exercise3 (Score: 12.0 / 12.0)

1. Task (Score: 12.0 / 12.0)

Lab 5

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考lab-5 (https://yuanyuyuan.github.io/itcm/lab-5.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 12/11(Wed.)**

In [1]:

```
name = "黃宇文"
student_id = "B06201029"
```

(Top)

Exercise 3

Analyse the convergence properties of the Jacobi and Gauss-Seidel methods for the solution of a linear system whose matrix is

$$\left[\begin{matrix}

```
\alpha &&0 &&1\\
0 &&\alpha &&0\\
1 &&0 &&\alpha
\end{matrix}\right],
\quad \quad
\alpha \in \mathbb{R}.$$
```

Let

$$A = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & \alpha \end{bmatrix}, \qquad \alpha \in \mathrm{R}.$$

Write $A = D + L + U$, where D is the diagonal matrix, L is the lower triangular matrix, U is the upper triangular matrix.

$$D = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \;, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \;, \quad U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To solve $Ax = b$, where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

Jacobi method

$$\begin{aligned}(D+L+U)x &= b\\ Dx_{n+1} &= b-(L+U)x_n\\ x_{n+1} &= D^{-1}(b-(L+U)x_n)\end{aligned}$$

Define $e_n = x_n - \bar{x}$, where $A\bar{x} = b$

$$\begin{aligned}e_{n+1} = x_{n+1} - \bar{x} &= -D^{-1}(L+U)x_n + D^{-1}b - \bar{x}\\ &= -D^{-1}(L+U)(x_n - \bar{x})\end{aligned}$$

Define $G = -D^{-1}(L+U)$

$$G = -D^{-1}(L+U) = -\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix}^{-1}\left[\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right]$$

$$= -\frac{1}{\alpha}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

$$\begin{aligned}\therefore e_{n+1} &= Ge_n\\ e_n &= G^n e_0\end{aligned}$$

If $\rho(G) < 1$, then $\lim_{n\to\infty}\|G^n\| = 0$.
Then, $\lim_{n\to\infty}\|G^n e_0\| = \lim_{n\to\infty}\|e_n\| = 0$

If G is diagonally dominant, then $\rho(G) < 1$.
But,

$$|a_{11}| = 0,\ |a_{12}| + |a_{13}| = \frac{1}{\alpha}\,|a_{11}| \not> |a_{12}| + |a_{13}|$$

Hence, G is not diagonally dominant.

It does not converge to unique solution.

Gauss-Seidal method

$$
\begin{aligned}
(D+L+U)x &= b \\
(D+L)x_{n+1} &= -Ux_n + b \\
x_{n+1} &= -(D+L)^{-1}Ux_n + (D+L)^{-1}b
\end{aligned}
$$

Define $e_n = x_n - \bar{x}$, where $A\bar{x} = b$

$$
\begin{aligned}
e_{n+1} = x_{n+1} - \bar{x} &= -(D+L)^{-1}Ux_n + (D+L)^{-1}b - \bar{x} \\
&= -(D+L)^{-1}U(x_n - \bar{x})
\end{aligned}
$$

Define $G = -(D+L)^{-1}U$

$$
G = -(D+L)^{-1}U = -\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 1 & 0 & \alpha \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
= -\begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha^2} & 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
= -\begin{pmatrix} 0 & 0 & \frac{1}{\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha^2} \end{pmatrix}
$$

If $\rho(G) < 1$, then $\lim_{n\to\infty} \|G^n\| = 0$.
Then, $\lim_{n\to\infty} \|G^n e_0\| = \lim_{n\to\infty} \|e_n\| = 0$

If G is diagonally dominant, then $\rho(G) < 1$.
But,

$$
|a_{11}| = 0, \ |a_{12}| + |a_{13}| = \frac{1}{\alpha} \ |a_{11}| \not> |a_{12}| + |a_{13}|
$$

Hence, G is not diagonally dominant.

It does not converge to unique solution.

In [ ]: