18-100 Introduction to Electrical and Computer Engineering

Lecture 16
Complex Numbers, Phasors, and Impedances

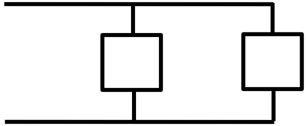
5-Feb	**W**	**Exam 1**	Pause for exam		N/A
10-Feb	M	L07: Capacitors, RC Time Constants, RC Circuits			Mark
12-Feb	W	L08: Inductors, RL Time Constants, 555	Lab3 : MOSFETs		Mark
17-Feb	M	L09: Binary, Logic Gates, Boolean Logic			Greg
19-Feb	W	L10: Latches, Registers, RAM, Flip-Flops	Lab4: Timer Lab		Greg
24-Feb	M	L11: Computers			Greg
26-Feb	W	L12: Op Amps	Lab5: Op Amps		Mark
3-Mar	M	SPRING BREAK			
5-Mar	W	SPRING BREAK	Pause for break		
10-Mar	M	L13: Arduino Programming Case Study			Greg
12-Mar	W	L14: Serial Communication Protocols	Lab 6: I2C		Greg
17-Mar	M	L15: Analog-to-Digital (ADC) and Digital-to-Analog (DAC) Conversion			Greg
19-Mar	W	L16: Complex Numbers, Phasors, and Impedance	Lab7: ADC		Mark
24-Mar	M	L17: Analog Filters, LC Circuits , Resonance			Mark
26-Mar	W	L18: Review/Exam Preview	Pause for exam		Greg
31-Mar	**M**	**Exam 2**			
2-Apr	W	L19: Time Varying Signals and Spectra (Trig)	Pause for Carnival		Mark
7-Apr	M	L20: Wireless Communication: Modulation to Protocols			Mark
9-Apr	W	L21: Crypto	Lab 8: Radio out		Greg
14-Apr	M	L22: IoT and Cloud			Greg
16-Apr	W	L23: Information Theory and Data Compression	Lab 9: Crypto out		Greg
21-Apr	M	L24: AI and ML			Greg
23-Apr	W	L25: Course wrap up	Crypto Due		Greg and Mark
Exams	**Period**	**Exam 3 (Scheduled by Registrar during the Final Exams Period)**			

Objectives of this Lecture

- Capacitors and Inductors
- Complex Numbers
- Great Mysteries of the Universe Unveiled
- Impedance
- Everything Old Is New Again

Inductors and Capacitors Are Opposites….

	Inductors	Capacitors
		
Cannot Change Instantly	Current $i_L\left(t=0^-\right)=i_L\left(t=0^+\right)$	Voltage $v_C\left(t=0^-\right)=v_C\left(t=0^+\right)$
Initially, When Current First Starts to Flow, It Looks Like a ….	Open 	Short 
Eventually, It Gets Fully Charged Up, and It Looks Like a ….	Short 	Open 

	Inductors	Capacitors
		
Series Equivalence 	$L_{eq} = L_1 + L_2 + \ldots. + L_N$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots. + \frac{1}{C_N}$
Parallel Equivalence 	$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots. + \frac{1}{L_N}$	$C_{eq} = C_1 + C_2 + \ldots. + C_N$
Time Constant	$\tau = L/R$	$\tau = RC$
"Ohm-ish" Law	$v_L = L\frac{di}{dt}$	$i_C = C\frac{dv}{dt}$

Budnik's Home Cooking Recipes for Solving **RC** Circuits

For Capacitors (No Sources)

1. You are given $v_C(t=0^-)$
2. Capacitor voltage cannot change instantly: $v_C(t=0^+) = v_C(t=0^-) = V_0$
3. If necessary, find the current $i_C(t=0^+)$ based on V_0
4. Without an external source: $i_C(t=\infty) = 0\text{A}$ and $v_C(t=\infty) = 0\text{V}$
5. Find time constant: $\tau = RC$
6. Both the current and voltage equations will have the form of: $i_C(t)=I_0 e^{-t/\tau}$ and $v_C(t)=V_0 e^{-t/\tau}$

For Capacitors (Constant Sources)

1. Calculate $v_C(t=0^-)$ as if capacitor is an open circuit
2. Capacitor voltage cannot change instantly: $v_C(t=0^+) = v_C(t=0^-) = V_0$
3. Eventually, capacitor looks like an open again, find $v_C(t=\infty)$
4. If necessary, find the current $i_C(t=0^+)$ and $i_C(t=\infty)$
5. To find the time constant, zero out all sources (0V and 0A): $\tau = RC$
6. Current and voltage each of the form: $v_C(t)=V_0 e^{-t/\tau}$ or $v_C(t)=(V_{HI}-V_{LO})(1 - e^{-t/\tau}) + V_{LO}$

Budnik's Home Cooking Recipes for Solving RL Circuits

For Inductors (No Sources)

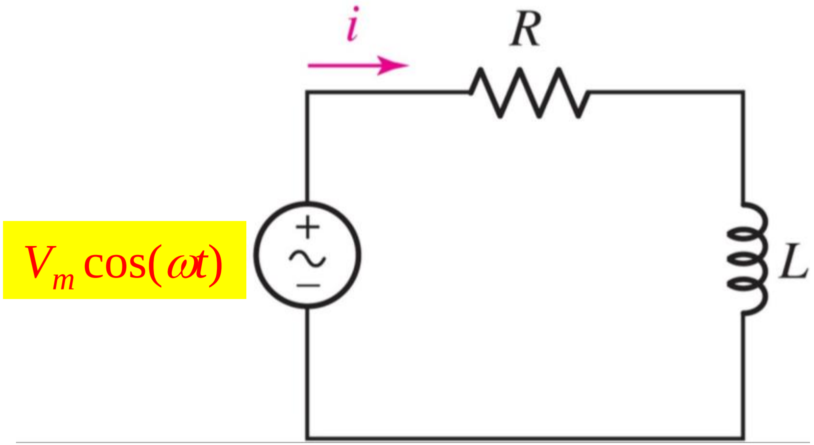
1. You are given $i_L(t=0^-)$
2. Inductor current cannot change instantly: $i_L(t=0^+) = i_L(t=0^-) = I_0$
3. If necessary, find the voltage $v_L(t=0^+)$ based on I_0
4. Without an external source: $i_L(t=\infty) = 0\text{A}$ and $v_L(t=\infty) = 0\text{V}$
5. Find time constant: $\tau = L/R$
6. Both the current and voltage equations will have the form of: $i_L(t)=I_0e^{-t/\tau}$ and $v_L(t)=V_0e^{-t/\tau}$

For Inductors (Constant Sources)

1. Calculate $i_L(t=0^-)$ as if inductor is a short circuit
2. Inductor current cannot change instantly: $i_L(t=0^+) = i_L(t=0^-) = I_0$
3. Eventually, inductor looks like a short again, find $i_L(t=\infty)$
4. If necessary, find the voltage $v_L(t=0^+)$ and $v_L(t=\infty)$
5. To find the time constant, zero out all sources (0V and 0A): $\tau = L/R$
6. Current and voltage each of the form: $i_L(t)=I_0e^{-t/\tau}$ or $i_L(t) = (I_{HI}-I_{LO})(1 - e^{-t/\tau}) + I_{LO}$

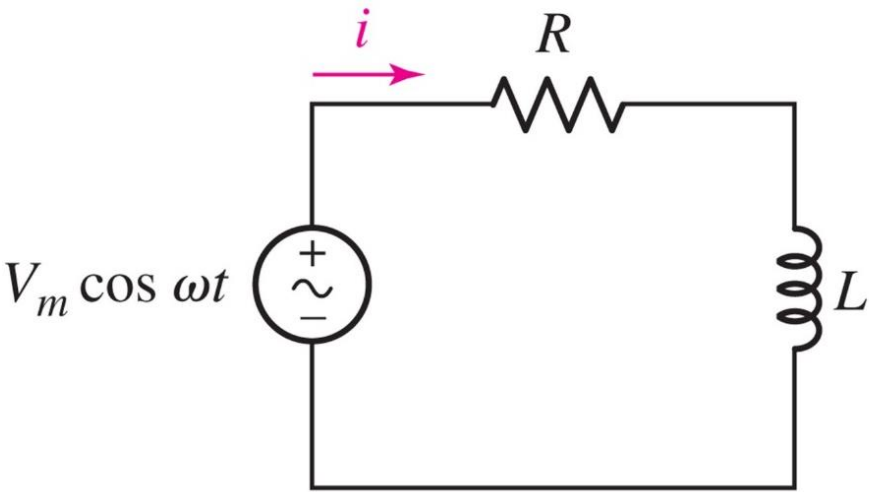
Steady State Response (NOT CONSTANT DC SOURCES)

- Ignore “start-up” and consider only the “steady-state” response



- The source is assumed to exist forever: $-\infty < t < +\infty$

Finding Steady-State Response (the hard way….)



1. Apply KVL:

$$V_m \cos(\omega t) = iR + L\frac{di}{dt}$$

2. Make a good guess:

$$i(t) = I_m \cos(\omega t + \theta)$$

3. Solve for the constants:

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{\omega L}{R}\right]\right)$$

Finding Steady-State Response (the hard way….)

