Probability

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1 Probability Spaces

1.1 Classical Probability Spaces

Probability theory [5, 7, 16] is defined using the notions of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ. In this paper, we will only consider *finite* sample spaces: we therefore define a sample space Ω as an arbitrary non-empty finite set, the space of events $\mathcal{E}$ as 2^{Ω}, the powerset of Ω, and the *probability measure* as a function $\mu : \mathcal{E} \to [0, 1]$ such that:

- $\mu(\Omega) = 1$, and
- for a collection of pairwise disjoint events E_i, the probability measures are additive $\mu(\bigcup E_i) = \sum \mu(E_i)$.

Example of a problem on a finite sample space (Two coin experiment) Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. The event that the first coin is "heads" is $\{HH, HT\}$; the event that the two coins land on opposite sides is $\{HT, TH\}$; the event that at least one coin is tails is $\{HT, TH, TT\}$. Depending on the assumptions regarding the coins, we can define several probability measures. Here is a possible one:

$$
\begin{array}{rclrcl}
\mu(\emptyset) & = & 0 & \mu(\{HT, TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT, TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH, TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH, HT, TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH, HT, TT\}) & = & 1/3 \\
\mu(\{HH, HT\}) & = & 1/3 & \mu(\{HH, TH, TT\}) & = & 1 \\
\mu(\{HH, TH\}) & = & 1 & \mu(\{HT, TH, TT\}) & = & 2/3 \\
\mu(\{HH, TT\}) & = & 1/3 & \mu(\{HH, HT, TH, TT\}) & = & 1
\end{array}
$$

Note that the probability measure for disjoint events such as $\{HT\}$ and $\{TH\}$ do indeed add.

1.2 Quantum Probability Spaces

The mathematical framework above assumes that one has complete knowledge of the events and their relationships. But even in many classical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot be determined with certainty. In the quantum case, this partial knowledge is compounded by the fact that not all quantum events can be observed simultaneously. Indeed, in the quantum world, there are non-commuting events which cannot even happen simultaneously. To accommodate these more complex situations, we abandon the sample space Ω and define and reason directly about events. A quantum probability space consist of just two components: a set of events $\mathcal{E}$ and a probability measure $\mu : \mathcal{E} \to [0, 1]$. We give an example before giving the formal definition.

Consider the two-qubit Hilbert space with computational basis $|0\rangle$ and $|1\rangle$ and states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The set of events associated with this Hilbert space consists of all projections including the empty projection $\mathbb{0}$ and the unit projection $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$:

$$\{\mathbb{0}, |0\rangle\langle 0|, |1\rangle\langle 1|, |+\rangle\langle +|, |-\rangle\langle -|, \ldots, \mathbb{1}\}$$

Each event is interpreted as a possible post post-measurement state of a quantum system as follows: given some arbitrary current quantum state $|\psi\rangle$ to be measured, the event $|0\rangle\langle 0|$ states that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ states that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ states that the post-measurement state will be $|+\rangle$; the event $|-\rangle\langle -|$ states that the post-measurement state will be $|-\rangle$; the event $\mathbb{1}$ states that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the event $\mathbb{0}$ states that the post-measurement state will be the empty state.

Irrespective of the current state $|\psi\rangle$ and irrespective of the particular experiment, the probability of event $\mathbb{0}$ will always be 0 (it is an impossible event) and the probability of event $\mathbb{1}$ will always be 1 (it is a certain event). The probabilities attached to other events will depend on the particular state in question. If the state is $|0\rangle$, the probability of event $|0\rangle\langle 0|$ is 1; the probability of event $|1\rangle\langle 1|$ is 0; the probability of event $|+\rangle\langle +|$ is $\frac{1}{2}$; and the probability of event $|-\rangle\langle -|$ is $\frac{1}{2}$. If the state is $|+\rangle$, the probability of each event $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ will be $\frac{1}{2}$; the probability of event $|+\rangle\langle +|$ is 1; and the probability of event $|-\rangle\langle -|$ is 0.

We now formalize a *quantum probability space* as follows [3, 6, 15, 1, 11]. We first assume an ambient Hilbert space $\mathcal{H}$ and define the set of events $\mathcal{E}$ as all *projections* on $\mathcal{H}$. Each quantum state $|\psi\rangle \in \mathcal{H} \backslash \{0\}$ induces a probability measure $\mu_\psi : \mathcal{E} \to [0,1]$ on the space of events defined for any event $E \in \mathcal{E}$ as follows[1]:

$$\mu_\psi(E) = \frac{\langle \psi | E\psi \rangle}{\langle \psi | \psi \rangle}$$

Similarly to the classical case, this probability measure must satisfy:

- $\mu(\mathbb{1}) = 1$, and
- for a collection of pairwise orthogonal E_i, we have $\mu(\sum_i E_i) = \sum_i \mu(E_i)$.

