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LAWS OF LARGE NUMBERS FOR NON-ADDITIVE PROBABILITIES

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Laws of Large Numbers for Non-additive Probabilities

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SUMMARY: We apply the concept of exchangeable random variables to the case of non-additive probability distributions exhibiting uncertainty aversion, and in the class generated by a convex core (convex non-additive probabilities, with a convex core). We are able to prove two versions of the law of large numbers (de Finetti's Theorems). By making use of two definitions of independence we prove two versions of the strong law of large numbers. It turns out that we cannot assure the convergence of the sample averages to a constant. We then model the case there is a "true" probability distribution behind the successive realizations of the uncertain random variable. In this case convergence occurs. This result is important because it renders true the intuition that it is possible "to learn" the "true" additive distribution behind an uncertain event if one repeatedly observes it (a sufficiently large number of times). We also provide a conjecture regarding the "learning" (or updating) process above, and prove a partial result for the case of Dempster-Shafer updating rule and binomial trials.

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Laws of Large Numbers for Non-additive Probabilities

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1. Introduction

In this paper, we explore the applicability of the Law of Large numbers to situations of *ambiguity*, or *Knightian uncertainty* (Knight(1921)). The classic paradigm of this type of uncertainty was presented by Ellsberg(1961): consider an urn containing red and blue balls, and a decision maker who must accept or decline lottery tickets that pay off fixed sums conditional on the ball being of a specified colour. A decision maker may behave differently when he has no information about the relative frequencies of red and blue, compared to a situation where he knows the frequencies to be 50%.

In the Bayes-Savage model, decision-makers are always represented by a utility function and a subjective probability (Savage(1954)). In the model studied in this paper, decision-makers in an ambiguous (Knightian-uncertain) environment, i. e. with little information about the functional form and parameters of the uncertainty they face, behave differently. Their preferences are represented by utility functions and a *non-additive* probability (Schmeidler(1982, 1989)). Intuitively, a non-additive probability is a probability measure where the probability of an event and the probability of its complementary event may sum to less than 1; the deviation from 1 may be thought of as a measure of the amount of uncertainty and induces cautious behaviour by the decision-maker. Thus, a probability zero event (intuitively, an event that almost never happens) need not be an event whose complement is of probability one (an event that will "definitely" not happen).

A closely related model of decision-making under Knightian uncertainty is that of Gilboa and Schmeidler(1989). In this alternative setting, preferences under uncertainty are represented by a set of additive probabilities, called the core, and the expected utility is computed as the minimum value of the expected utility for the additive distributions in the core. In general, when preferences

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exhibit uncertainty aversion, under mild assumptions a non-additive probability my be represented by a core. For the precise relationship between the approaches, see Gilboa and Schmeidler(1992a, 1992b, 1993).

These models of decision-making under Knightian uncertainty give qualitatively different results from the Bayes-Savage model when we consider situations with a single random variable. For example, in earlier research the present authors showed that Arrow's "local risk neutrality" result no longer holds in the same form (Dow and Werlang(1992)), that purely speculative trade can occur (Dow, Madrigal and Werlang(1989), that asset prices may be indeterminate (Simonsen and Werlang(1991)), and that there may be cooperation in the finitelly repeated version of the prisoners' dilemma (Dow and Werlang(1994)).

However, a very interesting class of problems concerns behaviour with many random variables. To be specific, we are interested in the question of whether uncertainty "disappears" when there are a large number of observations. In other words, we investigate the applicability of the Law of Large Numbers to situations of Knightian uncertainty. Two economic applications come immediately to mind.

The first economic application of the Law of Large Numbers is to portfolio diversification. If I hold a large portfolio containing many assets, and my beliefs about the distribution of individual asset returns are uncertain, will diversification cause this uncertainty to disappear?

The second application is to learning. If an agent observes a sequence of random variables, under what conditions will the uncertainty converge to zero over time?

The answer to the question of whether the Law of Large Numbers holds in this setting turns out to depend on the relationship between the uncertainty in the distributions of the random draws. To return to the Ellsberg urn paradigm, one can distinguish two very different situations:

Experiment (1): Balls are drawn (with replacement) from an urn with an uncertain fraction of red and blue balls.

Experiment(2): One ball each is drawn from a sequence of urns. The frequencies of red and blue balls in the urns are uncertain.

Clearly, the second experiment is in some sense more fundamental although the first is also of interest. It turns out that in the case of experiment (1), we can be sure that the agents' subjective beliefs about the frequencies becomes less uncertain over time and indeed eventually converge to a limit (which may be interpreted as the actual frequency in the urn) (Theorems 5 and 6). Thus, in the experiment (1) the Knightian uncertainty does disappear.

In the case of experiment (2), however, only a much weaker statement is possible (Theorems 3 and 4). This is perhaps unsurprising, since, intuitively, in order to have convergence, one should have a "true" underlying probability distribution on the events. However, the fact that we have a different urn in each stage destroys the notion that there is a "true" distribution behind the phenomenon.

Since the model in this paper is purely subjective (being a generalization of the Bayes-Savage model), we start by defining the concept of exchangeability (defined by de Finetti(1937)). Successive draws from an unknown distribution cannot properly be termed "independent" from the viewpoint of a decision-maker, since they all depend on the parameters of the common, unknown distribution. They are, rather, "exchangeable" in the sense that rearranging the observations in a different permutation would lead the decision-maker to make the same inferences. De Finetti's theorems (1937) shows that exchangeability ensures that the probabilities of any sequence of observations is the same as if the observations were generated by repeated independent draws from a given distribution (different from the original one, except in the case there is no correlation to start with). Thus, there is a Law of Large Numbers for exchangeable random variables. We start by exploring the analogue of this result in a Knightian-uncertain environment (Theorems 1 and 2). Then we turn to stronger independence conditions (Theorems 3, 4 5 and 6).

2. Laws of Large Numbers under Knightian Uncertainty

The concept of Knightian uncertainty is modelled here as in Schmeidler (1982, 1989), Gilboa(1987) and Schmeidler and Gilboa(1989, 1992a, 1992b). We are going to consider the case that we have the space of events as an infinite cartesian product of the same space of states of nature and events (Ω, A), endowed with the product algebra. We restrict ourselves to non-additive probabilities which are convex (i. e., $P(A \cup B) + P(A \cap B) \geq P(A) + P(B)$ for events A and B) and for which there exists a closed convex set C(P) of additive probabilities, known as the core of P, such that for all $q \in C(P)$, $q(A) \geq P(A)$ for every event A, and such that for all variables X which are bounded and measurable, one has $E_P[X] = \min_{q \in C(P)} E_q[X]$. On some conditions for non-additive probabilities to obey these properties, see Schmeidler and Gilboa(1989, 1992a, 1992b).

Definition If P is a non-additive probability defined in (Ω^n, A^n), we say it is *symmetric*, or *exchangeable* if for all permutations π of the set {1, ..., n} one has, for all events K, $P(K) = P(K^\pi)$, where K^π is obtained from K by changing the coordinates according to π:
$K^\pi = \{ (\omega_1, \ldots, \omega_n) \in \Omega^n | (\omega_1, \ldots, \omega_n) \in K^\pi \Leftrightarrow (\omega_{\pi^{-1}(1)}, \ldots, \omega_{\pi^{-1}(n)}) \in K\}$. On the other hand, if P is a non-additive probability defined in (Ω^∞, A^∞), we say it is symmetric or exchangeable if the restriction of P to any finite dimensional event subspace is symmetric. In the same way, for P in (Ω^n, A^n) we say that random variables (X_1, ..., X_n) defined in (Ω, A) are exchangeable, if for all events H (in $\Re^n$), and for all permutations π of the set {1, ..., n} we have that $P((X_1, \ldots, X_n) \in H) = P((X_{\pi(1)}, \ldots, X_{\pi(n)}) \in H^\pi)$, where H^π is defined analogously to K^π. Likewise, one defines exchangeability for variables on (Ω^∞, A^∞). It is easy to see that if (X_1, ..., X_n) are exchangeable, then except for a set of P-probability zero (to be precise, of marginal over Ω of P) we have, for all i and j: $X_i = X_j$. Thus, when (X_1, ..., X_n) are exchangeable we can always think of random variables X_i and X_j which are "identical", and an underlying probability distribution on (Ω^n, A^n) which is exchangeable. That is what is done from this point on in the text.

Definition We say that a set C of (additive) probabilities defined on (Ω^n, A^n) is *symmetric* if for all $q \in C$, and for all permutations π of the set $\{1, ..., n\}$ we have that $q^\pi \in C$, where q^π is the probability distribution defined by $q^\pi(K) = q(K^\pi)$. Again, it is possible to extend the definition to a set of probabilities in $(\Omega^\infty, A^\infty)$.

The first auxiliary result shows that the core of an exchangeable non-additive probability P is symmetric.

