Daily Problem: 31–Jan–2026

Problem Statement

Over the alphabet $\Sigma = \{a, b, c\}$, define

$$L \;=\; \{\, ab^n c^n : n \geq 0 \,\} \;\cup\; \{\, a^k w : k \geq 2 \text{ and } w \in \Sigma^* \,\}.$$

Show that L is not regular and yet it satisfies the pumping lemma (Jan 29 problem).

Important note about the pumping lemma statement. The *standard* pumping lemma for regular languages (the one usually stated as "for all $k \geq 0$") is:

$$\exists p\, \forall w \in L,\ |w| \geq p\ \exists x, y, z :\ w = xyz,\ |xy| \leq p,\ |y| \geq 1,\ \forall k \geq 0,\ xy^k z \in L.$$

For the language L above, the condition "$\forall k \geq 0$" is *too strong*: the language is a classic example of a *non-regular* language that nevertheless satisfies the *weaker pumping property with* $k \geq 1$ (pumping up, not pumping down). Many courses/notes use this weaker form when discussing why pumping-type arguments are not sufficient to prove regularity.

Accordingly, below we prove:

1. L is not regular, and
2. L satisfies the pumping property *for all* $k \geq 1$ (and we also explain why the $k = 0$ version fails).

Part 1: L is not regular

Assume for contradiction that L is regular. Consider the regular language

$$R \;=\; a\,b^*\,c^*.$$

Since regular languages are closed under intersection, $L \cap R$ would be regular.

We compute $L \cap R$.

- Every word of the form ab^nc^n lies in ab^*c^*, so $\{ab^nc^n : n \geq 0\} \subseteq L \cap R$.
- If a word lies in $\{a^kw : k \geq 2\}$, then it begins with at least two a's, i.e. it begins with aa. But every word in $R = ab^*c^*$ begins with *exactly one* leading a (followed only by b's then c's). Hence
$$\{a^kw : k \geq 2\} \cap R = \emptyset.$$

Therefore

$$L \cap R \;=\; \{\, ab^nc^n : n \geq 0 \,\}.$$

Now apply the left quotient by the fixed word a. Define

$$a^{-1}(L \cap R) \;=\; \{\, x \in \{b, c\}^* : ax \in L \cap R \,\}.$$

If $L \cap R$ were regular, then $a^{-1}(L \cap R)$ would also be regular (regular languages are closed under left quotients by fixed words).

But

$$a^{-1}(L \cap R) \;=\; \{\, b^nc^n : n \geq 0 \,\},$$

which is a standard non-regular language. This contradiction shows that L is not regular.

Part 2: L satisfies a pumping property (for all $k \geq 1$)

We prove the following "pumping" statement:

Claim (pumping for $k \geq 1$). There exists a constant $p \geq 1$ such that every word $w \in L$ with $|w| \geq p$ can be written as $w = xyz$ with

$$|xy| \leq p, \qquad |y| \geq 1,$$

and such that

$$xy^kz \in L \quad \text{for all integers } k \geq 1.$$

Proof. Take $p = 1$. Let $w \in L$ with $|w| \geq 1$. We consider two cases.

Case 1: $w \in \{ab^n c^n : n \geq 0\}$

Then w begins with the letter a. Choose the decomposition

$$x = \varepsilon, \qquad y = a, \qquad z = b^n c^n,$$

so $w = xyz$. Clearly $|xy| = 1 \leq p$ and $|y| = 1 \geq 1$.

Now let $k \geq 1$. Then

$$xy^k z \;=\; a^k b^n c^n.$$

If $k = 1$, we get $ab^n c^n \in L$ by definition. If $k \geq 2$, then $a^k b^n c^n$ begins with at least two a's, hence lies in the second component $a^k \Sigma^* \subseteq L$. Thus $xy^k z \in L$ for all $k \geq 1$.

Case 2: $w \in \{a^k u : k \geq 2,\ u \in \Sigma^*\}$

Then w begins with at least two a's, so we can write $w = au'$ for some $u' \in \Sigma^*$ and the word still begins with at least one a. Use the same decomposition:

$$x = \varepsilon, \qquad y = a, \qquad z = u'.$$

Again $|xy| = 1 \leq p$, $|y| = 1 \geq 1$, and for $k \geq 1$,

$$xy^k z = a^k u'.$$

For $k = 1$ this is $w \in L$. For $k \geq 2$, the word begins with at least two a's, so it is in the second component of L. Hence $xy^k z \in L$ for all $k \geq 1$.

In both cases, we found a valid decomposition with $|xy| \leq 1$ and $|y| \geq 1$ such that $xy^k z \in L$ for all $k \geq 1$. This proves the claim. $\square$

Why the standard $k \geq 0$ pumping lemma fails for this L

For completeness, note that if we require $xy^k z \in L$ for *all* $k \geq 0$, then taking $w = ab^p c^p$ (for any $p \geq 1$) forces $|xy| \leq p$, so y lies entirely within the prefix ab^p. Pumping down ($k = 0$) deletes at least one symbol from ab^p, producing a word that cannot have equal numbers of b's and c's and also cannot start with two a's. Hence it is not in L. Therefore L cannot satisfy the "$\forall k \geq 0$" form.

Conclusion

- L is **not regular**, via the intersection $L \cap (ab^*c^*) = \{ab^nc^n : n \geq 0\}$ and a left-quotient argument.
- Nevertheless, L **does satisfy a pumping property** (with pumping for all $k \geq 1$), showing that pumping-style conditions are not sufficient to characterize regularity.