Commutative, idempotent partially ordered monoids

A *unital quantale* is a suplattice with a compatible monoid structure. A quantale is called *idempotent* if it is idempotent as a monoid (every element is idempotent) (analogously for *commutativity*). Every suplattice can be regarded as a unital, commutative, idempotent suplattice, by chosing the join as operation. In this quantale the bottom element $\perp$ is the identity element. Does the converse hold, i. e.: Given a commutative, idempotent unital quantale (with an operation $+$) where $\perp$ is the identity, is $+$ the same as the join $\vee$?

According to the nLab it is known that $+$ is the meet if you replace $\perp$ by $\top$ in the condition (does anybody know a reference for this statement?).

Edit

As noticed below, this question is actually about partially order monoids, not quantales (I was confused). I changed the title such that it can be found more easily. The answer and the comment imply that the described situation is actually the situation of the trivial lattice consisting of a single element. But it generalizes to partially ordered monoids.

lattices monoids

edited Jun 2 '13 at 15:46

asked Jun 1 '13 at 15:14
The User
1,654 6 22

1 There's some imprecision here: the join can never be the quantale multiplication, because $x \vee -$ doesn't preserve all sups. Namely, it fails to preserve the empty sup (unless of course x is the bottom element). – Todd Trimble ♦ Jun 2 '13 at 2:40

Of course you are right, I misinterpreted an example in a paper and assumed it without rethinking it. Thanks. – The User Jun 2 '13 at 10:55

1 Answer

The answer is yes. Here is a proof.

Lemma 1. Let $\leq$ be the suplattice order. If $a \leq b$ the $c + a \leq c + b$.

Proof. $c + b = c + (a \vee b) = (c + a) \vee (c + b)$.

By the standard equivalence betwee join semilattices and idempotent commutative semigroups it suffices to prove:

Lemma 2. $a \leq b \iff a + b = b$.

Proof. If $a \leq b$ then $b = b + \perp \leq b + a \leq b + b = b$ using Lemma 1. So $b = a + b$.

If $a + b = b$ then $a = a + \perp \leq a + b = b$, again using Lemma 1. This completes the proof.

answered Jun 2 '13 at 1:51
Benjamin Steinberg
17.1k 2 42 100

Of course this has nothing to do with sup-lattices: it shows that an idempotent commutative monoids in posets, with identity the bottom element, are precisely posets with finite joins. This can of course be dualized (cf. the last sentence of the OP). – Todd Trimble ♦ Jun 2 '13 at 2:40

Thanks, that answers my question (modulo Todd's remark). – The User Jun 2 '13 at 10:57