# Homework 3

**Exercise 3(c):** Let $n$ be an integer and let $0 \leqslant d \leqslant n$ be a real number. Consider a random $n$-vertex graph $G$ formed by including each edge independently with probability $d/n$.

Prove that if $d = \omega(1)$, the average degree of $G$ is $(1 + o(1))d$ with probability $1 - o(1)$.

*Solution.* Let $X$ denote the number of edges of $G$. This is a binomial random variable with distribution $\mathrm{Bin}(\binom{n}{2}, p)$, where $p = d/n$. In particular, the expectation of $X$ is

$$\mathbb{E}[X] = p\binom{n}{2} = \frac{d}{n} \cdot \binom{n}{2} = \frac{d(n-1)}{2} = \frac{dn}{2} - \frac{d}{2}.$$

We first claim that with probability $1 - o(1)$, we have that $X = (1 + o(1))dn/2$. This follows from essentially any of the standard concentration results for the binomial distribution; for concreteness, we give an elementary proof using only Chebyshev's inequality.

Since $X$ is binomially distributed, its variance is given by

$$\mathrm{Var}(X) = p(1-p)\binom{n}{2} \leqslant p\binom{n}{2} \leqslant \frac{dn}{2}.$$

Chebyshev's inequality thus implies that for any $t > 0$, we have

$$\Pr\left(|X - \mathbb{E}[X]| \geqslant t \cdot \sqrt{\frac{dn}{2}}\right) \leqslant \frac{1}{t^2}. \tag{1}$$

We now note that

$$\begin{aligned}
\Pr\left(\left|X - \frac{dn}{2}\right| \geqslant d\sqrt{\frac{n}{2}}\right) &\leqslant \Pr\left(|X - \mathbb{E}[X]| \geqslant d\sqrt{\frac{n}{2}} - \frac{d}{2}\right) \\
&\leqslant \Pr\left(|X - \mathbb{E}[X]| \geqslant \frac{d}{2}\sqrt{\frac{n}{2}}\right) \\
&= \Pr\left(|X - \mathbb{E}[X]| \geqslant \frac{\sqrt{d}}{2}\sqrt{\frac{dn}{2}}\right) \\
&\leqslant \frac{4}{d},
\end{aligned}$$

where the first inequality uses the fact that $|\mathbb{E}[X] - \frac{dn}{2}| = \frac{d}{2}$, the second holds for $n \geqslant 2$ (which we are allowed to assume since we are working in the $n \to \infty$ limit), and the final holds by plugging in $t = \sqrt{d}/2$ into (1).

Recall that the average degree of any $n$-vertex graph $G$ is equal to $2e(G)/n$. Hence the average degree in our random graph is $2X/n$. Let $Y = 2X/n$ be this average degree, and note that the above implies

$$\Pr\left(|Y - d| \geqslant d\sqrt{\frac{2}{n}}\right) = \Pr\left(\left|X - \frac{dn}{2}\right| \geqslant d\sqrt{\frac{n}{2}}\right) \leqslant \frac{4}{d}.$$

To conclude the proof, we note that $d\sqrt{2/n} = o(d)$ as $n \to \infty$, and that $4/d = o(1)$ since we assume $d = \omega(1)$. Hence this implies that $Y = (1 + o(1))d$ with probability $1 - o(1)$. $\square$

**Exercise 5(a)** Prove that, for any fixed $s \geqslant 3$, we have

$$r(s,k) \geqslant k^{\frac{s-1}{2}-o(1)},$$

where the $o(1)$ term tends to 0 as $k \to \infty$.

