Part 2l: Poisson process on the line

<u>Textbook</u>: pp. 56-58

Introduction to Poisson processes

- The Poisson process is an important example of a point process.
- The number of point over the support is Poisson. Given the number of points, they are i.i.d. according to a given distribution.
- In the book the process is defined via its properties. We will go the other way around and prove the properties.
- At the end of the video we will discuss briefly the homogeneous Poisson process.

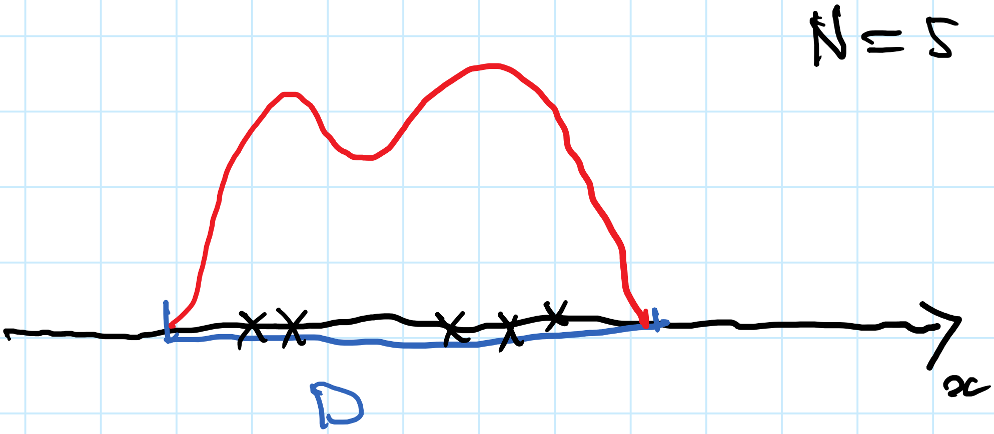
The definition of a Poisson process

Given a region D, a density f over D, and a rate Λ:

- The number of points X_i in the region is $N \sim \text{Poisson}(\Lambda)$.
- Given N, the points are i.i.d. with distribution f: $X_i \sim f$.

The Poisson process is the collection $\Pi = \{X_1, X_2, \dots, X_N\}$.

<u>Remark</u>: f may correspond to a σ-finite measure.



Simulation of a Poisson process

Algorithm 2.36 (Poisson process)

input:

an intensity function $\lambda\colon \mathbb{R}^d \to \mathbb{R}$

a set $D \subseteq \mathbb{R}^d$ with $\Lambda(D) < \infty$ where Λ is given by (2.5)

randomness used:

$N \sim \mathrm{Pois}(\Lambda(D))$

i.i.d. samples $X_i \sim \mathbb{1}_D \lambda(\cdot)/\Lambda(D)$ for $i = 1, 2, \ldots, N$

output:

a sample from the Poisson process on D with intensity λ

1: generate $N \sim \mathrm{Pois}(\Lambda(D))$
2: $\Pi \leftarrow \emptyset$
3: **for** $i = 1, 2, \ldots, N$ **do**
4: generate $X_i \sim \frac{1}{\Lambda(D)} \mathbb{1}_D \lambda(\cdot)$
5: $\Pi \leftarrow \Pi \cup \{X_i\}$
6: **end for**
7: return Π

The function $\mathbb{1}_D \lambda/\Lambda(D)$ in the algorithm, is given by

$$f(x) = \frac{\mathbb{1}_D \lambda(x)}{\Lambda(D)} = \begin{cases} \frac{\lambda(x)}{\Lambda(D)} & \text{if } x \in D \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

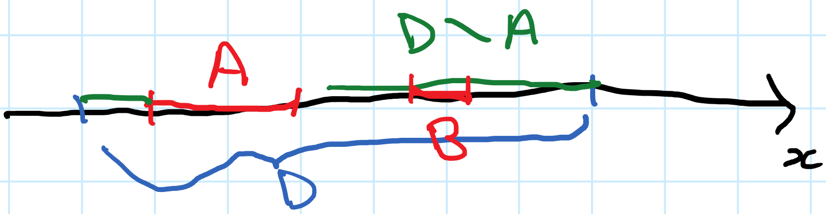
Basic properties of the Poisson process

Definition 2.34 A *Poisson process* on a set $D \subseteq \mathbb{R}^d$ with *intensity function* $\lambda: \mathbb{R}^d \to [0, \infty)$ is a random set $\Pi \subseteq D$ such that the following two conditions hold:

(a) If $A \subseteq D$, then $|\Pi \cap A| \sim \text{Pois}(\Lambda(A))$ where $|\Pi \cap A|$ is the number of points of Π in A and

$$\Lambda(A) = \int_A \lambda(x)\, dx. \tag{2.5}$$

(b) If $A, B \subseteq D$ are disjoint, then $|\Pi \cap A|$ and $|\Pi \cap B|$ are independent.



$N_A \sim \text{Poisson}(P(A) \Lambda)$

$N_{D \setminus A} \sim \text{Poisson}((1 - P(A)) \cdot \Lambda)$

> independent

$f_A(x) = \dfrac{f(x)}{P(A)}, \quad x \in A$

$f_{D \setminus A}(x) = \dfrac{f(x)}{1 - P(A)}, \quad x \in D \setminus A$

$\Pi = \Pi_A \cup \Pi_{D \setminus A}$

More properties of the Poisson process

Lemma 2.35

(a) Let Π be a Poisson process on D with intensity λ and let $A \subseteq D$. Then the restriction $\Pi \cap A$ of the process Π to the set A is a Poisson process on A with intensity λ.

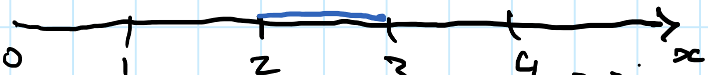
(b) Let Π_1 and Π_2 be independent Poisson processes on D_1 and D_2, respectively, and let $D_1 \cap D_2 = \emptyset$. Assume that both processes have the same intensity $\lambda \colon \mathbb{R}^d \to [0, \infty)$. Then $\Pi = \Pi_1 \cup \Pi_2$ is a Poisson process on $D = D_1 \cup D_2$ with intensity λ.

The homogeneous Poisson process

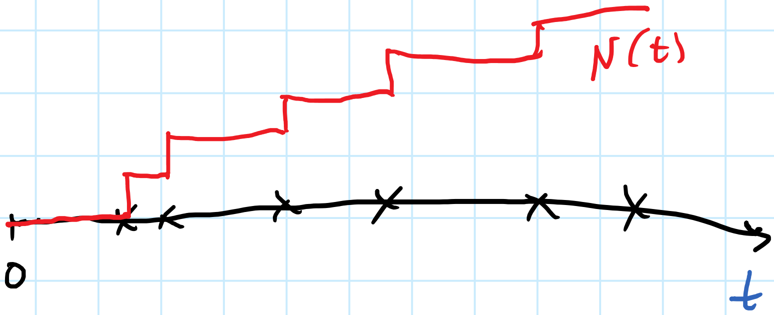
- If X is "distributed" according to the Lesbegue measure on the positive numbers then the process is called *homogenuous*, or simply a *Poisson process*.

$f(x) = 1, \quad 2 \le x < 3$

$\Lambda = \lambda$



- One may associate the point process with a counting process, which is also called the *Poisson process*.



$N(t) \sim \text{Poisson}(\lambda t)$

Independent increment