Hashing

Sets and Dictionaries

What do we use arrays for?

1 To keep a *collection* of elements of the same type in one place

- *E.g., all the words in the Collected Works of William Shakespeare*

"a"	"rose"	"by"	"any"	"name"	…	"Hamlet"

- The array is used as a **set**
  - the index where an element occurs doesn't matter much
- Main operations:
  - add an element
    - like uba_add for unbounded arrays
  - check if an element is in there
    - this is what search does (linear if unsorted, binary if sorted)
  - go through all elements
    - using a for-loop for example

What do we use arrays for?

2 As a *mapping* from indices to values

- *E.g., the monthly average high temperatures in Pittsburgh*

	0	1	2	3	4	5	6	7	8	9	10	11	12
High:		35	38	50	62	72	80	83	82	75			

0 = *unused*
1 = Jan
…
12 = Dec

- The array is used as a **dictionary**
  - each value is associated to a specific index
  - the indices are critical
- Main operations:
  - **insert**/update a value for a given index
    - *E.g., High[10] = 63 -- the average high for October is 63°F*
  - **lookup** the value associated to an index
    - *E.g., High[3] -- looks up the average high for March*

Dictionaries, beyond Arrays

- Generalize index-to-value mapping of arrays so that
  - index does not need to be a contiguous number starting at 0
  - in fact, index doesn't have to be a number at all

- A **dictionary** is a mapping from keys to values

key

entry if *e* contains *k*
value otherwise

$$k \longmapsto e$$

  - e.g.: mapping from month to high temperature (*value*)

    key "march" $\longmapsto$ 50 value

  - e.g.: mapping from student id to student record (*entry*)

    key "honk" $\longmapsto$ ("Honk", "!", "honk", "2019") entry

  - arrays: index *3* is the key, contents *A[3]* is the value

    key 3 $\longmapsto$ A[3] value

Dictionaries

key $k \longmapsto e$ **entry**

- Contains **at most** one entry associated to each key

Some keys may have **no** associated entry

- main operations:
  - create a **new** dictionary
  - **lookup** the entry associated with a key
    - or report that there is no entry for this key
  - **insert** (or update) an entry

(we will consider only these)

- many other operations of interest
  - delete an entry given its key
  - number of entries in the dictionary
  - print all entries, …

Dictionaries in the Wild

- Dictionaries are a primitive data structure in many languages
  - Like arrays in C0
  - E.g.,
    - Python
    - Javascript
    - PHP, …

Sample PHP session

Linux Terminal

```
# php -a
php > $A[0] = 3;
php > echo $A[0];
3
php > $A[15122] = 11;
php > echo $A[15122];
11
php > echo $A[3];
PHP Notice:  Undefined offset: 3 in php shell code on line 1
php > $A["hello world"] = 13;
```

- They are not primitive in low level languages like C and C0
  - We need to implement them and provide them as a library
  - This is also what we would do to write a Python interpreter

Implementing Dictionaries

- based on what we know so far …
  - worst-case complexity assuming the dictionary contains *n* entries

Binary search

Move other elements out of the way

Linear search

adding to an unbounded array

Linear search on list

Add to the front of the list

	unsorted array with (key, value) data	***(key, value) array sorted by key***	***linked list with (key, value) data***
lookup	*O(n)*	*O(log n)*	*O(n)*
insert	*O(1) amortized*	*O(n)*	*O(1)*

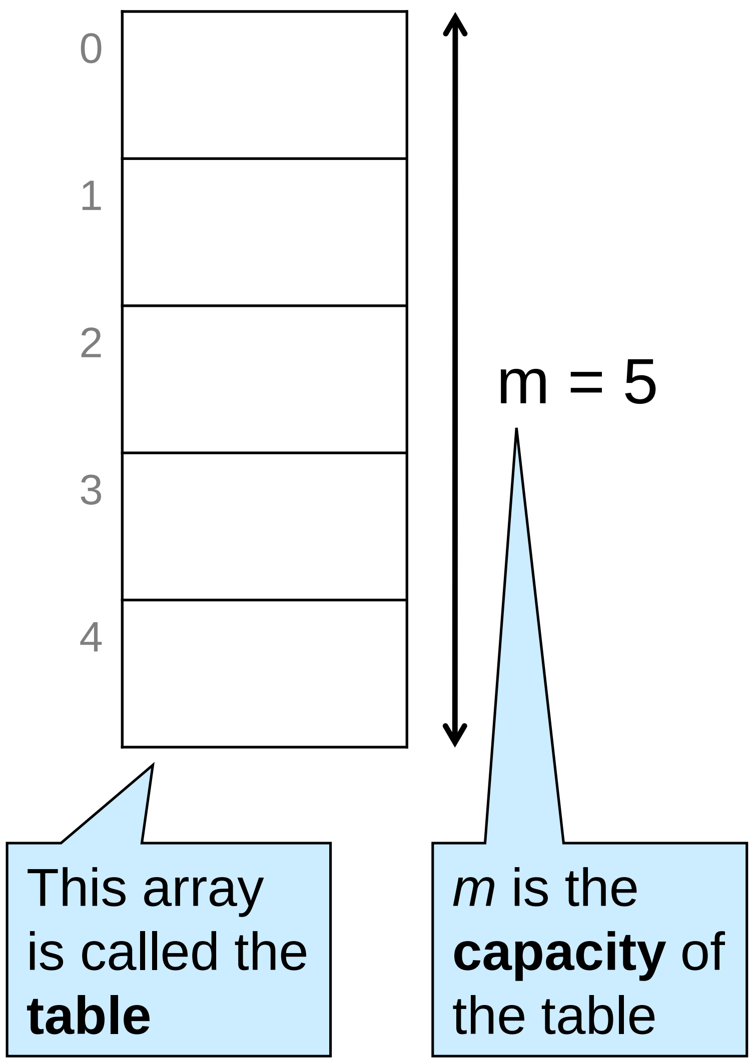
  - **Observation**: operations are fast when we know where to look
- **Goal**: efficient lookup and insert for large dictionaries
  - about O(1)

