Part 2h: Discrete Markov chains

Textbook: pp. 51-54

Introduction to Discrete Markov chains

- The support of X_j is a descrete set.
- The marginal distribution of X_j is a stochastic vector.
- The transition probability from X_{j-1} to X_j is a stochastic matrix.

Stochastic vectors

Definition 2.20 A vector $\pi \in \mathbb{R}^S$ is called a *probability vector*, if $\pi_x \geq 0$ for all $x \in S$ and $\sum_{x \in S} \pi_x = 1$.

$$S = \{1, 2, 3\}$$

$$\pi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\pi = \begin{pmatrix} 2/9 \\ 2/9 \\ 5/9 \end{pmatrix}$$

Definition 2.16 If the transition probabilities given by the right-hand side of (2.2) do not depend on the time j, the Markov chain X is called *time-homogeneous*.

Stochastic matrices

Definition 2.21 A matrix which satisfies the two conditions from lemma 2.19 is called a *stochastic matrix*.

Lemma 2.19 Let $P^{S\times S}$ be the transition matrix of a Markov chain with state space S. Then $P = (p_{xy})_{x,y\in S}$ has the following properties:

(a) $p_{xy} \geq 0$ for all $x, y \in S$.

(b) $\sum_{y\in S} p_{xy} = 1$ for all $x \in S$.

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/5 & 0 & 4/5 \end{pmatrix}$$

Simulation of a discrete Markov process

Algorithm 2.22 (Markov chains with discrete state space)

input:

a finite or countable state space S

a probability vector $\pi \in \mathbb{R}^S$

a stochastic matrix $P = (p_{xy})_{x,y\in S} \in \mathbb{R}^{S\times S}$

randomness used:

samples from discrete distributions on S

output:

a path of a Markov chain with initial distribution π and transition matrix P

1: generate $X_0 \in S$ with $P(X_0 = x) = \pi_x$ for all $x \in S$
2: output X_0
3: **for** $j = 1, 2, 3, \ldots$ **do**
4: generate $X_j \in S$ with $P(X_j = x) = p_{X_{j-1},x}$ for all $x \in S$
5: output X_j
6: **end for**

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/5 & 0 & 4/5 \end{pmatrix}, \quad \pi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$X_0 = 2$

$X_1 = 3$

$X_2 = 3$

$X_3 = 1$

$\vdots$

The time-lag transition matrix

Lemma 2.23 Let X be a time-homogeneous Markov chain with finite state space and transition matrix P. Then

$$P\left(X_{j+k}=y \,\middle|\, X_j=x\right)=\left(P^k\right)_{xy}$$

for all $j,k\in\mathbb{N}_0$ and $x,y\in S$, where $P^k=P\cdot P\cdots P$ is the kth power of the transition matrix P.

Proof:

$$\begin{aligned}
P(X_2=y\mid X_0=x) &= \sum_{x_1} P(X_2=y\mid X_1=x_1, X_0=x)\,P(X_1=x_1\mid X_0=x)\\
&= \sum_{x_1} P(X_2=y\mid X_1=x_1)\,P(X_1=x_1\mid X_0=x)\\
&= \sum_{x_1} P_{x_1,y}\cdot P_{x,x_1}\\
&= \sum_{x_1} P_{x,x_1}\,P_{x_1,y} = \left(P^2\right)_{xy}
\end{aligned}$$

$$\begin{aligned}
P(X_3=y\mid X_0=x) &= \sum_{x_2} P(X_3=y\mid X_2=x_2)\,P(X_2=x_2\mid X_0=x)\\
&= \sum_{x_2} P_{x_2,y}\,\left(P^2\right)_{x,x_2} = \left(P^3\right)_{xy}
\end{aligned}$$

$\vdots$

$$P(X_j = y \mid X_0 = x) = (P^j)_{xy}$$

The time-lag marginal distribution

Lemma 2.24 Let X be a time-homogeneous Markov chain with finite state space and transition matrix P and initial distribution π. Then we have

$$P(X_j = y) = (\pi^\top P^j)_y \tag{2.3}$$

for all $y \in S$.

$$P(X_j = y) = \sum_{x_0} P(X_j = y \mid X_0 = x_0) P(X_0 = x_0)$$

$$= \sum_{x_0} (P^j)_{x_0, y} \cdot \pi_{x_0}$$

$$= \sum_{x_0} \pi_{x_0} (P^j)_{x_0, y} = (\pi^\top P^j)_y$$

An example

Example 2.26 On the state space $S = \{1, 2, 3\}$, consider the Markov chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/5 & 0 & 4/5 \end{pmatrix}$$

and initial distribution $\alpha = (1, 0, 0)$.