MTH107 Advanced Linear Algebra

Chapter 0 (Recall MTH017/MTH007)

Maps

maps(functions)

$$f : A \to B, \quad a \mapsto f(a)$$

composition: Given two functions $f : B \to C$ and $g : A \to B$, the composition of f and g is denoted as:

$$(f \circ g)(x) = f(g(x)), \quad \text{for all } x \in A$$

$$a \mapsto c(a \mapsto b \mapsto c)$$

injective–one to one

if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

if $f(a_1) = f(a_2)$, then $a_1 = a_2$

surjective–onto

for $f : A \to B$

$$\forall b \in B, \exists a \in A \text{ such that } f(a) = b$$

bijetive–both inj and suj

for $f : A \to B$

$$\forall b \in B, \exists! a \in A \text{ such that } f(a) = b$$

(use $\exists!$expresses only exist one)

f: x–>y is a bijection if and only if $\exists$g:y—>x, s.t. f(g(y))=id_y and g(f(x))=id_x

, called the inclusion of A (into B)

restriction

That is:

if A ⊆ B and $f: B \to C$, we have a map for all a ∈ A:

$$f|_A : A \to C$$

:

$$f|_A(a) = f(a)$$

called the restriction of f to A

$$f|_A = f \circ \iota_A$$

because

$$f \circ \iota_A(a) = f(\iota_A(a)) = f(a) = f|_A(a)$$

the reason to define it:

- domains differ

For example, consider a map $g|_B : B \to C$, where the function takes each element x and maps it to $x + 2$. Let the sets be as follows:

- $A = \mathbb{N} \subseteq B = \mathbb{R}$
- $C = \mathbb{R}$ is the codomain of the function.

The map g is defined as:

$$g : B \to C, \quad g(x) = x + 2 \quad \text{for all } x \in B$$

Now, consider the restriction of g to A, denoted $g|_A : A \to C$, where:

$$g|_A : A \to C, \quad g|_A(x) = x + 2 \quad \text{for all } x \in A$$

Although both $g : \mathbb{R} \to \mathbb{R}$ and $g|_A : \mathbb{N} \to \mathbb{R}$ follow the same rule $x \mapsto x + 2$, they cannot be considered the same map because their domains differ.

(here, A ⊆ B)

- focus on the specific area within the whole area

set of all maps from A to B(f)

if a and b are sets we define $B^A = f : A - -- > B(map)$ the set of all maps from A to B

$$B^A = \{f : A \to B\}$$

$B^\emptyset = \{\emptyset \to B\}$ has only 1 element even if B=∅

- The set $B^\emptyset$ contains only the zero function. proof: suppose f,g ∈ $B^\emptyset = \{f : \emptyset \to B\}$, imagine f != g, we can derive $\exists x \in \emptyset$ s.t. f(x) != g(x), which is absurd, so f=g

Chapter 1: vector spaces

finite space

For finite spaces ,only R^n and C^n.

(For infinite spaces, there are many, e.g. a continuous set: [0,1] to ...)

R^n notification

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$$

real numbers

define 2 oprations on the set R^n

addition:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

- **Left-hand side**: $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$
- **Right-hand side**: The result of $x_i + y_i$ is in $\mathbb{R}$.

scalar multiplication: for lambda ∈ R,

$$\lambda \cdot (x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

- **Left-hand side**: $\lambda \in \mathbb{R}, (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
- **Right-hand side**: The result $\lambda x_i \in \mathbb{R}$, for each i.

these 2 oprations generalize the standard operations on R2 and R3

geometric realization

矢量三角形–addition

直线上–scalar multiplication

mention: for n greater than or equal to 4 we cannot visualize vectors in the real world but e can still use them to solove real world problems

if a and b are finite sets

to simplify notations we sometimes use a single letter for vectors:

x= $\vec{x}$ =(x_1,x_2,....)

R^n as a vector space

so we have(rn,+, mutiplication notation). the operations satisfies some useful properties that turn rn into a real vector space

relation between rn and + .

