Part 1i: Geometric interpretation

Textbook: pp. 26-30

Introduction to the geometric interpretation

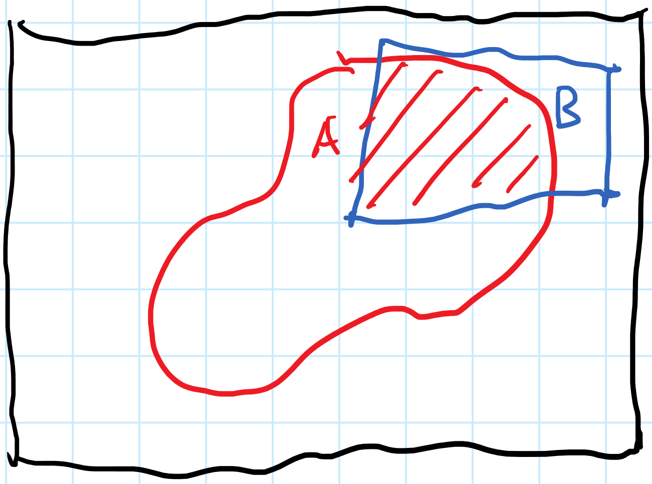
- The density of a distribution in R^d corresponds to a set in R^{d+1}.
- The envelope corresponds to a set that contains the density's set.
- Sampling a proposal relates to a sample from the uniform distribution over the envelope.
- The proposal is accepted if the proposal belongs to the set of the density.
- This produces a sample with a uniform distribution over the set of the density.
- The first coordinates of a sample from the uniform distribution over the set corresponding to the density produce a sample from the distribution.

The uniform distribution (generalization)

Definition 1.29 A random variable X with values in $\mathbb{R}^d$ is uniformly distributed on a set $A \subseteq \mathbb{R}^d$ with $0 < |A| < \infty$, if

$$P(X \in B) = \frac{|A \cap B|}{|A|}$$

for all $B \subseteq \mathbb{R}^d$. As for real intervals, we use the notation $X \sim \mathcal{U}(A)$ to indicate that X is uniformly distributed on A.



Lemma 1.30

Lemma 1.30 Let $A \subseteq \mathbb{R}^d$ be a set with volume $0 < |A| < \infty$. Then the uniform distribution $\mathcal{U}(A)$ has probability density $f = \mathbb{1}_A/|A|$ on $\mathbb{R}^d$.

Proof: $$P(B) = \frac{|A \cap B|}{|A|}$$

$$= \frac{1}{|A|} \int \mathbb{1}_{A \cap B}(\vec{x})\, d\vec{x}$$

$$= \int \mathbb{1}_B(\vec{x}) \frac{\mathbb{1}_A(\vec{x})}{|A|}\, d\vec{x}$$

$$= \int_B f(\vec{x})\, d\vec{x}$$

Lemma 1.31

Lemma 1.31 Let X be uniformly distributed on a set A, and let B be a set with $|A \cap B| > 0$. Then the conditional distribution $P_{X|X \in B}$ of X conditioned on the event $X \in B$ coincides with the uniform distribution on $A \cap B$.

Proof:

$$P(C|B) = \frac{P(C \cap B)}{P(B)}$$

$$= \frac{|C \cap B \cap A| / \cancel{|A|}}{|B \cap A| / \cancel{|A|}}$$

$$= \frac{|C \cap (B \cap A)|}{|B \cap A|}$$

Geometric interpretation of rejection sampling

Lemma 1.33 Let $f\colon \mathbb{R}^d \to [0,\infty)$ be a probability density and let

$$A = \{(x, y) \in \mathbb{R}^d \times [0,\infty) \mid 0 \le y < f(x)\} \subseteq \mathbb{R}^{d+1}.$$

Then $|A| = 1$ and the following two statements are equivalent:

(a) (X, Y) is uniformly distributed on A.

(b) X is distributed with density f on $\mathbb{R}^d$ and $Y = f(X)U$ where $U \sim \mathcal{U}[0, 1]$, independently of X.

$$|A| = \int_A d\vec{x}\, dy = \int \left[\int_0^{f(\vec{x})} dy\right] d\vec{x}$$

$$= \int f(\vec{x})\, d\vec{x} = 1$$

(a) => (b):
$$f_A(\vec{x}, y) = \frac{1_A(\vec{x}, y)}{|A|} = 1_A(\vec{x}, y)$$

$$f_{\vec{X}}(\vec{x}) = \int 1_A(\vec{x}, y)\, dy = \int_0^{f(\vec{x})} dy = f(\vec{x})$$

$$f_{Y|\vec{X}}(y|\vec{x}) = \frac{1_A(\vec{x}, y)}{f(\vec{x})}$$

$$= \begin{cases} 1/f(\vec{x}), & 0 \le y \le f(\vec{x}) \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow Y / \{\vec{X} = \vec{x}\} \sim U(0, f(\vec{x}))$$

$$\Rightarrow \frac{Y}{f(\vec{x})} = U \sim \text{Uniform}(0,1)$$

(b) $\Rightarrow$ (a): It is sufficient to prove that

$P((\vec{X},Y) \in B) = |A \cap B|$, for all

$B = C \times D$, $C \subset \mathbb{R}^d$, $D \subset \mathbb{R}$

$$P((\vec{X},Y) \in B) = P(\vec{X} \in C,\ Y \in D)$$

$$= \int_C P(Y \in D \mid \vec{X} = \vec{x})\, f(\vec{x})\, d\vec{x}$$

$$= \int_C \frac{|D \cap [0, f(\vec{x})]|}{f(\vec{x})} \cdot f(\vec{x})\, d\vec{x}$$