Lemma 1 If P is an exchangeable non-additive probability P, convex, defined on (Ω^n, A^n) (n possibly infinite), and C(P) is its core, C(P) is symmetric.
Proof. Suppose that there is $q \in C(P)$ and a permutation π of the set $\{1, ..., n\}$, with $q^\pi \notin C(P)$. This means that there is an event K with $q^\pi(K) < P(K)$. But P is exchangeable, so that $P(K) = P(K^\pi) \leq q(K^\pi)$. But this is a contradiction, since we have $q^\pi(K) = q(K^\pi)$.

QED.

The next auxiliary result is central to the proofs. It is shown that in the core of an exchangeable convex non-additive probability, where the core is convex, we have an exchangeable additive probability.

Lemma 2 Let P be an exchangeable convex non-additive probability defined on (Ω^n, A^n) (n possibly infinite, Ω finite), and C(P) its convex core. Then, there exists an additive probability q^P in C(P) which is exchangeable. Furthermore, it is the average of the extremal points of C(P).
Proof. Take n finite to begin with. Then the set of extremal points of C(P) is finite (because Ω is finite). Also, it is easy to check that if q is in the extremal set, then for any permutation π of the set $\{1, ..., n\}$ we have that q^π is extremal too. Therefore, if we take the average of the extremal points, call it q^P, it must be the case that $(q^P)^\pi = q^P$, for any permutation, that is, it is exchangeable. To see that the result is valid for n infinite, consider finite dimensional projections of increasing length. It is immediate to check that restrictions of projections to subspaces coincide with the projections on that subspace. Finally, with the weak topology C(P) is compact, and we take limits of the increasing finite dimensional projections.

QED.

We are ready to state and prove the first two results.

Theorem 1 (de Finettis's Weak Law of Large Numbers) Let $\{X_i\}_{i=1, 2, \dots}$ be an infinite sequence of bounded random variables in the space (Ω, A). Let S_k and S_h be, respectively, the average of k and h of the random variables. Assume that the random variables are exchangeable for P in $(\Omega^\infty, A^\infty)$, and that P is convex, with a closed convex core (C(P)), and such that for any bounded random variable Y, $E_P[Y] = \min_{q \in C(P)} E_q[Y]$. Then, given any $\varepsilon > 0$ and $\theta > 0$, there is k_0 such that for all k and h greater than or equal to k_0:

$$P(|S_k - S_h| > \varepsilon) < \theta.$$

[In other words, if one takes k and h large enough, the probability that the sample averages S_k and S_h are going to differ by more than ε can be made as small as it is wished.]

Proof. For q^P of the lemma above is an additive probability which is exchangeable in de Finetti's sense. But $P(|S_k - S_h| > \varepsilon) \leq q^P(|S_k - S_h| > \varepsilon)$. Hence, it is enough to prove the result for q^P. This is done in de Finetti(1937).

QED.

Theorem 2 (de Finettis's Strong Law of Large Numbers) Let $\{X_i\}_{i=1, 2, \dots}$ be an infinite sequence of bounded random variables in the space (Ω, A). Let S_k be the average of the k first random variables. Assume that the random variables are exchangeable for P in $(\Omega^\infty, A^\infty)$, and that P is convex, with a closed convex core (C(P)), and such that for any bounded random variable Y, $E_P[Y] = \min_{q \in C(P)} E_q[Y]$. Then, given any ε and $\theta > 0$, there is k_0 such that for all k greater than or equal to k_0:

$$P(\text{there is } \ell \geq 0 \text{ with } |S_k - S_{k+\ell}| > \varepsilon) < \theta.$$

[There is a sharp difference between this law of large numbers and the preceding one. In the first case, it is true that $P(|S_k - S_{k+\ell}| > \varepsilon) < \theta$ for any $\ell \geq 0$. However, the probability that *some* of the successive $S_{k+1}, S_{k+2}, \dots$ differs from S_k more than ε may be very large, while in the second result it is assured that this does not occur. Of course the second result is stronger than the first.]

Proof. Analogous to the proof of the theorem above.

QED.

Recall that these two laws show that there is zero probability of non convergence. With non-additive probabilities this is different from the fact that there should be convergence with probability 1. It may even be that convergence also occurs with probability zero. Theorem 5 in the text shows that convergence may occur with probability 1, for a particular class of exchangeable random variables.

