Part 1j: Transformation of random variables

<u>Textbook</u>: pp. 30-33

Introduction to the transformation of random vectors

- A generalization of the inverse transform.
- Describes the distribution of a transformation of a random vector, given the distribution of the source vector.
- If the source distribution can be simulated then its transformation is a simulation from the image distribution.
- Requires that the source distribution has a density and that the transformation is bijective and differentiable with continuous partial derivatives.

The Jacobian matrix

Definition 1.35 Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be differentiable. Then the *Jacobian matrix* $D\varphi$ is the $d \times d$ matrix consisting of the partial derivatives of φ: for $i, j = 1, 2, \ldots, d$ we have $D\varphi(x)_{ij} = \frac{\partial \varphi_i}{\partial x_j}(x)$.

$\underline{d=1}$

$$y = g(x) \iff w(y) = x$$

$$J_w(y) = \left(\frac{\partial}{\partial y} w(y)\right)$$

$\underline{d=2}$

$$\begin{aligned} y_1 &= g_1(x_1, x_2) \\ y_2 &= g_2(x_1, x_2) \end{aligned} \iff \begin{aligned} w_1(y_1, y_2) &= x_1 \\ w_2(y_1, y_2) &= x_2 \end{aligned}$$

$$J_w(y_1, y_2) = \begin{pmatrix} \frac{\partial}{\partial y_1} w_1(y_1, y_2) & \frac{\partial}{\partial y_2} w_1(y_1, y_2) \\ \frac{\partial}{\partial y_1} w_2(y_1, y_2) & \frac{\partial}{\partial y_2} w_2(y_1, y_2) \end{pmatrix}$$

Theorem 1.34 (transformation of random variables) Let $A, B \subseteq \mathbb{R}^d$ be open sets, $\varphi : A \to B$ be bijective and differentiable with continuous partial derivatives, and let X be a random variable with values in A. Furthermore let $g : B \to [0, \infty)$ be a probability density and define $f : \mathbb{R}^d \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(\varphi(x)) \cdot |\det D\varphi(x)| & \text{if } x \in A \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Then f is a probability density and the random variable X has density f if and only if $\varphi(X)$ has density g.

$\underline{d=1}$

$X \sim f_X(x)$, $g: \mathbb{R} \to \mathbb{R}$

$Y = g(X) \sim ?$

$f_Y(y) = f_X(w(y)) \cdot |w'(y)|$

$\underline{d=2}$

$\vec{X} \sim f_{\vec{X}}(\vec{x})$, $g: \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{Y} = g(\vec{X}) \sim ?$

$f_{\vec{Y}}(\vec{y}) = f_X(w_1(\vec{y}), w_2(\vec{y})) \cdot |\det J_w(\vec{y})|$

$$\det J_w(\vec{y}) = \det \begin{pmatrix} \frac{\partial}{\partial y_1} w_1(\vec{y}) & \frac{\partial}{\partial y_2} w_1(\vec{y}) \\ \frac{\partial}{\partial y_1} w_2(\vec{y}) & \frac{\partial}{\partial y_2} w_2(\vec{y}) \end{pmatrix}$$

$$\frac{\partial}{\partial y_1} W_2(\vec{y}) \qquad \frac{\partial}{\partial y_2} W_2(\vec{y})$$

$$= \left(\frac{\partial}{\partial y_1} W_1(\vec{y})\right)\left(\frac{\partial}{\partial y_2} W_2(\vec{y})\right) - \left(\frac{\partial}{\partial y_1} W_2(\vec{y})\right)\left(\frac{\partial}{\partial y_2} W_1(\vec{y})\right)$$

Example 1.38

$X \sim \text{Uniform}(0,1)$ and $Y = X^{2/3}$. What is the density of Y?

$f_X(x) = 1 \quad , \quad 0 \le x \le 1$

$g(x) = x^{2/3} \iff w(y) = y^{3/2}$

$w'(y) = \frac{3}{2} y^{1/2}$

$f_Y(y) = 1 \times \frac{3}{2} y^{1/2} = \frac{3}{2}\sqrt{y} \quad , \quad 0 \le y \le 1$

Example: Box–Muller transform

(a) Generate $\Theta \sim \mathcal{U}[0, 2\pi]$ and $U \sim \mathcal{U}[0, 1]$ independently.

(b) Let $R = \sqrt{-2\log(U)}$.

(c) Let $(X, Y) = \varphi(R, \Theta) = (R\cos(\Theta), R\sin(\Theta))$.

Then (X, Y) are standard normal and independent.

$$\vec{X} = (X_1, X_2), \quad X_1 \sim U(0, 2\pi), \quad X_2 \sim U(0,1)$$

independent

$$g_1(x_1, x_2) = \sqrt{-2\log x_2} \cdot \cos(x_1) = y_1$$

$$g_2(x_1, x_2) = \sqrt{-2\log x_2} \cdot \sin(x_1) = y_2$$

$$y_1^2 + y_2^2 = -2\log x_2 (\cos^2 x_1 + \sin^2 x_1) = -2\log x_2$$

$$\frac{y_2}{y_1} = \frac{\sin(x_1)}{\cos(x_1)} = \tan(x_1)$$

$$x_1 = w_1(y_1, y_2) = \operatorname{atan}(y_2/y_1)$$

$$x_2 = w_2(y_1, y_2) = e^{-\frac{1}{2}(y_1^2 + y_2^2)}$$

$$J_w(y_1, y_2) = \begin{pmatrix} \frac{1}{1+(y_2/y_1)^2} \cdot \left(-\frac{y_2}{y_1^2}\right) & \frac{1}{1+(y_2/y_1)^2} \cdot \frac{1}{y_1} \\ -y_1 e^{-\frac{1}{2}(y_1^2+y_2^2)} & -y_2 e^{-\frac{1}{2}(y_1^2+y_2^2)} \end{pmatrix}$$

$$|\det(J_W(y_1, y_2))| = e^{-\frac{1}{2}(y_1^2 + y_2^2)}$$

$$f_{\vec{X}}(x_1, x_2) = \frac{1}{2\pi} \cdot 1, \quad 0 \le x_1 \le 2\pi, \; 0 \le x_2 \le 1$$

$$f_{\vec{Y}}(y_1, y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(y_1^2 + y_2^2)}, \quad (y_1, y_2) \in \mathbb{R}^2$$

$\Rightarrow \; Y_1, Y_2 \sim N(0,1)$, independent