1. (a) Using the fact that $r(k) < 4^k$, prove that $r(k;q) < 4^{4^{4^{\cdot^{\cdot^{\cdot^{k}}}}}}$, where the number of 4s is $\lceil \log_2 q \rceil$.

   (b) Prove Theorem 2.1.5 in the notes. In particular, derive the bound $r(k;q) < q^{qk}$, which is much stronger than that in part (a).

2. Prove that $r(3;q) \leqslant \lceil e \cdot q! \rceil$, where e is Euler's constant.

3. Prove that, for any fixed k, the limit

$$\lim_{q \to \infty} r(k;q)^{1/q}$$

   exists. Conclude that Open problem 2.3.2 from the notes is a well-posed question.

   Hint: Use Fekete's lemma. If you've never heard of Fekete's lemma, look it up and try to prove it before using it!

4. The proof of Lemma 3.1.1 in the lecture notes is not 100% correct, as mentioned in Footnote 1. In this problem you will correct this.

   Let G satisfy the assumptions of Lemma 3.1.1, and let $t = N/M$. Let H be a random induced subgraph of G obtained by picking exactly t vertices of G, uniformly at random (i.e. each of the $\binom{N}{t}$ choices is equally likely). Prove that with positive probability, H has no independent set of order k, and hence

$$r(s,k) > t = \frac{N}{M}.$$

5. In the approach using Lemma 3.2.1, we lower-bound $r(k;q)$ by picking $q-2$ random homomorphisms to some K_k-free graph G, and using the last two colors to randomly color all remaining edges. Instead, we could have used $q-r$ random homomorphisms (for some $r < q$), and r random colors for the remaining edges. Prove that picking $r = 2$ gives the strongest bounds, hence this extra generality ends up not being useful.

6. Let $f, g_1, \ldots, g_q : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that there exist $\varepsilon, \delta > 0$ such that whenever $x, y \in \mathbb{R}$ satisfy $f(x) - f(y) \geqslant \varepsilon$, then

$$\max_{i \in [\![q]\!]} (g_i(x) - g_i(y)) \geqslant \delta.$$

   Prove that if $g_1, \ldots, g_q$ are all bounded, then f is bounded as well.

$\star$7. Prove that $r(3,3,3) = 17$.

$\star$ means that a problem is hard.

? means that a problem is open.

✥ means that a problem is on a topic beyond the scope of the course.

$\star$ 8. Prove Lemma 3.3.2 in the lecture notes. Use it to deduce Theorem 2.3.1, which remains the best known lower bound on $r(k;q)$ for fixed $q \geqslant 3$.

$\oplus$ 9. In this problem, you will see the original approach of Conlon–Ferber to the improved lower bounds on multicolor Ramsey numbers.

For a positive integer t, let $V_t \subseteq \mathbb{F}_2^t$ denote the subspace consisting of all vectors in $\mathbb{F}_2^t$ with an even number of entries equal to 1. Define a graph G_t with vertex set V_t by setting $x \sim y$ if $x \cdot y = 1$, where $x \cdot y = \sum_{i=1}^t x_i y_i$ denotes the usual dot product on $\mathbb{F}_2^t$.

(a) Prove that if t is even, then G_t is K_t-free.

(b) Prove that if t is odd, then G_t is K_{t+1}-free.

(c) Prove that every independent set in G_t is contained in a vector subspace of dimension at most $t/2$.

$\star$ (d) Prove that the number of independent sets in G_t of order at most t is at most $2^{\frac{5}{8}t^2+o(t^2)}$.

(e) Using the facts above and Lemma 3.2.1 from the notes, obtain a new proof that $r(k;q) \geqslant (2^{\frac{3}{8}q-\frac{1}{4}})^{k-o(k)}$ for $q \geqslant 3$.

$\star$ (f) Working with $t = 2k$, and randomly sampling a subset of V_t, obtain a different proof that $r(k;2) \geqslant 2^{\frac{k}{2}-o(k)}$.

? (g) In the proof of (f), you showed that a random induced subgraph of G_t, where $t = 2k$, has no clique or independent set of order k. Can you find an *explicit* description of such a subset (thus resolving Open problem 2.2.3)?