Homework 3

Exercise 3(c): Let n be an integer and let $0 \leqslant d \leqslant n$ be a real number. Consider a random n-vertex graph G formed by including each edge independently with probability d/n.

Prove that if $d = \omega(1)$, the average degree of G is $(1 + o(1))d$ with probability $1 - o(1)$.

Solution. Let X denote the number of edges of G. This is a binomial random variable with distribution $\mathrm{Bin}(\binom{n}{2}, p)$, where $p = d/n$. In particular, the expectation of X is

$$\mathbb{E}[X] = p\binom{n}{2} = \frac{d}{n} \cdot \binom{n}{2} = \frac{d(n-1)}{2} = \frac{dn}{2} - \frac{d}{2}.$$

We first claim that with probability $1 - o(1)$, we have that $X = (1 + o(1))dn/2$. This follows from essentially any of the standard concentration results for the binomial distribution; for concreteness, we give an elementary proof using only Chebyshev's inequality.

Since X is binomially distributed, its variance is given by

$$\mathrm{Var}(X) = p(1-p)\binom{n}{2} \leqslant p\binom{n}{2} \leqslant \frac{dn}{2}.$$

Chebyshev's inequality thus implies that for any $t > 0$, we have

$$\Pr\left(|X - \mathbb{E}[X]| \geqslant t \cdot \sqrt{\frac{dn}{2}}\right) \leqslant \frac{1}{t^2}. \tag{1}$$

We now note that

$$\begin{aligned}
\Pr\left(\left|X - \frac{dn}{2}\right| \geqslant d\sqrt{\frac{n}{2}}\right) &\leqslant \Pr\left(|X - \mathbb{E}[X]| \geqslant d\sqrt{\frac{n}{2}} - \frac{d}{2}\right) \\
&\leqslant \Pr\left(|X - \mathbb{E}[X]| \geqslant \frac{d}{2}\sqrt{\frac{n}{2}}\right) \\
&= \Pr\left(|X - \mathbb{E}[X]| \geqslant \frac{\sqrt{d}}{2}\sqrt{\frac{dn}{2}}\right) \\
&\leqslant \frac{4}{d},
\end{aligned}$$

where the first inequality uses the fact that $|\mathbb{E}[X] - \frac{dn}{2}| = \frac{d}{2}$, the second holds for $n \geqslant 2$ (which we are allowed to assume since we are working in the $n \to \infty$ limit), and the final holds by plugging in $t = \sqrt{d}/2$ into (1).

Recall that the average degree of any n-vertex graph G is equal to $2e(G)/n$. Hence the average degree in our random graph is $2X/n$. Let $Y = 2X/n$ be this average degree, and note that the above implies

$$\Pr\left(|Y - d| \geqslant d\sqrt{\frac{2}{n}}\right) = \Pr\left(\left|X - \frac{dn}{2}\right| \geqslant d\sqrt{\frac{n}{2}}\right) \leqslant \frac{4}{d}.$$

To conclude the proof, we note that $d\sqrt{2/n} = o(d)$ as $n \to \infty$, and that $4/d = o(1)$ since we assume $d = \omega(1)$. Hence this implies that $Y = (1 + o(1))d$ with probability $1 - o(1)$. $\square$

Exercise 5(a) Prove that, for any fixed $s \geqslant 3$, we have

$$r(s,k) \geqslant k^{\frac{s-1}{2}-o(1)},$$

where the $o(1)$ term tends to 0 as $k \to \infty$.

Solution. Let G be a random graph on $N := (\frac{k}{s\ln k})^{\frac{s-1}{2}}$ vertices, where each pair of vertices is included as an edge independently with probability $p := N^{-\frac{2}{s-1}}$. We begin by claiming that G is K_s-free with probability at least $\frac{5}{6}$. Indeed, any given set of s vertices forms a copy of K_s in G with probability $p^{\binom{s}{2}}$, and there are $\binom{N}{s}$ options for such a set of s vertices. Hence, by the union bound, we have that the probability that G contains a K_s is at most

$$\binom{N}{s}p^{\binom{s}{2}} \leqslant \frac{N^s}{s!}p^{\frac{s^2-s}{2}} = \frac{1}{s!}\left(Np^{\frac{s-1}{2}}\right)^s = \frac{1}{s!} \leqslant \frac{1}{6},$$

where we use our definition of p to see that $Np^{\frac{s-1}{2}} = 1$ and use the fact that $s \geqslant 3$ to conclude that $s! \geqslant 6$. Thus, G is K_s-free with probability at least $\frac{5}{6}$.

We now claim that G has no independent set of order k with probability at least $\frac{1}{2}$. Any set of k vertices forms an independent set with probability $(1-p)^{\binom{k}{2}}$, and there are $\binom{N}{k}$ choices for such a set. Applying the union bound, we find that the probability that G has an independent set of order k is at most

$$\binom{N}{k}(1-p)^{\binom{k}{2}} \leqslant \frac{N^k}{k!}(1-p)^{\frac{k^2-k}{2}} = \frac{1}{(1-p)^{\frac{k}{2}}k!} \cdot \left(N(1-p)^{\frac{k}{2}}\right)^k \tag{2}$$

Note that $p \to 0$ as $k \to \infty$, hence $p \leqslant \frac{1}{2}$ for sufficiently large k. Thus, for sufficiently large k, we have that

$$(1-p)^{\frac{k}{2}}k! \geqslant \left(\frac{1}{2}\right)^{\frac{k}{2}}k! \geqslant \left(\frac{1}{2}\right)^{\frac{k}{2}} \cdot \left(\frac{k}{2}\right)^{\frac{k}{2}} \geqslant \left(\frac{k}{4}\right)^{\frac{k}{2}} \geqslant 2,$$

where the second inequality uses the simple bound $k! \geqslant (k/2)^{k/2}$, and the final inequality also holds for sufficiently large k. On the other hand, using the bound $1-x \leqslant e^{-x}$, we have that

$$N(1-p)^{\frac{k}{2}} \leqslant Ne^{-p\frac{k}{2}} = \exp\left(\ln N - p\frac{k}{2}\right) = \exp\left(\ln N - \frac{k}{2}N^{-\frac{2}{s-1}}\right).$$

Note that, by our choice of N, we have that

$$\frac{k}{2}N^{-\frac{2}{s-1}} = \frac{k}{2} \cdot \frac{s\ln k}{k} = \frac{s\ln k}{2}$$

and that

$$\ln N \leqslant \ln\left(k^{\frac{s-1}{2}}\right) = \frac{(s-1)\ln k}{2}.$$

Therefore, $\ln N - \frac{k}{2}N^{-\frac{2}{s-1}} \leqslant -\frac{\ln k}{2} \leqslant 0$, so $N(1-p)^{\frac{k}{2}} \leqslant 1$. Plugging all of this back into (2), we find that the probability that G has an independent set of order k is at most

$$\frac{1}{(1-p)^{\frac{k}{2}}k!} \cdot \left(N(1-p)^{\frac{k}{2}}\right)^k \leqslant \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Putting this all together, we find that with probability at least $\frac{1}{2}$, G has no independent set of order k, and with probability at least $\frac{5}{6}$, G is K_s-free. Thus, with positive probability, G satisfies both these properties simultaneously, hence there exists an N-vertex graph which is K_s-free and has no independent set of order k. Therefore,

$$r(s,k) > N = \left(\frac{k}{s \ln k}\right)^{\frac{s-1}{2}} = k^{\frac{s-1}{2}-o(1)},$$

since for fixed s and for $k \to \infty$, we have that $s \ln k = k^{o(1)}$. $\square$