Part 2m: Poisson process in R^d

<u>Textbook</u>: pp. 58-67

The definition of a Poisson process

Given a region $D \in R^d$, a density f over D, and a rate Λ:

- The number of points X_i in the region is $N \sim \text{Poisson}(\Lambda)$.
- Given N, the points are i.i.d. with distribution f: $X_i \sim f$.

The Poisson process is the collection $\Pi = \{X_1, X_2, \dots, X_N\}$.

Remark: f may correspond to a σ-finite measure.

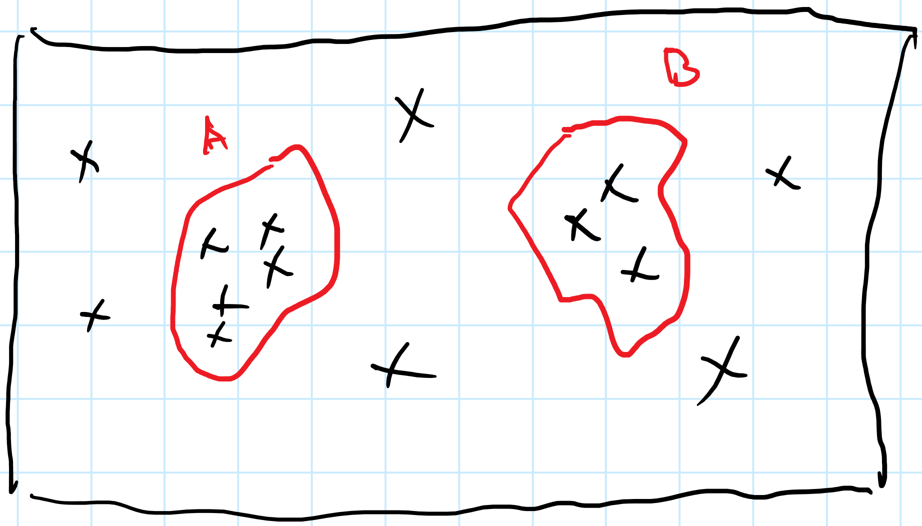
Basic properties of the Poisson process

Definition 2.34 A *Poisson process* on a set $D \subseteq \mathbb{R}^d$ with *intensity function* $\lambda : \mathbb{R}^d \to [0, \infty)$ is a random set $\Pi \subseteq D$ such that the following two conditions hold:

(a) If $A \subseteq D$, then $|\Pi \cap A| \sim \text{Pois}(\Lambda(A))$ where $|\Pi \cap A|$ is the number of points of Π in A and

$$\Lambda(A) = \int_A \lambda(x)\, dx. \tag{2.5}$$

(b) If $A, B \subseteq D$ are disjoint, then $|\Pi \cap A|$ and $|\Pi \cap B|$ are independent.



$(N_A \,,\, N_B \,,\, N_{D \setminus (A \cup B)})$

Thinning a Poisson process

Algorithm 2.41 (thinning method for Poisson processes)

input:
  intensity functions $\lambda, \tilde\lambda \colon \mathbb{R}^d \to \mathbb{R}$ with $\tilde\lambda < \lambda$
  a set $D \subseteq \mathbb{R}^d$ with $\Lambda(D) < \infty$ where Λ is given by (2.5)
randomness used:
  $N \sim \mathrm{Pois}(\Lambda(D))$
  i.i.d. samples $X_i \sim \mathbb{1}_D \lambda(\cdot)/\Lambda(D)$ for $i = 1, 2, \ldots, N$
  $U_i \sim \mathcal{U}[0, 1]$ i.i.d.
output:
  a sample from the Poisson process on D with intensity $\tilde\lambda$

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generate N ~ Pois(Λ(D))
Π̃ ← Ø
for i = 1, 2, ..., N do
    generate X_i ~ (1/Λ(D)) 1_D λ(·)
    generate U_i ~ U[0, 1]
    if U_i ≤ λ̃(X_i)/λ(X_i) then
        Π̃ ← Π̃ ∪ {X_i}
    end if
end for
return Π̃
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$$\Pi = \{X_1, X_2, \ldots, X_N$$

$$Y_1, Y_2, \ldots, Y_N, \qquad Y_i = \begin{cases} 1 & U_i \le \dfrac{\tilde\lambda(X_i)}{\lambda(X_i)} \\ 0 & > \end{cases}$$

$$\tilde\Pi = \{X_i : Y_i = 1, \ 1 \le i \le N\}$$

$$\tilde N = \#\tilde\Pi \sim \mathrm{Poisson}\left(P(Y=1) \cdot \Lambda(D)\right)$$

Proposition 2.42

Proposition 2.42 Let Π be a Poisson process on $D \subseteq \mathbb{R}^d$ with intensity $\lambda\colon \mathbb{R}^d \to [0,\infty)$. Let $\tilde{\lambda} : \mathbb{R}^d \to [0,\infty)$ such that $\tilde{\lambda}(x) \leq \lambda(x)$ for all $x \in D$ and define a random subset $\tilde{\Pi} \subseteq \Pi$ by randomly including each point $x \in \Pi$ into $\tilde{\Pi}$ with probability $\tilde{\lambda}(x)/\lambda(x)$, independently of each other and of Π. Then $\tilde{\Pi}$ is a Poisson process with intensity function $\tilde{\lambda}$.