

1. Prove that, for any $r \geq 1$,

$$r(\underbrace{3, 3, \dots, 3}_{r \text{ times}}, k) = \Omega\left(\frac{k^{r+1}}{(\log k)^{C_r}}\right),$$

for some constant C_r depending only on r . Note that this matches the upper bound of Theorem 2.1.5 up to the logarithmic factor.

2. Prove that the graph Λ_q has no O’Nan configuration.
3. Let q be a prime power. Construct a graph Π_q with vertex set $V(\Pi_q) = \mathbb{F}_q^2$, in which two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if and only if $x_1x_2 + y_1y_2 = 1$.

- (a) Prove that Π_q is C_4 -free.
- ★(b) Prove that Π_q satisfies the assumptions of Lemma 4.3.1 with $\beta = \Theta(1/q)$ and $R = \Theta(q^{3/2})$.

Remark: You should feel free to assume this result without proof, or to prove it with logarithmic losses in the value of R . The only way I know how to prove this involves techniques (from spectral graph theory) that we will not cover in this class, but I believe there should be an “elementary” proof. Please let me know if you find one!

- (c) Use these results, plus a natural generalization of Lemma 3.1.1, to deduce a lower bound for the off-diagonal graph Ramsey number $r(C_4, K_k)$, namely

$$r(C_4, K_k) = \Omega\left(\frac{k^{\frac{3}{2}}}{(\log k)^C}\right)$$

for an absolute constant $C > 0$. What value of C can you get?

- ?(d) Improve the lower bound to $r(C_4, K_k) \geq k^{\frac{3}{2} + \varepsilon - o(1)}$ for any absolute constant $\varepsilon > 0$.
4. Let q be a prime power. We define a graph $\Gamma_q^{(5)}$ to be the natural five-dimensional analogue of Γ_q . Namely, $\Gamma_q^{(5)}$ is a bipartite graph with parts $P \cup L$, where P is identified with \mathbb{F}_q^5 , and L comprises all lines in \mathbb{F}_q^5 whose direction is of the form $(1, z, z^2, z^3, z^4)$ for some $z \in \mathbb{F}_q$.
- (a) Prove that $\Gamma_q^{(5)}$ is C_4 -free and C_{10} -free.
- (b) Define a natural analogue $G_q^{(5)}$ of G_q . It is natural to suppose that $G_q^{(5)}$ is C_5 -free with probability 1; show that this is *not* the case.
- Hint:* Show that $\Gamma_q^{(5)}$ is *not* C_8 -free. Use this to find a C_5 in $G_q^{(5)}$.

★ means that a problem is hard.

? means that a problem is open.

↔ means that a problem is on a topic beyond the scope of the course.

- ★★(c) Without further modifications, this approach cannot be used to lower-bound $r(C_5, K_k)$, since $G_q^{(5)}$ has copies of C_5 . Can you nonetheless salvage a similar approach, and obtain a lower bound on $r(C_5, K_k)$?
5. (a) Prove that $r(T; q) = O(qn)$ for every $q \geq 2$ and every n -vertex tree T .
 ★(b) Prove that $r(T; q) = \Theta(qn)$ for every $q \geq 2$ and every n -vertex tree T .
- ⊕6. A *subdivision* of a graph H is obtained from H by replacing every edge of H by a path of some length (not necessarily the same length for all edges, and paths of length 1 are allowed, so that H is a subdivision of itself). A famous conjecture of Hajós asserts that if $\chi(G) \geq k$, then G contains a subdivision of K_k as a subgraph.
- (a) Prove that Hajós' conjecture is true for $k \leq 3$.
 ★(b) Prove that Hajós' conjecture is true for $k = 4$.
 (c) Prove that Hajós' conjecture for $k = 5$ implies the four-color theorem. Conclude that it is probably pretty hard to prove the $k = 5$ case.
 (d) Prove that if Hajós' conjecture is true, then $r(k) \leq 3k^3$. Conclude that Hajós' conjecture is false.
 (e) Prove that if Hajós' conjecture is true, then $r(3, k) \leq 12k$. Conclude that Hajós' conjecture is false.
- ⊕7. A classical fact in graph theory is that there exist triangle-free graphs of arbitrarily high chromatic number. A standard proof, taught in most introductory graph theory courses, uses the *Mycielski construction*. In this exercise, you will see two alternative Ramsey-theoretic proofs.
- (a) For an integer N , let S_N be a graph with vertex set $\binom{[N]}{2}$, where we think of the vertices of S_N as ordered pairs (a, b) with $1 \leq a < b \leq N$. The edges of S_N consist of all pairs of the form $((a, b), (b, c))$ for $a < b < c$. Prove that S_N is triangle-free, and that $\chi(S_N) \rightarrow \infty$ as $N \rightarrow \infty$.
 (b) The graph G_q constructed in class is triangle-free; prove that $\chi(G_q) \rightarrow \infty$ as $q \rightarrow \infty$.
- ⊕8. (a) Let $K_{\mathbb{N}}$ denote the complete graph whose vertex set is \mathbb{N} . Prove the “infinite Ramsey theorem”: for any positive integer q , and any q -coloring of $K_{\mathbb{N}}$, there is an infinite monochromatic clique.
 ★(b) Prove that the finite and infinite Ramsey theorems are equivalent.
Hint: This fact is often called “compactness”, and you may want to use something else called compactness in the proof.
- ⊕9. Prove that there is an infinite set $S \subseteq \mathbb{N}$ such that for every $a, b \in S$, the number $a + b$ has an even number of prime factors (counted without multiplicity).