Measurement Theory

Yuval Lavie

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Part I
Motivation

We would like to generalize the concept of Length by defining the following function.

1 Definition : Generalization of Length.

Let:

1. Ω be a space
2. $A \subseteq \mathbb{R}$
3. $\gamma \in \mathbb{R}$
4. $B = \bigcup_{n \geq 1} B_i : B_i \subseteq \mathbb{R}, \forall j, k \in \mathbb{N}_1 : j \neq k \rightarrow B_j \cap B_k = \phi$

Then:

1. λ assigns non negative values to any subset of the space.
   $\lambda : \mathbb{P}(\Omega) \rightarrow R^+ \cup \{+\infty\}$
2. λ assigns the natural length to an interval
   $\forall a, b \in \mathbb{R} : b > a \rightarrow \lambda\Big[(a,b)\Big] = b - a$
3. λ is Invariant to translation
   $\forall A \forall \gamma \rightarrow \lambda\Big[A + \gamma\Big] = \lambda\Big[A\Big]$
4. λ assigns the sums of its values to disjoint sets
   $\lambda(B) = \sum_{n \geq 1} \lambda(B_i)$

2 A Negative result in measure theory

It is impossible to define a function that will act as a measure as defined above, one of the following options have to be considered.

1. λ only assigns non negative values to some subsets of the space.

   (a) $\lambda : \left[\mathbb{F} \subset \mathbb{P}(\Omega)\right] \to R^+ \cup \{+\infty\}$

2. λ is variant to translation
   $\forall A \forall \gamma \to \lambda\left[A + \gamma\right] \neq \lambda\left[A\right]$

3. λ does not assigns the sums of its values to disjoint sets
   $\lambda(B) \neq \sum\limits_{n \geq 1} \lambda(B_i)$

4. The axioms of Zermelo-Fraenkel set theory with the Axiom of uncountable choice may have to be changed.

Such function will never be sigma additive and we will choose the first option and try to minimize our domain.

Part II

Algebras and σ-Alegbras

3 Semi Algebra

Let Ω be a space then $\mathbb{F}$ is a family of subgroups of Ω and is called a Semi Algebra if and only if:

1. $\phi \in \mathbb{F}$

2. $\forall A, B \in \mathbb{F} \exists (C_i)_{i=1}^n \subseteq \mathbb{F}, C_i \cap C_j = \phi : A \setminus B = \bigcup\limits_{i=1}^{n} C_i$

3. $\forall A, B \in \mathbb{F} \to A \cup B \in \mathbb{F}$

4 Algebra

4.1 Definition

Let Ω be a space then $\mathbb{F}$ is a family of subgroups of Ω and is called an Algebra if and only if:

1. $\phi \in \mathbb{F}$

2. $\forall A \in \mathbb{F} \to A^c \in \mathbb{F}$

3. $\forall A, B \in \mathbb{F} \to A \cup B \in \mathbb{F}$

4.2 Properties

1. $A \cap B \in \mathbb{F}$

2. $A \setminus B \in \mathbb{F}$

3. $A \Delta B \in \mathbb{F} \iff (A \setminus B) \cup (B \setminus A) \in \mathbb{F}$

4.2.1 Additional Properties

1. $2 \leq |\mathbb{F}| \leq 2^{\Omega}$

2. $\Omega \in \mathbb{F}$

3. $\forall A_i \in \mathbb{F} : \bigcup\limits_{i=1}^{n} A_i \in \mathbb{F}$

4. $\forall A_i \in \mathbb{F} : \bigcap\limits_{i=1}^{n} A_i \in \mathbb{F}$

4.2.2 Proofs:

1. $A, B \in \mathbb{F} \overset{D.2}{\Longrightarrow} A^C, B^C \in \mathbb{F} \overset{D.3}{\Longrightarrow} A^C \cup B^C \in \mathbb{F} \overset{D.2}{\Longrightarrow} (A^C \cup B^C)^C = A \cap B$

