Processing Digital Images using *Singular Value Decomposition*

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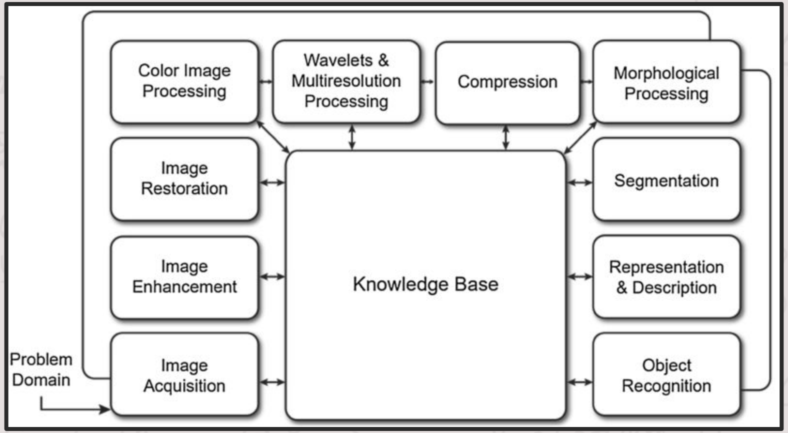
What is digital image processing?

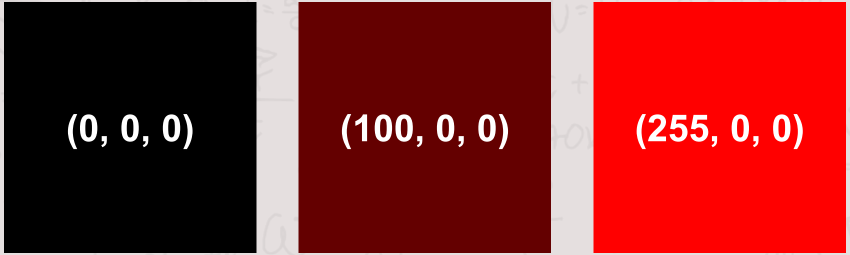
- Using mathematical models to **prepare digital image data** for storage, transmission, or representation
  - Retrieve important features
  - Enhance or restore images

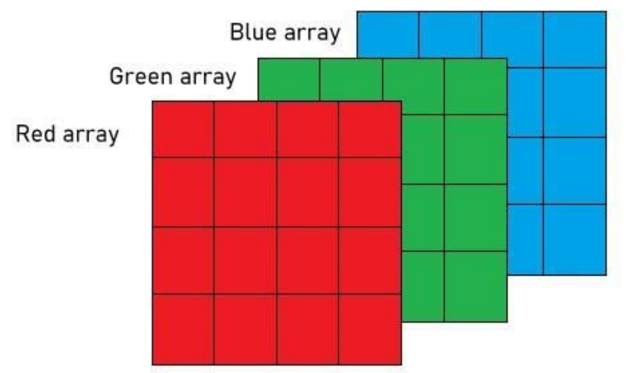
- Images can be transformed (rotation, addition, multiplication, filters)
- Images can be **compressed**
  - Reduces space occupied by an image



Images as Matrices

- A digital image can be represented by a **matrix of pixels**
- The color of **every pixel** corresponds to numerical values
  - The **RGB model** uses values from **0 to 255** for each color



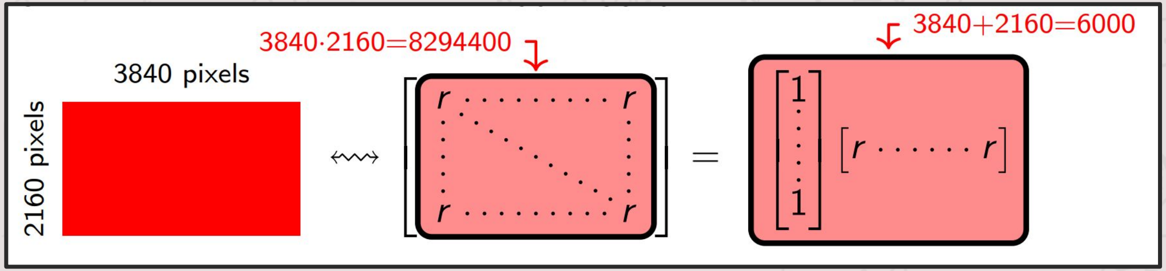


Arrrays stacked over each other to form a Digital Image.