The two laws of large numbers above assert that the successive averages get closer and closer the longer the sequence of trials. However, it does not say whether the averages themselves approach something. As it turns out, we may use the same arguments of de Finetti(1937) and characterize the distribution $\Phi(\xi) = \lim_{k\to\infty} q^P(S_k \leq \xi)$ (other versions of the results on exchangeable random variables can be seen in Feller(1966) and Loève(1963)). The point is that there is no meaning in this distribution, since the primitive uncertainty is P. In fact, in general the limit of S_k as k increases may be even a random variable with non-additive distribution. We turn now to add independence in the hope of understanding the process a little more.

There are two competing definitions of independence for convex non-additive probabilities.

Definition (Gilboa and Schmeidler(1989)) Let P_1 and P_2 be two convex non-additive probabilities on $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, respectively (the definition is trivially extended to more than 2). If the cores of P_1 and P_2 are $C(P_1)$ and $C(P_2)$, Schmeidler and Gilboa(1989) define the *product non-additive probability* $P_1 \otimes P_2$ (which corresponds to their definition of the *independent* combination of P_1 and P_2), as the non-additive probability defined on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ whose core is: $C(P_1 \otimes P_2) = co(\{q_1 \otimes q_2$, where $q_1 \in C(P_1)$ and $q_2 \in C(P_2)\})$, being co(T) the closed convex hull of the set T.

Definition (Hendon, Jacobsen, Sloth and Tranæs(1993)) Let the notation be as above. Hendon, Jacobsen, Sloth and Tranæs(1993) define the *minimal convex product non-additive probability* $P_1 \underline{\otimes} P_2$ (which corresponds to their definition of the *independent* combination of P_1 and P_2), as the non-additive probability whose core is: $C(P_1 \underline{\otimes} P_2) = \{q$ additive defined on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$, such that for all $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$, it follows that $q(E_1 \times E_2) \geq P_1(E_1)P_2(E_2)\}$. Clearly the core of this second definition contains the core of the first, which means that for all $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$, $P_1 \underline{\otimes} P_2(E) \leq P_1 \otimes P_2(E)$.

Lemma 3 Let P be an exchangeable convex non-additive probability defined on $(\Omega^n, \mathcal{A}^n)$ (n possibly infinite, Ω finite), and C(P) its convex core. Assume that P is a product non-additive probability, of P_i, i=1, ..., n, according to any of the two definitions above (that is to say, either $P=P_1\otimes P_2\otimes\cdots\otimes P_n$ or $P=P_1\underline{\otimes}P_2\underline{\otimes}\cdots\underline{\otimes}P_n$). Then $P_i = P_j$ for all i and j.

Proof. For any i=1, ..., n, and for any event E in the i-th copy of Ω, define the event $E^i = \Omega\times\cdots\times\Omega\times E\times\Omega\cdots\times\Omega$, where the set E is exactly in the i-th position of the n-fold cartesian product. By definition E^i is an element of the product algebra $\mathcal{A}\otimes\mathcal{A}\otimes\cdots\otimes\mathcal{A}$ (n times repeated) = $\mathcal{A}^n$. Consider first Schmeidler-Gilboa's definition. To start with, take n finite. $P_1\otimes P_2\otimes\cdots\otimes P_n(E^i) = \min_{q\in C(P1\otimes P2\otimes\ldots\otimes Pn)} q(\Omega\times\cdots\times\Omega\times E\times\Omega\cdots\times\Omega)$. But $C(P_1\otimes P_2\otimes\cdots\otimes P_n)$ is the closed convex hull of the independent combinations of the elements of the core of each P_i ($C(P_i)$). Thus, $q(\cdot)$ is a linear combination of a finite number of independent additive distributions in $(\Omega^n, \mathcal{A}^n)$, being each of the marginals on the i-th copy of Ω an element of $C(P_i)$. This is true because n and Ω are finite. Then the marginal of $q(\cdot)$ on the i-th copy of Ω is a finite linear combination of elements of $C(P_i)$. But since $C(P_i)$ is convex by assumption (all our non-additive probabilities are convex with convex cores), it follows that there is $q_i \in C(P_i)$ such that $q(\Omega\times\cdots\times\Omega\times E\times\Omega\cdots\times\Omega) = q_i(E)$, for all E in $\mathcal{A}$. Conversely, if $q_i \in C(P_i)$ we can obviously define an element in $C(P_1\otimes P_2\otimes\cdots\otimes P_n)$ with the marginal coinciding with q_i. Hence, $P_1\otimes P_2\otimes\cdots\otimes P_n(E^i) = \min_{qi\in C(Pi)} q_i(E) = P_i(E)$. By the definition of exchangeability, $P_1\otimes P_2\otimes\cdots\otimes P_n(E^i) = P_1\otimes P_2\otimes\cdots\otimes P_n(E^j)$, for all i and j. Therefore, for all i and j, $P_i(E) = P_j(E)$. If n is infinite, it is enough to consider finite marginals of increasing length of the product non-additive probability, and repeat the argument above. Now, take the definition of Hendon-Jacobsen-Sloth-Tranæs. $P_1\underline{\otimes}P_2\underline{\otimes}\cdots\underline{\otimes}P_n(E^i) = \min_{q\in C(P1\underline{\otimes}P2\underline{\otimes}\ldots\underline{\otimes}Pn)} q(\Omega\times\cdots\times\Omega\times E\times\Omega\cdots\times\Omega)$. But by the definition of $C(P_1\underline{\otimes}P_2\underline{\otimes}\cdots\underline{\otimes}P_n)$:

