Lecture 14

Inverse Functions

Wednesday, Feb. 19

In This Lecture...

- Inverses on the left, on the right, and on both sides at once!
- The shocking connection between inverses and bijectivity!

Definition 14.0: Inverse Functions

Let $f : X \to Y$ be a function.

- A **left inverse** for f is a function $g : Y \to X$ such that:

$$g \circ f = \mathrm{id}_X \text{ (for all } x \in X,\ g(f(x)) = x)$$

- A **right inverse** for f is a function $g : Y \to X$ such that:

$$f \circ g = \mathrm{id}_Y \text{ (for all } y \in Y,\ f(g(y)) = y)$$

- An **inverse** (or **two-sided inverse**) for f is a function $g : Y \to X$ such that:

$$g \text{ is a left inverse and a right inverse for } f$$

1 Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n$ for all $n \in \mathbb{Z}$. Find a left inverse for f.

Solution

We would like the left inverse of f to have the formula $g(n) = \frac{n}{2}$, but $\frac{n}{2}$ is only an integer if n is even. The workaround is to define g to be equal to some other integer when n is odd. We claim that the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

is a left inverse for f. This is true since for any $n \in \mathbb{Z}$, we have $g(f(n)) = g(2n)$, and as $2n$ is always even, this is equal to $\frac{2n}{2} = n$.

2 Let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. Find a right inverse for f.

Solution

Define the function $g : [0, \infty) \to \mathbb{R}$ by $g(x) = \sqrt{x}$. This is a well-defined function since the domain of g only has non-negative real numbers. For any $x \in [0, \infty)$, we have

$$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x,$$

so g is a right inverse for x.

(Note that the formula $g(x) = -\sqrt{x}$ would also define a right inverse for x.)

Theorem 14.1

Let $f : X \to Y$ be a function, and suppose f has a left inverse $\ell : Y \to X$ and a right inverse $r : Y \to X$. Then $\ell = r$.

(This means that a function with a left inverse and a right inverse has a unique two-sided inverse, which we denote by f^{-1}.)

Proof: Let $f : X \to Y$ be a function, $\ell : Y \to X$ be a left inverse of f, and $r : Y \to X$ be a right inverse of f. As ℓ is a left inverse of f, we have $\ell \circ f = \mathrm{id}_X$, and as r is a right inverse of f, we have $f \circ r = \mathrm{id}_Y$. Let $y \in Y$. We have

$$\begin{aligned}
\ell(y) &= \ell(\mathrm{id}_Y(y)) \\
&= \ell(f(r(y))) \\
&= \mathrm{id}_X(r(y)) \\
&= r(y).
\end{aligned}$$

As $\ell(y) = r(y)$ for all $y \in Y$, and the domains and codomains of ℓ and r match, we have $\ell = r$ by function extensionality. $\square$

Theorem 14.2

Let $f : X \to Y$ be a function. Then f has an inverse if and only if f is bijective.

Proof: ($\Rightarrow$) Suppose f has an inverse f^{-1}. To prove injectivity, let $a, b \in X$ with $f(a) = f(b)$. Applying the inverse to both sides, $f^{-1}(f(a)) = f^{-1}(f(b))$, so $a = b$ since f^{-1} is a left inverse, proving injectivity.

For surjectivity, let $y \in Y$. Taking $x = f^{-1}(y) \in X$, we have $f(x) = f(f^{-1}(y)) = y$ since f^{-1} is a right inverse, which proves surjectivity. As f is both injective and surjective, it is bijective.

($\Leftarrow$) Suppose f is bijective. Define a function $g : Y \to X$ by, for any $y \in Y$, taking $g(y)$ to be the unique element $x \in X$ satisfying $f(x) = y$. Note that such an element x must exist since g is surjective, and the element x must be unique since g is injective.

For any $x \in X$, let $y = f(x)$. We have $g(f(x)) = g(y) = x$, because x is the unique element of X with $f(x) = y$. This shows g is a left inverse of f. For any $y \in Y$, we have $f(g(y)) = f(x) = y$, where x is the unique element of X with $f(x) = y$. This shows g is a right inverse of f. As g is both a left inverse and a right inverse of f, it is the two-sided inverse of f. $\square$