$(R^n, +, \cdot)$

it means that + and · satisfy the following axioms:

1. $\forall x, y \in rn$: x+y=y+x. commutativity
2. $\forall x, y, z \in rn$: (x+y)+z=x+(y+z). associativity
3. x + 0 = 0 + x neutral element for addition
4. $x + (-x) = (-x) + x = 0$

——inverse for addition -x=y

5. $1 \cdot x = x$ Identity Element for Scalar Multiplication
6. $(\lambda\mu) \cdot x = \lambda \cdot (\mu \cdot x)$ compatibilily of multiplication
7. $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$ Distributivity of Scalar Multiplication Over Vector Addition
8. $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$ Distributivity of Scalar Addition

Cn complex Vector Space

$i^2 = -1$

Define $\mathbb{C}^n$ as the set of all ordered n-tuples of complex numbers:

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C},\ i = 1, 2, \ldots, n\}$$

Operations on C^n

We define **addition** and **scalar multiplication** on $\mathbb{C}^n$ in the same way as we did on $\mathbb{R}^n$, but using complex numbers:

Addition:

For two vectors $(z_1, z_2, \ldots, z_n)$ and $(w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$:

$$(z_1, z_2, \ldots, z_n) + (w_1, w_2, \ldots, w_n) = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$$

Scalar Multiplication:

For a scalar $\lambda \in \mathbb{C}$ and a vector $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$:

$$\lambda \cdot (z_1, z_2, \ldots, z_n) = (\lambda z_1, \lambda z_2, \ldots, \lambda z_n)$$

Thus, $(\mathbb{C}^n, +, \cdot)$ is a complex vector space because it satisfies the vector space axioms 1-8, where we replaced $\mathbb{R}^n$ with $\mathbb{C}^n$. Many of the results from MTH107 will hold regardless of whether we are using $\mathbb{R}$ or $\mathbb{C}$, so we will often use $\mathbb{F}$ to represent either $\mathbb{R}$ or $\mathbb{C}$.

For example, $\mathbb{F}^n$ is an $\mathbb{F}$-vector space, where $\mathbb{F}$ could be either $\mathbb{R}$ or $\mathbb{C}$.

Generalization (Not on the Exam):

Finite Field Example:

$\mathbb{F}_2 = \{0, 1\}$: the finite field with two elements.

Many of our results hold in a more general setting, where $\mathbb{F}$ is a **field**—a set in which we can perform addition, multiplication, subtraction, and division (except division by zero).

Examples of fields include:

- $\mathbb{R}$: the real numbers
- $\mathbb{C}$: the complex numbers
- $\mathbb{Q}$: the rational numbers
- $\mathbb{F}_2 = \{0, 1\}$: the finite field with two elements

Abstract Vector Space:

We can generalize this idea by replacing $\mathbb{F}^n$ with some abstract space V, define addition $+$ and scalar multiplication $\cdot$, and check if they satisfy the eight vector space axioms.

Definitions:

For a set V, **addition** on V is a map:

$$V \times V \to V$$

It maps an element set to their addition. For example, if $V = \mathbb{R}^2$:

$$(v, w) \mapsto v + w$$

Example: $(1, 2) + (3, 4) = (4, 6)$, which is also in V.

A **scalar multiplication** is a map:

$$F \times V \to V$$

For example, $(\lambda, v) \mapsto \lambda v$.

(remark:if V and W are sets, V× W = {(v,w)|v∈ V, w∈ W})

An F-vector space is a set V with an addition $+$ and scalar multiplication $\cdot$ by elements of F, such that $(V, +, \cdot)$ satisfies the vector space axioms 1-8, where $\mathbb{R}$ is replaced by F, and $\mathbb{R}^n$ is replaced by V.

Remarks and Examples:

- **Vectors**: Vectors are elements of V, denoted as $v \in V$.
- **Field** F: The choice of F matters! For example, we will see later that $\mathbb{C}^n$ is a complex vector space of dimension n, but is also a real vector space of dimension $2n$.

