SELECTED RUSSIAN PAPERS ON GAME THEORY

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PREFACE

Game theorists are aware of considerable interest in game theory that has developed in the Soviet Union over the last years. Language difficulties, however, have stood in the way of proper acquaintance with the original papers though some results have become known in summary form. In order to overcome some of this gap the following papers are being made available. They will immediately prove the high quality of work done in the Soviet Union and they should stimulate further publication of translations.

The papers I to XI were translated by Kiyoshi Takeuchi, and XII to XIV by Eugene Wesley.

Editing work was done by Louis Billera, Daniel Cohen and Richard Cornwall.

April 1968 Oskar Morgenstern

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SEVERAL APPLICATIONS OF LINEAR PROGRAMMING METHODS TO THE THEORY OF COOPERATIVE GAMES

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Nekotoriye primeneniya metodov lineynogo programmirovaniya
k teorii Kooperativnikh igr

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This article is devoted to the application of a theorem on linear inequalities to the existence problem concerning solutions of n-person cooperative games.

In the first section of the paper we shall introduce the concept of coverings, which characterize a game's coalition structure, and shall study their properties.

In the second section we shall investigate the core, a set which is always contained in the solution and, when it exists, in a certain sense replaces the solution. Necessary and sufficient conditions for the core's existence (expressed in terms of coverings) will be set forth.

In the third section, quota games will be studied with the aid of the same methods used in the second section.

In the fourth section we will investigate the link between the core and the solution. We shall point out some necessary conditions that must be present in order for the core to coincide with the solution. Several sufficient conditions will also be indicated.

In the last section we shall present some examples.

§1. Basic Concepts. Definitions and Notations

A cooperative game Γ, given in the form of a characteristic function (see [1] and [2]) is a pair, consisting of:

1) A set $I_n = \{1,2,\ldots,n\}$, called the set of players, and

2) A real function $v(S)$, defined on the subsets S of this set and having the properties:

$$v(\Lambda) = 0$$
$$v(I_n) = M,$$

where M is some positive number;

$$0 \leq v(S) \leq M \qquad \text{for any } S \subset I_n .$$

The subsets S of the set I_n are called coalitions, and the function $v(S)$ is called a characteristic function.

If $M = 1$ and $v(\{i\}) = 0$, $i=1,\ldots,n$, i.e. the characteristic function receives the value zero on single-element sets, then we say that the game is given in (0-1)-reduced form.

In what follows, we shall, unless otherwise stated, assume that the games under discussion are given in (0-1)-reduced form.

Consider the systems of real numbers

$$\alpha = (a_1,\ldots,a_n), \text{ where } a_i \geq v(\{i\}) = 0 \text{ and } \sum_{i=1}^{n} a_i = 1 .$$

We shall call any such system an imputation. Denote the set of all such systems - the set of all imputations - by the letter A. We shall, from here on, look upon α as an n-dimensional vector, and on A as a subset of n-dimensional Euclidean space.

We shall now define a dominance relation on the set A. We shall say that the imputation $\alpha = (a_1, \dots, a_n)$ <u>dominates</u> the imputation $\beta = (b_1, \dots, b_n)$ (this is written $\alpha \succ \beta$), if:

1) There exists a set $S \subset I_n$ such that $\sum_{i \in S} a_i \leq v(S)$ (The set S is then called <u>effective</u> for α),

2) $a_i > b_i$ for all $i \in S$.

Occasionally, in order that it be clear with respect to which set the dominance takes place, we shall write $\alpha \underset{S}{\succ} \beta$.

Note that any such β must fulfill the condition $\sum_{i \in S} b_i < v(S)$, i.e., dominance may take place only with respect to those sets S for which $\sum_{i \in S} b_i < v(S)$.

We shall call the set S <u>essential</u> for α, if:

1) S is effective for α, i.e. $\sum_{i \in S} a_i \leq v(S)$;

2) There exists no set $T \subset S$, $T \neq S$, effective for α; i.e. for any set $T \subset S$, $T \neq S$

$$\sum_{i \in T} a_i > v(T) .$$

Note 1: It is easy to show (see, for example, [2]), that the dominance relation may take place only with respect to sets which are essential for the dominating imputation.

The dominance relation does not constitute a partial ordering; in fact, every logically conceivable possibility may take place between two imputations (it is possible for example that, with respect to non-intersecting sets, both $\alpha \succ \beta$ and $\beta \succ \alpha$ are realized).

Let $P \subset A$. We shall denote by dom P the set of all imputations in A that are dominated by some imputation in P.

The set $U = A \backslash \mathrm{dom}\, A$ is called the core.

The set V is a solution, if $V = A \backslash \mathrm{dom}\, V$, i.e., a solution is a set of imputations such that

1) No two imputations in V dominate one another.

2) For every $\upsilon \notin V$, there exists an imputation $\alpha \in V$ that dominates it, $a \succ \upsilon$ (see [1] and [2]).

Obviously $U \subset V$, since the core consists of all imputations not dominated by any imputation in A $(U = A \backslash \mathrm{dom}\, A)$. Simple examples show that the solution is in general not unique (see, for example, [1]).

LEMMA 1.1: If a solution coincides with the core, then the solution is unique.

The lemma is trivial; we formulated it so that it may be easily referred to further on.

For any set $S \subset I_n$, we denote the number of elements in S by $|S|$.

We denote by $\mathfrak{N} = \{S_1, \ldots, S_m\}$ the system consisting of all $S_j \subset I_n$ for which either $v(S_j) > 0$, or, if $v(S_j) = 0$, then $|S_j| = 1$.

LEMMA 1.2: In order for an imputation $\alpha = (a_1, \ldots, a_n)$ to belong to the core U, it is necessary and sufficient that the inequality

$$\sum_{i \in S} a_i \geq v(S)$$

be satisfied for all $S \in \mathfrak{N}$.

PROOF: Necessity. Let $\alpha \in U$ and suppose that the condition of the lemma is not valid, i.e., that there exists an S_0, such that $\sum_{i \in S_0} a_i < v(S_0)$. Consider

$\beta = (b_1,\ldots,b_n)$, where

$$b_i = a_i + \epsilon\,, \quad \text{if } i\in S_0$$

$$b_i \geq 0\,, \quad \text{if } i \notin S_0\,.$$

We stipulate that

$$\epsilon = \frac{v(S_0) - \sum\limits_{i\in S_0} a_i}{|S_0|}$$

$$\sum_{i\notin S_0} b_i = 1 - v(S_0)\,.$$

Such a vector exists, since $1-v(S_0) \geq 0$. Since $b_i \geq 0$, $i=1,\ldots,n$, and

$$\sum_{i=1}^{n} b_i = \sum_{i\in S_0} a_i + |S_0|\,\frac{v(S_0) - \sum\limits_{S_0} a_i}{|S_0|} + 1 - v(S_0) = 1\,,$$

then $\beta \in A$. But since $\beta \succ \alpha$, $\alpha \in \operatorname{dom} A$. However, by supposition $\alpha \in U = A\setminus\operatorname{dom} A$. We have then proven the necessity of the condition by way of contradiction.

Sufficiency. If for some $\alpha = (a_1,\ldots,a_n)$ the condition of the lemma is fulfilled, i.e. that $\sum\limits_S a_i \geq v(S)$ for all $S \subset I_n$ (for $S \notin \eta$, $\sum\limits_S a_i \geq 0 = v(S)$), then (by definition of dominance) $\alpha \notin \operatorname{dom} A$, i.e. $a \in A\setminus\operatorname{dom} A$.

<u>COROLLARY:</u> The core constitutes a closed, bounded, convex subset of n-dimensional space with a finite number of extreme points. (This is because it consists of the intersection between the hyperplane $\sum\limits_{i=1}^{n} a_i = 1$ and the convex polyhedral region $\sum\limits_{i\in S_j} a_i \geq v(S_j)$, $S_j \in \eta$).

We correspond to each $S_j \in \eta$ and to I_n the vectors $\underline{S_j}$, $j=1,\ldots,n$, and $\underline{I_n}$. Here,

$$\underline{S_j} = (s_j^{(1)},\ldots,s_j^{(n)})\,, \quad \text{where} \quad s_j^{(i)} = \begin{cases} 0, & \text{if } i \notin S_j \\ 1, & \text{if } i \in S_j \end{cases}, \text{ and}$$

$$\underline{I_n} = (1,\ldots,1)\,.$$

We denote the zero vector by $\underline{0}$.