$$= \int_C |D \cap [0, f(\vec{x})]\, d\vec{x}$$

But also

$$|A \cap B| = \int_A 1_B(\vec{x}, y)\, dy\, d\vec{x}$$

$$= \int \int_0^{f(\vec{x})} 1_C(\vec{x})\, 1_D(y)\, dy\, d\vec{x}$$

$$= \int_C |D \cap [0, f(\vec{x})]| \, d\vec{x}$$

$$\Rightarrow \quad (X, Y) \sim \text{Uniform}(A)$$

Examples

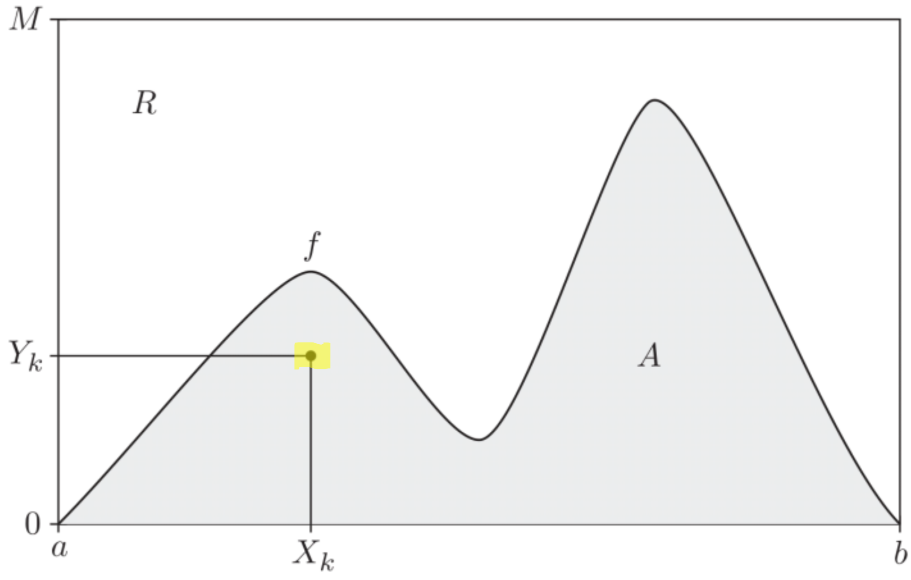


Figure 1.5 *Illustration of the rejection sampling method where the graph of the target density is contained in a rectangle* $R = [a, b] \times [0, M]$. *In this case the proposals are uniformly distributed on the rectangle R and a proposal is accepted if it falls into the shaded region.*

$$g(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$c \cdot g(x) = \frac{c}{b-a} = M \geq f(x)$$

Accept X if $Y = U \cdot M \leq f(X)$, $U \sim \text{Uniform}(0,1)$

$$\Longleftrightarrow \quad (X, Y) \sim \text{Uniform}\big((a,b) \times (0,M)\big)$$

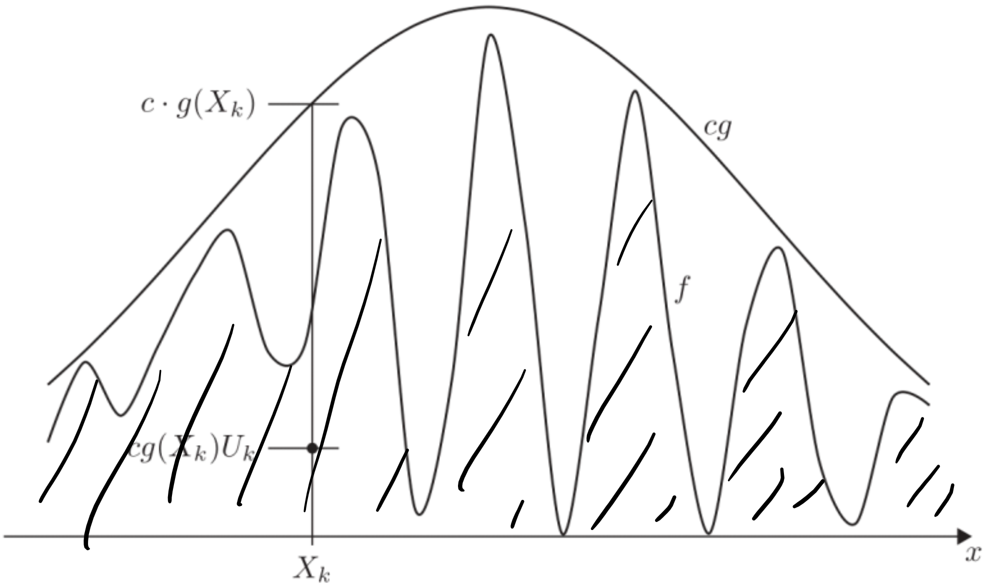


Figure 1.3 *Illustration of the envelope rejection sampling method from algorithm 1.22. The proposal (X_k, $cg(X_k)\,U_k$) is accepted, if it falls into the area underneath the graph of f. In Section 1.4.4 we will see that the proposal is distributed uniformly on the area under the graph of cg.*

$A = \{(x,y) : y \le c\,g(x)\}$, $|A| = c$

Sample from A and accept (x,y) if they belong to $B = \{(x,y) : y \le f(x)\}$