Off-diagonal book Ramsey numbers

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Abstract

The book graph $B_n^{(k)}$ consists of n copies of K_{k+1} joined along a common K_k. In the prequel to this paper, we studied the diagonal Ramsey number $r(B_n^{(k)}, B_n^{(k)})$. Here we consider the natural off-diagonal variant $r(B_{cn}^{(k)}, B_n^{(k)})$ for fixed $c \in (0, 1]$. In this more general setting, we show that an interesting dichotomy emerges: for very small c, a simple k-partite construction dictates the Ramsey function and all nearly-extremal colorings are close to being k-partite, while, for c bounded away from 0, random colorings of an appropriate density are asymptotically optimal and all nearly-extremal colorings are quasirandom. Our investigations also open up a range of questions about what happens for intermediate values of c.

1 Introduction

Given two graphs H_1 and H_2, their *Ramsey number* $r(H_1, H_2)$ is the smallest positive integer N such that every red/blue coloring of the edges of K_N is guaranteed to contain a red copy of H_1 or a blue copy of H_2. The main open problem in graph Ramsey theory is to determine the asymptotic order of $r(K_n, K_n)$. However, despite intense and longstanding interest, the lower and upper bounds $\sqrt{2}^n \leq r(K_n, K_n) \leq 4^n$ for this problem have remained largely unchanged since 1947 and 1935, respectively [11, 13].

Another major question in graph Ramsey theory, which has seen more progress, is to determine the growth rate of the *off-diagonal* Ramsey number $r(K_s, K_n)$, where we think of s as fixed and let n tend to infinity. The first non-trivial case is when $s = 3$, where it is known that

$$r(K_3, K_n) = \Theta\left(\frac{n^2}{\log n}\right),$$

with the upper bound due to Ajtai, Komlós, and Szemerédi [1] and the lower bound to Kim [17]. Subsequent work of Shearer [23], Bohman–Keevash [3], and Fiz Pontiveros–Griffiths–Morris [14] has led to a better understanding of the implicit constant, which is

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now known up to a factor of $4 + o(1)$. However, the successes in estimating $r(K_3, K_n)$ have not carried over to $r(K_s, K_n)$ for any other fixed s and a polynomial gap persists between the upper and lower bounds for all $s \geq 4$ (though see [18] for a promising approach to improving the lower bound).

The *book graph* $B_n^{(k)}$ is the graph obtained by gluing n copies of the clique K_{k+1} along a common K_k. The "book" terminology comes from the case $k = 2$, where $B_n^{(2)}$ consists of n triangles glued along a common edge. Continuing the analogy, each K_{k+1} is called a *page* of the book and the common K_k is called the *spine*. Ramsey numbers of books arise naturally in the study of $r(K_n, K_n)$; indeed, Ramsey's original proof [21] of the finiteness of $r(K_n, K_n)$ proceeds inductively by establishing the finiteness of certain book Ramsey numbers, while the Erdős–Szekeres bound [13] and its improvements [8, 22] are also best interpreted through the language of books. Because of this, Ramsey numbers of books have attracted a great deal of attention over the years, starting with papers of Erdős, Faudree, Rousseau, and Schelp [12] and of Thomason [25]. Both of these papers prove bounds of the form $2^k n - o_k(n) \leq r(B_n^{(k)}, B_n^{(k)}) \leq 4^k n$, where we think of k as fixed and $n \to \infty$, with Thomason conjecturing that the lower bound is closer to the truth. This was confirmed in a recent breakthrough result of the first author [9], who proved that, for every fixed k,

$$r(B_n^{(k)}, B_n^{(k)}) = 2^k n + o_k(n).$$

The original proof of this result relied heavily on an application of Szemerédi's celebrated regularity lemma, leading to rather poor control on the error term. In the prequel to this paper [10], we gave two alternative proofs of this result, one a simplified version of the first author's original proof and the other a proof which avoids the use of the full regularity lemma, allowing us to gain significantly better control over the error term (for a discussion of how further improvements might ultimately impinge on the estimation of $r(K_n, K_n)$, we refer the reader to [10]). We also proved a stability result, saying that extremal colorings for this Ramsey problem are quasirandom.

In this paper, we study a natural off-diagonal generalization of this problem. Specifically, we fix some $k \in \mathbb{N}$ and some $c \in (0, 1]$ and we wish to understand the asymptotics of the Ramsey number $r(B_{\lfloor cn \rfloor}^{(k)}, B_n^{(k)})$ as $n \to \infty$. Note that for $c = 1$ this is precisely the question considered above. Henceforth, we omit the floor signs and write $B_{cn}^{(k)}$ instead of $B_{\lfloor cn \rfloor}^{(k)}$.

Our results reveal that the behavior of the function $r(B_{cn}^{(k)}, B_n^{(k)})$ varies greatly as c moves from 0 to 1. As we shall see, for c sufficiently small, the behavior of this Ramsey number is determined by a simple block construction, while, for c sufficiently far from 0, its behavior is determined by a random coloring. There is also an intermediate range of c where our results say nothing, but where several interesting questions arise. We will say more about this in the concluding remarks.

To describe our results in detail, we begin by observing that for any positive integers k, m, and n with $m \leq n$, we have

$$r(B_m^{(k)}, B_n^{(k)}) \geq k(n + k - 1) + 1. \tag{1}$$

Indeed, let $N = k(n+k-1)$. We partition the vertices of K_N into k blocks, each of size $n+k-1$. We color all edges inside a block blue and all edges between blocks red. Then any blue $B_n^{(k)}$ must appear inside a block, which it cannot, since $B_n^{(k)}$ has $n+k$ vertices. On the other hand, since the red graph is k-partite, it does not contain any red K_{k+1} and so cannot contain a red $B_m^{(k)}$.

This simple inequality is a special case of a more general lower bound, usually attributed to Chvátal and Harary [7], that $r(H_1, H_2) \geq (\chi(H_1) - 1)(|V(H_2)| - 1) + 1$ provided H_2 is connected. Although this lower bound is usually far from optimal,[1] it is tight for certain sparse graphs. The study of when it is tight goes under the name of *Ramsey goodness*, a term introduced by Burr and Erdős [5] in their first systematic investigation of the concept. One of the central results in the field of Ramsey goodness is due to Nikiforov and Rousseau [19], who proved an extremely general theorem about when this lower bound is tight. As a very special case of their theorem, one has the following result; see also [15] for a new proof with better quantitative bounds.

Theorem 1.1 (Nikiforov–Rousseau [20, Theorem 2.12])**.** *For every $k \geq 2$, there exists some $c_0 \in (0,1)$ such that the following holds. For any $0 < c \leq c_0$ and n sufficiently large,*

$$r(B_{cn}^{(k)}, B_n^{(k)}) = k(n+k-1) + 1.$$

Moreover, Nikiforov and Rousseau's proof shows that the unique coloring on $k(n+k-1)$ vertices with no red $B_{cn}^{(k)}$ and no blue $B_n^{(k)}$ is the coloring we described, where the red graph is a balanced complete k-partite graph (meaning that all the parts have orders as equal as possible). By adapting their proof, we are able to prove a corresponding structural stability result, which says that any coloring on $N = (k + o(1))n$ vertices is either "close" to being balanced complete k-partite in red or contains monochromatic books with substantially more pages than what is guaranteed by Theorem 1.1. Note that if N is sufficiently large and congruent to 1 modulo k, then Theorem 1.1 says that any red/blue coloring of $E(K_N)$ contains a red K_k with at least $\frac{c}{k}(N-1) - c(k-1)$ extensions to a red K_{k+1} or a blue K_k with at least $\frac{1}{k}(N-1) - (k-1)$ extensions to a blue K_{k+1}.

Theorem 1.2. *For every $k \geq 2$ and every $\theta > 0$, there exist $c, \gamma \in (0,1)$ such that the following holds for any sufficiently large N and any red/blue coloring of $E(K_N)$. Either one can recolor at most θN^2 edges to turn the red graph into a balanced complete k-partite graph or else the coloring contains one of the following:*

- *At least γN^k red K_k, each with at least $(\frac{c}{k} + \gamma)N$ extensions to a red K_{k+1}, or*
- *At least γN^k blue K_k, each with at least $(\frac{1}{k} + \gamma)N$ extensions to a blue K_{k+1}.*

Informally, this theorem says that either the coloring is close to complete k-partite in red or else a constant fraction of the k-tuples span a clique that forms the spine of a monochromatic book with at least γN more pages than what is guaranteed by the Ramsey bound alone.

[1] For example, for $H_1 = H_2 = K_n$, it gives a lower bound of $r(K_n, K_n) = \Omega(n^2)$, whereas the truth is $2^{\Theta(n)}$.

However, once c is sufficiently far from 0, the deterministic construction that yields (1) stops being optimal. Indeed, as in the diagonal case, we can get another lower bound on $r(B_{cn}^{(k)}, B_n^{(k)})$ by considering random colorings. More precisely, let us fix $k \in \mathbb{N}$ and $c \in (0, 1]$ and define

$$p = \frac{1}{c^{1/k} + 1} \in \left[\tfrac{1}{2}, 1\right).$$

Then it is a simple exercise to see that if we set $N = (p^{-k} - o(1))n$ and color every edge of K_N blue with probability p and red with probability $1 - p$, then this coloring will w.h.p.[2] contain no blue $B_n^{(k)}$ and no red $B_{cn}^{(k)}$. This implies that for any $k \in \mathbb{N}$ and any $c \in (0, 1]$,

$$r(B_{cn}^{(k)}, B_n^{(k)}) \geq \left(c^{1/k} + 1\right)^k n - o_k(n),$$

while the lower bound in (1) is that $r(B_{cn}^{(k)}, B_n^{(k)}) \geq (k + o(1))n$. If $c > ((1 + o(1)) \log k/k)^k$, then the quantity $(c^{1/k} + 1)^k$ is larger than $k + o(1)$, where the logarithm is to base e. Thus, once c is sufficiently far from 0, the bound in (1) is smaller than the random bound.

Our next main result shows that the random bound actually becomes asymptotically tight at this point.

Theorem 1.3. *For every $k \geq 2$, there exists some $c_1 = c_1(k) \in (0, 1]$ such that, for any fixed $c_1 \leq c \leq 1$,*

$$r(B_{cn}^{(k)}, B_n^{(k)}) = \left(c^{1/k} + 1\right)^k n + o_k(n).$$

Moreover, one may take $c_1(k) = ((1 + o(1)) \log k/k)^k$.

Our third main result is a corresponding structural stability theorem, which says that all near-extremal Ramsey colorings (i.e., colorings on roughly $(c^{1/k} + 1)^k n$ vertices) must either contain a monochromatic book substantially larger than what is guaranteed by Theorem 1.3 or be "random-like". The latter possibility is captured by the notion of *quasirandomness*, introduced by Chung, Graham, and Wilson [6]. For parameters $p, \theta \in (0, 1)$, a red/blue coloring of $E(K_N)$ is said to be (p, θ)-quasirandom if, for every pair of disjoint sets $X, Y \subseteq V(K_N)$, we have that

$$|e_B(X, Y) - p|X||Y|| \leq \theta N^2,$$

where $e_B(X, Y)$ denotes the number of blue edges between X and Y. Note that since the colors are complementary, this is equivalent to the analogous condition requiring that $e_R(X, Y)$ is within θN^2 of $(1-p)|X||Y|$. In their seminal paper, Chung, Graham, and Wilson, building on previous results of Thomason [25], showed that this condition is essentially equivalent to a large number of other conditions, all of which encapsulate some intuitive idea of what it means for a coloring to be similar to a random coloring with blue density p. With this notion in hand, we can state our structural stability result.

Theorem 1.4. *For every $p \in [\frac{1}{2}, 1)$, there exists some $k_0 \in \mathbb{N}$ such that the following holds for every $k \geq k_0$. For every $\theta > 0$, there exists some $\gamma > 0$ such that if a red/blue coloring of $E(K_N)$ is not (p, θ)-quasirandom, then it contains one of the following:*

[2] As usual, we say that an event E happens with high probability (w.h.p.) if $\Pr(E) \to 1$ as $n \to \infty$, where the implicit parameter n will be clear from context.

- *At least γN^k red K_k, each with at least $((1-p)^k+\gamma)N$ extensions to a red K_{k+1}, or*
- *At least γN^k blue K_k, each with at least $(p^k+\gamma)N$ extensions to a blue K_{k+1}.*

Remark. As stated, this theorem does not mention the "off-diagonalness" parameter c from the previous theorem. But c can easily be recovered as $((1-p)/p)^k$ and the theorem can then be restated to be about blue books with slightly more than n pages or red books with slightly more than cn pages. However, since p is what matters while c plays no real role in the argument, we instead choose to use this language and avoid c entirely.

In Theorem 5.6, we also prove a converse to Theorem 1.4, which implies that for p fixed and k sufficiently large in terms of p, a coloring of K_N (or, more accurately, a sequence of colorings with N tending to infinity) is $(p, o(1))$-quasirandom if and only if all but $o(N^k)$ red K_k have at most $((1-p)^k + o(1))N$ extensions to a red K_{k+1} and all but $o(N^k)$ blue K_k have at most $(p^k + o(1))N$ extensions to a blue K_{k+1}. Thus, we derive a new equivalent formulation for $(p, o(1))$-quasirandomness.

The rest of the paper is organized as follows. In Section 2, we quote (mostly without proof) a number of key results that we will use repeatedly. In Section 3, we establish Theorem 1.2, the stability result for small c. We prove Theorem 1.3, that the random bound is asymptotically tight once c is not too small, in Section 4 and Theorem 1.4, that extremal colorings are quasirandom in this range, in Section 5. We end with some concluding remarks and open problems.