General properties of vector space

Let V be a vector space over F

We will prove some properties of V using only the defination (axioms 1-8)

Proposition

1. The zero(additive identity) is unique. That is: $\exists ! 0 \in V s.t. 0+V=V, \forall v \in V$

proof: Suppose we have 2 zero elements: 0 and 0'

0=0'+0=0+0'=0'

2. $\forall v \in V$ there exists a unique additive inverse

Suppppose w and w' are 2 inverses for v, w=0+w=(v+w')+w=v+(w'+w)=v+(w+w')=(v+w)+w'=0+w'=w'

3. $\forall v \in V, O_F \cdot V = O_V$

proof: 0_F·V= $(0_F + 0_F)$·V=$0_F v + 0_F v$

let w be the inverse of $0_F \cdot v$(use proposition 2)

then 0_V=$0_F \cdot v + w$=$(0_F v + 0_F v) + w = 0_F v + (0_F v + w) = 0_F v$

4. $\forall x \in F : x \cdot O_V = O_F$

proof: $x \cdot 0_V = x \cdot (0_V + 0_V) = x \cdot 0_V + x \cdot 0_V$

so $0_V = x \cdot 0_V$

Examples:

For any field F,

1. **Trivial Vector Space**: the set $V = \{0\}$ is a trivial F-vector space with addition $0 + 0 = 0$ and scalar multiplication $\lambda \cdot 0 = 0$.

2. **Finite-Dimensional Vector Space**: F^n is an F-vector space, and $F^0 = \{0\}$.

3. **Infinite-Dimensional Vector Space**: Let F^∞ be the space of infinite sequences, where:

$$F^\infty = \{(x_1, x_2, \ldots) \mid x_i \in F,\ i = 1, 2, \ldots\}$$

Addition and scalar multiplication are defined **component-wise**:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$

4. **Function Space**: Let S be a set, then:

$$F^S = \{f : S \to F\,(\text{maps from S to F})\}$$

Addition and scalar multiplication are defined **pointwise**. For functions $f, g \in F^S$ and $\lambda \in F$:

$$(f + g)(x) = f(x) + g(x) \quad \forall x \in S$$
$$(\lambda \cdot f)(x) = \lambda \cdot f(x) \quad \forall x \in S$$

F^S is an F-vector space

Chapter 2 Subspaces of Vector Spaces

- Def: Let $V(V, +_V, \cdot_V)$ be a vector space, a subset U of V is a subspace if and only if $U(U, +_V, \cdot_V)$ is a vector space.

$(\cdot_V|_{\mathbb{F}\times U}: \mathbb{F}\times U \to U,\ +_V|_{U\times U}: U\times U \to U$ i.e. $(\forall u_1, u_2 \in U,\ u_1 +_V u_2 \in U,\ \lambda \cdot_V u_1 \in U))$

- Propsition: $U \subseteq V$ is a subspace of V if and only if:
  - $0 \in U$
  - $\forall u_1, u_2 \in U,\ u_1 + u_2 \in U$
  - $\forall \lambda \in \mathbb{F},\ u_1 \in U,\ \lambda \cdot u_1 \in U$

- Eg. $\lambda \in \mathbb{F}$, $U := \{x_1, x_2, x_3, x_4 \mid x_3 = 5x_4 + \lambda\}$ is a subspace $\Leftrightarrow \lambda = 0$
  - Proof: $\Rightarrow$: $\because 0 \in U$,

    $\therefore$ When $x_3 = 0$, $\exists x_4 = 0$

    $\therefore 0 = \lambda$

    $\Leftarrow$: if $\lambda = 0$, let x and $y \in U$, let $a \in \mathbb{F}$

    $\because x_3 = 5x_4,\ y_3 = 5y_4$

    $\therefore x_3 + y_3 = 5(x_4 + y_4)$

    $\therefore x + y = (x_1 + y_1, x_2 + y_2, 5(x_4 + y_4), x_4 + y_4) \in U$

    $ax = (ax_1, ax_2, 5ax_3, ax_4) \in U$

    also, $0 \in U$

    so U is a subspace $\square$

- Eg. $\mathbb{R}_{\leq 0}$ is not a vector space $((-2)\cdot(-1) = 2 \notin \mathbb{R}_{\leq 0})$

