Lecture 2

Proofs

Friday, Jan. 17

In This Lecture...

- Investigating divisibility!
- Writing our first proofs!
- Learning what to do (and what not to do) in a proof!

Definition 2.0: Propositions and Proofs

A **proposition** is ______a statement that can be classified as true or false______ .

A **proof** is ______an argument that demonstrates the truth of a proposition______ .

Examples of propositions:

- "If a real number x satisfies $2x + 3 = 7$, then $x = 2$" is a proposition.
- "There are no real solutions to $2x + 3 = 7$" is a (false) proposition.
- "$2x + 3 = 7$" is not a proposition, as we are not making a true or false statement about the equation.
- "2 is a really cool number" is not a proposition, as it is a matter of opinion.

How do proofs work?

> A proof demonstrates that if certain starting *assumptions* are true, then a particular *conclusion* must follow from those assumptions. The proof establishes this with a sequence of logical deductions which start from the given assumptions and end with the conclusion. A good proof should be communicated clearly, so that the person reading your proof can understand each step unambiguously.

What are you allowed do in a proof?

- Restate a given assumption
- Make a deduction from known information
- Use a definition
- Show that something satisfies a definition
- Create and name a specific object (number, set, function, etc.)
- Apply a previously proven result
- Do scratch work (separately from the proof)
- Make a hypothetical assumption

Definition 2.1: Divisibility

Let $a, b \in \mathbb{Z}$. We say that a is **divisible** by b (or that a is a multiple of b, or that b divides a), and write $b \mid a$, if there is an integer k such that $a = kb$.

1 Prove that for any integer n, if $6 \mid n$, then $3 \mid n$.

Solution

Proof: Let $n \in \mathbb{Z}$ such that $6 \mid n$. By definition of divisibility, there is an integer k such that $n = 6k$. We can rewrite this equation as $n = 3(2k)$. Note that $2k$ is the product of two integers, and is thus also an integer. We have written n as a product of 3 and an integer, so by definition of divisibility, we conclude that $3 \mid n$. $\square$

Definition 2.2: Even and Odd

An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$ (that is, $2 \mid n$).
An integer n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

2 Prove that the square of an odd number is odd.

Solution

Proof: Let $n \in \mathbb{Z}$ be odd. By definition of odd, there exists a $k \in \mathbb{Z}$ such that $n = 2k + 1$. We compute

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Let $m = k^2 + 2k$, which is an integer since k is an integer. We have $n^2 = 2m + 1$, so n^2 is odd by definition. $\square$

3 Prove that if n is odd, then $n^2 - 3n$ is even.

Solution

Proof: Let $n \in \mathbb{Z}$ be odd. By definition of odd, there is a $k \in \mathbb{Z}$ such that $n = 2k + 1$. By the result of #2, we know that n^2 is odd, so there is an $m \in \mathbb{Z}$ such that $n^2 = 2m + 1$. We then have

$$\begin{aligned} n^2 - 3n &= (2m+1) - 3(2k+1) \\ &= 2m - 6k - 2 \\ &= 2(m - 3k - 1). \end{aligned}$$

Observe that $n^2 - 3n$ is 2 times the integer $m - 3k - 1$. Therefore $n^2 - 3n$ is even. $\square$

Theorem 2.3: AM–GM Inequality

If x and y are non-negative real numbers, then $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Proof: Let $x, y \in \mathbb{R}$ with $x \geq 0$ and $y \geq 0$. Note that $(x-y)^2$ is the square of a real number, and hence satisfies $(x-y)^2 \geq 0$. Rearranging the inequality,

$$\begin{aligned}
(x-y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 && \text{(expanding the square)} \\
x^2 + 2xy + y^2 &\geq 4xy && \text{(adding } 4xy \text{ to both sides)} \\
(x+y)^2 &\geq 4xy && \text{(factoring)} \\
\sqrt{(x+y)^2} &\geq \sqrt{4xy} && \text{(taking the square root of both sides)} \\
x+y &\geq 2\sqrt{xy} && \text{(simplifying roots, valid since } x+y \geq 0\text{)} \\
\frac{x+y}{2} &\geq \sqrt{xy} && \text{(dividing both sides by 2)}
\end{aligned}$$

We have obtained the inequality $\dfrac{x+y}{2} \geq \sqrt{xy}$, as desired. □