1. Apply KVL:

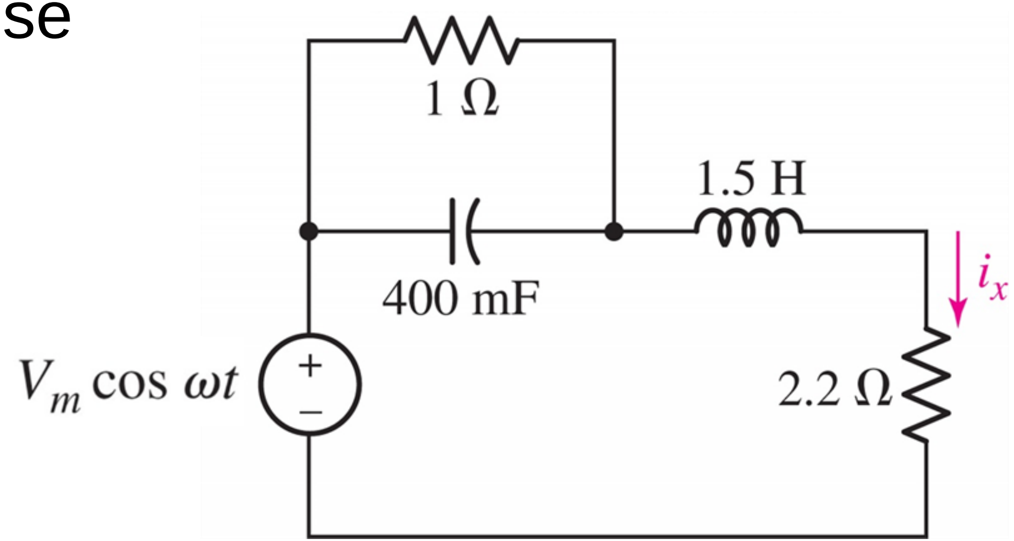
$$V_m \cos(\omega t) = iR + L\frac{di}{dt}$$

2. Make a good guess:

$$i(t) = I_m \cos(\omega t + \theta)$$

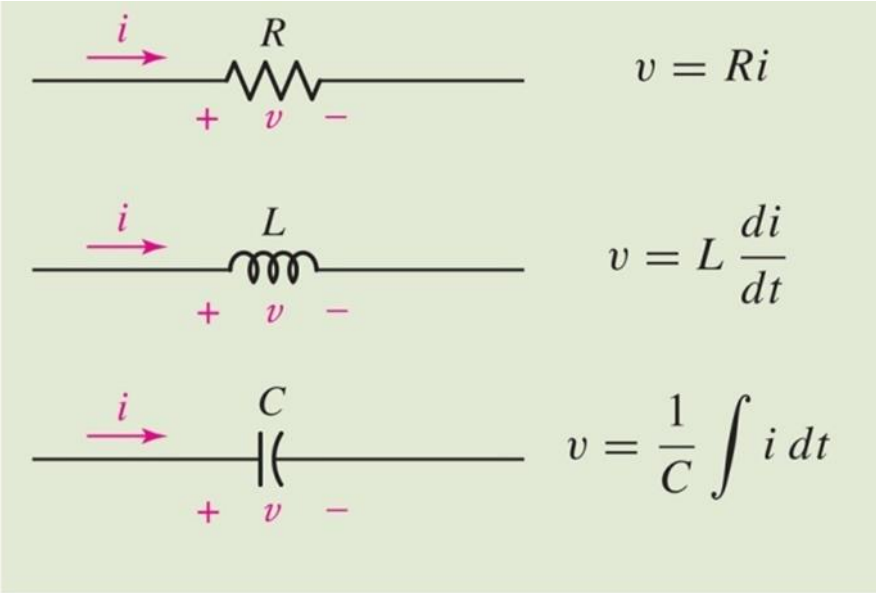
3. Solve for the constants:

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{\omega L}{R}\right]\right)$$



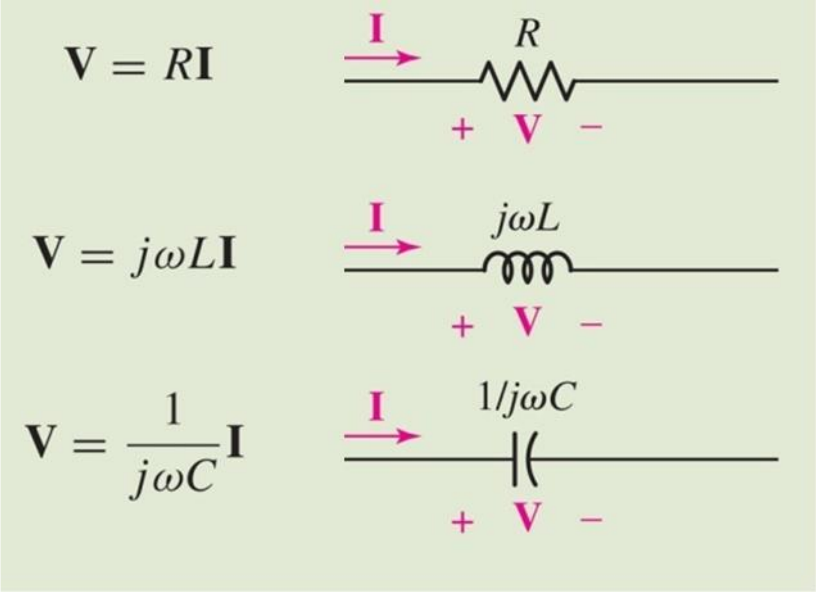
Time Domain vs. Frequency Domain

Time Domain



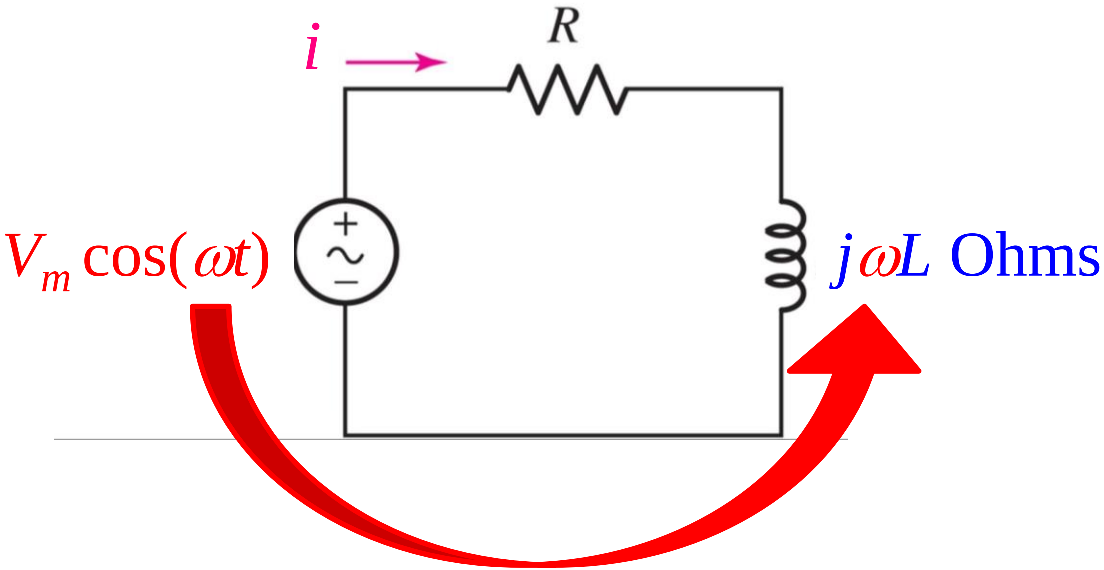
Uses “real” numbers, but requires calculus and/or differential equations

Frequency Domain



Uses “complex” numbers, but only requires algebra

Steady State Response (with Complex Numbers)

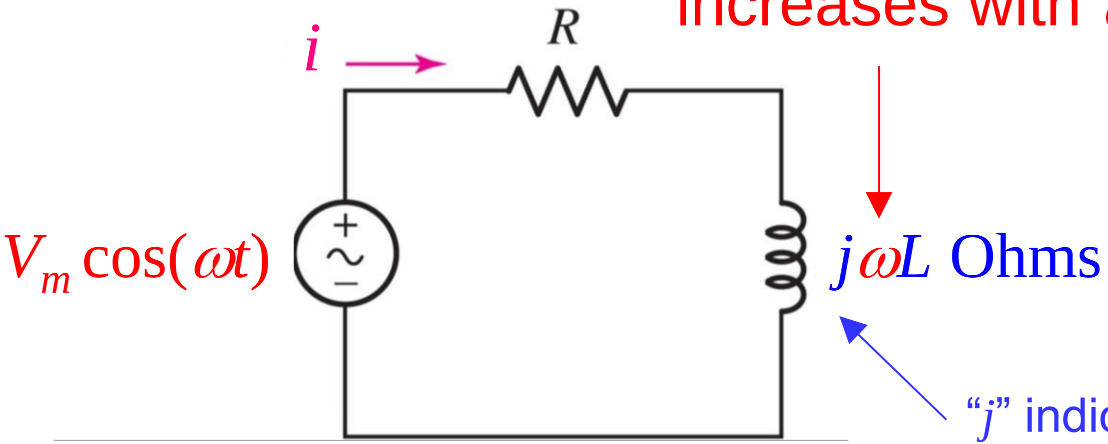


Inductors and capacitors
act like frequency dependent resistors

Steady State Response (with Complex Numbers)

Current through an inductor cannot change instantly

- Resistance to current flow increases with ω



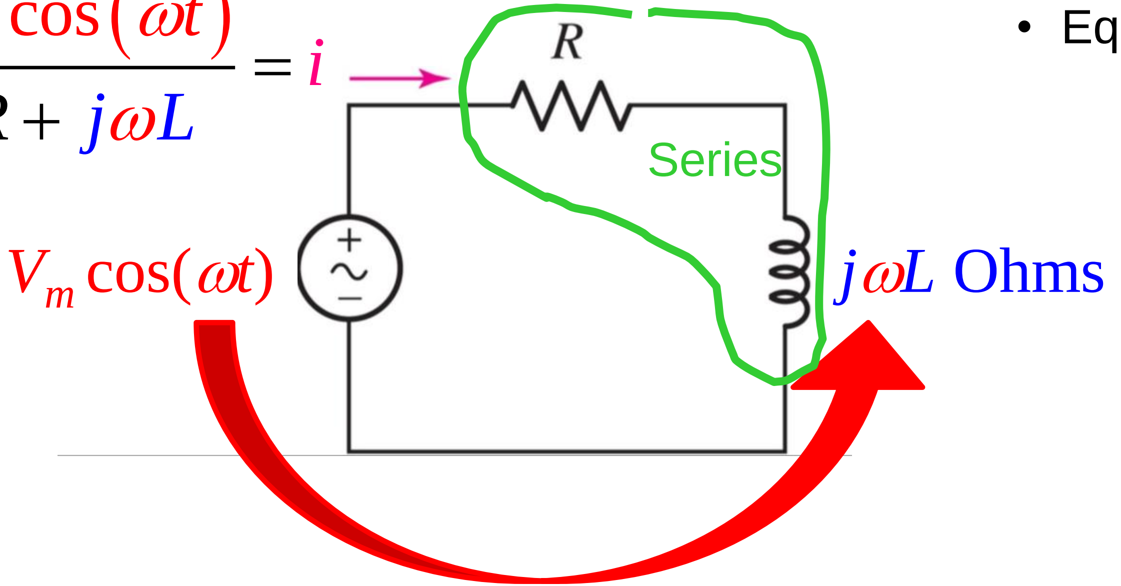
"j" indicates changes in the inductor current will lag behind the voltage changes