Image Compression

Purpose

- Images take up space
- **Factorization** allows matrices to be represented with **fewer values**
  - Diagonalization is **not applicable** to all matrices



Using SVD

- Image compression can be performed using **singular value decomposition (SVD)**, a form of factorization
  - Removes (**truncates**) some terms of the factorization
  - **Retains the most important features** of the matrix

Prerequisites for SVD

- **rank** = dim(Col A)
- **orthogonal matrix:** dot product of all pairs of vectors is 0
  - vectors of (mxn) orthogonal matrix P spans R^n
  - $P^T = P^{-1}$
  - **orthonormal** if the magnitude of each vector is 1

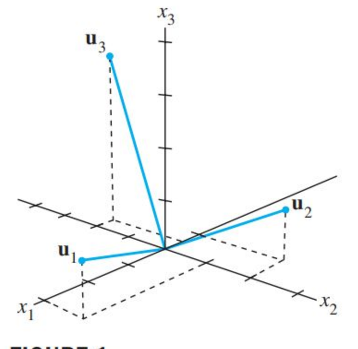
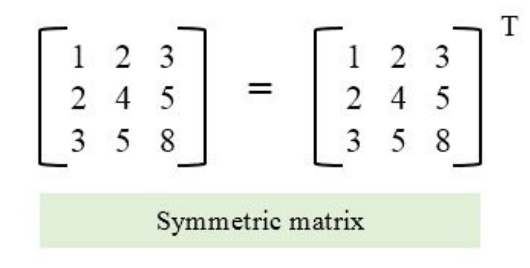


FIGURE 1

- **symmetric matrix:** square matrix such that $A^T = A$
  - eigenvectors from different eigenspaces are orthogonal
  - always orthogonally diagonalizable
    - exists an orthogonal matrix P ($P^{-1} = P^T$) and diagonal matrix D: $A = PDP^{-1} = PDP^T$
  - **A^TA is a symmetric matrix for any matrix A**



Prerequisites for SVD

- **Conjugate transpose ($A^\dagger$):** the transpose of a matrix with the elements replaced with its complex conjugate
  - For **real matrices**, $A^T = A^\dagger$

$$\underset{A}{\begin{bmatrix} 1+i & 1+2i & 1+3i \\ 2+i & 2+2i & 2+3i \\ 3+i & 3+2i & 3+3i \end{bmatrix}} \xrightarrow{\text{conjugate}} \underset{\overline{A}}{\begin{bmatrix} 1-i & 1-2i & 1-3i \\ 2-i & 2-2i & 2-3i \\ 3-i & 3-2i & 3-3i \end{bmatrix}} \xrightarrow{\text{Transpose}} \underset{\overline{A^T}}{\begin{bmatrix} 1-i & 2-1i & 3-1i \\ 1-2i & 2-2i & 3-2i \\ 1-3i & 2-3i & 3-3i \end{bmatrix}}$$

Singular Value Decomposition

- **SVD is a factorization**
  - Always possible!

- Uses **singular values** of A
  - Square roots of the eigenvalues of A^TA, denoted by σ_n
  - Arranged in decreasing order
    $\sigma_1 > \sigma_2 > \sigma_3 > \ldots > \sigma_r$

    $\sigma_1 = \sqrt{\lambda_1}$

m x n matrix
rank A = r

mxm orthonormal matrix of "left singular vectors"

nxn orthonormal matrix of "right singular vectors"

$$A = U\Sigma V^{\dagger}$$

mxn matrix:

$$\Sigma = \begin{array}{cl} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} & \leftarrow m - r \text{ rows} \\ \uparrow\, n - r \text{ columns} & \end{array}$$

D is an *rxr* diagonal matrix with the singular values of A on the diagonal

$$\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} \quad = \quad \mathbf{U} \quad \mathbf{\Sigma} \quad \mathbf{V}^\dagger$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}$$

The SVD of A can be expressed as a sum of matrices:

