Lecture 1

Number Sets

Wednesday, Jan. 15

In This Lecture...

- Defining sets using curly braces!
- Introducing different types of numbers!
- Stumbling into a weird paradox!

Definition 1.0: Sets

A **set** is an unordered collection of objects, called **elements** .

The notation $x \in A$ means x is an element of the set A .

The notation $x \notin A$ means x is not an element of the set A .

Sets are generally denoted by uppercase letters: A, B, C, X, Y, etc.

There are two main forms of notation for describing the elements of a set:

- **Roster notation**: Write a list of the elements of the set between curly braces $\{\}$

  Examples: $\{1, 2, 7\}$, $\{\pi, \text{Wednesday}, \text{banana}\}$, $\{0, 1, 2, 3, \dots\}$, $\{\{a, b\}, \{c, d, e\}\}$

- **Set-builder notation**: Define a set as the set of all objects satisfying a certain property.

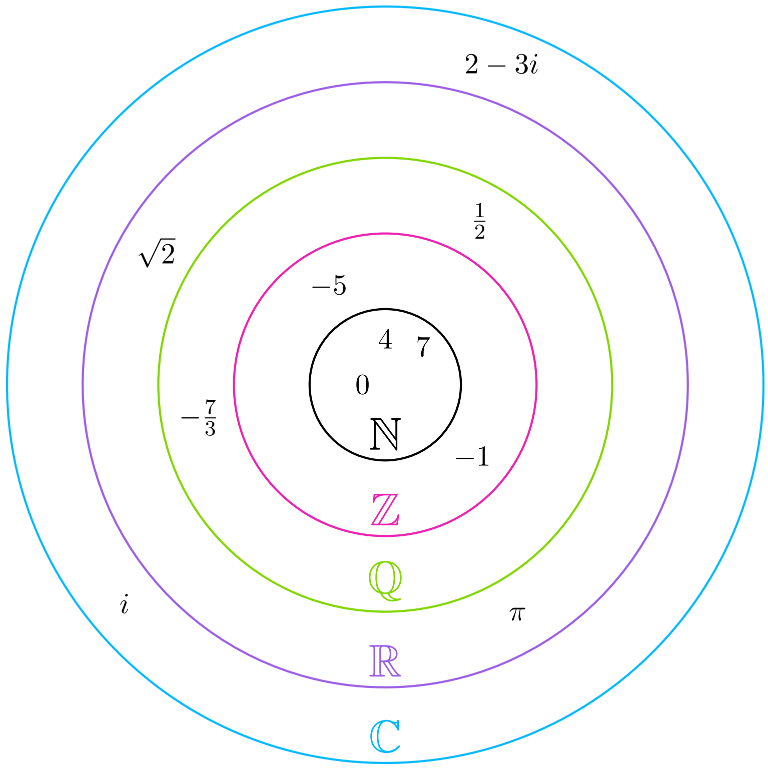
  Examples: $\{2x \mid x \text{ is an integer}\}$, $\{x \mid x \text{ is a day of the week}\}$, $\{x^2 \mid x \in \{1, 2, 7\}\}$

Definition 1.1: Number Sets

- **Natural numbers**: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- **Integers**: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- **Rational numbers**: $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ with } q \neq 0\}$
- **Real numbers**: $\mathbb{R} =$ all points on the number line
- **Complex numbers**: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

For a generic natural number or integer, the letters i, j, k, m, and especially n are customarily used; for a real number, x and y are common; for a complex number, z is the typical choice.

Relationship between number sets:



Definition 1.2: Intervals

Let $a, b \in \mathbb{R}$ with $a < b$. **Intervals** are specific sets of real numbers between a and b.

- Open interval: $(a, b) =$ $\{x \in \mathbb{R} \mid a < x < b\}$
- Closed interval: $[a, b] =$ $\{x \in \mathbb{R} \mid a \leq x \leq b\}$

Variations on this notation can be used to denote intervals which are open on one end and closed on the other, or unbounded intervals.

1 Write each set of real numbers as an interval, if possible.

(a) $\{x \in \mathbb{R} \mid 0 < x \leq 4\}$

Solution

We write this *half-open* interval as $(0, 4]$, using a hybrid of the open and closed interval notation.

(b) $\{x \in \mathbb{R} \mid x \leq 4\}$

Solution

To indicate that the interval has no lower bound, we write it as $(-\infty, 4]$. Note that we always use a parenthesis on the infinite end of the interval.

(c) $\{x \in \mathbb{R} \mid x^2 \leq 4\}$

Solution

For x^2 to be less than or equal to 4, x itself must be between -2 and 2 (inclusive). The interval of all possible values of x is $[-2, 2]$.

(d) $\{x \in \mathbb{R} \mid x^2 \geq 4\}$

Solution

For x^2 to be greater than or equal to 4, we must have either $x \leq -2$ or $x \geq 2$. These inequalities correspond to the intervals $(-\infty, -2]$ and $[2, \infty)$. As x can be in either interval, there is no way to write the set as a single interval.

(e) $\mathbb{R}$

Solution

The set $\mathbb{R}$ is unbounded in both the positive and negative directions, so we write it as $(-\infty, \infty)$ in interval notation.

2 Let S be the set of all sets, and let $R = \{X \in S \mid X \notin X\}$. Is $R \in R$?

Solution

The definition of the set R says that it's the collection of all sets X that do not contain themselves.

Suppose $R \in R$. Then by definition of R, it cannot contain itself, but this contradicts our assumption that $R \in R$.

Similarly, if we start by assuming that $R \notin R$ instead, then by definition of R, we have $R \in R$, which is another contradiction.

The assumptions $R \in R$ or $R \notin R$ both lead to contradictions, so neither one can be true! This is known as *Russell's Paradox*. The only way to resolve the paradox is to decide that objects like R and S are not valid sets. (It's possible to formulate the definition of a set more carefully than we did, using very specific rules to ensure that such sets are not allowed!)