

# The Bell–Kochen–Specker theorem

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## Abstract

Meyer, Kent and Clifton (MKC) claim to have nullified the Bell–Kochen–Specker (Bell–KS) theorem. It is true that they invalidate Kochen and Specker’s account of the theorem’s physical implications. However, they do not invalidate Bell’s point, that quantum mechanics is inconsistent with the classical assumption, that a measurement tells us about a property previously possessed by the system. This failure of classical ideas about measurement is, perhaps, the single most important implication of quantum mechanics. In a conventional colouring there are some remaining patches of white. MKC fill in these patches, but only at the price of introducing patches where the colouring becomes “pathologically” discontinuous. The discontinuities mean that the colours in these patches are empirically unknowable. We prove a general theorem which shows that their extent is at least as great as the patches of white in a conventional approach. The theorem applies, not only to the MKC colourings, but also to any other such attempt to circumvent the Bell–KS theorem (Pitowsky’s colourings, for example). We go on to discuss the implications. MKC do not nullify the Bell–KS theorem. They do, however, show that we did not, hitherto, properly understand the theorem. For that reason their results (and Pitowsky’s earlier results) are of major importance.

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## 1. Introduction

In ordinary language the word “measurement”

very strongly suggests the ascertaining of some pre-existing property of some thing, any instrument involved playing a purely passive role (Bell, 1987, p. 166).

The Bell–Kochen–Specker (Bell-KS) theorem (Bell, 1966; Kochen & Specker, 1967) shows that quantum mechanics is inconsistent with that natural idea.

Or so Bell thought. His conclusion has, however, been challenged: first by Pitowsky (1983, 1985), and then, using a different set-theoretic argument, by Meyer (1999), Kent (1999) and Clifton and Kent (2000), Meyer, Kent, and Clifton (MKC in the sequel). MKC’s argument was inspired by the previous work of Hales and Straus (1982) and Godsil and Zaks (1988). It has attracted much comment (see Mermin, 1999; Havlicek, Krenn, Summhammer, & Svozil, 2001; Simon, Brukner, & Zeilinger, 2001; Larsson, 2002; Appleby, 2001, 2002; Cabello, 2002; Breuer, 2002a,b).

MKC claim to have “nullified” the Bell-KS theorem. They mean by this that the theorem, though mathematically valid, is not physically significant. Pitowsky expresses himself less forcefully. He does not say, in so many words, that the Bell-KS theorem is entirely without significance. However, he clearly means to insinuate doubts.

Now there can be no question as to the importance of the results proved by Pitowsky and MKC (PMKC in the sequel). They clearly have some major consequences. However, we will argue that these consequences are less catastrophic than MKC think. They do not show that the Bell-KS theorem is without significance. They only show that we need to reassess its significance.

The question is complicated by the fact that the Bell-KS theorem was proved twice over: first by Bell (1966), and then again by Kochen and Specker (1967). They expressed strikingly different views as to the theorem’s physical significance. So the first thing one needs to ask is: what, exactly, is it that the MKC models are supposed to nullify?

In Section 8 we will argue that PMKC have indeed invalidated Kochen and Specker’s account of the theorem’s significance. We will also argue that Bell’s account contains a number of serious misconceptions. So it is true that PMKC nullify *some* of what was previously seen as the theorem’s significance. But they do not nullify it all. In particular, they do not nullify Bell’s main point, as stated above. It remains the case that quantum mechanics is inconsistent with classical ideas about measurement.

The fact that PMKC cannot have fully restored the ordinary, or classical concept of measurement becomes obvious, as soon as one reflects that their models are still hidden variables theories. A hidden variables theory is one in which the pre-existing values are, for some reason, concealed. But if the observables could be measured in the ordinary, classical sense, then the values would not be concealed. They would be open-to-view, as in classical physics. It follows that, in a hidden variables theory, there must necessarily be *some* breakdown of classical assumptions about measurement.

In Bohmian mechanics the mechanism responsible for concealing the values is the same as the mechanism which makes the theory contextual. Instead of the instrument playing a purely passive role, as it would classically, there is a complex interplay between system and instrument. The effect is to *create* a value, which did not exist before. As Bell puts it:

The result of a ‘spin measurement’, for example, depends in a very complicated way on the initial position  $\lambda$  of the particle and on the strength and geometry of the magnetic field. Thus, the result of the measurement does not actually tell us about some property previously possessed by the system, but about something which has come into being in the combination of system and apparatus. (Bell, 1987, p. 35)

(for a more detailed discussion of this point see Dewdney, Holland, & Kyprianidis, 1986, and Holland, 1993).

PMKC have discovered a completely different mechanism for concealing the values. This is an important discovery. It means, in particular, that Bell’s emphasis on the active role of the apparatus needs revision. Nevertheless, their models still are hidden variables theories. So Bell’s main point, that a measurement outcome “does not actually tell us about some property previously possessed by the system,” still stands.

The variables are hidden in the PMKC models because their colourings are violently, and even “pathologically” discontinuous (the relevance of continuity in this regard is noted by Mermin, 1999). Consider, for instance, the colouring described by Meyer (1999). This function is discontinuous at every point in its domain of definition. It is, in other words, as discontinuous as a function can possibly get.

In a conventional colouring, such as a political map of the Earth, the paints are applied in broad strokes to well-behaved regions having regular boundaries. This means that it is almost always possible, using finite precision measurements, to find out what country one is in. It is true that infinite precision would be needed if one was situated exactly on a boundary. However, there is zero probability of hitting such a point.

Meyer’s colouring is not like that. In the Meyer colouring it is as if, before applying the paints, one first mixes them, to the maximum extent possible, so that the colours become intermingled at the molecular level. It is (so to speak) a maximum entropy colouring. This makes the colours unobservable.

At least, the colours are not observable using finite precision measurements. MKC say that finite precision nullifies the KS theorem. But it would be nearer the truth if one said that finite precision *saves* the KS theorem.

Suppose one tries to find out the value of some particular vector  $\mathbf{n} \in S^2_{\mathbb{Q}}$  (where  $S^2_{\mathbb{Q}}$  is the rational unit 2-sphere, on which the Meyer colouring is defined) using a finite precision measurement. The finite precision means that the measurement actually reveals the value of some unknown vector  $\mathbf{n}'$ , with  $|\mathbf{n}' - \mathbf{n}| \leq \varepsilon$  for some positive  $\varepsilon$ . If the colouring were continuous at  $\mathbf{n}$ , and if  $\varepsilon$  were sufficiently small, then the value of  $\mathbf{n}'$  would be the same as the value of  $\mathbf{n}$ . However, the colouring is, in fact,

discontinuous. This means that, no matter how small the error  $\varepsilon$ , the measurement provides no more information about the value of  $\mathbf{n}$  than could be obtained by simply guessing a number at random (assuming that  $\varepsilon$  is not actually 0). The value is therefore unobservable, or hidden.

A procedure which leaves the experimenter in complete ignorance of the pre-existing values is clearly not a measurement in the classical sense. MKC focus on the point that, in their models, a measurement does always reveal the pre-existing value of *something*: namely, the value of the vector  $\mathbf{n}'$  representing the true alignment of the instrument. What they overlook is that the experimenter does not know the true alignment of the instrument. This means that, although the experimenter learns a value, s/he has no idea what it is a value *of*. Consequently, the experimenter does not acquire any actual knowledge.

A classical measurement is not *simply* a procedure which reveals a pre-existing value. Rather, it is a procedure which ascertains a pre-existing fact, of the form “observable  $A$  had value  $x$ ”. The specification of the observable  $A$  is no less essential than the specification of the value  $x$ . The problem with the PMKC models is that the observable  $A$  is not specified, so the experimenter only learns “... had value  $x$ ”. This statement is completely uninformative. It says no more than the statement “observable  $A$  had ...”. Indeed, it says no more than the completely empty statement “... had ...”.

What emerges from this is that PMKC have been asking the wrong question. The important question is not: “How much of  $S^2$  (the full unit 2-sphere) can be coloured *at all*?” But rather: “How much of  $S^2$  can be coloured in such a way that the colours are *empirically knowable*?”

In Sections 2–7 we address that question. We have seen that, in the case of Meyer’s model, *none* of the colours are empirically knowable. We need to consider whether another model might improve on that.

So as to have a standard of comparison we begin, in Section 2, by defining the concept of a regular KS-colouring. Intuitively, this is a colouring where the paint is applied in broad strokes, as in a political map of the Earth. We show that a regular KS-colouring must exclude a region having non-empty interior and subtending solid angle  $\geq 4\pi d_{\mathcal{R}}$ . Here  $d_{\mathcal{R}}$  is a fixed positive number whose value is determined, once and for all, by the principles of quantum mechanics.

We refer to functions of the same general kind as the PMKC colourings as pseudo-KS-colourings of  $S^2$ . In Section 3 we identify two conditions satisfied by every PMKC colouring (both the ones constructed by Pitowsky and the ones constructed by MKC). We argue that they would also have to be satisfied by any other pseudo-KS-colouring.

In Sections 4–5 we use these conditions to analyze the discontinuities of an arbitrary pseudo-KS-colouring. We show that  $S^2$  splits into an open set  $U$ , which is *regularly* KS-colourable, and a closed set  $D$  on which the discontinuities make the colours empirically unknowable. In the case of Meyer’s colouring  $U$  is empty and  $D = S^2$ . In the general case  $D$  might be smaller. However, it cannot be shrunk to nothing. The fact that  $U$  is regularly KS-colourable means that  $D$  must always have non-empty interior, and subtend solid angle  $\geq 4\pi d_{\mathcal{R}}$ .

In Section 6 we infer that the Bell-KS theorem is not nullified. A conventional colouring must exclude a region  $D$  which is simply not coloured at all. PMKC have found ways to extend the colouring into  $D$ . However, they only do so at the price of making the valuation so extremely discontinuous that the colours are empirically unknowable. From the point of view of a finite precision experimenter, who wants to ascertain the pre-existing values, this is not an improvement.

The fact that the colours cannot all be empirically knowable is also shown by Cabello (2002). However, the relation between Cabello's argument and ours may not be immediately apparent. In Section 7 we elucidate the relationship.

Finally, in Section 8 we assess the implications of PMKC's discoveries, and our counter-argument. PMKC clearly nullify *some* of what used to be seen as the Bell-KS theorem's significance. We will argue that they completely invalidate what KS say on the subject. They also invalidate some of the things said by Bell. In addition we present some further criticisms of Bell, which are only indirectly inspired by PMKC's argument. In short, PMKC make us recognize that we did not, in the past, fully understand what the theorem is telling us.

