Hedetniemi's conjecture is asymptotically false

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Abstract

Extending a recent breakthrough of Shitov, we prove that the chromatic number of the tensor product of two graphs can be a constant factor smaller than the minimum chromatic number of the two graphs. More precisely, we prove that there exists an absolute constant $\alpha > 0$ such that for all c sufficiently large, there exist graphs G and H with chromatic number at least $(1+\alpha)c$ for which $\chi(G \times H) \leq c$.

1 Introduction

If G and H are finite graphs, their *tensor product* $G \times H$ is the graph on $V(G) \times V(H)$ where vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 \sim g_2$ and $h_1 \sim h_2$. By composing with the projection maps to each coordinate, it is easy to check that the chromatic number satisfies

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}, \tag{1}$$

for all finite graphs G and H. In 1966, Hedetniemi [8] conjectured that equality always holds in (1). This conjecture has received a considerable amount of attention; for instance, it was proved if G and H are 4-colorable [3], if every vertex in G is contained in a large clique [1], or if G and H are Kneser graphs or hypergraphs [6]. Additionally, many natural variants of this conjecture have been studied. For instance, Hajnal [7] proved that the analogous conjecture is false for infinite graphs, while Zhu [16] proved that the analogous conjecture for fractional colorings is true. We refer the reader to the excellent surveys [11, 13, 15] for more information surrounding Hedetniemi's conjecture.

In a recent breakthrough, Shitov [12] disproved Hedetniemi's conjecture by demonstrating that for sufficiently large c, there exist graphs G and H with $\chi(G) > c, \chi(H) > c$, but $\chi(G \times H) \leq c$. Even more recently, Tardif and Zhu [14] proved that the gap between $\chi(G \times H)$ and $\min\{\chi(G), \chi(H)\}$ can be arbitrarily large, i.e. that for every integer d and

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for all c sufficiently large, there exist graphs G and H with $\chi(G) > c+d, \chi(H) > c+d$, but $\chi(G \times H) \leq c$. They also raised the question of whether the gap d can be made to be linear in c, and proved that this is possible under the additional assumption that Stahl's conjecture on multichromatic numbers is true.

In this paper, we modify Shitov's construction to answer Tardif and Zhu's question in the affirmative, showing that the ratio of $\chi(G \times H)$ and $\min\{\chi(G), \chi(H)\}$ is asymptotically bounded away from 1.

Theorem 1. *There is an absolute constant $\alpha \geq 10^{-111}$ such that for all sufficiently large c, there exist simple graphs G, H with $\chi(G) \geq (1+\alpha)c, \chi(H) \geq (1+\alpha)c$, and $\chi(G \times H) \leq c$.*

Equivalently, we may express Theorem 1 in terms of the *Poljak–Rödl function* [10], which is defined by

$$f(k) = \min_{\chi(G),\chi(H)\geq k} \chi(G \times H).$$

Then Hedetniemi's conjecture is equivalent to the statement that $f(k) = k$ for all k. The so-called weak version of Hedetniemi's conjecture simply asks whether $\lim_{k\to\infty} f(k) = \infty$, and is still open; however, Poljak and Rödl [10] proved that either $f(k) \to \infty$ or $f(k)$ is bounded by 9. With this notation, Theorem 1 is equivalent to the statement that

$$f(k) \leq (1-\alpha)k$$

for all sufficiently large k. For comparison, Tardif and Zhu [14] proved that $f(k) \leq k - (\log k)^{1/4-o(1)}$.

The proof of Theorem 1 closely mirrors Shitov's proof of the main result in [12]. Roughly speaking, the main innovation in our argument is that whereas Shitov constructs one uncolorable vertex $\nu \in V(H)$ to prove $\chi(H) \geq c+1$, we construct a large clique $\mathcal{N}$ of αc uncolorable vertices of H (see Lemma 7) to obtain the stronger lower bound $\chi(H) \geq (1+\alpha)c$.

For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial.

2 Definitions and Background

All graphs are assumed to not have multiple edges, but graphs not specified to be simple may contain loops. We will never refer to the chromatic number of a non-simple graph, as this is not a well-defined integer.

