exercise1 (Score: 13.0 / 14.0)

1. Test cell (Score: 2.0 / 2.0)
2. Test cell (Score: 1.0 / 1.0)
3. Test cell (Score: 1.0 / 1.0)
4. Test cell (Score: 1.0 / 1.0)
5. Test cell (Score: 0.0 / 1.0)
6. Test cell (Score: 3.0 / 3.0)
7. Test cell (Score: 2.0 / 2.0)
8. Task (Score: 3.0 / 3.0)

# Lab 5

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考lab-5 (https://yuanyuyuan.github.io/itcm/lab-5.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 12/11(Wed.)**

In [1]:

```
name = "黃宇文"
student_id = "B06201029"
```

# Exercise 1

### An $m \times m$ ***Hilbert matrix*** $H_m$ has entries $h_{ij} = 1/(i+j-1)$ for $1 \le i,j \le m$, and so it has the form

### $$\left [

$$\begin{matrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

\right ].$$

In [2]:

```python
import numpy as np
from numpy import linalg as LA
import matplotlib.pyplot as plt
```

## Part 1

### Generate the Hilbert matrix of order $m$, for $m = 2, 3, \ldots, 12$.

### For each $m$, compute the condition number of $H_m$, *ie* , in $p$-norm for $p = 1$ and $2$, and make a plot of the results.

## Part 1.1

Define the function of Hilbert matrix

In [3]:

(Top)

```
def hilbert_matrix(m):
    '''
    Return:
        2D np.array, the Hildert Matrix of order m
    '''
    # ===== 請實做程式 =====\

    H=np.zeros((m,m))
    for i in range(m):
        for j in range(m):
            H[i][j]=1/(i+j+1)
    return H
    # ====================
```

Test your function.

In [4]:

hilbert_matrix (Top)

```
print('H_2:\n', hilbert_matrix(2))
### BEGIN HIDDEN TESTS
assert np.mean(np.array(hilbert_matrix(3)) - np.array([[1, 1/2, 1/3], [1/2, 1/3, 1/4], [1/3, 1/4, 1/5]]))
< 1e-7
### END HIDDEN TESTS
```

```
H_2:
 [[1.         0.5       ]
 [0.5        0.33333333]]
```

## Part 1.2

Collect all Hilbert matrices into the list `H_m` for m = 2, 3, ..., 12.

In [5]:

(Top)

```
H_m = []
# ===== 請實做程式 =====
for m in range(2,13):
    H = hilbert_matrix(m)
    H_m.append(H)
# ====================
```

Check your Hilbert matrix list.

In [6]:

hilbert_matrices

```python
for i in range(len(H_m)):
    print('H_%d:' % (i+2))
    print(H_m[i])
    print()
### BEGIN HIDDEN TESTS
error = 0
for m in range(2, 13):
    error += LA.norm(hilbert_matrix(m) - np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)]))
assert error < 1e-16
### END HIDDEN TESTS
```

```
H_2:
[[1.         0.5       ]
 [0.5        0.33333333]]

H_3:
[[1.         0.5        0.33333333]
 [0.5        0.33333333 0.25      ]
 [0.33333333 0.25       0.2       ]]

H_4:
[[1.         0.5        0.33333333 0.25      ]
 [0.5        0.33333333 0.25       0.2       ]
 [0.33333333 0.25       0.2        0.16666667]
 [0.25       0.2        0.16666667 0.14285714]]

H_5:
[[1.         0.5        0.33333333 0.25       0.2       ]
 [0.5        0.33333333 0.25       0.2        0.16666667]
 [0.33333333 0.25       0.2        0.16666667 0.14285714]
 [0.25       0.2        0.16666667 0.14285714 0.125     ]
 [0.2        0.16666667 0.14285714 0.125      0.11111111]]

H_6:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125     ]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1       ]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909]]

H_7:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125     ]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1       ]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308]]

H_8:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125     ]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1       ]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667]]
```

```
H_9:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1       ]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625    ]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353]]

H_10:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1       ]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158]]

H_11:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1        0.09090909]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909 0.08333333]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333 0.07692308]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308 0.07142857]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857 0.06666667]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625     0.05882353]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353 0.05555556]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556 0.05263158]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158 0.05      ]
 [0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625
  0.05882353 0.05555556 0.05263158 0.05       0.04761905]]

H_12:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667 0.0625     0.05882353]
```

```
[0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
 0.07692308 0.07142857 0.06666667 0.0625     0.05882353 0.05555556]
[0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
 0.07142857 0.06666667 0.0625     0.05882353 0.05555556 0.05263158]
[0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
 0.06666667 0.0625     0.05882353 0.05555556 0.05263158 0.05      ]
[0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
 0.0625     0.05882353 0.05555556 0.05263158 0.05       0.04761905]
[0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625
 0.05882353 0.05555556 0.05263158 0.05       0.04761905 0.04545455]
[0.08333333 0.07692308 0.07142857 0.06666667 0.0625     0.05882353
 0.05555556 0.05263158 0.05       0.04761905 0.04545455 0.04347826]]
```