2. $B \in \mathbb{F} \overset{D.2}{\Longrightarrow} B^c \in \mathbb{F} \overset{P.1}{\Longrightarrow} A \cap B^c \in \mathbb{F} \iff A \setminus B \in \mathbb{F}$

3. $A \setminus B \in \mathbb{F}, B \setminus A \in \mathbb{F} \overset{D.3}{\Longrightarrow} (A \setminus B) \cup (B \setminus A) \in \mathbb{F}$

4.2.3 Examples

Let Ω be a space then

1. $\{\phi, \Omega\}$- is the smallest Algebra

2. $P(\Omega)$ - is the biggest Algebra

4.3 Intersection of Algebras is an Algebra.

Let $\{\mathbb{F}_i\}_{i \in I}$ be a family of Algebras on Ω then $\bigcap\limits_{i \in I} \mathbb{F}_i$ is an Algebra.

1. $\phi \in \bigcap\limits_{i \in I} \mathbb{F}_i$

2. $A \in \bigcap\limits_{i \in I} \mathbb{F}_i \implies A^c \in \bigcap\limits_{i \in I} \mathbb{F}_i$

3. $A, B \in \bigcap\limits_{i \in I} \mathbb{F}_i \implies A \cup B \in \bigcap\limits_{i \in I} \mathbb{F}_i$

4.4 Algebra generated by a family

Let Ω be a space and $\mathbb{M}$ be a family of subgroups of Ω and let A_i be a group of all algebras on Ω that contains $\mathbb{M}$ then $\mathbb{A}(\mathbb{M}) = \bigcap\limits_{i=1}^{n} A_i$ is the smallest Algebra that contains $\mathbb{M}$

5 σ-Algebra

5.1 Definition

Let Ω be a space then $\mathbb{F}$ is a family of subgroups of Ω and is called a σ-Algebra if and only if:

1. $\phi \in \mathbb{F}$
2. $\forall A \in \mathbb{F} \to A^c \in \mathbb{F}$
3. $\forall \{A_i\}_{i=1}^{\infty} \in \mathbb{F} \to \bigcup\limits_{i=1}^{\infty} A_i \in \mathbb{F}$

5.2 Properties

1. $\forall \{A_i\}_{i=1}^{\infty} \in \mathbb{F} \to \bigcap\limits_{i=1}^{\infty} A_i \in \mathbb{F}$
2. All properties and defintions on Algebra work on Sigma Algebra.

5.3 Borel Family σ-Algebra

Let $\Omega = R$ and $I_o = \{(a, b) : a < b, a, b \in \mathbb{R}\}$ be all the open intervals then $\mathbb{A}_\sigma(I_O)$ the sigma algebra generated by the set of all open intervals is called the Borel Set Family and is denoted $\mathbb{B}(\mathbb{R})$ and each subset is called A Borel Set.

6 Limits of series of sets

6.1 Lower limit of a series of sets (Limit Infimum)

Let $(A_n)_{n=1}^{\infty}$be a series of subgroups of Ω then:

- $lim\big[inf(A_n)\big] =: \{\omega \in \Omega : \exists m > N \to \omega \in \bigcap\limits_{n=m}^{\infty} A_n\}$
- $lim\big[inf(A_n)\big] = \bigcup\limits_{n=1}^{\infty} \left[\bigcap\limits_{m=n}^{\infty} A_m \right]$

6.2 Upper limit of a series of sets (Limit Supremum)

Let $(A_n)_{n=1}^{\infty}$be a series of subgroups of Ω then:

- $lim\big[sup(A_n)\big] = \bigcap\limits_{n=1}^{\infty} \left[\bigcup\limits_{m=n}^{\infty} A_m \right]$

