Erratum: Geometry of discrete quantum computing

Andrew J. Hanson
School of Informatics and Computing, Indiana University, Bloomington, IN 47405, U.S.A

Gerardo Ortiz
Department of Physics, Indiana University, Bloomington, IN 47405, U.S.A

Amr Sabry
School of Informatics and Computing, Indiana University, Bloomington, IN 47405, U.S.A

Yu-Tsung Tai
Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A
School of Informatics and Computing, Indiana University, Bloomington, IN 47405, U.S.A

Section 5.4 of our paper [3] requires a clarification and a correction.

Clarification: Unentangled vs. product states. In conventional quantum mechanics, using the field of complex numbers, a state is unentangled when it can be expressed as a product state or, equivalently, when equation (27) reports its purity to be 1 [1, 2]. When using finite Galois fields $\mathbb{F}_{p^2}$, for particular choices of p, it is possible for equation (27) to produce a purity of 1 for some entangled states. For example, the normalized entangled state $|\Psi\rangle = |011\rangle + (2+\mathrm{i})\,|100\rangle + |101\rangle + |110\rangle$ has $P_{\mathfrak{h}} = 1$ for $p = 7$. In addition, the process of determining whether a *given* state $|\Phi\rangle$ is a product state may depend on p.

Thus, in finite fields, the simplest way to calculate the number of unentangled states is to disregard equation (27) and count the number of product states. This is exactly how the counting in section 5.4 was done, but the paper did not point out that counting relying only on equation (27) might not lead to the same result.

Correction: Maximally entangled states. We present below a new version of section 5.4 that correctly counts maximally entangled states. The rest of the article is independent of this revision.

5.4. Maximal entanglement

Equation (27) for $P_{\mathfrak{h}}$ includes a normalization factor $\frac{1}{n}$. In the discrete case, this normalization factor is undefined when $p \mid n$. Equation (27) also includes a summation of n terms. In the discrete case, certainly when $p \mid n$ but also in other cases, this summation may vanish in the field even if the individual summands are non-zero. These anomalies are irrelevant for the classification of states with purity 1 as this is performed by directly checking the possibility of direct decomposition into product states, disregarding equation (27).

For maximally entangled states, the purity calculation in conventional quantum mechanics using equation (27) produces 0. Given the above observations, in a discrete field, equation (27) may be undefined or may report a purity of 0 even for product states. In the discrete case, we therefore check for maximally entangled states using the following equation,

$$\forall j, \forall \mu, \langle \sigma^j_\mu \rangle^2 = 0 \ . \tag{40}$$

which avoids the normalization factor and simply checks that each summand is 0.

****

In conventional quantum mechanics, a state $|\Psi\rangle$ is maximally entangled when equation (27) for $P_{\mathfrak{h}}$ reports its purity to be 0. In this case, the $\frac{1}{n}$ normalization factor in $P_{\mathfrak{h}}$ does not affect the result and each term in the summation vanishes individually. In the discrete case, to avoid the singular cases when $p \mid n$, we must use $P_{\mathfrak{h}}$

$$\forall j, \forall \mu, \langle \sigma^j_\mu \rangle^2 = 0 \ . \tag{41}$$

Because equation (41) avoids wrapping around when adding expectation values, it defines maximally entangled states over finite fields.

scaling does not affect the purity and the properties of the real and complex numbers imply that the individual terms of the summation are each 0.

==> scale purity; has no effect; new purity = n iff old = 1 for product ==> scaling fails when p | n ==> sum of terms = 0 because of wrapping around

In the discrete case,

states whose computed purity is a multiple of the field characteristic p will be incorrectly labeled as maximally entangled by equation (27).

This occurs trivially when $p \mid n$. For example, the normalized $n = 3$-qubit product state $|\Psi\rangle$ over $\mathbb{F}_{3^2}$:

$$|\Psi\rangle = |0\rangle \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \ .$$

In this case the quantity $nP_{\mathfrak{h}} = 3P_{\mathfrak{h}} = 0$ despite the fact that $|\Psi\rangle$ is a product state.

Computing some examples for various n and small values of p, one can verify explicitly that unit-norm product states for $n = 2$, $p = \{3, 7, 11, 19, \ldots\}$ occur with frequency

$$(p + 1)p^2(p - 1)^2 = \{144, 14\,112, 145\,200, 2339\,280, \ldots\} \ ,$$

and for general n, $(p + 1)p^n(p - 1)^n$.

The irreducible state counts are reduced by $(p + 1)$, giving

$$p^2(p - 1)^2 = \{36, 1764, 12\,100, 116\,964, \ldots\} \ ,$$

and in general for n-qubits, there are $p^n\,(p - 1)^n$ instances of product pure states.

Repeating the computation, we find maximally entanglement states with frequencies for two qubits of

$$p\,(p - 1)\,(p + 1)^2 = \{96, 2688, 15\,840, 136\,800, \ldots\} \ .$$

The irreducible state counts for maximal entanglement are reduced by $(p + 1)$, giving for $n = 2$

$$p\left(p^2 - 1\right) = \{24, 336, 1320, 6840, \ldots\} \ .$$

For three-qubits, there are $p^3\left(p^4 - 1\right)(p + 1)$ instances of pure maximally entangled states, while the general formula for four-qubit states remains unclear.

Acknowledgments

We would like to thank John Gardiner for pointing out the need for this correction.

References

[1] Howard Barnum, Emanuel Knill, Gerardo Ortiz, Rolando Somma, and Lorenza Viola. A subsystem-independent generalization of entanglement. *Phys. Rev. Lett.*, 92:107902, Mar 2004.
[2] Howard Barnum, Emanuel Knill, Gerardo Ortiz, and Lorenza Viola. Generalizations of entanglement based on coherent states and convex sets. *Phys. Rev. A*, 68:032308, Sep 2003.
[3] Andrew J Hanson, Gerardo Ortiz, Amr Sabry, and Yu-Tsung Tai. Geometry of discrete quantum computing. *Journal of Physics A: Mathematical and Theoretical*, 46(18):185301, 2013.