Lecture 10

Functions

Friday, Feb. 7

In This Lecture...

- Becoming a master of your domain (and codomain)!
- Defining functions well!
- Making graphs!

Definition 10.0: Functions

Let A and B be sets. A **function** f from X to Y, denoted $f : X \to Y$, is:

an assignment of a unique value $f(x) \in Y$ to each $x \in X$.

$$(\forall x \in X, \exists! y \in Y, y = f(x))$$

The **domain** of the function is the set X.

The **codomain** of the function is the set Y.

The letters f, g, and h typically stand for functions.

Examples of functions:

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$
- $g : \{1, 2, 3\} \to \mathbb{Q}$ defined by $g(1) = \frac{2}{3}$, $g(2) = -\frac{22}{7}$, and $g(3) = 0$
- $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $h(A) = A \cup \{1, 2, 3\}$
- $s : \mathbb{R} \to \mathbb{R}$ defined by $s(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

Definition 10.1: Well-defined Functions

A function $f : X \to Y$ is called **well-defined** if it actually satisfies the definition of a function. This can be broken down into three conditions.

- **Totality:** A value $f(x)$ is specified for each $x \in X$.
- **Existence:** $f(x) \in Y$ exists for each $x \in X$.
- **Uniqueness:** There is only one value $f(x)$ for each $x \in X$.

1 Each "function" below is not well-defined. Explain why.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = \dfrac{n}{2}$ when n is even.

Solution

This definition fails totality. A value $f(n)$ is not specified when n is odd.

(b) $g : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ where $g(A)$ is the smallest element of A.

Solution

This definition fails existence. The value $g(\varnothing)$ does not exist since the empty set doesn't have a smallest element.

(c) $h : \mathbb{Q} \to \mathbb{Z}$ defined by $h\left(\dfrac{a}{b}\right) = a$.

Solution

This definition fails uniqueness. We have $h\left(\frac{1}{2}\right) = 1$, while $h\left(\frac{2}{4}\right) = 2$. The values of the function should be equal since $\frac{1}{2} = \frac{2}{4}$.

Definition 10.2: Function Extensionality

Two functions f and g are equal if both of the following conditions hold:

- f and g have the same domain and codomain.
- For all elements x in the domain, $f(x) = g(x)$.

2 Show that $f = g$, where $f, g : \mathbb{Z} \to \mathbb{Z}$ are the functions defined by:

$$f(n) = 2\left\lfloor \frac{n}{2} \right\rfloor, \qquad g(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

Here $\lfloor \cdot \rfloor$ is the *floor function*, which gives the greatest integer less than or equal to its input (it "rounds down" to an integer).

Solution

Proof: The functions f and g both have domain $\mathbb{Z}$ and codomain $\mathbb{Z}$, so it suffices to show that $f(n) = g(n)$ for all $n \in \mathbb{Z}$ to prove that $f = g$.

Let $n \in \mathbb{Z}$. We will analyze two cases:

- *Case 1:* n is even

  In this case, $\frac{n}{2}$ is an integer, so

  $$f(n) = 2\left\lfloor \frac{n}{2} \right\rfloor = 2\left(\frac{n}{2}\right) = n = g(n).$$

- *Case 2:* n is odd

In this case, $\frac{n}{2}$ is $\frac{1}{2}$ more than an integer, so

$$f(n) = 2\left\lfloor \frac{n}{2} \right\rfloor = 2\left(\frac{n}{2} - \frac{1}{2}\right) = n - 1 = g(n).$$

We have $f(n) = g(n)$ for all $n \in \mathbb{Z}$, completing the proof that f and g are equal by function extensionality. $\square$

Definition 10.3: Graphs

Let $f : X \to Y$ be a function. The **graph** of f is the set $\mathrm{Gr}(f) \subseteq X \times Y$ defined by:

$$\mathrm{Gr}(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$$

3 Find the graph of each function.

(a) $f : \{1, 2, 3\} \to \{a, b, c, d\}$ defined by $f(1) = a$, $f(2) = b$, and $f(3) = d$.

Solution

To find the graph of f, we can simply list all ordered pairs of inputs and outputs:

$$\mathrm{Gr}(f) = \{(1, a), (2, b), (3, d)\}$$

(b) $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = 2^n - 1$.

Solution

The domain of g is infinite, so its graph must be expressed using set-builder notation or as an infinite list:

$$\mathrm{Gr}(g) = \{(n, 2^n - 1) \mid n \in \mathbb{N}\} = \{(0, 0), (1, 1), (2, 3), (3, 7), (4, 15), \dots\}$$