Probability

April 14, 2016

1 Probability Spaces

1.1 Classical Probability Spaces

Textbook probability theory [2, 3, 5] is defined using the notions of a *sample space* Ω, a space of *events* $\mathcal{F}$, and a *probability measure* μ. In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set and the space of events $\mathcal{F}$ as, 2^Ω, the powerset of Ω. A *probability measure* is a function $\mu : \mathcal{F} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i, we have $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example 1 (Two coin experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

1.2 Quantum Probability Spaces

A classical model decides the occurrence or non-occurence of all events simultaneously which is inconsistent with quantum mechanics. Indeed, in the quantum world, there are (non-commuting) events which cannot happen simultaneously. To accommodate this situation, we completely abandon the sample space Ω and define and reason directly about events. Thus a quantum probability space will consist of just two components: a set of events $\mathcal{A}$ and a probability measure $\phi : \mathcal{A} \to [0, 1]$. These components are defined as follows.

We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{A}$ as *projections* on $\mathcal{H}$. Similarly to the classical case, a probability measure is a function $\phi : \mathcal{A} \to [0, 1]$ satisfying:

- $\phi(\mathbb{1}) = 1$, and
- for all $A \in \mathcal{A}$, we have $\phi(A^*A) \geq 0$.

As an example, let $P_1, P_2, \ldots, P_k$ be mutually orthogonal projections on $\mathcal{H}$ with sum $\mathbb{1}$ and define the event space $\mathcal{A}$ to be the linear span of these operators:

$$\mathcal{A} = \{\sum_{j=1}^{k} \lambda_j P_j \mid \lambda_i, \ldots, \lambda_k \in \mathbb{C}\}.$$

Each state $|\psi\rangle$ of the Hilbert space induces a probability measure $\phi_\psi : \mathcal{A} \to [0,1]$ defined as follows:

$$\phi_\psi(A) = \langle\psi|A|\psi\rangle\rangle$$

Concrete example: consider the two qubit Hilbert space with computational bases $|0\rangle$ and $|1\rangle$ and consider the following families of projections:

- Family I: $|0\rangle\langle 0|$, $|1\rangle\langle 1|$
- Family II: $|+\rangle\langle +|$, $|-\rangle\langle -|$

and consider the two states ...

1.3 Plan

We see that a quantum probability space embeds several infinite precision assumptions, both in the Hilbert space itself and in the range of the probability measure. We will now study the consequences of restricting each occurrence of $\mathbb{R}$ or $\mathbb{C}$ to some finite structure.

References

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