Inverse Problems in Imaging
Lecture 11

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UCL, Term 2

Outline

1. Worked Example : In-Painting
   - Inpainting as an inverse problem
   - PDE methods for Inpainting
   - Dictionary Approach to Inpainting

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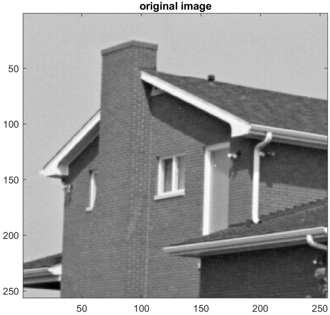
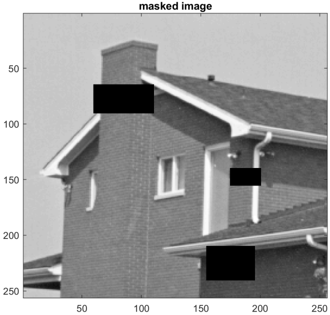
Image Inpainting
Introduction

Inpainting is defined as the recovery of missing pixels values.





We want to put this into an inverse Problems framework
Define this as

$$\boldsymbol{f}_{\mathrm{recon}} = \arg\min_{\boldsymbol{f}} \left[\Phi(\boldsymbol{f}, \boldsymbol{g}) = \frac{1}{2} \left\| \boldsymbol{g} - \mathrm{I}_{\Omega} \boldsymbol{f} \right\|^2 + \alpha \Psi(\boldsymbol{f}) \right] \tag{11.1}$$

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Suppose that the original image has n_{pix} pixel and there are n_{missing} missing pixels.
Let us define Ω as the set of $M = n_{\text{pix}} - n_{\text{missing}}$ pixels that are present and $\overline{\Omega}$ as the set of missing pixels, so that $\Xi = \Omega \cup \overline{\Omega}$ is the set representing the complete image.
Then I_Ω is a $M \times n_{\text{pix}}$ matrix got by deleting rows corresponding to missing pixels from the identity matrix.

Image Inpainting

Forward Model

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$$\mathrm{I}_{\Omega} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example for $n_{\text{pix}} = 6$ and $M = 4$. The missing pixels are $i_{\text{missing}} = [3, 4]$.

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Question : What is the SVD of this matrix ?

Image Inpainting

Inverse Solution

Let's first assume we will use a quadratic prior $\Psi(\boldsymbol{f}) = \frac{1}{2}\boldsymbol{f}^{\mathrm{T}}\Gamma\boldsymbol{f}$
The normal equations for eq. 11.1 become

$$\left(\mathsf{I}_\Omega^{\mathrm{T}}\mathsf{I}_\Omega + \alpha\Gamma\right)\boldsymbol{f} = \mathsf{I}_\Omega^{\mathrm{T}}\boldsymbol{g}$$

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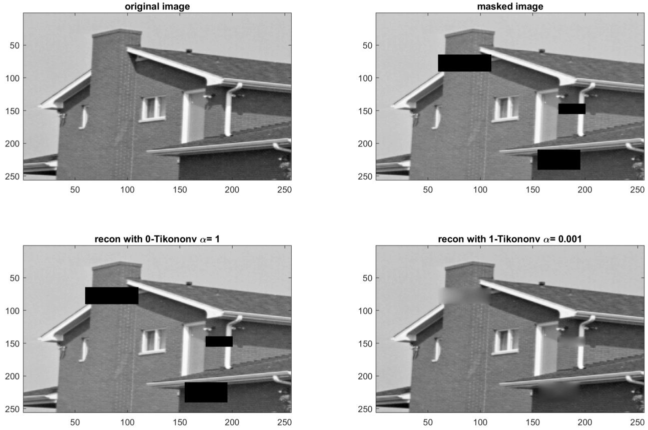
Clearly, if Γ is diagonal, it cannot generate any values in the missing pixels!

