Part 2k: The Poisson distribution

Textbook: pp. 58-67

Introduction to Poisson processes

- Poisson processes are important models for points processes.
- (In the context of simulations, they are a side issue).
- Here we summarize some of the properties of the Poisson distribution.
- In the next video we consider the Poisson process on the line and in the third video we will extend the discussion to the Poisson process on the plain.

The Poisson distribution

$X \sim \text{Poisson}(\lambda)$

$P_X(x) = e^{-\lambda} \dfrac{\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$

$\mathbb{E}(X) = \lambda$

$\text{Var}(X) = \lambda$

$M_X(s) = e^{\lambda(e^s - 1)}, \quad s \in \mathbb{R}$

A sum of independent Poisson random variables

If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent then
$X + Y \sim \text{Poisson}(\lambda + \mu)$

$$M_{X+Y}(s) = M_X(s) \cdot M_Y(s)$$

$$= e^{\lambda(e^s - 1)} \cdot e^{\mu(e^s - 1)}$$

$$= e^{(\lambda+\mu)(e^s - 1)}$$

$$\Rightarrow X + Y \sim \text{Poisson}(\lambda + \mu)$$

Conditional Poisson random variables

If $X \sim \text{Poisson}(\lambda)$ *and* $Y \sim \text{Poisson}(\mu)$ are independent then

$X \mid \{X+Y=s\} \sim \text{Binomial}\left(s, \frac{\lambda}{\lambda+\mu}\right)$

$$P(X=x, S=s) = P(X=x, Y=s-x)$$

$$= P(X=x)\,P(Y=s-x)$$

$$= e^{-\lambda}\frac{\lambda^x}{x!} \cdot e^{-\mu}\frac{\mu^{s-x}}{(s-x)!}$$

$$P(S=s) = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^s}{s!}$$

$$P(X=x \mid S=s) = \frac{s!}{x!(s-x)!}\,\frac{\lambda^x \mu^{s-x}}{(\lambda+\mu)^s}$$

$$= \binom{s}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{s-x}, \quad x \in \{0, 1, \ldots, s\}$$

$$X \mid \{S=s\} \sim \text{Binomial}\left(s, \frac{\lambda}{\lambda+\mu}\right)$$

The thinning of the Poisson distribution

Assume that $N \sim \text{Poisson}(\lambda)$ and $X_1, X_2, \dots \sim \text{Binomial}(1, p)$ are all independent .

Define the compound variables: $N_1 = \sum_{i=1}^{N} X_i$, $N_0 = \sum_{i=1}^{N} (1 - X_i)$.

Then:

- N_1, N_0 are independent,
- $N_1 \sim \text{Poisson}(\lambda p)$ and
- $N_0 \sim \text{Poisson}(\lambda(1-p))$.

$$\mathbb{E}\left(e^{s_1 N + s_0 N_0} \mid N = n\right)$$

$$= \mathbb{E}\left(e^{\sum_{i=1}^{n} s_1 X_i + s_0(1-X_i)}\right)$$

$$= \prod_{i=1}^{n} \mathbb{E}\left(e^{s_1 X_i + s_0(1-X_i)}\right)$$

$$= \left(pe^{s_1} + (1-p)e^{s_0}\right)^n$$

$$\mathbb{E}\left(e^{s_1 N_1 + s_0 N_0}\right) = \mathbb{E}\,\mathbb{E}\left(e^{s_1 N_1 + s_0 N_0} \mid N\right)$$

$$= \sum_{n=0}^{\infty} \left(pe^{s_1} + (1-p)e^{s_0}\right)^n e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} e^{\lambda(pe^{s_1} + (1-p)e^{s_0})}$$

$$= e^{\lambda p(e^{s_1} - 1)} \cdot e^{\lambda(1-p)(e^{s_0} - 1)}$$

$$N_1 \sim \text{Poisson}(p\lambda)$$
$$N_0 \sim \text{Poisson}((1-p)\lambda)$$
> independent

The Poisson distribution as a limit

Assume that $X_i^{(n)} \sim \text{Binomial}\left(1, p_i^{(n)}\right)$, $1 \le i \le n$, are independent and define $N^{(n)} = \sum_{i=1}^{n} X_i^{(n)}$. If:

- $\sum_{i=1}^{n} p_i^{(n)} \to \lambda$, $0 < \lambda < \infty$,
- $\sum_{i=1}^{n} \left(p_i^{(n)}\right)^2 \to 0$

Then $N^{(n)} \rightsquigarrow \text{Poisson}(\lambda)$.

$$M_{N^{(n)}}(s) = \prod_{i=1}^{n} \left(p_i^{(n)} e^s + 1 - p_i^{(n)}\right)$$

$$\log M_{N^{(n)}}(s) = \sum_{i=1}^{n} \log\left(1 + p_i^{(n)}(e^s - 1)\right)$$

$$\approx \sum_{i=1}^{n} \left[p_i^{(n)}(e^s - 1) + \frac{1}{2}\left(p_i^{(n)}\right)^2 (e^s - 1)^2\right]$$

$$\to \lambda(e^s - 1)$$

$$M_{N^{(n)}}(s) \to e^{\lambda(e^s - 1)} \Rightarrow N^{(n)} \rightsquigarrow \text{Poisson}(\lambda)$$