However, none of this detracts from the theorem's importance. The failure of classical assumptions about measurement is arguably the single most revolutionary feature of quantum mechanics.

## 2. Regular KS-colourings

We begin by showing that there is certainly no question of nullifying the Bell-KS theorem by means of the kind of well-behaved colouring one sees in a political map of the Earth. The results proved in this section will also play an important role in our subsequent analysis of the PMKC models.

The Bell-KS theorem states that there is no valuation  $f: S^2 \rightarrow \{0, 1\}$  (where  $S^2$  is the unit 2-sphere) such that  $f(-\mathbf{n}) = f(\mathbf{n})$  for all  $\mathbf{n}$  and

$$f(\mathbf{n}_1) + f(\mathbf{n}_2) + f(\mathbf{n}_3) = 2, \quad (1)$$

for every triad (every triplet of orthogonal unit vectors)  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ .

The problem exposed by PMKC is that this statement, as it stands, is very weak. It asserts that one cannot KS-colour *absolutely* all of  $S^2$ . But it does not place any stronger constraint on the maximum size of a KS-colourable set. It consequently leaves PMKC free to argue that one can KS-colour *effectively* all of  $S^2$ , in some suitably defined sense of the word "effectively".

In this section we make a first step in the direction of strengthening the theorem, by establishing constraints on the sizes of some special kinds of KS-colourable set. We will have occasion to consider

- (1) The class  $\mathcal{B}$  consisting of all KS-colourable Borel sets (i.e. all KS-colourable sets which are measurable with respect to the usual rotationally invariant measure on  $S^2$ ).
- (2) The class  $\mathcal{C}$  consisting of all closed KS-colourable sets.
- (3) The class  $\mathcal{O}$  consisting of all open KS-colourable sets.

However, we are mainly interested in the class  $\mathcal{R} \subseteq \mathcal{O}$  consisting of all *regularly* KS-colourable sets. Intuitively, a regular KS-colouring is one which exhibits the same kind of “good” behaviour one sees in a political map of the Earth. Formally, a KS-colouring  $f: U \rightarrow \{0, 1\}$  is regular if  $U$  is open and  $f$  is almost everywhere continuous on  $U$ .

Before proceeding further we ought to remove a potential ambiguity. We consider a function  $f: K \rightarrow \{0, 1\}$  defined on a subset  $K \subset S^2$  to be a KS-colouring if and only if

- (1)  $f(-\mathbf{n}) = f(\mathbf{n})$  whenever  $-\mathbf{n}$  and  $\mathbf{n}$  both  $\in K$ .
- (2)  $f(\mathbf{n}_1) + f(\mathbf{n}_2) + f(\mathbf{n}_3) = 2$  for every triad  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \in K$ .
- (3)  $f(\mathbf{n}_1) + f(\mathbf{n}_2) \geq 1$  for every orthogonal pair  $\mathbf{n}_1, \mathbf{n}_2 \in K$ .

We do *not* require that linear combinations of pairs of orthogonal vectors both  $f$ -evaluating to 1 should also  $f$ -evaluate to 1 (except, of course, when that is implied by condition 2). The reason for not imposing this additional requirement will appear in Section 3.

We begin by proving.

**Lemma 1.** *If  $K \in \mathcal{C}, \mathcal{O}$  or  $\mathcal{R}$  then the complement  $S^2 - K$  has non-empty interior.*

**Remark.** In other words  $K$  must exclude at least one non-empty disk of the form  $\{\mathbf{n} \in S^2: \cos^{-1}(\mathbf{n} \cdot \mathbf{n}_0) < r\}$  with centre  $\mathbf{n}_0$  and radius  $r > 0$ .

**Proof.** If  $K \in \mathcal{C}$  the statement is immediate, since  $S^2 - K$  is then open (being the complement of a closed set) and non-empty (in view of the Bell-KS theorem).

If, on the other hand,  $K \in \mathcal{O}$  or  $\mathcal{R}$  we must work a little harder. Choose some finite, KS-uncolourable set  $\{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_L\}$ . The fact that  $K$  is KS-colourable means that at least one of these vectors must belong to  $S^2 - K$ . We may assume the labelling is such that, for some  $l \geq 1$ ,  $\mathbf{n}_i \in S^2 - K$  if  $i \leq l$  and  $\mathbf{n}_i \in K$  if  $i > l$ .

We now adopt the method of proof by contradiction. Suppose that the interior of  $S^2 - K$  was empty. Then we could find a rotation  $R$  such that the rotated vectors  $\{R\mathbf{n}_1, R\mathbf{n}_2, \dots, R\mathbf{n}_L\}$  all  $\in K$ .

To see this note, first of all, that the fact that  $K$  is open means that we can choose  $\delta > 0$  such that  $\{\mathbf{m} \in S^2: \cos^{-1}(\mathbf{m} \cdot \mathbf{n}_i) < \delta\} \subseteq K$  for  $i > l$  (if it should happen that  $l = L$  then  $\delta$  can be chosen arbitrarily,  $= \pi/2$  say).

By assumption,  $S^2 - K$  has empty interior. This means each vector  $\in S^2 - K$  is a limit point of  $K$ . So there exists a rotation  $R_1$ , through an angle  $\varepsilon < \delta$ , such that  $R_1\mathbf{n}_l \in K$ . The fact that  $\varepsilon < \delta$  means that  $R_1$  does not move any of the vectors  $\mathbf{n}_{l+1}, \dots, \mathbf{n}_L$  out of  $K$ . Consequently the rotated set  $\{R_1\mathbf{n}_1, R_1\mathbf{n}_2, \dots, R_1\mathbf{n}_L\}$  contains  $l_1 < l$  vectors  $\in S^2 - K$ . If  $l_1 \neq 0$  we may repeat the argument, to obtain a new rotated set  $\{R_2R_1\mathbf{n}_1, R_2R_1\mathbf{n}_2, \dots, R_2R_1\mathbf{n}_L\}$  containing  $l_2 < l_1 < l$  vectors  $\in S^2 - K$ . After a sufficient number  $p$  of iterations we reach the point where  $l_p = 0$ . Define  $R = R_p \dots R_2R_1$ . Then  $\{R\mathbf{n}_1, R\mathbf{n}_2, \dots, R\mathbf{n}_L\} \subseteq K$ , as claimed.

This gives us a contradiction: for the set  $\{R\mathbf{n}_1, R\mathbf{n}_2, \dots, R\mathbf{n}_L\}$  is KS-uncolourable, and so cannot be contained in  $K$ . Consequently, the interior of  $S^2 - K$  cannot be empty.  $\square$

Our second result concerns the maximum area of a KS-colourable set. Define

$$d_{\mathcal{B}} = 1 - \sup_{B \in \mathcal{B}} (\mu(B)), \quad (2)$$

$$d_{\mathcal{C}} = 1 - \sup_{C \in \mathcal{C}} (\mu(C)), \quad (3)$$

$$d_{\mathcal{O}} = 1 - \sup_{U \in \mathcal{O}} (\mu(U)), \quad (4)$$

$$d_{\mathcal{R}} = 1 - \sup_{U \in \mathcal{R}} (\mu(U)), \quad (5)$$

where  $\mathcal{B}, \mathcal{C}, \mathcal{O}$  and  $\mathcal{R}$  are the classes of KS-colourable sets defined above, and where  $\mu$  is the usual rotationally invariant measure on  $S^2$ , normalized so that  $\mu(S^2) = 1$  (in other words, the solid angle scaled by  $1/4\pi$ ). We will refer to these numbers as deficits. They tell us the size of the region excluded by a colouring of maximal extent.

In Appendix A we prove.

**Lemma 2.**  $d_{\mathcal{R}}$  is strictly  $> 0$ .

We know that  $d_{\mathcal{R}} \geq d_{\mathcal{O}} \geq d_{\mathcal{B}}$  and  $d_{\mathcal{C}} \geq d_{\mathcal{B}}$  (because  $\mathcal{R} \subseteq \mathcal{O} \subseteq \mathcal{B}$  and  $\mathcal{C} \subseteq \mathcal{B}$ ). It can also be shown (see, for example, Halmos, 1961, Chapter 10) that each Borel set  $B$  contains a sequence of closed subsets  $C_n$  such that  $\mu(B) = \lim_{n \rightarrow \infty} (\mu(C_n))$ . Consequently  $d_{\mathcal{C}} \leq d_{\mathcal{B}}$ . Putting these facts together we deduce

$$d_{\mathcal{R}} \geq d_{\mathcal{O}} \geq d_{\mathcal{C}} = d_{\mathcal{B}} > 0. \quad (6)$$

It would be interesting to know whether the two inequalities at the left-hand end of the chain are actually strict. That is a question which requires further investigation.

It would also be interesting to know the values of the deficits. It is easy<sup>1</sup> to construct a regular KS-colouring which covers 87% of  $S^2$ . So we know that  $d_{\mathcal{R}}$  (and therefore  $d_{\mathcal{O}}$  and  $d_{\mathcal{B}}$ ) must be  $\leq 0.13$ . In Appendix A we show that  $d_{\mathcal{R}}$  is bounded from below by an integral defined in terms of a finite KS-uncolourable set. For the sets which have been described in the literature this integral is rather small. For instance, in the case of the Conway–Kochen set (Peres, 1993; Bub, 1997) the integral is  $\lesssim 0.01$  (see Appendix A). So the possibility is not excluded that  $\sim 99\%$  of the total solid angle can be covered with a KS-colourable Borel set (in terms of the Earth this would correspond to colouring everything except a region about the size of Australia).

One should, however, note that the integral is sensitive to the angular separation of the rays in the corresponding KS-uncolourable set. It follows that, if one could find a set containing many fewer rays than the Conway–Kochen set, this would be

<sup>1</sup>Simply assign 0 to the two polar caps defined by  $|\tan \theta| < 1$  and 1 to the equatorial region defined by  $|\tan \theta| > \sqrt{2}$  (where  $\theta$  is the usual polar angle). It is easily verified that this gives a regular KS-colouring covering a region having  $\mu$ -measure  $= 1 - 1/\sqrt{2} + 1/\sqrt{3}$ .

likely to give a substantially larger lower bound. In any case, the integral is only a bound on  $d_{\mathcal{B}}$ , not the actual value. Lastly, it is possible that the deficits are larger in higher dimensions.

The results we prove are enough to establish that the Bell-KS theorem is not nullified. However, it remains an open question, quite how close the theorem gets to being nullified. This too is a point that requires further investigation.