1.1 Notation and Terminology

If X and Y are two vertex subsets of a graph, let $e(X, Y)$ denote the number of pairs in $X \times Y$ that are edges. We will often normalize this and consider the *edge density*

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

If we consider a red/blue coloring of the edges of a graph, then $e_B(X, Y)$ and $e_R(X, Y)$ will denote the number of pairs in $X \times Y$ that are blue and red edges, respectively. Similarly, d_B and d_R will denote the blue and red edge densities, respectively. Finally, for a vertex v and a set Y, we will sometimes abuse notation and write $d(v, Y)$ for $d(\{v\}, Y)$ and similarly for d_B and d_R.

An *equitable partition* of a graph G is a partition of the vertex set $V(G) = V_1 \sqcup \cdots \sqcup V_m$ with $||V_i| - |V_j|| \leq 1$ for all $1 \leq i, j \leq m$. A pair of vertex subsets (X, Y) is said to be *ε-regular* if, for every $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$, $|Y'| \geq \varepsilon|Y|$, we have

$$|d(X, Y) - d(X', Y')| \leq \varepsilon.$$

Note that we do not require X and Y to be disjoint. In particular, we say that a single vertex subset X is *ε-regular* if the pair (X, X) is ε-regular. We will often need a simple fact, known as the *hereditary property* of regularity, which asserts that for any $0 < \alpha \leq 1$, if

(X,Y) is ε-regular and $X' \subseteq X$, $Y' \subseteq Y$ satisfy $|X'| \geq \alpha|X|$, $|Y'| \geq \alpha|Y|$, then (X',Y') is $(\max\{\varepsilon/\alpha, 2\varepsilon\})$-regular.

For real numbers a, b, we denote by $a \pm b$ any quantity in the interval $[a-b, a+b]$. All logarithms are base e unless otherwise specified. For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial. In this vein, whenever we have an equitable partition of a vertex set, we will always assume that all of the parts have exactly the same size, rather than being off by at most one. Because the number of vertices in our graphs will always be "sufficiently large", this has no effect on our final results.

2 Results from earlier work

In this section, we collect some useful tools for the study of book Ramsey numbers, all of which have appeared in previous works. We begin with several results from the theory around Szemerédi's regularity lemma and then quote two simple analytic inequalities.

2.1 Tools from regularity

We begin with a strengthened form of Szemerédi's regularity lemma taken from our first paper [10, Lemma 2.1].

Lemma 2.1. *For every $\varepsilon > 0$ and $M_0 \in \mathbb{N}$, there is some $M = M(\varepsilon, M_0) \geq M_0$ such that, for every graph G with at least M_0 vertices, there is an equitable partition $V(G) = V_1 \sqcup \cdots \sqcup V_m$ into $M_0 \leq m \leq M$ parts such that the following hold:*

1. *Each part V_i is ε-regular and*
2. *For every $1 \leq i \leq m$, there are at most εm values $1 \leq j \leq m$ such that the pair (V_i, V_j) is not ε-regular.*

To complement the regularity lemma, we will also need a standard counting lemma (see, e.g., [26, Theorem 3.27]).

Lemma 2.2. *Suppose that $V_1, \ldots, V_k$ are (not necessarily distinct) subsets of a graph G such that all pairs (V_i, V_j) are ε-regular. Then the number of labeled copies of K_k whose ith vertex is in V_i for all i is*

$$\left(\prod_{1\leq i<j\leq k} d(V_i, V_j) \pm \varepsilon\binom{k}{2}\right)\prod_{i=1}^{k}|V_i|,$$

where d denotes the edge density in G.

We will frequently use the following consequence of the counting lemma, proved in [10, Corollary 2.6], designed to count monochromatic extensions of cliques and thus estimate the size of monochromatic books.

Lemma 2.3. *Fix $k \geq 2$ and let $\eta, \alpha \in (0,1)$ be parameters with $\eta \leq \alpha^3/k^2$. Suppose $U_1, \ldots, U_k$ are (not necessarily distinct) vertex sets in a graph G and suppose that all pairs (U_i, U_j) are η-regular with $\prod_{1 \leq i < j \leq k} d(U_i, U_j) \geq \alpha$. Let Q be a randomly chosen copy of K_k with one vertex in each U_i, for $1 \leq i \leq k$, and say that a vertex u extends Q if u is adjacent to every vertex of Q. Then, for any $u \in V(G)$,*

$$\Pr(u \text{ extends } Q) \geq \prod_{i=1}^{k} d(u, U_i) - 4\alpha.$$

The final result in this subsection is actually a simple consequence of Markov's inequality and so does not require any regularity tools to prove. Nonetheless, we will always use it in conjunction with Lemmas 2.2 and 2.3, which is why we include it here. Both the statement and proof are very similar to [10, Lemma 5.2].

Lemma 2.4. *Let $\kappa, \xi \in (0,1)$, let $0 < \nu < \xi$, and suppose that $\mathcal{Q}$ is a set of at least κN^k copies of K_k in an N-vertex graph. Suppose that a uniformly random $Q \in \mathcal{Q}$ has at least ξN extensions to a K_{k+1} in expectation. Then the graph contains at least $(\xi - \nu)\kappa N^k$ copies of K_k, each with at least νN extensions.*

Proof. Let X be the random variable counting the number of extensions of a random $Q \in \mathcal{Q}$ and let $Y = N - X$. Then Y is a non-negative random variable with $\mathbb{E}[Y] = N - \mathbb{E}[X] \leq (1-\xi)N$. By Markov's inequality,

$$\Pr(X \leq \nu N) = \Pr\left(Y \geq (1-\nu)N\right) \leq \frac{\mathbb{E}[Y]}{(1-\nu)N} \leq \frac{(1-\xi)N}{(1-\nu)N} = \frac{1-\xi}{1-\nu}.$$

Thus,

$$\Pr(X \geq \nu N) \geq 1 - \frac{1-\xi}{1-\nu} = \frac{\xi - \nu}{1-\nu} \geq \xi - \nu,$$

which implies that the number of $Q \in \mathcal{Q}$ with at least νN extensions is at least $(\xi - \nu)|\mathcal{Q}| \geq (\xi - \nu)\kappa N^k$, as desired. □

2.2 Analytic inequalities

The following lemma is a multiplicative form of Jensen's inequality and is a simple consequence of the standard version. For a proof, see [10, Lemma A.1].

Lemma 2.5 (Multiplicative Jensen inequality)**.** *Suppose $0 < a < b$ are real numbers and $x_1, \ldots, x_k \in (a,b)$. Let $f : (a,b) \to \mathbb{R}$ be a function such that $y \mapsto f(e^y)$ is strictly convex on the interval* $(\log a, \log b)$*. Then, for any $z \in (a^k, b^k)$, subject to the constraint $\prod_{i=1}^{k} x_i = z$,*

$$\frac{1}{k} \sum_{i=1}^{k} f(x_i)$$

is minimized when all the x_i are equal (and thus equal to $z^{1/k}$).

The following theorem is the well-known "defect" or "stability" version of Jensen's inequality. For a proof, see [24, Problem 6.5].

Theorem 2.6 (Hölder's Defect Formula)**.** *Suppose $\varphi : [a, b] \to \mathbb{R}$ is a twice-differentiable function with $\varphi''(y) \geq m > 0$ for all $y \in (a, b)$. For any $y_1, \ldots, y_k \in [a, b]$, let*

$$\mu = \frac{1}{k}\sum_{i=1}^{k} y_i \qquad \text{and} \qquad \sigma^2 = \frac{1}{k}\sum_{i=1}^{k}(y_i - \mu)^2$$

be the empirical mean and variance of $\{y_1, \ldots, y_k\}$. Then

$$\frac{1}{k}\sum_{i=1}^{k}\varphi(y_i) - \varphi(\mu) \geq \frac{m\sigma^2}{2}.$$

3 The k-partite case

In this section, we analyze what happens when c is very small. Recall, from the introduction, that a simple k-partite construction yields a lower bound for $r(B_{cn}^{(k)}, B_n^{(k)})$ and, by a result of Nikiforov and Rousseau [20], this construction is tight for c sufficiently small.

Theorem 1.1 (Nikiforov–Rousseau, [20, Theorem 2.12])**.** *For every $k \geq 2$, there exists some $c_0 \in (0, 1)$ such that, for any $0 < c \leq c_0$ and n sufficiently large,*

$$r(B_{cn}^{(k)}, B_n^{(k)}) = k(n + k - 1) + 1.$$

Our aim here is to adapt the methods of [20] to prove a stability version of this theorem, our Theorem 1.2. We first make the following definition.

Definition 3.1. For $c, \gamma > 0$, we say that a red/blue coloring of $E(K_N)$ contains (c, γ)-*many books* if it contains

- At least γN^k red K_k, each with at least $(\frac{c}{k} + \gamma)N$ extensions to a red K_{k+1}, or
- At least γN^k blue K_k, each with at least $(\frac{1}{k} + \gamma)N$ extensions to a blue K_{k+1}.

With this definition in place, we may restate Theorem 1.2 as follows.

Theorem 1.2'. *For every $k \geq 2$ and every $\theta > 0$, there exist $c, \gamma \in (0, 1)$ such that the following holds. If a red/blue coloring of $E(K_N)$ does not have (c, γ)-many books, then one can recolor at most θN^2 edges to turn the red graph into a balanced complete k-partite graph.*

As well as referring to Section 2, we will need two further results in our proof. The first is a classical theorem of Andrásfai, Erdős, and Sós [2] (see also [4] for a simpler proof).

Theorem 3.2 (Andrásfai–Erdős–Sós [2])**.** *For every $k \geq 2$, there exists $\rho > 0$ such that if G is a K_{k+1}-free graph on m vertices with minimum degree greater than $(1 - \frac{1}{k} - \rho)m$, then G is k-partite. Moreover, one may take $\rho = 1/(3k^2 - k)$.*

This is a stability version of Turán's theorem. Indeed, Turán's theorem implies that if a graph on m vertices has minimum degree at least $(1-\frac{1}{k})m$, then it contains a copy of K_{k+1}, while the Andrásfai–Erdős–Sós theorem says that as long as the minimum degree is not too far below $(1-\frac{1}{k})m$, every K_{k+1}-free graph must be k-partite.

We will also need the following fact about bipartite graphs, which is a simple consequence of a double-counting technique first introduced by Kővári, Sós, and Turán [16].

Lemma 3.3. *Let $k \geq 2$ and $d \in (0,1)$ and let $\zeta = (d/4)^k$. Let H be a bipartite graph with parts A, B, where $|B| \geq 2k/d$, and suppose that H has at least $d|A||B|$ edges. Let $\mathcal{H}$ be a k-uniform hypergraph with vertex set B and at least $(1-\zeta)\binom{|B|}{k}$ edges. Then there are at least $\zeta\binom{|B|}{k}$ edges of $\mathcal{H}$ such that the vertices of each such edge have at least $\zeta|A|$ common neighbors in A.*

Proof. For a k-tuple $Q \in \binom{B}{k}$, let $\operatorname{ext}(Q)$ denote the number of common neighbors of Q in A. We double-count the number of stars $K_{1,k}$ in H whose central vertex is in A to find that

$$\sum_{Q\in\binom{B}{k}} \operatorname{ext}(Q) = \sum_{a\in A}\binom{\deg(a)}{k} \geq |A|\binom{\frac{1}{|A|}\sum_{a\in A}\deg(a)}{k} \geq |A|\binom{d|B|}{k},$$

where the first inequality follows from convexity. If we split the left-hand side into a sum over tuples Q which are non-edges of $\mathcal{H}$, a sum over tuples Q that are edges of $\mathcal{H}$ with fewer than $\zeta|A|$ extensions, and the remainder, we find that

$$\begin{aligned}|A|\binom{d|B|}{k} &\leq \sum_{Q\notin E(\mathcal{H})}\operatorname{ext}(Q) + \sum_{\substack{Q\in E(\mathcal{H})\\ \operatorname{ext}(Q)<\zeta|A|}}\operatorname{ext}(Q) + \sum_{\substack{Q\in E(\mathcal{H})\\ \operatorname{ext}(Q)\geq\zeta|A|}}\operatorname{ext}(Q)\\ &\leq \zeta|A|\binom{|B|}{k} + \zeta|A|\binom{|B|}{k} + |A||\{Q\in E(\mathcal{H}) : \operatorname{ext}(Q)\geq\zeta|A|\}|.\end{aligned}$$

Therefore, the number of edges of $\mathcal{H}$ with at least $\zeta|A|$ common neighbors is at least $\binom{d|B|}{k} - 2\zeta\binom{|B|}{k}$. We note that

$$\frac{\binom{d|B|}{k}}{\binom{|B|}{k}} = \frac{d|B|}{|B|}\cdot\frac{d|B|-1}{|B|-1}\cdots\frac{d|B|-(k-1)}{|B|-(k-1)} \geq \left(\frac{d}{2}\right)^k = 2^k\zeta,$$

where we used our assumption that $|B| \geq 2k/d$. Thus, the number of edges of $\mathcal{H}$ with at least $\zeta|A|$ common neighbors in A is at least

$$\binom{d|B|}{k} - 2\zeta\binom{|B|}{k} \geq (2^k\zeta - 2\zeta)\binom{|B|}{k} \geq \zeta\binom{|B|}{k}. \qquad \square$$

With these preliminaries in place, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix some $k \geq 2$, $\theta \in (0,1)$, and a red/blue coloring of $E(K_N)$. Let $\sigma = (\theta/(12k^4))^k$, $M_0 = k$, $\delta = \sigma^2$, and $\eta = \delta^{k^2}$. Let $M = M(\eta, M_0)$ be the parameter from Lemma 2.1 and let $c = \delta^{k^2}/M^4$ and $\gamma = \delta^{2k^2}/M^{2k}$. We apply Lemma 2.1 to the red graph in our coloring with parameters M_0 and η. This yields an equitable partition $V(K_N) = V_1 \sqcup \cdots \sqcup V_m$ with $M_0 \leq m \leq M$ such that each part V_i is η-regular in red and, for each i, there are at most ηm values of $j \neq i$ for which (V_i, V_j) is not η-regular in red.

