Noncontextual Hidden Variables and Physical Measurements

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Meyer [Phys. Rev. Lett. **83**, 3751 (1999)] recently showed that the Kochen-Specker theorem, which demonstrates the impossibility of a deterministic hidden variable description of ideal spin measurements on a spin 1 particle, can be effectively nullified if only finite precision measurements are considered. We generalize this result: one can ascribe consistent outcomes to a dense subset of the set of projection valued measurements, or to a dense subset of the set of positive operator valued measurements, on any finite dimensional system. Hence no Kochen-Specker-like contradiction can rule out hidden variable theories indistinguishable from quantum theory by finite precision measurements in either class.

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The experimental evidence against local hidden variable theories is compelling and fundamental theoretical arguments weigh heavily against those nonlocal hidden variable theories proposed to date, but the question of hidden variables is still interesting for at least two reasons.

First, we want to distinguish strongly held theoretical beliefs from established facts. Since there are reasonably sound (though clearly not universally persuasive) theoretical arguments against *every* interpretation of quantum theory proposed to date, we need to be particularly cautious about distinguishing belief from fact where quantum foundations are concerned. In particular, we still need to pin down precisely what types of hidden variable theories can or cannot be excluded by particular theoretical arguments.

Second, recent discoveries in quantum information theory and quantum computing give new interest to some foundational questions. For example, we would like to know in precisely what senses a quantum state carries information and can be regarded as a computer, and how it differs in these respects from classical analogs. From this perspective, as Meyer [1] has emphasized, questions about the viability of hidden variable models for a particular process translate into questions about the classical simulability of some particular aspect of quantum behavior and are interesting independently of the plausibility of the relevant models as physical theories.

Either way, we need to distinguish arguments based on idealized measurements, which can be specified precisely, from arguments based on realistic physical measurements, which are always of finite precision. Since we cannot precisely specify measurements, it is conceivable that all the measurements we can carry out actually belong to some subset of the full class of measurements allowed by the quantum formalism. That subset must be dense, assuming that any finite degree of precision can be attained in principle, but it need not have positive measure. Any no-go theorem relying on a model of measurement thus has a potential loophole which can be closed only if the theorem still holds when measurements are restricted to a dense subset. From the point of view of quantum computation, the precision attainable in a measurement is a computational resource: specifying infinite precision requires infinite resources and prevents any useful comparison with discrete classical computation.

We consider here whether the predictions of quantum theory can be replicated by a hidden variable model in which the outcomes of measurements are predetermined by truth values associated to the relevant operators. We first review the case of infinite precision with standard von Neumann, i.e., projection valued, measurements. The question then is whether there is a consistent way of ascribing truth values $p(P) \in \{0, 1\}$ to the projections in such a way as to determine a unique outcome for any projection valued measurement. That is, is it possible to find a truth function p such that if $\{P_i\}$ is a projective decomposition of the identity then precisely one of the P_i has truth value 1? To put it formally, does there exist a truth function p such that

$$\sum_i p(P_i) = 1 \quad \text{if} \sum_i P_i = I\,? \tag{1}$$

The Kochen-Specker (KS) theorem [2–4] shows that the answer is "no" for systems whose Hilbert space has dimension greater than two. The general result follows from the result for projections in three-dimensional real space, and so can be proved by exhibiting finite sets of projections in R^3 for which a truth function satisfying (1) is demonstrably impossible. Kochen and Specker gave the first example [3] of such a set, and some simpler examples were later found by Peres [5,6]. An independent proof was given by Bell [2], who noted that, by an argument of Gleason's [7], (1) implies a minimal finite separation between projectors with truth values 1 and 0, which is impossible since both values must be attained.

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We could ask the same question about positive operator valued measurements. That is, does there exist a truth function p on the positive operators such that

$$\sum_i p(A_i) = 1 \quad \text{if} \sum_i A_i = I\,? \tag{2}$$

Obviously, since projection valued measurements are special cases of positive operator valued measurements, the KS theorem still applies.

Returning to the case of projections, we want to know whether the restriction to finite precision could make a difference. Our hypothesis, recall, is that a finite precision measurement could correspond to a measurement of some particular projective decomposition in the precision range, a decomposition whose projections do indeed have hidden preassigned truth values. The relevant question then is: Is there a physically sensible truth function p defined on a dense subset S_1 of the space of all projections such that (1) holds on a dense subset PD_1 of the space of all projective decompositions? This requires in particular that the set S_2 of projections belonging to decompositions in PD_1 must satisfy $S_2 \subseteq S_1$. By physically sensible, we mean that the subsets S_3 and S_4 of projections P in S_2 for which $p(P) = 1$ and $p(P) = 0$, respectively, are both dense in the space of all projections, so as to avoid the possibility of contradiction by experiments of sufficiently high precision.

We first consider the case treated in the proof of the KS theorem, one-dimensional projections on R^3. The possibility of hidden variable models evading the KS theorem was first considered by Pitowsky [8]. Meyer [1] has given a very pretty example of a truth function p defined on the subset $S^2 \cap Q^3$ of projections defined by rational vectors, which satisfies (1) for all orthogonal triples. Meyer's elegant proofs [1], using the earlier work of Godsil and Zaks [9], show that all the necessary denseness conditions hold and hence that the KS proof is indeed nullified if we restrict attention to finite precision measurements.

Meyer's result shows that the KS theorem cannot be directly applied in the finite precision case. However, it does not imply that the theorem itself is false or that no similar no-go theorem can be found. Even in the case of three-dimensional systems, this requires an example of a truth function that satisfies (1) for a dense subset of the triads of projections on C^3 rather than R^3. A more complete argument requires an example of a physically sensible truth function satisfying (1) for a dense subset of the projective decompositions of the identity on C^n. More generally still, since all physical measurements are actually positive operator valued, a complete defense against KS-like arguments requires an example of a physically sensible truth function satisfying (2) for a dense subset of the positive operator decompositions of the identity on C^n.

We give such examples here. First, some notation. Define the one-dimensional projection $P_{r_1,\ldots,r_{2n}}$ on C^n to be the projection onto the vector $N(r_1 + ir_2, \ldots, r_{2n-1} + ir_{2n})$, where the r_i are real and not all zero and the normalization constant obeys $N^{-2} = \sum_{i=1}^{2n} r_i^2$. Call $P_{r_1,\ldots,r_{2n}}$ *true* if all the r_i are rational and nonzero and if, writing $r_i = p_i/q_i$, we have that q_1 is divisible by 3 and none of the other q_i are. Here, and throughout, any fractions we write are taken to be in lowest terms. Call an n-tuple $\{Q_1, \ldots, Q_n\}$ of orthogonal one-dimensional projections *suitable* if at least one of the Q_i is true. If P belongs to a suitable n-tuple but is not true, call P *false*. (Note that a projection need not be either true or false.) Define $p(P) = 1$ if P is true and $p(P) = 0$ if P is false.

Lemma 1: A suitable n-tuple contains precisely one true projection.

Proof: If P and Q are both true projections, the corresponding vectors have an inner product of the form $(a/9) + (p/q) + i(r/s)$, where 3 is not a factor of q. The real part thus cannot vanish, so P and Q cannot be orthogonal.

Lemma 2: *The true projections are dense in the space of all one-dimensional projections.*

Proof: Given any one-dimensional projection $P_{r_1,\ldots,r_{2n}}$ we can find an arbitrarily close approximation $P_{r'_1,\ldots,r'_{2n}}$ with rational $r'_i = p'_i/q'_i$. If q'_1 is not divisible by 3, we can find an arbitrarily close rational approximation $r''_1 = p''_1/q''_1$ to r'_1 with q''_1 divisible by 3, for example, by taking $p''_1 = 3Np'_1 + 1$ and $q''_1 = 3Nq'_1$ for a sufficiently large integer N. Similarly, if any of the q'_i for $i > 1$ are divisible by 3, we can find arbitrarily close rational approximations $r''_i = p''_i/q''_i$ to r'_i with q''_i not divisible by 3, for example, by taking $p''_1 = Np'_1$ and $q''_1 = Nq'_1 + 1$ for a sufficiently large integer N.

Lemma 3: *The suitable n-tuples are dense in the space of all n-tuples of orthogonal projections.*

Proof: Given any n-tuple $\{P_1, \ldots, P_n\}$, choose one of the projections, say P_1. As above, we can find an arbitrarily close approximation to P_1 by a true projection Q. Let U be a rotation in SU(n) which rotates P_1 to Q such that $|U - I| = \{\mathrm{Tr}[(U - I)(U^\dagger - I)]\}^{1/2}$ attains the minimal value for such rotations. The compactness of SU(n) ensures that such a U exists, though it need not be unique, and the minimal value tends to zero as Q tends towards P. The projections $\{UP_1, \ldots, UP_n\}$ form a suitable n-tuple, and this construction gives n-tuples of this type arbitrarily close to the original.

Lemma 4: *The false projections are dense in the space of all one-dimensional projections.*

Proof: Given any projection P, choose an n-tuple to which it belongs, and let Q be another projection in that n-tuple. By the construction above, we can find arbitrarily close n-tuples in which the projections approximating Q are true. The projections approximating P are thus false.

This concludes the argument for measurements defined by n-tuples of one-dimensional projections. For completeness, though, we also consider degenerate von Neumann measurements, corresponding to decompositions of the identity into general orthogonal projections. The construction above generalizes quite simply. Fixing the basis as before, we can write each projection as a matrix: $P = N(a_{ij} + ib_{ij})_{i,j=1}^{n}$, where the a_{ij} and b_{ij} are real and N is some normalization constant. Consistently with our earlier definitions for one-dimensional projections, we can define P to be true if it can be written in this form with all the a_{ij} and b_{ij} rational and nonzero and if a_{11} is then the only one which, when written in lowest terms, has a denominator divisible by 9. Clearly if P and Q are both true then $\mathrm{Tr}(PQ) \neq 0$, so they cannot be orthogonal. We can thus define suitable projective decompositions and false projections as above, and all the earlier arguments run through with trivial modifications.

At this stage a comment on measurement theory is required. The KS theorem assumes the traditional von Neumann definition of measurement, in which measurement projects the quantum state onto an eigenspace of the relevant observable. In more realistic modern treatments, a measurement causes an action on the quantum state by positive operators, which may but need not be close to projections. One could, indeed, realistically base measurement theory only on positive operator valued measurements in which the positive operators are not projections, for example, stipulating that all positive operators involved must be of maximal rank. If so, the original KS theorem becomes irrelevant, though it can easily be modified to deal with these cases. It seems more natural, though, either to allow any precisely specified positive operator decomposition, whether or not it includes projections, or else to consider general finite precision positive operator valued measurements. If all precisely specified positive operators are included, then of course the KS theorem applies. On the other hand, as we now show, the finite precision loophole also exists for positive operator measurements.

We need new definitions for positive operators. Again fixing a basis, we can write a positive operator as a matrix: $A = (a_{ij} + ib_{ij})_{i,j=1}^{n}$, where the a_{ij} and b_{ij} are real, so that $a_{ij} = a_{ji}$ and $b_{ij} = -b_{ji}$. We say that A is *true* if $a_{11} = r_1 + r_2\sqrt{2}$, with r_1 and r_2 both rational and r_2 positive, and that a projective decomposition $I = \sum_i A_i$ of the identity into positive operators is *suitable* if precisely one of the A_i is true. A is *false* if it belongs to a suitable decomposition but is not true. Under this definition, every A is either true or false. Define the truth function p by setting $p(A) = 1$ if A is true and $p(A) = 0$ if A is false. Clearly p satisfies (2) on suitable decompositions. Clearly, too, true and false operators are dense in the space of positive operators, and suitable decompositions are dense in the space of all positive operator decompositions. Hence the desired result holds.

Note that these last definitions, restricted to projections, give another example of a physically sensible truth function satisfying (1). The two different constructions perhaps help to illustrate the large scope for examples of this sort. There is nothing particularly special about either of our constructions or those of Ref. [1]: the possibility of closing the finite precision loophole by any KS-type argument can be refuted in many different ways.

It follows from the above examples that noncontextual hidden variable theories cannot be excluded by theoretical arguments of the KS type once the imprecision in real world experiments is taken into account. This does not, of course, imply that such theories are very plausible or that the particular constructions we give are capable of producing a physically interesting hidden variable theory. Nor does the discussion affect the situation regarding local hidden variable theories, which can be refuted by experiment, modulo reasonable assumptions [10–12].

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