### 3. Pseudo-KS-colourings

We refer to functions like the ones constructed by PMKC as pseudo-KS-colourings of  $S^2$ . In this section we identify two conditions which the PMKC colourings all satisfy (both the ones constructed by Pitowsky and the ones constructed by MKC). We argue that they would also have to be satisfied by any other pseudo-KS-colouring.

We aim to give a completely general analysis, applying to any pseudo-KS-colouring. But let us begin by looking at the particular colourings constructed by PMKC.

The clearest and most succinct description of the Pitowsky colourings is in Pitowsky (1985). Pitowsky (1983) contains important additional material concerning the measure-theoretic aspects. The interested reader should also consult Pitowsky's (1982a,b) discussion of the Bell inequalities, and comments by Mermin (1982) and Macdonald (1982).

Pitowsky's approach is to define a function  $f: S^2 \rightarrow \{0, 1\}$  on the whole of  $S^2$ . This means he has to relax the requirement, that  $f$  should sum to 2 on every triad. Instead, he imposes the weaker requirement, that  $f$  should sum to 2 on *almost* every triad (in a sense of the word “almost” which is not the standard measure-theoretic sense—see below).

Specifically, Pitowsky shows that there exist functions  $f: S^2 \rightarrow \{0, 1\}$  such that

- (1)  $f(-\mathbf{n}) = f(\mathbf{n})$  for all  $\mathbf{n}$ .
- (2) For all  $\mathbf{n}$ , there are at most countably many orthogonal pairs  $\mathbf{m}, \mathbf{l} \in \mathbf{n}^\perp$  for which

$$f(\mathbf{n}) + f(\mathbf{m}) + f(\mathbf{l}) \neq 2, \quad (7)$$

(where  $\mathbf{n}^\perp$  is the orthogonal complement of  $\mathbf{n}$ ).

Let  $T$  be the space of all triads, and let  $N \subset T$  be the set of “wrongly” coloured triads. A Pitowsky colouring has the property that  $p(N|\mathbf{n}) = 0$  for all  $\mathbf{n} \in S^2$  (where  $p(N|\mathbf{n})$  is the conditional probability of selecting a “wrongly” coloured triad, given that  $\mathbf{n}$  is one of its elements). Pitowsky infers that  $p(N) = 0$ , so that almost all triads are “correctly” coloured.

It should be stressed that there is a problem with this argument. Although  $p(N|\mathbf{n})$  is well-defined (in terms of the usual invariant measure on the circle), there is a serious difficulty with the definition of  $p(N)$ . This is because  $N$  is not a Borel set (as follows from a variant of Lemma 2 in the last section, applying to  $T$  instead of  $S^2$ ). Pitowsky consequently has to rely on “a strange concept



of probability which violates the axiom of additivity” (Pitowsky, 1982b). For that reason his results have attracted less attention than the subsequent work of MKC.<sup>2</sup>

MKC’s achievement was to find a way of obviating this difficulty. Their argument is formulated exclusively in terms of the standard theory of probability, as formalized by the Kolmogorov axioms. It therefore compels us to take Pitowsky’s suggestion, that the Bell-KS theorem may not have the implications usually imputed to it, much more seriously.

Pitowsky assigns values to the whole of  $S^2$ . Meyer’s (1999) key insight is that we do not need to colour the whole of  $S^2$  in order to account for the observations. We might, for instance, be living in a world where the only physically possible alignments are those specified by vectors  $\mathbf{n} \in S^2_{\mathbb{Q}}$  (i.e. unit vectors  $\mathbf{n}$  whose components are all rational).

An experimenter may, indeed, set out with the *intention* of measuring in a direction  $\mathbf{n} \notin S^2_{\mathbb{Q}}$ . However, the finite precision of real laboratory measurements means that we can never exclude the possibility that s/he actually measures along a slightly different direction  $\mathbf{n}'$  which does  $\in S^2_{\mathbb{Q}}$ . The set  $S^2_{\mathbb{Q}}$  is KS-colourable. Furthermore, the set of rational triads is dense in the space of all real triads. Meyer concludes that, since in this model a measurement does always reveal the pre-existing value of the vector which is *actually* measured, the Bell-KS theorem is nullified.

Kent (1999) subsequently extended Meyer’s results to higher dimensional spaces. Clifton and Kent (2000) then showed that, in the case of finite dimensional systems, models of the same general type can reproduce all the statistical predictions of quantum mechanics, in so far as these are verifiable by finite precision measurements (though it should be noted that the Clifton–Kent models are still incomplete in that they do not specify the dynamical behaviour of the system).

We want to give an argument which applies quite generally, not only to the particular constructions of Pitowsky and MKC, but also to any other model which might conceivably be thought to nullify the Bell-KS theorem. We therefore need to identify some minimal conditions which a function  $f: K \rightarrow \{0, 1\}$  must satisfy, if it is to count as a pseudo-KS-colouring of  $S^2$ .

Our first such condition is the following:

**Condition 1.** The domain  $K$  is a dense subset of  $S^2$ .

We consider this condition to be necessary because, if it is not satisfied, there is a non-empty disk which is not coloured at all. It should be noted that we do not

<sup>2</sup>Nevertheless, they are still worth considering: not only for reasons of completeness, but also because of their own intrinsic interest. In particular, Pitowsky’s attempt to extract deep physical meaning from the *arcana* of axiomatic set theory is, to our mind, intriguing.

Pitowsky’s non-standard concept of probability is certainly a source of serious difficulty. On the other hand, there is some force to his contention that, since  $p(N|\mathbf{n})$  is well defined, it ought to be possible to make sense of  $p(N)$  also. Perhaps it is true that one cannot make sense of  $p(N)$  while remaining within the framework of the Kolmogorov axioms. But that could be countered by asking whether there is any cogent physical reason which compels us to accept the Kolmogorov axioms (see Pitowsky, 1982a,b).

require that  $K$  is countable (as in the MKC colourings). In particular, the possibility is not excluded that  $K = S^2$  (as in the Pitowsky colourings).

We do not require  $f$  to be a KS-colouring of  $K$ . Nor do we require it to be Borel-measurable. We do, however, require.

**Condition 2.** Suppose that  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  is a triad of vectors  $\in S^2$  (they need not  $\in K$ ), and suppose that, for each  $r$ ,  $U_r$  is an open neighbourhood of  $\mathbf{n}_r$  with the property  $K \cap U_r \subseteq f^{-1}(\{a_r\})$  for  $a_r = 0$  or  $1$ . Then

$$a_1 + a_2 + a_3 = 2.$$

Similarly, if  $\mathbf{n}_1, \mathbf{n}_2$  is an orthogonal pair of vectors  $\in S^2$  (not necessarily  $\in K$ ), and if, for each  $r$ ,  $\mathbf{n}_r$  has an open neighbourhood  $U_r$  such that  $K \cap U_r \subseteq f^{-1}(\{a_r\})$  for  $a_r = 0$  or  $1$ , then

$$a_1 + a_2 \geq 1.$$

The reader may verify that the colourings described by PMKC satisfy this condition. We consider it to be necessary because, if it is not satisfied, there exist finite precision measurements whose outcomes are guaranteed to conflict with the predictions of quantum mechanics.<sup>3</sup>

Let us stress that these are minimal conditions, which every pseudo-KS-colouring must satisfy. They are not intended to fully characterize the concept.

#### 4. The phenomenological colouring

The mathematical properties of a pseudo-KS-colouring are best visualized using a three-coloured variant of the usual chromatic metaphor. We will call this the phenomenological colouring. Intuitively, it describes what is seen when the sphere is viewed through finite resolution eyes.

Consider some fixed pseudo-KS-colouring  $f: K \rightarrow \{0, 1\}$ . Let  $K_0 = f^{-1}(\{0\})$  be the set of points  $f$ -evaluating to  $0$ , and let  $K_1 = f^{-1}(\{1\})$  be the set of points  $f$ -evaluating to  $1$ .

We begin by defining a true, or intrinsic colouring. We take points  $\in K_0$  to be intrinsically blue, and points  $\in K_1$  to be intrinsically red. Points  $\notin K$  (if any) we take to be intrinsically white.

Suppose, now, that one has an open region which entirely consists of points intrinsically blue or white. We take it that such a region, seen through finite resolution eyes, would appear *phenomenologically blue* (all of it, including the points which are intrinsically white). Similarly, we take it that an open region which entirely consists of points intrinsically red or white would appear *phenomenologically red* (all of it, including the points which are intrinsically white).

<sup>3</sup>This is the reason we adopted the less restrictive definition of a KS-colourable set in Section 2. It is not obvious that the same would be true of the version of condition 2 corresponding to the more restrictive definition.

Suppose, on the other hand, that each neighbourhood of  $\mathbf{n}$  contains at least one intrinsically blue point, and at least one intrinsically red point. Then we take  $\mathbf{n}$  to be *phenomenologically black* (irrespective of whether it is in fact intrinsically blue, red or white).

One can think of the phenomenological colours as arising through the mixing of the true or intrinsic paints.<sup>4</sup> However, this mixing analogy, though helpful on an intuitive level, should not be taken too far. For instance, a single intrinsically blue point surrounded by a region which is otherwise intrinsically pure red counts, on our definition, as phenomenologically black (not red as, one might argue, it would actually appear). Our aim in this is not really to describe the actual visual appearance of a surface which has been stippled with differently coloured microscopic dots. We simply want to present a convenient way to picture the abstract mathematical properties of a pseudo-KS-colouring.

## 5. The discontinuity region

We now use the phenomenological colouring to investigate the discontinuities of  $f$ .

Let us begin by defining the phenomenological colouring in more formal terms. Let  $\overline{K}_0$  (respectively  $\overline{K}_1$ ) be the closure of  $K_0$  (respectively  $K_1$ ) considered as a subset of  $S^2$ . Then  $\overline{K}_0 \cup \overline{K}_1 = \overline{K} = S^2$ . Now define  $U_0 = S^2 - \overline{K}_1$ ,  $U_1 = S^2 - \overline{K}_0$  and  $D = \overline{K}_0 \cap \overline{K}_1$ . Then  $U_0$ ,  $U_1$ ,  $D$  partition  $S^2$  into three pairwise disjoint subsets. Furthermore,  $U_0$ ,  $U_1$  are open and  $D$  is closed.

The phenomenological colouring  $\tilde{f}: S^2 \rightarrow \{0, 1, -1\}$  may now be defined formally, by

$$\tilde{f}(\mathbf{n}) = \begin{cases} 0, & \text{if } \mathbf{n} \in U_0, \\ 1, & \text{if } \mathbf{n} \in U_1, \\ -1, & \text{if } \mathbf{n} \in D. \end{cases} \quad (8)$$

Thus,  $\mathbf{n}$  is phenomenologically blue, red or black depending on whether it  $\tilde{f}$ -evaluates to 0, 1, or  $-1$ , respectively.