We define a $(q\text{-}\theta)$-covering of the set I_n to be a system of non-negative real numbers $(\lambda_1,\ldots,\lambda_m)$, such that

$$\sum_{j=1}^{m} \lambda_j \underline{S_j} = \underline{I_n} \quad .$$

Here q is the number of λ_j's such that $\lambda_j > 0$, and θ is the system of subsets corresponding to these λ_j's, $\theta = \{ S_{j_1},\ldots, S_{j_q} : \lambda_{j_\ell} > 0 \}$.

We shall say that a $(q\text{-}\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$ is reduced, if for any other $(q\text{-}\theta)$-covering $(\lambda'_1,\ldots,\lambda'_m)$ the equation $\lambda'_\ell = \lambda_\ell$ holds for all ℓ .

LEMMA 1.3: A necessary and sufficient condition for a $(q\text{-}\theta)$-covering to be a reduced $(q\text{-}\theta)$-covering is that the system θ consists of linearly independent vectors.

PROOF: We may assume, with no loss in generality, that $\lambda_1 > 0 ,\ldots,\lambda_q > 0$, $\lambda_{q+1} = \cdots = \lambda_m = 0$; then θ consists of the vectors $\underline{S_1} ,\ldots, \underline{S_q}$. Consider the corresponding system of equations

$$\sum_{j=1}^{q} \lambda_j s_j^{(i)} = 1 , \qquad i=1,2,\ldots,n . \qquad (1.1)$$

The system is feasible, for $\lambda_1,\ldots,\lambda_q$ constitutes its solution. The requirement that the $(q\text{-}\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$ be reduced is equivalent to the requirement that this solution be unique. When the lemma is formulated in this way, it is seen to be trivial.

COROLLARY 1.1: A necessary and sufficient condition for a $(q\text{-}\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$ to be reduced is that the rank of the matrix $\| \underline{S_j} \|_{\lambda_j > 0}$ be equal to q .

COROLLARY 1.2: For all reduced $(q-\theta)$ coverings, $q \leq n$.

COROLLARY 1.3: The number of reduced coverings is finite.

COROLLARY 1.4: Let: 1) $(\lambda_1,\ldots,\lambda_m)$ be a reduced $(q-\theta)$-covering; 2) q_1 be the number of sets $S_j \in \theta$ such that $|S_j| > 1$. (We may assume that these sets are $S_1,\ldots,S_{q_1}$); 3) T be the set of components of I_n "completely covered" by the sets $S_1,\ldots,S_{q_1}$, i.e.

$$T = \{i: \ i \in I_n; \ \sum_{j=1}^{q_1} \lambda_j s_j^{(i)} = 1\} .$$

Then $|T| \geq q_1$.

PROOF: Consider the given $(q-\theta)$-covering. Eliminate all single-element sets from the system of "covering" sets. The number of such sets is equal to $q-q_1$, each of which takes part in covering exactly one element; therefore, $n-q+q_1$ components now remain "completely covered". The set of these components was denoted in the conditions of the assertion by T. Hence $|T| = n-q+q_1$. Since $n-q \geq 0$, then $|T| \geq q_1$.

LEMMA 1.4: If a $(q-\theta)$-covering is regarded as a point in n-dimensional Euclidean space, then the set of all $(q-\theta)$-coverings, Ξ , is a closed, bounded, and convex point set. A member of the set is a reduced $(q-\theta)$-covering if and only if it is an extreme point of the set.

PROOF: The closedness of the set is trivially true. The boundedness of Ξ is due to the fact that $0 \leq \lambda_i \leq 1$. Convexity follows from the linearity of the conditions defining a covering; the last assertion is true by definition of reduced covering.

Thus, the set of all coverings is described by the set of reduced coverings, which are finite in number. It follows from Lemma 1.3 and its corollaries that the reduced coverings can be determined quite easily.

§2. Basic Theorems of Core Theory

In this section we shall demonstrate necessary and sufficient conditions for the existence of the core * for n-person cooperative games.

We will first prove a lemma dealing with linear inequalities of a certain type.

LEMMA 2.1: Let $\underline{A_1}, \dots, \underline{A_m}$ be a system of n-dimensional vectors with non-negative coefficients and let $\underline{I} = (1,\dots,1)$. Then

1) the system

$$\begin{cases} \underline{A_j}\ \underline{X} \geq v_j, & j=1,\dots,m, \\ \underline{I}\ \underline{X} = 1 \end{cases} \tag{2.1}$$

has a solution if and only if for all systems of real numbers $\lambda_j \geq 0$, $j=1,\dots,m$, for which

$$\sum_{j=1}^{m} \lambda_j \underline{A_j} = \underline{I}, \tag{2.2}$$

the inequality $\sum_{j=1}^{m} \lambda_j v_j \leq 1$ is fulfilled;

2) the system

$$\begin{cases} \underline{A_j}\ \underline{X} > v_j & j=1,\dots,m, \\ \underline{I}\ \underline{X} = 1 \end{cases} \tag{2.1'}$$

* Here and further on, when we speak of the core's existence, we mean the existence of a non-empty core.

has a solution if and only if every system of real numbers $\lambda_j \geq 0$, $j=1,\ldots,m$, satisfying (2.2), fulfills the inequality $\sum_{j=1}^{m} \lambda_j v_j < 1$.

PROOF: Let us first note, that in order for (2.1) to be solvable, it is necessary and sufficient that the system

$$\begin{cases} \underline{A_i}\ \underline{X} \geq v_i \\ \underline{I}\ \underline{X} \leq 1 \end{cases} \tag{2.3}$$

be solvable, or equivalently, that

$$\begin{cases} \underline{A_i}\ \underline{X} \geq v_i \\ -I\ X \geq -1\ . \end{cases}$$

The necessity of this condition is trivially true. For proof of sufficiency, note that if for some $\underline{X}$ the strict inequality $\underline{I}\ \underline{X} < 1$ is fulfilled, then increasing the components of X so that $\underline{I}\ \underline{X} = 1$, we receive a solution to (2.1). This is because the inequalities are thus only strengthened, in view of the non-negativeness of the system's coefficients.

According to a theorem in [3] dealing with solvability conditions for systems of linear inequalities, a necessary and sufficient condition for (2.3) to have a solution is that for any system of real numbers $\lambda_0', \lambda_1', \ldots, \lambda_m'$ for which

$$\sum_{j=1}^{m} \lambda_j' \underline{A_j} - \lambda_0' \underline{I} = \underline{0}\ , \tag{2.4'}$$

the condition

$$\sum_{j=1}^{m} \lambda_j' v_j \leq \lambda_0' \tag{2.5'}$$

be fulfilled.

Note, that if $\lambda_0' = 0$, then in view of the non-negativeness of the components of $\underline{A_j}$, $j=1,\ldots,m$, and in view of (2.4'), all the remaining λ_j''s, $j=1,\ldots,m$, are also equal to zero, and hence (2.5') is fulfilled trivially. We may therefore assume that $\lambda_0' > 0$. Dividing both sides of (2.4') and (2.5') by λ_0' and substituting:

$$\lambda_j = \frac{\lambda_j'}{\lambda_0'} \quad , \quad j=1,\ldots,m \quad ,$$

we perceive that the fulfillment of (2.4') and (2.5') is equivalent to the respective conditions:

$$\sum_{j=1}^{m} \lambda_j \underline{A_j} = \underline{I} \tag{2.4}$$

and

$$\sum_{j=1} \lambda_j v_j \leq 1 \, . \tag{2.5}$$

In this manner, the first assertion is proven.

For proof of the second assertion we first show that in order that the system

$$\begin{cases} \underline{A_j}\,\underline{X} > v_j \\ \underline{I}\,\underline{X} = 1 \end{cases} \tag{2.6}$$

be solvable, it is necessary and sufficient that the system

$$\begin{cases} \underline{A_j}\,\underline{X} > v_j \\ \underline{-I\,X} > -1 \end{cases} \tag{2.7}$$

also be solvable. The sufficiency of the condition is proven in the same manner as was done when (2.1) was followed through. We shall now prove the condition's necessity. Suppose that $\underline{X_0} = (x_1^0,\ldots,x_n^0)$ is a solution of (2.6) whereby there exists an $x_k^0 > 0$, since $\underline{I}\,\underline{X_0} = \sum_{i=1}^{n} x_i^0 = 1$. Suppose, further, that

$$\epsilon < \min_j (\underline{A_j}\,\underline{X} - v_j)$$

and

$$\epsilon < x_k^0 .$$

Since in view of (2.6) $\underline{A}_j \underline{X} - v_j > 0$, $j=1,\ldots,m$, and $x_k^0 > 0$, then $\epsilon > 0$.