· Sum of subspaces

·Motivation: $U \cup W$ is a subspace $\iff U \subseteq W$ or $W \subseteq U$, so we want to produce a generalized subspace of V containing both U and W. (Here: $U \subseteq V, W \subseteq V$). Because if $A \subseteq C$, $B \subseteq C$ are 2 subsets ~~and~~ then $A \cup B \subseteq C$ is ~~a~~ the smallest subset of C containing both A and B, We want to produce the smallest subspace of V containing both U and W.

·Def: $U_1, U_2, \cdots U_n$ are subspaces of V, their sum is $U_1 + \cdots + U_n$, $U_1 + \cdots + U_n = \{u_1 + \cdots + u_n \mid u_i \in U_i, \forall i\}$. This is a subset of V containing all possible sums of elements of U.

is the smallest subspace of V containing each of

Statement:

Prove: The sum $U1 + \cdots + Un$ is the smallest subspace of V containing each of the subspaces $U1, U2, \ldots, Un$.

Proof:

Step 1: Show $U1 + \cdots + Un$ is a subspace.

By definition, $U1, U2, \ldots, Un$ are subspaces of V.

We want to show that their sum $U1 + \cdots + Un$, which is defined as the set of all sums of elements from the subspaces Ui (i.e., $U1 + \cdots + Un = \{u1 + u2 + \cdots + un \mid ui \in Ui \text{ for each } i\}$), is also a subspace of V.

Closure under addition:
Let $v, w \in U1 + \cdots + Un$. Then

$$v = v1 + v2 + \cdots + vn \quad \text{and} \quad w = w1 + w2 + \cdots + wn,$$

where $vi \in Ui$ and $wi \in Ui$ for all i.
Then

$$v + w = (v1 + w1) + (v2 + w2) + \cdots + (vn + wn).$$

Since each Ui is a subspace, $vi + wi \in Ui$ for all i, and therefore $v + w \in U1 + \cdots + Un$. Hence, the sum is closed under addition.

Closure under scalar multiplication:
Let $\lambda \in F$ (the field over which V is defined) and $v \in U1 + \cdots + Un$, where

$$v = v1 + v2 + \cdots + vn \quad \text{with each } vi \in Ui.$$

Then

$$\lambda v = \lambda(v1 + v2 + \cdots + vn) = (\lambda v1) + (\lambda v2) + \cdots + (\lambda vn).$$

Since each Ui is a subspace, $\lambda vi \in Ui$, so $\lambda v \in U1 + \cdots + Un$. Thus, the sum is closed under scalar multiplication.

Contains zero vector:
Since each Ui is a subspace, $0 \in Ui$ for all i. Therefore, $0 = 0 + 0 + \cdots + 0 \in U1 + \cdots + Un$, so the sum contains the zero vector.

Thus, $U1 + \cdots + Un$ is a subspace of V.

Step 2: Show $U1 + \cdots + Un$ contains all Ui.

By the definition of the sum, each element of $U1 + \cdots + Un$ is a sum of elements from the subspaces Ui. In particular, for each i, the subspace $Ui \subseteq U1 + \cdots + Un$ since any $ui \in Ui$ can be written as $ui + 0 + \cdots + 0$, which is in the sum.

Thus, $Ui \subseteq U1 + \cdots + Un$ for all i.

Step 3: Show that $U1 + \cdots + Un$ is the smallest subspace containing each Ui.

Let W be any subspace of V that contains each Ui, i.e., $U1 \subseteq W, U2 \subseteq W, \ldots, Un \subseteq W$.

Since subspaces are closed under addition, any sum of elements from $U1, U2, \ldots, Un$ must also be contained in W.

Therefore, every element of $U1 + \cdots + Un$ is contained in W, implying that $U1 + \cdots + Un \subseteq W$.

Conclusion:

We have shown that $U1 + \cdots + Un$ is a subspace of V that contains each Ui, and it is the smallest such subspace.