This topological gambit enables us to escape all the difficulties arising from the fact that  $f$  is not assumed to be Borel-measurable.  $K_0$ ,  $K_1$  may or may not be Borel sets. However,  $U_0$ ,  $U_1$  (being open) and  $D$  (being closed) are guaranteed to be Borel sets. Consequently,  $\tilde{f}$  is guaranteed to be Borel-measurable.

Let  $U = U_0 \cup U_1$ . Then the true colouring  $f$  is continuous at every point of  $K \cap U$ , and discontinuous at every point of  $K \cap D$ . We will therefore refer to  $U$  as the continuity region, and to  $D$  as the discontinuity region.

The significance of these sets is that they describe the extent to which the true colours are empirically knowable.

<sup>4</sup>In practice a mixture of blue and red paint gives purple, not black. We, however, are considering paint that is ideally blue (so that it is a perfect absorber for wavelengths  $\geq 491$  nm) and paint that is ideally red (so that it is a perfect absorber for wavelengths  $\leq 647$  nm).

Suppose  $\mathbf{k} \in K \cap U$ . Then every vector  $\in K$  sufficiently close to  $\mathbf{k}$  has the same intrinsic colour as  $\mathbf{k}$ . Consequently, a sufficiently accurate finite precision measurement in the approximate direction of  $\mathbf{k}$  is guaranteed to reveal that colour.

Suppose, on the other hand, that  $\mathbf{k} \in K \cap D$ . Then each neighbourhood of  $\mathbf{k}$ , no matter how small, contains infinitely many other vectors  $\in K$  having the opposite intrinsic colour. This means that the true colour of  $\mathbf{k}$  cannot be reliably ascertained by finite precision measurement in the direction  $\mathbf{k}$ .

It is obvious that  $D$  cannot be empty ( $S^2$  is connected so it cannot be the disjoint union of two non-empty open sets  $U_0$  and  $U_1$ ). That would not be a serious problem if  $D$  was in some sense negligible (even a political map of the Earth is discontinuous on the boundary lines). However,  $D$  is, in fact, non-negligible, as the following theorem shows.

**Theorem 1.** *Let  $f: K \rightarrow \{0, 1\}$  be a pseudo-KS-colouring of  $S^2$ , with continuity region  $U$ , and discontinuity region  $D$ . Let  $\tilde{f}_0: U \rightarrow \{0, 1\}$  be the restriction of the phenomenological colouring to  $U$ . Then*

- (1)  $\tilde{f}_0$  is a regular KS-colouring of  $U$ .
- (2)  $D$  has non-empty interior.
- (3)  $\mu(D) \geq d_{\mathcal{R}}$ .

**Remark.**  $d_{\mathcal{R}}$  is the deficit defined in Eq. (5).

**Proof.** We only need prove the first of these statements. The other two statements will then be immediate consequences of results proved in Section 2.

We note, first of all, that  $\tilde{f}_0$  is continuous (because the sets  $\tilde{f}_0^{-1}(\{0\}) = U_0$  and  $\tilde{f}_0^{-1}(\{1\}) = U_1$  are both open).

Now let  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  be any triad of vectors  $\in U$ . Since  $\tilde{f}_0$  is continuous we can choose, for each  $r$ , an open neighbourhood  $V_r$  of  $\mathbf{n}_r$  such that  $V_r \subseteq U$  and  $\tilde{f}_0(\mathbf{m}) = \tilde{f}_0(\mathbf{n}_r)$  for all  $\mathbf{m} \in V_r$ . Consequently  $f(\mathbf{m}) = \tilde{f}_0(\mathbf{n}_r)$  for all  $\mathbf{m} \in K \cap V_r$ . It now follows from condition 2 in Section 3 that

$$\tilde{f}_0(\mathbf{n}_1) + \tilde{f}_0(\mathbf{n}_2) + \tilde{f}_0(\mathbf{n}_3) = 2. \quad (9)$$

In the same way we can show

$$\tilde{f}_0(\mathbf{n}_1) + \tilde{f}_0(\mathbf{n}_2) \geq 1, \quad (10)$$

for any pair of orthogonal vectors  $\mathbf{n}_1, \mathbf{n}_2 \in U$ .

This shows that  $\tilde{f}_0$  is a KS-colouring. The regularity is a consequence of the fact that  $\tilde{f}_0$  is continuous.  $\square$

Theorem 1 is the central result of this paper. It shows that, in so far as the aim is to maximize the area on which the true colours are *empirically knowable*, pseudo-KS-colourings do no better than regular ones.

In the remainder of this section we examine the internal structure of  $D$ . It should already be apparent that the discontinuities might fairly be described as “pathological”. However, in the general case the situation is more complicated than it is with Meyer’s colouring.

Let  $D_i$  be the interior of  $D$ . In  $D_i$  the valuation shows exactly the same kind of “pathologically” discontinuous behaviour as the Meyer colouring. Each  $\mathbf{n} \in D_i$  is surrounded by a disk which is pure phenomenological black. This means that the intrinsic blues and reds are completely intermixed, everywhere, at the molecular level so to speak. In the case of the Meyer colouring  $D_i = S^2$ . In the general case it may be smaller. But it must always happen that  $\mu(D_i) > 0$ .

If the boundary of  $D$  is sufficiently well-behaved—if, for example, it consists of a finite set of closed  $C^1$  curves—then  $\mu(D_i) = \mu(D)$ . However, in the general case it might happen that  $\mu(D_i) < \mu(D)$ . For instance,  $D$  might have an infinitely complex lace-like structure similar to the Mandelbrot set (see the colour plates in, for example, Devaney, 1989). We therefore need to examine the nature of the discontinuities outside  $D_i$ . For that we can appeal to the following lemma.

**Lemma 3.** *Let  $f: K \rightarrow \{0, 1\}$  be a pseudo-KS-colouring with discontinuity region  $D$ . Let  $S(\mathbf{n}, \varepsilon) = \{\mathbf{m} \in S^2: \cos^{-1}(\mathbf{m} \cdot \mathbf{n}) < \varepsilon\}$  be the disk with centre  $\mathbf{n}$  and angular radius  $\varepsilon$ . Then the limit*

$$\lim_{\varepsilon \rightarrow 0} \left( \frac{\mu(S(\mathbf{n}, \varepsilon) \cap D)}{\mu(S(\mathbf{n}, \varepsilon))} \right)$$

*exists and  $= 1$  for almost all  $\mathbf{n} \in D$ .*

**Proof.** See Appendix B.  $\square$

Let  $D_m$  be the subset of  $D$  on which the limit exists and  $= 1$ . It is easily seen that  $D_i \subseteq D_m$ . We can think of  $D_m$  as the measure-theoretic interior. In some ways it gives a better idea of the interior, intuitively conceived, than the set  $D_i$ , defined topologically.

If  $\mathbf{n} \in D_m - D_i$  the disk  $S(\mathbf{n}, \varepsilon)$  is not completely pure black for any positive  $\varepsilon$  (otherwise  $\mathbf{n}$  would  $\in D_i$ ). However, if  $\varepsilon$  is sufficiently small the disk is *nearly* pure black. Furthermore, the proportion of blackness comes arbitrarily close to 1 as  $\varepsilon \rightarrow 0$ . This means that the mixing of intrinsic blues and reds in the vicinity of  $\mathbf{n}$  is not entirely complete. There are some microscopic specks of phenomenological blue or red. However, the mixing is *nearly* complete—implying that the intrinsic valuation  $f$  exhibits *nearly* the same degree of “pathological” discontinuity here that it does on  $D_i$ .

The intrinsic colours of vectors  $\in K \cap (D_m - D_i)$  cannot be ascertained by finite precision measurement. Let us note that the same is *effectively* true of the vectors in the specks of phenomenological blue or red buried deep in the interstices of the region  $D_m - D_i$ . Strictly speaking these specks are contained in  $U$ , not  $D$ . However, as one pushes in closer and closer to a vector  $\mathbf{n} \in D_m - D_i$  the specks become smaller and smaller. Consequently, the precision needed to ascertain the true colours of the vectors in them grows without bound. There will, for instance, come a point when the precision needed is so large that it would take a disk-pack the size of the observable universe to store the bit-string specifying a single vector  $\mathbf{k}$  to that degree of exactitude. We may say that the true colours of such vectors are unobservable FAPP (unobservable “for all practical purposes”).

We may thus visualize  $D$  as consisting of three concentric layers:

- (1) An inner core  $D_i$  on which  $f$  has the same “pathologically” discontinuous character as Meyer’s colouring.
- (2) An intermediate mantle  $D_m - D_i$  on which  $f$  has nearly the same “pathologically” discontinuous character as Meyer’s colouring.
- (3) An outer crust  $D - D_m$  on which the discontinuities may be comparatively mild.

The crust  $D - D_m$  has  $\mu$ -measure zero. So we may conclude that  $f$  is “pathologically” discontinuous over almost the whole of  $D$ .

Let us make one final point. Even the mildest of discontinuities is enough to frustrate the finite precision experimenter. Consider two colourings (not KS-colourings)  $g_b$  and  $g_r$  each of which assigns blue to everything north of the equator, and red to everything south of it. However, the equator itself is coloured blue by  $g_b$  and red by  $g_r$ . Then  $g_b$  and  $g_r$  are empirically indistinguishable.

It is tempting to think of a line discontinuity as somehow “harmless”. This is true in the sense that the line has  $\mu$ -measure zero (implying that there is zero probability of landing on it). But it is still the case that the line’s true colour is unobservable.

## 6. The Bell-KS theorem is not nullified

In a regular KS-colouring there are some remaining patches of white, which are simply not coloured at all. A pseudo-KS-colouring replaces these patches of white with patches of black, on which the colours are defined, but empirically unknowable. From the point of view of a finite precision experimenter, who wants to ascertain the intrinsic colour of a specified point, this is not an improvement.

Conventional models, such as the Bohm theory, and unconventional models, such as the ones proposed by PMKC, make completely different statements about what is going on “behind the scenes”. In a conventional model the pre-existing values are concealed because the apparatus actively manufactures new values. In a PMKC model, by contrast, the values are concealed because the valuation is “pathologically” discontinuous.

The difference is important. The PMKC models have major implications for the way we understand the Bell-KS theorem, as we discuss in Section 8. However, these implications do not include the statement, that the theorem is nullified. Bell’s point, that a quantum measurement “does not actually tell us about some property previously possessed by the system” (Bell, 1987, p. 35) remains intact.

