

## Daily Problem: 28–Jan–2026

### Problem Statement

Let  $n \in \mathbb{N}$ . Over the alphabet  $\{a, b\}$ , define

$$L_n = \{ w \in \{a, b\}^* \mid \#_a(w) - \#_b(w) \text{ is divisible by } n \}.$$

Show that

$$\bigcap_{p \text{ prime}} L_p = \{ w \in \{a, b\}^* \mid \#_a(w) = \#_b(w) \}.$$

### Proof

For any word  $w \in \{a, b\}^*$ , write

$$A(w) = \#_a(w), \quad B(w) = \#_b(w), \quad d(w) = A(w) - B(w).$$

Then by definition,

$$w \in L_n \iff d(w) \equiv 0 \pmod{n}.$$

**( $\subseteq$ ) If  $w \in \bigcap_{p \text{ prime}} L_p$ , then  $A(w) = B(w)$**

Assume  $w \in \bigcap_{p \text{ prime}} L_p$ . Then for every prime  $p$ ,

$$w \in L_p \implies d(w) \equiv 0 \pmod{p},$$

so  $p \mid d(w)$  for every prime  $p$ .

We claim this forces  $d(w) = 0$ . Suppose for contradiction that  $d(w) \neq 0$ . Then  $|d(w)| \geq 1$ . Choose a prime  $p$  such that  $p > |d(w)|$  (this is always possible since there are infinitely many primes). If  $p \mid d(w)$ , then  $|d(w)| \geq p$ , which contradicts  $p > |d(w)|$ . Hence  $p \nmid d(w)$ , contradicting the fact that every prime divides  $d(w)$ .

Therefore  $d(w) = 0$ , i.e.  $A(w) = B(w)$ .

( $\supseteq$ ) **If**  $A(w) = B(w)$ , **then**  $w \in \bigcap_{p \text{ prime}} L_p$

Assume  $A(w) = B(w)$ . Then  $d(w) = A(w) - B(w) = 0$ . Since 0 is divisible by every prime  $p$ , we have

$$d(w) \equiv 0 \pmod{p} \quad \text{for every prime } p,$$

so  $w \in L_p$  for every prime  $p$ . Hence

$$w \in \bigcap_{p \text{ prime}} L_p.$$

## Conclusion

We have shown both inclusions, and therefore

$$\bigcap_{p \text{ prime}} L_p = \{ w \in \{a, b\}^* \mid \#_a(w) = \#_b(w) \}.$$