt{I}$ (that is, $x^n \in I$ for $x \in R$ and some integer $n \geq 1$ implies $x \in I$). Let $\mathrm{RIdl}(R)$ denote the set of radical ideals of R. This forms a lattice with respect to inclusion, which is complete since the intersection of an arbitrary set of radical ideals is again radical. The join of an arbitrary family $\{I_j\} \subseteq \mathrm{RIdl}(R)$ and the pairwise meet of $I, J \in \mathrm{RIdl}(R)$ are given in terms of the usual ideal sum and product by

$$\bigvee I_j = \sqrt{\sum I_j}, \qquad I \wedge J = I \cap J = \sqrt{I \cdot J}.$$

With these descriptions, one can verify that the "infinite distributive law"

$$J \wedge \left(\bigvee I_j\right) = \sqrt{J \cdot \sqrt{\sum I_j}} = \sqrt{\sum J I_j} = \sqrt{\sum \sqrt{J I_j}} = \bigvee (J \wedge I_j)$$

holds for all J and $\{I_j\}$ as above. Thus $\mathrm{RIdl}(R)$ is a frame.

We define the *localic (Zariski) spectrum* of R to be the locale $\mathrm{LSpec}(R)$ whose corresponding frame is $\Omega(\mathrm{LSpec}(R)) = \mathrm{RIdl}(R)$. A morphism $f\colon R \to S$ in cRing induces a function $\mathrm{RIdl}(f)\colon \mathrm{RIdl}(R) \to \mathrm{RIdl}(S)$ via $I \mapsto \sqrt{S \cdot f(I)}$, which one may verify to be a morphism of frames; we denote the corresponding morphism of locales by $\mathrm{LSpec}(f)\colon \mathrm{LSpec}(S) \to \mathrm{LSpec}(R)$. In this way, the localic spectrum forms a functor

$$\mathrm{LSpec}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Loc}\,.$$

Readers familiar with the ("spatial") Zariski spectrum will recognize that $\mathrm{RIdl}(R)$ is isomorphic to the lattice of open sets of the Zariski topology on $\mathrm{Spec}(R)$ (see [23, Lemma 2.1], for instance). Indeed, the spatial Zariski spectrum $\mathrm{Spec}(R)$ is isomorphic to the *space of points* [29, II.1.3] of the locale $\mathrm{LSpec}(R)$; see [29, V.3.2]. However, the definition of LSpec is entirely constructive, while one must invoke the Axiom of Choice (or at least the Boolean

Prime Ideal Theorem) to verify that $\mathrm{Spec}(R)$ is nonempty for nonzero rings R. For this reason, localic spectra are preferred in the setting of constructive mathematics.

Given the emphasis in the locale theory literature on constructive proofs, we feel that it is appropriate for us to provide a proof of the obstruction to localic spectra in Theorem 4.9 below that avoids the use of points (thereby requiring the Axiom of Choice). Such an argument was given in the proof of [8, Corollary 6.1]. The proof is complicated by the fact that the Zariski spectrum (either spatial or localic) generally does not preserve limits out of $\mathsf{Ring}^{\mathrm{op}}$. This was handled in [8] by noting that the Zariski spectrum *does* preserve limits when restricted to finite-dimensional algebras over the algebraically closed field $\mathbb{C}$. We follow a different approach here, which nevertheless imitates this line of reasoning. The major idea behind the proof of this theorem is to establish that a particular diagram of localic spectra that has trivial limit. We now begin the technical preparations for this result, culminating in Proposition 4.8.

In the following we let fRing denote the full subcategory of Ring whose objects are those rings isomorphic to a ring of integer-valued functions $\mathbb{Z}^X = \mathsf{Set}(X, \mathbb{Z})$ on a finite set X. Letting fSet denote the category of finite sets, we have functors

$$\begin{aligned} \mathsf{Set}(-, \mathbb{Z})&\colon \mathsf{fSet}^{\mathrm{op}} \to \mathsf{fRing}, \\ \mathsf{Ring}(-, \mathbb{Z})&\colon \mathsf{fRing}^{\mathrm{op}} \to \mathsf{fSet}\,. \end{aligned}$$

In fact, these functors are mutually quasi-inverse, forming a contravariant equivalence between the categories fRing and fSet. This may be deduced as in [27, Theorem 4.7]. (Alternatively, the desired fact may be derived directly from [27, Theorem 4.7], by first extending scalars from $\mathbb{Z}$ to $\mathbb{Q}$, and noting that the $\mathbb{Q}$-algebra homomorphisms $\mathbb{Q}^X \to \mathbb{Q}^Y$ are uniquely determined by the images of the idempotents, so that each such homomorphism is obtained by extension of scalars from some ring homomorphism $\mathbb{Z}^X \to \mathbb{Z}^Y$.)