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \end{bmatrix}$$

$$A = \sigma_1(\mathbf{u_1})\mathbf{v_1}^\dagger + \sigma_2(\mathbf{u_2})\mathbf{v_2}^\dagger + \ldots + \sigma_n(\mathbf{u_n})\mathbf{v_2}^\dagger$$

Note: *Singular values are arranged in decreasing order. Each successive term creates smaller, more precise changes compared to its predecessor.*

Finding the SVD

$A = U\Sigma V^{\dagger}$

1. Compute the orthogonal diagonalization of A^TA

a. A^TA is symmetric and is always orthogonally diagonalizable

b. Find eigenvalues and orthonormal set of eigenvectors for A^TA

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 14 & 11 \\ 11 & 14 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$(14 - \lambda)^2 - (11)^2 = 0$

$\lambda^2 - 28\lambda + 75 = 0$

$(\lambda - 25)(\lambda - 3) = 0$

Order eigenvalues in decreasing order.

$\lambda_1 = 25$

$\lambda_2 = 3$

$\lambda_1 = 25$:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\lambda_2 = 3$:

$$\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

```
>> A
A =

   1   2
   3   1
   2   3

>> [P D] = eig(A'*A)
P =

  -0.7071   0.7071
   0.7071   0.7071

D =

Diagonal Matrix

   3    0
   0   25
```

$A = U\Sigma V^\dagger$

Finding the SVD (cont.)

2. Construct V and Σ

a. Eigenvectors are right singular vectors of A, the columns of V.
b. Singular values are the square root of the eigenvalues → diagonal entries of D.
c. Σ is the same size as A, *3x2,* with D in upper left corner and 0's to fill in the gaps.

$$V = [\,\mathbf{v}_1\ \mathbf{v}_2\,] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$V^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\lambda_1 = 25 \rightarrow \sigma_1 = \sqrt{25} = 5$
$\lambda_2 = 3 \rightarrow \sigma_1 = \sqrt{3}$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

2x2

A is *3x2*
so, Σ is *3x2*

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$A = U\Sigma V^{\dagger}$

Finding the SVD (cont.)

3. Construct U (*mxm* matrix of left singular vectors)

a. The first r columns of U are normalized vectors found from $A\mathbf{v}_1 \ldots A\mathbf{v}_r$

b. Remaining columns are an extension of the set to an orthonormal basis for R^m, R^3

$$\mathbf{u}_1 = (1/\sigma_1)A\mathbf{v}_1 = \begin{bmatrix} \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

To find the 3rd vector in U, find a unit vector that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.

In this case, the cross product of $\mathbf{u}_1$ and $\mathbf{u}_2$ was calculated.

$$\mathbf{u}_2 = (1/\sigma_2)A\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{bmatrix} \frac{7}{5\sqrt{3}} \\ \frac{1}{5\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, \quad U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{bmatrix} \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} \\ \frac{4}{5\sqrt{2}} & -\frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Finding the SVD (cont.)

$$A = U \, \Sigma \, V^{\dagger}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} \\ \frac{4}{5\sqrt{2}} & -\frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

```
>> A
A =

   1   2
   3   1
   2   3

>> [U S V] = svd(A)
U =

  -0.4243   0.4082  -0.8083
  -0.5657  -0.8165  -0.1155
  -0.7071   0.4082   0.5774

S =

Diagonal Matrix

   5.0000        0
        0   1.7321
        0        0

V =

  -0.7071  -0.7071
  -0.7071   0.7071
```

Using SVD to approximate

$$\begin{array}{cccc} \mathbf{A} & = \quad \mathbf{U} & \mathbf{\Sigma} & \mathbf{V}^\dagger \end{array}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} \\ \frac{4}{5\sqrt{2}} & -\frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \sigma_1(\mathbf{u_1})\mathbf{v_1}^\dagger + \sigma_2(\mathbf{u_2})\mathbf{v_2}^\dagger + \ldots + \sigma_n(\mathbf{u_n})\mathbf{v_n}^\dagger$$

Recall that singular values are arranged in decreasing order. A can be approximated by dropping or truncating the “least important” terms, starting with the last term. A has a rank of 2 and has 2 singular values; include only one term for the rank-1 approximation:

$$A \approx \sigma_1(\mathbf{u_1})\mathbf{v_1}^\dagger = 5 \begin{bmatrix} \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1.5 & 1.5 \\ 2 & 2 \\ 2.5 & 2.5 \end{bmatrix}$$

SVD and Image Compression

Recall:

- Digital images can be represented by a matrix
- SVD **can be performed on any matrix**

Thus, **digital images can be factored using SVD** and can therefore be **represented as a sum of matrices.**

From this, an approximation for an image can be found by truncating less influential terms. This approximation, known as a compression, **allows images to be stored using less space**, while retaining the important features of the image.