Dictionaries with Sparse Numerical Keys

Example

A dictionary that maps zip codes (keys) to neighborhood names (values) for the students in this room

- zip codes are 5-digit numbers -- e.g., 15213
  - use a 100,000-element array with indices as keys?
  - possibly, but most of the space will be wasted:
    - only about 200 students in the room
    - only some 43,000 zip codes are currently in use
- Use a much smaller *m*-element array
    - here m=5
  - reduce a key to an index in the range [0,m)
    - here reduce a zip code to an index between 0 to 4
    - do zipcode % 5



- This is the first step towards a **hash table**

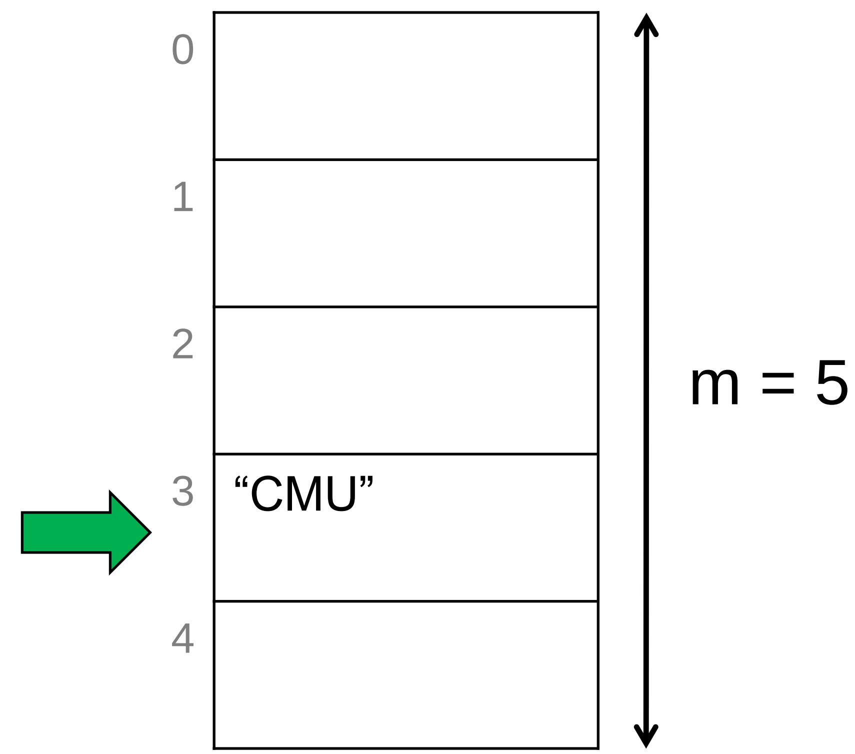
Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

- We now perform a sequence of insertions and lookups

key value

- insert (15213, “CMU”)
  - compute table index as 15213 % 5 = 3
    - insert “CMU” at index 3

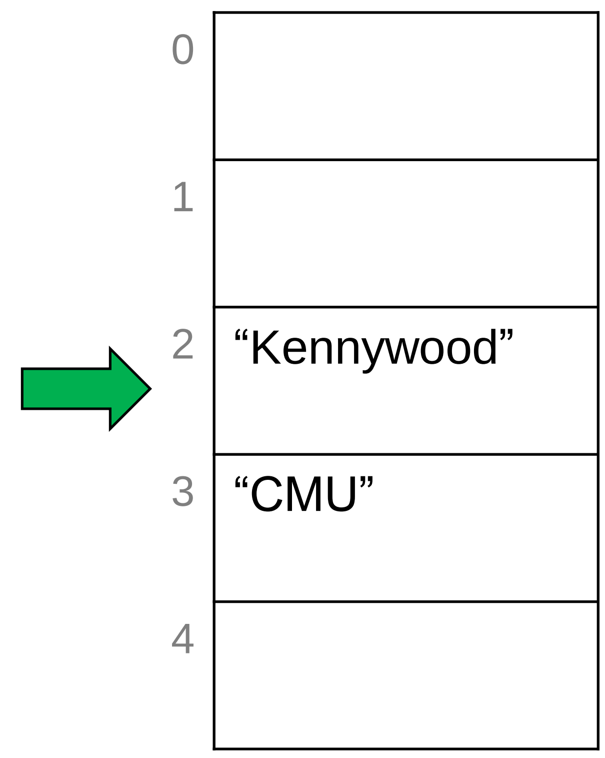


Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key | value

- insert (15122, “Kennywood”)
  - compute table index as 15122 % 5 = 2
    - insert “Kennywood” at index 2



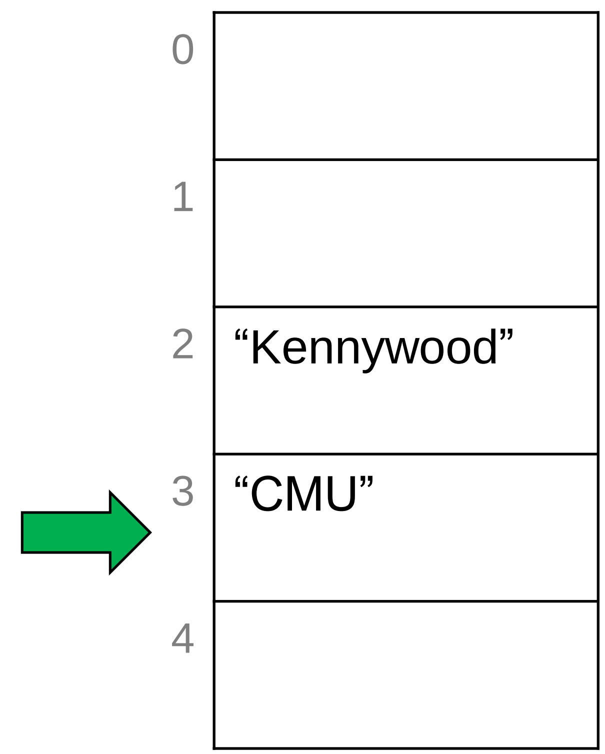
Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- lookup 15213
  - compute table index as 15213 % 5 = 3
    - return contents of index 3
      - “CMU”