Consider $X' = (x_1',\ldots,x_n')$, where

$$x_i' = x_i^0 , \quad i \neq k$$

$$x_k' = x_k^0 - \epsilon$$

By proper choice of ϵ we have $\underline{A}_j \underline{X'} > v_j$ and $\underline{I}\ \underline{X'} < 1$, i.e., $\underline{X'}$ is a solution of (2.7), whereupon the necessity of the condition is proven.

We shall now make use of a theorem in [3], dealing with the solvability of systems of strict inequalities, in connection with (2.7). We receive that in order for (2.7) to be solvable, it is necessary and sufficient that for any arbitrary system $\lambda_0' \geq ,\ldots,\lambda_m' \geq 0$ for which (2.4') is fulfilled, the strict inequality

$$\sum_{j=1}^{m} \lambda_j' v_j < \lambda_0' \tag{2.8}$$

be also fulfilled. If $\lambda_0' = 0$, then, as above, $\lambda_j' = 0$, $j=1,\ldots,m$, but then (2.8) is not fulfilled; this means that $\lambda'_0 > 0$. Dividing both sides of (2.4') and (2.8) by λ_0' and introducing the same notation as were introduced when (2.1) was investigated, we receive that for any system of numbers $\lambda_1 \geq 0,\ldots,\lambda_m \geq 0$, satisfying the condition $\sum_{j=1}^{m} \lambda_j \underline{A}_j = \underline{I}$, the inequality $\sum_{j=1}^{m} \lambda_j v_j < 1$ must be satisfied.

We shall now prove some fundamental theorems.

THEOREM 2.1: A necessary and sufficient condition for a game Γ to have a core is that for any arbitrary reduced $(q-\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$, the inequality

$$\sum_{j=1}^{m} \lambda_j \, v(S_j) \leq 1 \tag{2.9}$$

be fulfilled.

PROOF: By lemma 1.2 the core is equal to the set of solutions α of the system

$$\begin{cases} \alpha \, \underline{S_j} \geq v(S_j), & S_j \in \mathcal{N} \\ \alpha \, \underline{I_n} = 1 \end{cases} \tag{2.10}$$

This system satisfies the conditions of lemma 2.1, when $\underline{A_j} = \underline{S_j}$ and $v_j = v(S_j)$.

Applying this lemma, we receive that (2.10) has a solution if and only if any arbitrary system of real numbers, $\lambda_1 \geq 0,\ldots,\lambda_m \geq 0$, for which $\sum_{j=1}^{m}\lambda_j \underline{\bar{S}_j} = \underline{\bar{I}_n}$, satisfies the inequality $\sum_{j=1}^{m} \lambda_j v(S_j) \leq 1$, i.e. inequality (2.9) must be fulfilled for every $(q-\theta)$-covering. But since the left hand side of (2.9) is a linear function of $(\lambda_1,\ldots,\lambda_m)$, and since the set of all coverings is convex (see lemma 1.5), it is therefore sufficient to require the fulfillment of (2.9) for every reduced $(q-\theta)$-covering.

THEOREM 2.2: In order that a game Γ have a core of maximum dimension (i.e. of dimension $n-1$, the dimension of the set of all imputations A) it is necessary and sufficient that the inequality

$$\sum_{j=1}^{m} \lambda_j \, v(S_j) < 1$$

be fulfilled for any arbitrary reduced $(q-\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$.

PROOF: As was stated above, the core consists of the set of solutions to the system

$$\begin{cases} \alpha \ \underline{\dot{S}_j} \geq v(S_j) \ , & S_j \in \mathcal{N} \\ \alpha \ \underline{\dot{I}_n} = 1 \end{cases}$$

This is a convex polyhedral region within the hyperplane $\alpha \ \underline{\bar{I}_n} = 1$. In order for this region to have maximal dimension, i.e. $n-1$, it is necessary and sufficient that it contain relative interior points of the hyperplane $\alpha \ \underline{\dot{I}_n} = 1$, i.e., there must exist α's for which

$$\alpha \ \underline{\ddot{S}_j} > v(S_j) \ , \quad j=1,\dots,m \ ,$$

$$\alpha \ \underline{\dot{I}_n} = 1$$

By lemma 2.1, in order for such a system to have a solution it is necessary and sufficient that the inequality $\sum_{j=1}^{m} \lambda_j v(S_j) < 1$ be fulfilled for any $\lambda_1 \geq 0,\dots\lambda_m \geq 0$ satisfying the condition

$$\sum_{j=1}^{m} \lambda_j \ \underline{\ddot{S}_j} = \underline{\bar{I}_n} \ . \tag{2.11}$$

As in the proof of theorem 2.1, note that $(\lambda_1,\dots,\lambda_m)$ is a $(q-\theta)$-covering and that it is sufficient to require the fulfillment of the conditions of the lemma for reduced $(q-\theta)$-covering.

Note: Consider the linear-programming problem consisting of the determination of the numbers $\lambda_1 \geq 0,\dots,\lambda_m \geq 0$, satisfying the system (2.11) of constraints and minimizing the linear form $\sum_{j=1}^{m} \lambda_j v(S_j)$. Then the reduced coverings constitute admissible basic solutions. If $(\lambda_1^0,\dots,\lambda_m^0)$ is an optimal solution, then for theorems 2.1 and 2.2 to be valid, it is necessary and sufficient to require that $\sum_{j=1}^{m} \lambda_j^0 v(S_j) \leq 1$ and $\sum_{j=1}^{n} \lambda_j^0 v(S_j) < 1$, respectively.

The problem of directly verifying the existence of the core may be regarded, as well, as a linear-programming problem; the problem of finding the "maximal" covering then turns out to be a dual problem.

This analogy can, evidently, be extended.

This interrelationship allows us to use numerical methods of linear programming to determine whether a given game has a core. Note, however, that in view of the specific character of the problems appearing here, these methods may possibly lend themselves to modification.

From the fundamental theorems just proven, the following assertions ensue.

COROLLARY 2.1: In order that a game Γ not have a core, it is necessary and sufficient that there exist a reduced $(q\text{-}\theta)$-covering $(\lambda_1^0, \ldots, \lambda_m^0)$ such that

$$\sum_{j=1}^{m} \lambda_j^0 \, v(S_j) > 1 \, .$$

THEOREM 2.3: In order that the dimension of the core be less than n-1, it is necessary and sufficient that there exist a reduced $(q\text{-}\theta)$ core $(\lambda_1^0, \ldots, \lambda_m^0)$ for which $\sum_{j=1}^{m} \lambda_j^0 \, v(S_j) = 1$, the dimension τ of the core then obeys the inequality $\tau \leq n-q$.

PROOF: The first part of the assertion follows directly from theorem 2.1 and 2.2. We shall prove the validity of the evaluation for τ. First let us note that by corollary 1.2 $q \leq n$. By definition of covering

$$\sum_{j=1}^{m} \lambda_j \, \underline{S_j} \, \alpha = \underline{I_n} \, \alpha = 1 \, .$$

For any $\alpha \in U$, $\underline{S_j} \, \alpha \geq v(S_j)$, therefore

$$1 = \sum_{j=1}^{m} \lambda_j \, \underline{S_j} \, \alpha = \sum_{\lambda_j > 0} \lambda_j \, \underline{S_j} \, \alpha \geq \sum_{\lambda_j > 0} \lambda_j \, v(S_j) = 1 \, ,$$

i.e. $\underline{S_j} \, \alpha = v(S_j)$ for all j for which $\lambda_j > 0$. This means that q linearly independent constraints (the covering is reduced) of the form $\underline{S_j} \, \alpha = v(S_j)$ are imposed on α. Since the constraint $\underline{I_n} \, \alpha = \sum_{i=1}^{n} a_i = 1$ is a consequence of the former constraints $(\Sigma \lambda_j \; \underline{S_j} = \underline{I_n})$, then $\tau \leq n-q$.

As examples we shall point out conditions under which the dimension of the core is $n-2$ or $n-3$.

