

# Rejection Sampling

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## Rejection Sampling Method

It's always not easy to withdraw samples from posterior  $\pi(\theta|x)$  distribution. Most of the time are not familiar with the functional form of the posterior distribution.

Suppose we want to take a sample from the posterior distribution  $\pi(\theta|x)$ . Then we will find another probability distribution  $p(\theta)$  which has the following properties

1. Easy to withdraw samples from
2. Resembles the posterior distribution
3. For all parameter  $\theta$  and a constant  $k$ ,  $\pi(\theta|x) \leq kp(\theta)$

## Algorithm

1. Take a sample from the distribution  $p(\theta)$  and a Uniform Random Variable  $U$
2. If  $U < \frac{\pi(\theta|x)}{k \cdot p(\theta)}$  then accept the sample
3. If  $U > \frac{\pi(\theta|x)}{k \cdot p(\theta)}$  then reject the sample

The Performance of the Rejection Sampling Method is measured by Acceptance Rate.

## Example

Suppose we want to withdraw samples from normal distribution with mean  $\mu$  and variance  $\sigma$ , which is equivalent to get samples from standard Normal distribution, because we just have to do a simple linear transformation to get a distribution with mean  $\mu$  and variance  $\sigma$ . So we will be using the standard Normal distribution

Now we are taking proposal density or in some literature mentioned as candidate density  $p(\theta)$  as an exponential distribution with mean 1, while we know that exponential random variable is always positive and hence we will be taking the absolute value of the Standard Normal random variable, and then multiply it by -1 by generating a uniform random variable  $U$ . Whenever  $U$  is less than 0.5

$$p(\theta) = e^{-\theta} \pi(\theta|x) = \frac{2}{2\pi} e^{-\frac{1}{2}(\theta)^2} 1_{x \geq 0}$$

Then

$\frac{\pi(\theta|x)}{p(\theta)}$  is the ratio of the posterior distribution and the candidate density. It will be at maximum at  $\theta = 1$  thus  $k = \sqrt{2e/\pi} \approx 1.32$

Then steps for generating samples from the posterior distribution are as follows:

1. Take a sample from exponential distribution with mean 1 and a uniform random variable  $U$
2. If  $U \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}$  i.e.  $U \leq e^{-(1-\theta)^2/2}$  then accept the sample
3. Generate another uniform random variable  $U$ , if  $U$  is less than 0.5 then multiply the sample by -1

```

nsample = 10000                                # number of samples
sample = c()                                    # empty vector to store samples
count = 0                                       # count of samples accepted
while(length(sample) < nsample){                # loop until we have nsample samples
  U = runif(1)                                  # generate a uniform random variable
  count = count + 1                             # increment count
  theta = rexp(1)                               # generate a random variable from exponential distribution
  U2 = runif(1)                                 # generate a uniform random variable
  if(U <= exp((-1-theta)^2)/2)){ # if U is less than the ratio of the posterior distribution and the c
    if(U2 <= 0.5){                             # if U2 is less than 0.5 then multiply the sample by -1
      theta = -theta                           # multiply the sample by -1
    }

    sample = c(sample, theta)                  # add the sample to the vector
  }
}
cat("Acceptance Rate: ", count/nsample)

## Acceptance Rate:  1.3285

plot(density(sample))

```

