Erratum: Geometry of discrete quantum computing

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Section 5.4 of our paper [3] requires a clarification and a correction.

Clarification: Unentangled vs. product states. In conventional quantum mechanics, using the field of complex numbers, a state is unentangled when it can be expressed as a product state or, equivalently, when equation (27) reports its purity to be 1 [1, 2]. When using finite Galois fields $\mathbb{F}_{p^2}$, for particular choices of p, it is possible for equation (27) to produce a purity of 1 for some entangled states. For example, the normalized entangled state $|\Psi\rangle = |011\rangle + (2 + \mathrm{i})\,|100\rangle + |101\rangle + |110\rangle$ has $P_{\mathfrak{h}} = 1$ for $p = 7$. In addition, the process of determining whether a *given* state $|\Phi\rangle$ is a product state may depend on p.

Thus, in finite fields, the simplest way to calculate the number of unentangled states is to disregard equation (27) and count the number of product states. This is exactly how the counting in section 5.4 was done, but the paper did not point out that counting relying only on equation (27) might not lead to the same result.

Correction: Maximally entangled states. We present below a new version of section 5.4 that correctly counts maximally entangled states. The rest of the article is independent of this revision.

5.4. Maximal entanglement

In conventional quantum mechanics, a state $|\Psi\rangle$ is maximally entangled when equation (27) reports its purity to be 0. Therefore, in the discrete case, states whose computed purity is a multiple of the field characteristic p will be incorrectly labeled as maximally entangled by equation (27). This occurs trivially when $n \mid p$.

For example, consider the normalized $n = 3$-qubit product state $|\Psi\rangle$ over $\mathbb{F}_{3^2}$:

$$|\Psi\rangle = |0\rangle \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \ .$$

In this case the quantity $nP_{\mathfrak{h}} = 3P_{\mathfrak{h}} = 0$

In the continuous case, the zero purity actually means every summand is 0, i.e.,

$$\forall j, \forall \mu, \langle \sigma^j_\mu \rangle = 0 \ . \tag{40}$$

Because equation (40) avoids wrapping around when adding expectation values, it defines maximally entangled states over finite fields.

Computing some examples for various n and small values of p, one can verify explicitly that unit-norm product states for $n = 2$, $p = \{3, 7, 11, 19, \ldots\}$ occur with frequency

$$(p + 1)p^2(p - 1)^2 = \{144, 14\,112, 145\,200, 2339\,280, \ldots\} \ ,$$

and for general n, $(p + 1)p^n(p - 1)^n$.

The irreducible state counts are reduced by $(p + 1)$, giving

$$p^2(p - 1)^2 = \{36, 1764, 12\,100, 116\,964, \ldots\} \ ,$$

and in general for n-qubits, there are $p^n\,(p - 1)^n$ instances of product pure states.

Repeating the computation, we find maximally entanglement states with frequencies for two qubits of

$$p\,(p - 1)\,(p + 1)^2 = \{96, 2688, 15\,840, 136\,800, \ldots\} \ .$$

The irreducible state counts for maximal entanglement are reduced by $(p+1)$, giving for $n=2$

$$p\left(p^2-1\right)=\{24,336,1320,6840,\ldots\}\ .$$

For three-qubits, there are $p^3\left(p^4-1\right)(p+1)$ instances of pure maximally entangled states, while the general formula for four-qubit states remains unclear.

Acknowledgments

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References

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[2] Howard Barnum, Emanuel Knill, Gerardo Ortiz, and Lorenza Viola. Generalizations of entanglement based on coherent states and convex sets. *Phys. Rev. A*, 68:032308, Sep 2003.
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