1 Newton Method

Use Taylor for $f(x_{k+1})$ at x_k:

$$f(x_{k+1}) = f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}(x_{k+1} - x_k)^T H(x_k)(x_{k+1} - x_k) \quad (1)$$

where the Hessian Matrix

$$H(x_k)_{i,j} = \frac{\partial f(x)}{\partial x_i \partial x_j}|_{x=x_k}$$

We want to find the minimum $f(x)$, if x_{k+1} make $f(x)$ minimal

$$\nabla f(x_{k+1}) = 0 \quad (2)$$

we derivate for the (1) equation

$$0 = \nabla f(x_{k+1}) = \nabla f(x_k) + H(x_k)(x_{k+1} - x_k) \quad (3)$$

we can write this equation as below

$$y_k = \nabla f(x_{k+1}) - \nabla f(x_k) = H(x_k)(x_{k+1} - x_k) = H(x_k)\delta_k \quad (4)$$

Take (2) equation into (4), we obtain

$$x_{k+1} = x_k - H(x_k)^{-1}\nabla f(x_k) \quad (5)$$

2 BFGS

General, it's hard to compute the inverse of Hessian.So we propose a method to compute a new matrix G_k instead of the inverse of Hessian. According to the (4), we have

$$\delta_k = x_{k+1} - x_k = G_k y_k = G_k \nabla f(x_{k+1}) - \nabla f(x_k) \quad (6)$$

$$G_{k+1} = G_k + P_k + Q_k \quad (7)$$

where

$$P_k y_k = \delta_k$$

and

$$Q_k y_k = -G_k y_k$$

and

$$y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$$

we can set

$$P_k = \frac{\delta_k \delta_k^T}{\delta_k^T y_k} \quad (8)$$

$$Q_k = -\frac{G_k y_k y_k^T G_k}{y_k^T G_k y_k} \tag{9}$$

which satisfied the (7). Consequently,

$$G_{k+1} = G_k + \frac{\delta_k \delta_k^T}{\delta_k^T y_k} - \frac{G_k y_k y_k^T G_k}{y_k^T G_k y_k} \tag{10}$$