value



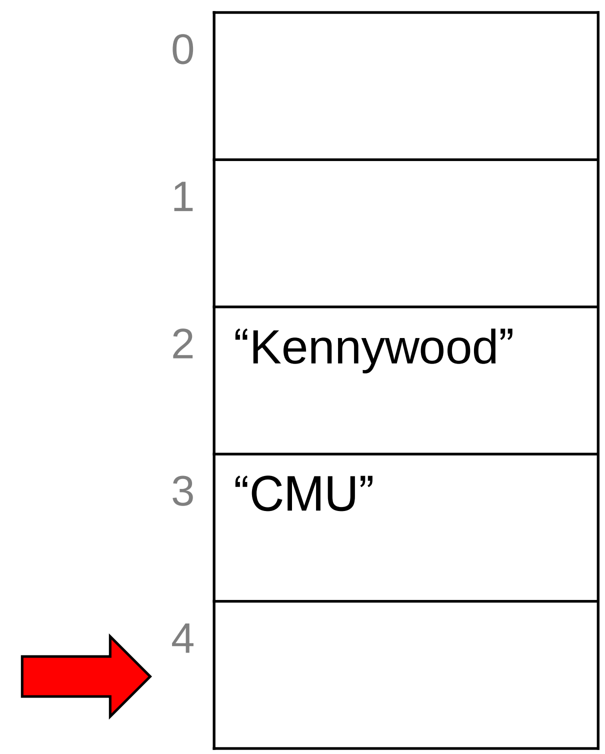
Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- lookup 15219
  - compute table index as 15219 % 5 = 4
    - nothing at index 4
    - report there is no value for 15219

no value



Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- lookup 15217
  - compute table index as 15217 % 5 = 2
    - return contents of index 2
      - “Kennywood”

value

0	
1	
2	“Kennywood”
3	“CMU”
4	

- This is **incorrect**!
  - we never inserted an entry with key 15217
  - it should signal there is no value

We need to store **both** the **key** and the **value** -- the whole **entry**

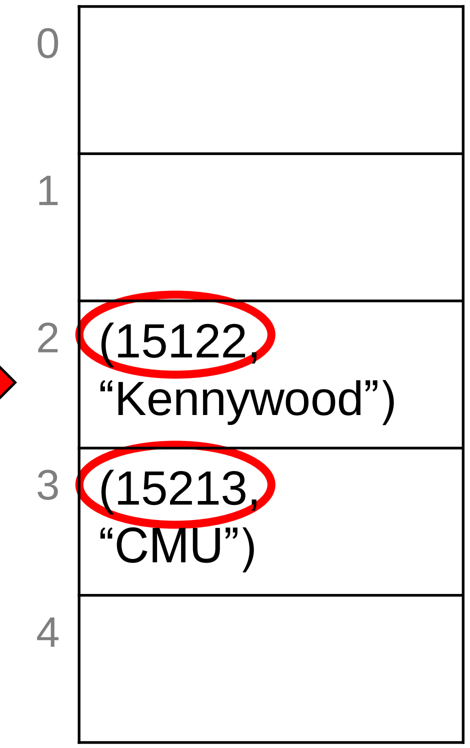
Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- ○ lookup 15217
  - ➢ compute table index as 15217 % 5 = 2
    - ❑ check the key at index 2
      15122 ≠ 15217
    - ❑ entry at index 2 is not about this key

no value for 15217



- lookup now returns a whole entry

Example

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- insert (15217, “Squirrel Hill”)
  - compute table index as 15217 % 5 = 2
    - there is an entry in there
    - check its key

      15122 ≠ 15217 ✗
    - entry at index 2 is not about this key

0	
1	
2	(15122, “Kennywood”)
3	(15213, “CMU”)
4	

- We have a **collision**
  - different entries map to the same index

Dealing with Collisions

Two common approaches

- **Open addressing**
  - if a table index is taken, store the new entry at a predictable index nearby
    - **linear probing**: use next free index (modulo m)
    - **quadratic probing**: try table index + 1, then +4, then +9, etc.
- **Separate chaining**
  - do not store the entries in the table itself but in **buckets**
    - bucket for a table index contains all the entries that map to that index
    - buckets are commonly implemented as **chains**
      - a chain is a NULL-terminated linked list

Collisions are Unvoidable

- If n > m
  - **pigeonhole principle**
    - *“If we have n pigeons and m holes and n > m, one hole will have more than one pigeon”*
  - This is a certainty