Inductors and capacitors act like frequency dependent resistors

Steady State Response (with Complex Numbers)

Use our models:
- Ohm's Law
- KCL
- KVL
- Equivalent circuits

$$\frac{V_m \cos(\omega t)}{R + j\omega L} = i$$



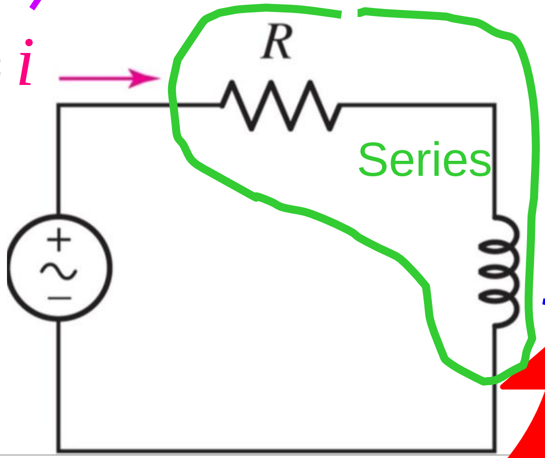
Inductors and capacitors
act like frequency dependent resistors

Steady State Response (with Complex Numbers)

Calculate current as a complex number, and transform back into a sinusoid!!!

Use our models:
- Ohm's Law
- KCL
- KVL
- Equivalent circuits

$$\frac{a+bj}{R+j\omega L} = i$$



$V_m \cos(\omega t)$

$j\omega L$ Ohms

Every sinusoid can be expressed as a complex number

Inductors and capacitors act like frequency dependent resistors

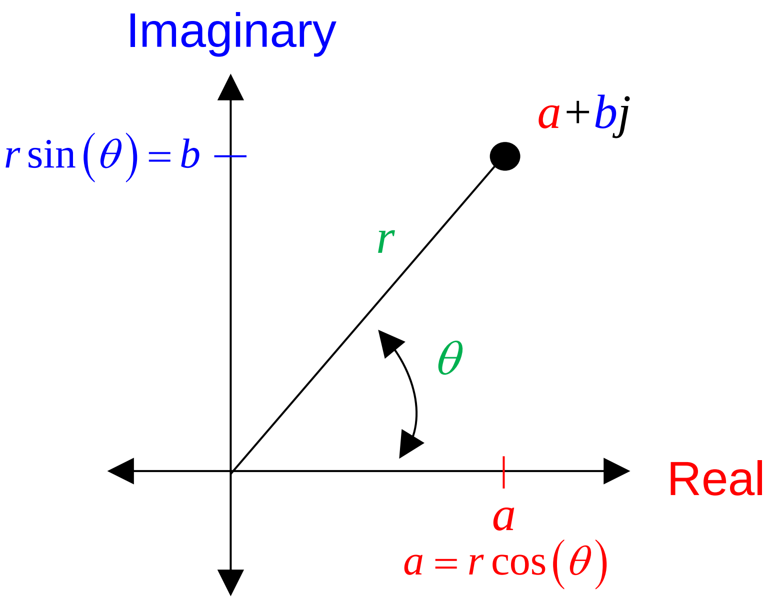
Complex Numbers Are Real!

- Cartesian form

$$z = a + b\sqrt{-1} = a + bj$$

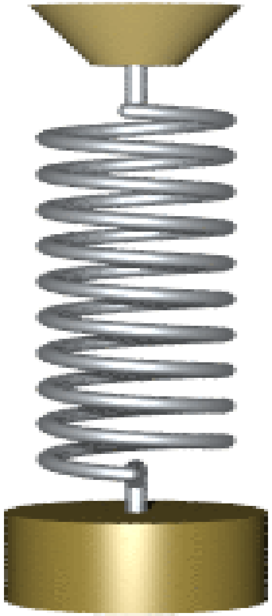
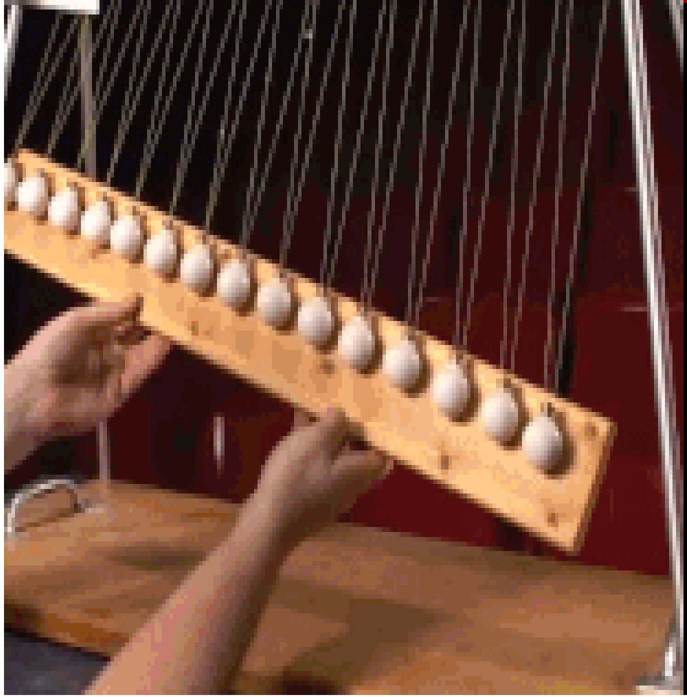
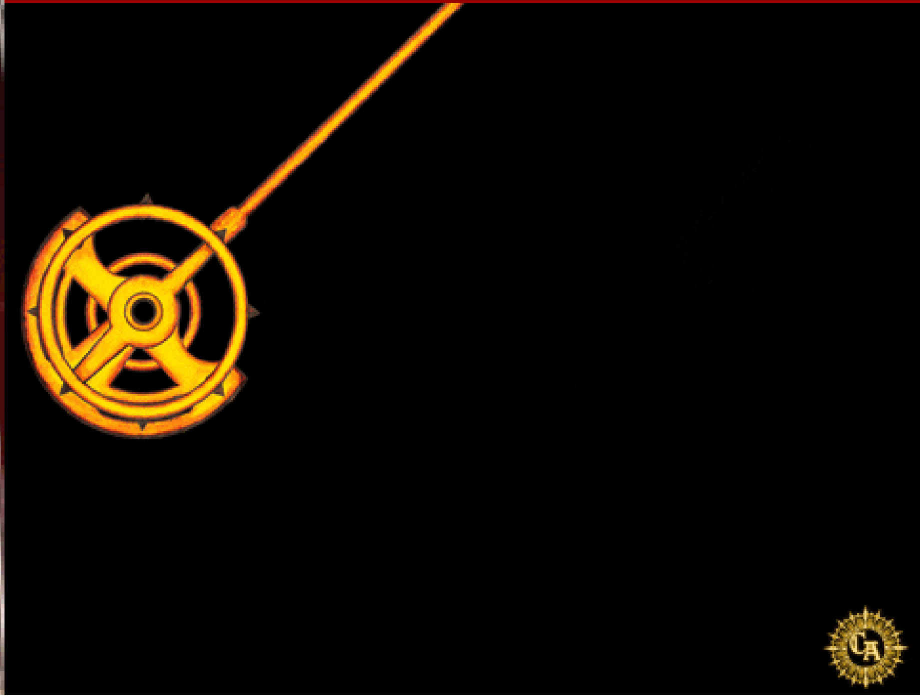
- Polar form

$$z = r\angle\theta$$



- Complex numbers occur all the time in the “real” world
- Complex numbers are just another way to “name” a sinusoidal signal