$q(\Omega\times\cdots\times\Omega\times E\times\Omega\cdots\times\Omega) \geq P_1(\Omega)\cdots P_{i-1}(\Omega)P_i(E)P_{i+1}(\Omega)\cdots\cdots P_n(\Omega) = P_i(E)$. Thus, the marginal of q on the i-th copy of Ω is an element of $C(P_i)$. Conversely, consider $q_i \in C(P_i)$. As before, it is immediate to define an element of $C(P_1\underline{\otimes}P_2\underline{\otimes}\cdots\underline{\otimes}P_n)$ with the marginal coinciding with q_i. Hence, the same argument as above holds, and exchangeability assures that the non-additive distributions P_i and P_j are equal for all i and j.

QED.

The lemma above shows that the interpretation of the exchangeability hypothesis in the non-additive case is similar to the additive case. In a certain sense, exchangeability is the generalization of the assumption of iid distributions to allow for correlation. We may now analyse the behaviour of sample averages of $P=Q\otimes Q\otimes\cdots$ ad infinitum (or $P=Q\underline{\otimes}Q\underline{\otimes}\cdots$).

Theorem 3 (Strong Law of Large Numbers for Independent and Identically Distributed Random Variables - Independence HJST) Let $\{X_i\}_{i=1,2,\ldots}$ be an infinite sequence of bounded random variables in the space (Ω, A). Let S_k be the average of the k first random variables. Assume that the random variables are independent (according to the definition of Hendon-Jacobsen-Sloth-Tranæs) and exchangeable for $P=Q\underline{\otimes}Q\underline{\otimes}\cdots$ in $(\Omega^\infty, A^\infty)$, and that P is convex, with a closed convex core (C(P)), and such that for any bounded random variableY, $E_P[Y] = \min_{q\in C(P)} E_q[Y]$. Then, the probability P that the limit points of the sequence $\{S_k\}$ are outside the closed interval $[E_Q(X_1), -E_Q(-X_1)]$ is zero.

Proof. It is enough to consider any probability in the core of the form $q = p\otimes p\otimes\cdots$ (p defined on (Ω, A)), and apply the usual strong law of large numbers.

QED.

As it was mentioned before, this result is interesting, but it allows a lot of freedom, because it may be that the P-probability that the accumulation points of the sample means $\{S_k\}$ lye in $[E_Q(X_1), -E_Q(-X_1)]$ is also zero. We are able to prove a stronger result, for the definition of independence of Gilboa-Schmeidler. We do not know whether this stronger theorem is true for the definition of independence of Hendon-Jacobsen-Sloth-Tranæs.

Theorem 4 (Strong Law of Large Numbers for Independent and Identically Distributed Random Variables - Independence GS) Let $\{X_i\}_{i=1,2,\ldots}$ be an infinite sequence of bounded random variables in the space (Ω, A). Let S_k be the average of the k first random variables. Assume that the random variables are independent (according to the definition of Gilboa-Schmeidler) and exchangeable for $P=Q\otimes Q\otimes\cdots$ in $(\Omega^\infty, A^\infty)$, and that P is convex, with a closed convex core (C(P)), and such that for any bounded random variable Y, $E_P[Y] = \min_{q\in C(P)} E_q[Y]$. Then:

(i) the probability P that the accumulation points of the sequence $\{S_k\}$ are in the closed interval $[E_Q(X_1), -E_Q(-X_1)]$ is one;

(ii) the probability P that the accumulation points of the sequence $\{S_k\}$ are in any set T strictly contained in $[E_Q(X_1), -E_Q(-X_1)]$ is zero, which means that this closed interval is the smaller set with this property.