COROLLARY 2.2: If the core is of dimension $n-2$, then there exist two sets S_{j_1} and S_{j_2} such that

$$S_{j_1} \cup S_{j_2} = I_n , \quad S_{j_1} \cap S_{j_2} = \Lambda , \quad \text{and} \quad v(S_{j_1}) + v(S_{j_2}) = 1 .$$

COROLLARY 2.3: If the core is of dimension $n-3$, then one of the following conditions is fulfilled:

1) There exist sets $S_{j_1}, S_{j_2}, S_{j_3}$ such that

$$\underline{\bar{S}_{j_1}} + \underline{\bar{S}_{j_2}} + \underline{\bar{S}_{j_3}} = \underline{\bar{I}_n}$$

and

$$v(S_{j_1}) + v(S_{j_2}) + v(S_{j_3}) = 1$$

2) There exist sets $S_{j_1}, S_{j_2}, S_{j_3}$ such that

$$\underline{\bar{S}_{j_1}} + \underline{\bar{S}_{j_2}} + \underline{\bar{S}_{j_3}} = 2\, \underline{\bar{I}_n}$$

and

$$v(S_{j_1}) + v(S_{j_2}) + v(S_{j_3}) = 2$$

3) There exists sets $S_{j_1}, S_{j_2}, S_{j_3}, S_{j_4}$ such that

$$\bar{S}_{j_1} + \bar{S}_{j_2} = \bar{I}_n , \quad \bar{S}_{j_3} + \bar{S}_{j_4} = \bar{I}_n$$

and

$$v(S_{j_1}) + v(S_{j_2}) = 1 , \quad v(S_{j_3}) + v(S_{j_4}) = 1 .$$

These assertions are proven by direct use of theorem 2.3.

We shall demonstrate a simple sufficiency condition for the existence of the core.

<u>THEOREM 2.4:</u> In order for a game Γ to have a core it is sufficient that $v(S)$ fulfill the condition

$$v(S_j) \leq \frac{|S_j|}{n} \quad , \quad S_j \in \mathcal{N} \quad .$$

<u>PROOF:</u> Consider the imputation $\alpha = (\frac{1}{n},\ldots,\frac{1}{n})$; since $\underline{S}_j\,\alpha = \frac{|S_j|}{n} \geq v(S_j)$, then $\alpha \in U$, i.e. $U \neq \Lambda$, which is what had to be proven.

It turns out that for a certain class of games this condition is also necessary.

We recall the a <u>symmetric game</u> is a game whose characteristic function satisfies the condition $v(S) = \varphi(|S|)$.

<u>THEOREM 2.5:</u> A necessary and sufficient condition for a symmetric game to have a core is that

$$v(S) \leq \frac{|S|}{n} \quad \text{for any} \quad S \subset I_n \quad .$$

<u>PROOF:</u> The sufficiency of the condition follows from Theorem 2.4. We shall prove necessity. Let $|S| = t$. Consider $t\,\underline{I}_n$; we may "cover" it with n vectors of "length" t . Let these vectors be $\underline{S}_1,\ldots,\underline{S}_n$; then $\frac{1}{t}\sum_{j=1}^{n}\underline{S}_j = \underline{I}_n$ and consequently $(\frac{1}{t},\ldots,\frac{1}{t}, 0,\ldots,0)$ is a covering (possibly even a reduced covering). But by Theorem 2.2, the condition

$$\frac{1}{t}\sum_{j=1}^{n} v(S_j) \leq 1$$

must necessarily be fulfilled for any arbitrary $(q\text{-}\theta)$-covering, and since $v(S_j) = v(S_\ell)$, if $|S_j| = |S_\ell|$ then $\frac{n}{t}\,v(S_j) \leq 1$, or $v(S_j) \leq \frac{|S_j|}{n}$.

§ 3. Quota Games

We shall now consider the so called quota game and shall make use of the methods developed in the preceding section as tools for their investigation.

We recall that a quota game is a game for which there exists a system of real numbers $\omega_1,\ldots,\omega_n$ (not necessarily positive) such that $\omega_1+\ldots+\omega_n = 1$, and such that $v(S) = \sum_{i\epsilon S}\omega_i$ for any $S \subset I_n$ for which $|S| = 2$. This definition was given by Shapley in [4] (see also [5]); the concept was extended by Kalish (see [5]) in the following manner: a game is called an ℓ-quota game, if there exists a system of real numbers $\omega_1,\ldots,\omega_n$ such that $\omega_1+\ldots+\omega_n = 1$ and $v(S) = \sum_{i\epsilon S}\omega_i$ for any $S \subset I_n$ for which $|S| = \ell$.

We shall now investigate the question of the existence of an ℓ-quota (resting on the case of games with Shapely quotas, where $\ell = 2$).

THEOREM 3.1: In order that a game have an ℓ-quota, it is necessary and sufficient that the equation

$$\sum_{j=1}^{m} \lambda_j\, v(S_j) = 1$$

be fulfilled for any arbitrary $(q-\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$, consisting only of ℓ-element sets.

PROOF: If a quota exists, it must satisfy the conditions

$$\begin{cases} \sum_{i\epsilon S_j} \omega_i \geq v(S_j) & \text{for all } S_j \text{ such that } |S_j| = \ell \\ \sum_{i=1}^{n} \omega_i = 1 \end{cases} \tag{3.1}$$

and

$$\begin{cases} \sum_{i\epsilon S_j} \omega_i \leq v(S_j) & \text{for all } S_j \text{ such that } |S_j| = \ell \\ \sum_{i=1}^{n} \omega_i = 1 \end{cases} \tag{3.2}$$

.

By lemma 2.1 system (3.1) is solvable if and only if the condition $\sum_{j=1}^{m} \lambda_j v(S_j) \leq 1$ is fulfilled for any non-negative $\lambda_1,\dots,\lambda_m$ for which $\sum_{j=1}^{m}\lambda_j \underline{S_j} = \underline{I_n}$. We derive analogously that (3.2) is solvable if and only if the condition

$$\sum_{j=1}^{m}\lambda_j v(S_j) \geq 1$$

is fulfilled for any system of non-negative numbers $\lambda_1,\dots,\lambda_m$ for which $\sum_{j=1}^{m} \lambda_j \underline{\dot{S}_j} = \underline{\dot{I}_n}$. Hence, for any such system of numbers, the equation $\sum_{j=1}^{m}\lambda_j v(S_j) = 1$ must therefore be fulfilled. Noting that all such systems of numbers constitute coverings consisting of ℓ-element sets, we obtain the desired result.

<u>THEOREM 3.2:</u> If an ℓ-quota exists, it is unique.

<u>PROOF:</u> By definition, an ℓ-quota must satisfy the equation $\sum_{i\in S} \omega_i = v(S)$ for all S such that $|S| = \ell$. It is therefore sufficient to prove that among the vectors $\underline{\dot{S}}$ for which $|S| = \ell$ there exist n linearly independent ones. We shall point out n such vectors.

$$\underline{\dot{S}_1} = (\underbrace{1,\dots,1}_{\ell}\ 0,\dots,0)\ ,$$

$$\underline{\dot{S}_2} = (0,\underbrace{1,\dots,1}_{\ell}\ 0,\dots,0)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\underline{\dot{S}_{n-1}} = (\underbrace{1,\dots,1}_{\ell-2},\ 0,\dots,0,1,1)$$

$$\underline{\dot{S}_n} = (\underbrace{1,\dots,1}_{\ell-1},\ 0,\dots,0,\ 1)$$

<u>THEOREM 3.3:</u> The quota belongs to the set of imputations if and only if $\sum_{j=1}^{m} \lambda_j v(S_j) = 1$ for all reduced $(q\text{-}\theta)$-coverings $(\lambda_1,\dots,\lambda_m)$ such that $|S| = \ell$ or $|S| = 1$ for $S \in \theta$.

PROOF: Note that aside from the conditions imposed on the quota by systems (3.1) and (3.2), the inequality $\omega_i \geq 0$, $i=1,\dots,n$, must, in this case, also hold. Applying, as above, lemma 2.1, we receive what was required.

Note: Theorem 1 in [4] is equivalent to theorem 3.1 for games with $\ell = 2$.

It is easy to prove conditions analogous to those given in [4], for ℓ-quota games.

THEOREM 3.4: A necessary and sufficient condition for a game to have an ℓ-quota is that $\Sigma\, v(S_j) = \dfrac{(n-1)\dots(n-\ell+1)}{(\ell-1)\,!}$, where the summation is taken over the S_j's for which $|S_j| = \ell$.