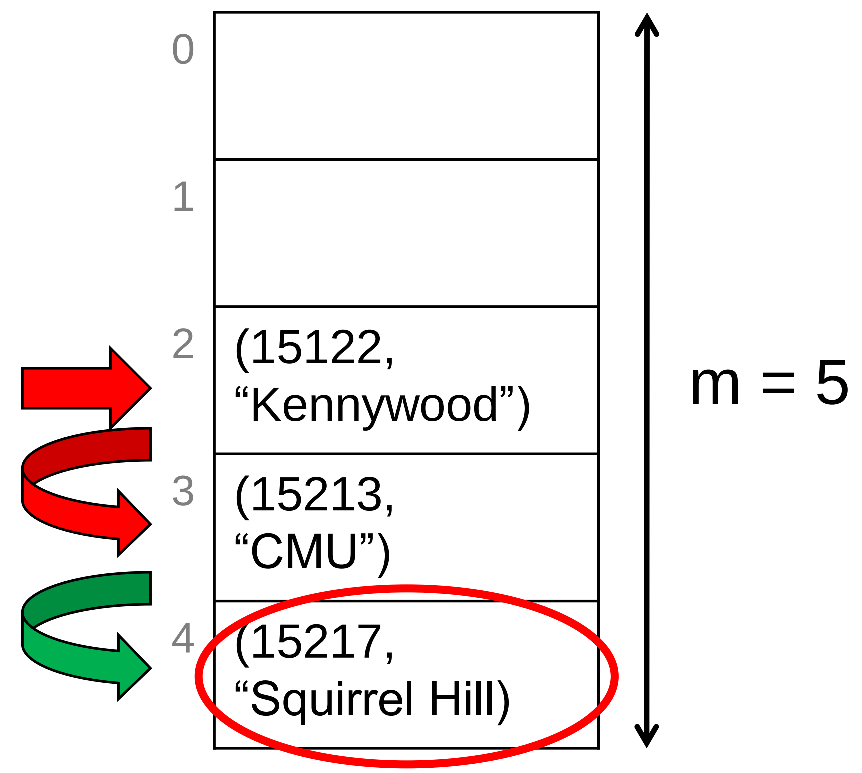
- If n > 1
  - **birthday paradox**
    - *“Given 25 people picked at random, the probability that 2 of them share the same birthday is > 50%”*
  - This is a probabilistic result

Example, continued with linear probing

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- insert (15217, “Squirrel Hill”)
  - compute table index as 15217 % 5 = 2
    - there is an entry in there
    - check its key: 15122 ≠ 15217 ✗
  - try next index, 3
    - there is an entry in there
    - check its key: 15213 ≠ 15217 ✗
  - try next index, 4
    - there is no entry in there ✓
    - insert (15217, “Squirrel Hill”) at index 4



Example, continued with linear probing

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

- Lookup 15217
  - compute table index as 15217 % 5 = 2
    - there is an entry in there
    - check its key: 15122 ≠ 15217 ✗
  - try next index, 3
    - there is an entry in there
    - check its key: 15213 ≠ 15217 ✗
  - try next index, 4
    - there is an entry in there
    - check its key: 15217 = 15217 ✓
    - return (15217, “Squirrel Hill”)

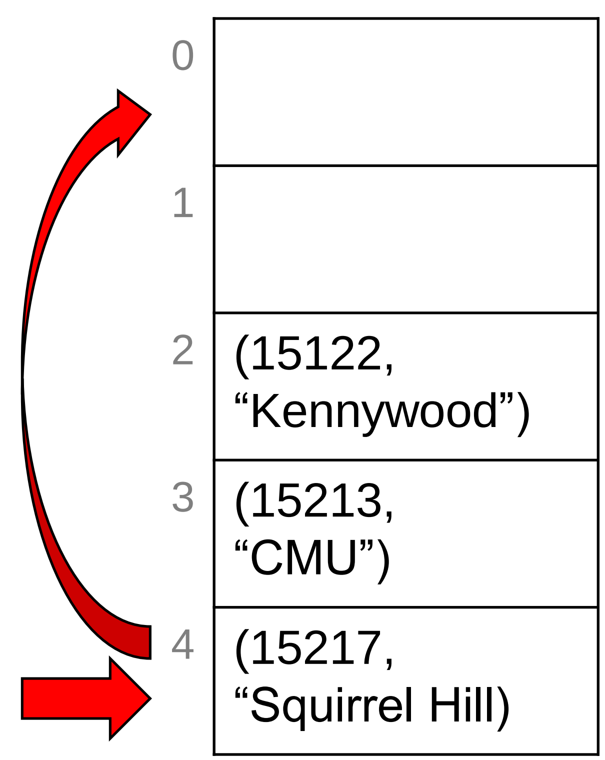
0	
1	
2	(15122, “Kennywood”)
3	(15213, “CMU”)
4	(15217, “Squirrel Hill)

Example, continued with linear probing

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

key

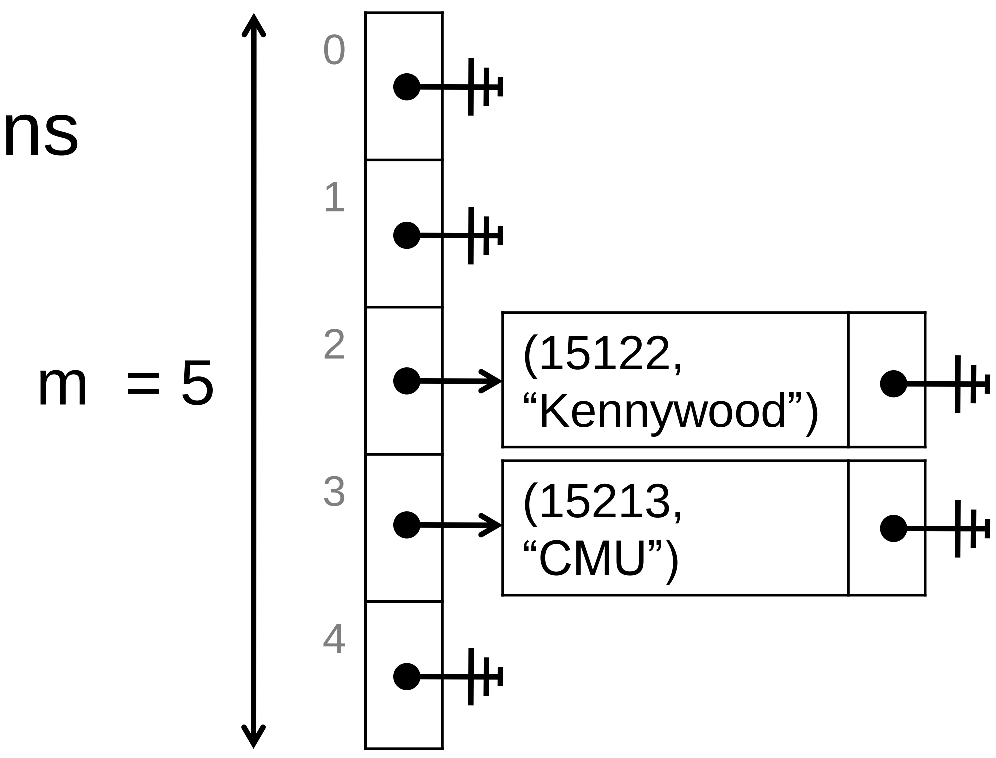
- Lookup 15219
  - compute table index as 15219 % 5 = 4
    - there is an entry in there
    - check its key: 15217 ≠ 15219 ✘
  - try next index, 5 % 5 = 0
    - there is no entry in there
    - report there is no entry for 15219 ✘