We also recall that the well-known contravariant equivalence between fSet and the category fBool of finite Boolean algebras; see, for instance, [29, Example VI.4.5(a)]. This is implemented by the mutually quasi-inverse functors

$$\begin{aligned} \mathsf{Set}(-, \mathbf{2})&\colon \mathsf{fSet}^{\mathrm{op}} \to \mathsf{fBool}, \\ \mathsf{Bool}(-, \mathbf{2})&\colon \mathsf{fBool}^{\mathrm{op}} \to \mathsf{fSet}\,. \end{aligned}$$

These two dualities involving fSet are related as follows. For a finite set X, the Boolean algebra of idempotents of $\mathbb{Z}^X$ is isomorphic to $\mathbf{2}^X$ and is consequently finite. This isomorphism is natural in X, giving a natural isomorphism $\mathsf{Set}(-, \mathbf{2}) \cong \mathrm{Idpt} \circ \mathsf{Set}(-, \mathbb{Z})$ as functors $\mathsf{fSet}^{\mathrm{op}} \to \mathsf{fBool}$. Precomposing with the quasi-inverse $\mathsf{Ring}(-, \mathbb{Z})$, we also obtain a natural isomorphism $\mathsf{Set}(-, \mathbf{2}) \circ \mathsf{Ring}(-, \mathbb{Z}) \cong \mathrm{Idpt}^{\mathrm{op}}$ as functors $\mathsf{fRing}^{\mathrm{op}} \to \mathsf{fBool}^{\mathrm{op}}$.

Definition 4.3. For the set $\mathcal{S} = \{P_i \mid i = 1, \dots, 28\} \subseteq \mathrm{Idpt}(\mathbb{M}_3(\mathbb{Z}))$ shown to be Kochen-Specker uncolorable in the proof of Theorem 3.10, we set $\mathcal{S}_3 = \mathcal{S}$ and following Remark 2.17 we inductively define Kochen-Specker uncolorable sets $\mathcal{S}_{n+1} = (\mathcal{S}_n)^+ \subseteq \mathrm{Idpt}(\mathbb{M}_{n+1}(\mathbb{Z}))$. For each integer $n \geq 3$, let $\mathcal{S}_n^c \subseteq \mathrm{Idpt}(\mathbb{M}_n(\mathbb{Z}))$ denote the smallest set of nonzero idempotents containing $\mathcal{S}_n$ such that every maximal orthogonal set of idempotents in $\mathcal{S}_n$ is contained in an orthogonal set of (nonzero) idempotents $E_1, \dots, E_m \in \mathcal{S}_n^c$ with $\sum E_i = I$. Now let $\mathcal{D}_n$ be the finite set formed by taking all commutative subrings of $\mathbb{M}_n(\mathbb{Z})$ of the form $\bigoplus_{i=1}^m \mathbb{Z}E_i$

where $\{E_i\}_{i=1}^m \subseteq \mathcal{S}_n^c$ is an orthogonal set of idempotents satisfying $\sum E_i = I$, along with the intersections of these subrings (which are also spanned by idempotents). We consider $\mathcal{D}_n$ both as a subcategory of Ring whose morphisms are the inclusions of subrings in $\mathbb{M}_n(\mathbb{Z})$, and also as a diagram in Ring via the "inclusion" functor $\mathcal{D}_n \to \mathsf{Ring}$.

Conscientious readers may recall that the proof of Theorem 3.10 does not provide full verification of the uncolorability of $\mathcal{S}$, but rather of the larger (finite) set $\mathcal{S}' \supseteq \mathcal{S}$ that is closed under conjugation by permutation matrices. For such readers who prefer not to complete the verification that $\mathcal{S}$ itself is uncolorable, we note that the same inductive process beginning with $\mathcal{S}_3 = \mathcal{S}'$ will yield finite uncolorable sets $\mathcal{S}_n$ of idempotents in $\mathbb{M}_n(\mathbb{Z})$ for all $n \geq 3$. We also take this opportunity to remark that, while it is tempting to replace $(\mathcal{S}_n)^{scs}$ with the partial Boolean subalgebra of $\mathrm{Idpt}(\mathbb{M}_n(\mathbb{Z}))$ that is generated by $\mathcal{S}_n$, it is not clear to us that this set is finite; thus we have used the more technical approach above to ensure finiteness of the diagram $\mathcal{D}_n$.

