Erratum: Geometry of discrete quantum computing

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We correct Sec. 5.4 of our paper [1]. There are several related mistakes that all stem from an incorrect use of Eq. (27) in a discrete setting with modular arithmetic. Eq. (27) defines purity from which the definitions of unentangled, partially entangled, and maximally entangled states were derived in the field of complex numbers. In finite fields, the definitions are more subtle.

Purity, Product States, and Unentangled States. The paper correctly identifies unentangled states with product states. In the continuous case, a state has maximal purity iff it is a product state. In the discrete case, product states have maximal purity, but the converse does not necessarily hold.

Purity and Maximally Entangled States. The paper defines maximally entangled states as those whose purity is 0. This definition needs to be modified in the discrete case to avoid identifying as maximally entangled states whose purity is a multiple of the field characteristic.

The results in other sections do not depend on Eq. (27): the corrected version of Sec. 5.4 below has no influence on the rest of the article.

5.4. Maximal entanglement

In the case of finite fields, we need to modify this definition because modular arithmetic confuses some of the identifications.

For instance, although product states and maximal purity can be identified when using complex arithmetic, there exist, in the discrete case, non-product states that also have maximal purity.

The first step in adapting Eq. (27) to the discrete case is to eliminate the $1/n$ factor to avoid dividing by zero:

$$\overline{P_{\mathfrak{h}}} = \sum_{j=1}^{n} \overline{P_{\mathfrak{h}}^{j}}, \qquad \text{with } \overline{P_{\mathfrak{h}}^{j}} = \sum_{\mu=x,y,z} \langle \sigma_{\mu}^{j} \rangle^2 \ , \tag{40}$$

where $\overline{P_{\mathfrak{h}}} \in \mathbb{F}_p$.

For example, consider the normalized product state $|\Psi_0\rangle$ and the normalized entangled state $|\Psi_1\rangle$ over $\mathbb{F}_{3^2}$:

$$|\Psi_0\rangle \ = \ |0\rangle \otimes \frac{|0\rangle - |1\rangle}{1-\mathrm{i}} \otimes \frac{|0\rangle - |1\rangle}{1-\mathrm{i}} \ , \tag{41}$$

$$|\Psi_1\rangle \ = \ |011\rangle + |100\rangle + |101\rangle + |110\rangle \ . \tag{42}$$

In both cases $\overline{P_{\mathfrak{h}}^{j}} = 1$ for all j: the states are not distinguished by purity.

$|\Psi_0\rangle$ also serves an example that a product state has $\overline{P_{\mathfrak{h}}} = 0$ because of modular arithmetic. In the continuous case, $\overline{P_{\mathfrak{h}}} = 0$ is equivalent to

$$\forall j, \forall \mu, \langle \sigma_{\mu}^{j} \rangle = 0 \ . \tag{43}$$

Eq. (43) defines maximally entangled states over $\mathbb{F}_{p^2}$ as well. From this definition, the Bell states are maximally entangled, while a normalized state $|\Psi_2\rangle$ over $\mathbb{F}_{3^2}$ is not maximally entangled although $\overline{P_{\mathfrak{h}}^{j}} = 0$ for all j, where

$$|\Psi_2\rangle \ = \ |01\rangle + |10\rangle + (-1-\mathrm{i})\,|11\rangle \ . \tag{44}$$

Computing some examples for various n and small values of p, one can verify explicitly that unit-norm product states for $n = 2$, $p = \{3, 7, 11, 19, \ldots\}$ occur with frequency

$$(p+1)p^2(p-1)^2 = \{144, 14\,112, 145\,200, 2339\,280, \ldots\} \ , \tag{45}$$

and for general n, $(p+1)p^n(p-1)^n$.

The irreducible state counts are reduced by $(p+1)$, giving

$$p^2(p-1)^2 = \{36, 1764, 12\,100, 116\,964, \ldots\} \ , \tag{46}$$

and in general for n-qubits, there are $p^n\,(p-1)^n$ instances of product pure states.

Repeating the computation, we find maximally entanglement states with frequencies for two qubits of

$$p\left(p^2-1\right)(p+1) = \{96, 2688, 15\,840, 136\,800, \ldots\} \ . \tag{47}$$

The irreducible state counts for maximal entanglement are reduced by $(p+1)$, giving for $n = 2$

$$p\left(p^2-1\right) = \{24, 336, 1320, 6840, \ldots\} \ . \tag{48}$$

For three-qubits, there are $p^3\left(p^4-1\right)(p+1)$ instances of pure maximally entangled states. For four-qubits and $p = 3$, there are $2195\,538\,048$ instances of pure maximally entangled states, while the general formula for four-qubit states remains unclear.

Yu-Tsung says:

$$\begin{aligned} 2195\,538\,048 &= 4 \cdot 548\,884\,512 \ , \\ 548\,884\,512 &= 2^5 \cdot 3^7 \cdot 11 \cdot 23 \cdot 31 \ . \end{aligned}$$

The irreducible state counts for maximal entanglement for $n = 4$ and $p = 7$ is about 8.22×10^{14}.

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References

[1] Andrew J Hanson, Gerardo Ortiz, Amr Sabry, and Yu-Tsung Tai. Geometry of discrete quantum computing. *Journal of Physics A: Mathematical and Theoretical*, 46(18):185301, 2013.