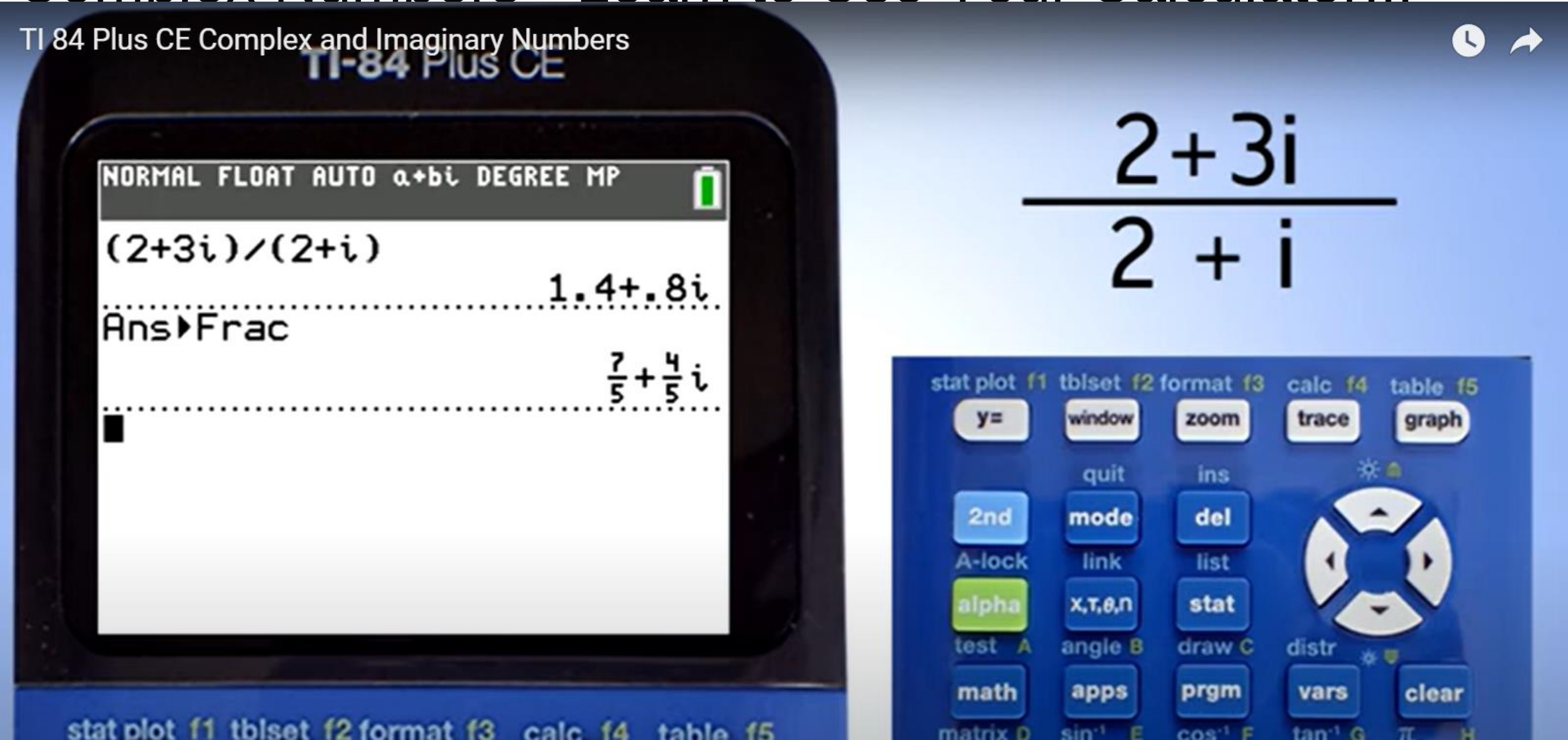
$$z = r\cos(\omega t + \theta)$$

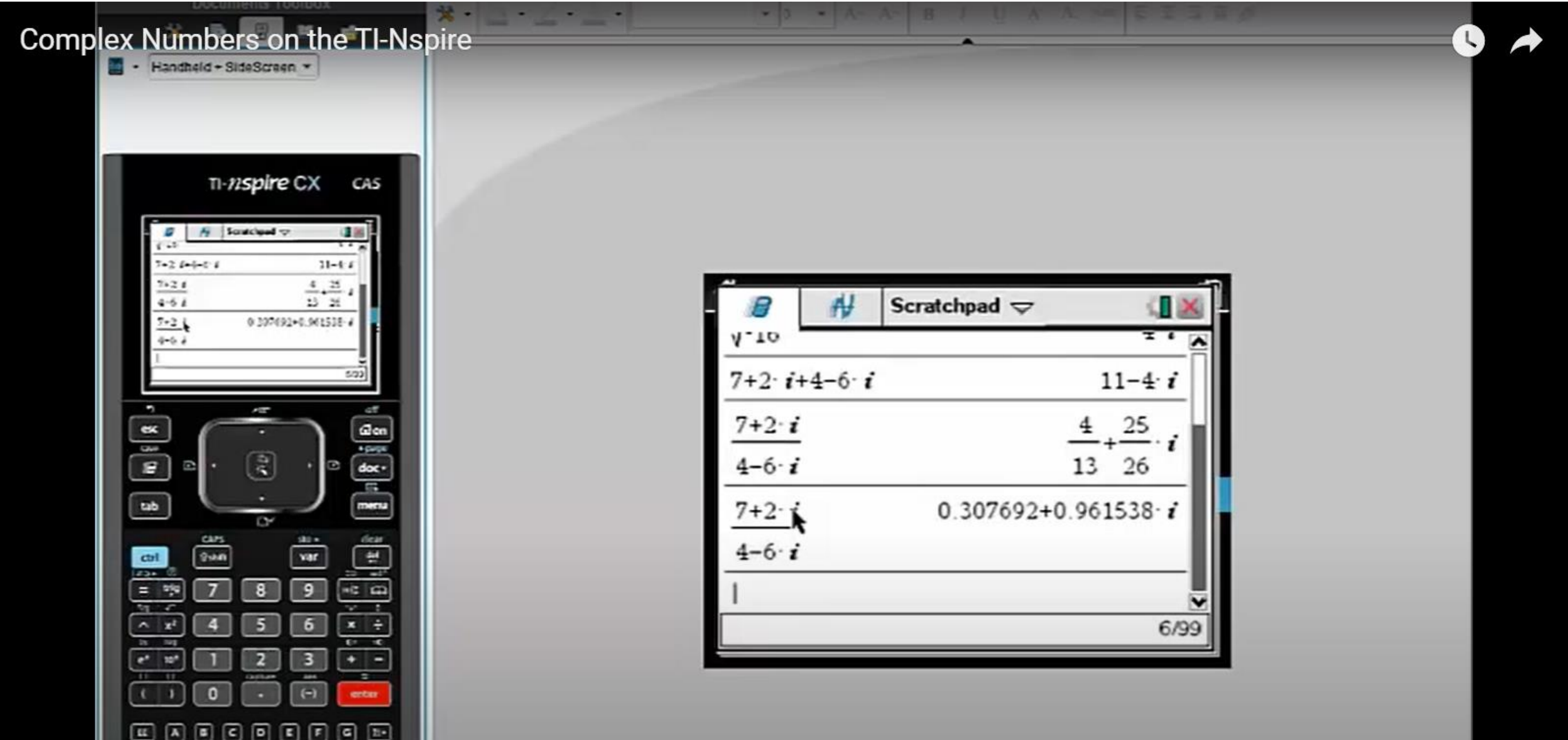
- Complex numbers occur all the time in the “real” world
- Complex numbers are just another way to “name” a sinusoidal signal

$$z = r\cos\left(\omega t + \theta\right) \quad = \quad r\angle\theta \quad = \quad a + bj$$

Complex Numbers - Learn to Use Your Calculator!!!



Complex Numbers - Learn to Use Your Calculator!!!



Google

ti 84 plus ce user manal pdf

All Images Videos Shopping Web Forums Books ⋮ More

These are results for ti 84 plus ce user ***manual*** pdf

Search instead for ti 84 plus ce user manal pdf

TI Education

https://education.ti.com › guidebook › details › ti-84-plu...

TI-84 Plus CE Guidebooks - Texas Instruments

TI-84 Plus CE Guidebooks. Download Item, **PDF**, eGuide, Version, Size (KB). Graphing Calculator Apps **Guide** for the **TI-84 Plus CE** (English) · View: · View, 5.7 ...

TI Education

https://education.ti.com › download › ed-tech › TI... PDF

Getting Started with the TI-84 Plus CE Graphing Calculator

Dec 23, 2024 — Use this **guide** to learn more about these features and other essential tools of your.

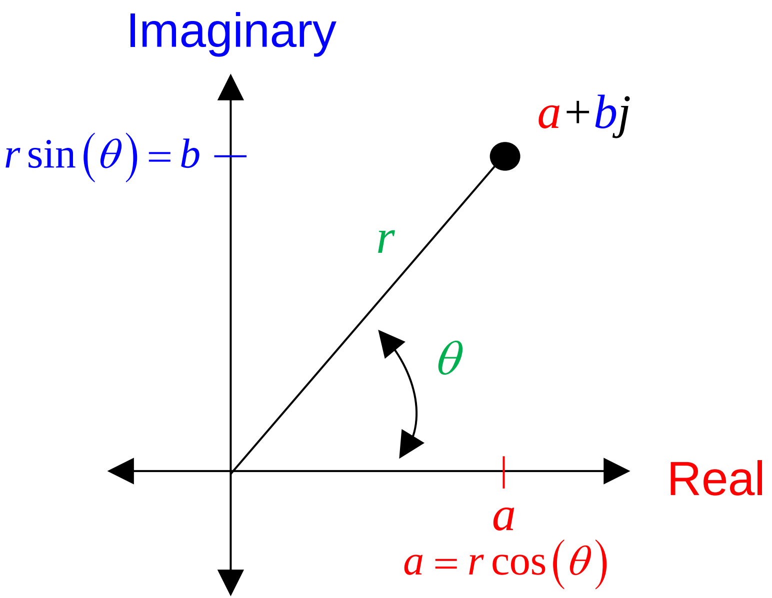
Complex Numbers Are Real!

- Cartesian form

$$z = a + b\sqrt{-1} = a + bj$$

- Polar form

$$z = r\angle\theta$$



- Complex numbers occur all the time in the “real” world
- Complex numbers are just another way to “name” a sinusoidal signal

$$z = r\cos(\omega t+\theta) \quad = \quad r\angle\theta \quad = \quad a+bj$$

Complex Numbers and Algebra 2

Khan Academy

- Intro to Imaginary Numbers (5:20)
- Simplifying roots of negative numbers (4:04)
- Intro to complex numbers (4:44)
- Classifying complex numbers (4:39)
- Plotting numbers on the complex plane (1:14)
- Adding complex numbers (1:11)
- Subtracting complex numbers (1:53)
- Multiplying complex numbers (5:32)
- Intro to complex number conjugates (8:04)
- Dividing complex numbers (4:58)

Addition and Subtraction of Complex Numbers (Combine Like Terms)

$$
\begin{aligned}
(1+2j)+(3+4j) &= (1+3)+(2j+4j) \\
&= 4+6j
\end{aligned}
$$

$$
\begin{aligned}
(1+2j)-(3+4j) &= (1-3)+(2j-4j) \\
&= -2-2j
\end{aligned}
$$

Multiplication of Complex Numbers (FOIL - First, Outer, Inner, Last)

$$(1+2j)(3+4j) \quad = \quad (1)(3)+(1)(4j)+(2j)(3)+(2j)(4j)$$

$$= \quad 3+4j+6j+8j^2$$

$$= \quad 3+10j+8\left(\sqrt{-1}\right)^2$$

$$= \quad 3+10j+8(-1) \quad = \quad -5+10j$$

Division of Complex Numbers

$$\frac{-1+5j}{2+3j} = \left(\frac{-1+5j}{2+3j}\right)\left(\frac{2-3j}{2-3j}\right) = \frac{(-1+5j)(2-3j)}{(2+3j)(2-3j)}$$

$$= \frac{(-1)(2)+(-1)(-3j)+(5j)(2)+(5j)(-3j)}{(2)(2)+(2)(-3j)+(3j)(2)+(3j)(-3j)}$$

$$= \frac{-2+3j+10j-15j^2}{4-6j+6j-9j^2}$$

$$= \frac{-2+13j-15(-1)}{4-9(-1)} = \frac{13+13j}{13} = 1+j$$

If you get it right, the denominator will be **real**

Phasors: Polar Form of Complex Numbers

Khan Academy

- Absolute value [magnitude] of complex numbers (3:33)
- Complex numbers with the same absolute value [magnitude] (2:52)
- Absolute value & angle of complex numbers (13:04)
- Polar & rectangular forms of complex numbers (12:16)
- Converting a complex number from polar to rectangular form (2:39)
- Multiplying complex numbers in polar form (2:27)
- Dividing complex numbers in polar form (3:10)
- Complex numbers for electrical and computer engineers (8:36)
- Euler's formula (8:42)

Converting from Cartesian to Phasor (Polar) Form

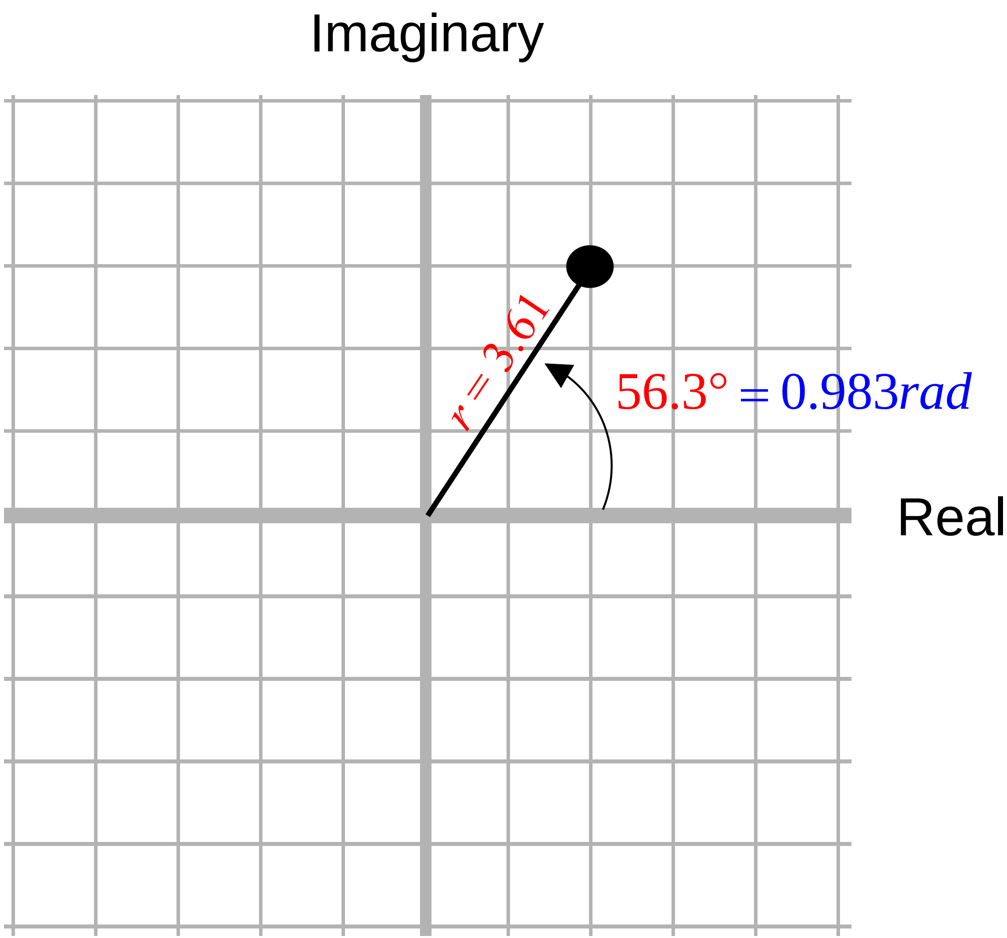
- $2 + j3 = 3.61 \angle 56.3°$

$$= 3.61e^{j0.983\text{rad}} = 3.61e^{j0.983}$$

$$r = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{3}{2}\right) = \tan^{-1}(1.5)$$

$$= 56.3° = 0.983rad$$



Converting from Cartesian to Phasor (Polar) Form

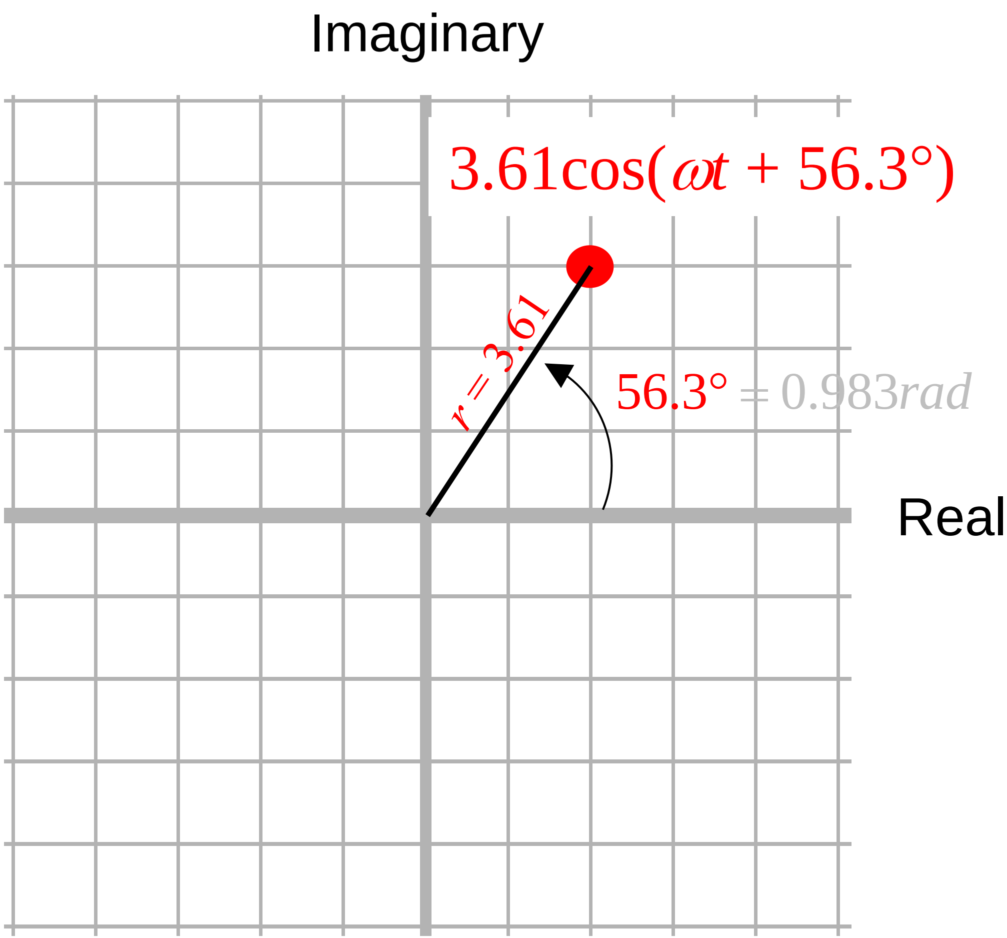
- $2 + j3 = 3.61 \angle 56.3°$

$$= 3.61e^{j0.983\text{rad}} = 3.61e^{j0.983}$$

$$r = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{3}{2}\right) = \tan^{-1}(1.5)$$

$$= 56.3° = 0.983rad$$

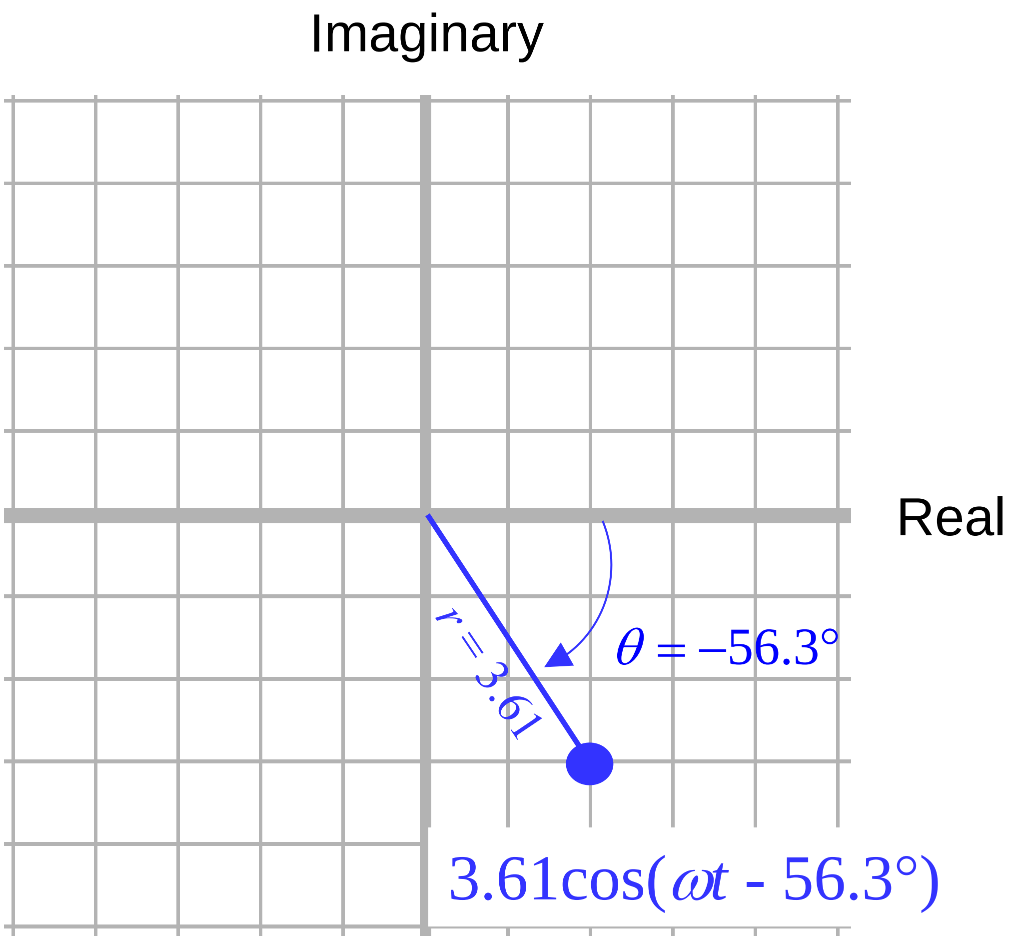


Converting from Cartesian to Phasor (Polar) Form

- $2 - j3 = 3.61 \angle -56.3°$

$$r = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (-3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{-3}{2}\right) = \tan^{-1}(-1.5) = -56.3°$$