Example, continued with separate chaining

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

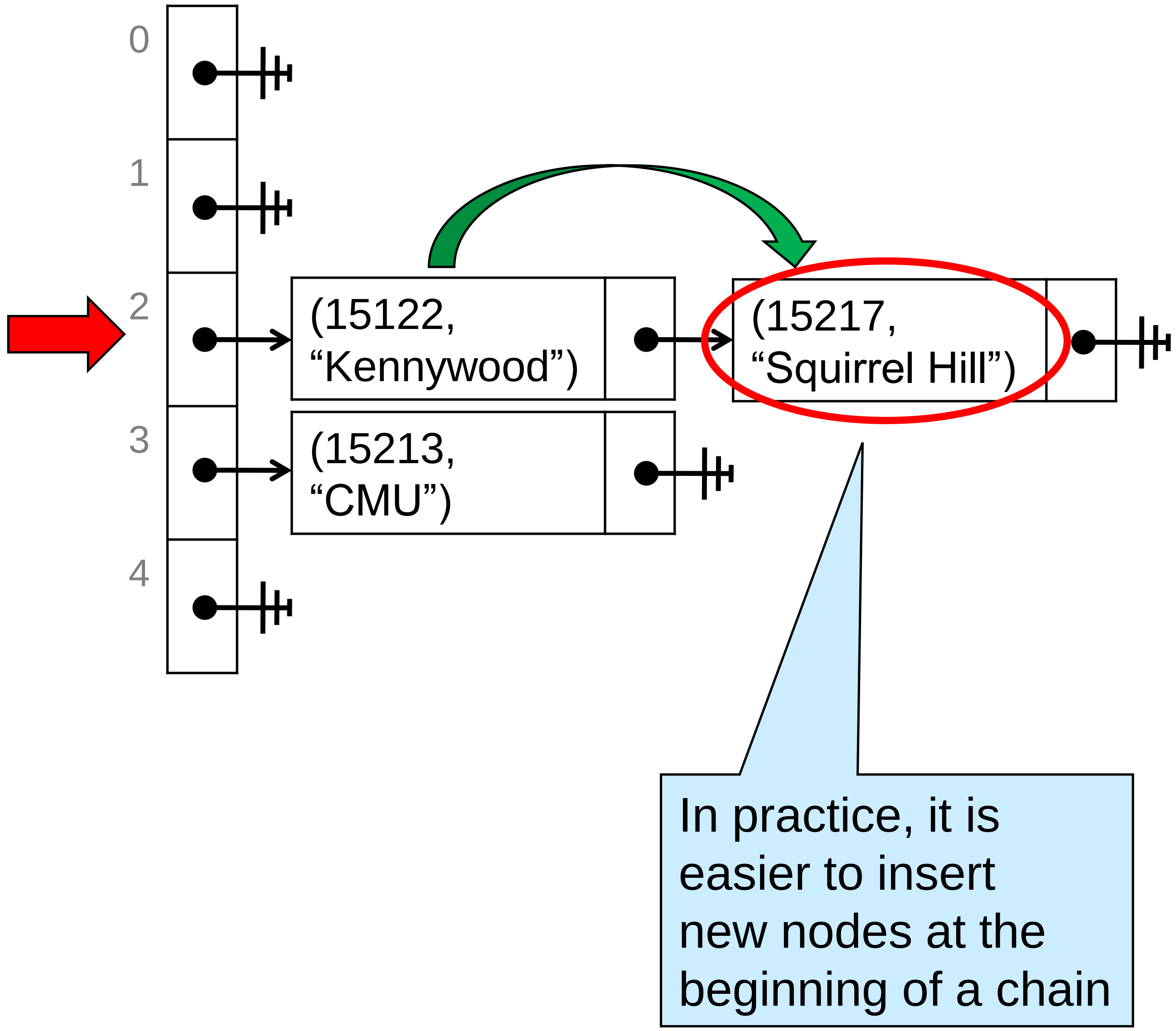
- Each table position contains a chain
  - a NULL-terminated linked list of entries
  - the chain at index *i* contains all entries that map to *i*