First suppose that some part, say V_1, has internal red density at least δ. By the counting lemma, Lemma 2.2, we see that V_1 contains at least $\frac{1}{(k+1)!}(\delta^{\binom{k+1}{2}} - \eta\binom{k+1}{2})|V_1|^{k+1}$ red K_{k+1}. Since each red K_{k+1} contains exactly $k+1$ red K_k, this implies that an average k-tuple of vertices in V_1 lies in at least

$$\frac{\frac{k+1}{(k+1)!}(\delta^{\binom{k+1}{2}} - \eta\binom{k+1}{2})|V_1|^{k+1}}{\binom{|V_1|}{k}} \geq \left(\delta^{\binom{k+1}{2}} - \eta\binom{k+1}{2}\right)|V_1| =: \xi M|V_1|$$

red K_{k+1}. If we also define $\kappa = (\delta^{\binom{k}{2}} - \eta\binom{k}{2})/(k!M^k)$, then Lemma 2.2 implies that V_1 contains at least κN^k red K_k, with an average one having at least ξN extensions to a red K_{k+1}, where we use the fact that $|V_1| \geq N/M$ since the partition is equitable and has $m \leq M$ parts. If we now set $\nu = \xi/2$ and apply Lemma 2.4, we conclude that V_1 contains at least $(\xi\kappa/2)N^k$ red K_k, each with at least $(\xi/2)N$ extensions to a red K_{k+1}. By our choice of parameters,

$$\xi = \frac{1}{M}\left(\delta^{\binom{k+1}{2}} - \eta\binom{k+1}{2}\right) \geq \frac{\delta^{k^2}}{2M}$$

and, therefore, $\xi/2 \geq c/k + \gamma$. Similarly, $\kappa \geq \delta^{k^2}/M^k$ and, therefore, $\xi\kappa/2 \geq \gamma$. Thus, we find that in this case the coloring contains (c,γ)-many books.

Therefore, we may assume that all V_i have $d_R(V_i) < \delta$. We build a reduced graph G with vertex set $v_1, \ldots, v_m$ and declare $\{v_i, v_j\} \in E(G)$ if (V_i, V_j) is η-regular and $d_R(V_i, V_j) \geq \delta$. Suppose that some vertex of G, say v_1, has degree less than $(1 - \frac{1}{k} - \sigma)m$. Since at most ηm non-neighbors of v_1 can come from irregular pairs, we find that $d_B(V_1, V_i) \geq 1 - \delta$ for at least $(\frac{1}{k} + \sigma - \eta)m$ choices of $i \in [m]$. Let $I \subseteq [m]$ be this set of i. Since $d_B(V_1) \geq 1 - \delta$ and $\delta \leq 1/k^2$, we see that, for $\alpha = k\delta/4$,

$$d_B(V_1)^{\binom{k}{2}} \geq (1-\delta)^{\binom{k}{2}} \geq 1 - \binom{k}{2}\delta \geq \alpha.$$

Moreover, we have that $\eta < \alpha^3/k^2$ by our choice of η. Therefore, we may apply Lemma 2.3, which implies that if Q is a randomly chosen blue K_k in V_1 and u is some vertex in K_N, then $\Pr(u \text{ extends } Q) \geq d(u, V_1)^k - 4\alpha$. In particular, if we sum this up over all $u \in \bigcup_{i \in I} V_i$, we find that the expected number of blue extensions of Q is at least

$$\sum_{i \in I}\sum_{u \in V_i}(d(u,V_1)^k - 4\alpha) \geq |I|\frac{N}{m}\left((1-\delta)^k - 4\alpha\right) \geq \left(\frac{1}{k} + \sigma - \eta\right)\left((1-\delta)^k - 4\alpha\right)N,$$

where the first inequality follows from the convexity of the function $x \mapsto x^k$. Using $\eta < \sigma/2$, we have that

$$\left(\frac{1}{k} + \sigma - \eta\right)\left((1-\delta)^k - 4\alpha\right) \geq \left(\frac{1}{k} + \frac{\sigma}{2}\right)(1 - 2k\delta) \geq \frac{1}{k} + \frac{\sigma}{2} - 2k\delta,$$

where the last step follows from the bound $1/k + \sigma \leq 1/k + 1/k \leq 1$. By our choice of $\delta = \sigma^2 \leq \sigma/(8k)$, we see that the expected number of blue extensions of Q is at least $(\frac{1}{k}+\frac{\sigma}{4})N$. Moreover, the number of choices for Q is at least $\frac{1}{k!}((1-\delta)^{\binom{k}{2}} - \eta\binom{k}{2})(N/M)^k \geq \kappa N^k$. Therefore, if we apply Lemma 2.4 with parameters κ, $\xi' = \frac{1}{k} + \frac{\sigma}{4}$, and $\nu' = \frac{1}{k} + \gamma$, then we find that the coloring contains (c, γ)-many books.

Therefore, we may assume that every vertex in G has degree greater than $(1 - \frac{1}{k} - \sigma)m$, so, by Theorem 3.2 and the fact that $\sigma < 1/(3k^2 - k)$, we see that either G contains a K_{k+1} or G is k-partite. Assume first that there is a K_{k+1} in G. By relabeling the vertices, we may assume that $v_1, \ldots, v_{k+1}$ span a clique. By the counting lemma, Lemma 2.2, we have that $V_1, \ldots, V_k$ span at least $(\delta^{\binom{k}{2}} - \eta\binom{k}{2})(N/m)^k$ red K_k and $V_1, \ldots, V_{k+1}$ span at least $(\delta^{\binom{k+1}{2}} - \eta\binom{k+1}{2})(N/m)^{k+1}$ red K_{k+1}. Hence, by essentially the same computation as before, we see that we have a set of at least κN^k red K_k with at least ξN extensions on average and so our coloring has (c, γ)-many books.

Thus, we may assume that G is k-partite. Let this k-partition of $V(G)$ be $A_1 \sqcup \cdots \sqcup A_k$. Note that $|A_\ell| \leq (\frac{1}{k} + \sigma)m$ for every ℓ, since the minimum degree of G is at least $(1 - \frac{1}{k} - \sigma)m$ and each A_ℓ is an independent set in G. This in turn implies that $|A_\ell| \geq (\frac{1}{k} - k\sigma)m$ for every ℓ, since $|A_\ell| = m - \sum_{\ell' \neq \ell} |A_{\ell'}| \geq (\frac{1}{k} - k\sigma)m$. We lift this partition to a partition of the vertices of K_N into k parts $X_1, \ldots, X_k$ by letting $X_\ell = \bigcup_{v_i \in A_\ell} V_i$, noting that our observations above imply that $|X_\ell| = (\frac{1}{k} \pm k\sigma)N$ for all ℓ. We claim that each X_ℓ contains at most $\frac{3\delta}{2}\binom{|X_\ell|}{2}$ red edges. Indeed, observe that if v_i, v_j are two (not necessarily distinct) vertices of G that are in the same part A_ℓ, then they must be non-adjacent in G. This means that either (V_i, V_j) is an irregular pair or $d_R(V_i, V_j) < \delta$. There are at most ηm^2 irregular pairs, so the irregular pairs can contribute at most $\eta N^2 \leq 4k^2\eta|X_\ell|^2 \leq 10k^2\eta\binom{|X_\ell|}{2}$ red edges inside X_ℓ, where we used that $|X_\ell| \geq (\frac{1}{k} - k\sigma)N \geq N/(2k)$. All other parts inside each X_ℓ have red density at most δ, so the total number of red edges inside X_ℓ is at most $\delta\binom{|X_\ell|}{2} + 10k^2\eta\binom{|X_\ell|}{2} \leq \frac{3\delta}{2}\binom{|X_\ell|}{2}$, since $\eta \leq \delta/(20k^2)$. This implies that the number of ordered pairs of (not necessarily distinct) vertices in X_ℓ which do not form a blue edge is at most $2\delta|X_\ell|^2$.

This already implies that the red graph can be made k-partite by recoloring at most $2\delta N^2$ edges, so it only remains to show that by recoloring a small number of additional edges, we can make the red graph balanced complete k-partite. For this, suppose that $d_B(X_1, X_2) \geq \theta/k^2$. If we sample (with repetition) a random k-tuple Q of vertices from X_2, then the probability that it does not form a blue clique is at most $\binom{k}{2} \cdot 2\delta \leq k^2\delta$, since each pair of vertices does not span a blue edge with probability at most 2δ. Moreover, the expected number of common blue neighbors of Q inside X_2 is at least $(1-2\delta)^k|X_2| - k \geq (1 - 2k\delta)|X_2|$, by convexity. By applying Markov's inequality as in the proof of Lemma 2.4, the probability that Q has fewer than $(1-\sqrt{\delta})|X_2|$ common blue neighbors in X_2 is at most $2k\sqrt{\delta}$. Therefore, the probability that Q is a blue clique with at least $(1 - \sqrt{\delta})|X_2|$ common blue neighbors in X_2 is at least $1 - k^2\delta - 2k\sqrt{\delta} \geq 1 - 3k\sqrt{\delta}$, since $\sqrt{\delta} \leq 1/k$. Let $\mathcal{H}$ be the k-uniform hypergraph with vertex set X_2 whose edges are all blue K_k in X_2 with at least $(1-\sqrt{\delta})|X_2|$ common blue neighbors in X_2. Then $\mathcal{H}$ has at least $(1-3k\sqrt{\delta})\binom{|X_2|}{k}$ edges.

We now apply Lemma 3.3 to the hypergraph $\mathcal{H}$ and to the bipartite graph of blue edges between X_1 and X_2, which has edge density $d \geq \theta/k^2$ by assumption. We have that $(d/4)^k \geq (\theta/(4k^2))^k \geq 3k\sigma = 3k\sqrt{\delta}$ by our choice of σ and δ, so we may indeed apply Lemma 3.3 to conclude that at least $(\theta/(4k^2))^k\binom{|X_2|}{k}$ of the edges of $\mathcal{H}$ have at least $(\theta/(4k^2))^k|X_1|$ common blue neighbors in X_1. This yields at least

$$\left(\frac{\theta}{4k^2}\right)^k \binom{|X_2|}{k} \geq \left(\frac{\theta|X_2|}{4k^3}\right)^k \geq \left(\frac{\theta}{8k^4}\right)^k N^k \geq \gamma N^k$$

blue K_k, each of which has at least

$$\begin{aligned}
(1-\sqrt{\delta})|X_2| + \left(\frac{\theta}{4k^2}\right)^k |X_1| &\geq \left(1 - \sqrt{\delta} + \left(\frac{\theta}{4k^2}\right)^k\right)\left(\frac{1}{k} - k\sigma\right) N \\
&\geq \left(\frac{1}{k} + \left(\frac{\theta}{4k^3}\right)^k - \sqrt{\delta} - 2k\sigma\right) N \\
&\geq \left(\frac{1}{k} + \left(\frac{\theta}{4k^3}\right)^k - 3k\sigma\right) N \\
&\geq \left(\frac{1}{k} + \gamma\right) N
\end{aligned}$$

extensions to a blue K_{k+1}, where in both computations we used the fact that $|X_1|, |X_2| \geq (\frac{1}{k} - k\sigma)N \geq N/(2k)$, as well as our choices of $\sqrt{\delta} = \sigma = (\theta/(12k^4))^k$. Thus, in this case, we have found (c, γ)-many books, a contradiction.

Hence, we may assume that $d_B(X_1, X_2) < \theta/k^2$. By the same argument, all the blue densities between different parts X_ℓ can be assumed to be at most θ/k^2. Since we have already argued that the red density inside each part is at most 2δ, we see that, by recoloring at most $(\binom{k}{2}\theta/k^2 + 2k\delta)N^2$ edges, we can make the red graph complete k-partite. Finally, we recall that each part X_ℓ has size $|X_\ell| = (\frac{1}{k} \pm k\sigma)N$. Therefore, by moving at most $k^2\sigma N$ arbitrary vertices into a new part, we see that we can make our partition equitable. We then recolor the edges incident with any moved vertex to obtain a balanced complete k-partite red graph. Doing so entails recoloring at most $k^2\sigma N^2$ additional edges. Thus, in total, we recolor at most

$$\left(\binom{k}{2}\frac{\theta}{k^2} + 2k\delta + k^2\sigma\right) N^2 \leq \left(\frac{\theta}{2} + 3k^2\sigma\right) \leq \theta N^2$$

edges, where we used that $\delta \leq \sigma$ and $\sigma \leq (\theta/(12k^4))^k \leq \theta/(6k^2)$. $\square$

4 An upper bound matching the random bound

In this section, we prove Theorem 1.3, which says that when c is not too small, the random lower bound for $r(B_{cn}^{(k)}, B_n^{(k)})$ is asymptotically tight.

To prove this theorem, we will mimic our simplified proof of the diagonal result from [10, Section 3], though it needs to be adapted to the off-diagonal setting in several ways. For instance, the following result generalizes a key analytic inequality from the diagonal case [10, Lemma 3.4].

Lemma 4.1. *For every $p \in (0,1)$, there exists some $k_1 \in \mathbb{N}$ such that if $k \geq k_1$ and $x_1, \ldots, x_k \in [0,1]$, then*

$$p^{1-k} \prod_{i=1}^{k} x_i + \frac{(1-p)^{1-k}}{k} \sum_{i=1}^{k} (1-x_i)^k \geq 1.$$

Moreover, one may take

$$k_1(p) = \begin{cases} 6 & \text{if } p \geq 1 - 5/(4e) \\ 1 + \frac{5 - \log\log\frac{1}{1-p} + \log(-\log\log\frac{1}{1-p})}{\log\frac{1}{1-p}} & \text{otherwise.} \end{cases}$$

Proof. First suppose that $x_j \leq \frac{1}{k}$ for some $j \in [k]$. Then we have that

$$\frac{(1-p)^{1-k}}{k} \sum_{i=1}^{k} (1-x_i)^k \geq \frac{(1-p)^{1-k}}{k} (1-x_j)^k \geq \frac{(1-p)^{1-k}}{k} \left(1 - \frac{1}{k}\right)^k \geq \frac{(1-p)^{1-k}}{e^2 k} =: f(p,k),$$

where we used the inequality $1-x \geq e^{-2x}$ for $x \in [0, \frac{1}{2}]$. If $p \geq 1-5/(4e)$, then $1-p \leq 5/(4e)$, so $f(p,k) \geq (4/5)^{k-1} e^{k-3}/k$. Once $k \geq 6$, this last expression is at least 1, so in the case where $p \geq 1-5/(4e)$, we may take $k_1(p) = 6$.