Converting from Cartesian to Phasor (Polar) Form

- $2 - j3 = 3.61 \angle -56.3°$

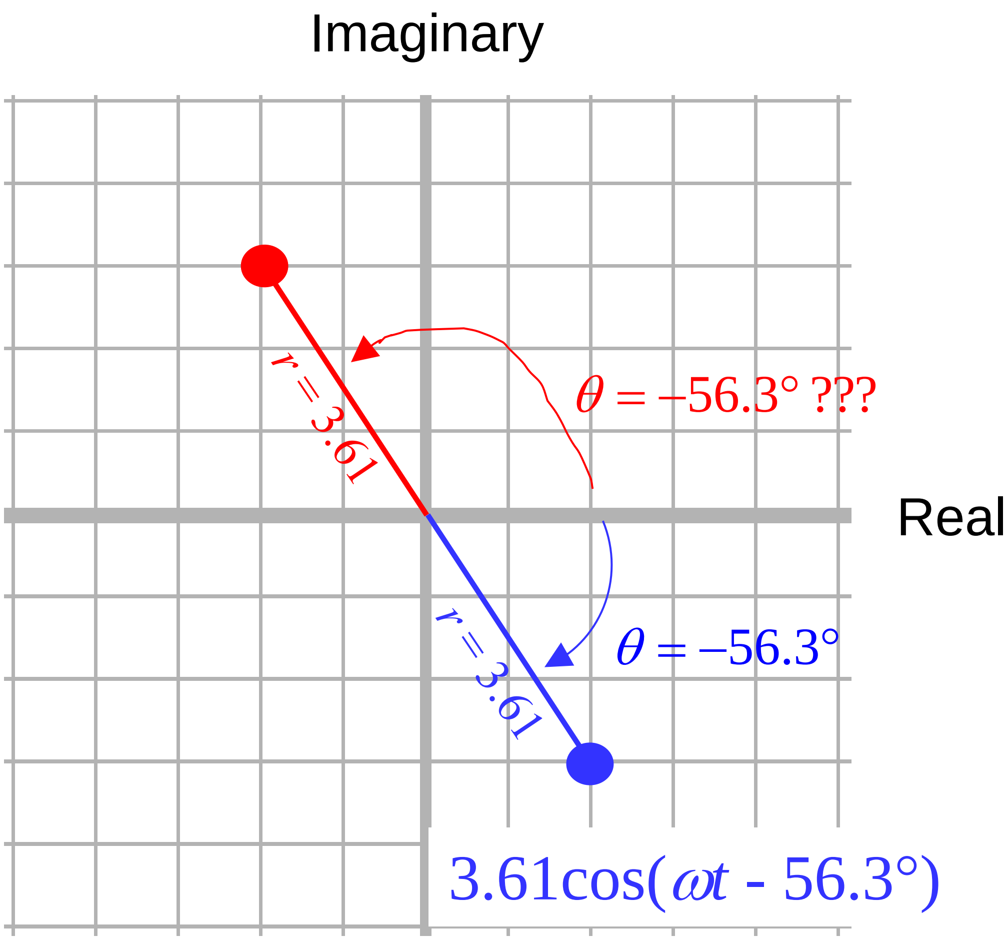
$$r = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (-3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{-3}{2}\right) = \tan^{-1}(-1.5) = -56.3°$$

- $-2 + j3 =$

$$r = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{3}{-2}\right) = \tan^{-1}(-1.5) = -56.3°$$



Converting from Cartesian to Phasor (Polar) Form

- $2 - j3 = 3.61 \angle -56.3°$

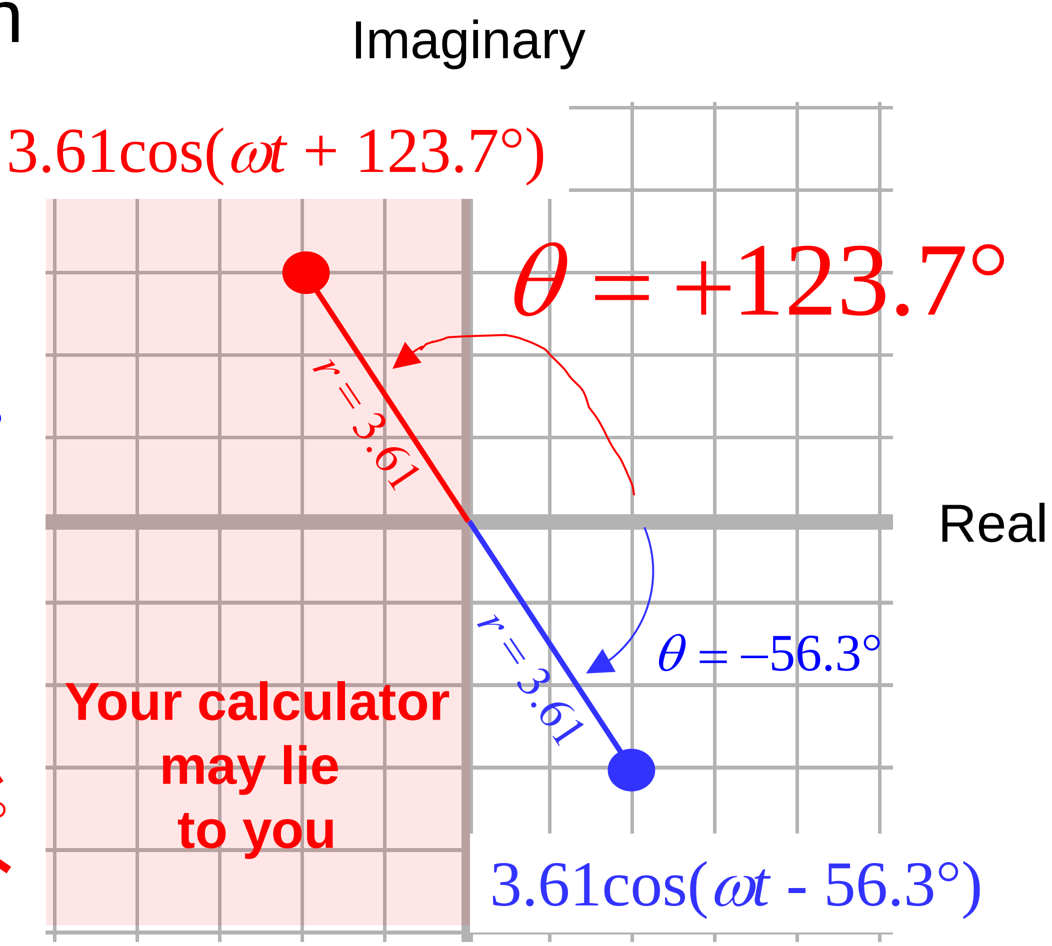
$$r = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (-3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{-3}{2}\right) = \tan^{-1}(-1.5) = -56.3°$$

- $-2 + j3 =$

$$r = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{3}{-2}\right) = \tan^{-1}(-1.5) = \cancel{-56.3°}$$



Multiplication and Division with Phasors

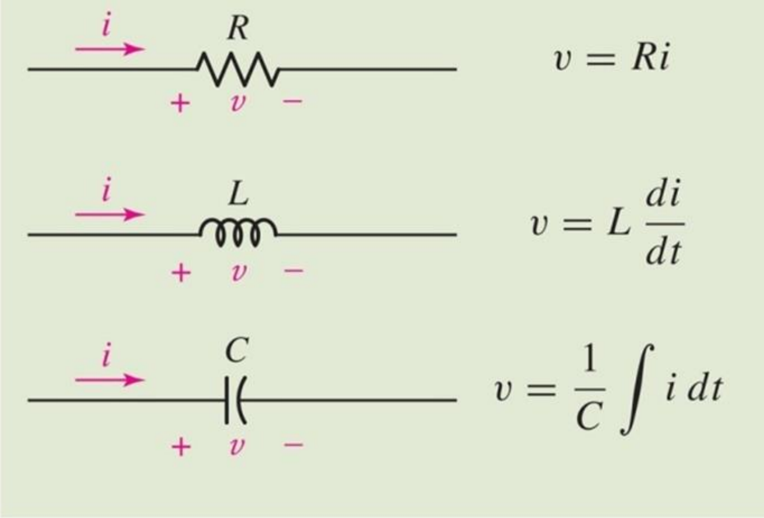
$$(3\angle +30°)(4\angle +45°) = (3)(4)\angle(30° + 45°) = 12\angle(75°)$$

$$\frac{-1+5j}{2+3j} = \frac{5.099\angle 101.3°}{3.606\angle 56.3°} = \frac{5.099}{3.606}\angle(101.3° - 56.3°) = \sqrt{2}\angle 45°$$

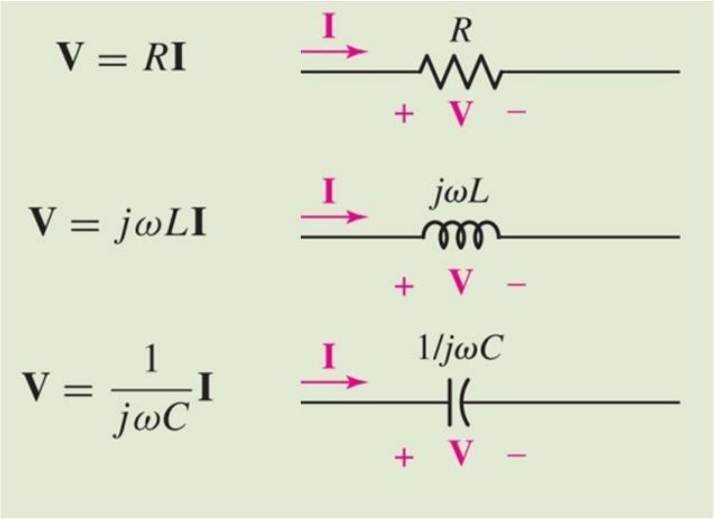
Time Domain vs. Frequency Domain

$$2.82\cos\left(\omega t + 45^\circ\right) = 2 + j2 = 2.82\angle 45^\circ$$

Time Domain



Frequency Domain



Uses “real” numbers,
but requires calculus and differential equations

Uses “complex” numbers,
but only requires algebra

Phasor Representation

$$V_m \cos(2\pi f t + \theta) = V_m \cos(\omega t + \theta) = V_m \angle \theta$$

$$I_m \cos(2\pi f t + \theta) = I_m \cos(\omega t + \theta) = I_m \angle \theta$$