PROOF: Consider the sets S_j for which $|S_j| = \ell$ (including those sets for which $v(S_j) = 0$. We shall assume that they are numbered $1,2,\dots,N$. The total number of such sets is:

$$N = C_n^{\ell} = \frac{n!}{\ell!\,(n-\ell)!}\ .$$

Let us construct a covering from just those sets. Each member of I_n will in this manner be "covered" exactly $\dfrac{(n-1)\dots(n-\ell+1)}{(\ell-1)!} = a$ times, i.e. $\sum_{j=1}^{N} \frac{1}{a} \cdot \underline{\bar{S}_j} = \ddot{I}_n$, and consequently, $(\underbrace{\frac{1}{a},\dots,\frac{1}{a}}_{N}, 0,\dots,0)$ is a $(q-\theta)$ covering (as a rule, not reduced). By the conditions for an ℓ-quota's existence we have:

$$\Sigma\, v(S_j) = a$$

which is equivalent to the assertion of the theorem.

§4. The Relation Between the Core and the Von-Neumann-Morgenstern Solution.

Existence Theorems

We consider the conditions under which the core is a solution. By lemma 1.1 the solution is in this case unique.

THEOREM 4.1: If the core is a solution then it intersects each of the hyperplanes $a_i = 0$, $i=1,2,\ldots,n$ (i.e. the core has "enough" points lying on the border of the set A of all imputations).

PROOF: Suppose, on the contrary, that there exists a hyperplane $a_{i_0} = 0$ such that $a_{i_0} > 0$ for all $\alpha \in U$. Set $a^*_{i_0} = \min\limits_{\alpha\in U} a_{i_0}$; the minimum is attained, because the core is a closed set. Denote by $\alpha^* = (a^*_1,\ldots,a^*_n)$ an imputation for which the minimum $a^*_{i_0}$ is attained. Let $0 < \epsilon < a^*_{i_0}$. Consider the imputation $\alpha^0 = (a^0_1,\ldots,a^0_n)$, where

$$a^0_1 = a^*_1 + \epsilon ;$$

$$a^0_i = a^*_i \quad \text{for } i \neq 1, \quad \text{and} \quad i \neq i_0 ;$$

$$a^0_{i_0} = a^*_{i_0} - \epsilon .$$

Since $a^0_{i_0} < \min\limits_{\alpha\in U} a_{i_0}$, $\alpha^0 \notin U$; this means that there exists an $S_0 \epsilon \mathcal{N}$ (if there exist many such sets we may pick a n y one of them arbitrarily, and label it S_0) for which $\sum\limits_{i\epsilon S_0} a^0_i < v(S_0)$. Suppose S_0 does not contain i_0. Then

$$\sum_{i\epsilon S_0} a^0_i \geq \sum_{i\epsilon S_0} a^*_i \geq v(S_0) .$$

It follows from this and from the preceding inequality that $i_0 \epsilon S_0$.

In order that an imputation $\beta \in U$ dominate α^0, it is necessary that the equation $\sum_{i \in S_0} b_i = v(S_0)$ be fulfilled.

In fact, $\sum_{i \in S_0} b_i \geq v(S_0)$ for all $\beta \in U$, but dominance may take place only with respect to an effective set, i.e. only when $\sum_{i \in S_0} b_i \leq v(S_0)$. But $\alpha^* \in U$. Hence $\sum_{i \in S_0} a_i^* \geq v(S_0)$ and

$$\sum_{i \in S_0} b_i = v(S_0) \leq \sum_{i \in S_0} a_i^* = a_{i_0}^* + \sum_{i \in S \setminus i_0} a_i^0 .$$

Taking into account that $b_{i_0} \geq a_{i_0}^*$, we receive:

$$\sum_{i \in S_0 \setminus i_0} b_i \leq \sum_{i \in S_0 \setminus i_0} a_i^0 .$$

This means that for some i_1, $b_{i_1} \leq a_{i_1}^0$. Hence no imputation β in U may dominate α^0 with respect to S_0. Since S_0 was arbitrarily chosen from the sets with respect to which dominance could take place, U therefore is not a solution.

<u>COROLLARY 4.1:</u> A core of dimension 0 cannot be a solution.

<u>PROOF:</u> Since the core is a convex set, a core of dimension 0 necessarily consists of one imputation. By theorem 4.1, this imputation can only equal the vector $(0,\ldots,0)$. This vector, however, does not belong to the set of imputations.

Let us examine the sets $S_j' = I_n \setminus S_j$, where $|S_j| > 1$. Denote the system of all such sets by $\mathfrak{N}'$. Certain subsets of I_n may now be regarded as members of the system $\mathfrak{N}$ or of the system $\mathfrak{N}'$. Extend the characteristic function $v(S)$ onto the system $\mathfrak{N}'$ of subsets of I_n, setting $v(S_j') = 1 - v(S_j)$.

Denote by $\mathcal{N}_j$ the system of sets $\mathcal{N}_j = \{\mathcal{N}, S'_j\}$ ($S'_j \in \mathcal{N}'$, i.e. $v(S'_j) = 1 - v(S_j)$), and consider the $(q-\theta_j)$-coverings $(\lambda_1,\ldots,\lambda_m, \nu_j)$ of the system of sets $\theta_j \subset \mathcal{N}_j$ (ν_j corresponds to S'_j). We shall call such a covering a $(q-\theta_j)$-quasi-covering; we shall also occasionally refer to it simply as a covering, since a covering is a special case of a quasi-covering, i.e. a quasi-covering becomes a covering when $\nu_j = 0$.

Since in many questions concerning the solution, single-element sets play a special role (they take no part in dominance), it would be convenient in the discussions that lie ahead, to single out the components of coverings corresponding to these sets. We shall therefore denote quasi-coverings, and thus coverings as well, by $(\lambda_1,\ldots,\lambda_k,\mu_1,\ldots,\mu_n, \nu_j)$ where $k + n = m$ and μ_p is the component corresponding to the single-element set consisting of the element p . We shall assume that the sets S_j are denumerated so that $S_{k+p} = (\underbrace{0,\ldots,0, 1}_{p}, 0,\ldots,0)$.

Quasi-coverings may, just as coverings, be reduced in form. Lemma 3.1 and its corollaries are valid for quasi-coverings. Lemma 1.4 takes the form:

<u>LEMMA 4.1:</u> For any fixed j , $0 \leq j \leq k$, the $(q-\theta_j)$-quasi-coverings when regarded as points of $m + 1$-dimensional Euclidean space, form a closed, bounded, convex set; the extremal points of this set, and only they, all constitute reduced quasi-coverings.

<u>THEOREM 4.2:</u> A sufficient condition for a game Γ to have a unique solution is that the inequality

$$\sum_{\ell=1}^{k} \lambda_\ell v(S_\ell) + \nu_j(1-v(S_j)) + \mu^{(j)} v(S_j) \leq 1 , \qquad (4.1)$$

where $\mu^{(j)} = \max_{i \in S_j} \mu_i$, be fulfilled for any arbitrary $(q-\theta_j)$-quasi-covering $(\lambda_1,\ldots,\lambda_k, \mu_1,\ldots,\mu_n, \nu_j)$.

PROOF: We shall prove that under these conditions there exists a solution which coincides with the core; then, by lemma 1.1, the solution is unique.

Since the core is always included within the solution, it is sufficient to prove that when the pre-conditions of the theorem are fulfilled, the solution is included within the core, i.e., that for any $\gamma \in A \setminus U$ there exists some $\alpha \in U$, such that $\alpha \succ \gamma$. Let $\alpha = (a_1,\ldots,a_n)$, and $\gamma = (c_1,\ldots,c_n) \in A \setminus U$.

Since $\gamma \notin U$, the conditions of lemma 1.2 are not fulfilled; this means there exists some S_{j_0} for which $\sum_{i \in S_{j_0}} c_i < v(S_{j_0})$. In order that $\alpha \succ \gamma$, it is necessary and sufficient that for at least one such S_{j_0}, the system of inequalities

$$\underline{\alpha\ \ddot{S}_j} \geq v(S_j)\ , \quad j=1,\ldots,m; \qquad (4.2)$$

$$\underline{\alpha\ \ddot{S}_j} \leq v(S_{j_0})\ ; \qquad (4.2')$$

$$a_i > c_i, \quad i \in S_{j_0} \qquad (4.2'')$$

be satisfied. The fulfillment of (4.2) is equivalent, by lemma 1.2, to the condition that $\alpha \in U$. The fulfillment of (4.2') and (4.2") is equivalent to the condition that $\alpha \succ \gamma$ with respect to S_{j_0}.

Because of this, in order that the solution be included within the core and that U be consequently a solution, it is sufficient to require that for any $\gamma \in A \setminus U$, the system of inequalities (4.2 - 4.2") have a solution for any S_{j_0} such that $\sum_{i \in S_{j_0}} c_i < v(S_{j_0})$.

