Part 2a: Simulating statistical models

Textbook: pp. 41.

Introduction to the simulation of statistical models

- Statistical models describe the theoretical distributions for data, data that emerges in real applications.
- These models are typically more complex than the examples that were considered in Part 1.
- However, the methods from Part 1 can be used as building blocks in the simulation of the more complex models.
- In the rest of Part b, that is based on Ch 2, we will discuss widely used statistical models.

Multivariate normal distribution

A standard model for data that may be collected in an experiment or an observational study:

Definition 2.1 Let $\mu \in \mathbb{R}^d$ be a vector and $\Sigma \in \mathbb{R}^{d \times d}$ be a symmetric, positive definite matrix. Then a random vector $X \in \mathbb{R}^d$ is normally distributed with mean μ and covariance matrix Σ, if the distribution of X has density $f: \mathbb{R}^d \to \mathbb{R}$ given by

$$f(x) = \frac{1}{(2\pi)^{d/2} \, |\det \Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1} (x - \mu) \right) \tag{2.1}$$

for all $x \in \mathbb{R}^d$.

Hierarchical models

The model is structured in layers. The distribution in a layer depends on the evaluation of the previous layer.

- In Bayesian models (discussed in Section 4.3) the distribution of the data depends on the value of one or more random parameters.
- In mixture models the distribution of samples depends on the random choice of mixture component.
- In Markov chains (discussed in Section 2.3) the distribution of the value at time t depends on the value of the Markov chain at time $t-1$.

$$f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x|y)$$

Markov chains

A relatively simple model of dependence:

- Flexible enough to fit many applications.
- Simple enough to allow efficient computation and extensive mathematical theory.

Definition 2.13 A stochastic process $X = (X_j)_{j \in \mathbb{N}_0}$ with values in a set S is a *Markov chain*, if

$$\begin{aligned} &P\left(X_j \in A_j \,\middle|\, X_{j-1} \in A_{j-1}, X_{j-2} \in A_{j-2}, \ldots, X_0 \in A_0\right) \\ &\quad = P\left(X_j \in A_j \,\middle|\, X_{j-1} \in A_{j-1}\right) \end{aligned} \tag{2.2}$$

for all $A_0, A_1, \ldots, A_j \subseteq S$ and all $j \in \mathbb{N}$, that is if the distribution of X_j depends on $X_0, \ldots, X_{j-2}$ only through X_{j-1}. The set S is called the *state space* of X. The distribution of X_0 is called the *initial distribution* of X.

The Poisson process

A point process in a space. The process is govern by the Poisson distribution.

Definition 2.34 A *Poisson process* on a set $D \subseteq \mathbb{R}^d$ with *intensity function* $\lambda\colon \mathbb{R}^d \to [0, \infty)$ is a random set $\Pi \subseteq D$ such that the following two conditions hold:

(a) If $A \subseteq D$, then $|\Pi \cap A| \sim \text{Pois}(\Lambda(A))$ where $|\Pi \cap A|$ is the number of points of Π in A and

$$\Lambda(A) = \int_A \lambda(x)\, dx. \tag{2.5}$$

(b) If $A, B \subseteq D$ are disjoint, then $|\Pi \cap A|$ and $|\Pi \cap B|$ are independent.