Probability

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1 Classical Conventional Probability Spaces

Textbook probability theory is defined using the notions of *sample spaces*, *events*, and *measures* [3, 2, 5].

1.1 Sample Space Ω

In this paper, we will only consider **finite** sample spaces. We therefore define a sample space Ω as a non-empty finite set.

Example 1 (A Classical Sample Space.). Consider an experiment that tosses three coins. A possible outcome of the experiment is HHT which means that the first and second coins landed with "heads" as the face-up side and that the third coin landed with "tails" as the face-up side. There are clearly a total of eight possible outcomes, and this collection constitutes the sample space:

$$\Omega_C = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Example 2 (A Quantum Sample Space.). Consider a quantum system composed of three electrons. By the postulates of quantum mechanics, an experiment designed to measure whether the spin of each electron along the x axis is left (L) or right (R) can only result in one of eight outcomes:

$$\Omega_H = \{LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR\}$$

1.2 Events $\mathcal{F}$

The space of events $\mathcal{F}$ associated with a sample space Ω is 2^{Ω}, the powerset of Ω. In other words, every subset of Ω is a possible event.

Example 3 (Some classical events.). The following are events associated with Ω_C:

- E_0, exactly zero coins are H, is the set $\{TTT\}$.
- E_1, exactly one coin is H, is the set $\{HTT, THT, TTH\}$.
- E_2, exactly two coins are H, is the set $\{HHT, HTH, THH\}$.
- E_3, exactly three coins are H, is the set $\{HHH\}$.
- $E_{>0}$, at least one coin is H, is the set $\{HHH, HHT, HTH, HTT, THH, THT, TTH\}$.

As the examples illustrate, events are *indirect* questions built from elementary elements of the sample space using logical connectives. Also note that some events may be disjoint and that some events may be expressed as combinations of other events. For example, we have $E_{>0} = E_1 \cup E_2 \cup E_3$ and each of these four events is disjoint from event E_0.

Example 4 (Some quantum events.). The following are events associated with Ω_H:

- F_0, exactly zero electrons are spinning L, is the set $\{RRR\}$.
- F_1, exactly one electron is spinning L, is the set $\{LRR, RLR, RRL\}$.
- F_2, exactly two electrons are spinning L, is the set $\{LLR, LRL, RLL\}$.
- F_3, exactly three electrons are spinning L, is the set $\{LLL\}$.
- $F_{>0}$, at least one electron is spinning L, is the set $\{LLL, LLR, LRL, LRR, RLL, RLR, RRL\}$.

As the examples illustrate, quantum events are, at first glance, similar to classical events. There are however some subtle differences that we point out in the next section.

1.3 Measures $\mathbb{P}$

The last ingredient of a probability space is a probability measure $\mathbb{P} : \mathcal{F} \to [0,1]$ that assigns to each event a real number in the closed interval $[0,1]$ subject to the following conditions:

- $\mathbb{P}(\Omega) = 1$, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) = \Sigma_i \, \mathbb{P}(A_i)$.

Example 5 (Classical probability measure). There are 2^8 events associated with Ω_C. A possible probability measure for these events is:

$$\mathbb{P}(E) = \begin{cases} 1 & \text{if } E = \Omega \\ 0 & \text{otherwise} \end{cases}$$

Yu-Tsung says: Actually, the above $\mathbb{P}$ is not a probability measure because

$$\mathbb{P}(\Omega) = 1 \neq 0 + 0 = \mathbb{P}(E_0) + \mathbb{P}(E_{>0})$$

A more interesting measure is defined recursively as follows:

$$\begin{aligned}
\mathbb{P}(\emptyset) &= 0 \\
\mathbb{P}(\{HHH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E) \\
\mathbb{P}(\{HHT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{HTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E) \\
\mathbb{P}(\{HTT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{THH\} \cup E) &= \tfrac{1}{5} + \mathbb{P}(E) \\
\mathbb{P}(\{THT\} \cup E) &= \mathbb{P}(E) \\
\mathbb{P}(\{TTH\} \cup E) &= \tfrac{3}{10} + \mathbb{P}(E) \\
\mathbb{P}(\{TTT\} \cup E) &= \mathbb{P}(E)
\end{aligned}$$

Yu-Tsung says: Should write like

$$\begin{array}{rcl}
\mathbb{P}(\emptyset) & = & 0 \\
\mathbb{P}(\{HHH\} \cup E) & = & \frac{1}{5} + \mathbb{P}(E), \text{ if } HHH \notin E \\
\mathbb{P}(\{HHT\} \cup E) & = & \mathbb{P}(E), \text{ if } HHT \notin E \\
\mathbb{P}(\{HTH\} \cup E) & = & \frac{3}{10} + \mathbb{P}(E), \text{ if } HTH \notin E \\
\mathbb{P}(\{HTT\} \cup E) & = & \mathbb{P}(E), \text{ if } HTT \notin E \\
\mathbb{P}(\{THH\} \cup E) & = & \frac{1}{5} + \mathbb{P}(E), \text{ if } THH \notin E \\
\mathbb{P}(\{THT\} \cup E) & = & \mathbb{P}(E), \text{ if } THT \notin E \\
\mathbb{P}(\{TTH\} \cup E) & = & \frac{3}{10} + \mathbb{P}(E), \text{ if } TTH \notin E \\
\mathbb{P}(\{TTT\} \cup E) & = & \mathbb{P}(E), \text{ if } TTT \notin E
\end{array}$$