For $p < 1 - 5/(4e)$, let $\lambda = \lambda(p) = \log\frac{1}{1-p}$ and

$$k_1(p) = 1 + \frac{5 - \log\lambda + \log\log\frac{1}{\lambda}}{\lambda} = 1 + \frac{5 - \log\log\frac{1}{1-p} + \log(-\log\log\frac{1}{1-p})}{\log\frac{1}{1-p}}.$$

We claim that if $k \geq k_1(p)$, then $f(p,k) \geq 1$. By differentiating, we see that $f(p,k)$ is monotonically increasing in k for $k \geq 1/\log\frac{1}{1-p} = 1/\lambda$. Since $p < 1-5/(4e)$, we have that $5 - \log\lambda + \log\log\frac{1}{\lambda} > 1$ and so we are in the monotonicity regime. It therefore suffices to prove the statement for $k = k_1(p)$. Note now that

$$(1-p)^{1-k_1(p)} = (1-p)^{(5-\log\lambda+\log\log\frac{1}{\lambda})/\log(1-p)} = \frac{e^5 \log\frac{1}{\lambda}}{\lambda}$$

and let $g(p) = 1 + \lambda/\log\frac{1}{\lambda} + 5/\log\frac{1}{\lambda} + \log\log\frac{1}{\lambda}/\log\frac{1}{\lambda}$. Then we have

$$f(p, k_1(p)) = \frac{e^5 \log\frac{1}{\lambda}}{\lambda} \cdot \frac{1}{e^2 k_1(p)} = \frac{e^3 \log\frac{1}{\lambda}}{\lambda + 5 + \log\frac{1}{\lambda} + \log\log\frac{1}{\lambda}} = \frac{e^3}{g(p)}.$$

Thus, to prove that $f(p,k_1(p)) \geq 1$, it suffices to prove that $g(p) \leq e^3$ for all $p < 1-5/(4e)$. By differentiating, one can check that $g(p)$ is monotonically increasing in $p \in [0, 1-5/(4e)]$. Thus, it suffices to check that $g(1-5/(4e)) \leq e^3$. But $g(1-5/(4e)) \approx 18.4 < e^3$, so $f(p,k_1(p)) \geq 1$, as claimed. Hence, from now on, we may assume that all the x_i are in $(\frac{1}{k}, 1]$.

For the moment, let's assume that all the x_i are in $(\frac{1}{k}, 1)$. Note that the function $\varphi : y \mapsto (1-e^y)^k$ is strictly convex on the interval $(\log \frac{1}{k}, 0)$. By the multiplicative Jensen inequality, Lemma 2.5, this implies that, subject to the constraint $\prod_{i=1}^k x_i = z$, the term $\frac{1}{k}\sum_{i=1}^k (1-x_i)^k$ is minimized when all the x_i are equal to $z^{1/k}$. Therefore,

$$p^{1-k}\prod_{i=1}^k x_i + \frac{(1-p)^{1-k}}{k}\sum_{i=1}^k (1-x_i)^k \geq p^{1-k}z + (1-p)^{1-k}(1-z^{1/k})^k.$$

So it suffices to minimize this expression as a function of z. Changing variables to $w = z^{1/k}$, it suffices to minimize

$$\psi(w) = p^{1-k}w^k + (1-p)^{1-k}(1-w)^k$$

as a function of w. By differentiating, we find that ψ is minimized at $w = p$, where $\psi(p) = 1$. This proves the desired result as long as all the x_i are in $[0, 1)$. By continuity, the result then extends to all $x_i \in [0, 1]$. $\square$

Definition 4.2. Fix parameters $k \in \mathbb{N}$ and $\eta, \delta \in (0, 1)$ and suppose that we are given a red/blue coloring of $E(K_N)$. Then a k-tuple of pairwise disjoint vertex sets $C_1, \ldots, C_k \subseteq V(K_N)$ is called a (k, η, δ)*-red-blocked configuration* if the following properties are satisfied:

1. Each C_i is η-regular with itself,
2. Each C_i has internal red density at least δ, and
3. For all $i \neq j$, the pair (C_i, C_j) is η-regular and has blue density at least δ.

Similarly, we say that $C_1, \ldots, C_k$ is a (k, η, δ)*-blue-blocked configuration* if properties (1–3) hold, but with the roles of red and blue interchanged.

The reason we care about these configurations is that, for appropriate choices of the parameters η and δ, they automatically contain the spines of the required monochromatic books. This idea (or, rather, the version of it when red and blue play symmetric roles) already appears implicitly in the work of the first author [9], but was made much more explicit in the prequel to this paper [10]. The precise statement we will need here is given by the next lemma.

Lemma 4.3. *For every $p \in [\frac{1}{2}, 1)$, there is $k_2 \in \mathbb{N}$ such that the following holds. Let $k \geq k_2$, $c = ((1-p)/p)^k$, and $0 < \varepsilon < \frac{1}{2}$ and suppose $0 < \delta \leq (1-p)\varepsilon$ and $0 < \eta \leq \delta^{4k^2}$. Suppose that the edges of K_N with $N = (p^{-k}+\varepsilon)n$ are red/blue colored and this coloring contains either a (k, η, δ)-red-blocked configuration or a (k, η, δ)-blue-blocked configuration. Then, in either case, the coloring contains either a red $B_{cn}^{(k)}$ or a blue $B_n^{(k)}$. Moreover, one may take $k_2(p) = k_1(1-p)$, where k_1 is the constant from Lemma 4.1.*

Proof. We tackle the two cases separately: suppose first that the coloring has a (k,η,δ)-red-blocked configuration, say $C_1,\dots,C_k$. By the counting lemma, Lemma 2.2, we know that the number of blue K_k with one vertex in each C_i is at least

$$\left(\prod_{1\le i<j\le k} d_B(C_i,C_j) - \eta\binom{k}{2}\right)\prod_{i=1}^k |C_i| \ge \left(\delta^{\binom{k}{2}} - \eta\binom{k}{2}\right)\prod_{i=1}^k |C_i| > 0,$$

so there is at least one blue K_k spanning $C_1,\dots,C_k$. By exactly the same computation, we see that each C_i contains at least one red K_k.

For a vertex v and $i \in [k]$, let $x_i(v) \coloneqq d_B(v, C_i) \in [0,1]$. We observe that from the definition in Lemma 4.1, we have that $k_1(1-p) \ge k_1(p)$ for all $p \ge \frac{1}{2}$. Therefore, Lemma 4.1 implies that since $k \ge k_2 \ge k_1(p)$, we have that

$$p\left(p^{-k}\prod_{i=1}^k x_i(v)\right) + (1-p)\left(\frac{(1-p)^{-k}}{k}\sum_{i=1}^k (1-x_i(v))^k\right) \ge 1$$

for all $v \in V$. Summing this fact up over all v, we find that

$$p\left(p^{-k}\sum_{v\in V}\prod_{i=1}^k x_i(v)\right) + (1-p)\left(\frac{(1-p)^{-k}}{k}\sum_{i=1}^k\sum_{v\in V}(1-x_i(v))^k\right) \ge N. \tag{2}$$

This says that a $(p,1-p)$-weighted average of two numbers is at least N, which means that at least one of them is at least N. Suppose first that the first term is at least N, i.e., that

$$\sum_{v\in V}\prod_{i=1}^k x_i(v) \ge p^k N.$$

Let Q be a uniformly random blue K_k spanning $C_1,\dots,C_k$, which must exist by our computations above. Let $\alpha = \delta^{k^2} \le \prod_{i<j} d_B(C_i,C_j)$ and observe that $\eta \le \delta^{4k^2} = \alpha^4 \le \alpha^3/k^2$. Thus, for any v, we can apply Lemma 2.3 to conclude that the probability v extends Q to a blue K_{k+1} is at least $\prod_i x_i(v) - 4\alpha$. Therefore, the expected number of extensions of Q to a blue K_{k+1} is at least

$$\begin{aligned}\sum_{v\in V}\left(\prod_{i=1}^k x_i(v) - 4\alpha\right) &\ge (p^k - 4\alpha)N \\ &\ge (p^k-4\alpha)(p^{-k}+\varepsilon)n \\ &\ge (1 + p^k\varepsilon - 8\alpha p^{-k})n \\ &\ge n, \end{aligned} \tag{3}$$

where (3) uses that $\alpha = \delta^{k^2} \le ((1-p)\varepsilon)^{k^2} \le (p\varepsilon)^{k^2} \le p^{2k}\varepsilon/8$. Therefore, Q has at least n extensions in expectation, so there must exist some blue K_k with at least n extensions, i.e., a blue $B_n^{(k)}$.

Now assume that the other term in the weighted average in (2) is at least N, i.e., that

$$\frac{1}{k}\sum_{i=1}^{k}\sum_{v\in V}(1-x_i(v))^k \geq (1-p)^k N.$$

Then there must exist some i for which

$$\sum_{v\in V}(1-x_i(v))^k \geq (1-p)^k N.$$

Therefore, if Q is a random red K_k inside this C_i, then, by Lemma 2.3, the expected number of extensions of Q is at least[3]

$$\begin{aligned}\sum_{v\in V}\left[(1-x_i(v))^k-4\alpha\right] &\geq \left[(1-p)^k-4\alpha\right](p^{-k}+\varepsilon)n\\ &\geq \left(\left(\frac{1-p}{p}\right)^k+(1-p)^k\varepsilon-8\alpha p^{-k}\right)n\\ &\geq cn, \end{aligned} \tag{4}$$

where we use the fact that $c=((1-p)/p)^k$ and that

$$\alpha=\delta^{k^2}=((1-p)\varepsilon)^{k^2}\leq p^k(1-p)^k\varepsilon/8,$$

since $1-p\leq p$. Thus, the expected number of red extensions of a red K_k in C_i is at least cn, so there must exist a red $B_{cn}^{(k)}$. This concludes the proof under the assumption that the coloring contains a (k,η,δ)-red-blocked configuration.

Now, we instead assume that the coloring contains a (k,η,δ)-blue-blocked configuration and aim to conclude the same result; the proof is more or less identical. As before, we find that there is at least one red K_k spanning $C_1,\ldots,C_k$ and that each C_i contains at least one blue K_k. For a vertex v and $i\in[k]$, let $y_i(v)=d_R(v,C_i)\in[0,1]$ and write $q=1-p$. Since $k\geq k_2=k_1(q)$, we can sum the result of applying Lemma 4.1 over all $v\in V$ to find that

$$q\left(q^{-k}\sum_{v\in V}\prod_{i=1}^{k}y_i(v)\right)+(1-q)\left(\frac{(1-q)^{-k}}{k}\sum_{i=1}^{k}\sum_{v\in V}(1-y_i(v))^k\right)\geq N.$$

As before, this is a $(q,1-q)$-weighted average of two terms, which means that one of the terms must be at least N. Suppose first that the first term is at least N, i.e., that

$$\sum_{v\in V}\prod_{i=1}^{k}y_i(v)\geq q^k N.$$

[3]Strictly speaking, if $v\in C_i$, then $d_R(v,C_i)\neq 1-x_i(v)$, as v has no edge to itself. However, this tiny loss can be absorbed into the error terms and the result does not change.

If Q is a uniformly random red K_k spanning $C_1, \dots, C_k$ and $\alpha = \delta^{k^2}$, then, as before, we find that the expected number of extensions of Q to a red K_{k+1} is at least

$$\sum_{v \in V} \left(\prod_{i=1}^{k} y_i(v) - 4\alpha \right) \geq (q^k - 4\alpha)N \geq ((1-p)^k - 4\alpha)(p^{-k} + \varepsilon)n \geq cn,$$

by the computation in (4). Therefore, in this case, there must exist some red K_k with at least cn red extensions, giving the desired red $B_{cn}^{(k)}$. So we may assume instead that

$$\frac{1}{k} \sum_{i=1}^{k} \sum_{v \in V} (1 - y_i(v))^k \geq (1-q)^k N,$$

which implies that for some $i \in [k]$,

$$\sum_{v \in V} (1 - y_i(v))^k \geq p^k N.$$

Thus, if Q is a random blue K_k inside this C_i, we find that the expected number of blue extensions of Q is at least

$$\sum_{v \in V} \left[(1 - y_i(v))^k - 4\alpha \right] \geq (p^k - 4\alpha)N \geq (p^k - 4\alpha)(p^{-k} + \varepsilon)n \geq n,$$

by the same computation as in (3). This gives us our blue $B_n^{(k)}$, completing the proof. $\square$

With this result in hand, we can now prove Theorem 1.3.

Proof of Theorem 1.3. Given an integer $k \geq 2$, let $c_1(k)$ be the infimum of $c \in (0, 1]$ such that $k_2((c^{1/k} + 1)^{-1}) \geq k$, where k_2 is the constant from Lemma 4.3. Note that we declare this infimum to equal 1 if no $c \in (0, 1]$ satisfies this condition (as happens for $k = 2$). In this case, there is nothing to prove, since Theorem 1.3 for $c = 1$ is already known [9]. We now fix $c \in [c_1, 1]$ and $p = 1/(c^{1/k} + 1) \in (\frac{1}{2}, 1]$, noting that we have $k \geq k_2(p)$.