Proof. For the definition of independence of Gilboa-Schmeidler, it turns out that it is simpler to prove that a set has P-probability 1. If we can prove that for all elements of the core C(P) which are independent, i. e., are of the form $q_1 \otimes q_2 \otimes \cdots$ with $q_i \in C(Q)$, it is true that the probability of the accumulation points lying in the set $[E_Q(X_1), -E_Q(-X_1)]$ is 1, then the probability under any other of the elements of the core is also 1, since they are convex combinations of the independent distributions. Hence, let us restrict attention to proving this property for the independent elements of the core, and we are done. Consider a generic independent element of the core, $\mathbf{q}=q_1 \otimes q_2 \otimes \cdots$ with $q_i \in C(Q)$ for all i. The random variables $\{X_i\}_{i=1, 2, \ldots}$ are fixed, and under $\mathbf{q}$ they are independent. Then, $Y_i = X_i - E_{q_i}X_i$ form a sequence of random variables such that $E[Y_k \mid Y_1, Y_2, \ldots, Y_{k-1}] = 0$ under $\mathbf{q}$, for all $k \geq 2$. Define Σ_k to be the sum of the first k random variables Y_i. Then, by Feller(1966, pp. 238, Thm. 2), if it is true that the infinite sum $\Sigma_{i\geq 1}(1/i^2) \cdot E_{qi}[Y_i^2]$ is finite, then with $\mathbf{q}$-probability 1: Σ_k/k tends to 0. But the infinite sum of the variances is finite, because the random variables are bounded, and $\Sigma_{i\geq 1}(1/i^2)$ is finite. Therefore, we have that the set for which Σ_k/k tends to 0 has probability 1 according to $\mathbf{q}$. If we recall the definiiton of Y_i, $\Sigma_k/k = S_k - (1/k) \cdot \Sigma_{i=1,\ldots,k} E_{qi}[Y_i]$ it is immediate to check that the set of accumulation points of the sample averages (S_k) is contained in the closed interval $[E_Q(X_1), -E_Q(-X_1)]$, because we have that $E_Q(X_1) = \min_{q \in C(Q)} E_q[X_1]$, and $-E_Q(-X_1) = \max_{q \in C(Q)} E_q[X_1]$. In other words, we proved that the set of states of the world for which the accumulation points of the sequence $\{S_k\}_{k\geq 1}$ are in the closed interval above is 1 for <u>any</u> independent distribution on the core of P. Thus, by the argument at the beginning of the proof we proved (i). To see that (ii) is true also, we have to notice that if there is a set $T \subset [E_Q(X_1), -E_Q(-X_1)]$ which is different from the interval, then there is some $q' \in C(Q)$ with $E_{q'}X_1 \in [E_Q(X_1), -E_Q(-X_1)]$ and $E_{q'}X_1 \notin T$. Thus, by considering the probability $\mathbf{q'} = q' \otimes q' \otimes \cdots$ and applying the usual strong law of large numbers, T has $\mathbf{q'}$-probability zero, and hence P-probability zero, since $\mathbf{q'} \in C(P)$.

QED.

The fact that we cannot prove a stronger result should not come as a surprise, under the weak hypotheses of theorems 3 and 4. Think about the following two experiments. Experiment number 1 is a repeated draw with replacement of a ball from the same urn with unknown number of red and blue balls. Experiment number 2 is a repeated draw of a ball from urns which only have red and blue balls. The difference is that in the second experiment there is a different urn in each stage. If the total number of the balls is the same in all the urns, say 100, and there is no other information available, both experiments could be modelled, in principle, by means of repeated realizations of a random variable (say it is 0 if the ball is red and it is 1 if the ball is red) with a non-additive probability distribution.

Let us try to distinguish them in probability terms. In the first case the urn is fixed, so that one knows that there is a "true" (possibly interpreted by some as "objective") probability distribution behind the event of drawing a ball, although this "true" probability distribution is not known. The second experiment does not have such a probability. We argue that the second experiment is suitably formalized by the independence of the non-additive iid probabilities, as in theorems 3 and 4. However, the way to model independence in order to capture the intuition of the first experiment is through a non-additive probability whose core contains only (additive) iid probabilities, and that is what we do in our last result below. Note that distributions in this category are not independent according to the definitions Gilboa-Schmeidler or Hendon-Jacobsen-Sloth-Tranæs, but are exchangeable.

