Computing Integrals Monte Carlo Way

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Rejection Sampling Method

Its always not easy to withdraw samples from posterior $\pi(\theta|x)$ distribution.Most of the time are not familiar with the functional form of the posterior distribution.

Supoose we wanna take a sample from the posterior distribution $\pi(\theta|x)$ Then we will find another probability distribution $p(\theta)$ which have the following properties

1. Easy to withdraw samples from
2. Resembles the posterior distribution
3. For all parameter θ and a constant k , $\pi(\theta|x) \leq kp(\theta)$

Algorithm

1. Take a sample from the from the distribution $p(\theta)$ and a Uniform Random Variable U
2. If $U < \frac{\pi(\theta|x)}{k \cdot p(\theta)}$ then accept the sample
3. If $U > \frac{\pi(\theta|x)}{k \cdot p(\theta)}$ then reject the sample

The Performance of the Rejection Sampling Method is measured by Acceptance Rate.

Example

Suppose we want to withdraw samples from normal distribution with mean μ and variance σ,which equivalent to get samples from standard Normal distribution, because we just have to do a simple linear transformation to get a distribution with mean μ and variance σ.So we will be using the standard Normal distribution

Now we are taking proposaldensity or in some literature mentioned as candidate density $p(\theta)$ as an exponential distribution with mean 1, while we know that exponential random variable is always positive and hence we will be taking the absolute value of the Standard Normal random variable, and then multiply iy by -1 by generating a uniform random variable U.Whenever U is less than 0.5

$$p(\theta) = e^{-\theta}\pi(\theta|x) = \frac{2}{2\pi}e^{-\frac{1}{2}(\theta)^2}1_{x\geq 0}$$

Then

$\frac{\pi(\theta|x)}{p(\theta)}$ is the ratio of the posterior distribution and the candidate density.It will be at maximum at $\theta = 1$ thus k = $\sqrt{2e/\pi} \approx 1.32$

Then steps for generating samples from the posterior distribution are as follows 1. Take a sample from exponential distribution with mean 1 and a uniform random variable U 2. If $U \leq \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{\theta^2}{2}}}{\sqrt{2e/\pi}e^{-\theta}}$ $i.e$ $U \leq e^{-(1-\theta)^2/2}$ then accept the sample 3. Generate another uniform random variable U, if U is less than 0.5 then multiply the sample by -1

```r
nsample = 10000
sample = c()
count = 0
while(length(sample) < nsample){
  U = runif(1)
  count = count + 1
  theta = rexp(1)
  U2 = runif(1)
  if(U <= exp((-(1-theta)^2)/2)){
    if(U2 <= 0.5){
      theta = -theta
    }

    sample = c(sample, theta)
  }
}
cat("Acceptance Rate: ", count/nsample)
```

```
## Acceptance Rate:  1.3135
```

```r
plot(density(sample))
```