Lecture 13

Function Composition

Monday, Feb. 17

In This Lecture...

- Creating beautiful compositions (of functions)!
- Being very careful about the order in which things are written!
- Determining how properties of functions work for compositions!

Definition 13.0: Function Composition

Let $f : X \to Y$ and $g : Y \to Z$ be functions. The **composition** of g and f is the function $g \circ f :$ $X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

Even though composition is written in the order $g \circ f$, the inner function f is applied first!

1 Find formulas for the compositions $g \circ f$ and $f \circ g$, if possible.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 + 2n$ and $g : \mathbb{Z} \to \mathbb{Q}$ defined by $g(n) = \dfrac{n}{2}$.

Solution

The composition $g \circ f$ is a function with domain $\mathbb{Z}$ and codomain $\mathbb{Q}$. For any $n \in \mathbb{Z}$, we have

$$(g \circ f)(n) = g(f(n)) = g(n^2 + 2n) = \frac{n^2}{2} + n.$$

The composition $f \circ g$ is not defined, since the codomain of g (which is $\mathbb{Q}$) doesn't match the domain of f (which is $\mathbb{Z}$).

(b) $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $f(A) = A \cup \{0\}$ and $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $g(A) = A \setminus \{0\}$

Solution

For the composition $g \circ f$ and any $A \in \mathcal{P}(\mathbb{N})$, we have

$$(g \circ f)(A) = g(f(A)) = g(A \cup \{0\}) = (A \cup \{0\}) \setminus \{0\}.$$

Here, we are adding the element 0 to the set and then removing it. The net result is the same as simply removing 0, meaning $(g \circ f)(A) = A \setminus \{0\}$.

For the composition $f \circ g$ and any $A \in \mathcal{P}(\mathbb{N})$, we have

$$(f \circ g)(A) = f(g(A)) = f(A \setminus \{0\}) = (A \setminus \{0\}) \cup \{0\}.$$

This time, we are removing the element 0 from the set and then putting it in. The net result is the same as adding 0 to the set, meaning $(f \circ g)(A) = A \cup \{0\}$.

Definition 13.1: Identity Function

Let X be a set. The **identity function** on X is the function $\mathrm{id}_X : X \to X$ defined by:

$$\mathrm{id}_X(x) = x \text{ for all } x \in X$$

2 Prove that for any function $f : X \to Y$, $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$.

Solution

Proof: Let $f : X \to Y$ be a function. First, we will show that $f \circ \mathrm{id}_X = f$. From $\mathrm{id}_X : X \to X$ and $f : X \to Y$, we see that the composition is a function with domain X and codomain Y, which match the domain and codomain of f. For any $x \in X$, we have

$$(f \circ \mathrm{id}_X)(x) = f(\mathrm{id}_X(x)) = f(x)$$

by definition of the identity function, which establishes $f \circ \mathrm{id}_X = f$.

The proof that $\mathrm{id}_Y \circ f = f$ is similar. We have $f : X \to Y$ and $\mathrm{id}_Y : Y \to Y$, so the composition is a function with domain X and codomain Y, the same as f. For any $x \in X$, we have

$$(\mathrm{id}_Y \circ f)(x) = \mathrm{id}_Y(f(x)) = f(x)$$

by definition of the identity function, giving us $\mathrm{id}_Y \circ f = f$. $\square$

3 Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Prove that if f and g are injective, then $g \circ f$ is injective.

Solution

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be injections. To show that $g \circ f$ is injective, let $a, b \in X$ with $(g \circ f)(a) = (g \circ f)(b)$. We can write this as $g(f(a)) = g(f(b))$. As g is injective, this implies $f(a) = f(b)$, and as f is injective, we have $a = b$. Thus $g \circ f$ is injective. $\square$

(b) Prove that if f and g are surjective, then $g \circ f$ is surjective.

Solution

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be surjections. To show that $g \circ f$ is surjective, let $z \in Z$. As g is surjective, there exists a $y \in Y$ such that $g(y) = z$. As f is surjective, there exists an $x \in X$ such that $f(x) = y$. Putting these together, we have

$$(g \circ f)(x) = g(f(x)) = g(y) = z,$$

proving that $g \circ f$ is surjective. $\square$

(c) Prove that if f and g are bijective, then $g \circ f$ is bijective.

Solution

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Then f and g are both injective and both surjective. By part (a), the composition $g \circ f$ is injective, and by part (b), the composition $g \circ f$ is surjective, so $g \circ f$ is bijective. $\square$