Example, continued with separate chaining

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

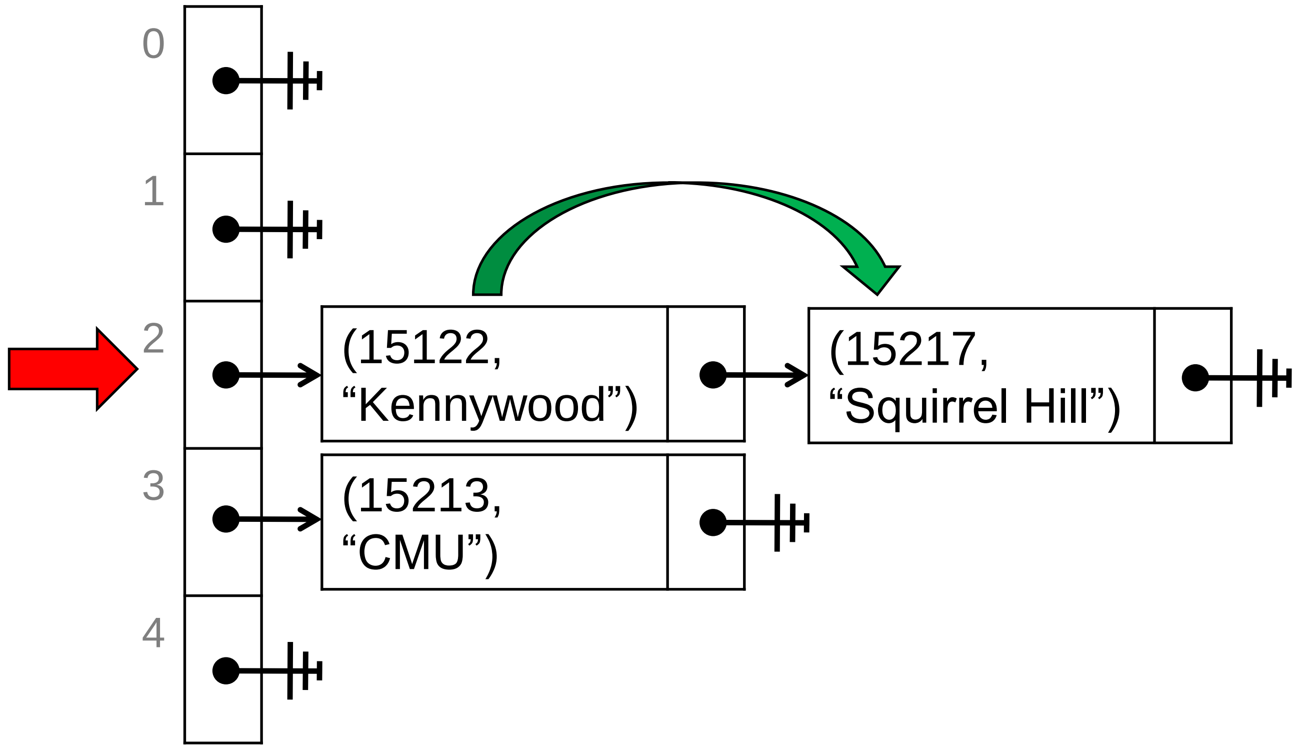
- insert (15217, “Squirrel Hill”)
  - compute table index as 15217 % 5 = 2
    - points to a chain node
    - check its key: 15122 ≠ 15217 ✗
  - try next node
    - there is no next node ✓
    - create new node and insert (15217, “Squirrel Hill”) in it



Example, continued with separate chaining

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

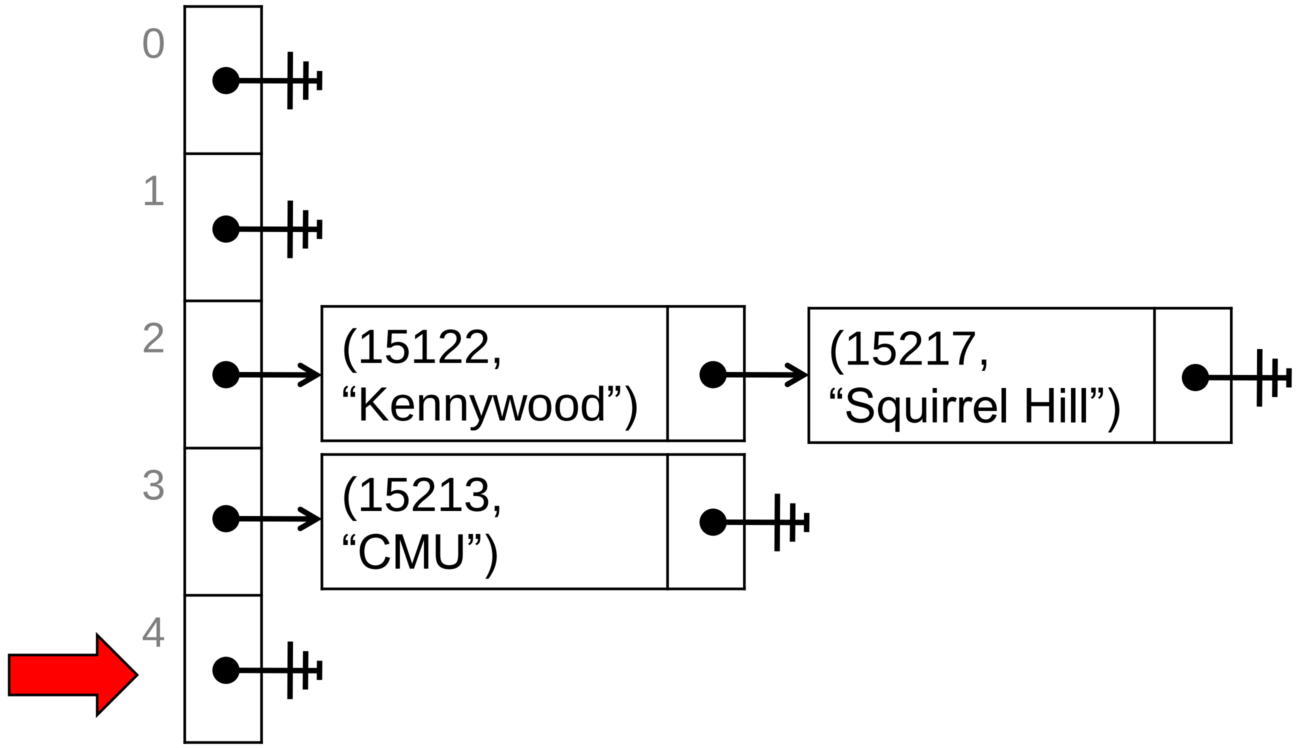
- lookup 15217
  - compute table index as 15217 % 5 = 2
    - points to a chain node
    - check its key: 15122 ≠ 15217 ✗
  - try next node
    - check its key: 15217 = 15217 ✓
    - return (15217, “Squirrel Hill”)