Therefore, $U1 + \cdots + Un$ is the smallest subspace of V containing each of the subspaces $U1, U2, \ldots, Un$, completing the proof.

1. Subspaces and Their Operations:

Intersection and Union of Subspaces:

$U \cap W$ is the largest subspace contained in both U and W.

$U + W$ (the sum of subspaces) is the smallest subspace containing both U and W.

If $U \cap W = \{0\}$, then $U + W$ is a direct sum, denoted $U \oplus W$.

Direct Sum of Subspaces:

A sum of subspaces $U1 + U2 + \cdots + Un$ is a **direct sum** (denoted $U1 \oplus U2 \oplus \cdots \oplus Un$) if every element in the sum can be written uniquely as $u1 + u2 + \cdots + un$ where $ui \in Ui$.

The **uniqueness** of the decomposition is key to defining a direct sum.

Theorem: $U1 + \cdots + Un$ is a direct sum if and only if the only way to express 0 as $u1 + \cdots + un$ is when $u1 = \cdots = un = 0$.

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2. Linear Combinations and Span:

A **linear combination** of vectors $v1, \ldots, vn \in V$ is any expression of the form $\lambda 1 v1 + \lambda 2 v2 + \cdots + \lambda n vn$, where $\lambda i \in F$.

The **span** of a set of vectors $v1, \ldots, vn \in V$, denoted $\mathrm{span}(v1, \ldots, vn)$, is the set of all possible linear combinations of these vectors. It is the smallest subspace containing all the vi.

Theorem: The span of vectors $v1, \ldots, vn$ is the smallest subspace of V containing all the vi.

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3. Linear Independence:

A set of vectors $v1, \ldots, vn \in V$ is **linearly independent** if the only solution to the equation $\lambda 1 v1 + \cdots + \lambda n vn = 0$ is $\lambda 1 = \lambda 2 = \cdots = \lambda n = 0$.

If a vector $vi \in \mathrm{span}(v1, \ldots, vi{-}1, vi{+}1, \ldots, vn)$, the set is linearly **dependent**.

Remark: In a vector space $\mathbb{F}^n$, any list of $m > n$ vectors must be linearly dependent.

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4. Complementary Subspaces:

If $V = U \oplus W$, then W is called the **complementary subspace** of U in V.

Counterexamples exist where, even if $U1 + W = U2 + W$, it does not necessarily imply $U1 = U2$.

(Finite dimensional) Spaces Basis Dimention.

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- A vector space is called finite-dimetional ~~space~~ if some list of vectors in it span the spans the space.
  - eg. $\mathbb{F}^n$ is a finite-dimentional space for every $n \in \mathbb{N}$.
- Polynomial $P(\mathbb{F})$ — infinite dimensional

$$p: \mathbb{F} \longrightarrow \mathbb{F}$$
$$z \longmapsto p(z).$$

$a_0, \cdots, a_m \in \mathbb{F}$, $z \in \mathbb{F}$

- $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$. — has degree m $(a_m \neq 0)$; has degree $-\infty$ if $p(z)=0$.
- $P_n(\mathbb{F})$: degree $\leq n$
- The set of polynomials $\{1, x, x^2, \cdots, x^n\}$ forms a basis for $P_n(\mathbb{F})$
- $P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$, $P_n(\mathbb{F})$ is finite-dimentional.

- If V is spanning with n elements, its subspace U is spanning at most n elements $\longrightarrow$ any subspace $U \subseteq V$ is finite-dimentional if V is finite-dimentional

BASES. — Spanning; Linearly Independent.

eg. $(1,2)$ $(3,5)$ is a basis.

① $\lambda(1,2) + n(3,5) = 0$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} \lambda \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1 \neq 0 \Rightarrow \begin{pmatrix} \lambda \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \Rightarrow \text{Linearly independent}$$

② Let $v(x,y)$, $v = \lambda(1,2) + n(3,5)$.

