Computer Science Department, Ben Gurion University of the Negev

Optimization methods with Applications - Spring 2020

# Home Assignment 4

Due on June ???th.

1. **Equality constrained optimization**
   In this section we will find the maximal surface area of a box given the sum of its edges' length. The optimization problem is given by

$$\max_{\mathbf{x}\in\mathbb{R}^3} \{x_1x_2 + x_2x_3 + x_1x_3\} \quad s.t. \quad \left\{ \; x_1 + x_2 + x_3 = 3 \right. . \tag{1}$$

   (a) Find a critical point for the problem (1) using the Lagrange multiplier method.

   (b) Show that this critical point is a maximum point. For this show that the Hessian of the Lagrangian is negative ($\mathbf{y}^\top \nabla^2 \mathcal{L} \mathbf{y} < 0$) for vectors $\mathbf{y} \neq 0$ who satisfy $\mathbf{y}^\top \mathbf{1} = y_1 + y_2 + y_3 = 0$.

2. **General constrained optimization**
   Assume that we have the following problem.

$$\min_{\mathbf{x}\in\mathbb{R}^2} \left\{(x_1 + x_2)^2 - 10(x_1 + x_2)\right\} \quad s.t. \quad \left\{ \begin{array}{c} 3x_1 + x_2 = 6 \\ x_1^2 + x_2^2 \leq 5 \\ -x_1 \leq 0 \end{array} \right. . \tag{2}$$

   (a) Find a critical point $\mathbf{x}^*$ using the Lagrange multipliers method for which the fewest inequality constraints are active. Show that the KKT conditions hold.

   (b) Using second order conditions, determine whether the point $\mathbf{x}^*$ you found in the previous section is a minimum or maximum.

   (c) Write the unconstrained minimization problem that corresponds to the problem (2) above using the penalty method with $\rho(x) = x^2$ as a penalty function.

   (d) Use the penalty method to get the minimizer $\mathbf{x}^*$ up to two digits of accuracy. Solve the optimization problems using steepest descent with Armijo linesearch. Use penalty parameters $\mu = 0.01, 0.1, 1, 10, 100$.

3. **Box-constrained optimization**

In this question we will write the **coordinate descent** method for quadratic box-constrained minimization. Assume the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \frac{1}{2}\mathbf{x}^\top H\mathbf{x} - \mathbf{x}^\top\mathbf{g} \right\} \quad s.t. \quad \mathbf{a} \le \mathbf{x} \le \mathbf{b}, \tag{3}$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $\mathbf{g} \in \mathbb{R}^n$, and $\mathbf{a} < \mathbf{b} \in \mathbb{R}^n$ are the lower and upper bounds on the solution $\mathbf{x}$.

(a) Give a closed form solution for the scalar box constrained minimization problem

$$\min_{x\in\mathbb{R}} \left\{ \frac{1}{2}hx^2 - gx \right\} \quad s.t. \quad a \le x \le b, \qquad \text{for } a < b, h > 0.$$

(b) In the coordinate descent algorithm we sweep over all the variables $x_i$ one by one, and for each solve the box-constrained minimization problem for the scalar variable $x_i$, given that the rest are known. Show that the minimization for each scalar $x_i$ is given by

$$\min_{x_i\in\mathbb{R}} \left\{ \frac{1}{2}h_{ii}x_i^2 + \left[ \left( \sum_{j\neq i} h_{ij}x_j \right) - g_i \right] x_i \right\} \quad s.t. \quad a_i \le x_i \le b_i.$$

Use the previous section to show the expression for the update of the coordinate descent method for this problem.

(c) Use the previous section to write a program for solving (3) using the coordinate descent algorithm.

(d) Solve the problem (3) for the following parameters using the projected CD method up to three digits of accuracy:

$$H = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & 5 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 18 \\ 6 \\ -12 \\ -6 \\ 18 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}.$$