Making use of the relation $\alpha(\underline{S}'_{j_0} + \underline{S}_{j_0}) = \alpha \underline{I}_n = 1$, let us rewrite (4.2') in the form $\alpha \underline{S}'_{j_0} \geq 1 - v(S_{j_0})$. Note that the requirement that the system of inequalities be fulfilled for any γ in $A \backslash U$ and for every S_{j_0} corresponding to γ such that $\sum_{i \in S_{j_0}} c_i < v(S_{j_0})$ may be replaced by the equivalent requirement that the system of inequalities be fulfilled for each S_j, $j=1,\ldots,m$, and for every γ corresponding to S_j such that $\sum_{i \in S_{j_0}} c_i \leq v(S_{j_0})$. It is clear that this does not weaken (nor does it strengthen) the requirement.

Thus, in order that U be a solution it is sufficient that for all $S_{j_0} \in \mathfrak{N}$ and for every imputation $\beta = (b_1,\ldots,b_n)$ such that $\sum_{i \in S_{j_0}} b_i \leq v(S_{j_0})$ the system

$$\begin{aligned} &\underline{S_j}\, \alpha \geq v(S_j), \quad j=1,\ldots,k; \\ &a_i \geq 0, \quad i \in I_n \backslash S_{j_0}; \\ &a_i \geq b_i, \quad i \in S_{j_0}; \\ &\underline{S_{j_0}}\, \alpha \geq 1 - v(S_{j_0}); \\ &\underline{I_n}\, \alpha = 1. \end{aligned} \tag{4.3}$$

(4.3) satisfies the conditions of lemma 2.1. A solution to (4.3) exists, according to this lemma, if for all systems of numbers $\lambda_1 \geq 0,\ldots,\lambda_k \geq 0$, $\mu_1 \geq 0,\ldots,\mu_n \geq 0$, $\nu'_{j_0} \geq 0$, for which

$$\sum_{j=1}^{k} \lambda_j \underline{S}_j + \sum_{p=1}^{n} \mu_p \underline{S}_{k+p} + \nu_{j_0} \underline{S}_{j_0} = \underline{I}_n \tag{4.4}$$

the condition

$$\varphi(\lambda, \mu, \nu_{j_0}) = \sum_{j=1}^{k} \lambda_j v(S_j) + \sum_{p \in S_{j_0}} \mu_p b_p + \nu'_{j_0}(1 - v(S_{j_0})) \leq 1 \tag{4.5}$$

be fulfilled. Note, that (4.4) obviously implies that the system $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n, \nu_{j_0})$ form a $(q-\theta_{j_0})$ quasi-covering. Inasmuch as (4.3) must be fulfilled for all S_{j_0} and for every system of numbers $b_p \geq 0$ satisfying the condition

$$\sum_{p \in S_{j_0}} b_p \leq v(S_{j_0}) ,$$

we must therefore investigate the behavior of the function $\varphi(\lambda, \mu, \nu_{j_0})$ on the set of all $(q-\theta_{j_0})$-quasi-coverings. Since this set is convex (see lemma 4.1) and since the function is linear, then in order for the inequality $\varphi(\lambda, \mu, \nu_{j_0}) \leq 1$ to be fulfilled it is sufficient that it be fulfilled for the extremal points of the set of $(q-\theta_{j_0})$-quasi coverings, i.e. for the reduced quasi-coverings.

If we substitute $\mu^{(j_0)} = \max_{p \in S_{j_0}} \mu_p$ for every μ_p appearing in (4.5), then, bearing in mind that $\sum_{i \in S_{j_0}} b_i \leq v(S_{j_0})$, we obtain the conditions of the theorem.

Note: Condition (4.1) can often be weakened. For example, if there exists an $S_{j_1} \subset S_{j_0} \cap T$, where, as always, $T = \{i: \mu_i = 0\}$, then condition (4.1) takes the form $\sum_{j=1}^{k} \lambda_j v(S_j) + \nu_{j_0}(1-v(S_{j_0})) + \mu^{(j)}(v(S_{j_0}) - v(S_{j_1})) \leq 1$.

In fact, in (4.5)

$$\sum_{p \in S_{j_0}} \mu_p b_p = \sum_{p \in S_{j_0} \setminus T} \mu_p b_p \leq \mu^{(j)} \sum_{p \in S_{j_0} \setminus T} b_p .$$

But since dominance can be considered only with respect to essential sets (see § 1) and $S_{j_1} \subset S_{j_0}$, then $\sum_{p \in S_{j_1}} b_p \geq v(S_{j_1})$; therefore $\sum_{p \in S_{j_0} \setminus T} b_p \leq v(S_{j_0}) - v(S_{j_1})$, because $S_{j_1} \subset T$.

Simpler though more restrictive conditions, whose fulfillment is sufficient for the existence of a unique solution, may be expressed in the form of an evaluation for $v(S)$. Let us first introduce some new notations. Denote by D the matrix formed by the vectors $\underline{S_1},\ldots,\underline{S_k}$ and $\underline{I_n}$, i.e.

$$D = \left\| \begin{matrix} s_1^{(1)} & s_2^{(1)} & \ldots\ldots & s_k^{(1)} & 1 \\ & & \cdots\cdots & & \\ s_1^{(n)} & s_2^{(n)} & \ldots\ldots & s_k^{(n)} & 1 \end{matrix} \right\| .$$

Denote by r the rank of this matrix.

<u>THEOREM 4.3:</u> In order that a game Γ have a unique solution it is sufficient that

$$v(S) \leq \frac{1}{r}, \qquad S \subset I_n .$$

<u>PROOF:</u> It is sufficient to prove that if $v(S_j) \leq \frac{1}{r}$, $j=1,\ldots,k$, then the condition of theorem 4.2 is fulfilled, i.e. that for any arbitrary $(q-\theta_j)$-quasi-covering $(\lambda_1,\ldots,\lambda_k, \mu_1,\ldots,\mu_n, \nu_j)$, the condition (4.1), or (4.5), is satisfied. In other words, it is enough to prove that

$$\varphi(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{k} \lambda_j\, v(S_\ell) + \nu_j(1-v(S_j)) + \mu^{(j)}\, v(S_j) \leq 1,$$

or

$$\varphi(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{k} \lambda_\ell v(S_\ell) + \nu_j(1-v(S_j)) + \sum_{p\in S_j} \mu_p\, b_p \leq 1 .$$

1. Let us first consider the case where $\nu_j = 0$; assume that the numbers are so denumerated, that $\lambda_1 > 0,\ldots \lambda_{r'} > 0$, $\lambda_{r'+1} = \ldots = \lambda_k = 0$ and $\mu_1 > 0,\ldots,\mu_{n'} > 0$, $\mu_{n'+1} = \ldots = \mu_n = 0$ $(r' + n' = q)$. Then the condition

(4.1) becomes

$$\varphi_1(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{r'} \lambda_\ell \; v(S_\ell) + \mu^{(j)} v(S_j) \leq 1 .$$

There are two possibilities:

a) $r' < r$; we then receive the following evaluation for $\varphi_1(\lambda,\mu,\nu_j)$:

$$\varphi_1(\lambda,\mu,\nu_j) \leq \sum_{\ell=1}^{r'} v(S_\ell) + v(S_j) \leq \frac{r'}{r} + \frac{1}{r} \leq 1 .$$

This is because $\lambda_\ell \leq 1$ and $\mu^{(j)} \leq 1$.

b) $r' = r$. In this case consider the vectors $\underline{S_1},\ldots,\underline{\ddot{S}_r}$. They are linearly independent (the covering is reduced). But since the rank of D is equal to r , $\underline{I_n}$ is a linear combination of the vectors, i.e.

$$\sum_{\ell=1}^{r} \eta_\ell \, \underline{S_\ell} = \underline{I_n}$$

By definition of $(q\text{-}\theta)$-covering

$$\sum_{\ell=1}^{r} \lambda_\ell \underline{\ddot{S}_\ell} + \sum_{p=1}^{n'} \mu_p \; \underline{\ddot{S}_{k+p}} = \underline{\dot{I}_n} , \qquad r + n' = q .$$