## 7. Cabello’s argument

Cabello (2002), in an important paper, has given an argument which is closely related to ours. Suppose an experimenter makes a finite precision measurement in the direction  $\mathbf{k} \in K$  with alignment uncertainty  $\varepsilon$ . Let  $p(\mathbf{k}, \varepsilon)$  be the probability that the

measurement reveals the true colour of  $\mathbf{k}$ . Classically, one would assume

$$\lim_{\varepsilon \rightarrow 0} (p(\mathbf{k}, \varepsilon)) = 1. \quad (11)$$

Cabello, however, shows<sup>5</sup> that the MKC models are inconsistent with that natural assumption.

Cabello's argument thus establishes, by a different route, the point we made in Section 5: namely, that the colours in  $K$  cannot all be reliably ascertainable by finite precision measurement.

It would be interesting to see if Cabello's approach could be extended. It would, for instance, be interesting to investigate the size of the set<sup>6</sup> on which  $\lim_{\varepsilon \rightarrow 0} (p(\mathbf{k}, \varepsilon))$  either fails to exist, or exists but is  $\ll 1$ .

## 8. The physical significance of the Bell-KS theorem

The PMKC models show that the significance of the Bell-KS theorem is primarily epistemological: it concerns the nature and extent of the knowledge acquired by measurement. In this section we examine how far that proposition departs from the views of Bell and KS. We also explain why, in our view, the theorem matters: why it deserves its status as one of the key foundational results of quantum mechanics.

For a long time it was widely (though not universally) believed that quantum mechanics is just plain inconsistent with the classical picture, of particles moving along sharply-defined, objective trajectories. However, Bohm's (1952) rediscovery of de Broglie's pilot wave theory showed that that is incorrect. As Bell (1987) puts it: "in 1952 I saw the impossible done" (p. 160).

The Bell-KS theorem was conceived in response to that event. Bell and KS, in their different ways, were both trying to establish that a hidden variables theory does not really mark the restoration of classical physics. However, their attitudes could hardly have been more divergent.

<sup>5</sup>We should note that Cabello states the conclusion differently. As he sees it his argument shows that the MKC models "lead to experimentally testable predictions that are in contradiction with those of quantum mechanics". However, if one examines the last paragraph in Section IV of his paper, and endnote 28 of his paper, it becomes clear that such a contradiction only arises if MKCs own assumptions are supplemented with the additional assumption that "successive tests with increasing precision will give us the true colours with a higher probability".

We are unable to detect any mathematical error in Clifton & Kent's (2000) proof that their models do reproduce the experimentally testable predictions of quantum mechanics. So we conclude that what Cabello actually does is to show, by *reductio ad absurdum*, that the MKC models are inconsistent with his additional assumption (and consequently with our Eq. (11)).

<sup>6</sup>Let us note that our argument in Sections 5–6, though it suggests, does not logically imply any statement regarding the size of this set. Nor does it tacitly depend on such a statement.

Suppose that, for some  $\varepsilon$  and some non-empty open  $V \subseteq D$ ,  $p(\mathbf{k}, \varepsilon) = 1$  for all  $\mathbf{k} \in K \cap V$ . This would mean, in effect, that measurements to finite precision  $\varepsilon$  are really infinite precision measurements so far as the true colours are concerned. But an experimenter could not exploit this fact to acquire information about the true colours because s/he could not distinguish an instrument guaranteed to reveal the true colour of  $\mathbf{k}$  from another instrument guaranteed to reveal the true colour of a nearby vector  $\mathbf{k}'$ .

Let us begin by examining KS's "take" on the theorem. KS (1967) describe the theorem as "a proof of the non-existence of hidden variables". This is unfortunate, for it is rather obviously no such thing. However, one finds on closer inspection that they do not really mean that. The intuition which drives their work is the perception that there is a kind of logico-mathematical symmetry to quantum mechanics, which a hidden variables theory must violate. They try to capture that intuition by means of a formal criterion. Specifically, they maintain that a "successful" hidden variables theory must assign, to each quantum observable  $\hat{A}$ , a real function  $f_{\hat{A}}$  defined on a phase space  $\Omega$  with the property

$$f_{g(\hat{A})} = g \circ f_{\hat{A}}, \quad (12)$$

for each Borel function  $g$ . They use the Bell-KS theorem to infer that no such assignment is possible and, consequently, that "successful" hidden variables theories do not exist.

We will not say much more about this because it is an implication that PMKC clearly do invalidate. It is true that functions satisfying Eq. (12) cannot be assigned to every quantum observable. However, Clifton & Kent (2000) demonstrate, by explicit construction, that such functions can be assigned to all the observables in a dense subset. KS concede in advance that this is enough to invalidate their argument because they accept

that in fact it is not physically meaningful to assume that there are a continuum number of quantum mechanical propositions (Kochen & Specker, 1967, p. 70).

Before moving on let us say that, although KS's specific proposal has been shown not to work, there might be some substance to their underlying intuition. It certainly seems to this writer (on an intuitive level) that there is a kind of symmetry to quantum mechanics, which hidden variables spoil. It might be worth trying to find a more satisfactory way to capture that intuition formally.

Bell approaches the problem from a completely different angle. KS are interested in questions of abstract logico-mathematical structure. Bell, by contrast, is motivated by a strong philosophical objection to the Copenhagen interpretation. He particularly objects to the fact that the Copenhagen interpretation accords primacy to a concept, "observation", which, besides being subjective, is not even sharply defined (see, for example, Bell, 1987, p. 174). It is probably fair to say that he does not regard the de Broglie-Bohm theory as a satisfactory solution to the interpretation problem. But he certainly sees it as an improvement on the Copenhagen interpretation. This means that, where KS are looking for reasons to rule out the hidden variables idea, Bell is looking for clues which may guide us to a fully satisfactory, fully objective interpretation of quantum mechanics.

As Bell sees it the theorem shows (Bell, 1987, pp. 8–9 and 164–166) that measurement outcomes must depend, not only on the observable measured and the hidden state of the system, but also on the complete experimental set-up (or measurement context). However, that does not (he thinks) represent any kind of objection to the hidden variables concept. It simply means that hidden variables theories are not classical theories, and so cannot be expected to obey the classical



rules. Specifically (*ibid.*, pp. 2, 9, 35, 165, 166) he sees contextuality as the manifestation, in hidden variables terms, of Bohr's (1949) point concerning

the impossibility of any sharp distinction between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.

Bell's anxiety that the theorem should not be seen as an impossibility proof is so great that it seriously unbalances his exposition. As Mermin (1993) notes he seems, in places, almost to suggest that the theorem is "silly". However, the following passage shows that he does not really think it silly:<sup>7</sup>

This word ['measurement'] very strongly suggests the ascertaining of some pre-existing property of some thing, any instrument involved playing a purely passive role. Quantum experiments are just not like that, as we learned especially from Bohr. The results have to be regarded as the joint product of 'system' and 'apparatus', the complete experimental set-up. ... I am convinced that the word 'measurement' has now been so abused that the field would be significantly advanced by banning its use altogether, in favour for example of the word 'experiment'. (Bell, 1987, p. 166)

In other words: the Bell-KS theorem shows quantum mechanics to be so extremely inconsistent with the ordinary idea of a measurement that it would be better not to use the word "measurement" at all. So Bell can hardly be accused of understating the theorem's significance (at least in this passage).

In Sections 1–7 we showed that Bell's most important point (that a quantum measurement does not, as a general rule, ascertain "some pre-existing property of some thing") remains valid. We now need to examine the rest of what he says in the light of PMKCs discoveries.

When Bell wrote the above passage he had in mind the way that spin measurements work in the de Broglie–Bohm theory (Bell, 1987, pp. 35 and 163; Dewdney et al., 1986; Holland, 1993, Bohm & Hiley, 1993). In that theory the apparatus interacts with the system, so as to manufacture a value which did not previously exist. The unobservability of the Bohmian spin is a consequence of this. So it appeared to Bell that there is a deep connection between the point that

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<sup>7</sup>Bell does not explicitly mention the Bell-KS theorem in this passage. For some reason he seeks to minimize his own contribution throughout the paper (Bell, 1982) from which it is taken. For instance, two pages earlier (Bell, 1987, p. 164) he introduces his contextuality theorem as: "... the Gleason–Jauch proof. I was told of it by J. M. Jauch in 1963. Not all of the powerful mathematical theorem of Gleason is required, but only a corollary which is easily proved by itself. (The idea was later rediscovered by Kochen and Specker; see also Belinfante and Fine and Teller)"—as though he himself had nothing to do with it. He gives a long list of names—Gleason, Jauch, Kochen, Specker, Belinfante, Fine, Teller—but omits to mention his own (though it will be found that one of the numbered citations is to Bell, 1966). Quite why Bell should want to pass his theorem off, first as something Jauch told him, and then as something we learned from Bohr, is not entirely clear. Most probably it is connected with his desire to refute KS's allegation, that the result is an impossibility proof.

quantum measurements do not, as a general rule, ascertain pre-existing properties, and Bohr's point, concerning the importance of the complete experimental set-up.

This part of Bell's analysis clearly does need revision. In the PMKC models the apparatus does passively reveal a value which was already there, as in classical physics. It is true that the experimenter does not acquire any *knowledge* thereby. So Bell's point, concerning our ability to *ascertain* the pre-existing values, still stands. However, this is not because the apparatus actively manufactures completely different values. Instead, it is because the valuation is "pathologically" discontinuous.

One might be tempted to conclude that Bohr and Bell were just wrong about the importance of the complete experimental set-up. However, that would be incorrect. The indivisibility of the system-apparatus complex, on which Bohr so strongly insists, is still a feature of the PMKC models. For instance, we showed in [Appleby \(2002\)](#) that, in the case of a system comprising three spin- $\frac{1}{2}$  particles, the very existence of a property typically depends on the complete experimental set-up<sup>8</sup> (also see the discussion of sequential measurements on a single spin-1 particle in [Appleby, 2001](#)).

So the PMKC models do not invalidate either of Bell's two main points concerning the significance of the Bell-KS theorem. They do, however, have a major impact on the logic of Bell's analysis. As Bell sees it his epistemological proposition (concerning our ability to ascertain the pre-existing values) is a direct consequence of his Bohrian proposition (concerning the importance of the complete experimental set-up). It now appears that he is misled by what turn out to be merely accidental features of the de Broglie-Bohm theory. In the general case the two propositions are independent of one another.

Of the two of them, it appears to us that the epistemological proposition is the more important—which is why we said, at the beginning of this section, that the implications of the Bell-KS theorem are *primarily* epistemological. But let us now qualify that by saying that the Bohrian proposition is by no means unimportant. In particular, it is connected with the non-locality of the MKC models ([Appleby, 2002](#)).

