Part 2j: Continuous Markov chains

<u>Textbook</u>: pp. 56-58

Introduction to continuous Markov chains

- The support of X_j is a continuous set, usually a subset of R^d.
- The marginal distribution of X_j is a density over the support.
- The transition probability from X_{j-1} to X_j is a stochastic kernel.

Stochastic kernels

Definition 2.27 A *transition kernel* is a map $P(\cdot, \cdot)$ such that:

(a) $P(x, A) \geq 0$ for all $x \in \mathbb{R}^d$ and all $A \subseteq \mathbb{R}^d$; and

(b) $P(x, \cdot)$ is a probability distribution on $\mathbb{R}^d$ for all $x \in \mathbb{R}^d$.

Definition 2.28 A *transition density* is a map $p \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that:

(a) $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}^d$; and

(b) $\int_{\mathbb{R}^d} p(x, y)\, dy = 1$ for all $x \in \mathbb{R}^d$.

If the Markov chain X can be described by a transition density, then we have

$$P(X_j \in A | X_{j-1} = x) = \int_A p(x, y)\, dy$$

for all $x \in \mathbb{R}^d$.

Simulation of a continuous Markov process

Algorithm 2.31 (Markov chains with continuous state space)

input:

a probability density $\pi\colon \mathbb{R}^d \to [0, \infty)$

a transition density $p\colon \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$

randomness used:

one sample $X_0 \sim \pi$

samples from the densities $p(x, \cdot)$ for $x \in \mathbb{R}^d$

output:

a path of a Markov chain with initial distribution π and transition matrix P

1: generate $X_0 \sim \pi$
2: output X_0
3: **for** $j = 1, 2, 3, \ldots$ **do**
4: generate $X_j \sim p(X_{j-1}, \cdot)$
5: output X_j
6: **end for**

The stationary distribution and convergence

Definition 2.30 A probability density $\pi: \mathbb{R}^d \to [0, \infty)$ is a *stationary density* for a Markov chain on the state space $\mathbb{R}^d$ with transition density p, if it satisfies

$$\int_S \pi(x) p(x, y)\, dx = \pi(y)$$

for all $y \in \mathbb{R}^d$.

- Under appropriate regularity conditions of the transition density $p(x, y)$ one can prove the convergence of the marginal distributions to the stationary distribution.

An example

Example 2.29 On $S = \mathbb{R}$, let $X_0 = 0$ and

$$X_j = \frac{1}{2} X_{j-1} + \varepsilon_j$$

for all $j \in \mathbb{N}$, where $\varepsilon_j \sim \mathcal{N}(0, 1)$ i.i.d. is a Markov chain with state space $S = \mathbb{R}$.

$AR(1), \quad a_1 = \frac{1}{2}, \quad \sigma^2 = 1$

$\pi = N(0, \tau^2)$

$$Var(X_j) = Var\left(\frac{1}{2} X_{j-1}\right) + Var(\varepsilon_j)$$

$$\Rightarrow \tau^2 = \frac{1}{4}\tau^2 + 1$$

$$\Rightarrow \tau^2 = \frac{1}{1 - 1/4} = \frac{4}{3}$$

$$X_{j-1} \sim N\left(0, \frac{4}{3}\right) \Rightarrow X_j \sim N\left(0, \frac{4}{3}\right)$$