Example, continued with separate chaining

insert (15213, “CMU”)
insert (15122, “Kennywood”)
lookup 15213
lookup 15219
lookup 15217
insert (15217, “Squirrel Hill”)
lookup 15217
lookup 15219

- lookup 15219
  - compute table index as 15219 % 5 = 4
    - there is no chain node
    - report there is no entry for 15219



Cost Analysis

Setup

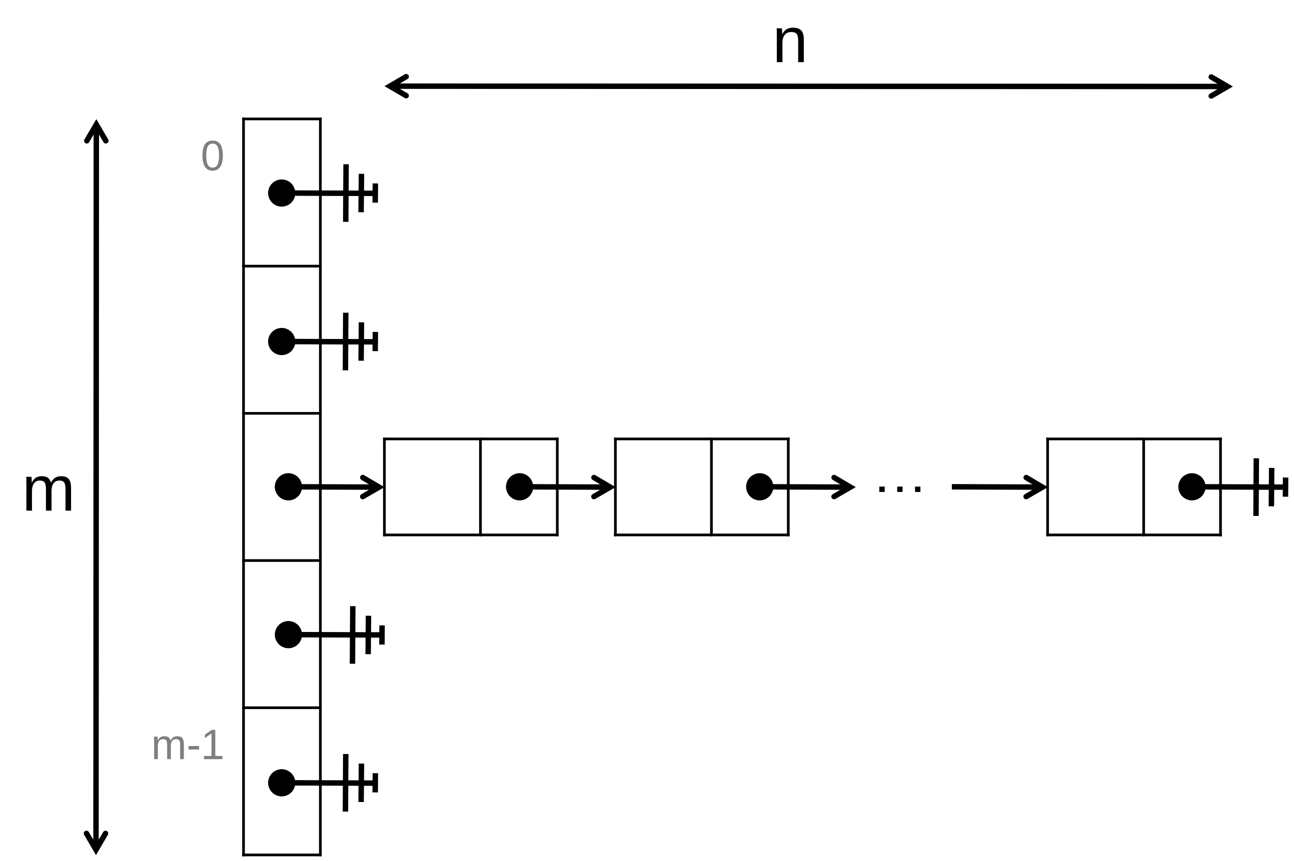
- Assume
  - the dictionary contains *n* entries
  - the table has capacity *m*
  - collisions are resolved using separate chaining
    - the analysis for open addressing requires more advanced math
      - but it yields similar findings

- What is the cost of **lookup** and **insert**?

  - Observe that **insert** costs *at least* as much as **lookup**
    - we need to check if an entry with that key is already in the dictionary
      - if so, replace that entry (update)
      - if not, add a new node to the chain (proper insert)