## Part 1.3

Plot the condition number of $H_m$ for $m = 2, 3, \ldots, 12$

Collect all condition numbers in 1-norm of `H_m` into a list `one_norm`

In [7]:

(Top)

```python
one_norm = []
# ===== 請實做程式 =====
for m in range(2,13):
    one_norm.append(np.sum(hilbert_matrix(m)))
# ====================
```

In [8]:

kappa_one_norm

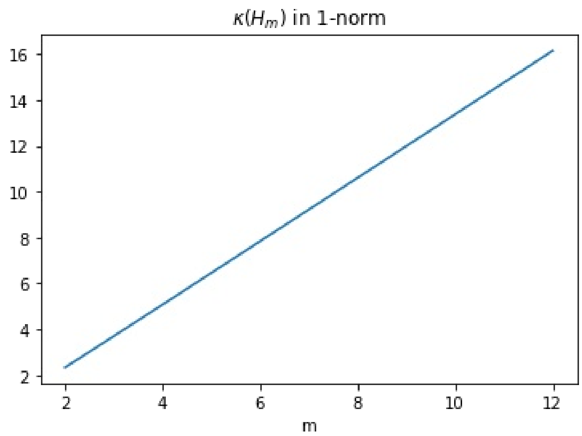
(Top)

```python
print('one_norm:\n', one_norm)
### BEGIN HIDDEN TESTS
assert len(one_norm) == 11
### END HIDDEN TESTS
```

```
one_norm:
 [2.3333333333333335, 3.6999999999999997, 5.076190476190476, 6.456349206349206, 7.83852813852
8138, 9.22187257187257, 10.605949605949606, 11.99051683610507, 13.375428063508556, 14.7605899
1747846, 16.145939989027887]
```

In [9]:

```python
plt.plot(range(2,13), one_norm)
plt.xlabel('m')
plt.title(r'$\kappa(H_m)$ in 1-norm')
plt.show()
```




Collect all condition numbers in 2-norm of `H_m` into a list `two_norm`

In [10]:

(Top)

```
two_norm = []
# ===== 請實做程式 =====
for m in range(2,13):
    two_norm.append(LA.norm(hilbert_matrix(m), 'fro'))
# ====================
```

In [11]:

kappa_two_norm

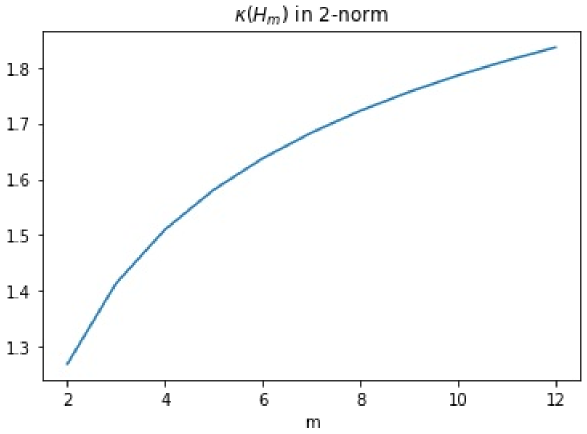
(Top)

```
print('two_norm:\n', two_norm)
### BEGIN HIDDEN TESTS
assert len(two_norm) == 11
### END HIDDEN TESTS
```

```
two_norm:
 [1.2692955176439846, 1.413624183909335, 1.5097340998183075, 1.580906263272022, 1.63702239330
239, 1.6831324888435926, 1.7221431395612752, 1.755871909716653, 1.7855271226510334, 1.8119508
00908123, 1.835752037381468]
```

In [12]:

```
plt.plot(range(2,13), two_norm)
plt.xlabel('m')
plt.title(r'$\kappa(H_m)$ in 2-norm')
plt.show()
```




## Part 2

### Now generate the $m$-vector $b_m = H_m x$ also, where $x$ is the $m$-vector with all of its components equal to $1$.

### Use Gaussian elimination to solve the resulting linear system $H_m x = b_m$ with $H_m$ and $b$ given above, obtaining an approximate solution $\tilde{x}$.

### Part 2.1

Construct the $m$-vector $b_m$ for $m = 2, 3, \ldots, 12$. Store all 1D `np.array` $b_m$ into the list `b_m`.

