Lecture 11

Images and Preimages

Monday, Feb. 10

In This Lecture...

- Imagining images!
- Predicting preimages!

Definition 11.0: Images and Preimages

Let $f : X \to Y$, and let $A \subseteq X$ and $B \subseteq Y$.

- The **image** of A under f is the set $f[A] \subseteq Y$ defined by:

$$f[A] = \{f(x) \mid x \in A\}$$

  The image of the function f is $\underline{\quad f[X] \quad}$.

- The **preimage** of B under f is the set $f^{-1}[B] \subseteq X$ defined by:

$$f^{-1}[B] = \{x \in X \mid f(x) \in B\}$$

Images and preimages are sometimes written using parentheses instead of square brackets. When using this notation, it should be clear whether f is being applied to a single element or a subset of its domain.

1 Let $f : \{1, 2, 3, 4\} \to \{5, 6, 7, 8, 9\}$ be defined by $f(1) = 5$, $f(2) = 6$, $f(3) = 6$, and $f(4) = 8$. Find each image or preimage.

(a) $f[\{1, 2\}]$

Solution

We have $f(1) = 5$ and $f(2) = 6$, so $f[\{1, 2\}] = \{f(1), f(2)\} = \{5, 6\}$.

(b) $f[\{1, 2, 3, 4\}]$

Solution

We have $f[\{1, 2, 3, 4\}] = \{f(1), f(2), f(3), f(4)\} = \{5, 6, 8\}$.

(c) $f^{-1}[\{6\}]$

Solution

This is the set of all elements x where $f(x) = 6$, namely $f^{-1}[\{6\}] = \{2, 3\}$.

(d) $f^{-1}[\{7, 9\}]$

Solution

This is the set of all elements x where $f(x) = 7$ or $f(x) = 9$. There are no such elements in the set $\{1, 2, 3, 4\}$, so $f^{-1}[\{7, 9\}] = \varnothing$.

2 Let $f : X \to Y$ be a function, and let $A, B \subseteq X$.

(a) Prove that $f[A \cap B] \subseteq f[A] \cap f[B]$.

Solution

Proof: Let $y \in f[A \cap B]$. By definition of image, there exists an element $x \in A \cap B$ such that $y = f(x)$. By definition of intersection, we have $x \in A$ and $x \in B$. Then $y = f(x)$ where x is an element of A, meaning $y \in f[A]$. Similarly, $y = f(x)$ where x is an element of B, meaning $y \in f[B]$. Thus $y \in f[A] \cap f[B]$, proving that $f[A] \subseteq f[B]$. $\square$

(b) Prove that $f[A \cap B] \neq f[A] \cap f[B]$ in general.

Solution

We want to find two sets A and B where $f[A \cap B] \neq f[A] \cap f[B]$. In the previous part, we showed that $f[A \cap B] \subseteq f[A] \cap f[B]$, so in our counterexample, we should have an element of $f[A] \cap f[B]$ that's not in $f[A \cap B]$.

Proof: Let $f : \{1, 2\} \to \{3\}$ be defined by $f(1) = 3$ and $f(2) = 3$. Let $A = \{1\}$ and $B = \{2\}$. We have $A \cap B = \varnothing$, so $f[A \cap B] = \varnothing$ as well. However, $f[A] = \{3\}$ and $f[B] = \{3\}$, so $f[A] \cap f[B] = \{3\}$, which is not equal to $f[A \cap B]$. $\square$

3 Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$ for all $x \in \mathbb{R}$.

(a) Find $f[(-1, 1)]$.

Solution

By definition of image, $f[(-1, 1)] = \{f(x) \mid x \in (-1, 1)\}$. For a real number x, the condition $-1 < x < 1$ is equivalent to $0 \leq x^2 < 1$, which are the possible values of the function $f(x) = x^2$. Thus $f[(-1, 1)] = [0, 1)$.

(Note that this is only a brief explanation of what the image is. Proving this rigorously requires a double containment proof.)

(b) Find $f^{-1}[(1, 4)]$.

Solution

By definition of preimage, $f^{-1}[(1, 4)] = \{x \in \mathbb{R} \mid 1 < f(x) < 4\}$. As $f(x) = x^2$, the condition to be the preimage is $1 < x^2 < 4$. The solutions to these inequalities are $-2 < x < -1$ or $1 < x < 2$. Therefore, $f^{-1}[(1, 4)] = (-2, -1) \cup (1, 2)$.