Yu-Tsung says: Claim: If $\sum_i |\psi_i\rangle\langle\psi_i|$ is a projection, then they are orthogonal.
Proof: If $\sum_i |\psi_i\rangle\langle\psi_i|$ is a projection, then

$$\begin{aligned} |\psi_j\rangle &= \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\psi_j\rangle \\ &= \sum_i |\psi_i\rangle\langle\psi_i|\psi_j\rangle \end{aligned}$$

Therefore, we have

$$0 = \sum_{i\neq j} |\psi_i\rangle\langle\psi_i|\psi_j\rangle$$

1.3 Plan

Several assumptions are woven in the definition of a quantum probability space:

[1] Recently, people extend the domain of μ_ψ to all operators $\mathcal{A}$ on $\mathcal{H}$ and consider $\mu_\psi : \mathcal{A} \to \mathbb{C}$ [11, 17]. When an operator $A \in \mathcal{A}$ is Hermitian, $\mu_\psi(A)$ is the expectation value of A. We does not take this approach because we want to focus only on probability.

- the Hilbert space $\mathcal{H}$;
- the real interval $[0,1]$;
- the fact that each state induces a probability measure, i.e., the Born rule [4, 12];
- the fact that every probability measure is induced by a state, i.e., Gleason's theorem [6, 13, 15].

In the remainder of the paper, we examine each of these assumptions and consider variations motivated by computation of numerical quantities in a world with limited resources. In particular, we will consider a variant of the Hilbert space over finite fields $\mathbb{F}_{p^2}^d$ [10, 9, 8].

For the probability values, we will consider set-valued probability measures [2, 14], in particular $\mathscr{L}_2 = \{\text{impossible}, \text{possible}\}$, where impossible is a singleton set,$\{0\}$, and possible is an open interval, $(0,\infty)$. Surprisingly, some combinations of space and probability will result in an unique probability measure. In these cases, there is no need to discuss whether there is a Born rule, because we do not have enough probability to correspond to every state.

If there may be more than one probability measure, we will discuss whether there is a Born rule to generate a probability measure from a state. When the space is $\mathbb{C}^d$, we will try to induced a Born rule from the conventional Born rule. When the space is $\mathbb{F}_{p^2}^d$, there is no natural way to induce a probability measure from a state, so we will set some conditions for a Born rule $\tilde{\pi}$:

- Given a pure state $|\Psi\rangle \in \mathbb{F}_{p^2}^{d}{}^*$, a Born-rule $\tilde{\pi}$ should give a probability $\tilde{\pi}_\Psi$;
- $\langle\Psi|\Phi\rangle = 0 \Leftrightarrow \tilde{\pi}_\Psi(|\Phi\rangle) = \tilde{0}$, where $\tilde{0}$ is 0 for $[0,1]$ and $\tilde{0}$ is impossible for $\mathscr{L}_2 = \{\text{impossible}, \text{possible}\}$.
- $\tilde{\pi}_\Psi(|\Phi\rangle) = \tilde{\pi}_{\mathbf{U}|\Psi\rangle}(\mathbf{U}|\Phi\rangle)$,where $|\Psi\rangle, |\Phi\rangle \in \mathbb{F}_{p^2}^{d}{}^*$ and $\mathbf{U}$ is any unitary map, i.e., $\mathbf{U}^\dagger\mathbf{U} = \mathbb{1}$.

Notice that when the space is $\mathbb{C}^d$, every Born rule we consider will satisfy these three conditions.

Finally, if there is a Born rule, we will see whether every probability measure is induced by a state, and establish Gleason's theorem. Notice that Gleason's theorem only hold when $d \geq 3$ in CQT so that we may expect the situation for $d \geq 3$ and $d = 2$ are different.

The results can be summarized in the following table:

State space $\mathcal{H}$	Probability values	How many probability measure?	Is there a nature Born rule?	How many possible Born rule if there is no nature one?	How many probability measure not come from any possible Born rules?
$\mathbb{C}^2$	$[0,1]$	≥ 2	Yes		≥ 1
$\mathbb{C}^d$ for $d \geq 3$	$[0,1]$	≥ 2	Yes		0
$\mathbb{C}^d$	$\mathscr{L}_2$	≥ 2	Yes		≥ 1
$\mathbb{F}_{3^2}^d$ for $3 \geq d \geq 2$	$[0,1]$	≥ 2	No	1	≥ 1
$\mathbb{F}_{p^2}^2$ for $p \geq 7$	$[0,1]$	≥ 2	No	∞	≥ 1
$\mathbb{F}_{7^2}^3$	$[0,1]$	1	No	0	
$\mathbb{F}_{p^2}^{d}{}^*$for $d \geq 3$ except $d = p = 3$	$[0,1]$	≥ 1 (Whether ≥ 2 or 1?)	No	0	
$\mathbb{F}_{p^2}^{d}{}^*$	$\mathscr{L}_2$	≥ 2	No	1	≥ 1

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