6.3 Lemmas

1. $\uparrow (A_n)_{n=1}^{\infty} \to lim\big[inf(A_n)\big] = lim\big[sup(A_n)\big] = \overset{\infty}{\underset{n=1}{\cup}} A_n$
2. $\downarrow (A_n)_{n=1}^{\infty} \to lim\big[inf(A_n)\big] = lim\big[sup(A_n)\big] = \overset{\infty}{\underset{n=1}{\cap}} A_n$
3. $(A_n)_{n=1}^{\infty} : \forall i, j \in n : A_i \cap A_j = \phi \to lim\big[inf(A_n)\big] = lim\big[sup(A_n)\big] = \phi$
4. A series $(A_n)_{n=1}^{\infty}$ converges if and only if $lim\big[inf(A_n)\big] = lim\big[sup(A_n)\big]$
5. $lim\big[sup(A_n)\big] \in \mathbb{F}$

Part III
Measures

6.4 Definitions

6.4.1 Additivity

Let $f : M \to R^{+} \cup \{\infty\}$ then f is called Additive if $\forall A, B, A \cup B \in M : A \cap B = \phi :$

1. $f(A \cup B) = f(A) + f(B)$

6.4.2 Sigma Additivity (σ Additivity)

Let $f : M \to R^{+} \cup \{\infty\} : M \subseteq P(\Omega)$ then f is called Sigma Additive if $\forall (A_n)_{n=1}^{\infty} \in M :$

1. $\forall i \neq j \to A_i \cap A_j = \phi$
2. $\overset{\infty}{\underset{n=1}{\cup}} A_n \in M$

then:

1. $\sum_{n=1}^{\infty} f(A_n) < \infty$
2. $f\left(\overset{\infty}{\underset{n=1}{\cup}} A_n \right) = \sum_{n=1}^{\infty} f(A_n)$

6.4.3 Sub Additivity

Let $f : M \to R^{+} \cup \{\infty\}$ then f is called Sub Additive if $\forall A, B, A \cup B \in M$

1. $f(A \cup B) \leq f(A) + f(B)$

6.4.4 Sigma Sub Additivity (σ Sub Additivity)

Let $f : M \to R^+ \cup \{\infty\} : M \subseteq P(\Omega)$ then f is called Sigma Sub Additive if $\forall (A_n)_{n=1}^{\infty} \in M$:

1. $\forall i \neq j \to A_i \cap A_j = \phi$
2. $\bigcup\limits_{n=1}^{\infty} A_n \in M$

then:

1. $-\infty < \sum\limits_{n=1}^{\infty} f(A_n) < \infty$
2. $f\left(\bigcup\limits_{n=1}^{\infty} A_n\right) \leq \sum\limits_{n=1}^{\infty} f(A_n)$

6.5 Measure Definitions

6.5.1 Measureable Space / Measureable Set

1. Let Ω be a space and $\mathbb{F}$ be a σ-algebra on Ω then:
2. $(\Omega, \mathbb{F})$ is called a Measurable Space.
3. $A \in \mathbb{F}$ is called a Measureable Set.

6.5.2 Measure Space

Let $(\Omega, \mathbb{F}, \mu)$ be a Measure Space if:

1. Ω - Any set
2. $\mathbb{F} \subseteq \mathbb{P}(\Omega)$ - σ-Algebra on Ω, $\mathbb{F}$ decides whether a subgroup is measureable or not.
3. $\mu : \mathbb{F} \to R^+ \cup \{\infty\}$ - A measure on $\mathbb{F}$, μ decides the measure of a measurable subgroup.

