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LETTER TO THE EDITOR

Bell–Kochen–Specker theorem for 20 vectors

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Abstract. An example of the Bell–Kochen–Specker argument is given for 20 rays in four dimensions.

The Bell–Kochen–Specker theorem offers a simple proof that quantum mechanics contradicts the claim that individual constituents of an ensemble have 'hidden values' [1, 2]. The first explicit demonstration of the theorem [3] required 117 rays in $\mathbb{R}^3$, but this has been reduced to 31 rays in $\mathbb{R}^3$ [4] and 24 in $\mathbb{R}^4$ [5]. In this letter, we provide an ensemble of 20 elements which demonstrates the theorem. The proof is a refinement of that of Peres [5].

In $\mathbb{R}^4$, select 20 rays through the origin. Let each ray be named by a coordinate x, y, z, w, but treat opposite directions as picking out the same ray. The claim that each ray may be consistently associated with a 'hidden value' of either one or zero is refutable in quantum mechanics.

To demonstrate the refutation, we first assume an assignment of 'hidden values' to a specified set of rays. Then, by invoking a simple rule of quantum mechanics, we derive a contradiction. The simple rule, derived by Penrose [6], is that given four mutually-orthogonal rays 1, 2, 3, 4 in $\mathbb{R}^4$ (a four-clique), we may write $f(1)+f(2)+f(3)+f(4)=1$, where $f(\,)$ has value one or zero. Table 1 demonstrates a set of 20 vectors with 11 four-cliques for which the set of associated equations cannot be satisfied. We note in table 1 that the sum of the left-hand side is *odd*, yet, since each vector contributes either one or zero an *even* number of times to the sum of the right-hand side, the equations are not simultaneously consistent.

Table 1. Inconsistent equations derived from mutually-orthogonal rays. Each ray occurs twice or four times.

$$1 = f(1,0,0,0) + f(0,1,0,0) + f(0,0,1,0) + f(0,0,0,1)$$
$$1 = f(1,0,0,0) + f(0,1,0,0) + f(0,0,1,1) + f(0,0,1,-1)$$
$$1 = f(1,0,0,0) + f(0,0,1,0) + f(0,1,0,1) + f(0,1,0,-1)$$
$$1 = f(1,0,0,0) + f(0,0,0,1) + f(0,1,1,0) + f(0,1,-1,0)$$

$$1 = f(-1,1,1,1) + f(1,-1,1,1) + f(1,1,-1,1) + f(1,1,1,-1)$$
$$1 = f(-1,1,1,1) + f(1,1,-1,1) + f(1,0,1,0) + f(0,1,0,-1)$$
$$1 = f(1,-1,1,1) + f(1,1,-1,1) + f(0,1,1,0) + f(1,0,0,-1)$$
$$1 = f(1,1,-1,1) + f(1,1,1,-1) + f(0,0,1,1) + f(1,-1,0,0)$$

$$1 = f(0,1,-1,0) + f(1,0,0,-1) + f(1,1,1,1) + f(1,-1,-1,1)$$
$$1 = f(0,0,1,-1) + f(1,-1,0,0) + f(1,1,1,1) + f(1,1,-1,-1)$$
$$1 = f(1,0,1,0) + f(0,1,0,1) + f(1,1,-1,-1) + f(1,-1,-1,1)$$

There are 192 distinct sets of 11 equations derivable from subsets of Peres' 24 ray set [5]. However, none of the latter is a subset of Penrose's 'dodecahedron' [6].

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