However, many if clauses seems confusing, and it seems easier to write like this:

$$\begin{array}{rcl}
\mathbb{P}(\{HHH\}) & = & \frac{1}{5} \\
\mathbb{P}(\{HHT\}) & = & 0 \\
\mathbb{P}(\{HTH\}) & = & \frac{3}{10} \\
\mathbb{P}(\{HTT\}) & = & 0 \\
\mathbb{P}(\{THH\}) & = & \frac{1}{5} \\
\mathbb{P}(\{THT\}) & = & 0 \\
\mathbb{P}(\{TTH\}) & = & \frac{3}{10} \\
\mathbb{P}(\{TTT\}) & = & 0 \\
\mathbb{P}(E) & = & \sum_{\omega \in E} \mathbb{P}(\{\omega\})
\end{array}$$

Because this is a *classical* situation, the probability assignments can be understood *locally* and *non-contextually*. In other words, we can reason about each coin separately and perform experiments on it ignoring the rest of the context. If we were to perform such experiments we may find that for the first coin, the probability of either outcome H or T is $\frac{1}{2}$; for coin two, the probabilities are skewed a little with the probability of outcome H being $\frac{2}{5}$ and the probability of outcome T being $\frac{3}{5}$; and that coin 3 is a fake double-headed coin where the probability of outcome H is 1 and the probability of outcome T is 0. The reader may check that these local observations are consistent with the probability measure above.

Example 6. [Quantum probability measure] Like in the classical case, there are 2^8 events. But as Mermin explains in a simple example [4], here is a possible probability measure:

$$\begin{array}{rcl}
\mathbb{P}_{xxx}(\{LLL\}) & = & \frac{1}{4} \\
\mathbb{P}_{xxx}(\{LLR\}) & = & 0 \\
\mathbb{P}_{xxx}(\{LRL\}) & = & 0 \\
\mathbb{P}_{xxx}(\{LRR\}) & = & \frac{1}{4} \\
\mathbb{P}_{xxx}(\{RLL\}) & = & 0 \\
\mathbb{P}_{xxx}(\{RLR\}) & = & \frac{1}{4} \\
\mathbb{P}_{xxx}(\{RRL\}) & = & \frac{1}{4} \\
\mathbb{P}_{xxx}(\{RRR\}) & = & 0 \\
\mathbb{P}_{xxx}(E) & = & \sum_{\omega \in E} \mathbb{P}_{xxx}(\{\omega\})
\end{array}$$

In contrast with the classical example previously, the probability of each electron is not independent, i.e., we cannot find three probability space for each subsystem such that

$$\mathbb{P}_{xxx}(\{abc\}) = \mathbb{P}(\{a\})\mathbb{P}(\{b\})\mathbb{P}(\{c\}) \ .$$

Correlated variables are also common in the classical world. For example, if the first coin is head or tail decided by whether the temperature is higher than a particular degree and the second coin is decided by whether Coca Cola is sold more than a particular amount, then we know these two coins are correlated. Einstein, Podolsky, and Rosen (EPR) [1] suggested any correlated quantum probability results may be interpreted classically. For example, the nature might roll an tetrahedron die with $\{LLL, LRR, RLR, RRL\}$ in its four faces. If LRR were face-down, the measurement result for three electrons would be L, R, and R, respectively. Furthermore, this tetrahedron die might not be accessed by human for some reasons so that we think we are handling a system constituting three electrons.

In order to verify EPR's idea, we need to analysis the idea of probability more carefully. Imagining somebody tosses a coin behind a veil, we heard the coin is tossed, but cannot see the result. Although the probability for us is still $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}$, we know the coin behind the veil must be either head or tail face-up.

In classical probability, there is no difference between the coins have not been tossed or the coins have been tossed but we do not know. However, there is a difference when we consider the quantum probability. Because the three electrons can be specially separated, and each electron can be measured along the x axis separately, if the nature actually rolled the tetrahedron die, this die should be rolled before the electrons are separated and measured. In another word, there should be a period that the human observer would not know the measurement results for each electron along the x axis, but the nature should know. The result for j-th electron is denoted by $w\left(\sigma_x^j\right)$ so that $w\left(\sigma_x^1\right) w\left(\sigma_x^2\right) w\left(\sigma_x^3\right) \in \{LLL, LRR, RLR, RRL\}$, i.e., the number of L in $w\left(\sigma_x^1\right)$, $w\left(\sigma_x^2\right)$, and $w\left(\sigma_x^3\right)$ should be odd.

The interesting part is that three electrons cannot be measured the spin only along the x axis, but also along the y axis with the result down (D) or up (U). We only consider to measure even number of electrons along the y axis, and the measure of non-singleton set should be computed by $\mathbb{P}_{ijk}(E) = \sum_{\omega \in E} \mathbb{P}_{ijk}(\{\omega\})$.

$$\begin{array}{rclrclrcl}
\mathbb{P}_{xyy}(\{LDD\}) &=& 0 & \mathbb{P}_{yxy}(\{DLD\}) &=& 0 & \mathbb{P}_{yyx}(\{DDL\}) &=& 0 \\
\mathbb{P}_{xyy}(\{LDU\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{DLU\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{DDR\}) &=& \frac{1}{4} \\
\mathbb{P}_{xyy}(\{LUD\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{DRD\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{DUL\}) &=& \frac{1}{4} \\
\mathbb{P}_{xyy}(\{LUU\}) &=& 0 & \mathbb{P}_{yxy}(\{DRU\}) &=& 0 & \mathbb{P}_{yyx}(\{DUR\}) &=& 0 \\
\mathbb{P}_{xyy}(\{RDD\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{ULD\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{UDL\}) &=& \frac{1}{4} \\
\mathbb{P}_{xyy}(\{RDU\}) &=& 0 & \mathbb{P}_{yxy}(\{ULU\}) &=& 0 & \mathbb{P}_{yyx}(\{UDR\}) &=& 0 \\
\mathbb{P}_{xyy}(\{RUD\}) &=& 0 & \mathbb{P}_{yxy}(\{URD\}) &=& 0 & \mathbb{P}_{yyx}(\{UUL\}) &=& 0 \\
\mathbb{P}_{xyy}(\{RUU\}) &=& \frac{1}{4} & \mathbb{P}_{yxy}(\{URU\}) &=& \frac{1}{4} & \mathbb{P}_{yyx}(\{UUR\}) &=& \frac{1}{4}
\end{array}$$

Similarly, there would be predetermined $w\left(\sigma_x^j\right)$ and $w\left(\sigma_y^j\right)$ for these probabilities. The number of L or D in $\left\{w\left(\sigma_x^1\right), w\left(\sigma_y^2\right), w\left(\sigma_y^3\right)\right\}$, $\left\{w\left(\sigma_y^1\right), w\left(\sigma_x^2\right), w\left(\sigma_y^3\right)\right\}$, and $\left\{w\left(\sigma_y^1\right), w\left(\sigma_y^2\right), w\left(\sigma_x^3\right)\right\}$ should be even. If we look these 9 letters carefully, we can find every $w\left(\sigma_x^j\right)$ appear once and every $w\left(\sigma_y^j\right)$ appear twice. Hence, the number of L in $w\left(\sigma_x^1\right)$, $w\left(\sigma_x^2\right)$, and $w\left(\sigma_x^3\right)$ should be even. This contradict to the conclusion in our last paragraph. Therefore, EPR's assumption is wrong and the correlation of quantum probability cannot be interpreted by a classical probability.

1.4 Finite Precision of Measurements

In a laboratory setting or a computational setting, there are neither uncountable entities nor uncomputable entities. We are thus looking at alternative probability spaces which do not depend on the real numbers and revisit the mysteries of quantum mechanics in that setting. In other words, is it possible that at least part of the quantum mysteries related to probability and measurement are due to the reliance on uncomputable probability values?

Following previous work on probability, we will replace the closed interval $[0, 1]$ by the *finite set* $S = \{\textbf{possible}, \textbf{impossible}\}$ and adapt the definition of probability measure as follows.

A set-valued probability measure $\mathbb{P} : \mathcal{F} \to S$ assigns to each event either the tag **possible** or the tag **impossible** subject to the following conditions:

- $\mathbb{P}(\Omega) =$ **possible**, and
- For any collection of pairwise disjoint events A_i, we have $\mathbb{P}(\bigcup_i A_i) =$ **possible** if any event A_i is **possible** and **impossible** otherwise.

2 Conventional Quantum Mechanics

Attempting to modify the probability measure to be set-valued, while keeping the rest of the mathematical framework of quantum mechanics intact leads to a contradiction. More precisely, it is not possible to maintain infinite precision probability amplitudes in the presence of set-valued probabilities without violating essential aspects of quantum theory.

...explain and give theorem

3 Discrete Quantum Theory

The next question to ask is therefore whether the infinite precision of probability amplitudes is itself justified. If all measurements are finite and all probabilities are computable, then it is plausible that the internal mathematical representation of quantum states should also be based on countable computable entities.

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References

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