Daily Problem: 21–Jan–2026

Problem Statement

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Prove that $L(M)$ is infinite if and only if the transition graph of M contains a directed cycle that is reachable from the start state and from which some accepting state is reachable (i.e., the cycle is both reachable and co-reachable).

Proof

We prove both directions of the statement.

($\Rightarrow$) If such a cycle exists, then $L(M)$ is infinite

Suppose there exists a state $p \in Q$ such that:

- p is reachable from q_0,
- there is a nonempty string v with $\delta(p, v) = p$ (so v forms a directed cycle at p),
- some accepting state $f \in F$ is reachable from p.

Let u be a string taking q_0 to p, and let w be a string taking p to f. Then for every $k \geq 0$, the string

$$uv^k w$$

leads from q_0 to p, loops around the cycle k times, and then reaches the accepting state f. Since all these strings are distinct (their lengths differ), $L(M)$ contains infinitely many strings. Hence $L(M)$ is infinite.

($\Leftarrow$) If $L(M)$ is infinite, then such a cycle must exist

Assume $L(M)$ is infinite. Let $|Q| = n$. Because $L(M)$ is infinite, there exists an accepted string x with length at least n. Consider the sequence of states the DFA enters while processing x:

$$q_0, q_1, q_2, \ldots, q_{|x|}$$

where q_0 is the start state and $q_{|x|} \in F$. This sequence has $|x| + 1 \geq n + 1$ states but only n distinct possible states, so by the pigeonhole principle there exist indices $0 \leq i < j \leq |x|$ such that

$$q_i = q_j.$$

Let the input x be written as $x = abc$, where:

- a is the prefix taking q_0 to q_i,
- b is the substring read from steps $i + 1$ to j,
- c is the remainder of the input.

Since $q_i = q_j$, reading b from q_i returns to the same state, so b labels a directed cycle at q_i. Moreover, since x is accepted, reading c from q_i eventually reaches some accepting state. Thus:

- the cycle is reachable from q_0 (via a), and
- there is a path to an accepting state (via c).

Hence a reachable and co-reachable cycle exists.

Conclusion

We have shown both directions, so we conclude that:

$L(M)$ is infinite $\iff$ the transition graph of M contains a reachable and co-reachable directe