1. We saw that $r_o(k;2) = r(k;2)$ and that $r_o(k;q) \leqslant r(k;q)$ for all k, q. Prove an inequality in the other direction, namely that

$$r(k;q) \leqslant r_o(r_o(\cdots r_o(r_o(k;q);q-1)\cdots;3);2).$$

for any $q \geqslant 3$.

2. (a) Prove that

$$r_o(k;2k-1) = 2k$$

for all $k \geqslant 2$.

$\star$(b) Determine $r_o(k;q)$ exactly for all $q \geqslant 2k$.

$\star$(c) For any fixed $\alpha \in (0,2)$, determine

$$\lim_{k\to\infty} \frac{r_o(k;\alpha k)}{k}.$$

3. Let $1 \leqslant \ell \leqslant q-1$ be integers, and let $\binom{[\![q]\!]}{\ell}$ denote the collection of all ℓ-element subsets of $[\![q]\!]$. A (q,ℓ)-*set coloring* is a function $\chi : E(K_N) \to \binom{[\![q]\!]}{\ell}$; in other words, rather than assigning every edge of K_N a single color out of q options, we assign every edge a list of ℓ colors from a palette of size q. We say that $v_1, \ldots, v_k \in V(K_N)$ form a *color-intersecting clique* if there is a color that appears in all of the $\binom{k}{2}$ lists associated to the edges they span, that is, if $\bigcap_{1\leqslant i<j\leqslant k} \chi(v_iv_j) \neq \varnothing$. The *set coloring Ramsey number* $r_s(k;(q,\ell))$ is the least N such that every (q,ℓ)-set coloring of $E(K_N)$ contains a color-intersecting clique of order k.

(a) Prove that $r_s(k;(q,1)) = r(k;q)$.

(b) Prove that $r_s(k;(q,\ell)) \leqslant r_s(k;(q,\ell-1))$ for any $2 \leqslant \ell \leqslant q-1$. Conclude that $r_s(k;(q,\ell)) \leqslant r(k;q)$ for all $1 \leqslant \ell \leqslant q-1$.

(c) Prove that $r_s(k;(q,q-1)) = r_o(k;q)$.

(d) Combining parts (a) and (c) with our known bounds on $r(k;q)$ and $r_o(k;q)$, conclude the following. There exist absolute constants c, C such that for any $k \geqslant q \geqslant 2$, we have

$$2^{ckq} \leqslant r_s(k;(q,1)) \leqslant 2^{Ckq\log q} \qquad \text{and} \qquad 2^{\frac{ck}{q}} \leqslant r_s(k;(q,q-1)) \leqslant 2^{\frac{Ck}{q}\log q}.$$

In other words, at both extremes $\ell = 1$ and $\ell = q-1$, we have a $\Theta(\log q)$ gap between the upper and lower bounds.

(e) Prove that, for every $\varepsilon > 0$ there exists some $B > 0$ such that the following holds. If $\ell \geqslant \varepsilon q$, then $r_s(k;(q,\ell)) \leqslant 2^{Bkq}$.

(f) Using Theorem 8.1.4, prove the following. For every $x \geqslant 1$, there exists $D > 0$ such that

$$r_s(k;(q,q-x)) \leqslant 2^{\frac{Dk}{q}\log q}.$$

Note that this bound is much stronger than that given in (e).

⋆ (g) Prove that, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that the following holds. If $\varepsilon q \leqslant \ell \leqslant (1-\varepsilon)q$, then

$$r_s(k; (q, \ell)) \geqslant 2^{\delta k q}.$$

This shows that the bound in (e) is tight up to the value of B when $\varepsilon q \leqslant \ell \leqslant (1-\varepsilon)q$. On the other hand, (f) shows that the upper bound $\ell \leqslant (1-\varepsilon)q$ cannot be entirely removed.

4. (a) Prove that Theorem 8.2.4 is equivalent to the following statement. For every $C > 0, k \in \mathbb{N}$, the following holds for sufficiently large N. Consider a coloring $\chi : E(K_N) \to \{\text{red, blue}\}$, and suppose that χ contains no monochromatic clique of order $C \log N$. Then for every coloring $\psi : E(K_k) \to \{\text{red, blue}\}$, there is a k-vertex subset S of K_N such that the restriction of χ to S equals ψ (up to permutations of the vertices).

   (b) State and prove a generalization of (a) to colorings with more than two colors.

⊕ 5. Prove that if G is a k-universal graph, then G has at least $2^{(k-1)/2}$ vertices.