

1. (a) Prove that for any positive integer q , there exists a positive integer $N = N(q)$ such that the following holds. For any q -coloring of $\llbracket N \rrbracket$, there exist $x, y, z \in \llbracket N \rrbracket$ such that $x, y, z, x + y, x + z, y + z, x + y + z$ all receive the same color.
 - (b) Generalize the previous part as follows. Prove that for all positive integers q, t , there exists a positive integer $N = N(q, t)$ such that the following holds. For any q -coloring of $\llbracket N \rrbracket$, there exist $x_1, \dots, x_t \in \llbracket N \rrbracket$ such that the sums $\sum_{i \in S} x_i$ all receive the same color, for all non-empty $S \subseteq \llbracket t \rrbracket$.
 - (c) Prove that in part (b), one can moreover ensure that the numbers x_1, \dots, x_t are all distinct.
- ⊕ 2. In class, we proved that $r(k) < 4^k$ using the Erdős–Szekeres argument. Ramsey’s original proof used a *different* argument, which yielded the worse bound $r(k) \leq k!$. Find a natural argument yielding this bound. (That is, don’t simply quote or rederive the Erdős–Szekeres argument!)
3. (a) Prove that $r(3, 3) = 6$, $r(3, 4) = 9$, and $r(4, 4) \leq 18$.
 - ★(b) Prove that $r(4, 4) = 18$.
 - ?(c) The best known bounds on $r(5, 5)$ are $43 \leq r(5, 5) \leq 48$. Can you improve either of these bounds?
 4. (a) By more carefully analyzing the proof of Theorem 2.7 in the notes, prove that

$$r(k) > \left(\frac{1}{e\sqrt{2}} - o(1) \right) k 2^{k/2}.$$

- ★(b) Improve this bound by a constant factor.
 - ?(c) Improve this bound by a super-constant factor.
5. Given two graphs G, H , their *lexicographic product* $G \cdot H$ is defined as follows. Its vertex set is $V(G \cdot H) = V(G) \times V(H)$, and two vertices $(a, b), (c, d)$ are adjacent if either $ac \in E(G)$ or $a = c$ and $bd \in E(H)$.
 - (a) Compute the size of the largest clique and the largest independent set in $G \cdot H$.
 - (b) Prove that the Ramsey number $r(k)$ satisfies $r(k) > k^{\log_2(5)}$ for all even k .
[Note that this already disproves Turán’s belief that $r(k)$ may grow only quadratically as a function of k .]
 - ★(c) Using the same approach, find an *explicit* construction of a coloring witnessing that $r(k)$ grows super-polynomially in k . In other words, for any $C > 0$ and any sufficiently large k , find an explicit 2-coloring of $E(K_N)$, where $N = k^C$, with no monochromatic clique of order k .

★ means that a problem is hard.

? means that a problem is open.

⊕ means that a problem is not directly related to the topic of the course.

- ★(d) By carefully working through the dependencies in (c), prove via an explicit coloring that

$$r(k) > k^{c \frac{\log \log \log k}{\log \log \log \log k}}$$

for some absolute constant $c > 0$. Can you further improve this bound?

- ?(e) Can you use such an approach to resolve Open problem 2.8?