6.5.3 Infinite Measure

Let $(\Omega, \mathbb{F})$ be a Measureable Space then an infinite measure on $\mathbb{F}$ is an expanded real valued function defined as:

- $\mu : \mathbb{F} \to R^+ \cup \{+\infty\}$

and has the properties:

1. μ assigns non negative values to subsets of $\mathbb{F}$

   $\forall A \in \mathbb{F} : \mu(A) \geq 0$

2. μ assigns ZERO to the empty set

   $\mu(\phi) = 0$

3. μ is Sigma Additive

   $\forall (A_i)_{n=1}^{\infty} : \forall i, j \in \mathbb{N}_1, i \neq j : A_i \cap A_j = \phi \to \mu\big(\bigcup_{n=1}^{\infty} A_n \big) = \sum_{n=1}^{\infty} \mu(A_n)$

6.5.4 Finite Measure

Let $(\Omega, \mathbb{F})$ be a Measureable Space then a finite measure on $\mathbb{F}$is an expanded real valued function defined as like 3.5.3 with an additional property:

1. $\mu(\Omega) < \infty$

6.5.5 Outer Measure / Exterior Measure

Let Ω be a space then an exterior measure is defined as follows:

1. μ assigns non negative values to subsets of $\mathbb{P}(\Omega)$

   $\mu^* : \mathbb{P}(\Omega) \to R^+ \cup \{+\infty\}$

2. μ assigns ZERO to the empty set

   $\mu(\phi) = 0$

3. μ is Sigma Sub Additive

   $\forall (A_i)_{n=1}^{\infty} : \forall i, j \in \mathbb{N}_1, i \neq j : A_i \cap A_j = \phi \to \mu\big(\bigcup_{n=1}^{\infty} A_n \big) \leq \sum_{n=1}^{\infty} \mu(A_n)$

4. μ is monotone

   $\forall A \subseteq B : \mu^*(A) \leq u^*(B)$

6.5.6 Set Cover

Let Ω be a space and $\mathbb{M}$ be a family of subsets of Ω then $\mathbb{M}$is called a covering set of Ω if:

1. $\phi \in \mathbb{M}$

2. $\exists (C_n)_{n=1}^{\infty} : \bigcup\limits_{n=1}^{\infty} C_n = \Omega$

Let $A \subseteq \Omega$ then a series $(C_n)_{n=1}^{\infty}$ covers A if $A \subseteq \bigcup\limits_{n=1}^{\infty} C_n$

6.5.7 Lebesgue Outer Measure

Let $\Omega = \mathbb{R}$ and $I_o = \{(a,b) : a, b \in \mathbb{R}, a \leq b\}$ is a covering set for $\mathbb{R}, l\big[(a,b)\big] = b - a$

1. $\forall E \subseteq \mathbb{R} \rightarrow T_E = \left\{ \sum\limits_{n=1}^{\infty} l\big(C_n\big) : (C_n)_{n=1}^{\infty} \text{ covers } E \right\}$
2. $\lambda^*(E) = inf[T_E]$

6.5.8 Translation of subsets

Let $E \subseteq \mathbb{R}, \gamma \in \mathbb{R}$ then

1. $E = \{x | x \in E\}$
2. $E + \gamma = \{x + \gamma | x \in E\}$

7 From outer measure to a measure

7.1 μ^*- Measurable set

Let Ω be a space then the subset $E \subseteq \Omega$ is Measurable with respect to an outer measure (μ^*- Measurable) if and only if

- $\forall A \subseteq \Omega : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C)$

8 Measurable function

Let $(\Omega, \mathbb{F})$ and $(\mathbb{R}, \mathbb{B})$ be Measurable spaces and $f : \Omega \rightarrow \mathbb{R}$

- f is $(\Omega, \mathbb{F})$ - Measurable $\iff \forall \alpha \in \mathbb{R} \rightarrow \{\omega \in \Omega : f(\omega) \leq \alpha\} \in \mathbb{F}$
- f is $(\Omega, \mathbb{F})$ - Measurable $\iff \forall \alpha \in \mathbb{R} \rightarrow f^{-1}\Big[(-\infty, \alpha]\Big] \in \mathbb{F}$
- We denote $f : (\Omega, \mathbb{F}) \rightarrow (\mathbb{R}, \mathbb{B})$ to emphasize the dependency on the Sigma Algebras.