Finally, the PMKC models show that the Bell-KS theorem (in its epistemological aspect) only applies to finite precision measurements. MKC misconstrue this implication of their models: for they say that finite precision *nullifies* the Bell-KS theorem. In fact, finite precision *saves* the theorem. It is infinite precision measurements that would nullify the theorem, if we could perform them (nullify its epistemological implications, that is—there would still be the Bohrian aspect).

This is a very interesting result, which certainly puts quantum mechanics in a different light. It means that it is only the finite precision of real laboratory instruments which prevents us giving a *non*-hidden variables interpretation of quantum mechanics (what Bell calls an exposed variables interpretation—see below). However, one should not attach too much significance to this point. Even if infinite precision instruments existed our finite precision brains would be incapable of

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<sup>8</sup>It seems that a similar phenomenon occurs in [Palmer's \(1995, 2004\)](#) models. Palmer's models are of some independent interest, and should be examined by anyone concerned with these questions.

assimilating the information they provided. Moreover, even a quite modest number of significant figures would take us below the Planck length—in which case it would very likely be quantum mechanics itself that got nullified (and the PMKC models along with it).

In Sections 1–7 we defended Bell against MKC. The discussion in this section redresses the balance. PMKC may not have actually nullified Bell's argument. But they have certainly exposed some serious defects.

For the sake of completeness we now present two further criticisms of Bell, which are only indirectly motivated by PMKCs arguments. We noted earlier that Bell is sympathetic to the de Broglie–Bohm theory. This leads him to underplay the significance of his contextuality theorem. Much of the confusion which has afflicted this subject is attributable to that.

The Bell-KS theorem shows that it is not typically possible to ascertain the pre-existing value of an arbitrary observable  $\hat{A}$  simply by making a measurement in the approximate direction of  $\hat{A}$ . There does, of course, remain the possibility that one might ascertain the value by some other, more sophisticated means.<sup>9</sup> So the theorem does not conclusively demonstrate that there is no non-hidden, or exposed variables interpretation of quantum mechanics, in which the pre-existing values are all open to empirical view, as in classical physics. Nevertheless, it may excite the suspicion that no such interpretation is possible.

Bell, however, does not want to accept that conclusion. This is because he is sympathetic to the de Broglie–Bohm theory, and he fears that if the pre-existing values were ineluctably hidden it would mean that the theory was devoid of empirical interest. As he puts it:

Pragmatically minded people can well ask *why bother about hidden entities that have no effect on anything?* (Bell, 1987, p. 92, footnote 24, his italics)

It should be noted that Bell is not just worried about what pragmatically minded people might ask. He feels the force of this objection himself. He therefore tries to find some way of circumventing the constraint imposed by his contextuality theorem.

In effect, Bell himself tries to nullify his own theorem. His strategy is to amputate the spin observables on which the proof is based.<sup>10</sup> He thereby arrives at a kind of

<sup>9</sup>See, for example, Valentini's (2002a) ideas, concerning the possibility of detecting matter in a non-equilibrium state at the sub-quantum level.

Another possibility is that one might, perhaps, be able to construct a theory in which the pre-existing value of an arbitrary observable  $\hat{A}$  could be ascertained by measuring in certain *specific* experimental contexts. For instance, one might perhaps be able to ascertain the pre-existing value of  $\hat{A}$  by measuring  $\hat{A}$  jointly with  $\hat{B}_1, \hat{B}_2$  (but not by measuring it jointly with  $\hat{C}_1, \hat{C}_2$  or  $\hat{D}_1, \hat{D}_2$  or ...). The reason that Bohmian spins are unobservable is that, in the Bohm theory, the outcome of a spin measurement is sensitive to numerous microscopic features of the system + apparatus complex which are experimentally uncontrollable. But one could perhaps envisage a theory in which the relevant features of system + apparatus are experimentally controllable. This suggestion might seem implausible. The point is, however, that the Bell-KS theorem is not, as it stands, sufficient to exclude the possibility.

<sup>10</sup>There are some similarities between this and MKCs approach. Bell and MKC both attempt to circumvent the theorem by only assigning values to a restricted class of observables—a dense subset of  $S^2$  in the case of MKC, positions in the case of Bell.

“stripped-down” version of the de Broglie–Bohm theory, in which the only beables are the wave-function itself and the particle positions (Bell, 1987, pp. 10, 34–35, 127–133, 160–163). In this picture “the particle does not ‘spin’, although the experimental phenomena associated with spin are reproduced” (*ibid*, p. 35).

Bell is under the impression that, in the de Broglie–Bohm theory, measurements of position are always non-contextual. So he thinks that a particle’s true position is empirically observable. He considers that a particle has no other intrinsic properties, apart from its position. So it appears to him that *every* property is empirically observable. He consequently thinks it “absurd” to describe the de Broglie–Bohm theory as a hidden variables theory (Bell, 1987, p. 201; also see *ibid*, pp. 92, 128, 162–163). He suggests that the term “exposed variables” would be more appropriate (*ibid*, p. 128). It is an opinion that is widely shared in the Bohmian community (see, for example, Bohm & Hiley, 1993, p. 2 and Holland, 1993, pp. 106–107).

The first objection to this argument is that it is not in fact true that measurements of de Broglie–Bohm position are always non-contextual. This is shown by the discovery (shortly after Bell’s death) of Englert, Scully, Süssmann, & Wlater’s (1992, 1993) “surreal” de Broglie–Bohm trajectories (also see Dürr, Füsseder, Goldstein, & Zanghi, 1993; Dewdney, Hardy, & Squires, 1993; Aharonov & Vaidman, 1996; Scully, 1998). In this phenomenon a particle’s trajectory is recorded by an array of detectors. If the detectors are only read after a non-zero time interval, then it can happen that the particle is recorded as having been in one place when it was in fact somewhere entirely different. As Dewdney et al. (1993) note, this is “yet another illustration of the contextuality of measurements”.

The second objection is, to our mind, even more telling. In Bell’s “stripped-down” version of the de Broglie–Bohm theory objective reality is ascribed to the entire trajectory  $x(t)$ . So Bell is wrong to think that the instantaneous position  $x$  is a particle’s only intrinsic property. The time derivative  $dx/dt$  must also be considered an intrinsic property. And de Broglie–Bohm velocities are *generically* hidden. Except in special cases they are completely different from the empirical velocities, found by measurement (Bohm, 1952; Bohm & Hiley, 1993; Holland, 1993; Appleby, 1999a).

Bell says that it is in the “‘hidden’(!) variables” that “one finds an image of the visible world” (Bell, 1987, p. 201). This is perfectly true, if by “visible world” is meant the macroscopic bodies of our ordinary experience. In the classical limit the variables are no longer hidden. For instance, if one observes a bus as it journeys intermittently down a London street, then it is the de Broglie–Bohm trajectory that registers in one’s brain. However, that is a consequence of the interaction between the bus and its thermal environment (Bohm & Hiley, 1993, Chapter 8 and Appleby, 1999b). Undecohered de Broglie–Bohm velocities tend to be strikingly, and even grotesquely at variance with anything that is actually observed. For instance, the electron in an  $ns$  state of a Hydrogen atom is, according to the de Broglie–Bohm theory, always at rest.

For these reasons it appears to us that Bell’s attempt to circumvent his contextuality theorem is no more successful than PMKCs. Of course, we have only been considering the de Broglie–Bohm theory. There remains the possibility that there exists some other theory in which positions and velocities are both empirically

ascertainable (although it does not seem likely, given Clifton's (2000) discussion of KS obstructions in the Weyl algebra).

Nevertheless, Bell and PMKC raise an important question. The non-existence of an exposed variables interpretation of quantum mechanics is often regarded as obvious. However, the kind of semi-intuitive reasoning on which that opinion is based is not a substitute for formal argument, starting from the fundamental principles of quantum mechanics. In any case a little reflection suffices to show that the point is, in fact, very far from obvious. It certainly did not seem obvious to Bell. Also, if the point really were as obvious as is often supposed, then PMKC's discovery, that it critically depends on the impossibility of performing infinite precision measurements, would not have come as such a surprise.

This brings us to our final criticism of Bell. Kochen and Specker see the Bell-KS theorem as an impossibility proof. That, of course, is wrong: Bohm commits no actual fallacy. However, it appears to us that Bell goes much too far in the opposite direction. Bell describes non-locality as a "real problem" (Bell, 1987, p. 172). By contrast, he sees contextuality as no kind of problem at all. In fact, he appears to regard the idea, that it should be seen as a problem, as "silly" (see Mermin, 1993). It seems to us that this dismissive response is just as inappropriate as Kochen and Specker's overly assertive one.

Bell takes this relaxed attitude because he thinks that contextuality only limits our ability to ascertain pre-existing spins. Once one appreciates that it also limits our ability to ascertain pre-existing velocities and (in certain circumstances) pre-existing positions, then it becomes clear that contextuality does represent a serious problem.

The problem is not necessarily insuperable (see footnote 9). But it seems to be, certainly hard, and perhaps impossible to construct a contextual theory which is not also a hidden variables theory. The trouble with that is that hidden variables theories are, in a sense, metaphysical. Bell makes the point very clearly when he asks why we should "bother about hidden entities that have no effect on anything" (Bell, 1987, p. 92). Englert et al. (1993) make the same point when they say (in connection with the "surreal" de Broglie–Bohm trajectories mentioned above) "if the [de Broglie–Bohm] trajectories ... have no relation to the phenomena, in particular to the detected path of the particle, then their reality remains metaphysical, just like the reality of the ether of Maxwellian electrodynamics".

Of course, a quantity does not need to be directly observable in order to be physically relevant. In conventional quantum mechanics the state vector of an individual system is not directly observable. Nevertheless, it plays an essential role in the theory: without it quantum mechanics could not function as a predictive physical theory. However, the pre-existing values posited by a hidden variables theory are not like that. There does not seem to be anything that can be calculated using such values that cannot be calculated equally well without them. They seem gratuitous. That is what is meant by calling them metaphysical.

We should acknowledge there are some points to be made on the other side. It is probably fair to say that Bell, in spite of what he explicitly says, is not unaware of the considerations just adduced. However, they are, for him, outweighed by his aversion to the "vagueness" and "subjectivity" of the once orthodox Copenhagen

interpretation (Bell, 1987, p. 160). Bell feels that the de Broglie–Bohm theory, though metaphysical, is at least *clear*. One need not be a committed Bohmian to acknowledge the force of that argument. Moreover, the fact that the hidden variables idea has, until now, proved to be devoid of predictive power does not necessarily mean that it will always remain so (see, for example, Valentini, 1991a,b, 2002a,b; t’Hooft, 1999; Faraggi & Matone, 2000; Bertoldi, Faraggi, & Matone, 2000).