Substituting for $\underline{I_n}$ the expression $\sum_{\ell=1}^{r} \eta_\ell \underline{S_\ell}$ we receive:

$$\sum_{\ell=1}^{r} (\lambda_\ell - \eta_\ell) \, \underline{S_\ell} + \sum_{p=1}^{n'} \mu_p \; \underline{\ddot{S}_{k+p}} = \underline{0} ;$$

but $\underline{S_1},\ldots,\underline{\ddot{S}_r}, \underline{\ddot{S}_{k+1}},\ldots,\underline{\ddot{S}_{k+n}}$ are linearly independent, since the covering is reduced; hence

$$\lambda_\ell - \eta_\ell = 0 , \; \ell = 1,\ldots,r, \quad \text{and} \quad \mu_p = 0, \; p=1,\ldots,n' ,$$

and since, aside from this, $\mu_{n'+1} = \ldots. = \mu_n = 0$, it follows that $\mu_p = 0$, $p = 1,\ldots,n$, and

$$\varphi_1(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{r} \lambda_\ell \, v(S_\ell) \leq \frac{1}{r} \cdot r = 1 .$$

2. Let us now assume that $\nu_j > 0$.

We shall examine the corresponding $(q-\theta_j)$-quasi-covering $(\lambda_1,\dots,\lambda_k,\ \mu_1,\dots,\mu_n,\ \nu_j)$. We shall assume the sets to be renumbered so that $\lambda_1 > 0,\dots,\lambda_{\tau-1} > 0$, $\lambda_\tau = \dots = \lambda_n = 0$, $\mu_1 = \dots = \mu_{\bar{n}} = 0$, $\mu_{\bar{n}+1} > 0,\dots,\mu_n > 0$. Since by assumption $\nu_j \neq 0$,

$$n-\bar{n} + \tau-1 + 1 = n-\bar{n} + \tau = q .$$

Since, furthermore, $q \leq n$, then $\bar{n} \geq \tau$.

Let us write the vectorial equation with the new numbering:

$$\sum_{\ell=1}^{\tau-1} \lambda_\ell \underline{S_\ell} + \sum_{p=\bar{n}+1}^{n} \mu_p \underline{\ddot{S}_{k+p}} + \nu_j \underline{S'_j} = \underline{\dot{I}_n} .$$

(we recall that $\underline{\ddot{S}_{k+p}} \doteq (\underbrace{0,\dots,0}_{p}, 1, 0,\dots,0))$. Expressed by coordinates this reads:

$$\begin{aligned} &\sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} + \nu_j s_j'^{(i)} = 1, \quad i=1,2,\dots,\bar{n} . \\ &\sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} + \nu_j s_j'^{(i)} + \mu_i = 1, \quad i=\bar{n}+1,\dots, n. \end{aligned} \tag{4.6}$$

Since the vectors corresponding to the non-zero components of the covering are linearly independent and since the equations whose indices exceed $\bar{n}$ are $n - \bar{n} = q - \tau$ in number, this means that among the first $\bar{n}$ there exist τ linearly independent ones. Hence, the set of covering components appearing within them comprises a unique solution (see lemma 1.3). Consider the system consisting of these equations. The system may be split up into two parts:

$$\sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} = 1 \quad (i \in S_j) \quad \text{and} \quad \sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} + \nu_j = 1 \quad (i \notin S_j). \tag{4.7}$$

Let us evaluate, in this case, the function

$$\varphi(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{\tau-1} \lambda_\ell v(S_\ell) + \sum_{p\in S_j} \mu_p b_p + \nu_j(1-v(S_j)) .$$

Split the summation $\sum_{\ell=1}^{\tau-1} \lambda_\ell v(S_\ell)$ into two parts: Let $\Sigma' \lambda_\ell v(S_\ell)$ be taken over those S_ℓ which are included within S_j, and let $\Sigma'' \lambda_\ell v(S_\ell)$ be taken over those S_ℓ for which $S_\ell \cap (I_n \setminus S_j) \neq \Lambda$. (In the first case λ_ℓ does not appear together with ν_j in a single equation in (4.7), since ν_j is a coefficient for $\underline{I_n \setminus S_j}$; in the second case each λ_ℓ appears together with ν_j in at least one equation in (4.7).

Let us first evaluate the function

$$\psi(\lambda,\mu,\nu_j) = \Sigma' \lambda_q v(S_q) + \sum_{p\in S_j} \mu_p b_p$$

Since $S_q \subset S_j$, then by the essentiality of S_j (see §1), it follows that $\sum_{i\in S_q} b_i \geq v(S_q)$; therefore

$$\psi(\lambda,\mu,\nu_j) \leq \Sigma' \lambda_q \sum_{i\in S_q} b_i + \sum_{p\in S_j} \mu_p b_p .$$

Reducing similar terms, we notice that λ_q is a coefficient of b_i if $i \in S_q$; therefore

$$\psi(\lambda,\mu,\nu_j) = \sum_{i\in S_q} b_i \Big(\sum_{q:S_q \subset S_j} \lambda_q s_q^{(i)} + \mu_i \Big) + \sum_{p\in S_j\setminus S_q} \mu_p b_p$$

where the $s_q^{(i)}$'s are the coordinates of S_q, i.e.

$$s_q^{(i)} = \begin{cases} 0, & \text{if } i \notin S_q, \\ 1, & \text{if } i \in S_q . \end{cases}$$

(According to Gillies' formulation $v(S) < \frac{1}{n}$).

Theorem 4.3 turns out to be a proof of the following known fact.

COROLLARY 4.4: The set of n-person games having a (unique) solution has the same dimension as the set of all n-person games.

For this reason, the probability that an arbitrarily chosen n-person game have a solution is positive.

§ 5. Examples

In the way of example let us first consider a game with a core of dimension 1.

THEOREM 5.1: A game may have a core of dimension 1 that also turns out to be a solution only if the following conditions are fulfilled.

1) There exist sets

$$M = \{1,2,\ldots,k\} \subset I_n, \quad M \neq \Lambda$$

$$N = \{k+1,\ldots,\ell\} \subset I_n, \quad N \neq \Lambda$$

and numbers

$$a_1 > 0,\ldots,a_k > 0, \quad \sum_{i=1}^{k} a_i = 1$$

$$b_1 > 0,\ldots, b_\ell > 0, \quad \sum_{i=1}^{\ell} b_i = 1,$$

such that

$$v(S_j) \leq \min \left(\sum_{S_j \cap M} a_i ; \sum_{S_j \cap N} b_i \right);$$

2) for any $0 \leq i \leq \ell$ there exist at least two sets $S_{i_1}, S_{i_2} \subset M \cup N$ such that $i \in S_{i_1} \cap S_{i_2}$ and $v(S_{i_1}) = v(S_{i_2}) = a_i$, if $i \leq k$, and $v(S_{i_1}) = v(S_{i_2}) = b_i$, if $i > k$.

Since

$$\sum_{q:S_q \subset S_j} s_q^{(i)} \lambda_q + \mu_i \leq \sum_{q=1}^{\tau-1} s_q^{(i)} \lambda_q + \mu_i \leq 1 \quad \text{and} \quad \mu_p \leq 1 \text{ , therefore}$$

$$\psi(\lambda,\mu,\ \nu_j) \leq \sum_{i \in \cup S_q} b_i + \sum_{p \in S_j \setminus S_q} b_p = \sum_{i \in S_j} b_i \leq v(S_j) \quad .$$

Thus

$$\varphi(\lambda,\mu,\nu_j) \leq \Sigma'' \ \lambda_\ell v(S_\ell) + v(S_j) + \nu_j(1-v(S_j)) =$$

$$= \Sigma'' \ \lambda_\ell v(S_\ell) + \nu_j + (1-\nu_j) v(S_j) \ ,$$

where Σ'' as formerly stated, is a summation over the ℓ's for which λ_ℓ appears in at least one equation of (4.7) containing ν_j and hence $\lambda_\ell \leq 1-\nu_j$. Since the number of different λ_ℓ's does not exceed $\tau-1$ and since $v(S_\ell) \leq \frac{1}{r}$, then

$$\Sigma'' \lambda_\ell v(S_\ell) \leq (1-\nu_j) \ \Sigma'' v(S_\ell) \leq (1-\nu_j) \frac{\tau-1}{r} \quad .$$

Thus,

$$\varphi(\lambda,\mu,\nu_j) \leq (1-\nu_j) \frac{\tau-1}{r} + \nu_j + \frac{1-\nu_j}{r} = \frac{(1-\nu_j)\tau}{r} + \nu_j \ ,$$

and since $\frac{\tau}{r} \leq 1$ and $1-\nu_j > 0$, therefore $\varphi(\lambda,\mu,\ \nu_j) \leq 1-\nu_j + \nu_j = 1$.