Solve $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} \lambda \\ n \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \lambda \\ n \end{pmatrix} = \begin{pmatrix} x \\ y-3x \end{pmatrix}$$

$$\Rightarrow \begin{cases} n = 3x - y \\ \lambda = \cancel{5x} + 2y \end{cases} \Rightarrow \text{We are able to find } \lambda \text{ and } n \text{ so the family is spanning. } \forall x, y$$

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Lemma Let V be a finite dimentional vector space. If there exists a basis, then all basis have the same number of elements.

Proof: If $(v_1, \dots v_m)$ and $(w_1, \dots w_n)$ are 2 basis,
by the Thm, $m \leq n$ ($v_1, \dots v_m$ is linearly independent, $w_1, \dots w_n$ are spanning).
but also $n \leq m$ ($w_1, \dots w_n$ is linearly independent, $v_1, \dots v_m$ are spanning)
$\Rightarrow m = n$.

eg. • $P(\mathbb{F})$: $1, z, z^2, \dots z^{m+1}, \dots$ is a basis, called the standard basis.
• ϕ is a basis for $\{0\}$.

Thm: [Criterion for basis].
[$v_1, \dots v_n$ is a basis] $\Leftrightarrow$ [$\forall v \in V$, the vector equation $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ has a unique solution $(\lambda_1, \dots \lambda_n) \in \mathbb{F}^n$]
i.e. [V can be written in a unique way as a linear combination of $v_1, \dots v_n$]

* if $\beta = (v_1, \dots, v_n)$ is a basis of V, then we have a bijection:

$$[\cdot]_\beta : V \to F^n$$
$$v \mapsto (\lambda_1, \dots \lambda_n)$$

eg. $P(\mathbb{F}_m) \xrightarrow{\simeq} \mathbb{F}^{m+1}$

$a_0 + \dots + a_m z^m \mapsto (a_0, a_1, \dots, a_m)$.

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Construct basis
- Start from many vectors that spanning and delete to Linearly Independent
- Start from linearly independant basis and add to spanning. ~~bas~~

Thm If U is a subspace of V, then there exists a direct sum completement.
$U \oplus W = V$, such W is not unique unless $U = \{0\}$ or $U = V$
$(W = V)$ $(W = \{0\})$.

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Key Steps in the Proof:

Subspaces and Basis Setup:

Assume U and W are subspaces of the vector space V.

Let $V1, \ldots, Vm$ be a basis for the intersection $U \cap W$.

Let $V1', \ldots, Vn'$ be a basis for U, but none of these vectors are in $U \cap W$.

Let $W1', \ldots, Wk'$ be a basis for W, but none of these vectors are in $U \cap W$.

Constructing the Basis for $U + W$:

The next step is to claim that the set:

$$\{V1, \ldots, Vm, V1', \ldots, Vn', W1', \ldots, Wk'\}$$

forms a basis for $U + W$.

To prove this, the proof first establishes **spanning**. Take an arbitrary vector $v \in U + W$. Since $v \in U + W$, it can be written as $v = u + w$, where $u \in U$ and $w \in W$. Each of these components can be expressed as a linear combination of their respective bases in U and W.

Linear Independence:

To show that the constructed set is linearly independent, assume:

$$0 = a1V1 + \cdots + amVm + b1V1' + \cdots + bnVn' + c1W1' + \cdots + ckWk'.$$

By the properties of U and W, we deduce that all coefficients $b1 = b2 = \cdots = bn = 0$ and $c1 = c2 = \cdots = ck = 0$. This shows that all coefficients must be zero, hence the vectors are linearly independent.

Conclusion:

Since the constructed set both spans $U + W$ and is linearly independent, it forms a basis for $U + W$.

The number of vectors in this basis is $m + n + k$, where $m = \dim(U \cap W)$, $n = \dim(U) - \dim(U \cap W)$, and $k = \dim(W) - \dim(U \cap W)$.

Therefore, the formula for the dimension of the sum of two subspaces is:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Chapter 3 Linear Maps (Linear Transformation)

A **linear map** $T : V \to W$ between vector spaces V and W (over the same field F) satisfies two properties:

Additivity: $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.