Fix $0 < \varepsilon < \frac{1}{2}$ and suppose we are given a red/blue coloring of $E(K_N)$ where $N = (p^{-k} + \varepsilon)n$. Our goal is to prove that if n is sufficiently large in terms of ε, then this coloring contains a red $B_{cn}^{(k)}$ or a blue $B_n^{(k)}$. To do this, we fix parameters $\delta = (1-p)^{2k}\varepsilon/(4k)$ and $\eta = \min\{\delta^{4k^2}, (1-p)/(4k)\}$ depending on c, k, and ε.

We apply Lemma 2.1 to the red graph from our coloring with parameters η and $M_0 = 1/\eta$ to obtain an equitable partition $V(K_N) = V_1 \sqcup \cdots \sqcup V_m$, where each V_i is η-regular and, for each i, there are at most ηm values $1 \leq j \leq m$ such that the pair (V_i, V_j) is not η-regular. Moreover, $M_0 \leq m \leq M = M(\eta, M_0)$. Note that since the colors are complementary, the same properties also hold for the blue graph. Call a part V_i *blue* if $d_B(V_i) \geq \frac{1}{2}$ and *red* otherwise.

Suppose first that at least $m' \geq pm$ of the parts are blue and rename the parts so that $V_1, \dots, V_{m'}$ are these blue parts. We build a reduced graph G whose vertex set is $v_1, \dots, v_{m'}$

by making $\{v_i, v_j\}$ an edge if and only if (V_i, V_j) is η-regular and $d_R(V_i, V_j) \geq \delta$. Suppose that some vertex in G, say v_1, has degree at most $(1 - p^{k-1} - \eta/p)m' - 1$. Since v_1 has at most $\eta m \leq \eta m'/p$ non-neighbors coming from irregular pairs (V_1, V_j), this means that there are at least $p^{k-1}m'$ parts V_j such that (V_1, V_j) is η-regular and $d_B(V_1, V_j) \geq 1 - \delta$. Let J be the set of all these indices j and $U = \bigcup_{j \in J} V_j$ be the union of all of these V_j. We then have

$$e_B(V_1, U) = \sum_{j \in J} e_B(V_1, V_j) \geq \sum_{j \in J} (1 - \delta)|V_1||V_j| = (1 - \delta)|V_1||U|. \tag{5}$$

Let $V_1' \subseteq V_1$ denote the set of vertices $v \in V_1$ with $e_B(v, U) \geq (1 - 2\delta)|U|$. Then we may write

$$e_B(V_1, U) = \sum_{v \in V_1'} e_B(v, U) + \sum_{v \in V_1 \setminus V_1'} e_B(v, U) \leq |V_1'||U| + (1 - 2\delta)|V_1 \setminus V_1'||U|. \tag{6}$$

Combining inequalities (5) and (6), we find that $|V_1'| \geq \frac{1}{2}|V_1|$, where every vertex in V_1' has blue density at least $1 - 2\delta$ into U. Moreover, since $\eta < \frac{1}{2}$, we may apply the η-regularity of V_1 to conclude that the internal blue density of V_1' is at least $\frac{1}{2} - \eta \geq \frac{1}{3}$, while the hereditary property of regularity implies that V_1' is 2η-regular. Then the counting lemma, Lemma 2.2, implies that V_1' contains at least

$$\frac{1}{k!}\left(d_B(V_1')^{\binom{k}{2}} - 2\eta\binom{k}{2}\right)|V_1'|^k \geq \frac{1}{k!}\left(3^{-\binom{k}{2}} - 2\eta\binom{k}{2}\right)|V_1'|^k > 0$$

blue K_k, so that V_1' contains at least one blue K_k. Every vertex of this blue K_k has at least $(1-2\delta)|U|$ blue neighbors in U, so the blue K_k has at least $(1-2k\delta)|U|$ blue extensions into U. Moreover, since we assumed that $|J| \geq p^{k-1}m' \geq p^k m$ and the partition is equitable, we find that $|U| \geq p^k N$. Therefore,

$$\begin{aligned}
(1 - 2k\delta)|U| &\geq (1 - 2k\delta)p^k(p^{-k} + \varepsilon)n \\
&= (1 - 2k\delta)(1 + p^k\varepsilon)n \\
&\geq (1 + p^k\varepsilon - 4k\delta)n \\
&\geq n,
\end{aligned}$$

since our choice of δ yields $4k\delta = (1 - p)^{2k}\varepsilon \leq p^k\varepsilon$. Thus, we find that any blue K_k inside V_1' must have at least n blue extensions, giving us our blue $B_n^{(k)}$.

So we may assume that every vertex in G has degree at least $(1 - p^{k-1} - \eta/p)m'$. Recall from the proof of Lemma 4.1 that $f(1-p, k) = p^{1-k}/(e^2k) \geq 1$ for $k \geq k_1(1-p)$. Since we assume that $k \geq k_2(p) = k_1(1-p)$, this implies that

$$p^{k-1} \leq \frac{1}{e^2k} \leq \frac{1}{3(k-1)}.$$

Additionally, by our choice of $\eta \leq (1-p)/(4k) \leq p/(4k)$, we know that

$$\frac{\eta}{p} \leq \frac{1}{3(k-1)}.$$

The previous two inequalities imply that

$$1 - p^{k-1} - \frac{\eta}{p} > 1 - \frac{1}{k-1},$$

so that G contains a K_k by Turán's theorem. Let $v_{i_1}, \dots, v_{i_k}$ be the vertices of this K_k and let $C_j = V_{i_j}$ for $1 \le j \le k$. Then we claim that $C_1, \dots, C_k$ is a (k, η, δ)-blue-blocked configuration. The fact that each C_i is η-regular follows immediately from Lemma 2.1 and the fact that $d_B(C_i) \ge \delta$ follows from the fact that we assumed $d_B(C_i) \ge \frac{1}{2}$. Finally, the definition of edges in G implies that (C_i, C_j) is η-regular with $d_R(C_i, C_j) \ge \delta$ for all $i \ne j$. Thus, our coloring contains a (k, η, δ)-blue-blocked configuration with $\delta \le (1-p)\varepsilon$ and $\eta \le \delta^{4k^2}$, so Lemma 4.3 implies that the coloring contains either a red $B_{cn}^{(k)}$ or a blue $B_n^{(k)}$.

We have now finished the proof if at least pm of the parts V_i are blue. Therefore, we may assume instead that at least $m'' \ge (1-p)m$ of the parts are red and again rename the parts so that these red parts are $V_1, \dots, V_{m''}$. We construct a reduced graph G on vertices $v_1, \dots, v_{m''}$ by connecting v_i to v_j if (V_i, V_j) is η-regular with $d_B(V_i, V_j) \ge \delta$. Suppose that some vertex in G, say v_1, has degree at most $(1 - (1-p)^{k-1} - \eta/(1-p))m'' - 1$. As before, v_1 has at most $\eta m \le \eta m''/(1-p)$ non-neighbors coming from irregular pairs. Thus, if we let J denote the set of indices j for which (V_1, V_j) is η-regular with $d_R(V_1, V_j) \ge 1 - \delta$, then we find that $|J| \ge (1-p)^{k-1} m'' \ge (1-p)^k m$. Thus, if $U = \bigcup_{j \in J} V_j$, then we see that $|U| \ge (1-p)^k N$, since the partition is equitable. Next, as above, we let $V_1' \subseteq V_1$ denote the set of vertices $v \in V_1$ with $e_R(v, U) \ge (1 - 2\delta)|U|$ and find that $|V_1'| \ge \frac{1}{2}|V_1|$. Therefore, as above, we know that V_1' contains at least one red K_k and this red K_k has at least $(1 - 2k\delta)|U|$ red extensions in U. Moreover,

$$\begin{aligned}
(1 - 2k\delta)|U| &\ge (1 - 2k\delta)(1-p)^k N \\
&= (1 - 2k\delta)(1-p)^k (p^{-k} + \varepsilon) n \\
&= (1 - 2k\delta)(c + (1-p)^k \varepsilon) n \qquad (7) \\
&\ge (c + (1-p)^k \varepsilon - 4k\delta) n \\
&\ge cn, \qquad (8)
\end{aligned}$$

where in (7) we used the definition of p, which implies that $((1-p)/p)^k = c$, and in (8) we used our choice of δ to see that $\delta \le (1-p)^k \varepsilon/(4k)$. Thus, in this case, we can find a red $B_{cn}^{(k)}$.

We may therefore assume that every vertex in G has degree at least $(1 - (1-p)^{k-1} - \eta/(1-p))m''$. As before, we know that, since $k \ge k_2(p)$,

$$(1-p)^{k-1} \le p^{k-1} \le \frac{1}{3(k-1)}$$

and our choice of $\eta \le (1-p)/(4k)$ implies that

$$\frac{\eta}{1-p} \le \frac{1}{3(k-1)}.$$

Thus, by Turán's theorem, G must contain a K_k, with vertices $v_{i_1}, \ldots, v_{i_k}$. If we let $C_j = V_{i_j}$, then $C_1, \ldots, C_k$ will be a (k, η, δ)-red-blocked configuration, by the definition of edges in G and the assumption that N is sufficiently large in terms of ε. Thus, by Lemma 4.3, we can again conclude that the coloring contains either a red $B_{cn}^{(k)}$ or a blue $B_n^{(k)}$.

To finish, we note that as claimed, we may take $c_1(k) \le ((1 + o(1)) \log k/k)^k$. Indeed, for any c and k, let $p(c, k) = (c^{1/k} + 1)^{-1}$ and $y = y(c, k) = 1/\log[1/p(c, k)]$. Then, from Lemmas 4.1 and 4.3, we see that $k_2(p(c, k)) = 1 + y(5 + \log y + \log\log y)$. Thus, if $y \le (1+o(1))k/\log k$, then $k \ge k_2(p(c, k))$. Since $y = 1/\log(1+c^{1/k})$, this condition is equivalent to $c^{1/k} \ge \exp((1+o(1)) \log k/k) - 1 = (1+o(1)) \log k/k$, which yields the desired bound. $\square$

5 Quasirandomness

In the previous section, we showed that for a certain range of c and k, the Ramsey number $r(B_{cn}^{(k)}, B_n^{(k)})$ is, as $n \to \infty$, asymptotically equal to the lower bound coming from a p-random construction. In this section, we strengthen this result, showing that all colorings whose number of vertices is close to the Ramsey number must either be quasirandom or else contain substantially larger books than the Ramsey property implies. Here is the restatement of Theorem 1.4 in terms of (c, γ)-many books that we will prove.

Theorem 1.4'. *For every $p \in [\frac{1}{2}, 1)$, there exists some $k_0 \in \mathbb{N}$ such that the following holds for every $k \ge k_0$. For every $\theta > 0$, there exists some $\gamma > 0$ such that if a red/blue coloring of $E(K_N)$ is not (p, θ)-quasirandom, then it contains (c, γ)-many books for $c = ((1-p)/p)^k$.*

To prove Theorem 1.4, we will need a few technical lemmas. At a high level, the proof closely follows the proof of the main quasirandomness theorem in [10, Section 5], as follows. First, we prove a strengthening of Lemma 4.1, which can be thought of as a stability version of that result; it says that if our vector $(x_1, \ldots, x_k)$ is bounded in ℓ_∞ away from the minimizing point $(p, \ldots, p)$, then the value of the function in Lemma 4.1 is bounded away from its minimum of 1. Using this, we can strengthen Lemma 4.3 to say that not only does a blocked configuration imply the existence of the desired monochromatic book, but in fact it implies the existence of a larger book unless every part of the blocked configuration is ε-regular to the entire vertex set. Therefore, assuming our coloring does not contain many blue $B_{(p^k+\gamma)N}^{(k)}$ or red $B_{((1-p)^k+\gamma)N}^{(k)}$, we will be able to repeatedly pull out vertex subsets that are ε-regular to the entire vertex set until we have almost partitioned all the vertices into such subsets. At that point, we can use the structure coming from this partition to deduce that the coloring is (p, θ)-quasirandom, as desired.

We begin with the strengthening of Lemma 4.1.

Lemma 5.1. *For $p \in (0, 1)$, let $k_1 = k_1(p)$ be as in Lemma 4.1. Then, for every integer $k \ge k_1$ and any $\varepsilon_0 > 0$, there exists some $\delta_0 > 0$ such that if $x_1, \ldots, x_k \in [0, 1]$ are numbers with $|x_j - p| \ge \varepsilon_0$ for some j, then*

$$p^{1-k} \prod_{i=1}^{k} x_i + \frac{(1-p)^{1-k}}{k} \sum_{i=1}^{k} (1 - x_i)^k \ge 1 + \delta_0.$$

Proof. Let

$$F(x_1, \dots, x_k) = p^{1-k} \prod_{i=1}^{k} x_i + \frac{(1-p)^{1-k}}{k} \sum_{i=1}^{k} (1-x_i)^k$$

and $\varphi(y) = (1-e^y)^k$. The goal is to apply Hölder's defect formula, Theorem 2.6, using the strict convexity of the function φ. However, φ is only strictly convex on the interval $(\log \frac{1}{k}, 0)$ and, in order to apply Theorem 2.6, we in fact need a positive lower bound on φ'', but no such bound exists for the whole interval $(\log \frac{1}{k}, 0)$. Because of this, we need to separately analyze the cases where all the variables are inside a large subinterval of $(\frac{1}{k}, 1)$ and when one of them is outside such a subinterval.

First, suppose that one of the variables, say x_1, is in the interval $[0, \frac{1+\varepsilon_1}{k}]$, for some small constant $\varepsilon_1 > 0$. Then we have that

$$F(x_1, \dots, x_k) \geq \frac{(1-p)^{1-k}}{k} (1-x_1)^k \geq \frac{(1-p)^{1-k}}{k} \left(1 - \frac{1+\varepsilon_1}{k}\right)^k.$$

From the proof of Lemma 4.1, we see that this quantity is strictly larger than 1 for all $k \geq k_1(p)$, so, by choosing δ_0 appropriately, we see that $F(x_1, \dots, x_k) \geq 1 + \delta_0$ in this case. We may therefore assume from now on that all the variables are at least $\frac{1+\varepsilon_1}{k}$.