If H is a finite graph, two maps $\phi_1, \phi_2 : V(H) \to [c]$ are called *co-proper* if $\phi_1(u) \neq \phi_2(v)$ whenever $u \sim v$ in H. The *exponential graph* $\mathcal{E}_c(H)$ is the graph on vertex set $[c]^{V(H)}$ where two vertices ϕ_1, ϕ_2 are adjacent if and only if they are co-proper. Observe that a map ϕ is co-proper with itself if and only if ϕ is a proper coloring of H. Therefore, at most one of H and $\mathcal{E}_c(H)$ has loops. Moreover, both will be simple exactly when H is simple and $\chi(H) > c$. To avoid confusion, we will follow the convention of calling vertices of $\mathcal{E}_c(H)$ *maps* and proper colorings of $\mathcal{E}_c(H)$ *colorings*.

A pair of the form $(G, \mathcal{E}_c(G))$ is a natural candidate for counterexamples to Hedetniemi's conjecture. Indeed, El-Zahar and Sauer [3] observed that if $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$ for some H, then this also holds for $H = \mathcal{E}_c(G)$ where $c = \chi(G) - 1$. In any case, it is easy to establish an upper bound on $\chi(G \times H)$ when $H = \mathcal{E}_c(G)$.

Lemma 2. *For any graph H and any integer $c \geq 1$,*

$$\chi(H \times \mathcal{E}_c(H)) \leq c.$$

Proof. We construct a coloring $\Phi : V(H \times \mathcal{E}_c(H)) \to [c]$ by declaring

$$\Phi(h, \phi) = \phi(h) \in [c]$$

for every $h \in V(H)$ and $\phi : V(H) \to [c]$. To see that this is a proper coloring of $H \times \mathcal{E}_c(H)$, suppose that $((h_1, \phi_1), (h_2, \phi_2))$ is an edge in $H \times \mathcal{E}_c(H)$. By the definition of the tensor product, this implies that $(h_1, h_2) \in E(H)$ and that ϕ_1, ϕ_2 are co-proper. By the definition of co-proper maps, this implies that $\phi_1(h_1) \neq \phi_2(h_2)$. Thus, (h_1, ϕ_1) and (h_2, ϕ_2) receive different colors under Φ, so Φ is a proper c-coloring. □

If Ψ is a proper $(c+t)$-coloring of an exponential graph $\mathcal{E}_c(H)$, where $t \geq 0$, we call $\{1, \ldots, c\}$ the *primary colors* and $\{c+1, \ldots, c+t\}$ the *secondary colors*. We say that a proper $(c+t)$-coloring Ψ of $\mathcal{E}_c(H)$ is *standard* if for every $\phi \in V(\mathcal{E}_c(H))$,

$$\Psi(\phi) \in \operatorname{im}(\phi) \cup \{c+1, \ldots, c+t\}.$$

In other words, a proper $(c+t)$-coloring is standard if it only assigns a primary color b only to maps ϕ which have b in their image.

Lemma 3. *If $\chi(\mathcal{E}_c(H)) \leq c + t$, then $\mathcal{E}_c(H)$ has a standard $(c+t)$-coloring.*

Proof. Let Ψ be a proper $(c+t)$-coloring, and let ϕ_i be the constant map $v \mapsto i$ in $\mathcal{E}_c(H)$. Since the maps ϕ_i are pairwise co-proper, they form a c-clique inside $\mathcal{E}_c(H)$ and Ψ assigns them different colors.

We may permute the colors of Ψ so that $\Psi(\phi_i) = i$. We claim that such a Ψ is standard. Indeed, if $\phi \in V(\mathcal{E}_c(H))$, ϕ is co-proper to all constant maps ϕ_i where i is a primary color not in $\operatorname{im}(\phi)$. Since this map gets color i, we see that ϕ is not colored i. It follows that $\Psi(\phi) \in \operatorname{im}(\phi) \cup \{c+1, \ldots, c+t\}$ as desired. □

If G is a finite simple graph, write G° for the graph obtained by adding loops to every vertex of G. We also write $G \subseteq H$ if G is a subgraph of H.

Lemma 4. *Suppose H is a graph on n vertices. Then for any integer $c \geq 2n$, the independence number of the exponential graph satisfies*

$$\alpha(\mathcal{E}_c(H)) \leq nc^{n-1}.$$

Proof. First, observe that if H and H' have the same vertex set, and if $H \subseteq H'$, then $\mathcal{E}_c(H') \subseteq \mathcal{E}_c(H)$. This is because every pair of co-proper maps on H' are also co-proper on H, as the edge set of H is a subset of that of H'. Thus, if we let K_n° denote the complete graph on n vertices where every vertex has a loop, then we see that $\mathcal{E}_c(K_n^\circ) \subseteq \mathcal{E}_c(H)$, so it suffices to upper-bound $\alpha(\mathcal{E}_c(K_n^\circ))$.