Scalar Multiplication: $T(\lambda v) = \lambda T(v)$ for all $v \in V$ and $\lambda \in F$.

Examples:

A common example of a linear map is a matrix acting on vectors in $\mathbb{R}^n$. For instance, the transformation $T(x) = Ax$ where A is an $n \times n$ matrix is a linear map.

Another example is the differentiation operator $D : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$, where $D(f) = f'$. This is a linear transformation because $D(f + g) = D(f) + D(g)$ and $D(\lambda f) = \lambda D(f)$.

1. Lemma:

Let $T : V \to W$ be a linear map.

- If $T(0) = 0W$ (the zero vector in W) and for any vector $v \in V$, we have $T(v) = T(v)$, then the following are true:

  - **Proof**: The linear map preserves the additive identity, so $T(0V) = T(0V + 0V) = T(0V) + T(0V)$, which implies that $0W = T(0V)$, showing that T maps zero to zero.
  - Similarly, scalar multiplication is preserved: $T(-v) = T((-1) \cdot v) = (-1) \cdot T(v)$.

2. Examples of Linear Maps:

- A linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is not always bijective, particularly when not defined over the same field.
- **Example**: The zero map $O : V \to W$, which maps every element of V to $0W$, is always a linear map.

3. Spaces of Linear Maps:

- For two vector spaces V and W, $L(V, W)$ is the space of all linear maps from V to W.
  - **Lemma**: For any $v \in V$ and $w \in W$, the zero map $0 \in L(V, W)$.

4. Linear Operators and Examples:

- **Identity Map**: Denoted as $\mathrm{Id}V$, is the map that sends every element of V to itself. This map is linear but depends on the field over which it operates (not necessarily the same for all spaces).
- **Differentiation Operator** D:
  - D is a linear operator acting on the space of polynomials, mapping a polynomial $p(x)$ to its derivative $p'(x)$.
  - The operator extends to $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions.
  - **Check**: $D \in L(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))$, meaning the differentiation operator is indeed a linear map within these function spaces.

5. Properties of Linear Maps:

- If $T : V \to W$ is a linear map, then T respects both vector addition and scalar multiplication, which is a foundational property of linear maps.

Constructing Linear Maps:

The construction of a linear map is straightforward when we know the basis of the vector spaces involved.

Assumption: Suppose $v1, v2, \ldots, vn$ is a basis of the vector space V and $w1, w2, \ldots, wn$ are vectors in another vector space W.

Define a linear map $T : V \to W$ such that $T(vi) = wi$ for each basis vector $vi \in V$.

The proof of **existence** relies on the fact that any vector $v \in V$ can be expressed as a linear combination of the basis vectors:

$$v = \lambda 1 v1 + \lambda 2 v2 + \cdots + \lambda n vn.$$

Applying the linear map T, we get:

$$T(v) = T(\lambda 1 v1 + \lambda 2 v2 + \cdots + \lambda n vn) = \lambda 1 T(v1) + \lambda 2 T(v2) + \cdots + \lambda n T(vn),$$

which equals:

$$T(v) = \lambda 1 w1 + \lambda 2 w2 + \cdots + \lambda n wn.$$

This shows that T is a well-defined linear map.

Uniqueness of Linear Maps:

To prove **uniqueness**, assume there are two linear maps $T : V \to W$ and $S : V \to W$ such that $T(vi) = S(vi)$ for all $vi \in V$.

For any vector $v \in V$, we know $v = \lambda 1 v1 + \cdots + \lambda n vn$.

By linearity of both T and S, we have:

$$T(v) = \lambda 1 T(v1) + \cdots + \lambda n T(vn)$$

and

$$S(v) = \lambda 1 S(v1) + \cdots + \lambda n S(vn).$$

Since $T(vi) = S(vi)$, it follows that $T(v) = S(v)$ for all $v \in V$, proving that T and S must be the same map. Therefore, the linear map is unique.