In [13]:

(Top)

```python
'''
Hint:
    b_m = ?
'''
# ===== 請實做程式 =====
b_m = []

for i in range(len(H_m)):
    b_m.append(H_m[i] @ np.ones((i+2,1)))
# ====================
```

Print b_m

In [14]:

b_m (Top)

```python
for i in range(len(b_m)):
    print('b_%d:' % (i+2))
    print(b_m[i])
    print()
### BEGIN HIDDEN TESTS
error = 0
for m in range(2,13):
    error += LA.norm(b_m[m-2] - np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)])@np.ones(m
))
assert error < 1e-16
### END HIDDEN TESTS
```

```
b_2:
[[1.5       ]
 [0.83333333]]

b_3:
[[1.83333333]
 [1.08333333]
 [0.78333333]]

b_4:
[[2.08333333]
 [1.28333333]
 [0.95      ]
 [0.75952381]]

b_5:
[[2.28333333]
 [1.45      ]
 [1.09285714]
 [0.88452381]
 [0.74563492]]

b_6:
[[2.45      ]
 [1.59285714]
 [1.21785714]
 [0.99563492]
 [0.84563492]
 [0.73654401]]

b_7:
[[2.59285714]
 [1.71785714]
 [1.32896825]
 [1.09563492]
 [0.93654401]
 [0.81987734]
 [0.73013376]]

b_8:
[[2.71785714]
 [1.82896825]
```

```
 [1.42896825]
 [1.18654401]
 [1.01987734]
 [0.89680042]
 [0.80156233]
 [0.72537185]]

b_9:
[[2.82896825]
 [1.92896825]
 [1.51987734]
 [1.26987734]
 [1.09680042]
 [0.96822899]
 [0.86822899]
 [0.78787185]
 [0.72169538]]

b_10:
[[2.92896825]
 [2.01987734]
 [1.60321068]
 [1.34680042]
 [1.16822899]
 [1.03489566]
 [0.93072899]
 [0.84669538]
 [0.77725094]
 [0.7187714 ]]

b_11:
[[3.01987734]
 [2.10321068]
 [1.68013376]
 [1.41822899]
 [1.23489566]
 [1.09739566]
 [0.98955252]
 [0.90225094]
 [0.82988251]
 [0.7687714 ]
 [0.71639045]]

b_12:
[[3.10321068]
 [2.18013376]
 [1.75156233]
 [1.48489566]
 [1.29739566]
 [1.15621919]
 [1.04510808]
 [0.95488251]
 [0.87988251]
 [0.81639045]
 [0.761845  ]
 [0.71441417]]


---------------------------------------------------------------------------
AssertionError                            Traceback (most recent call last)
<ipython-input-14-03ed81c14c8f> in <module>
      7 for m in range(2,13):
      8     error += LA.norm(b_m[m-2] - np.array([[1/(i + j + 1) for j in range(m)] for i in
range(m)])@np.ones(m))
----> 9 assert error < 1e-16
     10 ### END HIDDEN TESTS

AssertionError:
```

### Part 2.2

Implement the function of **Gaussian elimination**.

(Note that you need to implement it by hand, simply using some package functions is not allowed.)

In [15]:

(Top)

```
def gaussian_elimination(
    A,
    b
):
    '''
    Arguments:
        A : 2D np.array
        b : 1D np.array

    Return:
        x : 1D np.array, solution to Ax=b
    '''

    # ===== 請實做程式 =====
    U = A
    m = len(A)
    L = np.diag(np.ones(m))
    x = np.zeros([m,1])
    new_b = np.copy(b)

    for k in range(m-1):
        new_L = np.diag(np.ones(m))
        for j in range(k+1,m):
            new_L[j,k] = -U[j,k] / U[k,k]
            U[j, k:] = U[j, k:] + new_L[j,k] * U[k,k:]
            new_b[j,0] = new_b[j,0] + new_L[j,k] * new_b[k,0]
        L = new_L@L

    for i in range(m):
        if i == 0:
            x[-1,0] = new_b[-1,0] / U[-1,-1]
        else:
            x[-1-i,0] = (new_b[-1-i,0] - U[-1-i,-i:]@x[-i:,0]) / U[-1-i,-1-i]

    return x
    # ===================
```

Store all approximate solutions $\tilde{x}$ of $H_m$ into a list `x_m` for $m = 2, 3, \ldots, 12$

In [16]:

```
x_m = []
for i in range(len(H_m)):
    x = gaussian_elimination(H_m[i], b_m[i])
    x_m.append(x)
```

---

# Part 3

### Investigate the error behavior of the computed solution $\tilde{x}$.

### (i) Compute the ∞-norm of the residual $r = b - H_m\tilde{x}$.

### (ii) Compute the error $\delta x = \tilde{x} - x$, where $x$ is the vector of all ones.

### (iii) How large can you take $m$ before there is no significant digits in the solution ?

## Part 3.1

Compute the ∞-norm of the residual $r_m = b_m - H_m\tilde{x}$ for $m = 2, 3, \ldots, 12$. And store the values into the list `r_m` .