Theorem 5 (Strong Law of Large Numbers - Core of the Distribution Formed Only by IID Probabilities) Let $\{X_i\}_{i=1,2,\ldots}$ be an infinite sequence of bounded random variables in the space (Ω, A). Let S_k be the average of the k first random variables. Assume that the distribution Q in $(\Omega^\infty, A^\infty)$ of the random variables is convex, with a closed convex core (C(Q)) consists only of additive probabilities of the form $q = p\otimes p\otimes\cdots$ (p defined on (Ω, A), and $p\in C$ - that is to say that C is the projection of C(Q) in any of the copies of (Ω, A)), and such that for any bounded random variable Y, $E_Q[Y] = \min_{q\in C(Q)} E_q[Y]$. Then:

(i) the probability Q that the limit points of the sequence $\{S_k\}$ are outside the closed interval $[E_Q(X_1), -E_Q(-X_1)]$ is zero;

(ii) there is a collection of subsets of Ω^∞, $\{\Omega_p\}_{p\in C}$ such that:

(iia) $Q(\cup_{p\in C}\Omega_p) = 1$;
(iib) if $\omega\in\Omega_p$, then $\lim_{k\to\infty} S_k = E_pX_1$ (which implies that the Q-probability of convergence and, furthermore, of convergence to some value in $[E_Q(X_1), -E_Q(-X_1)]$ is one);
(iii) any set strictly contained in the interval $[E_Q(X_1), -E_Q(-X_1)]$ has Q-probability zero of being the set of accumulation points of the sequence $\{S_k\}$.
Proof. By the strong law of large numbers, the set Ω_p for which there is convergence of S_k to E_pX_1, when k goes to infinite is of q-probability 1, where $q=p\otimes p\otimes\cdots$. That is, $q(\Omega_p) = 1$. But $Q(\cup_{p'\in C}\Omega_{p'}) = \min_{q\in C(Q)} q(\cup_{p'\in C}\Omega_{p'})$, and we have that $q(\cup_{p'\in C}\Omega_{p'}) \geq q(\Omega_p) = 1$ for all q in C(Q). Thus, $Q(\cup_{p'\in C}\Omega_{p'}) = 1$. Hence, (i), (iia) and (iib) are proven. To prove (iii) we have to notice that if there is a set $T \subset [E_Q(X_1), -E_Q(-X_1)]$ which is different from the interval, then there is some $p'\in C$ with $E_{p'}X_1\in[E_Q(X_1), -E_Q(-X_1)]$ and $E_{p'}X_1\notin T$. Thus, by considering the probability $q' = p'\otimes p'\otimes\cdots$ and applying the usual strong law of large numbers, T has q'-probability zero, and hence Q-probability zero, since $q'\in C(Q)$.

QED.

Here we see that there is room to "learn" what is the "true" distribution. If the "true" distribution is p^*, then the likelihood that the realized averages converge to $E_{p^*}X_1$ is 1 (under p^*), and that is what the data will show the observers.

Consider the case of the first experiment, and interpret the random variable X_i as 1 if a red ball is drawn in the i-th draw, and zero if the ball is blue. Notice that, for all i, we have that the expected value of X_i is given by $E_Q[X_i] = E_Q[X_1] = \min_{p\in C} E_p[X_1]$, so that when there is a draw from the urn we would evaluate the likelihood of a red ball occurring according to the non-additive probability on (Ω, A) whose core is C. As soon as we have a history of draws, we are able to compute S_k for $k\geq1$ and we may use this knowledge to update our expectation. For the updating rule of Dempster-Shafer (see Gilboa and Schmeidler(1993) for axiomatizations of updating rules, and Gilboa and Schmeidler(1992a, 1992b) for more discussion about them), we are able to prove the following theorem. Recall that $E_{Q'}(X_i) = Q'(X_i=1)$ for any possible i and Q' non-additive, due to the definition of the Choquet integration, and the fact that X_i can only attain values 0 and 1. Also, if Q' is a generic non-additive probability and B a set such that $Q'(B^c) < 1$, then the Dempster-Shafer updating rule is given by: $Q'(A|B) = (Q'(A \cup B^c) - Q'(B^c)) / (1 - Q'(B^c))$.