We recall the notion of a sublocale. A *nucleus* on a frame F is a function $j\colon F \to F$ that preserves pairwise meets satisfying both $u \leq j(u)$ for all $u \in F$ and $j \circ j = j$. The subset $F/j = \{u \in F \mid j(u) = u\}$ of all j-fixed elements of F forms a frame, and $j\colon F \to F/j$ is a surjective frame homomorphism. For a locale L and a nucleus j on $\Omega(L)$, we denote by L_j the locale whose frame of opens is $\Omega(L_j) = \Omega(L)/j$, equipped with the corresponding locale morphism $L_j \to L$. A *sublocale* of L is precisely a locale of the form L_j. We recall that a sublocale L_j of L is called:

- *dense* if $j(0) = 0$, and
- *closed* if $j(-) = v \vee (-)$ for some fixed $v \in \Omega(L)$.

Every sublocale is dense in a unique closed sublocle (taking $v = j(0)$ above). For further details on the above facts, see [29, II.2].

Given a morphism $f\colon L \to S$ in Loc, the frame morphism $f^*\colon \Omega(S) \to \Omega(L)$ viewed as a map of partially ordered sets has a right adjoint $f_*\colon \Omega(L) \to \Omega(S)$, defined explicitly by $f_*(u) = \bigvee\{v \in \Omega(L) \mid f^*(v) \leq u\}$; see [29, Theorem I.4.2]. These maps satisfy the relations $f^* f_* f^* = f^*$ and $f_* f^* f_* = f_*$. The composite $f_* f^*\colon \Omega(S) \to \Omega(S)$ is a nucleus [29, II.2.2], and we will call the corresponding sublocale $\mathrm{im}(f) = S_{f_* f^*} \to S$ the *image* of f. Notice from the equation $f^* = f^* f_* f^*$ that f factors as $L \to \mathrm{im}(f) \to S$.

A locale L is *compact* if, whenever $1 = \bigvee_{i\in I} u_i$ for a set of opens $\{u_i \mid i \in I\}$ in L, it follows that $1 = \bigvee_{i\in J} u_I$ for some finite subset $J \subseteq I$.

Lemma 4.4. *Let $f\colon L \to S$ be a morphism in* Loc.

1. *If L is compact, then* $\mathrm{im}(f)$ *is compact.*
2. $\mathrm{im}(f)$ *is a dense sublocale of S if and only if $f_*(0) = 0$.*
3. *If $g\colon S \to T$ is another morphism in* Loc *and both f and g have dense images, then $g \circ f$ has dense image in T.*

Proof. (1) For the nucleus $j = f_* f^*$ on $\Omega(S)$, to see that the sublocale S_j is compact, suppose that $1 = \bigvee_{i\in I} u_i$ is a join of opens $u_i \in \Omega(S)$ satisfying $j(u_i) = u_i$ for all i. Applying f^*, we have $1 = f^*(1) = \bigvee f^*(u_i)$ in $\Omega(L)$. Compactness of L implies that $1 = \bigvee_{i\in J} f^*(u_i) = f^*(\bigvee_{i\in J} u_i)$ for some finite subset $J \subseteq I$, whence $1 = f_*(1) = f_*(f^*(\bigvee_{i\in J} u_i)) = \bigvee_{i\in J} f_* f^*(u_i) = \bigvee_{i\in J} u_i$. So $\mathrm{im}(f) = S_j$ is compact as desired.

(2) Note that any morphism $f\colon L \to S$ satisfies $f^*(0) = 0$, so that $f_* f^*(0) = f_*(0)$. Thus $\mathrm{im}(f) = S_{f_* f^*}$ is dense in S if and only if $0 = f_* f^*(0) = f_*(0)$.

(3) Noting that $(g \circ f)^* = f^* g^* \colon \Omega(T) \to \Omega(L)$, by the behavior of the composition of adjoints we have $(g \circ f)_* = f_* g_*$. According to part (2), if both f and g have dense image then $(g \circ f)_*(0) = f_*(g_*(0)) = 0$, making the image of the composite dense. □

Let KRLoc denote the full subcategory of Loc whose objects are compact, *regular* locales [29, III.1.1]. The inclusion functor $\mathsf{KRLoc} \to \mathsf{Loc}$ has a left adjoint $\beta\colon \mathsf{Loc} \to \mathsf{KRLoc}$, the "localic Stone-Čech compactification," as in [29, IV.2.1]. Being a right adjoint functor, β preserves colimits.

