Daily Problem: 24–Jan–2026

Problem Statement

Given a regular language L specified by a DFA, construct a DFA for the reverse language $\mathrm{rev}(L)$. Then show that there exists a family of languages L_n for which:

- L_n has a DFA of size $O(n)$, but
- any DFA recognizing $\mathrm{rev}(L_n)$ must have size at least 2^n.

Part 1: Constructing a DFA for the Reversed Language

Suppose we are given a DFA

$$M = (Q, \Sigma, \delta, q_0, F)$$

recognizing L. Our goal is to construct a DFA that recognizes $\mathrm{rev}(L)$.

Reversing transitions of a DFA turns it into an NFA with multiple initial states. We then convert that NFA into an equivalent DFA using subset construction.

Construction

Define an NFA $N = (Q, \Sigma, \delta_N, I_N, F_N)$ by:

- The set of states is Q,
- The initial states are all former accepting states: $I_N = F$,

- The accepting states are the original start state: $F_N = \{q_0\}$,
- The transition function is reversed: for all $p, q \in Q$ and $a \in \Sigma$,

$$q \in \delta_N(p, a) \iff \delta(q, a) = p.$$

It is well-known that N recognizes $\mathrm{rev}(L)$.

Now apply the subset construction to convert N into an equivalent DFA

$$D = (2^Q, \Sigma, \delta_D, I_D, F_D),$$

where:

- $I_D = I_N = F$,
- For any subset $S \subseteq Q$ and $a \in \Sigma$,

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a),$$

- $F_D = \{S \subseteq Q \mid q_0 \in S\}$.

Correctness Argument

Any string $w \in \Sigma^*$ is accepted by D iff there exists a path in N from some $f \in F$ to q_0 labeled w, which happens iff there is a path from q_0 to some $f \in F$ labeled $\mathrm{rev}(w)$, which is equivalent to $\mathrm{rev}(w) \in L$. Hence D recognizes $\mathrm{rev}(L)$.

Conclusion for Part 1

Given a DFA for L, we can effectively construct a DFA for $\mathrm{rev}(L)$ by reversing transitions, swapping initial/accepting roles, and applying subset construction.

Part 2: Exponential Blowup in the Reversal

We now prove that reversal can cause an exponential increase in DFA size.

Language Family Definition

Let $\Sigma = \{0, 1\}$. Define for each $n \in \mathbb{N}$ the language:

$$L_n = \{w \in \Sigma^* \mid \text{the } n\text{-th symbol from the right of } w \text{ is } 1\}.$$

From a previous problem it is known that:

- L_n has a DFA of size $O(n)$,
- Any DFA for L_n requires at most $O(n)$ states.

We now study the reversal.

Understanding $\mathrm{rev}(L_n)$

A string w belongs to L_n iff we can write $w = x1y$ with $|y| = n - 1$. Thus for the reversed string $\mathrm{rev}(w)$, this becomes:

$$\mathrm{rev}(w) = \mathrm{rev}(y)1\,\mathrm{rev}(x),$$

and so the condition becomes:

$$\mathrm{rev}(w) \in \mathrm{rev}(L_n) \iff \text{the } n\text{-th symbol from the } \textit{left} \text{ of } \mathrm{rev}(w) \text{ is } 1.$$

Therefore:

$$\mathrm{rev}(L_n) = \{w \in \Sigma^* \mid |w| \geq n \text{ and the } n\text{-th symbol from the left is } 1\}.$$

So membership depends on exactly which prefix of length n was read.

Lower Bound on DFA Size for $\mathrm{rev}(L_n)$

Consider all binary strings $u \in \Sigma^n$. There are 2^n of them.

Claim. Any DFA recognizing $\mathrm{rev}(L_n)$ must have at least 2^n states.

Proof. Suppose a DFA D recognizes $\mathrm{rev}(L_n)$. For any $u \in \Sigma^n$, consider the state reached after reading u from the start state. We show that different strings must lead to different states.

Let $u \neq v \in \Sigma^n$. Then u and v differ at some position. Without loss of generality, suppose that at the earliest differing position i, $u[i] = 1$ and

$v[i] = 0$. Consider any continuation $z \in \Sigma^*$ (possibly empty). The string uz belongs to $\text{rev}(L_n)$ if and only if the n-th symbol from the left in u is 1, i.e. $u[1]$, while vz is not in $\text{rev}(L_n)$. Hence the DFA must distinguish u and v, which means their target states must differ.

Since all 2^n strings of length n must reach distinct states, the DFA needs at least 2^n states.

Conclusion for Part 2

We have exhibited a family $\{L_n\}_{n\in\mathbb{N}}$ such that:

$$|DFA(L_n)| = O(n) \quad \text{but} \quad |DFA(\text{rev}(L_n))| \geq 2^n.$$

Final Conclusion

We showed:

1. Reversal of a DFA can be computed systematically by reversing transitions, swapping initial/accepting roles, and determinizing.

2. Reversal can cause an exponential blowup in the size of the smallest DFA.

3. Specifically, for the family L_n described above, the minimal DFA for L_n has size $O(n)$ while the minimal DFA for $\text{rev}(L_n)$ has size at least 2^n.

This demonstrates that language reversal is regularity-preserving but not size-preserving for deterministic automata.