In [17]:

(Top)

```
r_m = []
# ===== 請實做程式 =====
for i in range(2,13):
    r = b_m[i-2] - hilbert_matrix(i) @ x_m[i-2]
    r_m.append(LA.norm(r,np.inf))
# ====================
```

In [18]:

infty_norm (Top)

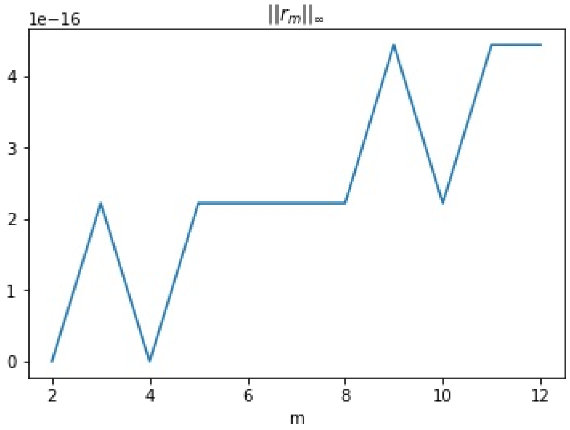
```
print('r_m:\n', r_m)
### BEGIN HIDDEN TESTS
assert np.sum(r_m) < 1e-12
### END HIDDEN TESTS
```

```
r_m:
 [0.0, 2.220446049250313e-16, 0.0, 2.220446049250313e-16, 2.220446049250313e-16, 2.2204460492
50313e-16, 2.220446049250313e-16, 4.440892098500626e-16, 2.220446049250313e-16, 4.44089209850
0626e-16, 4.440892098500626e-16]
```

Plot the figure of the $\infty$-norm of the residual for $m = 2, 3, \ldots, 12$

In [19]:

```
plt.plot(range(2,13), r_m)
plt.xlabel('m')
plt.title(r'$||r_m||_\infty$')
plt.show()
```




## Part 3.2

Compute the error $\delta x = \tilde{x} - x$, where $x$ is the vector of all ones. And store the values into the list `delta_x` .

In [20]:

(Top)

```
delta_x = []
# ===== 請實做程式 =====
for i in range(2,13):
    delta_x.append(x_m[i-2]-np.ones((i,1)))
# ====================
```

Collect all errors $\delta x$ in 2-norm into the list `delta_x_two_norm` for $m = 2, 3, \ldots, 12$

In [21]:

(Top)

```python
delta_x_two_norm = []
# ===== 請實做程式 =====
for i in range(2,13):
    delta_x_two_norm.append(LA.norm(delta_x[i-2], 'fro'))
# ====================
```

In [22]:

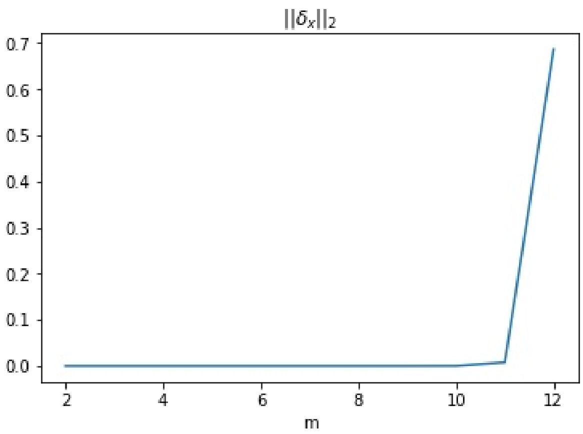
delta_x_two_norm (Top)

```python
print('delta_x_two_norm =', delta_x_two_norm)
### BEGIN HIDDEN TESTS
assert (len(delta_x_two_norm) == 11) and (np.mean(delta_x_two_norm) <= 0.1)
### END HIDDEN TESTS
```

```
delta_x_two_norm = [8.005932084973442e-16, 1.762179615616027e-14, 1.531093756529614e-13, 3.47
26989976779822e-12, 3.380198688368993e-10, 9.687978241052163e-09, 4.244691764807721e-07, 1.73
93602379023114e-07, 0.00014527351138776998, 0.007757231295806999, 0.6867691718060317]
```

In [23]:

```python
plt.plot(range(2,13), delta_x_two_norm)
plt.xlabel('m')
plt.title(r'$||\delta_x||_2$')
plt.show()
```




(Top)

### Part 3.3

How large can you take $m$ before there is no significant digits in the solution ?

When $m = 10$, the graph starts rising gradually. \ When $m = 11$, the graph starts rocketing. \ Hence, $m$ should take the values which are not greater than $10$.

In [ ]: