Daily Problem: 30–Jan–2026

Problem Statement

Over the alphabet $\Sigma = \{a, b, c\}$, define

$$L \;=\; \{\, ab^n c^n : n \geq 0 \,\} \;\cup\; \{\, a^k w : k \geq 2 \text{ and } w \in \Sigma^* \,\}.$$

Show that L is *not* regular and yet it *satisfies the pumping lemma* (the Jan 29 pumping lemma).

Remark on the pumping lemma. The standard pumping lemma statement (as in Jan 29) requires that for the chosen decomposition $w = xyz$, we have $xy^kz \in L$ for *all* $k \geq 0$, including $k = 0$. For the language L as written above, the "$k = 0$" requirement causes a problem for words of the form ab^nc^n with $n \geq 1$: any decomposition with $|xy| \leq p$ forces y to lie inside the initial ab^* prefix, and pumping down to $k = 0$ deletes something from that prefix, producing a word that begins with at most one a but no longer has equal numbers of b's and c's, hence falls outside L.

So, *with the standard lemma including* $k = 0$, this L does *not* satisfy the pumping lemma.

However, many course notes state a weaker "pumping" condition with $k \geq 1$ (only pumping up), and under that weaker version the language L *does* satisfy pumping. Below we prove:

- L is not regular (this is independent of the pumping issue), and
- L satisfies the pumping property for all $k \geq 1$.

Part 1: L is not regular

Assume for contradiction that L is regular. Consider the regular language

$$R \;=\; a\,b^*\,c^*.$$

Since regular languages are closed under intersection, $L \cap R$ would be regular.

We compute $L \cap R$.

- If $w \in \{ab^nc^n : n \geq 0\}$, then $w \in R$ automatically, so all such words are in $L \cap R$.
- If $w \in \{a^k u : k \geq 2, u \in \Sigma^*\}$ and also $w \in R = ab^*c^*$, then w must begin with exactly one a (because every word in R begins with exactly one a followed by only b's and then only c's). This is impossible when $k \geq 2$.

Hence

$$L \cap R \;=\; \{\, ab^nc^n : n \geq 0 \,\}.$$

Now take the left quotient by a. Define

$$a^{-1}(L \cap R) \;=\; \{\, x \in \{b,c\}^* : ax \in L \cap R \,\}.$$

If $L \cap R$ were regular, then $a^{-1}(L \cap R)$ would also be regular (regular languages are closed under left quotient by a fixed word).

But

$$a^{-1}(L \cap R) \;=\; \{\, b^nc^n : n \geq 0 \,\},$$

which is a standard non-regular language. This contradiction shows that L is not regular.

Part 2: L satisfies a pumping property (for all $k \geq 1$)

We prove the following pumping statement:

Claim. There exists $p \geq 1$ such that for every $w \in L$ with $|w| \geq p$, there is a decomposition $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$ such that

$$xy^kz \in L \quad \text{for all integers } k \geq 1.$$

Proof. Take $p = 2$. Let $w \in L$ with $|w| \geq 2$. We split into cases.

Case 1: $w \in \{a^k u : k \geq 2,\, u \in \Sigma^*\}$

Then w begins with at least two a's, so we can write $w = aa\,t$ for some $t \in \Sigma^*$. Choose

$$x = \varepsilon, \qquad y = aa, \qquad z = t.$$

Then $|xy| = 2 \leq p$ and $|y| = 2 \geq 1$. For any $k \geq 1$,

$$xy^k z \;=\; (aa)^k\, t,$$

which begins with at least two a's (indeed $2k \geq 2$), hence $xy^k z \in \{a^k u : k \geq 2\} \subseteq L$. So pumping works for all $k \geq 1$.

Case 2: $w \in \{ab^n c^n : n \geq 0\}$

If $n = 0$, then $w = a$ has length $1 < 2 = p$, so this case is irrelevant because the pumping condition only applies to $|w| \geq p$. So assume $n \geq 1$. Then $w = ab^n c^n$ has length at least 3, hence at least p.

Choose

$$x = a, \qquad y = b, \qquad z = b^{n-1}c^n.$$

Then $|xy| = 2 \leq p$ and $|y| = 1 \geq 1$. For every $k \geq 1$,

$$xy^k z \;=\; a\,b^k\,b^{n-1}c^n \;=\; a\,b^{n-1+k}c^n.$$

If $k = 1$, this is exactly $ab^n c^n \in L$. If $k \geq 2$, then $ab^{n-1+k}c^n$ begins with a single a but is still in L because it is *not required* to stay in the first component: it suffices to be in L, and when $k \geq 2$ we have

$$a\,b^{n-1+k}c^n \in a^2\Sigma^* \subseteq L$$

only if it begins with aa. But it does not; it begins with only one a. Hence this decomposition pumps correctly only for $k = 1$, not for all $k \geq 1$.

Therefore, for the language L as given, no fixed p can guarantee a decomposition that pumps *all* words of the form $ab^n c^n$ for all $k \geq 1$ while keeping $|xy| \leq p$, unless the pumping requirement is weakened further (e.g. allowing the decomposition to depend on k, or restricting the range of k).

This confirms the earlier remark: the language L as stated does not satisfy the standard pumping lemma (nor the usual "all $k \geq 1$" pumping-up variant) for the $ab^n c^n$ component.

$\square$

Conclusion

- L is not regular, since intersecting with ab^*c^* isolates $\{ab^nc^n : n \geq 0\}$, and left-quotienting by a yields $\{b^nc^n : n \geq 0\}$, which is not regular.
- With the standard pumping lemma statement (Jan 29, requiring $k \geq 0$), the language L as written cannot satisfy the pumping lemma because pumping down ($k = 0$) breaks the ab^nc^n part and cannot land in the $a^k\Sigma^*$ part (which requires at least two leading a's).

If you paste your exact course statement for the pumping lemma (some notes use a slightly different form), I can rewrite Part 2 to match it exactly.