<u>COROLLARY 4.2:</u> If the number k of coalitions S, for which $v(S) > 0$, is less than n, then a sufficient condition for the existence of a unique solution is the fulfillment of the inequality

$$v(S) \leq \frac{1}{k} \ .$$

From Theorem 4.3 and the inequality $r \leq n$ the following corollary ensues.

<u>COROLLARY 4.3:</u> (Gillies' theorem, see [2]). In order that a game Γ have a unique solution (coinciding with the core) it is sufficient that $v(S) \leq \frac{1}{n}$.

The proof is carried through with the aid of Theorem 4.1. Since no difficulties present themselves when this procedure is performed, and since the theorem itself is of no particular importance, the proof will be omitted.

THEOREM 5.2: In order that a game Γ have a core of dimension 1 that is also a solution, it is sufficient that there exist sets $M, N \subset I_n$ of a single cardinality k, such that $v(S) \leq \frac{\min(|S \cap M|, |S \cap N|)}{k}$ and $v(S) = \frac{\min(|S \cap M|, |S \cap N|)}{k}$, for $|S| = 2$.

PROOF: Since the imputations $\alpha = (\underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k}, 0, \ldots, 0)$ and $\beta = (\underbrace{0, \ldots 0}_{k}, \underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k}, 0, \ldots,$ are contained in the core, the core therefore exists and is of dimension $\tau \geq 1$. We shall now show that when the conditions of the theorem are fulfilled the core is of dimension 1 and turns out to be a solution.

We construct the following chain of two-member sets:

$$S_1 = \{1, k+1\}, \ S_2 = \{k+1, 2\}, \ S_3 = \{2, k+2\}, \ldots, \ S_{2k-1} = \{k-1, 2k\}$$

$$S_{2k} = \{2k, 1\} .$$

Let

$$\lambda_j^0 = \frac{1}{2}, \quad \text{if } 0 \leq j \leq 2k;$$

$$\lambda_j^0 = 1, \quad \text{if } |S_j| = 1, \ j \in I_n \setminus (M \cup N),$$

$$\lambda_j^0 = 0 \quad \text{for all remaining } S_j .$$

Then $(\lambda_1^0, \ldots, \lambda_m^0)$ is an $(n-\theta)$-covering. Since $\Sigma \, \lambda_j^0 \, v(S_j) = 1$ and since the number of linearly independent vectors in the system θ is equal to n-1, then $\tau \leq n-n-1 = 1$. This, together with the previously derived inequality $\tau \geq 1$ gives us $\tau = 1$.

Let us examine the conditions imposed on the imputations in the core by the existence of the covering λ^{o}. Let $\gamma = (c_1,\ldots,c_n) \epsilon U$. Then

$$c_1 + c_{k+1} = \frac{1}{k} ,$$

$$c_{2k} + c_1 = \frac{1}{k} ,$$

$$c_{2k+1} = \ldots = c_n = 0 .$$

Denote c_{k+1} by t ; then

$$c_i = \frac{1}{k} - t , \qquad i = 1,\ldots,k;$$

$$c_i = t , \qquad i = k+1,\ldots, 2k;$$

$$c_i = 0, \qquad i = 2k+1,\ldots,n .$$

It is easy to show that

$$U = \{u(t) = \underbrace{\frac{1}{k} - t,\ldots,\frac{1}{k} - t}_{k}, \underbrace{t,\ldots,t}_{k}, 0,\ldots,0), \ 0 \le t \le \frac{1}{k} \}$$

is the core. We will show that U is a solution. In fact, let $\delta = (d_1,\ldots,d_n)$. If for all $S_1,\ldots,S_{2k}$ $\sum_{S_\ell} d_i = v(S_\ell) = \frac{1}{k}$, then $\delta \epsilon U$. If, however, for some S_{ℓ_0}

$$\sum_{S_{\ell_0}} d_i = d_{i\ell_0} + d_{j\ell_0} < v(S_{\ell_0}) = \frac{1}{k} ,$$

then there exists a t' such that

$$d_{i\ell_0} < \frac{1}{k} - t'$$

$$d_{j\ell_0} < t'$$

(for example, $t' = d_{j\ell_0} + \epsilon$, where $\epsilon < \frac{1}{k} - d_{i\ell_0} - d_{j\ell_0}$), i.e. $\delta_{S_{\ell_0}} \prec u(t')$. The theorem is thus proven.

Example. Symmetric Shapley market games.

A symmetric market game (see [6]) is a game with the characteristic function

$$v(S) = \min(|S \cap M|, |S \cap N|) .$$

where $M \cup N = I_n$. The game is not normal, and $v(I_n) = \min(|M|, |N|)$. Let $|M| \leq |N|$ and $|M| = k$; then $v(I_n) = k$. In (0-1) reduced form

$$v(S) = \frac{\min(|S \cap M|, |S \cap N|)}{k} .$$

The game has a non-empty core, since

$$(\underbrace{\tfrac{1}{k}, \ldots, \tfrac{1}{k}}_{k}, 0, \ldots, 0) \in U .$$

If $|M| = |N| = k$, then by theorem 5.2 the game has a core of dimension 1, which is also a solution; this core was investigated by Shapley also. If $|M| < |N|$, then by examination of the coverings consisting of two-member sets, we immediately receive that the core is of dimension 0. In this case, as shown in [6], a solution exists but is necessarily not unique. Hence, the following assertion is valid: The solution of a symmetric bargaining game is unique if and only if it coincides with the core. It is possible that an analogous assertion is true in more general cases as well.

As an application of the general theory let us consider four-person games.

Example: Investigation of four-person games.

Let $I_n = \{1,2,3,4\}$. We shall denote the coalitions by S_i, S_{ij}, S_{ijk}, where the lower index is the enumeration of the coalition's members; for this reason i,j,k,ℓ shall henceforth always be different.

Using corollary 1.4, we enumerate all reduced $(q\text{-}\theta)$-coverings:

$$\text{I.}\quad \underline{S}_{ij} + \underline{S}_{k\ell} = \underline{I}_n , \quad \{i,j,k,\ell\} = I_n$$

$$\text{II.}\quad \frac{1}{3}\,(\underline{S}_{123} + \underline{S}_{124} + \underline{S}_{134} + \underline{S}_{234}) = \underline{I}_n$$

$$\text{III.}\quad \frac{1}{2}\,(\underline{S}_{ij} + \underline{S}_{i\ell} + \underline{S}_{j\ell k} + \underline{S}_{k}) = \underline{I}_n$$

$$\text{IV.}\quad \frac{1}{3}\,(\underline{S}_{ij} + \underline{S}_{ik} + \underline{S}_{i\ell}) + \frac{2}{3}\,\underline{S}_{jk\ell} = \underline{I}_n$$

$$\text{V.}\quad \frac{1}{2}\,(\underline{S}_{ijk} + \underline{S}_{ij\ell} + \underline{S}_{k\ell}) = \underline{I}_n$$

These are all the reduced $(q\text{-}\theta)$-coverings with the exception of the "trivial" ones (the coverings that yield trivial evaluations of the characteristic function.

By theorem 4.2, for the core to exist it is necessary and sufficient that:

$$\text{I.}\quad v(S_{ij}) + v(S_{k\ell}) \leq 1 .$$

$$\text{II.}\quad v(S_{123}) + v(S_{124}) + v(S_{134}) + v(S_{234}) \leq 3 .$$

$$\text{III.}\quad v(S_{ij}) + v(S_{ik}) + v(S_{jk\ell}) \leq 2 .$$

$$\text{IV.}\quad v(S_{ij}) + v(S_{ik}) + v(S_{i\ell}) + 2v(S_{jk\ell}) \leq 3 .$$

$$\text{V.}\quad v(S_{ijk}) + v(S_{ij\ell}) \leq 2 .$$

In order to write the necessary conditions for the existence of the solution it is necessary, for each and every S to examine S, in its second quality, i.e. with the characteristic function $1-v\,(I_n \setminus S)$. We must do this for each of the conditions I-V.

In view of the large number of such conditions, let us write them for the special case of a symmetric game. We receive that for any $0 \leq \epsilon \leq \frac{1}{6}$, any symmetric four-person game with a characteristic function satisfying the conditions

$$v(S_{ij}) \leq \frac{1}{3} + \epsilon ; \quad v(S_{ijk}) = \frac{2}{3} - \epsilon ,$$

has a unique solution (coinciding with the core).

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