Orthogonal projections with $\sum P_i = I$, proving that $i \neq j \Rightarrow P_j P_i = 0$

I am reading *Introduction to Quantum Computing* by Kaye, Laflamme, and Mosca. As an exercise, they write

"Prove that if the operators P_i satisfy $P_i^* = P_i$ and $P_i^2 = P_i$, then $P_i P_j = 0$ for $i \neq j$."

In the context of this problem, it has been assumed that $I = \sum_{i=1}^n P_i$, where I suppose that n could be infinite. I have shown that this is true in the trivial case $n = 2$, but the general case has been eluding me. How should I attack this?

(linear-algebra) (quantum-mechanics)

edited Mar 8 '12 at 0:12 Jonas Meyer 37.6k 5 116 200

asked Mar 7 '12 at 23:11 Nick Thompson 2,545 7 15

3 Answers

For each j,

$$P_j = P_j I P_j = P_j \left(\sum_{k=1}^n P_k \right) P_j = \sum_{k=1}^n P_j P_k P_j = P_j + \sum_{k \neq j} P_j P_k P_j,$$

so $\sum_{k \neq j} P_j P_k P_j = 0$. For each $i \neq j$, $P_j P_i P_j = (P_i P_j)^* P_i P_j$ is a positive operator, and a sum of positive operators is positive, so $-P_j P_i P_j = \sum_{k \neq i,j} P_j P_k P_j$ is also positive. This is only possible if $P_j P_i P_j = 0$. Since $\|P_i P_j\|^2 = \|(P_i P_j)^* P_i P_j\| = \|P_j P_i P_j\|$, it follows that $P_i P_j = 0$.

edited Mar 8 '12 at 0:18

answered Mar 7 '12 at 23:28 Jonas Meyer 37.6k 5 116 200

Thank you very much! – Nick Thompson Mar 8 '12 at 0:13

Because of the properties you state, $\|P_j x\|^2 = (x, P_j x) = (P_j x, x)$.
Therefore,

$$\|x\|^2 = (\sum_j P_j x, x) = \sum_j \|P_j x\|^2.$$

Apply this identity to $x = P_k y$, and use the fact that $P_k^2 = P_k$:

$$\|P_k y\|^2 = \sum_{j \neq k} \|P_j P_k y\|^2 + \|P_k y\|^2.$$

The only way this can happen is $P_j P_k y = 0$ for all $j \neq k$.

answered Dec 3 '14 at 22:31 TrialAndError 31.7k 3 9 48

Here is a slight variant of Jonas' argument.

Assume that $p_1, \ldots, p_n$ are projection elements of a unital C^* -algebra A, where $n \in \mathbb{N}_{\geq 2}$, such that

$$\sum_{k=1}^{n} p_k = 1_A.$$

Choose distinct $i, j \in [n]$, where $[n] \stackrel{\text{df}}{=} \mathbb{N}_{\leq n}$. Then

$$\begin{aligned} p_i &= p_i 1_A \\ &= p_i \sum_{k \in [n]} p_k \\ &= \sum_{k \in [n]} p_i p_k \\ &= p_i^2 + p_i p_j + \sum_{k \in [n] \setminus \{i,j\}} p_i p_k \\ &= p_i + p_i p_j + \sum_{k \in [n] \setminus \{i,j\}} p_i p_k. \end{aligned}$$

It follows that

$$p_i p_j p_j^* = p_i p_j = - \sum_{k \in [n] \setminus \{i,j\}} p_i p_k = - \sum_{k \in [n] \setminus \{i,j\}} p_i p_k p_k^*,$$

and consequently,

$$(\spadesuit) \qquad (p_i p_j)(p_i p_j)^* = p_i p_j p_j^* p_i^* = - \sum_{k \in [n] \setminus \{i,j\}} p_i p_k p_k^* p_i^* = - \sum_{k \in [n] \setminus \{i,j\}} (p_i p_k)(p_i p_k)^*.$$

On the extreme left of $(\spadesuit)$, we have a positive element, while on the extreme right of $(\spadesuit)$, we have a negative element. This can only mean that both extremes are zero, so $(p_i p_j)(p_i p_j)^* = 0_A$. Hence,

$$\|p_i p_j\|_A^2 = \|(p_i p_j)(p_i p_j)^*\|_A = \|0_A\|_A = 0,$$

or equivalently, $p_i p_j = 0_A$.

answered Jun 25 '15 at 2:05

Transcendental

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