Lecture 12

Injections and Surjections

Friday, Feb. 14

In This Lecture...

- Understanding injections, surjections, and bijections! (Who comes up with these words, anyway?)
- Proving injectivity, surjectivity, and bijectivity!

Definition 12.0: Injectivity

A function $f : X \to Y$ is **injective** if $\forall a, b \in X, (f(a) = f(b) \Rightarrow a = b)$.
That is, every $y \in Y$ has at most one $x \in X$ with $y = f(x)$.

1 Determine whether each function is an injection.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

Solution

This function is not an injection.

Proof: Let $a = 2$ and $b = -2$. We have $f(a) = (2)^2 = 4$ and $f(b) = (-2)^2 = 4$, so $f(a) = f(b)$, even though $a \neq b$. Thus f is not injective. $\square$

(b) $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x^2$

Solution

This function is an injection.

Proof: Let $a, b \in \mathbb{N}$ with $g(a) = g(b)$. By definition of g, we have $a^2 = b^2$. Rewriting the equation, $a^2 - b^2 = 0$, which can be factored as $(a + b)(a - b) = 0$. Thus $a + b = 0$ or $a - b = 0$. If $a + b = 0$, we must have $a = 0$ and $b = 0$ since a and b are natural numbers (and hence non-negative). If $a - b = 0$, then $a = b$. In either case, we have $a = b$, proving that g is injective. $\square$

Proof Strategy: Proving Injectivity

To prove that a function $f : X \to Y$ is injective,

1. Let $a, b \in X$ be arbitrary elements with $f(a) = f(b)$.
2. Show that $a = b$.

Definition 12.1: Surjectivity

A function $f : X \to Y$ is **surjective** if $\underline{\quad \forall y \in Y, \exists x \in X, y = f(x) \quad}$.
That is, $\underline{\quad \text{every } y \in Y \text{ has at least one } x \in X \text{ with } y = f(x) \quad}$.

2 Determine whether each function is a surjection.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$

Solution

This function is a surjection.

Proof: Let $y \in \mathbb{R}$ be arbitrary. Define $x = \frac{y}{2}$. Then $f(x) = 2\left(\frac{y}{2}\right) = y$, demonstrating that f is surjective. $\square$

(b) $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = 2x$

Solution

This function is not a surjection.

Proof: Let $y = 1$. Assume for the sake of contradiction that there is an $x \in \mathbb{N}$ with $g(x) = y$. Then $2x = 1$, so $x = \frac{1}{2}$, contradicting our assumption that x is a natural number. Thus $y \neq f(x)$ for all $x \in \mathbb{N}$, showing that g is not surjective. $\square$

Proof Strategy: Proving Surjectivity

To prove that a function $f : X \to Y$ is surjective,

1. Let $y \in Y$ be arbitrary.
2. Find an $x \in X$ (based on y) such that $f(x) = y$.

Definition 12.2: Bijectivity

A function $f : X \to Y$ is **bijective** if $\underline{\quad f \text{ is both injective and surjective} \quad}$.
That is, $\underline{\quad \text{every } y \in Y \text{ has exactly one } x \in X \text{ with } y = f(x) \quad}$.

3 Let $f : \mathcal{P}(\mathbb{N}) \setminus \{\varnothing\} \to \mathbb{N}$ be the function where $f(A)$ is the smallest element of A. Is f injective? Surjective? Bijective?

Solution

f is not injective, because $f(\{0\}) = f(\{0, 1\}) = 0$.

f is surjective, because for any $n \in \mathbb{N}$, we have $f(\{n\}) = n$.

f is not bijective, because it is not injective.