Lemma 4.5. *Let L be a locale, and let $i\colon L \to \beta L$ be the canonical map. Then the image of i is dense in βL.*

Proof. Let L' denote the sublocale of βL that is the closure of the image of i. Because βL is regular, we find that the closed sublocale L' is both regular and compact [29, Proposition III.1.2]. Thus the canonical map $L \to \beta L$ factors through the compact regular sublocale L' of βL. From the universal property of βL, we deduce that in fact $L' = \beta L$, so that i has image equal to βL. □

We let $*$ denote the terminal object in Loc, whose frame of opens in $\Omega(*) = \{0, 1\}$. This is the localic representation of the space consisting of a single point.

Proposition 4.6. *The restriction of the functor $\beta \circ \mathrm{LSpec}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{KRLoc}$ to the full subcategory $\mathsf{fRing}^{\mathrm{op}}$ is naturally isomorphic to the composite of $\mathrm{Idpt}^{\mathrm{op}}\colon \mathsf{fRing}^{\mathrm{op}} \to \mathsf{fBool}^{\mathrm{op}}$ with the inclusion of the full subcategory $\mathsf{fBool}^{\mathrm{op}} \hookrightarrow \mathsf{KRLoc}$.*

Proof. Consider the morphism $f\colon * \to \mathrm{LSpec}(\mathbb{Z})$ in Loc corresponding to the frame morphism $f^*\colon \mathrm{RIdl}(\mathbb{Z}) \to \{0, 1\}$ given by $0 \mapsto 0$ and $I \mapsto 1$ for any nonzero radical ideal I of $\mathbb{Z}$. The image of f is the sublocle of $\mathrm{LSpec}(\mathbb{Z})$ with corresponding nucleus $j = f_* f^*$ on $\mathrm{RIdl}(\mathbb{Z})$ given by $j(0) = 0$ and $j(I) = \mathbb{Z}$ for $I \neq 0$, so that f has dense image. (This is a localic representation of inclusion of the generic point of the topological space $\mathrm{Spec}(\mathbb{Z})$.) By Lemma 4.5, the canonical map $\mathrm{LSpec}(\mathbb{Z}) \to \beta\,\mathrm{LSpec}(\mathbb{Z})$ also has dense image. Now from Lemma 4.4 we may deduce that the image of the composite $* \to \mathrm{LSpec}(\mathbb{Z}) \to \beta\,\mathrm{LSpec}(\mathbb{Z})$ is both dense and compact. Because $\beta\,\mathrm{LSpec}(\mathbb{Z})$ is regular, the compact image of this composite is also closed [29, Proposition III.1.2]. So the image of f is both dense and closed, making $\mathrm{LSpec}(Z) = \mathrm{im}(f) = \mathrm{LSpec}(Z)_j$. Thus $\Omega(\mathrm{LSpec}(\mathbb{Z})) = \{0, 1\}$, and f is an isomorphism.

Recalling that β is a left adjoint, which consequently preserves coproducts, the fact that the given map $* \to \beta\,\mathrm{LSpec}(\mathbb{Z})$ is an isomorphism extends directly to show that the map $\coprod_X * \to \coprod_X \beta\,\mathrm{LSpec}(\mathbb{Z}) \cong \beta(\coprod_X \mathrm{LSpec}(\mathbb{Z})) \cong \beta\,\mathrm{LSpec}(\mathbb{Z}^X)$, which is evidently natural in X, is also an isomorphism. Denoting copowers as $X \otimes * = \coprod_X *$, we have a natural isomorphism $- \otimes * \cong \beta \circ \mathrm{LSpec} \circ \mathsf{Set}(-, \mathbb{Z})$ of functors $\mathsf{fSet} \to \mathsf{Loc}$.