By definition, the vertex set of $\mathcal{E}_c(K_n^\circ)$ is the set of maps $[c]^n$, and two vertices are adjacent in $\mathcal{E}_c(K_n^\circ)$ if and only if their images are disjoint. Let I be an independent set of $\mathcal{E}_c(K_n^\circ)$ and let $\mathcal{F} = \{\mathrm{im}(\phi) \mid \phi \in I\}$. We can partition $\mathcal{F}$ into layers $\mathcal{F}_\ell$ according to the size of $\mathrm{im}(\phi)$, where $\mathcal{F}_\ell = \mathcal{F} \cap \binom{[c]}{\ell}$. By virtue of the fact that I is an independent set, every pair of images of elements in I must intersect, so $\mathcal{F}_\ell$ is an intersecting family in $\binom{[c]}{\ell}$. As $\ell \le n \le c/2$, the Erdős–Ko–Rado theorem [5] applies to give

$$|\mathcal{F}_\ell| \le \binom{c-1}{\ell-1}.$$

It follows that if a_ℓ is the number of surjective maps $[n] \twoheadrightarrow [\ell]$, then

$$|I| \le \sum_{\ell=1}^{n} |\mathcal{F}_\ell| \cdot a_\ell \le \sum_{\ell=1}^{n} \binom{c-1}{\ell-1} a_\ell.$$

The right hand side is exactly the number of maps in $[c]^n$ which contain 1 in their image. The number of such maps is at most $n \cdot c^{n-1}$, since there are n ways to pick a vertex to send to 1 and at most c^{n-1} ways to color the rest. Thus, $|I| \le nc^{n-1}$ for all independent sets I, as desired. $\square$

We remark that for fixed n, Lemma 4 is tight for $H = K_n^\circ$ up to an additive error of $O(c^{n-2})$.

3 Tethered colors

The main technical lemma of Shitov's argument shows that for every standard c-coloring Ψ of $\mathcal{E}_c(H)$ there is a "central" vertex $v \in V(H)$ for which the color of a map $\phi \in \mathcal{E}_c(H)$ must appear in a ball around v. We make the following general definition, which for $r = 3$ is equivalent to Shitov's notion of "v-robustness."

Definition. Given a standard $(c+t)$-coloring Ψ of $\mathcal{E}_c(H)$, we say that a primary color $b \in [c]$ is r-*tethered* to a vertex $v \in V(H)$ if for every $\phi \in \Psi^{-1}(b)$ there exists a vertex $w \in V(H)$ with $\phi(w) = b$ and $\mathrm{dist}(v, w) \le r$.

Lemma 5 below can be thought of as an analog of a stablility result for the Erdős–Ko–Rado theorem. The Erdős–Ko–Rado theorem [5] states that if $n \ge 2k$ and every pair of a family of k-subsets of an n-set intersects, then there are at most $\binom{n-1}{k-1}$ such subsets. Stability results (see e.g. [2]) tell us additionally that for an intersecting family with size close to the maximum $\binom{n-1}{k-1}$, there exists a particular element in most of the sets.

An independent set in $\mathcal{E}_c(H)$ is a family of pairwise non-co-proper maps, and we think of "co-proper maps" as an analog of "disjoint sets," and being non-co-proper as "intersecting" on a particular edge. Thus, the two lemmas below will show that for every large independent set I in $\mathcal{E}_c(H)$, there is a particular vertex v for which most of the elements of I intersect on an edge close to v.

We will show that all sufficiently large primary color classes are 1-tethered to some vertex (depending on the color), and that most of these classes are 2-tethered to the same vertex.

Lemma 5. *If H is a graph on n vertices, Ψ is a standard $(c+t)$-coloring of $\mathcal{E}_c(H)$, and b is a primary color of Ψ such that $|\Psi^{-1}(b)| > n^3c^{n-2}$, then b is 1-tethered to some vertex $v \in V(H)$.*

Proof. For each $v \in V(H)$, let I_v be the set of maps in $\Psi^{-1}(b)$ for which $\phi(v) = b$. Since b is a primary color and Ψ is a standard coloring, every $\phi \in \Psi^{-1}(b)$ is in some I_v. By the pigeonhole principle, there is some vertex v for which $|I_v| \geq \frac{1}{n}|\Psi^{-1}(b)| > n^2c^{n-2}$. We claim that b is 1-tethered to v.