But, after making due allowance for these qualifications, and the qualifications in footnote 9, contextuality is clearly a problem, in just the same sense that Bell considers non-locality to be a “real problem” (Bell, 1987, p. 172). Furthermore, it is a problem the PMKC models do nothing to obviate. It is true that the PMKC models are not contextual in the same way as more conventional theories, such as the de Broglie–Bohm theory. But the essential difficulty remains. The postulated beables are still metaphysical.

Indeed, it appears to us that it is the Bell-KS theorem which encapsulates the really fundamental problem. Non-locality simply provides a particularly graphic illustration of this more basic point, that the postulated level of objective reality is systematically concealed from view.

Suppose, *per impossibile*, that we could perform infinite precision measurements. Then it can be seen from the results proved in Appleby (2002) that the MKC models would violate *signal* locality.<sup>11</sup> In that case non-locality would no longer be a conceptual problem. It would be an empirical prediction. The prediction might be confirmed (implying that relativity is wrong) or disconfirmed (implying that the MKC models are wrong). Either way, we would not be involved in a cosmic conspiracy.

But as it is we believe non-locality to be unobservable. A theory which is profoundly non-local at the level of the underlying beables somehow contrives to be completely local at the level of the observable phenomena. This is certainly objectionable. However, it is only one illustration—albeit a very striking illustration—of the more general point, that the postulated beables are highly metaphysical.

## 9. Conclusion

PMKC have made a most important contribution to this subject. However, it is not important for the reason MKC think. The PMKC models do not nullify the Bell-KS theorem. Instead, they give us a deeper and more accurate insight into what the theorem is really telling us.

We have argued that the Bell-KS theorem has a primarily epistemological significance. It concerns the knowledge we acquire by measurement. So what one needs to ask is not: “how much of  $S^2$  can be coloured *at all*?” But rather: “how much of  $S^2$  can be coloured in such a way that the colours are *empirically knowable*?” Once

<sup>11</sup>cf. Valentini’s (1991a,b, 2002a,b); speculation, that violations of signal locality might be observable in many other hidden variables theories.



that is understood it can be seen that there is no question of the theorem being nullified.

It can also be seen that the theorem encapsulates the essential distinction between quantum and classical. Quantum mechanics does not (as was once thought) require us to abandon the classical picture, of particles having sharply-defined, fully objective properties. However, it seems that we do have to abandon the assumption, that the properties are empirically knowable. We can, if we like, retain the belief: but only at the price of making it metaphysical. To Bell's "pragmatically minded people" a belief of that kind seems empty.

To some extent these points were already recognized by Bell and by others. However, Bell's account, as we have seen, is vitiated by a number of misconceptions. PMKC's achievement is to devise models in which the essential meaning of the theorem emerges in a particularly pure form. This greatly clarifies the issue.

Finally, let us note that the point, that the distinction between classical and quantum is partly epistemological in character, accords with the current interest in quantum information.<sup>12</sup> In particular, it accords with the suggestion of Fuchs (2001, 2002) and Caves, Fuchs, & Schack (2002) that quantum mechanics can partly (and perhaps even mostly) be seen as a "law of thought".

In this paper we have tried to avoid taking sides in the interpretational dispute. But, now that we have reached the end, let us say that we share the pragmatically minded person's distaste for metaphysical theories. On the other hand, we also share Bell's distaste for the vagueness and subjectivity of the Copenhagen interpretation. It appears to us that what Bell says on that score is amply justified. We therefore find ourselves impaled on the horns of a very unpleasant dilemma.

We do not profess to know how the dilemma can be resolved. But one possibility would be to improve the Copenhagen interpretation to the point where it was no longer vague, and no longer subjective. Or, at any rate, not so offensively vague, and not so offensively subjective. Fuchs's "law of thought" idea strikes us as very promising in that respect. However, it would take a great deal of work before that promise could be fulfilled.

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<sup>12</sup>In this connection let us note that MKC suggest that their models may have implications for quantum computation. However, they are misled by their belief that "once the assumption of infinite precision is relaxed" non-relativistic quantum mechanics can be simulated classically (Clifton & Kent, 2000, p. 2103). As we have seen (in Section 8) their models actually suggest the exact opposite: namely, that it is only if infinite precision measurements *were* possible that quantum mechanics might be simulated classically (in which case it would also violate *signal* locality). As Meyer himself remarks (citing Schönhage, 1979 and Freedman, 1998) it would not be surprising if a quantum computer performed no better than an *infinite* precision classical machine.

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## Appendix A. Proof of Lemma 2

In this appendix we prove that  $d_{\mathcal{B}} > 0$ , as stated in Lemma 2. We conclude with a few remarks concerning its actual magnitude.

Let  $\{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_{2M}\}$  be a KS-uncolourable set<sup>13</sup> of unit vectors with the property  $\mathbf{n}_{M+i} = -\mathbf{n}_i$  for  $i = 1, \dots, M$ . Let  $\theta_0$  be the minimum angular separation of the vectors in this set:

$$\theta_0 = \min_{1 \leq i, j \leq 2M} (\cos^{-1}(\mathbf{n}_i \cdot \mathbf{n}_j)). \quad (\text{A.1})$$

For each  $i$ , surround  $\mathbf{n}_i$  with a circular patch  $E_i$  of radius  $\theta_0/2$ :

$$E_i = \{\mathbf{m} \in S^2 : \cos^{-1}(\mathbf{m} \cdot \mathbf{n}_i) \leq \theta_0/2\}. \quad (\text{A.2})$$

Let  $B$  be a KS-colourable set  $\in \mathcal{B}$ , and let  $B^c = S^2 - B$  be its complement. We may assume, without loss of generality, that  $B$  (and consequently  $B^c$ ) is invariant under the parity operation (since, if  $B$  is not invariant, we can replace it with another set  $\in \mathcal{B}$  which is invariant, and whose measure is the same or larger).

By construction the sets  $E_i$  are non-overlapping with the possible exception of a set of measure zero on their boundaries. Consequently

$$\mu(B^c) \geq \sum_{i=1}^{2M} \mu(E_i \cap B^c). \quad (\text{A.3})$$

We now define, for each  $i$ , a function  $g_i: S^2 \rightarrow E_i$  by

$$g_i(\mathbf{m}) = e^{(\theta_0/2)\mathbf{m} \cdot \mathbf{L}} \mathbf{n}_i, \quad (\text{A.4})$$

for each  $\mathbf{m} \in S^2$ . Here  $L_1, L_2, L_3$  are the generators of  $SO(3)$ . Thus  $g_i$  maps  $\mathbf{m}$  onto the vector obtained by rotating  $\mathbf{n}_i$  through the (fixed) angle  $\theta_0/2$  about the (variable) axis  $\mathbf{m}$ . It is easily seen that, as  $\mathbf{m}$  ranges over  $S^2$ , the interior of  $E_i$  is covered twice, and the boundary once. Consequently

$$\mu(E_i \cap B^c) = \frac{1}{2} \int_{\tilde{B}_i^c} J_i(\mathbf{m}) d\mu, \quad (\text{A.5})$$

where  $J_i$  is the Jacobian of  $g_i$  and  $\tilde{B}_i^c = g_i^{-1}(E_i \cap B^c)$ .

Now let

$$J(\mathbf{m}) = \min_{1 \leq i \leq 2M} (J_i(\mathbf{m})). \quad (\text{A.6})$$

<sup>13</sup>That is, a set which is not KS-colourable in the sense of the definition in Section 2.



Eqs. (A.3) and (A.5) then imply

$$\mu(B^c) \geq \frac{1}{2} \int_{\bigcup_{i=1}^M \tilde{B}_i^c} J(\mathbf{m}) d\mu + \frac{1}{2} \int_{\bigcup_{i=M+1}^{2M} \tilde{B}_i^c} J(\mathbf{m}) d\mu. \quad (\text{A.7})$$

We now observe that, for each fixed value of  $\mathbf{m}$ , the set  $\{g_1(\mathbf{m}), \dots, g_{2M}(\mathbf{m})\}$ , being obtained by rotating the KS-uncolourable set  $\{\mathbf{n}_1, \dots, \mathbf{n}_{2M}\}$ , must itself be KS-uncolourable. Since  $B$  is KS-colourable this means that, for each  $\mathbf{m}$ , there must exist some  $1 \leq i \leq M$  such that  $g_i(\mathbf{m})$  and  $g_{i+M}(\mathbf{m})$  both  $\in B^c$ . Consequently  $\bigcup_{i=1}^M \tilde{B}_i^c = \bigcup_{i=M+1}^{2M} \tilde{B}_i^c = S^2$ . Hence

$$\mu(B^c) \geq \int_{S^2} J(\mathbf{m}) d\mu. \quad (\text{A.8})$$

We deduce that

$$\mu(B) \leq 1 - \int_{S^2} J(\mathbf{m}) d\mu, \quad (\text{A.9})$$

for every KS-colourable set  $B \in \mathcal{B}$ . Finally, we note that  $J(\mathbf{m})$  is a continuous, non-negative function which is not identically 0. This fact, together with Eq. (2), implies

$$d_{\mathcal{B}} \geq \int_{S^2} J(\mathbf{m}) d\mu > 0. \quad (\text{A.10})$$

This proves Lemma 2.

Finally, let us briefly consider the size of this integral. An exact calculation, though straightforward, would be somewhat tedious. We therefore confine ourselves to noting that it follows from the definition of  $J$  that, for all  $i$ ,

$$\int_{S^2} J(\mathbf{m}) d\mu \leq \int_{S^2} J_i(\mathbf{m}) d\mu = 2\mu(E_i) = 2\sin^2\left(\frac{\theta_0}{4}\right). \quad (\text{A.11})$$

For the Conway–Kochen set (Peres, 1993; Bub, 1997) one has  $\theta_0 = 18.4^\circ$  (the angle between the directions  $(0, 1, 2)$  and  $(0, 2, 2)$ ), implying that for this set

$$\int_{S^2} J(\mathbf{m}) d\mu < 0.013. \quad (\text{A.12})$$

If the Conway–Kochen set maximizes the integral, and if  $d_{\mathcal{B}}$  is of the same order as the lower bound set by Inequality (A.10), it would follow that  $d_{\mathcal{B}} \lesssim 0.01$ . However, it would require further investigation to tell whether that is actually the case.  $\square$

## Appendix B. Proof of Lemma 3

The lemma is a consequence of the Lebesgue-Vitali theorem (see, for example, Shilov & Gurevich, 1966, Chapter 10).