Results for zero and first order Tikhonov regularisation



Code: `InpaintPrior/inpaint1`

Recall that the 1^{st}-order Tikhonov prior corresponds to $\Gamma = -\nabla^2$.
The normal equations correspond to solving a PDE

$$\left(\mathsf{I}_\Omega^{\mathrm{T}} \mathsf{I}_\Omega - \alpha \nabla^2\right) \boldsymbol{f} = \mathsf{I}_\Omega^{\mathrm{T}} \boldsymbol{g} \tag{11.2}$$

$$\equiv \qquad \nabla^2 \boldsymbol{f} = \frac{1}{\alpha} \boldsymbol{f} \tag{11.3}$$

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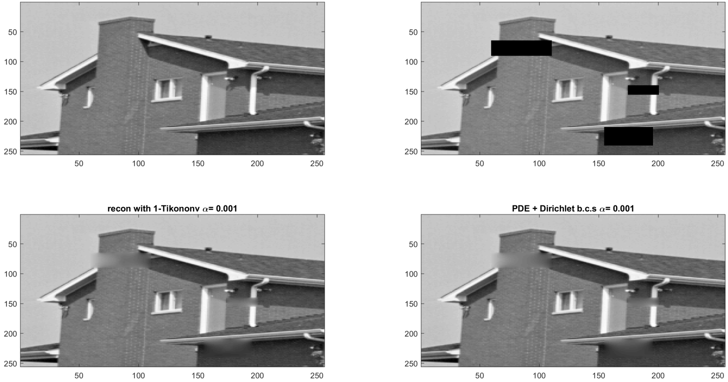
Since we know $\boldsymbol{f}$ in Ω, it makes sense to state this problem only for the missing pixels. We can state this as

$$\begin{aligned} \nabla^2\boldsymbol{f} &= \frac{1}{\alpha}\boldsymbol{f}, \qquad \boldsymbol{f} \in \overline{\Omega} \\ \text{subject to } \boldsymbol{f} &= \boldsymbol{g} \qquad \text{on } \partial\Omega \end{aligned} \tag{11.4}$$

This is a diffusion equation with Dirichlet boundary conditions.



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$$\begin{aligned} \frac{\partial \boldsymbol{f}}{\partial t} &= -\alpha \mathcal{L}(\boldsymbol{f})\boldsymbol{f}, \qquad \boldsymbol{f} \in \overline{\Omega} \\ \text{subject to } \boldsymbol{f} &= \boldsymbol{g} \qquad \text{on } \partial\Omega \end{aligned} \tag{11.5}$$

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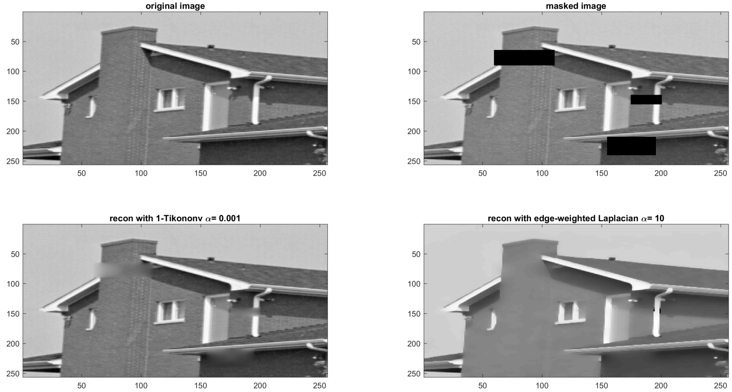
Other forms of PDE have been considered, e.g. higher order derivatives; see e.g R. D. Adam, P. Peter, J. Weickert: "Denoising by inpainting" *In* F. Lauze, Y. Dong, A. B. Dahl (Eds.): Scale Space and Variational Methods in Computer Vision. Lecture Notes in Computer Science, Vol. 10302, 121-132, Springer, Cham, 2017.



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One simple approach is to assume that the samples are from a Normal distribution $\boldsymbol{f}_i \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{f}}, \mathrm{C}_{\boldsymbol{f}})$ and to take the dictionary as the set of eigenfunctions of $\{\boldsymbol{v}_k; k = 1 \ldots n_v\}$ of $\mathrm{C}_{\boldsymbol{f}}$ up to a chosen cutoff n_v.

