The Erdős–Simonovits compactness conjecture needs more assumptions

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For a graph H and an integer n, let $\mathrm{ex}(n,H)$ denote the *extremal number* of H, namely the maximum number of edges in an n-vertex H-free graph. Similarly, if $\mathcal{F}$ is a family of graphs, then $\mathrm{ex}(n,\mathcal{F})$ denotes the maximum number of edges in an n-vertex graph containing no copy of any $H \in \mathcal{F}$.

Thanks to the Erdős–Stone–Simonovits theorem [1, 3], it is known that

$$\mathrm{ex}(n,\mathcal{F}) = \left(1 - \frac{1}{\chi(\mathcal{F})-1} + o(1)\right)\binom{n}{2},$$

where $\chi(\mathcal{F}) \coloneqq \min_{H\in\mathcal{F}} \chi(H)$. In particular, this gives a precise asymptotic for $\mathrm{ex}(n,\mathcal{F})$ whenever $\mathcal{F}$ contains no bipartite graph. However, extremal numbers of bipartite graphs remain poorly understood, despite decades of intensive research. For more on this topic, see e.g. the survey of Füredi and Simonovits [4].

A central open problem in the field is the Erdős–Simonovits compactness conjecture.

Conjecture ([2, Conjecture 1])**.** *For every finite collection $\mathcal{F}$ of graphs, there exists some $H \in \mathcal{F}$ and some $c > 0$ so that*

$$\mathrm{ex}(n,\mathcal{F}) \geq c \cdot \mathrm{ex}(n,H)$$

for all n.

This is the form the conjecture is stated in [2], as well as in later sources such as [4]. However, there is a simple counterexample to this statement, as pointed out to me by Jordan Lefkowitz. Recall that $2K_2$ denotes a matching of two edges.

Observation (Lefkowitz)**.** *Let $\mathcal{F} = \{K_{1,2}, 2K_2\}$. Then $\mathrm{ex}(n,H) = \Theta(n)$ for all $H \in \mathcal{F}$, while $\mathrm{ex}(n,\mathcal{F}) = 1$. In other words, $\mathcal{F}$ is a counterexample to the compactness conjecture.*

Proof. The claim that $\mathrm{ex}(n,\mathcal{F}) = 1$ follows from the fact that any graph with at least two edges contains either a vertex of degree at least 2 (and thus a copy of $K_{1,2}$) or two vertex-disjoint edges (and thus a copy of $2K_2$). Moreover, for $H \in \mathcal{F}$, the upper bound $\mathrm{ex}(n,H) = O(n)$ follows from the simple fact that every forest has linear extremal number (see e.g. [4, Theorem 2.32]). For the lower bound on $\mathrm{ex}(n,K_{1,2})$ simply consider the perfect matching $\lfloor \frac{n}{2} \rfloor K_2$, and for the lower bound on $\mathrm{ex}(n,2K_2)$ consider the star $K_{1,n-1}$. $\square$

There are (at least) two natural strengthenings of the compactness conjecture which rule out this simple counterexample. The first is to assume that no graph in $\mathcal{F}$ is a forest, and the second is to assume that every graph in $\mathcal{F}$ is connected. It is not clear which of these (if any) is the "correct" statement of the compactness conjecture.

References

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