Let  $\phi$  be an integrable function defined on a measure space  $(X, S, \mu)$  with  $\sigma$ -ring  $S$  and countably additive measure  $\mu$ . Then it can be shown (Shilov & Gurevich, 1966,

pp. 220–221) that, for almost all  $x_0 \in X$ , the limit

$$\lim_{\delta \rightarrow 0} \left( \frac{1}{\mu(V_\delta(x_0))} \int_{V_\delta(x_0)} |\phi(x) - \phi(x_0)| d\mu \right) \quad (\text{B.13})$$

exists and  $= 0$ , provided that for each  $\delta > 0$ ,  $V_\delta(x_0)$  is a Vitali set containing  $x_0$  and having  $\mu$ -measure  $< \delta$  (for the definition of a Vitali set see Shilov & Gurevich, 1966, p. 209).

A straightforward modification of Banach's elegant argument to show that the set of cubes is a Vitali system for  $\mathbb{R}^n$  (Shilov & Gurevich, 1966, pp. 216–218) establishes that the set of disks  $S(\mathbf{n}, \varepsilon)$  is a Vitali system for  $S^2$ .

The result is now immediate if we take  $\phi$  in Eq. (B.13) to be the indicator function of  $D$  (i.e. the function which is 1 on  $D$  and 0 on its complement).  $\square$

## References

- Aharonov, Y., & Vaidman, L. (1996). About position measurements which do not show the Bohmian particle position. In Cushing, J. T., Fine, A., & Goldstein, S. (Eds.), *Bohmian mechanics and quantum theory: an appraisal*. Dordrecht: Kluwer.
- Appleby, D. M. (1999a). Generic Bohmian trajectories of an isolated particle. *Foundations of Physics*, 29, 1863–1883.
- Appleby, D. M. (1999b). Bohmian trajectories post-decoherence. *Foundations of Physics*, 29, 1885–1916.
- Appleby, D. M. (2001). Nullification of the nullification. quant-ph/0109034
- Appleby, D. M. (2002). Existential contextuality and the models of Meyer, Kent and Clifton. *Physical Review A*, 65, 022105.
- Bell, J. S. (1966). On the problem of hidden variables in quantum mechanics. *Reviews of Modern Physics*, 38, 447–452 (Reprinted in Bell (1987)).
- Bell, J. S. (1982). On the impossible pilot wave. *Foundations of Physics*, 12, 989–999 (Reprinted in Bell (1987)).
- Bell, J. S. (1987). *Speakable and unspeakable in quantum mechanics*. Cambridge: Cambridge University Press.
- Bertoldi, G., Faraggi, A. E., & Matone, M. (2000). Equivalence principle, higher-dimensional Möbius group and the hidden antisymmetric tensor of quantum mechanics. *Classical and Quantum Gravity*, 17, 3965–4005.
- Bohm D. (1952). A suggested interpretation of the quantum theory in terms of “hidden variables” I and II. *Physical Review*, 85, 166–179 and 180–193.
- Bohm, D., & Hiley, B. J. (1993). *The undivided universe*. London: Routledge.
- Bohr, N. (1949). Discussion with Einstein on epistemological problems in atomic physics. In P. A. Schilp (Ed.), *Albert Einstein: philosopher-scientist* (pp. 199–242). La Salle: Open Court.
- Breuer, T. (2002a). The Kochen–Specker theorem for finite precision spin 1-measurements. *Physical Review Letters*, 88, 240402.
- Breuer, T. (2002b). A Kochen–Specker theorem for unsharp spin 1 observables. In T. Placek & J. Butterfield (Eds.), *Proceedings of the Nato advanced research workshop on modality, probability, and Bell's theorems* (pp. 195–203). Dordrecht: Kluwer.
- Bub, J. (1997). *Interpreting the quantum world*. Cambridge: Cambridge University Press.
- Cabello, A. (2002). Finite-precision measurement does not nullify the Kochen–Specker theorem. *Physical Review A*, 65, 052101.
- Caves, C. M., Fuchs, C. A., & Schack, R. (2002). Quantum probabilities as Bayesian probabilities. *Physical Review A*, 65, 022305.
- Clifton, R. (2000). Complementarity between position and momentum as a consequence of Kochen–Specker arguments. *Physics Letters A*, 271, 1–7.

- Clifton, R., & Kent, A. (2000). Simulating quantum mechanics by non-contextual hidden variables. *Proceedings of the Royal Society of London A*, 456, 2101–2114.
- Devaney, R. L. (1989). *An introduction to chaotic dynamical systems*. Reading, MA: Addison-Wesley.
- Dewdney, C., Hardy, L., & Squires, E. J. (1993). How late measurements of quantum trajectories can fool a detector. *Physics Letters A*, 184, 6–11.
- Dewdney, C., Holland, P. R., & Kyprianidis, A. (1986). What happens in a spin measurement? *Physics Letters A*, 119, 259–267.
- Dürr, D., Füsseder, W., Goldstein, S., & Zanghi, N. (1993). Comment on “Surrealistic Bohm trajectories”. *Zeitschrift für Naturforschung*, 48a, 1261–1262.
- Englert, B.-G., Scully, M. O., Süssmann, G., & Walter, H. (1992). Surrealistic Bohm Trajectories. *Zeitschrift für Naturforschung*, 47a, 1175–1186.
- Englert, B.-G., Scully, M. O., Süssmann, G., & Walter, H. (1993). Reply to comment on “Surrealistic Bohm trajectories”. *Zeitschrift für Naturforschung*, 48a, 1263–1264.
- Faraggi, A. E., & Matone, M. (2000). The equivalence postulate of quantum mechanics. *International Journal of Modern Physics A*, 15, 1869–2017.
- Freedman, M. H. (1998). Topological views on computational complexity. *Proceedings of the international congress of mathematicians Berlin 1998, Documenta Mathematica J. DMV, Extra. Vol. ICM, II* (pp. 453–464).
- Fuchs, C. A. (2001). Notes on a Paulian idea: foundational, historical, anecdotal and forward-looking thoughts on the quantum. quant-ph/0105039.
- Fuchs, C. A. (2002). Quantum mechanics as quantum information (and only a little more). quant-ph/0205039.
- Godsil, C. D. & Zaks, J. (1988). Colouring the sphere. University of Waterloo Research Report No. CORR 88-12.
- Hales, A. W., & Straus, E. G. (1982). Projective colorings. *Pacific Journal of Mathematics*, 99, 31–43.
- Halmos, P. R. (1961). *Measure theory (graduate texts in mathematics no. 18)*. New York, Heidelberg, Berlin: Springer.
- Havlicek, H., Krenn, G., Summhammer, J., & Svozil, K. (2001). Colouring the rational quantum sphere and the Kochen–Specker theorem. *Journal of Physics A*, 34, 3071–3077.
- Holland, P. R. (1993). *The quantum theory of motion*. Cambridge: Cambridge University Press.
- Kent, A. (1999). Noncontextual hidden variables and physical measurements. *Physical Review Letters*, 83, 3755–3757.
- Kochen, S., & Specker, E. P. (1967). *Journal of Mathematics and Mechanics*, 17, 59–87.
- Larsson, J.-Å (2002). A Kochen–Specker inequality. *Europhysics Letters*, 58, 799–805.
- Macdonald, A. L. (1982). Comment on “Resolution of the Einstein–Podolsky–Rosen and Bell paradoxes”. *Physical Review Letters*, 49, 1215.
- Mermin, N. D. (1982). Comment on “Resolution of the Einstein–Podolsky–Rosen and Bell paradoxes”. *Physical Review Letters*, 49, 1214.
- Mermin, N. D. (1993). Hidden variables and the two theorems of John Bell. *Reviews of Modern Physics*, 65, 803–815.
- Mermin, N. D. (1999). A Kochen–Specker theorem for imprecisely specified measurements. quant-ph/9912081.
- Meyer, D. A. (1999). Finite precision measurement nullifies the Kochen–Specker theorem. *Physical Review Letters*, 83, 3751–3754.
- Palmer, T. N. (1995). A local deterministic model of quantum spin measurement. *Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences*, 451, 585–608.
- Palmer, T. N. (2004). A granular permutation-based representation of complex numbers and quaternions: elements of a possible realistic quantum theory. *Proceedings of the Royal Society of London Series A—Mathematical, Physical and Engineering Sciences*, 460, 1039–1055.
- Peres, A. (1993). *Quantum theory: concepts and methods*. Dordrecht: Kluwer.
- Pitowsky, I. (1982a). Resolution of the Einstein–Podolsky–Rosen and Bell paradoxes. *Physical Review Letters* 48, 1299–1302.

- Pitowsky, I. (1982b). Response to comments by Mermin and Macdonald. *Physical Review Letters*, 49, 1216.
- Pitowsky, I. (1983). Deterministic model of spin and statistics. *Physical Review D*, 27, 2316–2326.
- Pitowsky, I. (1985). Discussion: quantum mechanics and value definiteness. *Philosophy of Science*, 52, 154–156.
- Schönhage, A. (1979). On the power of random access machines. In H. A. Maurer (Ed.), *Automata, languages and programming: sixth colloquium*, Graz, Austria, July 16–20, 1979 (Lecture Notes in Computer Science, Vol. 71) (pp. 520–529). Berlin, Heidelberg, New York: Springer.
- Scully, M. O. (1998). Do Bohm trajectories always provide a trustworthy physical picture of particle motion? *Physica Scripta*, T76, 41.
- Shilov, G. E., & Gurevich, B. L. (1966). *Integral, measure and derivative: a unified approach*. Englewood Cliffs, NJ: Prentice-Hall.
- Simon, C., Brukner, C., & Zeilinger, A. (2001). Hidden-variable theorems for real experiments. *Physical Review Letters*, 86, 4427–4430.
- 't Hooft, G. (1999). Quantum gravity as a dissipative deterministic system. *Classical and Quantum Gravity*, 16, 3263–3279.
- Valentini, A. (1991a). Signal-locality, uncertainty, and the subquantum H-theorem I. *Physics Letters A*, 156, 5–11.
- Valentini, A. (1991b). Signal-locality, uncertainty, and the subquantum H-theorem II. *Physics Letters A*, 158, 1–8.
- Valentini, A. (2002a). Subquantum information and computation. *Pramana—Journal of Physics*, 59, 269–277.
- Valentini, A. (2002b). Signal-locality in hidden-variable theories. *Physics Letters A*, 297, 273–278.