We will also see that we can represent resistors, inductors, and capacitors as **complex numbers**

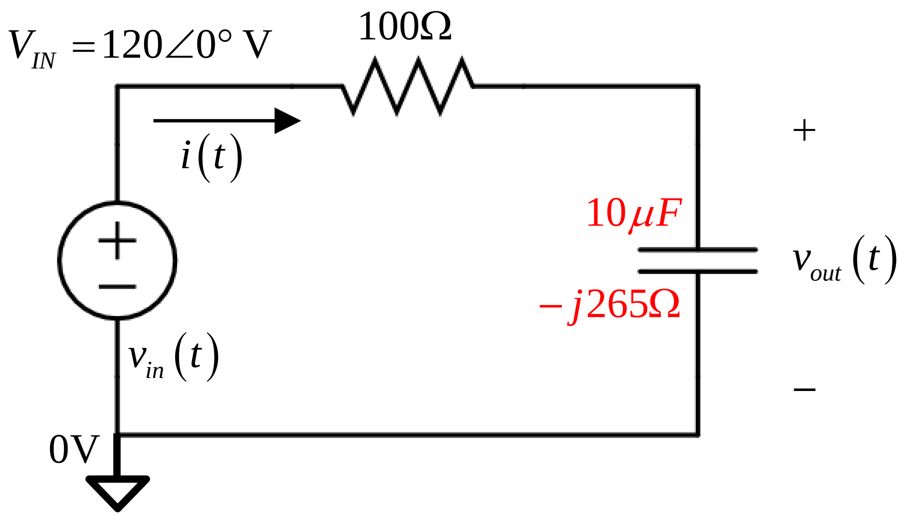
We will use the term **impedance** (Z) to refer to a “resistance” to current flow that varies with **frequency**

$$Z_{Resistor} = Z_R = R$$

$$Z_{Inductor} = Z_L = j\omega L$$

$$Z_{Capacitor} = Z_C = \frac{1}{j\omega C} = \left(\frac{1}{j\omega C}\right)\left(\frac{j}{j}\right) = \frac{j}{j^2\omega C} = \frac{j}{-1\omega C} = \frac{-j}{\omega C}$$

Circuit Analysis is Easy with Phasors



$$v_{in}(t) = 120\cos(377t + 0°)\text{ V} = 120\angle 0°\text{ V} = V_{IN}$$

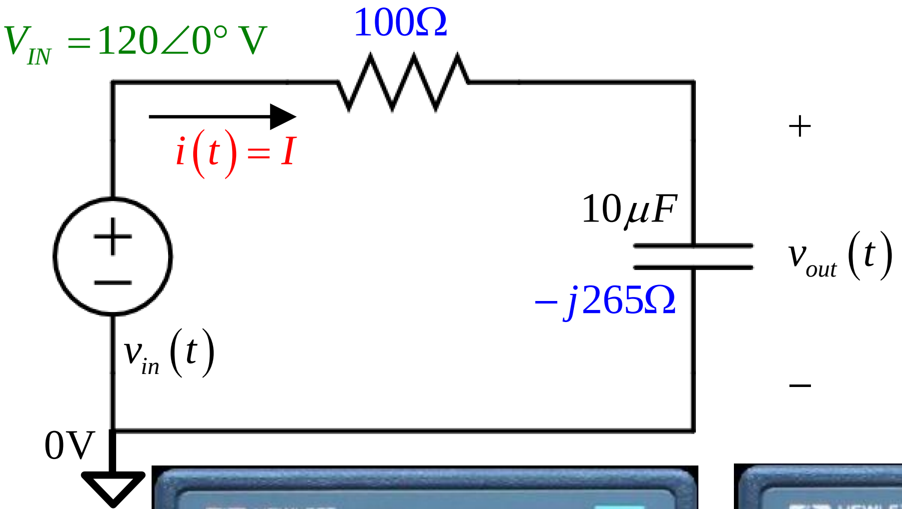
$$R = R = 100\ \Omega$$

$$Z_C = \frac{-j}{\omega C} = \frac{-j}{(377 rad/s)(10\mu F)} = -j265\ \Omega$$

10μF acts like a 265Ω resistor to a 377rad/s sinusoidal voltage,
AND it will result in a $90°$ phase shift
to any current that flows through it

Changing ω will change the resistance of the 1μF, but not the (-j) $90°$ phase shift

Circuit Analysis is Easy with Phasors

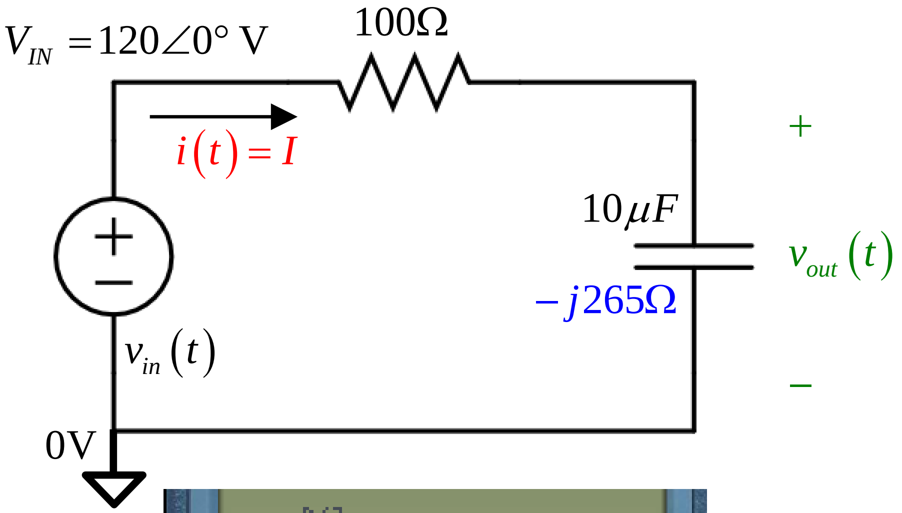


$$I = \frac{120\angle 0^\circ \text{ V}}{100\Omega - j265\Omega}$$

$$I = 0.1496 + j0.3964 \text{ A} \quad = \quad 424\angle 69.3^\circ \text{ mA}$$

$$i(t) = 424\cos(377t + 69.3^\circ) \text{ mA}$$

Circuit Analysis is Easy with Phasors



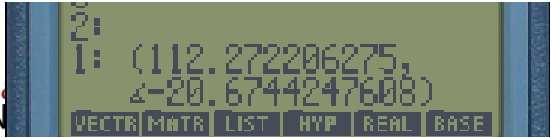
$$I = \frac{120\angle 0° \text{ V}}{100\Omega - j265\Omega}$$

$$I = 0.1496 + j0.3964 \text{ A} \quad = \quad 424\angle 69.3° \text{ mA}$$

$$i(t) = 424\cos(377t + 69.3°) \text{ mA}$$

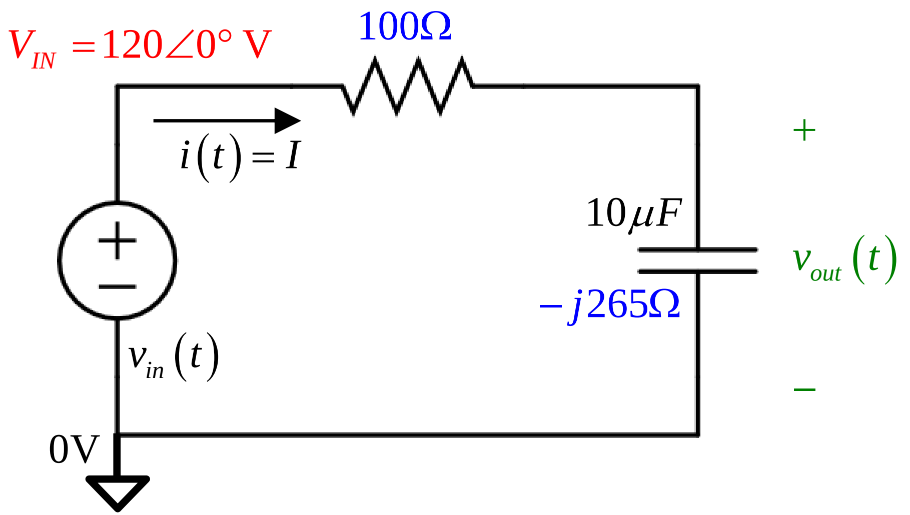
$$V_{OUT} = (424\angle 69.3° \text{ mA})(-j265) = 112\angle -20.7° \text{ V}$$

$$v_{out}(t) = 112\cos(377t - 20.7°) \text{ V}$$



Electrical ENGIN

Circuit Analysis is Easy with Phasors (Voltage Division)