Next, suppose that there exist values $x_1, \dots, x_{k-1} \in [\frac{1+\varepsilon_1}{k}, 1]$ such that $F(x_1, \dots, x_{k-1}, 1) = 1$. We observe that

$$\left.\frac{\partial F}{\partial x_k}\right|_{x_k=1} = \left[p^{1-k} \prod_{i=1}^{k-1} x_i - (1-p)^{1-k}(1-x_k)^{k-1}\right]_{x_k=1} = p^{1-k} \prod_{i=1}^{k-1} x_i > 0.$$

This implies that if we move from $x_k = 1$ to $x_k = 1 - \varepsilon_2$ for some sufficiently small ε_2, the value of F will decrease. Therefore, there will exist a vector $(x_1, \dots, x_k)$ for which $F(x_1, \dots, x_k) < 1$, contradicting Lemma 4.1 as long as $k \geq k_1(p)$. Thus, for every choice of $x_1, \dots, x_{k-1} \in [\frac{1+\varepsilon_1}{k}, 1]$, we have that $F(x_1, \dots, x_{k-1}, 1) > 1$. Since the space $[\frac{1+\varepsilon_1}{k}, 1]^{k-1} \times \{1\}$ is compact, we in fact find that $F(x_1, \dots, x_{k-1}, 1) \geq 1 + \delta_1'$ for all $x_1, \dots, x_{k-1} \in [\frac{1+\varepsilon_1}{k}, 1]$, for some sufficiently small δ_1' depending on p and k. Finally, by continuity of F, we have that $F(x_1, \dots, x_k) \geq 1 + \delta_1$ whenever $x_k \geq 1 - \varepsilon_2$ for some other $\delta_1, \varepsilon_2 > 0$. Since F is a symmetric function of its variables, the same conclusion holds if $x_i \geq 1 - \varepsilon_2$ for any i. Thus, as long as we take the δ_0 in the lemma statement to be smaller than δ_1, we can assume from now on that $x_i \in [\frac{1+\varepsilon_1}{k}, 1 - \varepsilon_2]$ for all i.

By Lemma 2.5, subject to the constraint $\prod_{i=1}^{k} x_i = z$, the term $\frac{1}{k} \sum_{i=1}^{k} (1-x_i)^k$ is minimized when $x_i = z^{1/k}$ for all i. As in the proof of Lemma 4.1, this shows that $F(x_1, \dots, x_k) \geq \psi(z^{1/k})$, where $\psi(w) = p^{1-k} w^k + (1-p)^{1-k}(1-w)^k$. The function ψ has a global minimum at $w = p$, where its value is 1. This shows that $F(x_1, \dots, x_k) \geq 1 + \delta_0$ if $|z^{1/k} - p| \geq \varepsilon_3$, for some $\varepsilon_3 > 0$ depending on p, k, and δ_0. Moreover, by picking δ_0 sufficiently small, we can make ε_3 as small as we wish. Therefore, we may now assume that $z^{1/k} = p \pm \varepsilon_3$, which implies that $\log(z^{1/k}) = (\log p) \pm \varepsilon_4$ for some $\varepsilon_4 > 0$, which can also be made arbitrarily small by picking δ_0 appropriately.

We are now ready to apply Hölder's defect formula. First, we observe that for $y \in [\log \frac{1+\varepsilon_1}{k}, \log(1-\varepsilon_2)]$, we have

$$\varphi''(y) = ke^y(1-e^y)^{k-2}(ke^y - 1) \geq k \cdot \frac{1+\varepsilon_1}{k} \cdot \varepsilon_2^{k-2} \cdot \varepsilon_1 =: m,$$

where m is a fixed, strictly positive constant. Let $y_i = \log x_i$ for $1 \leq i \leq k$, so that $\frac{1}{k}\sum_{i=1}^k y_i = \log(z^{1/k})$. We assumed that $|x_j - p| \geq \varepsilon_0$ for some j, which implies that $|y_j - \log p| \geq \varepsilon_0$ as well, since the derivative of $\log y$ is bounded below by 1 on the interval $(0, 1)$. Therefore, choosing δ_0 small enough that $\varepsilon_4 < \varepsilon_0$, we see that

$$\frac{1}{k}\sum_{i=1}^{k}(y_i - \log(z^{1/k}))^2 \geq \frac{1}{k}(y_j - \log(z^{1/k}))^2 \geq \frac{1}{k}(\varepsilon_0 - \varepsilon_4)^2,$$

since $\log(z^{1/k}) = (\log p) \pm \varepsilon_4$ and $|y_j - \log p| \geq \varepsilon_0$. Hence, by Theorem 2.6, we have that

$$\begin{aligned}
F(x_1, \ldots, x_k) &= p^{1-k}z + \frac{(1-p)^{1-k}}{k}\sum_{i=1}^{k}(1-x_i)^k \\
&= p^{1-k}z + (1-p)^{1-k} \cdot \frac{1}{k}\sum_{i=1}^{k}\varphi(y_i) \\
&\geq p^{1-k}z + \varphi(\log(z^{1/k})) + \frac{m}{2k}(\varepsilon_0 - \varepsilon_4)^2 \\
&= \psi(z^{1/k}) + \frac{m}{2k}(\varepsilon_0 - \varepsilon_4)^2 \\
&\geq 1 + \delta_0,
\end{aligned}$$

where we use the fact that $\psi(w) \geq 1$ for all $w \in [0, 1]$ and take δ_0 sufficiently small. □

Using Lemma 5.1, we can now prove the following strengthening of Lemma 4.3, which says that if we have a blocked configuration $C_1, \ldots, C_k$ and many vertices whose blue density into C_i is far from p, then we can find a substantially larger monochromatic book than what is guaranteed by Lemma 4.3.

Lemma 5.2. *Fix $p \in [\frac{1}{2}, 1)$ and let $k \geq k_2(p)$, where k_2 is the constant from Lemma 4.3. Suppose $0 < \varepsilon_0 < \frac{1}{4}$ and let $\delta_0 = \delta_0(\varepsilon_0)$ be the parameter from Lemma 5.1. Let $0 < \delta \leq (1-p)\delta_0\varepsilon_0$ and $0 < \eta \leq \delta^{4k^2}$ and suppose that $C_1, \ldots, C_k$ is either a (k, η, δ)-red-blocked configuration or a (k, η, δ)-blue-blocked configuration in a red/blue coloring of K_N. Define*

$$B_i = \{v \in K_N : |d_B(v, C_i) - p| \geq \varepsilon_0\}.$$

If $|B_i| \geq \varepsilon_0 N$ for some i, then the coloring contains a blue $B^{(k)}_{(p^k+\beta)N}$ or a red $B^{(k)}_{((1-p)^k+\beta)N}$, where $\beta = (1-p)^k\delta_0\varepsilon_0/2$. Moreover, if $|C_i| \geq \tau N$ for all i and some $\tau > 0$, then there exists some $0 < \gamma < \beta$ depending on ε_0, τ, and δ such that the coloring contains (c, γ)-many books for $c = ((1-p)/p)^k$.

Proof. We may assume without loss of generality that $|B_1| \geq \varepsilon_0 N$. As in the proof of Lemma 4.3, we need to split into two cases, depending on whether $C_1, \ldots, C_k$ is blue-blocked or red-blocked. We begin by assuming that it is (k, η, δ)-red-blocked.

First, as in the proof of Lemma 4.3, observe that each C_i contains at least one red K_k and there is at least one blue K_k spanning $C_1, \ldots, C_k$. Moreover, if we assume that $|C_i| \geq \tau N$ for all i, then Lemma 2.2 shows that the number of blue K_k spanning $C_1, \ldots, C_k$ is at least

$$\left(\prod_{1 \leq i < j \leq k} d_B(C_i, C_j) - \eta\binom{k}{2}\right) \prod_{i=1}^{k} |C_i| \geq \left(\delta^{\binom{k}{2}} - \eta\binom{k}{2}\right)(\tau N)^k \geq \left(\frac{\delta^{\binom{k}{2}}\tau^k}{2}\right) N^k$$

and similarly, with an additional factor of $1/k!$, for the number of red K_k inside each C_i.

For a vertex v and $i \in [k]$, let $x_i(v) = d_B(v, C_i)$. Lemma 4.1 implies that, for any $v \in V$,

$$p\left(p^{-k}\prod_{i=1}^{k} x_i(v)\right) + (1-p)\left(\frac{(1-p)^{-k}}{k}\sum_{i=1}^{k}(1 - x_i(v))^k\right) \geq 1.$$

Additionally, if $v \in B_1$, then $|x_i(v) - p| \geq \varepsilon_0$, so Lemma 5.1 implies that, for $v \in B_1$,

$$p\left(p^{-k}\prod_{i=1}^{k} x_i(v)\right) + (1-p)\left(\frac{(1-p)^{-k}}{k}\sum_{i=1}^{k}(1 - x_i(v))^k\right) \geq 1 + \delta_0.$$

Adding these two equations up over all $v \in V$ shows that

$$p\left(p^{-k}\sum_{v \in V}\prod_{i=1}^{k} x_i(v)\right) + (1-p)\left(\frac{(1-p)^{-k}}{k}\sum_{i=1}^{k}\sum_{v \in V}(1 - x_i(v))^k\right) \geq N + \delta_0|B_1| \geq (1 + \delta_0\varepsilon_0)N.$$

That is, a $(p, 1-p)$-weighted average of two quantities is at least $(1 + \delta_0\varepsilon_0)N$, which implies that one of the quantities must itself be at least $(1 + \delta_0\varepsilon_0)N$. Suppose first that

$$p^{-k}\sum_{v \in V}\prod_{i=1}^{k} x_i(v) \geq (1 + \delta_0\varepsilon_0)N.$$

Let Q be a random blue K_k spanning $C_1, \ldots, C_k$. Let $\alpha = \delta^{k^2} \leq \prod_{i<j} d_B(C_i, C_j)$, so that $\eta \leq \delta^{4k^2} = \alpha^4 \leq \alpha^3/k^2$. Therefore, applying Lemma 2.3 to each v and summing up the result, we find that the expected number of blue extensions of Q is at least

$$\sum_{v \in V}\left(\prod_{i=1}^{k} x_i(v) - 4\alpha\right) \geq (p^k + p^k\delta_0\varepsilon_0 - 4\alpha)N.$$

Next, observe that

$$4\alpha = 4\delta^{k^2} \leq \frac{\delta^k}{2} \leq \frac{((1-p)\delta_0\varepsilon_0)^k}{2} \leq \frac{(1-p)^k\delta_0\varepsilon_0}{2} \leq \frac{p^k\delta_0\varepsilon_0}{2}, \tag{9}$$

which implies that the expected number of blue extensions of Q is at least $(p^k + \beta)N$, where $\beta = (1-p)^k \delta_0 \varepsilon_0/2$. Thus, there exists a blue $B^{(k)}_{(p^k+\beta)N}$. Moreover, if we assume that $|C_i| \geq \tau N$ for all i, then our earlier computation shows that Q is chosen uniformly at random from a set of at least κN^k monochromatic cliques, where $\kappa = \delta^{\binom{k}{2}} \tau^k/2$. We may therefore apply Lemma 2.4 with $\xi = p^k + \beta$ and $\nu = p^k + \gamma$, for some appropriately chosen $0 < \gamma < \beta$, to conclude that in this case our coloring contains at least γN^k blue cliques, each with at least $(p^k + \gamma)N$ blue extensions.

Therefore, we may assume that the other term in the weighted average is the large one, i.e., that

$$\frac{(1-p)^{-k}}{k} \sum_{i=1}^{k} \sum_{v \in V} (1 - x_i(v))^k \geq (1 + \delta_0 \varepsilon_0)N,$$

which implies that, for some i,

$$\sum_{v \in V} (1 - x_i(v))^k \geq (1-p)^k (1 + \delta_0 \varepsilon_0) N.$$

Therefore, if Q is now a random red K_k inside this C_i, Lemma 2.3 implies that the expected number of red extensions of Q is at least

$$\sum_{v \in V} \left[(1 - x_i(v))^k - 4\alpha\right] \geq \left[(1-p)^k + (1-p)^k \delta_0 \varepsilon_0 - 4\alpha\right] N.$$

But, by (9), $4\alpha \leq (1-p)^k \delta_0 \varepsilon_0/2$, which implies that the expected number of red extensions of Q is at least $((1-p)^k + \beta)N$, as desired. As before, if we also assume that $|C_i| \geq \tau N$ for all i, then we may apply Lemma 2.4 with $\kappa = \delta^{\binom{k}{2}} \tau^k/2k!$, $\xi = (1-p)^k + \beta$, and $\nu = (1-p)^k + \gamma$ to find that our coloring contains at least γN^k red K_k, each with at least $((1-p)^k + \gamma)N$ red extensions for some appropriately chosen $\gamma \in (0, \beta)$. This concludes the proof of the lemma in the case where $C_1, \ldots, C_k$ is a (k, η, δ)-red-blocked configuration.

As in the proof of Lemma 4.3, the other case, where $C_1, \ldots, C_k$ is a (k, η, δ)-blue-blocked configuration, follows in an almost identical fashion. We define $y_i(v) = d_R(v, C_i)$ for all $v \in V$ and $i \in [k]$ and let $q = 1 - p$. We then apply Lemmas 4.1 and 5.1 with these y variables and with q instead of p. The remaining details are exactly the same. $\square$

Next, we strengthen Lemma 5.2 by showing that not only does every part of a blocked configuration have density roughly p to most vertices, but it is in fact (p, ε)-regular to the entire vertex set. Here, by saying that a pair of vertex subsets (X, Y) is (p, ε)-*regular*, we mean that $|d(X, Y) - p| \leq \varepsilon$ for every $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \varepsilon |X|$, $|Y'| \geq \varepsilon |Y|$. Note that (p, ε)-regularity is equivalent, up to a linear change in the parameters, to ε-regularity with density $p \pm \varepsilon$.