If not, there is some $\phi \in \Psi^{-1}(b)$ which does not take value b on any neighbor of v. Since the color class $\Psi^{-1}(b)$ is an independent set of $\mathcal{E}_c(H)$, ϕ is not co-proper with any element of I_v. We count the number of $\psi \in I_v$ which are not co-proper to ϕ. Such a map must have $\psi(v) = b$ (because it is in I_v) and $\psi(w) \in \mathrm{im}(\phi)$ for some $w \neq v$. This w can be picked in at most n ways, and $\psi(w)$ itself in at most n ways since $|\mathrm{im}(\phi)| \leq n$. Thus, the total number of ways to pick such a ψ from I_v is at most n^2c^{n-2}.

But every $\psi \in I_v$ must not be co-proper with ϕ, and we already computed that $|I_v| > n^2c^{n-2}$. This is a contradiction and we're done. □

Lemma 6. *If H is a graph on $n \geq 10$ vertices with maximum degree at most $D-1$, $c \geq 3^{2D}(nt+n^3)$, Ψ is a standard $(c+t)$-coloring of $\mathcal{E}_c(H)$, and*

$$x = \left\lceil \sqrt[2D]{(nt+n^3)c^{2D-1}} \right\rceil,$$

then there exists a vertex v to which at least $c-x$ primary colors are 2-tethered.

Proof. Call a primary color b *large* if $|\Psi^{-1}(b)| > n^3c^{n-2}$ and *small* otherwise.

By Lemma 5, every large primary color b is 1-tethered to some vertex v_b. If b is a large primary color, let $V_b = \{v_b\} \cup N(v_b)$ be the closed neighborhood of v_b, while if b is a small primary color, we declare $V_b = \varnothing$. Note that for every large primary color b and every $\phi \in \Psi^{-1}(b)$, we have $\phi(v) = b$ for some $v \in V_b$. Suppose for the sake of contradiction that every vertex v is in fewer than $c-x$ of the sets V_b. Since the degree of v_b is at most $D-1$, $|V_b| \leq D$ for all b.

Let S be the set of maps $\phi \in [c]^n$ which have the property that $\phi(v) \neq b$ whenever $v \in V_b$. Let $s(v)$ denote the number of primary colors b for which $v \notin V_b$, so $|S| = \prod_v s(v)$. By the above assumption, $x \leq s(v) \leq c$ for all v. Also,

$$\sum_{v \in V(H)} s(v) = nc - \sum_{b \in [c]} |V_b| \geq (n-D)c.$$

By a standard convexity argument, the product of $s(v)$ is minimized when their values are as far apart as possible under the above assumptions. Thus,

$$|S| = \prod_{v \in V(H)} s(v) \geq x^A c^{n-A},$$

where A satisfies

$$Ax + (n - A)c = (n - D)c,$$

implying that $A = cD/(c - x) < 2D$, since our assumption on c implies $x < \frac{c}{2}$. Therefore,

$$|S| \geq x^A c^{n-A} > x^{2D} c^{n-2D} \geq (nt + n^3)c^{n-1}.$$

On the other hand, we know that no $\phi \in S$ is in a large primary color class b, since such a ϕ does not take value b on V_b. Thus, S lies in the union of the small primary color classes and the secondary color classes. The sizes of the former we bound by $n^3 c^{n-2}$, and the latter by nc^{n-1} by Lemma 4, whereby

$$|S| \leq c \cdot (n^3 c^{n-2}) + t \cdot (nc^{n-1}) = (nt + n^3)c^{n-1}.$$

We have arrived at a contradiction, so there exists a vertex v which lies in at least $c - x$ of the sets V_b. Each of these primary colors b is 2-tethered to v, as desired. □

4 The Construction

If G and H are finite simple graphs, their *strong product* $G \boxtimes H$ is a simple graph on vertex set $V(G) \times V(H)$, where vertices (g_1, h_1) and (g_2, h_2) are adjacent if one of the following three conditions hold.

$$g_1 \sim g_2, h_1 \sim h_2 \qquad \text{or} \qquad g_1 \sim g_2, h_1 = h_2 \qquad \text{or} \qquad g_1 = g_2, h_1 \sim h_2.$$

Let G be a finite simple graph. We will be studying the exponential graphs of the two graphs G° and $G \boxtimes K_q$, for some $q \geq 2$. Note that there is a natural embedding $\iota : \mathcal{E}_c(G^\circ) \hookrightarrow \mathcal{E}_c(G \boxtimes K_q)$ where an element ϕ of $V(\mathcal{E}_c(G^\circ)) = [c]^{V(G)}$ is sent to the map $\phi^* : (g, i) \mapsto \phi(g)$ which ignores the K_q coordinate.