Theorem 6 (Updating with Dempster-Shafer in the Case of Binomial Trials, under the Hypotheses of Theorem 5) Consider the case that we have the observation of an event with two outcomes. Take the variable X_i as 1 if one of the outcomes occurred, and 0 otherwise. [Think of the event of drawing repeatedly the ball from the urn with fixed, but unknown probability distribution - experiment number 1.] Let $\{X_i\}_{i=1,2,\ldots}$ be the infinite sequence of trials (X_i means trial in stage i) in the case above, i. e. Ω with two outcomes, $A = \wp(\Omega)$ (that is, the power set of Ω). Let S_k be the average of the first k trials. Assume that the distribution Q in $(\Omega^\infty, A^\infty)$ of the trials is convex, with a closed convex core (C(Q)) consists only of additive probabilities of the form $q = \mathbf{p} \otimes \mathbf{p} \otimes \cdots$ with $\mathbf{p}$ defined on (Ω, A) as the distribution which gives probability p to $X_i = 1$ and 1-p to $X_i = 0$, $p \in [p_-, p^-]$ - that is to say that the interval $[p_-, p^-]$ is the projection of C(Q) in any of the copies of (Ω, A). Then, if $\lim_{k\to\infty} S_k = p^*$:

(i) $p^* \in]p_-, p^-[\Rightarrow \lim_{k\to\infty} E_Q[X_{k+1}|S_k] = p^*$;

(ii) $p^* \in [0, p_-] \Rightarrow \lim_{k\to\infty} E_Q[X_{k+1}|S_k] = p_-$;

(iii)$p^* \in [p^-, 1] \Rightarrow \lim_{k\to\infty} E_Q[X_{k+1}|S_k] = p^-$.

Moreover, by combining this result with theorem 5, we see that:

$\lim_{k\to\infty} E[X_{k+1}|S_k] = \lim_{k\to\infty} S_k$ in a set of Q-probability 1.

[This theorem is exactly about "learning" the "true" distribution with the successive observation of the trials.]

Proof. The conditional expected value above is taken with respect to the Dempster-Shafer updated probability. This means that $E_Q[X_{k+1}|S_k] = Q[X_{k+1}=1|S_k]$. Thus, we compute in general $Q[X_{k+1}=1|S_k=r/k]$, and take the limit as k goes to infinity and r/k to p^*. Making the calculations, we have:

$$Q[X_{k+1}=1|S_k=r/k] = 1 - \left[\left(\max_{p\in[p_-,p^-]} p^r(1-p)^{k-r+1}\right)/\left(\max_{p\in[p_-,p^-]} p^r(1-p)^{k-r}\right)\right] \quad (*).$$

The behaviour of the function $f(p) = p^r(1-p)^s$ is the following: at 0 and 1 it attains its minimum, which is 0. There is only one maximum, attained at $p=r/(r+s)$, with value $f(r/(r+s)) = [r/(r+s)]^r \cdot [s/(r+s)]^s$. For values of p less than this number, it is strictly increasing, and for larger values, strictly decreasing. Thus, in case (i) lim $r/k = p^* \in]p_-, p^-[$, which means that for large enough k we have that both maxima in expression (*) above are attained in the corresponding value of $r/(r+s)$. This gives us:

$$Q[X_{k+1}=1 | S_k=r/k] = 1 - \left(k/(k+1)\right)^{k} \cdot \left(((k-r+1)/(k-r))^{k-r}\right) \cdot \left((k-r+1)/(k+1)\right) \quad (**).$$

Taking the limit when k goes to infinity and k-r also so that r/k tends to p*, the right hand side of (**) tend to:

$\lim_{k\to\infty,\ r/k\to p^*} Q[X_{k+1}=1 | S_k=r/k] = 1 - (1/e)\cdot e\cdot(1-p^*) = p^*$.

This shows (i). The other two cases are similar, but easier. The final remark on the statement of the theorem is immediate, given Theorem 5.

QED.

In fact, we conjecture that the following result is true: for any sequence of random variables $\{X_i\}_{i=1, 2, \ldots}$ in the hypotheses of Theorem 5 it must be the case that $\lim_{k\to\infty} E_Q[X_{k+1} | S_k] = \lim_{k\to\infty} S_k$ in a set of Q-probability 1. On the other hand, Theorem 6 above is not true if one replaces Dempster-Shafer rule by the analogue of Bayes rule, i. e. by the updating rule defined by $Q'(A|B) = Q'(A\cap B)/Q'(B)$. An interesting problem is to find the set of (fixed) updating rules (see Gilboa and Schmeidler(1993) for a detailed description of the set of updating rules) for which Theorem 6 would hold.

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