Lemma 5.3. *Fix $p \in [\frac{1}{2}, 1)$ and let $k \geq k_2(p)$. Suppose $0 < \varepsilon_1 < \frac{1}{4}$, $\varepsilon_0 = \varepsilon_1^2/2$, and let $\delta_0 = \delta_0(\varepsilon_0)$ be the parameter from Lemma 5.1. Let $0 < \delta \leq (1-p)\delta_0 \varepsilon_0$ and $0 < \eta \leq \varepsilon_1 2^{-4k^2} \delta^{4k^2}$ and suppose that $C_1, \ldots, C_k$ is either a (k, η, δ)-red-blocked or a (k, η, δ)-blue-blocked configuration in a red/blue coloring of K_N. If, for some i, the pair (C_i, V) is not*

(p,ε_1)-regular in blue, then the coloring contains a blue $B^{(k)}_{(p^k+\beta)N}$ or a red $B^{(k)}_{((1-p)^k+\beta)N}$, where $\beta = (1-p)^k\delta_0\varepsilon_0/2$. Moreover, if $|C_i| \geq \tau N$ for all i and some $\tau > 0$, then the coloring contains (c,γ)-many books for $c = ((1-p)/p)^k$ and some $0 < \gamma < \beta$ depending on ε_1, δ, and τ.

Proof. Without loss of generality, suppose that (C_1, V) is not (p,ε_1)-regular in blue. Then there exist $C_1' \subseteq C_1, D \subseteq V$ with $|C_1'| \geq \varepsilon_1|C_1|, |D| \geq \varepsilon_1 N$ such that $|d_B(C_1', D) - p| > \varepsilon_1$. Assume first that $d_B(C_1', D) \geq p + \varepsilon_1$. Let $D_1 \subseteq D$ denote the set of vertices $v \in D$ with $d_B(v, C_1') < p + \frac{\varepsilon_1}{2}$ and let $D_2 = D \setminus D_1$. Then we have that

$$(p+\varepsilon_1)\,|C_1'||D| \leq \sum_{v\in D_1} e_B(v, C_1') + \sum_{v \in D_2} e_B(v, C_1') \leq \left(p + \frac{\varepsilon_1}{2}\right)|C_1'||D| + |C_1'||D_2|,$$

which implies that $|D_2| \geq \frac{\varepsilon_1}{2}|D| \geq \frac{\varepsilon_1^2}{2}N = \varepsilon_0 N$, where each $v \in D_2$ has $d_B(v, C_1') \geq p + \frac{\varepsilon_1}{2}$. Now, consider the k-tuple of sets $C_1', C_2, \ldots, C_k$; by the hereditary property of regularity, we see that this is a (k, η', δ')-blocked configuration, where $\eta' = \eta/\varepsilon_1$ and $\delta' = \delta - \eta \geq \delta/2$. This implies that $\delta' \leq (1-p)\delta_0\varepsilon_0$ and $\eta' \leq (\delta')^{4k^2}$. Therefore, we may apply Lemma 5.2 to the (k,η',δ')-blocked configuration $C_1', C_2, \ldots, C_k$ to conclude that the coloring contains a blue $B^{(k)}_{(p^k+\beta)N}$ or a red $B^{(k)}_{((1-p)^k+\beta)N}$. Moreover, if we assume that $|C_i| \geq \tau N$ for all i, then $|C_i'| \geq \varepsilon_1 \tau N$ for all i, where $C_i' = C_i$ if $i \geq 2$. Thus, the second part of Lemma 5.2 implies that in this case the coloring contains (c,γ)-many books for $c = ((1-p)/p)^k$ and some $0 < \gamma < \beta$ depending on ε_1, δ, and τ.

To complete the proof of the lemma, we also need to check the case where $d_B(C_1', D) \leq p - \varepsilon_1$. However, the proof is essentially identical: we find a subset $D_2 \subseteq D$ such that every vertex $v \in D_2$ has $d_B(v, C_1') \leq p - \frac{\varepsilon_1}{2}$ and such that $|D_2| \geq \frac{\varepsilon_1}{2}|D|$ and then the rest of the proof is as above. □

Our next technical lemma gives the inductive step for our proof of Theorem 1.4. The proof mimics that of Theorem 1.3, except that the vertex set is split into parts that were already pulled out as regular and a part that has not yet been touched. Inside the untouched part, we build a reduced graph and use it to find either many large monochromatic books or a blocked configuration, at which point Lemma 5.3 implies that the induction can continue.

Lemma 5.4. *Fix $p \in [\frac{1}{2}, 1)$ and let $k \geq k_2(p)$. Fix $0 < \varepsilon \leq p/(20k)$ and suppose that the edges of the complete graph K_N with vertex set V have been red/blue colored. Suppose that $A_1, \ldots, A_\ell$ are disjoint subsets of V such that (A_i, V) is (p,ε^2)-regular for all i. Let $W = V \setminus (A_1 \cup \cdots \cup A_\ell)$ and suppose that $|W| \geq \varepsilon N$. Then either there is some $A_{\ell+1} \subseteq W$ such that $(A_{\ell+1}, V)$ is (p,ε^2)-regular or else the coloring contains (c,γ)-many books for $c = ((1-p)/p)^k$ and some $\gamma > 0$ depending on ε, p, and k.*

Proof. Let $\varepsilon_1 = \varepsilon^2$, $\varepsilon_0 = \varepsilon_1^2/2$, and $\delta_0 = \delta_0(\varepsilon_0)$ be the parameter from Lemma 5.1 and set $\delta = (1-p)\delta_0\varepsilon_0$, $\eta = \varepsilon^2 2^{-4k^2}\delta^{4k^2}$, $\beta = kp^{k-1}\varepsilon^2$, and $\beta' = 4\varepsilon$. We apply Lemma 2.1 to the subgraph induced on W, with parameters η and $M_0 = 1/\eta$, to obtain an equitable partition $W = W_1 \sqcup \cdots \sqcup W_m$, where $M_0 \leq m \leq M = M(\eta, M_0)$. Call a part W_i *blue* if $d_B(W_i) \geq \frac{1}{2}$

and *red* otherwise. As in the proof of Theorem 1.3, we first assume that at least $m' \geq pm$ of the parts are blue and rename them so that $W_1, \ldots, W_{m'}$ are the blue parts.

We build a reduced graph G on vertex set $w_1, \ldots, w_{m'}$, connecting w_{i_1} and w_{i_2} by an edge if (W_{i_1}, W_{i_2}) is η-regular and $d_R(W_{i_1}, W_{i_2}) \geq \delta$. Suppose that w_1 has at most $(1 - p^{k-1} - \beta'/p - \eta/p)m' - 1$ neighbors in G. Since w_1 has at most $\eta m \leq \eta m'/p$ non-neighbors coming from irregular pairs, this means that there are at least $(p^{k-1} + \beta'/p)m'$ parts W_j with $2 \leq j \leq m'$ such that (W_1, W_j) is η-regular and $d_B(W_1, W_j) \geq 1 - \delta$. Let J be the set of these indices j and set $U = \bigcup_{j \in J} W_j$. By the counting lemma, Lemma 2.2, W_1 contains at least $\frac{1}{k!}\left(2^{-\binom{k}{2}} - \eta\binom{k}{2}\right)|W_1|^k$ blue copies of K_k and

$$\frac{1}{k!}\left(2^{-\binom{k}{2}} - \eta\binom{k}{2}\right)|W_1|^k \geq \frac{2^{-k^2}}{k!}\left(\frac{|W|}{M}\right)^k \geq \left(\frac{\varepsilon N}{k2^k M}\right)^k,$$

where we use that $\eta \leq \delta^{4k^2} \leq \delta^{\binom{k}{2}}/\binom{k}{2}$ and that $2^{-\binom{k}{2}} - \delta^{\binom{k}{2}} > 2^{-k^2}$, along with our assumption that $|W| \geq \varepsilon N$. If we set $\kappa = (\varepsilon/k2^k M)^k$, then this implies that W_1 contains at least κN^k blue K_k. If we pick a random such blue K_k, then Lemma 2.3 with $\alpha = \delta^{k^2} \leq 2^{-\binom{k}{2}} \leq d_B(W_1)^{\binom{k}{2}}$ implies that its expected number of blue extensions inside U is at least

$$\sum_{u \in U}\left(d_B(u, W_1)^k - 4\alpha\right) \geq \left[(1-\delta)^k - \delta^k\right]|U| \geq (1 - 2k\delta)|U|,$$

where we first use Jensen's inequality applied to the convex function $x \mapsto x^k$ to lower bound $\sum_u d_R(u, W_1)^k$ by $(1-\delta)^k|U|$ and then use that $(1-\delta)^k \geq 1 - k\delta$ and $4\delta^{k^2} \leq \delta^k \leq k\delta$. Since we assumed that J was large, and since the partition is equitable, we find that

$$|U| \geq (p^{k-1} + \beta'/p)m'|W_j| \geq (p^k + \beta')|W|.$$

Thus, a random blue K_k inside W_1 has at least $(1 - 2k\delta)(p^k + \beta')|W|$ blue extensions in W.

Now, suppose that instead of just w_1 having low degree in G, we have a set of at least εm vertices $w_j \in V(G)$, each with at most $(1 - p^{k-1} - \beta'/p - \eta/p)m' - 1$ neighbors in G. Let S be the set of these j and $T = \bigcup_{j \in S} W_j$. By the above argument, for every $j \in S$, we have that W_j contains at least κN^k blue K_k, each with at least $(1 - 2k\delta)(p^k + \beta')|W|$ blue extensions on average into W. Moreover, we have that

$$|T| = |S||W_j| \geq \varepsilon m \frac{|W|}{m} = \varepsilon|W| \geq \varepsilon^2|V|.$$

We may therefore apply the (p, ε^2) regularity of (A_i, V) to conclude that $d_B(A_i, T) = p \pm \varepsilon^2$ for all i. Thus, if we pick $j \in S$ randomly, then $\mathbb{E}[d_B(W_j, A_i)] = p \pm \varepsilon^2$. Therefore, if we first sample $j \in S$ randomly and then pick a random blue K_k inside W_j, then Lemma 2.3 implies that this random blue K_k will have in expectation at least

$$\begin{aligned}\sum_{a\in A_i}\left(d_B(a,W_j)^k-4\delta^{k^2}\right) &\geq \left[\left(p-\varepsilon^2\right)^k-\delta^k\right]|A_i| \\ &\geq \left[p^k\left(1-\frac{k\varepsilon^2}{p}\right)-\delta^k\right]|A_i| \\ &\geq p^k\left(1-\frac{2k\varepsilon^2}{p}\right)|A_i|\end{aligned}$$

blue extensions into A_i, again by Jensen's inequality. This implies that this random K_k has in expectation at least $(1-2k\varepsilon^2/p)p^k|A_1\cup\cdots\cup A_\ell|$ extensions into $A_1\cup\cdots\cup A_\ell$. Adding up the extensions into this set and into W, its complement, shows that this random blue K_k has in expectation at least ξN blue extensions, where ξ is a weighted average of $(1-2k\varepsilon^2/p)p^k$ and $(1-2k\delta)(p^k+\beta')$, and where the latter quantity receives weight at least ε, since $|W|\geq\varepsilon N$. Thus,

$$\begin{aligned}\xi &\geq (1-\varepsilon)\left(1-\frac{2k\varepsilon^2}{p}\right)p^k+\varepsilon(1-2k\delta)(p^k+\beta') \\ &\geq \left(1-\frac{2k\varepsilon^2}{p}-\varepsilon\right)p^k+\varepsilon(1-2k\delta)(1+p^{-k}\beta')p^k \\ &\geq \left(1-\frac{2k\varepsilon^2}{p}-\varepsilon\right)p^k+\varepsilon\left(1+\frac{3k\varepsilon}{p}\right)p^k \\ &= p^k\left(1+\frac{k\varepsilon^2}{p}\right) \\ &= p^k+\beta,\end{aligned}$$

where we used the definition of β, the fact that $2k\delta<p^{-k}\beta'/4$, that $(1-x/4)(1+x)\geq 1+x/2$ for all $x\in[0,1]$, and that $p^{-k}\beta'\geq 6k\varepsilon/p$, which follows since $\beta'=4\varepsilon$ and, as in the proof of Lemma 4.1, $p^{1-k}\geq e^2k\geq\frac{3}{2}k$ for $k\geq k_2(p)$. Therefore, by Lemma 2.4, we can find at least γN^k blue K_k, each with at least $(p^k+\gamma)N$ blue extensions, for some $\gamma<\beta$ depending on ε and β and, thus, only on ε, p, and k.

Therefore, we may assume that in G, all but $\varepsilon m\leq\varepsilon m'/p$ of the vertices have degree at least $(1-p^{k-1}-\beta'/p-\eta/p)m'$. Hence, the average degree in G is at least

$$\left(1-\frac{\varepsilon}{p}\right)\left(1-p^{k-1}-\frac{\beta'}{p}-\frac{\eta}{p}\right)m'\geq\left(1-p^{k-1}-\frac{6\varepsilon}{p}\right)m'\geq\left(1-p^{k-1}-\frac{1}{3k}\right)m',$$

since $\beta'=4\varepsilon$, $\eta\leq\varepsilon$, and $\varepsilon\leq p/(20k)$. As in the proof of Theorem 1.3, the fact that $k\geq k_2(p)$ implies that $p^{k-1}\leq 1/(3k)$. Therefore, the average degree in G is greater than than $(1-1/(k-1))m'$, so, by Turán's theorem, G will contain a K_k. Let $w_{i_1},\ldots,w_{i_k}$ be the vertices of this K_k and let $C_j=W_{i_j}$ for $1\leq j\leq k$. Then, by the definition of G, we see that $C_1,\ldots,C_k$ is a (k,η,δ)-blue-blocked configuration with $|C_i|\geq\tau N$ for all i, where $\tau=\varepsilon/M$ depends only on ε, p, and k. Thus, by Lemma 5.3, we see that either the coloring contains

(c,γ)-many books for $c = ((1-p)/p)^k$ and some γ depending on ε, p, and k or else (C_j, V) is (p, ε^2)-regular for all j. In the latter case, we can set $A_{\ell+1} = C_1$ (or any other C_j) and get the desired result.