Image Compression Examples

$$A = \begin{bmatrix} 100 & 0 & 0 & 255 \\ 0 & 100 & 255 & 0 \\ 0 & 255 & 100 & 0 \\ 255 & 0 & 0 & 100 \end{bmatrix} = \overset{U}{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} 355 & 0 & 0 & 0 \\ 0 & 355 & 0 & 0 \\ 0 & 0 & 155 & 0 \\ 0 & 0 & 0 & 155 \end{bmatrix}} \overset{V^{\dagger}}{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^{\dagger}}$$

Image Compression Examples

A has rank 4 and can thus be represented as the sum of 4 matrices. The following images illustrate approximations where each successive image is the result of truncating the last term of the previous approximation.

Original *A*, rank-4

100	0	0	255
0	100	255	0
0	255	100	0
255	0	0	100

rank-3

100	0	0	255
0	178	178	0
0	178	178	0
255	0	0	100

rank-2

178	0	0	178
0	178	178	0
0	178	178	0
178	0	0	178

rank-1

178	0	0	178
0	0	0	0
0	0	0	0
178	0	0	178

Our Father (*Paternostro* 🍝)

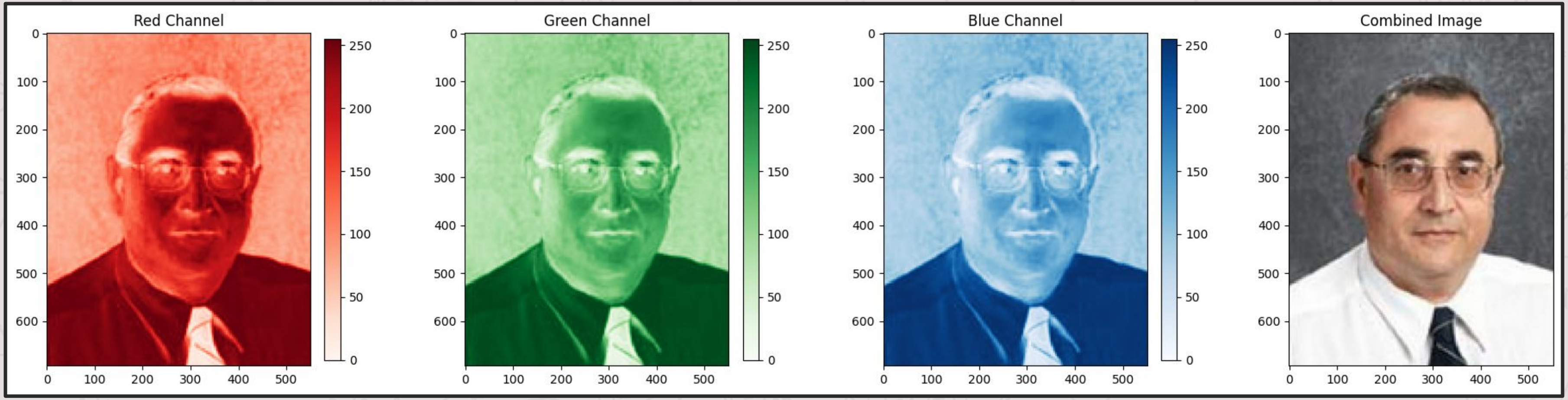
Being able to apply SVD on any matrix thus any image allows us to visualize SVD and rank-k approximation on a more familiar example.

Enter, Mr. E. Paterno. Better said, a 551 x pixel image of him, with **rank 551.**

"Smile, tomorrow will be worse."
~ Mr. Paterno, and copied by some Murphy guy.