1. Definition:

If $T, S \in L(V, W)$, where $L(V, W)$ is the space of linear maps from vector space V to W, then the composition $(T + S)(v) = T(v) + S(v)$ and the composition $(c \cdot T)(v) = c \cdot T(v)$ (where c is a scalar) define operations within $L(V, W)$.

2. Theorem:

$L(V, W)$ **is a vector space**:

> The space of all linear maps from V to W forms a vector space under the usual operations of addition of maps and scalar multiplication.

3. Proof:

To prove that $L(V, W)$ is a vector space, the proof starts by verifying that it satisfies the axioms of vector spaces, such as closure under addition and scalar multiplication, associativity, commutativity, etc.

Step-by-step verification: For two linear maps $T, S \in L(V, W)$, check that:

$$(T + S)(v) = T(v) + S(v)$$

satisfies linearity (additivity and scalar multiplication), ensuring that the sum of two linear maps is still a linear map.

4. Identity and Zero Map:

The **identity map** $\mathrm{Id}V$ in $L(V, V)$ is defined as $\mathrm{Id}V(v) = v$ for all $v \in V$. This map acts as the neutral element in the composition of linear maps.

The **zero map** in $L(V, W)$ is the map that sends every vector in V to the zero vector in W, i.e., $0(v) = 0W$. This map acts as the additive identity in $L(V, W)$.

5. Composition of Maps:

If $S, T \in L(V, W)$, then the composition $S \circ T$ (denoted as ST) is also a linear map.

The lemma discussed here shows that the composition of two linear maps is again a linear map, i.e., $ST \in L(V, W)$ for all $T, S \in L(V, W)$.

6. Dimensionality of $L(V, W)$:

The dimension of the space of linear maps $L(V, W)$ is given by $\dim(L(V, W)) = \dim(V) \times \dim(W)$.

> This result is derived by considering the basis of V and W and using the fact that a linear map is determined by its action on a basis of V, which can map to any combination of the basis vectors of W.

7. Final Conclusion:

The structure of $L(V, W)$ as a vector space is formalized, and it is shown that the space of linear maps between two vector spaces has a specific dimension that depends on the dimensions of the source and target spaces. The identity and zero maps are also fundamental

1. Injectivity and Null Space:

A **linear transformation** $T : V \to W$ is **injective** (one-to-one) if and only if its **null space** (kernel) is trivial. The null space is the set of vectors in V that T maps to the zero vector in W.

Definition of the Null Space:

$$\mathrm{null}(T) = \{v \in V \mid T(v) = 0W\}.$$

If $\mathrm{null}(T) = \{0\}$, then T is injective, meaning that no two distinct vectors in V map to the same vector in W.

The notes give the following theorem:

Theorem: If $T : V \to W$ is injective, then $\mathrm{null}(T) = \{0\}$.

Proof: Let $v \in \mathrm{null}(T)$, which means $T(v) = 0W$. Since T is injective, the only vector mapped to the zero vector is the zero vector itself. Thus, $v = 0$, proving that the null space contains only the zero vector.

2. The Relationship Between Injectivity and Spanning:

The notes explore whether a set of linearly independent vectors $v1, \ldots, vn$ in V that spans V under T also implies that the transformation is injective. It is confirmed that if T maps linearly independent vectors to linearly independent vectors, T preserves the injectivity condition.

3. Range (Image) of a Linear Transformation:

The **range** (or image) of a linear transformation T is the set of all vectors in W that can be written as $T(v)$ for some $v \in V$:

$$\mathrm{range}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

In simpler terms, the range is the subset of W that T maps onto. The range is always a subspace of W.

Theorem: The range of a linear transformation T is a subspace of W.

Proof: Let $w1, w2 \in \mathrm{range}(T)$, so $w1 = T(v1)$ and $w2 = T(v2)$ for some $v1, v2 \in V$. Since T is linear:

$$w1 + w2 = T(v1) + T(v2) = T(v1 + v2),$$

which shows that the sum of two vectors in the range is also in the range. Similarly, scalar multiplication is preserved, so the range is closed under addition and scalar multiplication, making it a subspace of W.