Now, we need to assume instead that at least $m'' \geq (1-p)m$ of the parts W_i are red. However, just as in the proof of Theorem 1.3, this proof is essentially identical: we first rule out the existence of too many low-degree vertices in the reduced graph by counting extensions to W and to $A_1 \cup \cdots \cup A_\ell$ and then apply Turán's theorem to find a K_k in the reduced graph, which completes the proof by Lemma 5.3. $\square$

By repeatedly applying Lemma 5.4 until W has fewer than εN vertices, we can partition K_N into a collection of subsets A_i such that (A_i, V) is (p, ε^2)-regular, plus a small remainder set $A_{\ell+1}$ about which we have no such information. Our final technical lemma shows that such a structural decomposition suffices to conclude that the coloring is (p, θ)-quasirandom.

Lemma 5.5. *Let $\varepsilon \leq \theta/3$. Suppose we have a partition*

$$V(K_N) = A_1 \sqcup \cdots \sqcup A_\ell \sqcup A_{\ell+1}$$

where (A_i, V) is (p, ε)-regular for each $1 \leq i \leq \ell$ and $|A_{\ell+1}| \leq \varepsilon N$. Then the coloring is (p, θ)-quasirandom.

Proof. Fix disjoint $X, Y \subseteq V(K_N)$. We need to check that

$$|e_B(X,Y) - p|X||Y|| \leq \theta N^2.$$

First, observe that if $|Y| \leq \varepsilon N$, then

$$|e_B(X,Y) - p|X||Y|| \leq |X||Y| \leq \varepsilon N^2 \leq \theta N^2.$$

Therefore, from now on, we may assume that $|Y| \geq \varepsilon N$. For $1 \leq i \leq \ell + 1$, let $X_i = A_i \cap X$ and define $I_X = \{1 \leq i \leq \ell : |X_i| \geq \varepsilon |A_i|\}$. Then we have that

$$\sum_{i \notin I_X} |X_i| \leq |A_{\ell+1}| + \varepsilon \sum_{i=1}^{\ell} |A_i| \leq 2\varepsilon N.$$

We now write

$$e_B(X,Y) - p|X||Y| = \sum_{i=1}^{\ell+1} \left(e_B(X_i, Y) - p|X_i||Y|\right).$$

We will split this sum into two parts, depending on whether $i \in I_X$ or not. First, suppose that $i \in I_X$. Then $|X_i| \geq \varepsilon |A_i|$ and $|Y| \geq \varepsilon |V|$, so we may apply the (p, ε)-regularity of (A_i, V) to conclude that

$$\sum_{i \in I_X} |e_B(X_i, Y) - p|X_i||Y|| = \sum_{i \in I_X} |d_B(X_i, Y) - p|\,|X_i||Y| \leq \sum_{i \in I_X} \varepsilon |X_i||Y| \leq \varepsilon |X||Y| \leq \varepsilon N^2.$$

On the other hand, since $\sum_{i\notin I_X}|X_i| \le 2\varepsilon N$, we have that

$$\sum_{i\notin I_X} |e_B(X_i,Y) - p|X_i||Y|| \le |Y| \sum_{i\notin I_X} |X_i| \le |Y|(2\varepsilon N) \le 2\varepsilon N^2.$$

Adding these together, we conclude that

$$|e_B(X,Y) - p|X||Y|| \le 3\varepsilon N^2 \le \theta N^2,$$

as desired. □

With all these pieces in place, the proof of Theorem 1.4 becomes quite straightforward.

Proof of Theorem 1.4. Fix $p \in [\frac{1}{2}, 1)$ and suppose $k \ge k_0 := k_2(p)$. Fix $\theta > 0$ and set $\varepsilon = \min\{\theta/3, p/(20k)\}$. Let $\gamma = \gamma(\theta, p, k)$ be the parameter from Lemma 5.4. Suppose we are given a coloring of K_N without (c, γ)-many books. We wish to prove that the coloring is (p, θ)-quasirandom. We inductively apply Lemma 5.4 to find a sequence $A_1, \dots, A_\ell$ of vertex subsets such that (A_i, V) is (p, ε^2)-regular for all i and, therefore, (p, ε)-regular for all i. We continue until the remainder set $A_{\ell+1} = V \setminus (A_1 \cup \cdots \cup A_\ell)$ satisfies $|A_{\ell+1}| \le \varepsilon N$, at which point the assumptions of Lemma 5.4 are no longer met. However, at this point, we can apply Lemma 5.5 to conclude that our coloring is indeed (p, θ)-quasirandom. □

5.1 The converse

In this section, we prove a converse to Theorem 1.4, which implies that not containing (c, γ)-many books is an equivalent characterization of p-quasirandomness.

Theorem 5.6. *Fix $k \ge 2$ and $p \in (0,1)$ and let $c = ((1-p)/p)^k$. Then, for every $\gamma' > 0$, there exists some $\theta > 0$ such that the following holds for every (p, θ)-quasirandom coloring of $E(K_N)$ with N sufficiently large. Apart from at most $\gamma' N^k$ exceptions, every red K_k has $(\frac{c}{k} \pm \gamma')N$ extensions to a red K_{k+1} and every blue K_k has $(\frac{1}{k} \pm \gamma')N$ extensions to a blue K_{k+1}. In particular, the coloring does not contain (c, γ')-many books.*

Remark. In this direction, there is no dependence between p and the range of k for which the result holds. As we know from the fact that the k-partite structure is the extremal structure for small c, such a dependence is necessary in the forward direction. However, here, all we are saying is that almost all monochromatic books in a quasirandom coloring are of essentially the correct size, that is, asymptotic to what they would be in a random coloring.

Proof. We will use the well-known result of Chung, Graham, and Wilson [6], that a quasirandom coloring contains roughly the correct count of any fixed monochromatic subgraph.

Specifically, for every $\delta > 0$, there is some $\theta > 0$, such that, in any (p, θ)-quasirandom coloring of $E(K_N)$,

$$B(K_k) \coloneqq \#(\text{blue } K_k) = p^{\binom{k}{2}} \binom{N}{k} \pm \delta N^k,$$
$$B(K_{k+1}) \coloneqq \#(\text{blue } K_{k+1}) = p^{\binom{k+1}{2}} \binom{N}{k+1} \pm \delta N^{k+1},$$
$$B(K_{k+2} - e) \coloneqq \#(\text{blue } K_{k+2} - e) = p^{\binom{k+2}{2}-1} \binom{N}{k+2} \binom{k+2}{2} \pm \delta N^{k+2},$$

where $K_{k+2} - e$ is the graph formed by deleting one edge from K_{k+2}; note that for this count we have an extra factor of $\binom{k+2}{2}$ to account for the fact that this graph is not vertex-transitive. On the other hand, we can observe that every blue copy of $K_{k+2} - e$ corresponds to two distinct extensions of a single blue K_k to a blue K_{k+1}. Therefore,

$$B(K_{k+2} - e) = \sum_Q \binom{\#(\text{blue extensions of } Q)}{2},$$

where the sum is over all blue K_k. Let $\text{ext}_B(Q)$ denote the number of blue extensions of Q. Then we can also observe that $\sum_Q \text{ext}_B(Q)$ counts the total number of ways of extending a blue K_k into a blue K_{k+1}, which is precisely $(k+1)B(K_{k+1})$, since each blue K_{k+1} contributes exactly $k+1$ terms to this sum.

Now, we consider the quantity

$$E = \sum_{Q \text{ a blue } K_k} (\text{ext}_B(Q) - p^k N)^2.$$

On the one hand, we have that if $\delta \geq 1/N$, then

$$\begin{aligned}
E &= \sum_Q \text{ext}_B(Q)^2 - 2p^k N \sum_Q \text{ext}_B(Q) + \sum_Q p^{2k} N^2 \\
&= \left(2 \sum_Q \binom{\text{ext}_B(Q)}{2} + \sum_Q \text{ext}_B(Q) \right) - 2p^k N(k+1) B(K_{k+1}) + p^{2k} N^2 B(K_k) \\
&= 2B(K_{k+2} - e) + (1 - 2p^k N)(k+1) B(K_{k+1}) + p^{2k} N^2 B(K_k) \\
&\leq 2p^{\binom{k+2}{2}-1} \binom{N}{k+2} \binom{k+2}{2} - 2p^k N(k+1) p^{\binom{k+1}{2}} \binom{N}{k+1} + p^{2k} N^2 p^{\binom{k}{2}} \binom{N}{k} + 5k\delta N^{k+2} \\
&= p^{\frac{k^2+3k}{2}} \binom{N}{k} (-N + k^2 + k) + 5k\delta N^{k+2} \\
&< 5k\delta N^{k+2}.
\end{aligned}$$

On the other hand, suppose there were at least $\gamma' N^k/2$ blue K_k with at least $(p^k + \gamma')N$ or at most $(p^k - \gamma')N$ blue extensions. Then, by only keeping these cliques in the sum defining

E, we would have that

$$E = \sum_{Q} (\mathrm{ext}_B(Q) - p^k N)^2 \geq \frac{\gamma' N^k}{2} (\gamma' N)^2 = \frac{(\gamma')^3}{2} N^{k+2}.$$

Therefore, if we pick $\delta < (\gamma')^3/10k$, we get a contradiction. The same argument with p replaced by $1-p$ and blue replaced by red shows that there are also at most $\gamma' N^k/2$ red K_k with at least $((1-p)^k + \gamma')N$ or at most $((1-p)^k - \gamma')N$ red extensions. This proves the theorem, since the total number of exceptional cliques is at most $\gamma' N^k$. $\square$

6 Concluding remarks

Putting together the main results of this paper, we obtain the following picture. For every $k \geq 2$, there exist two numbers $c_0(k), c_1(k) \in (0,1]$ such that if $0 < c \leq c_0$, then $r(B_{cn}^{(k)}, B_n^{(k)}) = k(n+k-1)+1$, while if $1 \geq c \geq c_1$, then $r(B_{cn}^{(k)}, B_n^{(k)}) = (c^{1/k}+1)^k n + o_k(n)$. Moreover, in both these regimes, there are stability results: there exist $c_0'(k) \leq c_0(k)$ and $c_1'(k) \geq c_1(k)$ such that for $0 < c \leq c_0'$, all the near-extremal colorings are close to k-partite, while for all $1 \geq c \geq c_1'$, all near-extremal colorings are quasirandom. Of course, the most natural question remaining is to understand what happens in the interval (c_0', c_1'), where our results say nothing. Note that this gap is real, since below c_0' all extremal colorings must be k-partite, whereas above c_1' all extremal colorings must be quasirandom.

This question about the gap really comprises at least two separate questions: what happens for fixed k and what happens as $k \to \infty$? To address the second question first, our results give some indication. Indeed, we have shown that both $c_0(k)$ and $c_1(k)$ tend to 0 as $k \to \infty$ and thus the gap interval shrinks as $k \to \infty$. More precisely, we have that

$$c_0(k) \leq \left((1+o(1)) \frac{\log k}{k} \right)^k \leq c_1(k) \leq \left((1+o(1)) \frac{\log k}{k} \right)^k.$$

Moreover, the results of [15] show that $1/c_0$ is at most single-exponential in a power of k. On the other hand, because we used the regularity lemma, our upper bound for $1/c_0'$ is only of tower-type. However, it seems likely that the methods of [15] could also be adapted to improve this.

The other question is what happens for fixed k. Here, our understanding is much more limited, even for the simplest case $k = 2$. In this case, Nikiforov and Rousseau [19] proved that $c_0(2) = 1/6$, in the sense that, for all $c < 1/6$ and all n sufficiently large, $r(B_{cn}^{(2)}, B_n^{(2)}) = k(n+1)+1$, whereas, for any $c > 1/6$ and all n sufficiently large, there is a construction showing that $r(B_{cn}^{(2)}, B_n^{(2)}) > k(n+1)+1$. Curiously, our results do not say anything nontrivial about $c_1(2)$, other than the fact that the random bound is correct for $c = 1$; in other words, we cannot prove that $c_1(2) < 1$ and in fact believe this to not be the case.

Conjecture 6.1. *For every $c < 1$, the random bound for $r(B_{cn}^{(2)}, B_n^{(2)})$ is not tight. In other words, there exists some $\beta = \beta(c) > 0$ such that $r(B_{cn}^{(2)}, B_n^{(2)}) \geq ((\sqrt{c}+1)^2 + \beta)n$ for all n sufficiently large.*

Of course, this conjecture is really only the tip of an iceberg, with the general open question being to understand $r(B_{cn}^{(2)}, B_n^{(2)})$ for $c \in (1/6, 1)$ and $n \to \infty$. There are many conjectures one could make about the behavior of this quantity as a function of c; for instance, perhaps there are a number of thresholds in the interval $(1/6, 1)$ at which new extremal structures emerge, each dictating the value of $r(B_{cn}^{(2)}, B_n^{(2)})$ until the next threshold. Because we know that the random bound is correct for $c = 1$ and that quasirandom colorings are the only extremal ones, such a sequence of extremal examples would need to converge, in some appropriate sense, to the quasirandom coloring as $c \to 1$. However, at the moment we are not even able to conjecture a single such extremal structure or threshold.

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