Lab 7

Name: Yuvraj Singh
Roll No: 23072021
M.Tech CSE

Problem: Consider a bin1 full of n red, n green and n yellow balls and another empty bin2. Now x balls are uniformly chosen from bin1 and placed in bin2.What is the probability that bin 2 now has at least y green balls. Write a program that takes n,x,y as inputs and outputs the upperbounds of the probabilty according to Markov, Cheybyshev and Chernoff (with different choice of Δ)

Solution:

Markov's Inequality: Markov's inequality provides an upper bound on the probability that a non-negative random variable X is greater than or equal to a specified positive constant. It states that for any non-negative random variable X and any $a > 0$, the probability that X is at least a times its mean $E[X]$ is at most the reciprocal of a.

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Chebyshev's Inequality: Chebyshev's inequality provides an upper bound on the probability that a random variable X deviates from its mean by more than a specified number of standard deviations. It states that for any random variable X with finite mean μ and finite non-zero variance σ^2, and any $k > 0$, the probability that X deviates from its mean by more than k standard deviations is at most $1/k^2$.

$$P(|X - \mu| \geq k) \leq \frac{\sigma}{k^2}$$

Chernoff Bounds: Chernoff bounds offer exponential tail bounds on the sum of independent random variables. In our case, we can utilize Chernoff bounds to establish an upper limit on the likelihood that the number of green balls in bin2 surpasses a predetermined threshold

$$P(X \geq (1+\delta)E[X]) \leq \frac{e^{-\delta E[X]}}{(1+\delta)^{(1+\delta)E[X]}}$$

Example: Let X be a random variable representing the outcome of rolling a fair six-sided die. The possible outcomes are $\{1, 2, 3, 4, 5, 6\}$, each with probability $\frac{1}{6}$.

$$E[X] = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5$$

$$E[X^2] = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$$

$$\text{Var}[X] = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \approx 2.9167$$

Markov's Inequality

$$P(X \geq 10) \leq \frac{E[X]}{10} = \frac{3.5}{10} = 0.35$$

Chebyshev's Inequality

$$P(|X - 3.5| \geq 10) \leq \frac{\text{Var}[X]}{10^2} = \frac{2.9167}{10^2} \approx 0.029167$$

Chernoff Bounds

$$P(X \geq (1+\delta)E[X]) \leq \frac{e^{-\delta E[X]}}{(1+\delta)^{(1+\delta)E[X]}}$$

Using $\delta = 1$:

$$P(X \geq 7) \leq \frac{e^{-3.5}}{(1+1)^{2*3.5}} = \frac{e^{-3.5}}{2^7} \approx 0.000235917$$

Implementation in C++

```cpp
#include <iostream>
#include <cmath>
using namespace std;


class Bounds {
public:
    double markovBound(int n, int x, int y) {
        double mean = static_cast<double>(x * n) / 3;
        return static_cast<double>(n) / (y * mean);
    }
    double chebyshevBound(int n, int x, int y) {
        double variance = static_cast<double>(x * n) * (2 * n) / 9;
        double mean = static_cast<double>(x * n) / 3;
        double stdDev = sqrt(variance);
        double k = (mean - static_cast<double>(y * n)) / stdDev;
        return 1 / (1 + pow(k, 2));
    }

    double chernoffBound(int n, int x, int y, double delta) {
        double mean = static_cast<double>(x * n) / 3;
        double k = (mean - static_cast<double>(y * n)) / mean;
        return exp(-k * delta) / pow(1 + delta, (1 + delta)*mean);
    }
};

int main() {
    int n, x, y;
    double delta;

    cout << "Enter the value of n, x, y: ";
    cin >> n >> x >> y;

    cout << "Enter the value of delta (for Chernoff bound): ";
    cin >> delta;
```

```

    Bounds bounds;

    cout<<"Markov's Upper Bound: "<<bounds.markovBound(n, x, y)<<endl;
    cout<<"Chebyshev's Upper Bound: "<<bounds.chebyshevBound(n, x, y)<<endl;
    cout<<"Chernoff's Upper Bound with delta = "<<delta<<": "<<bounds.cherno

    return 0;
}
```

Time Complexity: $O(1)$, given that we already know $E[X]$ and $Var[X]$

Output:

```
Enter the value of n, x, y: 10 5 3
Enter the value of delta (for Chernoff bound): 0.1
Markov's Upper Bound: 5.55556
Chebyshev's Upper Bound: 12.3457
Chernoff's Upper Bound with delta = 0.1: 0.0329086
```