*Solution.* Let $G$ be a random graph on $N := (\frac{k}{s \ln k})^{\frac{s-1}{2}}$ vertices, where each pair of vertices is included as an edge independently with probability $p := N^{-\frac{2}{s-1}}$. We begin by claiming that $G$ is $K_s$-free with probability at least $\frac{5}{6}$. Indeed, any given set of $s$ vertices forms a copy of $K_s$ in $G$ with probability $p^{\binom{s}{2}}$, and there are $\binom{N}{s}$ options for such a set of $s$ vertices. Hence, by the union bound, we have that the probability that $G$ contains a $K_s$ is at most

$$\binom{N}{s} p^{\binom{s}{2}} \leqslant \frac{N^s}{s!} p^{\frac{s^2-s}{2}} = \frac{1}{s!}\left(Np^{\frac{s-1}{2}}\right)^s = \frac{1}{s!} \leqslant \frac{1}{6},$$

where we use our definition of $p$ to see that $Np^{\frac{s-1}{2}} = 1$ and use the fact that $s \geqslant 3$ to conclude that $s! \geqslant 6$. Thus, $G$ is $K_s$-free with probability at least $\frac{5}{6}$.

We now claim that $G$ has no independent set of order $k$ with probability at least $\frac{1}{2}$. Any set of $k$ vertices forms an independent set with probability $(1-p)^{\binom{k}{2}}$, and there are $\binom{N}{k}$ choices for such a set. Applying the union bound, we find that the probability that $G$ has an independent set of order $k$ is at most

$$\binom{N}{k}(1-p)^{\binom{k}{2}} \leqslant \frac{N^k}{k!}(1-p)^{\frac{k^2-k}{2}} = \frac{1}{(1-p)^{\frac{k}{2}} k!} \cdot \left(N(1-p)^{\frac{k}{2}}\right)^k \tag{2}$$

Note that $p \to 0$ as $k \to \infty$, hence $p \leqslant \frac{1}{2}$ for sufficiently large $k$. Thus, for sufficiently large $k$, we have that

$$(1-p)^{\frac{k}{2}} k! \geqslant \left(\frac{1}{2}\right)^{\frac{k}{2}} k! \geqslant \left(\frac{1}{2}\right)^{\frac{k}{2}} \cdot \left(\frac{k}{2}\right)^{\frac{k}{2}} \geqslant \left(\frac{k}{4}\right)^{\frac{k}{2}} \geqslant 2,$$

where the second inequality uses the simple bound $k! \geqslant (k/2)^{k/2}$, and the final inequality also holds for sufficiently large $k$. On the other hand, using the bound $1 - x \leqslant e^{-x}$, we have that

$$N(1-p)^{\frac{k}{2}} \leqslant Ne^{-p\frac{k}{2}} = \exp\left(\ln N - p\frac{k}{2}\right) = \exp\left(\ln N - \frac{k}{2}N^{-\frac{2}{s-1}}\right).$$

Note that, by our choice of $N$, we have that

$$\frac{k}{2}N^{-\frac{2}{s-1}} = \frac{k}{2} \cdot \frac{s \ln k}{k} = \frac{s \ln k}{2}$$

and that

$$\ln N \leqslant \ln\left(k^{\frac{s-1}{2}}\right) = \frac{(s-1)\ln k}{2}.$$

Therefore, $\ln N - \frac{k}{2}N^{-\frac{2}{s-1}} \leqslant -\frac{\ln k}{2} \leqslant 0$, so $N(1-p)^{\frac{k}{2}} \leqslant 1$. Plugging all of this back into (2), we find that the probability that $G$ has an independent set of order $k$ is at most

$$\frac{1}{(1-p)^{\frac{k}{2}}k!} \cdot \left(N(1-p)^{\frac{k}{2}}\right)^k \leqslant \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Putting this all together, we find that with probability at least $\frac{1}{2}$, $G$ has no independent set of order $k$, and with probability at least $\frac{5}{6}$, $G$ is $K_s$-free. Thus, with positive probability, $G$ satisfies both these properties simultaneously, hence there exists an $N$-vertex graph which is $K_s$-free and has no independent set of order $k$. Therefore,

$$r(s,k) > N = \left(\frac{k}{s\ln k}\right)^{\frac{s-1}{2}} = k^{\frac{s-1}{2}-o(1)},$$

since for fixed $s$ and for $k \to \infty$, we have that $s \ln k = k^{o(1)}$. $\square$

# Homework 4

**Exercise 3(a)** Let $q$ be a prime power. Construct a graph $\Pi_q$ with vertex set $V(\Pi_q) = \mathbb{F}_q^2$, in which two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if and only if $x_1x_2 + y_1y_2 = 1$.

Prove that $\Pi_q$ is $C_4$-free.

*Solution.* Suppose for contradiction that we have a $C_4$ in $\Pi_q$, namely four distinct vertices $(x_1, y_1), \ldots, (x_4, y_4)$ which are pairwise adjacent in this cyclic order. Let

$$\ell_1 := \{(x, y) \in \mathbb{F}_q^2 : xx_1 + yy_1 = 1\} \qquad \text{and} \qquad \ell_3 := \{(x, y) \in \mathbb{F}_q^2 : xx_3 + yy_3 = 1\}.$$

Note that by definition, $\ell_1$ is precisely the neighborhood of $(x_1, y_1)$ in $\Pi_q$, and similarly $\ell_3$ is the neighborhood of $(x_3, y_3)$. Moreover, by construction, $\ell_1, \ell_3$ are both lines in $\mathbb{F}_q^2$.

However, we know that two lines in $\mathbb{F}_q^2$ intersect in at most one point. Formally, suppose that $(x, y) \in \ell_1 \cap \ell_3$. Then $(x, y)$ satisfies the two equations

$$\begin{aligned} xx_1 + yy_1 &= 1 \\ xx_3 + yy_3 &= 1. \end{aligned}$$

Subtracting the second equation from the first, we conclude that

$$x(x_3 - x_1) = y(y_1 - y_3).$$

We assumed that the points $(x_1, y_1)$ and $(x_3, y_3)$ were distinct, so either $x_1 \neq x_3$ or $y_1 \neq y_3$ (or both). Let us assume the first case happens; the second case is essentially identical. Since $x_1 \neq x_3$, we may divide the equation above by $x_3 - x_1$ to conclude that

$$x = \frac{y_1 - y_3}{x_3 - x_1} y. \tag{3}$$

Plugging this in to the equation $xx_1 + yy_1 = 1$, we find that

$$y \left( x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1 \right) = 1.$$

Note that there is at most one choice of $y$ satisfying this. Indeed, if $x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1 = 0$ then there is no solution to this equation, and if $x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1 \neq 0$ then the unique solution is $y = 1/(x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1)$. Plugging this back into (3) shows that, given the value of $y$, we can also determine the value of $x$.

In other words, we have proven that there is at most one point $(x, y)$ in the intersection $\ell_1 \cap \ell_3$. Therefore, the points $(x_1, y_1)$ and $(x_3, y_3)$ have at most one common neighbor in $\Pi_q$, as their common neighborhood is precisely $\ell_1 \cap \ell_3$. However, our starting assumption was that $(x_2, y_2)$ and $(x_4, y_4)$ are distinct points, both of which are common neighbors of $(x_1, y_1)$ and $(x_3, y_3)$; this is a contradiction. $\square$

# Homework 6

**Exercise 3(b)** Let $\widehat{K_k}$ denote the 1-*subdivision* of $K_k$. This is a graph on $k + \binom{k}{2}$ vertices, obtained by introducing a new vertex in the middle of every edge of $K_k$. Equivalently, it is obtained from $K_k$ by replacing every edge by a 2-edge path.

By applying Lemma 5.4.11 and being more careful, prove that $r(\widehat{K_k}) = O(k^2)$. Note that this bound is tight up to the implicit constant since $\widehat{K_k}$ has $\Theta(k^2)$ vertices.

*Solution.* By choosing the implicit constant in the big-$O$ appropriately, we may assume that $k$ is sufficiently large. We will assume that $k \geqslant 100$.

