Part 1e: The Ratio-of-Uniforms method

Textbook: pp. 33-36

Introduction to the Ratio-of-Uniforms method

- An application of the transformation method.
- Exploits a special case where both $\vec{X}$ and $\vec{Y} = \varphi(\vec{X}) = g(\vec{X})$ are uniformly distributed.
- The distributions are uniform (over different sets) since the source distribution is uniform and the determinant of the Jacobian is constant.
- The set of the image is associated with the graph of a density.

Ratio-of-Uniforms Method

Theorem 1.39 (ratio-of-uniforms method) Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be such that $Z = \int_{\mathbb{R}^d} f(x)\,dx < \infty$ and let X be uniformly distributed on the set

$$A = \left\{ (x_0, x_1, \ldots, x_d) \;\middle|\; x_0 > 0, \frac{x_0^{d+1}}{d+1} < f\left(\frac{x_1}{x_0}, \ldots, \frac{x_d}{x_0}\right) \right\} \subseteq \mathbb{R}_+ \times \mathbb{R}^d.$$

Then the vector

$$Y = \left(\frac{X_1}{X_0}, \ldots, \frac{X_d}{X_0}\right)$$

has density $\frac{1}{Z} f$ on $\mathbb{R}^d$.

Proof:

$$g_1(\vec{x}) = g_1(x_0) = \frac{Z}{d+1} x_0^{d+1} = y_0$$

$$g_i(\vec{x}) = \frac{x_i}{x_0} = y_i, \quad 1 \le i \le d$$

$$x_0 = w_0(\vec{y}) = w_0(y_0) = \left[\frac{d+1}{Z} \cdot y_0\right]^{\frac{1}{d+1}}$$

$$x_i = w_i(\vec{y}) = w_i(y_i, y_0) = y_i \cdot w_0(y_0), \quad 1 \le i \le d$$

$$J_w(\vec{y}) = \begin{pmatrix} w_0' & 0 & 0 & 0 & \cdots & 0 \\ y_1 w_0' & w_0 & 0 & 0 & \cdots & 0 \\ y_2 w_0' & 0 & w_0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ y_d \cdot w_0' & 0 & 0 & 0 & \cdots & w_0 \end{pmatrix}$$

$$\det J_w(\vec{y}) = w_0' \cdot w_0^d$$

$$= \left(\frac{d+1}{Z}\right)^{\frac{1}{d+1}} \cdot \frac{1}{d+1} y_0^{\frac{1}{d+1}-1} \times \left[\frac{d+1}{Z} y_0\right]^{\frac{d}{d+1}}$$

$$= \frac{1}{Z}$$

$$A = \left\{ (x_0, \dots, x_d) : \frac{x_0^{d+1}}{d+1} < f\left(\frac{x_1}{x_0}, \dots, \frac{x_d}{x_0}\right) \right\} \cap [0,1]^{d+1}$$

The joint distributio of $(X_0, \dots, X_d)$, conditional on the set A, is Uniform(A).

$$g(A) = B$$
$$= \left\{ (y_0, \dots, y_d) : 0 < y_0 < f(y_1, \dots, y_d) \right\}$$

The joint density of $(Y_0, \dots, Y_d)$ is $1_B(\vec{y}) \cdot \frac{1}{Z}$.

$\Rightarrow$ The marginal density of $(Y_1, \dots, Y_d)$ is f.

Example: The Cauchy distribution

Example 1.40 The Cauchy distribution has density

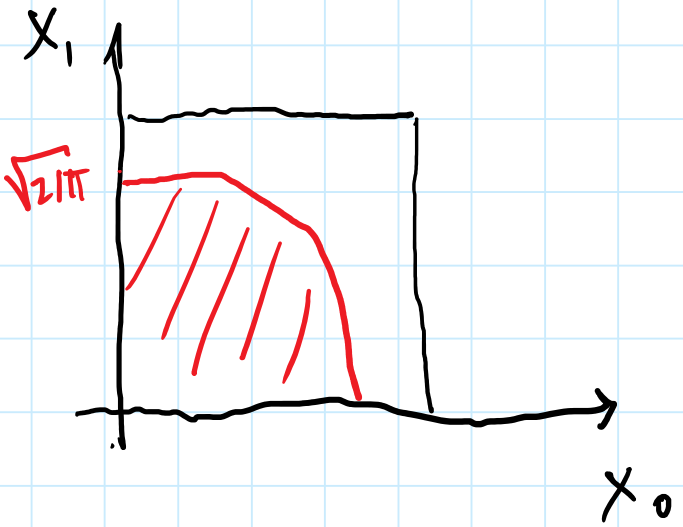
$$f(x) = \frac{1}{\pi(1+x^2)}.$$

$d = 1$, $0 < x_0, x_1 < 1$

$$A = \left\{ (x_0, x_1) : \frac{x_0^2}{2} < \frac{1}{\pi\left(1+\left(\frac{x_1}{x_0}\right)^2\right)} \right\}$$

$$= \left\{ (x_0, x_1) : \frac{\pi}{2} x_0^2 < \frac{x_0^2}{x_0^2 + x_1^2} \right\}$$

$$= \left\{ (x_0, x_1) : \sqrt{x_0^2 + x_1^2} \leq \sqrt{2/\pi} \right\}$$



$$Y = \frac{X_1}{X_0} \sim f$$