Contents

1 Preliminary

1.1 Set Theory

Definition 1.1 Number Sets

$\mathbb{N}$ = 1, 2, 3, 4, 5, ... is the set of all **Natural Numbers** or positive integers

$\mathbb{Z}$ = ..., -3, -2, -1, 0, 1, 2, 3, ... is the set of all **Integers**

$\mathbb{Q} = \{x\text{: } p/q \text{ for some } p, q \in \mathbb{Z}\}$ is the set of all **Rational Numbers**

$\mathbb{R} = \{x\text{: } x \text{ is a real number}\}$ is the set of all **Real Numbers**

$\mathbb{C}$ = is the set of all **Complex Numbers**

2 Vectors

2.1 Inner Products

2.2 Matrix Vector Multiplication

Definition 2.1 Matrix Column Vector Multiplication via Linear Combinations

Let A $\in \mathbb{R}^{m\times n}$ and let $\mathbf{x} \in \mathbb{R}^{n\times 1}$. We take linear combinations of the scalars of $\mathbf{x}$ with the columns of A.

$$\mathbf{b} = \sum_{k=1}^{n} x_k \cdot A(:,k)$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\mathbf{b} = \sum_{k=1}^{n} x_k \cdot A(:,k)$$

$$= x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Definition 2.2 Matrix Column Vector Multiplication via Dot Products

Let A $\in \mathbb{R}^{m\times n}$ and let $\mathbf{x} \in \mathbb{R}^{n\times 1}$. We take the dot products of the rows of A with the column vector $\mathbf{x}$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$b_i = \begin{bmatrix} \text{A(i, :)} \end{bmatrix}_{1\times n} \cdot \begin{bmatrix} \mathbf{x} \end{bmatrix}_{n\times 1}$$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}_{1\times n} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$= \sum_{k=1}^{n} a_{ik} \cdot x_k$$

$$= a_{i1} \cdot x_1 + a_{i2} \cdot x_2 + \cdots + a_{in} \cdot x_n$$

Definition 2.3 Row Vector Matrix Multiplication via Linear Combinations

Let A $\in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^{m\times 1}$. Then $\mathbf{x}^T A$ are the linear combinations of the scalars of $\mathbf{x}$ and the rows of A.

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}_{1\times m} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}_{1\times n}$$

$$\mathbf{b} = \sum_{k=1}^{n} x_k \cdot \begin{bmatrix} A(k,:) \end{bmatrix}_{1\times n}$$

$$= x_1 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \tag{1}$$

$$+ x_2 \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \tag{2}$$

$$+ \cdots + \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{3}$$

Definition 2.4 Row Vector Matrix Multiplication via Dot Products

Let A $\in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^{m\times 1}$. Then $\mathbf{x}^T A$ are the dot products of the columns of A with the row vector $\mathbf{x}$.

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}_{1\times m} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}_{1\times n}$$

$$\begin{aligned} b_j &= \mathbf{x}^T \cdot A(:,j) \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}_{1\times m} \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \\ &= \sum_{k=1}^{m} x_k \cdot a_{mj} \\ &= x_1 \cdot a_{1j} + x_2 \cdot a_{2j} + \cdots + x_1 \cdot a_{mj} \end{aligned}$$

3 Matrices

3.1 Outer Products

3.2 Matrix Multiplication

Definition 3.1 Matrix Matrix Multiplication via Linear Combinations of Columns

Let A $\in \mathbb{R}^{m\times p}$ and X $\in \mathbb{R}^{p\times n}$

The jth column of B are linear combinations of columns of A with scalars in jth column X.

$$\mathrm{B}(:, \mathrm{j}) = \mathrm{A} \cdot \mathrm{X}(:, \mathrm{j})$$

$$\mathrm{B}(:, \mathrm{j}) = \sum_{k=1}^{p}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix}_{m\times p} \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{pj} \end{bmatrix}_{p\times 1} = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}_{m\times 1}$$

$$= x_{1j} \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_{2j} \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_{pj} \begin{bmatrix} a_{1p} \\ a_{2p} \\ \vdots \\ a_{mp} \end{bmatrix}$$

Definition 3.2 Matrix Matrix Multiplication via Linear Combinations of Rows

Let X $\in \mathbb{R}^{m\times p}$ and A $\in \mathbb{R}^{p\times n}$

Definition 3.3 Matrix Matrix Multiplication via Dot Products

Let A $\in \mathbb{R}^{m\times p}$ and X $\in \mathbb{R}^{p\times n}$. Then the entries of B $\in \mathbb{R}^{m\times n}$ are the dot products of the rows of A with the columns of X.

$$b_{ij} = \mathrm{A}(\mathrm{i}, :) \cdot \mathrm{X}(:, \mathrm{j})$$

$$b_{ij} = \sum_{k=1}^{p} a_{ik} \cdot x_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix}_{m \times p} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \end{bmatrix}_{p \times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

$$\begin{aligned} b_{ij} &= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{bmatrix} \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{pj} \end{bmatrix} \\ &= a_{i1} \cdot x_{1j} + a_{i2} \cdot x_{2j} + \cdots + a_{ip} \cdot x_{pj} \end{aligned}$$

Definition 3.4 Matrix Matrix Multiplication via Outer Products

Let A $\in \mathbb{R}^{m \times p}$ and X $\in \mathbb{R}^{p \times n}$

4 Nonsingular Linear Systems Problem

4.1 Elimination