Let $N = 81k^2$, and fix a two-coloring of $E(K_N)$. We may assume without loss of generality that at least half the edges are red; let $G$ be the red graph, and note that the average degree $d$ of $G$ satisfies $d \geqslant \frac{N-1}{2} \geqslant \frac{N}{3}$.

Let $t = \log_3 k$, let $\Delta = 2$, and let $r = k + \binom{k}{2} \leqslant k^2$. Note that

$$\begin{aligned}
\frac{d^t}{N^{t-1}} - \binom{N}{\Delta}\left(\frac{r}{N}\right)^t &\geqslant N\left(\frac{d}{N}\right)^t - N^2\left(\frac{k^2}{N}\right)^t \\
&\geqslant N\left(\frac{1}{3}\right)^t - N^2\left(\frac{1}{81}\right)^t \\
&= \frac{N}{k} - \frac{N^2}{k^4} \\
&= 81k - 81^2 \\
&\geqslant k,
\end{aligned}$$

where the final step holds since we assumed $k \geqslant 100$. Therefore, we are in the position to apply Lemma 5.4.11 with the parameters above and with $s = k$. We conclude that there is a set $T \subseteq V(G)$ of size $|T| \geqslant k$ such that every pair of vertices in $T$ has at least $r = k + \binom{k}{2}$ common neighbors.

We now argue exactly as in the proof of Theorem 5.4.10. $\widehat{K_k}$ is a bipartite graph with one part of size $k$ (corresponding to the original vertices of $K_k$) and the other of size $\binom{k}{2}$ (corresponding to the original edges of $K_k$). We embed the part of size $k$ into $T$ arbitrarily. We then arbitrarily order the vertices in the part of size $\binom{k}{2}$. Each vertex $v$ in this part has exactly two neighbors in $\widehat{K_k}$, which were already embedded into $T$. By the way we constructed $T$, this pair of embedded vertices has at least $k + \binom{k}{2}$ common neighbors, and in particular at least one common neighbor that was not yet used in the embedding. We embed $v$ arbitrarily into one of these common neighbors, and continuing in this process we find a red copy of of $\widehat{K_k}$. □

**Exercise 7** Prove Theorem 6.2.3, the linear bound on multicolor Ramsey numbers of bounded-degree graphs.

*Solution.* First we pick some parameters depending on $\Delta$ and $q$. Let $\varepsilon = q^{-\Delta}/(2\Delta)$, which is chosen so that $\frac{1}{q} = (2\Delta\varepsilon)^{1/\Delta}$. Let $\delta(\varepsilon, q)$ be the constant from Lemma 6.2.1. Finally, let $C = 2/(\varepsilon\delta)$, and note that $C$ depends only on $\Delta$ and $q$.

Fix an $n$-vertex graph $H$ with maximum degree at most $\Delta$, and let $N = Cn$. Consider a $q$-coloring of $E(K_N)$, and let $G_1, \dots, G_q$ be the $q$ color classes. Applying Lemma 6.2.1, we find a subset $Q \subseteq V(K_N)$ with $|Q| \geqslant \delta N$ such that $G_1[Q], \dots, G_q[Q]$ are all $\varepsilon$-quasirandom. By the pigeonhole principle, among the edges in $Q$, at least a $\frac{1}{q}$ fraction have the same color. So we may pick some $i \in [\![q]\!]$ such that at least $\frac{1}{q}\binom{|Q|}{2}$ of the edges in $Q$ have color $i$; equivalently, this says that $d(G_i[Q]) \geqslant \frac{1}{q} = (2\Delta\varepsilon)^{1/\Delta}$. Note that

$$|Q| \geqslant \delta N = \delta C n = \frac{2n}{\varepsilon}.$$

Thus, we are in the setting of Lemma 6.1.3, which immediately tells us that $H$ is a subgraph of $G_i[Q]$. Thus, we have found a monochromatic copy of $H$ in color $i$, implying that $r(H) \leqslant N$. $\square$