On the other hand, as each $X \otimes *$ has its frame of opens isomorphic to $\mathbf{2}^X$, the copower functor $- \otimes *$ is naturally isomorphic to the functor $\mathsf{Set}(-, \mathbf{2})\colon \mathsf{fSet} \to \mathsf{fBool}^{\mathrm{op}} \hookrightarrow \mathsf{Loc}$. Thus we have the following diagram of functors that commutes up to natural equivalence:

$$\begin{tikzcd}
\mathsf{fSet} \arrow[r, "\mathsf{Set}(-,\mathbb{Z})"] \arrow[d, "\mathsf{Set}(-,\mathbf{2})"'] \arrow[dr, "-\otimes *"'] & \mathsf{fRing}^{\mathrm{op}} \arrow[d, "\beta\circ\mathrm{LSpec}"] \\
\mathsf{fBool}^{\mathrm{op}} \arrow[r, hook] & \mathsf{KRLoc}
\end{tikzcd}$$

Because the upper arrow is an equivalence with quasi-inverse $\mathsf{Ring}(-,\mathbb{Z})$, the functor $\beta\circ\mathrm{LSpec}$ is naturally isomorphic to the composite of $\mathsf{Set}(-,\mathbf{2})\circ\mathsf{Ring}(-,\mathbb{Z})\cong \mathrm{Idpt}^{\mathrm{op}}\colon \mathsf{fRing}^{\mathrm{op}}\to \mathsf{fBool}^{\mathrm{op}}$ with the inclusion $\mathsf{fBool}^{\mathrm{op}}\to\mathsf{KRLoc}$. □

The following lemma shows that the "finite discrete spaces" are closed under finite limits in the category of locales, as one expects by analogy with the category of topological spaces.

Lemma 4.7. *The natural inclusion* $\mathsf{fBool}^{\mathrm{op}}\hookrightarrow\mathsf{Loc}$ *preserves finite limits.*

Proof. Let DLat denote the category of (bounded) distributive lattices with (bounded) lattice homomorphisms. It is well-known that every element of a distributive lattice has at most one complement. We may consider Bool as a full subcategory of DLat whose objects are the *complemented* distributive lattices (i.e., every element has a complement). It is straightforward to verify that the set $C(L)$ of complemented elements of a distributive lattice L is closed under meets and joins, forming a distributive sublattice (which is a Boolean algebra). This construction clearly forms a functor $C\colon \mathsf{DLat}\to\mathsf{Bool}$; by construction, it is right adjoint to the forgetful functor $\mathsf{Bool}\to\mathsf{DLat}$. Thus Bool is closed under colimits in DLat, making $\mathsf{fBool}^{\mathrm{op}}$ closed under finite limits in $\mathsf{DLat}^{\mathrm{op}}$.

The functor $\mathrm{Idl}\colon \mathsf{DLat}\to\mathsf{Frm}$ sending a distributive lattice to its frame of ideals is left adjoint to the forgetful functor [29, Corollary II.2.11], so that its opposite $\mathrm{Idl}^{\mathrm{op}}\colon \mathsf{DLat}^{\mathrm{op}}\to\mathsf{Loc}$ is right adjoint to the forgetful functor $\mathsf{Loc}\to\mathsf{DLat}^{\mathrm{op}}$. In particular, the latter functor preserves limits.

Because every ideal of a finite Boolean algebra is principal (generated by the largest element in the ideal), the restriction of $\mathrm{Idl}^{\mathrm{op}}$ to $\mathsf{fBool}^{\mathrm{op}}$ is naturally isomorphic to the inclusion $\mathsf{fBool}^{\mathrm{op}}\hookrightarrow\mathsf{Loc}$. Considering that $\mathsf{fBool}^{\mathrm{op}}$ is closed under finite limits in $\mathsf{DLat}^{\mathrm{op}}$, we conclude that the inclusion $\mathsf{fBool}^{\mathrm{op}}\hookrightarrow\mathsf{Loc}$ preserves finite limits. □

We say that a locale is *trivial* if its frame of opens is a singleton (i.e., satisfies $0=1$), or equivalently, if it is an initial object of Loc. Trivial locales play the role of the empty space in pointless topology. If L is a locale with a morphism to a trivial locale, then the top and bottom elements of the frame of opens of L are also equal, making L a trivial locale.

Proposition 4.8. *For every integer $n\geq 3$, the finite diagram* $\mathrm{LSpec}(\mathcal{D}_n)$*, obtained by applying the functor* LSpec *to the diagram* $\mathcal{D}_n$ *in* Ring*, has trivial limit in* Loc*.*

Proof. By Proposition 4.6, the composite functor $\beta\circ\mathrm{LSpec}\colon \mathsf{fRing}^{\mathrm{op}}\to\mathsf{KRLoc}$ is naturally isomorphic to the composite of $\mathrm{Idpt}^{\mathrm{op}}\colon \mathsf{fRing}\to\mathsf{fBool}^{\mathrm{op}}$ with the inclusion $\mathsf{fBool}^{\mathrm{op}}\to\mathsf{KRLoc}$. Thus the diagram $\beta(\mathrm{LSpec}(\mathcal{D}_n))$ is opposite to a finite diagram of finite Boolean algebras. By Lemma 4.7, its colimit in Frm is a finite Boolean algebra and consequently is isomorphic to $\mathbf{2}^X$ for some finite set X. To see that that $X=\varnothing$, note that an element x of X would yield (via evaluation at x) a homomorphism of Boolean algebras $\mathbf{2}^X\to\mathbf{2}=\Omega(*)$, yielding a cocone from $\mathrm{Idpt}(\mathcal{D}_n)$ to $\mathbf{2}$ in fBool. Under these maps, each idempotent $P \in \mathcal{S}_n$ is assigned a value in $\{0,1\} = \mathrm{Idpt}(\mathbb{Z})$ that is independent of the choice of containing object R in $\mathcal{D}_n$. This yields a coloring of $\mathcal{S}_n$ by setting P to be black if $P \mapsto 0$ or white if $P \mapsto 1$. One may readily verify (as in Proposition 2.10) that this is a Kochen-Specker coloring of $\mathcal{S}_n$, contradicting the proof of Theorem 3.10 (along with Remark 2.17). Thus $X = \varnothing$, so that the colimit of $\mathrm{Idpt}(\mathcal{D}_n)$ is $\mathbf{2}^{\varnothing} = 0$. We conclude dually that the limit of $\beta(\mathrm{LSpec}(\mathcal{D}_n))$ in KRLoc (indeed, even in Loc) is trivial.

Now let L be the limit of the finite diagram $\mathrm{LSpec}(\mathcal{D}_n)$ in Loc. Then βL forms a cone over the diagram $\beta(\mathrm{LSpec}(\mathcal{D}_n))$, and we obtain locale morphisms $L \to \beta L \to \varprojlim \beta(\mathrm{LSpec}(\mathcal{D}_n))$. Because the third locale in this sequence is trivial, we deduce the desired result that L is trivial. □

Theorem 4.9. *Let $F\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Loc}$ be a functor whose restriction to $\mathsf{cRing}^{\mathrm{op}}$ is isomorphic to* $\mathrm{LSpec}\colon \mathsf{cRing}^{\mathrm{op}} \to \mathsf{Loc}$. *Then $F(\mathbb{M}_n(R))$ is trivial for every ring R and every integer $n \geq 3$.*

Proof. Fix a ring R and an integer $n \geq 3$. The existence of a ring homomorphism $\mathbb{M}_n(\mathbb{Z}) \to \mathbb{M}_n(R)$ and the the the fact that $\mathbb{M}_n(\mathbb{Z})$ is a cocone over $\mathcal{D}_n$ means that $\mathbb{M}_n(R)$ is a cocone over $\mathcal{D}_n$. Thus $F(M_n(R))$ has a cone over the diagram $F(\mathcal{D}_n) \cong \mathrm{LSpec}(\mathcal{D}_n)$, which must factor through a morphism $F(\mathbb{M}_n(R)) \to \varprojlim \mathrm{LSpec}(\mathcal{D}_n)$. The latter locale is trivial by Proposition 4.8, making $F(\mathbb{M}_n(R))$ trivial as desired. □

We also remark that as in [8], similar obstructions hold if we regard Spec as a functor from commutative rings into the any one of the categories of quantales, toposes, ringed toposes, ringed locales, or ringed spaces. Recall that a *quantale* $(Q, \bigvee, \&, 0, 1)$ is a complete upper-semilattice $(Q, \bigvee, 0, 1)$ and a (not necessarily commutative) multiplicative semigroup $(Q, \&)$ which satisfies the frame-like distributivity laws

$$a \& \left(\bigvee b_i\right) = \bigvee a \& b_i \quad \text{and} \quad \left(\bigvee b_i\right) \& a = \bigvee b_i \& a$$

for all $a \in Q$ and $\{b_i\} \subseteq Q$. A quantale Q is said to be *strong* if it satisfies $1 = 1\&1$, *unital* if there is a multiplicative identity $e \in Q$, and *two-sided* if it is unital with unit 1. The category Qu (respectively, Qu_e, Qu_1) has quantales (respectively, unital quantales, strong quantales) as its objects, and its morphisms are complete upper-semilattice homomorhpisms that preserve the product (respetively, as well as the unit, as well as the top element). Note that Frm forms a subcategory of Qu, and that all frames are commutative, two-sided quantales. Thus the localic spectrum LSpec may be considered as a functor into $\mathsf{Qu}^{\mathrm{op}}$ via the embedding $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}} \to \mathsf{Qu}^{\mathrm{op}}$.

Corollary 4.10. *Let $\mathcal{C}$ be any of the categories $\mathsf{Qu}_e^{\mathrm{op}}$, $\mathsf{Qu}_1^{\mathrm{op}}$, toposes, ringed toposes, ringed locales, or ringed spaces, and consider* Spec *as a functor $\mathsf{cRing}^{\mathrm{op}} \to \mathcal{C}$ in the usual way. Suppose that $F\colon \mathsf{Ring}^{\mathrm{op}} \to \mathcal{C}$ is a functor whose restriction to $\mathsf{cRing}^{\mathrm{op}}$ is isomorphic to* Spec. *Then $F(\mathbb{M}_n(R))$ is the trivial (initial) object of $\mathcal{C}$ for any ring R and any integer $n \geq 3$.*

Proof. Let $\mathcal{C}$ be either of the categories $\mathsf{Qu}_e^{\mathrm{op}}$ or $\mathsf{Qu}_1^{\mathrm{op}}$ above. Note as in [32] that $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}}$ forms a full subcategory of $\mathcal{C} = \mathsf{Qu}^{\mathrm{op}}$, and that KRLoc is closed under limits in $\mathcal{C}$ by [32, Corollary 4.4]. (While Lemmas 4.1 and 4.3 as well as Corollaries 4.2 and 4.4 of [32] are stated for involutive quantales, the proofs remain valid in the absence of an involution.) Now the proof of Proposition 4.8 still shows that the diagram $\beta(F(\mathcal{D}_n))$ has trivial limit when computed in $\mathcal{C}$, which still implies that $F(\mathcal{D}_n)$ has trivial limit when computed in $\mathcal{C}$. The proof for each of these categories of quantales now follows as in Theorem 4.9.

The proofs for the case when $\mathcal{C}$ is one of the categories of toposes, ringed toposes, ringed spaces, or ringed locales are direct analogues of those given in Corollaries 6.2 and 6.3 of [8]. $\square$

In closing, we note that readers seeking further open problems regarding contextuality in noncommutative algebraic geometry will find some in [39, Question 4.9], while readers interested in positive results on noncommutative spectrum functors are referred to [26] for one successful example.

Appendix A. Partial algebra morphisms from Kochen-Specker colorings, by Alexandru Chirvasitu

We prove here the following result relating idempotent colorings to morphisms of partial algebras.

Lemma A.1. *Let F be a perfect field and let A be a partial F-subalgebra of $\mathbb{M}_3 = \mathbb{M}_3(F)$. If there exists a Kochen-Specker coloring of $\mathrm{Idpt}(A)$, then for $K = \overline{F}$ the algebraic closure of F there exists a morphism of partial F-algebras $A \to K$. Consequently, p-$\mathrm{Spec}(R)$ is nonempty.*

Proof. A Kochen-Specker coloring provides a map φ from $\mathrm{Idpt}(A)$ to $\{0, 1\}$ compatible with addition of orthogonal idempotents, and we wish to extend this map to all of A.

Step 1: Reducing to semisimple operators. The fact that F is perfect ensures that we can decompose every $x \in A$ as a sum $x_s + x_n$, where $x_s \in A$ is semisimple, $x_n \in A$ is nilpotent, and each is a polynomial in x with no constant term. Because $x, y \in A$ commute if and only if x_s and x_n both commute with y_s and y_n, we can simply extend φ to the partial algebra generated by idempotents and nilpotent operators by sending the latter to zero.

If we had an extension of φ to the partial subalgebra $A_{ss} \subset A$ consisting of semisimple elements, then we could set $\varphi(x) = \varphi(x_n) + \varphi(x_s)$. The above observation that $x, y \in A$ commute if and only if x_s and x_n commute with y_s and y_n then ensures that this is well defined and a partial algebra morphism.

We may now assume that all elements of A are semisimple; this assumption will be in place throughout the rest of the proof.

The *support* $\mathrm{supp}(x) \in \mathrm{Idpt}(\mathbb{M}_3)$ of a semisimple element $x \in \mathbb{M}_3$ is the idempotent with the same range and kernel as x. For every $x \in A$ consider the element x_d (for 'diagonalizable') defined as $\sum t_i p_i$, where t_i are the distinct non-zero eigenvalues of x and p_i are the corresponding spectral idempotents.

The element x_d is expressible as a polynomial in x with no constant term (because p_i are so expressible) and hence belongs to A. Moreover, it is the unique element of $\mathbb{M}_3$ that is diagonalizable in $\mathbb{M}_3$, a polynomial in x with no constant term, and whose support is maximal among elements with this property.

It follows from the description of x that $x - x_d$ is either zero or *purely non-diagonalizable*, in the sense that $(x - x_d)_d$ vanishes (i.e. $x - x_d$ has no non-zero eigenvalues). Denote $x_{nd} = x - x_d$.

Step 2: Diagonalizable operators. The decomposition $x = x_d + x_{nd}$ is similar in spirit to the Jordan decomposition, and we can put it to similar use.

Any diagonalizable $x \in A$ breaks up uniquely as $\sum t_i p_i$ where $t_i \in F$ and $p_i \in \mathrm{Idpt}(A)$ are as above. Now simply set $\varphi(x) = \sum t_i \varphi(p_i)$. This is easily seen to be a partial algebra morphism from the partial subalgebra $A_d \subseteq A$ consisting of diagonalizable operators to $F \subseteq K$.

Step 3: Purely non-diagonalizable operators. Let $x \neq 0$ be such an operator and $\langle x\rangle$ the (non-unital) subalgebra of A that it generates. It is isomorphic to a field extension of F generated by any one of the non-zero eigenvalues of x, with unit $\mathrm{supp}(x)$. Define an arbitrary unital algebra morphism

$$\psi : \langle x\rangle \to K$$

and extend φ to $\langle x\rangle$ via $\varphi(x) = \psi(x)\varphi(\mathrm{supp}(x))$.

Step 4: Putting the ingredients together. For $x \in A$ set $\varphi(x) = \varphi(x_d) + \varphi(x_{nd})$.

Step 5: Checking that φ is a morphism. We have to check that φ as defined above preserves products and sums of commuting elements x, y, which we fix throughout the rest of the proof.

Because both x_d and x_{nd} can be expressed as polynomials with no constant term in x, an operator commutes with x if and only if it commutes with x_d and x_{nd}. Consequently, $x, y \in A$ commute if and only if x_d, x_{nd}, y_d and y_{nd} all commute.

If x and y are diagonalizable there is nothing to check, as we already know that $\varphi|_{A_d}$ is a morphism of partial algebras. So we may as well suppose $x_{nd} \neq 0$.

Now y commutes with the idempotent $e = \mathrm{supp}(x_{nd})$, and since φ annihilates exactly one of e and $1 - e$ we may as well restrict our attention to $e\mathbb{M}_3 e$ or $(1-e)\mathbb{M}_3(1-e)$, depending on whether $\varphi(e) = 1$ or $\varphi(1-e) = 1$ respectively.

There are two possibilities for e: either it has rank two and $\langle ex\rangle \subset e\mathbb{M}_3 e$ is a field L of degree two over F, or $e = 1$ and $\langle x\rangle$ is a field of degree three over F. This means that if $\varphi(1-e) = 1$ then $(1-e)\mathbb{M}_3(1-e)$ is (at most) one-dimensional with $(1-e)x$ and $(1-e)y$ both scalars therein, so there is nothing to check.

On the other hand, if $\varphi(e) = 1$ then $ey \in e\mathbb{M}_3 e$ commutes with ex and hence acts as L-linear endomorphisms of L. Putting $ex \in e\mathbb{M}_3 e$ in rational normal form will identify eF^n with eL and hence $e\mathbb{M}_3 e$ with $\mathrm{End}_F(L)$ in such a manner that $\langle ex\rangle$ gets identified with $L \subset \mathrm{End}_F(L)$ (acting on itself by multiplication). Since $ey \in \mathrm{End}_F(L)$ acts on L as L-linear endomorphisms (because it commutes with $\langle ex\rangle$) we must have $ey \in L = \langle ex\rangle$. We can now conclude from the fact that φ is a morphism when restricted to $\langle x\rangle$. $\square$

Remark A.2. Note that the proof only requires that the field K contain an isomorphic copy of every extension of F of degree either two or three. In particular, if F is finite then K can be taken to be its degree-six extension.

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