Lemma 7. *Fix a simple graph G with $n \geq 10$ vertices, maximum degree at most $D - 1$ and girth at least 8, and let $\alpha = n^{-1}4^{-2D}$. If q is sufficiently large in terms of n and if $c = (4 + 14\alpha)q$, then $\chi(\mathcal{E}_c(G \boxtimes K_q)) > (1 + \alpha)c$.*

Proof. Let $t = \alpha c$. Suppose for the sake of contradiction that $\chi(\mathcal{E}_c(G \boxtimes K_q)) \leq (1+\alpha)c = c+t$, so that by Lemma 3 there is a standard $(c + t)$-coloring Ψ of $\mathcal{E}_c(G \boxtimes K_q)$. Since $\mathcal{E}_c(G^\circ)$ is an induced subgraph of $\mathcal{E}_c(G \boxtimes K_q)$, Ψ induces a standard $(c + t)$-coloring Ψ° on $\mathcal{E}_c(G^\circ)$.

With our choices of t and c, the condition $c \geq 3^{2D}(nt + n^3)$ holds if q is sufficiently large, so Lemma 6 applies. Thus, there is a vertex $v \in V(G^\circ)$ to which at least $c - x$ primary colors of Ψ° are 2-tethered, where

$$x = \left\lceil \sqrt[2D]{(nt + n^3)c^{2D-1}} \right\rceil = ((\alpha n)^{\frac{1}{2D}} + o(1))c$$

as $q \to \infty$. By our choice of α, this means that $x = (\frac{1}{2} + o(1))c$. We find that

$$\begin{aligned} c - x &= (\tfrac{1}{2} + o(1))c = (2 + 7\alpha + o(1))q, \\ 2q + t + 1 &= (2 + 4\alpha + 14\alpha^2 + o(1))q. \end{aligned}$$

Observe that $14\alpha < 3$ by our assumption on n, so $c - x \geq 2q + t + 1$ for q large enough. Thus, there exist $t + 1$ primary colors $\sigma_1, \ldots, \sigma_{t+1} \notin \{1, \ldots, 2q\}$ which are 2-tethered to v in the coloring Ψ°.

We next pick a set $\mathcal{M} = \{\mu_{2q+1}, \ldots, \mu_c\}$ of vertices in $\mathcal{E}_c(G \boxtimes K_q)$. They are defined by

$$\mu_r(g, i) = \begin{cases} i & \operatorname{dist}(v, g) \in \{0, 2\}, \\ q + i & \operatorname{dist}(v, g) \in \{1, 3\}, \\ r & \text{otherwise}. \end{cases}$$

Since the girth of G is at least 8, there are no edges $(g, g') \in E(G)$ for which $\operatorname{dist}(v, g)$ and $\operatorname{dist}(v, g')$ are both at most 3 and have the same parity. Thus, μ_r and $\mu_{r'}$ are co-proper whenever $r \neq r'$, so $\mathcal{M}$ forms a clique of size $c - 2q$ in $\mathcal{E}_c(G \boxtimes K_q)$.

Since we chose $t = (4\alpha + 14\alpha^2)q$, if q is sufficiently large, we have

$$c - 4q = 14\alpha q \geq 3t + 2$$

and therefore $c - 4q - 2t - 1 \geq t + 1$. In particular, $t + 1$ of the colors $\{\Psi(\mu_r)\}_{r=2q+1}^{c}$ do not lie in the union $\{1, \ldots, 2q\} \cup \{\sigma_1, \ldots, \sigma_{t+1}\} \cup \{c + 1, \ldots, c + t\}$.

Let $\mathcal{M}' = \{\mu_{r_1}, \ldots, \mu_{r_{t+1}}\}$ be a set of $t + 1$ vertices of $\mathcal{M}$ with colors not among $\{1, \ldots, 2q\} \cup \{\sigma_1, \ldots, \sigma_{t+1}\} \cup \{c+1, \ldots, c+t\}$. Since Ψ is a standard coloring and $\operatorname{im}(\mu_{r_s}) = \{1, \ldots, 2q\} \cup \{r_s\}$, it follows that $\Psi(\mu_{r_s}) = r_s$ for each $s = 1, \ldots, t + 1$.