$$I = \frac{120\angle 0° \text{ V}}{100\Omega - j265\Omega}$$

$$I = 0.1496 + j0.3964 \text{ A} \quad = \quad 424\angle 69.3° \text{ mA}$$

$$i(t) = 424\cos(377t + 69.3°) \text{ mA}$$

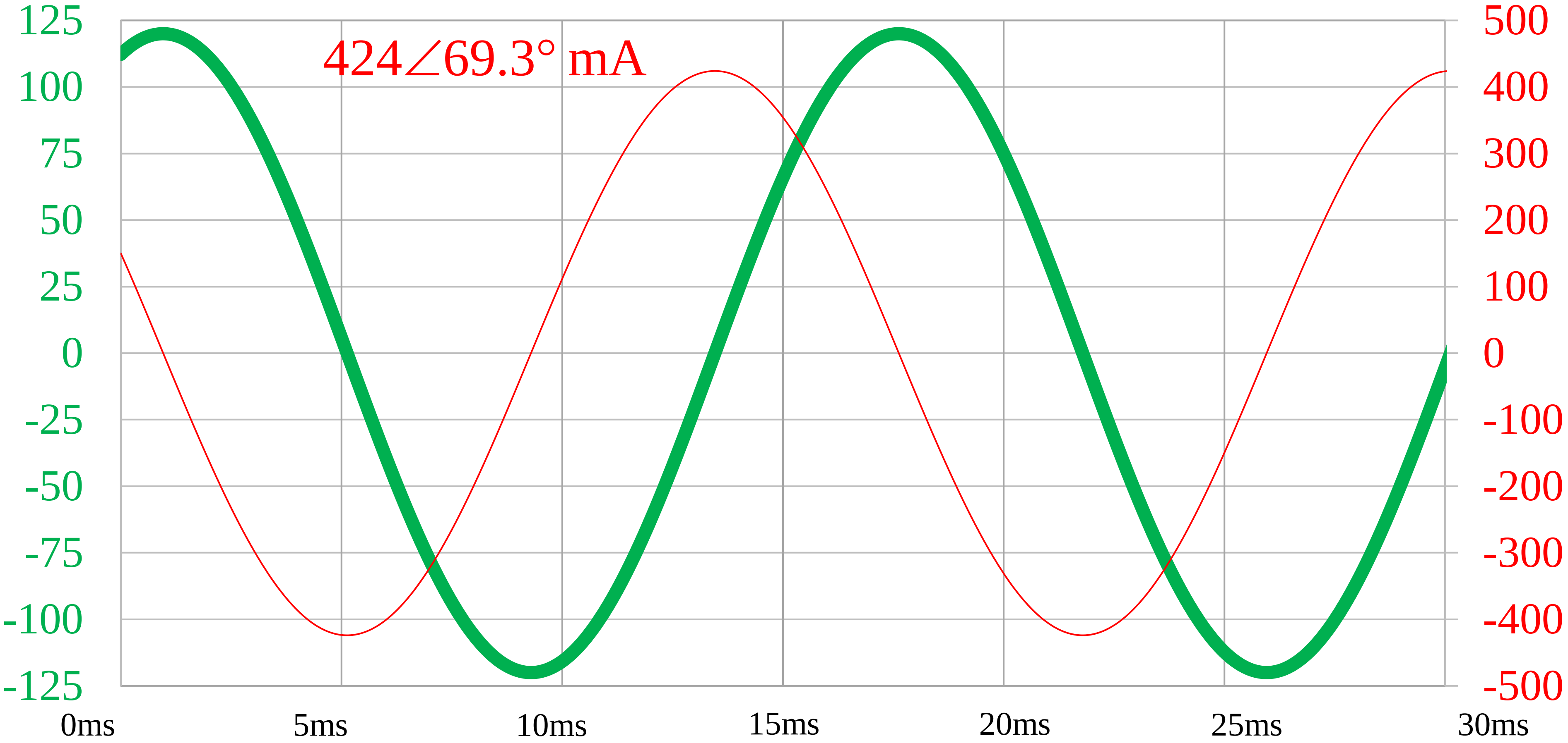
$$V_{OUT} = (424\angle 69.3° \text{ mA})(-j265) = 112\angle -20.7° \text{ V}$$

$$V_{OUT} = (120\angle 0° \text{ V})\left[\frac{(-j265\Omega)}{(100\Omega)+(-j265\Omega)}\right]$$

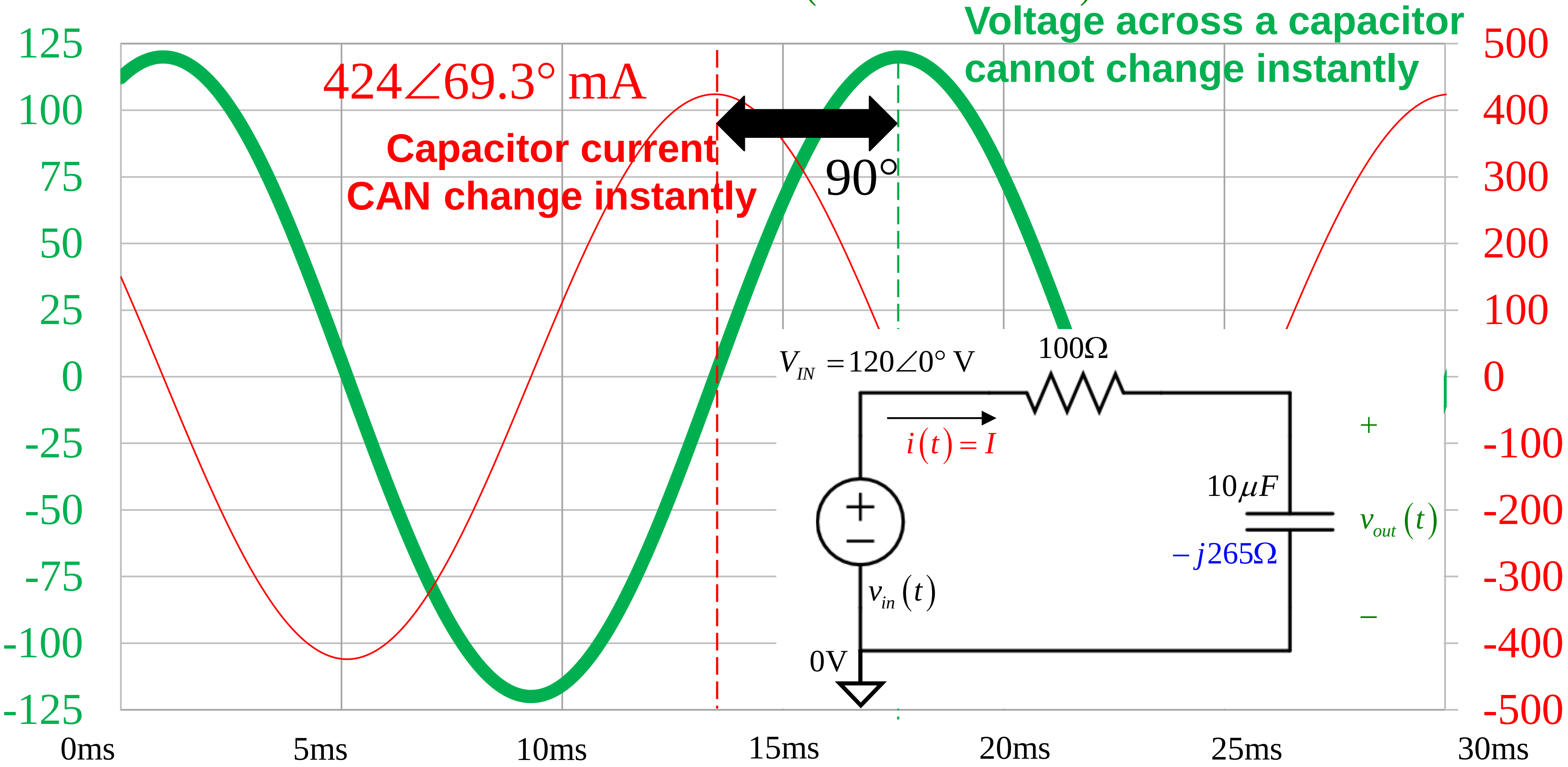
$$v_{out}(t) = 112\cos(377t - 20.7°) \text{ V}$$

$$v_{out}(t) = 112\angle -20.7° \text{ V} = 112\cos(377t - 20.7°) \text{ V}$$

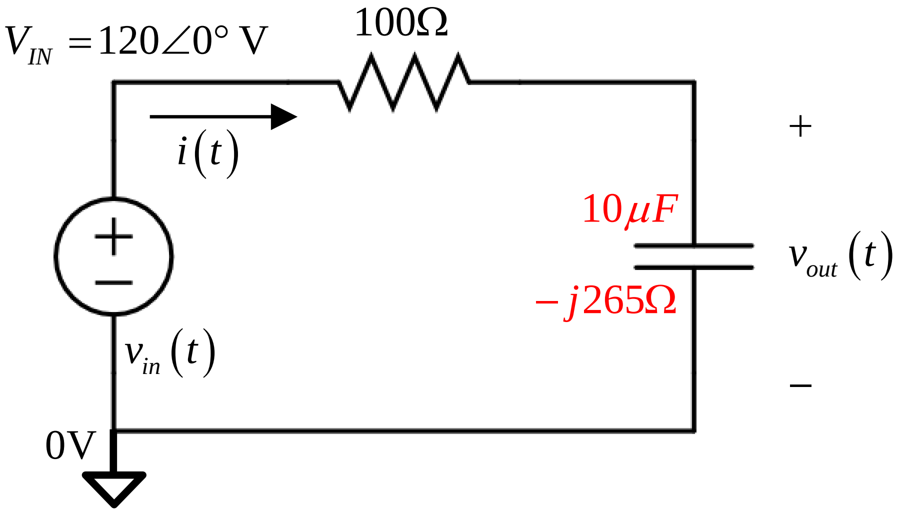
$$112\cos\left(377t - 20.7°\right)\ \text{V}$$



$$112\cos\left(377t - 20.7°\right)\text{ V}$$



Circuit Analysis is Easy with Phasors



$$v_{in}(t) = 120\cos(377t + 0°)\text{ V} = 120\angle 0° \text{ V} = V_{IN}$$

$$R = R = 100\ \Omega$$

$$Z_C = \frac{-j}{\omega C} = \frac{-j}{(377 rad/s)(10\mu F)} = -j265\ \Omega$$

10μF acts like a 265Ω resistor to a 377rad/s sinusoidal voltage,
AND it will result in a 90° phase shift
to any current that flows through it

Changing ω will change the resistance of the 1μF, but not the (-j) 90° phase shift

What Do You Need to Do Next?

1. Take the **Lecture 15 Quiz** on canvas**!**

2. Check out Piazza and Gradescope