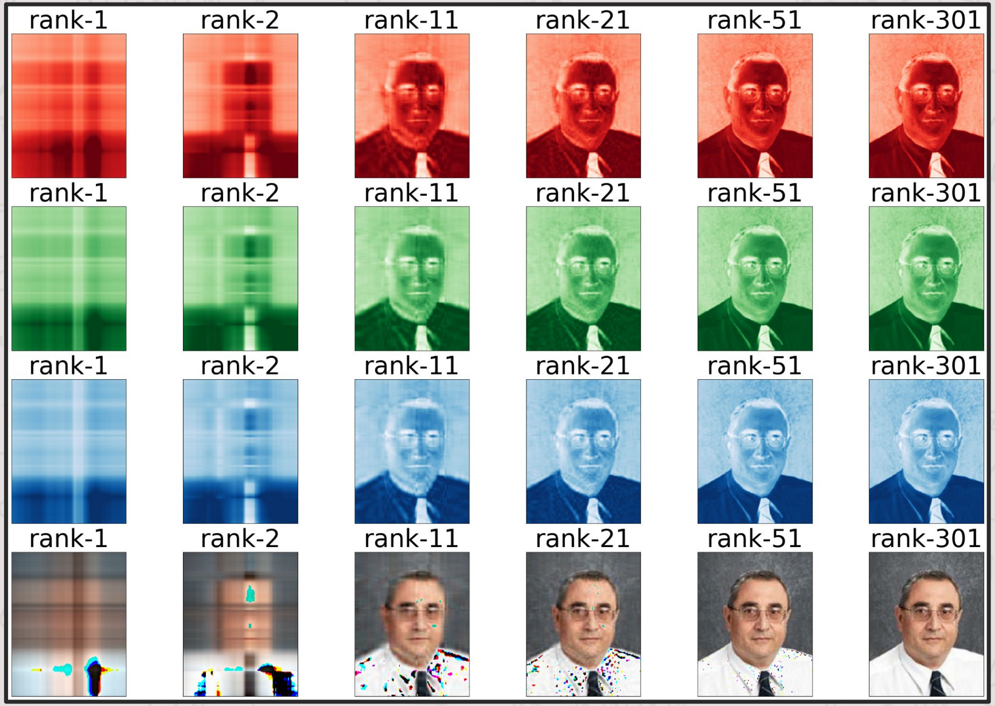
Our Father (*Paternostro* 🍝)

We can **split the image** into the **three color channels** that compose the original image. Note that all three matrices are able to be decomposed using SVD and can therefore be approximated.

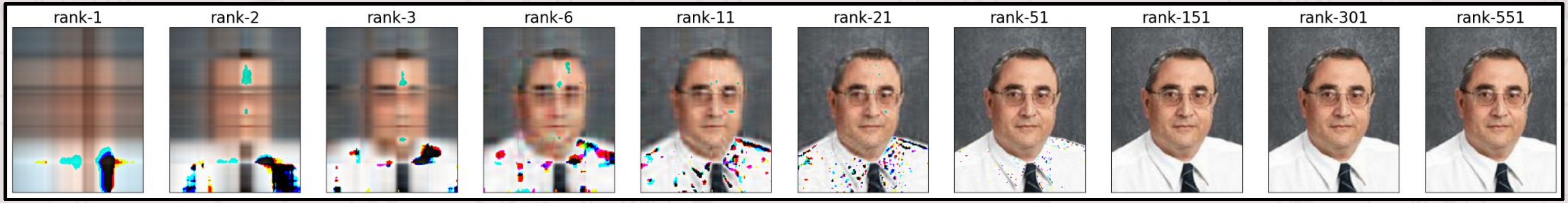


Our Father (*Paternostro* 🍝)

- Original image has **rank of 551**
  - Each matrix rewritten as the sum of 551 matrices
- Rank-*k* approximation includes only the first *k* terms.
  - Combine approximations → **rank-*k* approximation of original image**



Our Father (*Paternostro* 🍝)



Observe how the rank-*k* approximations **approach the likeness of the original image as *k* approaches *r*.**

The rank-151 approximation looks pretty accurate to the human eye. This approximation requires the sum of only 151 matrices instead of 551. **Hooray, we've saved space!**

Thanks!

Do you have any questions?