We define a set $\mathcal{N}$ of $t + 1$ other vertices $\nu_1, \ldots, \nu_{t+1} \in \mathcal{E}_c(G \boxtimes K_q)$, by

$$\nu_s(g, i) = \begin{cases} r_s & \operatorname{dist}(v, g) \leq 2, \\ \sigma_s & \text{otherwise}. \end{cases}$$

Recall that we chose $\{r_1, \ldots, r_{t+1}\}$ to be $t+1$ distinct colors disjoint from $\{\sigma_1, \ldots, \sigma_{t+1}\}$. Therefore, ν_s and $\nu_{s'}$ have disjoint images when $s \neq s'$, so $\mathcal{N}$ forms a clique of size $t + 1$ in $\mathcal{E}_c(G \boxtimes K_q)$. Also, the vertices ν_s are constant on the K_q coordinate, so they lie in the image of the embedding $\iota : \mathcal{E}_c(G^\circ) \hookrightarrow \mathcal{E}_c(G \boxtimes K_q)$ and correspond to vertices of $\mathcal{E}_c(G^\circ)$. We chose σ_s to be 2-tethered to v in Ψ°, so if $\Psi^\circ(\iota^{-1}(\nu_s)) = \sigma_s$, then $\iota^{-1}(\nu_s)(g) = \sigma_s$ for some $g \in V(G^\circ)$ with $\operatorname{dist}(v, g) \leq 2$. This contradicts the definition of ν_s, so $\Psi(\nu_s) = \Psi^\circ(\iota^{-1}(\nu_s)) \neq \sigma_s$.

Therefore, by the standardness of Ψ, $\Psi(\nu_s) \in \{r_s\} \cup \{c+1, \ldots, c+t\}$ for each $s = 1, \ldots, t+1$. There are only t secondary colors to use for the $t+1$ vertices of this clique, which implies that for some s, $\Psi(\nu_s) = r_s$. But $\Psi(\mu_{r_s}) = r_s$ as well, and μ_{r_s} and ν_s are co-proper in $\mathcal{E}_c(G \boxtimes K_q)$. This is the desired contradiction, completing the proof that $\chi(\mathcal{E}_c(G \boxtimes K_q)) > (1+\alpha)c$. □

All that remains is to find a G with large fractional chromatic number which satisfies the conditions of Lemma 7. A probabilistic argument of Erdős [4] shows that there exist graphs of arbitrarily large girth and fractional chromatic number; however, in order to obtain a larger value of α, we instead use the famous theorem of Lubotzky, Phillips, and Sarnak on the existence of Ramanujan graphs.

Theorem 8. *[9] If p_1 and p_2 are distinct primes congruent to $1 \pmod 4$ and p_2 is a quadratic residue modulo p_1, then there exists a p_1+1-regular graph G on $n = p_2(p_2^2-1)/2$ vertices with girth at least $2\log_{p_1} p_2$ and independence number at most $\frac{2\sqrt{p_1}}{p_1+1}n$.*

Using this result and the previous lemmas, we can prove Theorem 1.

Proof of Theorem 1. Take $p_1 = 73$ and $p_2 = 3323833$. It is easy to check that both are primes $1 \pmod 4$ and p_2 is a quadratic residue modulo p_1. Applying Theorem 8 with these two primes, there exists a simple graph G with $n = p_2(p_2^2-1)/2 \leq 10^{20}$ vertices, maximum degree less than $D = 75$, girth at least $\lceil 2\log_{p_1} p_2 \rceil = 8$, and independence number at most $\frac{2\sqrt{73}}{74}n$. Thus, the fractional chromatic number of G satisfies

$$\chi_f(G) \geq \frac{n}{\alpha(G)} \geq \frac{74}{2\sqrt{73}} > 4.1.$$

Let $\alpha = n^{-1}4^{-2D} \geq 10^{-111}$, let q be sufficiently large so that Lemma 7 applies, and let $c = (4+14\alpha)q$. We will only prove the theorem for c of this form; it can be proved for all c large enough by rounding off to the nearest such value.

It is easy to see that

$$\chi(G \boxtimes K_q) \geq \chi_f(G)\chi_f(K_q) \geq 4.1q > (1+\alpha)c.$$

By Lemma 7, we also know that $\chi(\mathcal{E}_c(G \boxtimes K_q)) > (1+\alpha)c$. Note that by definition, $G \boxtimes K_q$ is simple, and since it is not c-colorable, we see that $\mathcal{E}_c(G \boxtimes K_q)$ is also simple.

On the other hand, Lemma 2 shows that $\chi((G \boxtimes K_q) \times \mathcal{E}_c(G \boxtimes K_q)) \leq c$, which proves the theorem for some $\alpha \geq 10^{-111}$. □

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