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We estimate the mean $\boldsymbol{\mu}_{\boldsymbol{f}}$ and covariance $\mathrm{C}_{\boldsymbol{f}}$ from the sample set

$$\boldsymbol{\mu}_{\boldsymbol{f}} \simeq \frac{1}{n_{\text{samples}}} \sum_{i=1}^{n_{\text{samples}}} \boldsymbol{f}_i, \qquad \mathrm{C}_{\boldsymbol{f}} \simeq \frac{1}{n_{\text{samples}}} \sum_{i=1}^{n_{\text{samples}}} (\boldsymbol{f}_i - \boldsymbol{\mu}_{\boldsymbol{f}})(\boldsymbol{f}_i - \boldsymbol{\mu}_{\boldsymbol{f}})^{\mathrm{T}}$$

A problem dealing with images is that the size of the covariance ($n_{\text{pix}} \times n_{\text{pix}}$) is very large.
Statistics tells us that we need about 10 samples per matrix element to get a reliable estimate of the covariance.
For a $n_{\text{pix}} = 256^2$ image that gives us $n_{\text{samples}} = 2^{33} \simeq 10^{10}$!

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Fortunately, we assume that images are correlated and so $\mathrm{C}_{\boldsymbol{f}}$ is low-rank, so it can be approximated by its *principle components*

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We can find the components by taking the SVD (to limited order) of the $n_{\rm pix} \times n_{\rm samples}$ data matrix $\tilde{\mathsf{F}} = \mathsf{F} - \boldsymbol{\mu}_{\boldsymbol{f}} \mathbf{1}^{\rm T}$ where the $i^{\rm th}$ column of F is $\boldsymbol{f}_i$ and $\mathbf{1}$ is a vector of 1's of length $n_{\rm samples}$.

The problem to solve now is

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which leads to the normal equations

$$\left(\mathsf{I}_{\Omega}^{\mathrm{T}} \mathsf{I}_{\Omega} + \alpha \tilde{\mathsf{C}}_{\boldsymbol{f}}^{-1} \right) \boldsymbol{f} = \mathsf{I}_{\Omega}^{\mathrm{T}} \boldsymbol{g} - \alpha \tilde{\mathsf{C}}_{\boldsymbol{f}}^{-1} \boldsymbol{\mu}_{\boldsymbol{f}}$$

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Since $\tilde{\mathsf{C}}_{\boldsymbol{f}}^{-1}$ is low rank (by design) we replace it's inverse by the pseudo-inverse

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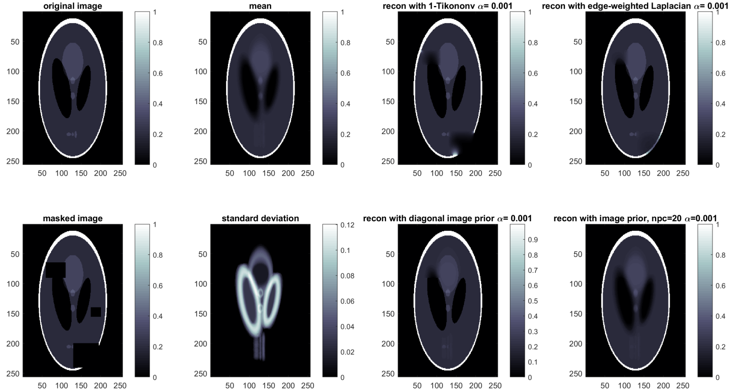
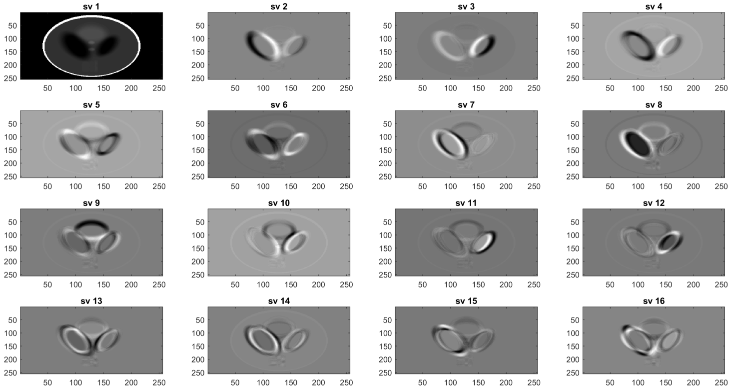
We can now solve the normal equations using a matrix-free solved like `pcg` where the Krylov iterations require only the masking operation $\mathsf{I}_\Omega^{\mathrm{T}} \mathsf{I}_\Omega$, and the projection onto the singular vectors (i.e. the dictionary) up to the order n_v.

Image Inpainting

Covariance Dictionary results



Code: `InpaintPrior/InpaintSL`



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