Part 2i: Convergence of Markov chains

<u>Textbook</u>: pp. 55-56

Introduction to convergence of Markov chains

- The convergence theorem in Markov chains deals with the convergence of the marginal distribution for an increasing lag in time.
- Under appropriate regularity conditions the convergence is to the stationary distribution.
- The convergence theorem is a basis for important simulation algorithms.

The stationary distribution

Definition 2.25 Let X be a time-homogeneous Markov chain with transition matrix P. A probability vector π is called a *stationary distribution* of X, if $\pi^\top P = \pi^\top$, that is if

$$\sum_{x \in S} \pi_x p_{xy} = \pi_y \tag{2.4}$$

for all $y \in S$.

- $\pi^T P = \pi^T \iff (P^T)\pi = \pi$
- P^T is not symmetric.
- P^T and P share the same eigenvalues.
- $P\vec{1} = \vec{1} \implies \lambda = 1$ is and eigenvalue of P^T.
- $\lambda = 1$ is the largest eigenvalue.
- The multiplicity of $\lambda = 1$, under appropriate regularity conditions, is one.

An example

Example 2.26 On the state space $S = \{1, 2, 3\}$, consider the Markov chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/5 & 0 & 4/5 \end{pmatrix}$$

and initial distribution $\alpha = (1, 0, 0)$.

$$\pi = \begin{pmatrix} 2/9 \\ 2/9 \\ 5/9 \end{pmatrix} \Rightarrow \pi^T P = \pi^T$$

A convergence theorem

A Markov chain is *ergodic* if it is both irreducible and aperiodic.

Theorem 1 (Fundamental Theorem of Markov chains) *If a discrete time, finite, and time-homogeneous Markov chain is ergodic then it will have a unique stationary distribution that assignns positive probability to every state and the chain, starting with any initial distribution, will attain the stationary distribution in the limit.*[1]

[1] The result extends also for chains with denumerable state spaces.

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