Worst Possible Layout

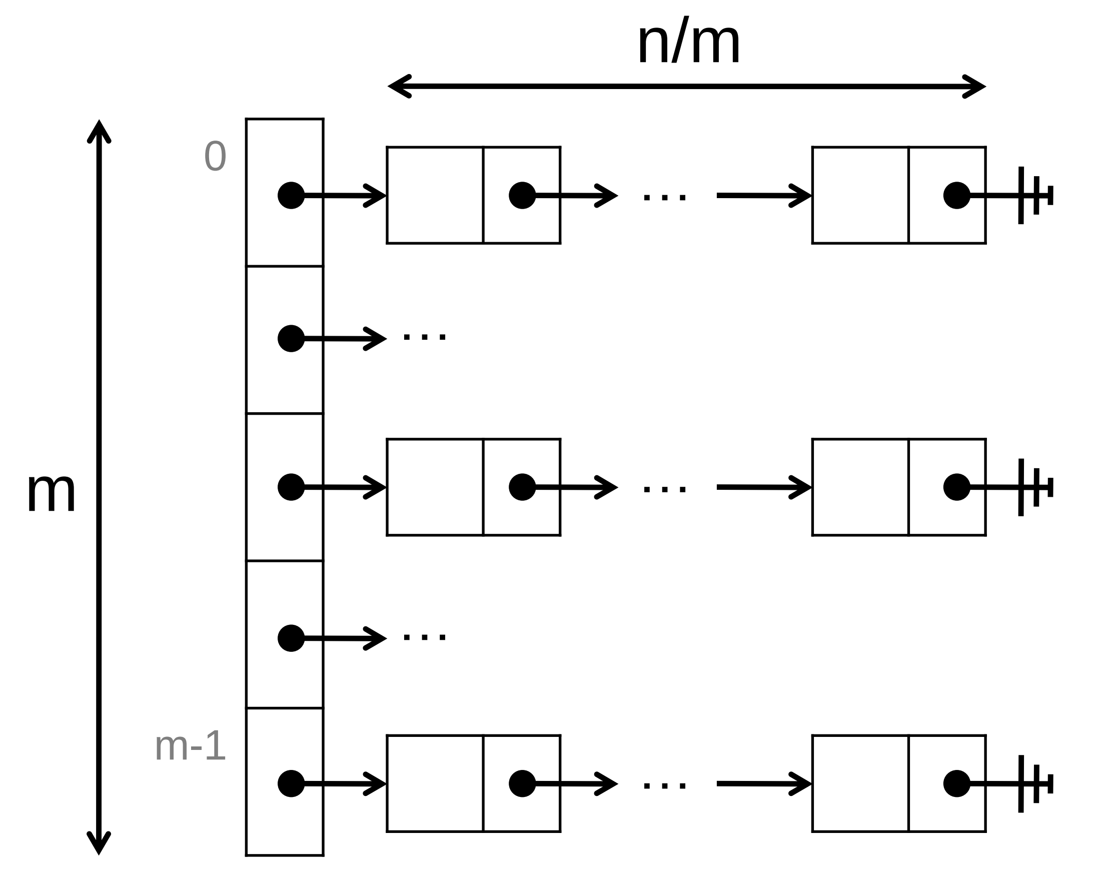
- All entries are in the same bucket
  - look for a key that belongs to this bucket but that is not in the dictionary



- Looking up a key has cost O(n)
  - find the bucket -- O(1)
  - going through all n nodes in the chain

Best Possible Layout

- All buckets have the same number of entries
  - all chains have the same length
    - n/m
  - n/m is called the **load factor** of the table
    - in general, the load factor is a fractional number, e.g., 1.2347
- Looking up a key has **worst-case** cost O(n/m)
  - find the bucket -- O(1)
  - go through all n/m nodes in the chain

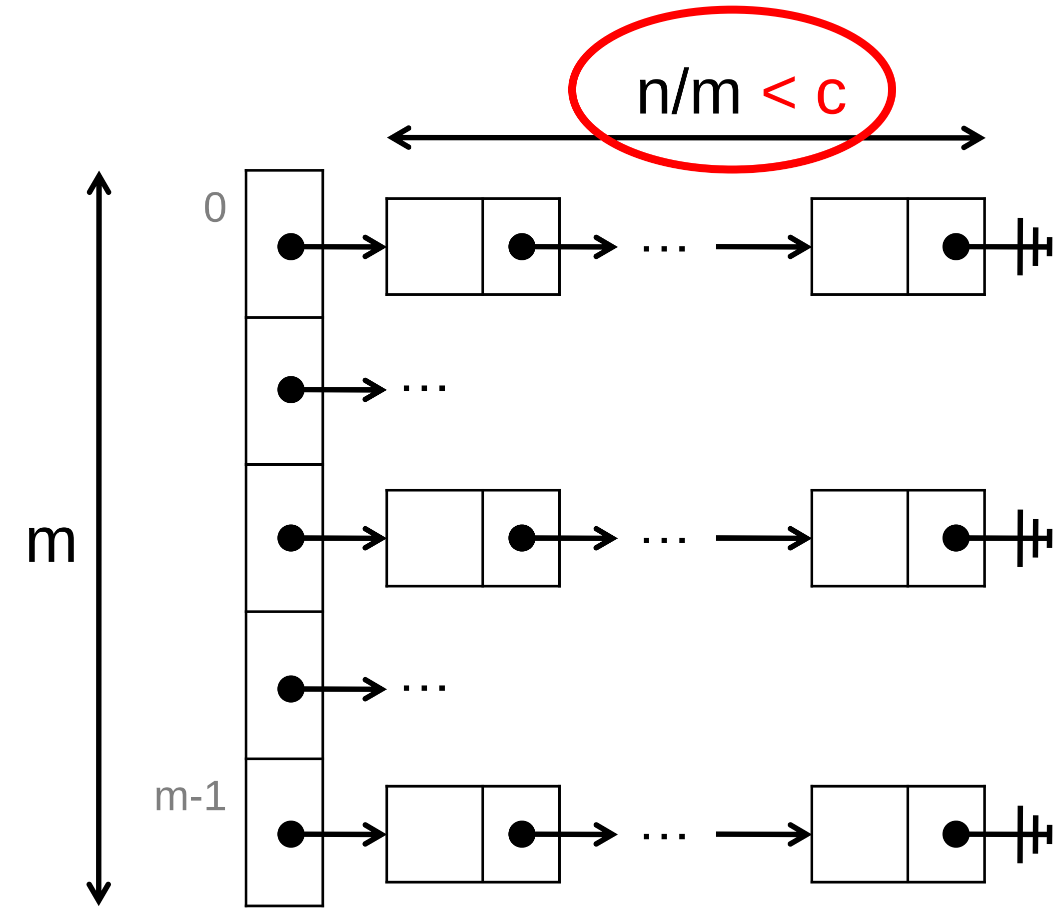


Best Possible Layout

Cost is O(n/m)

- Can we arrange so that n/m is **about constant**?
  - Yes! Resize the table when n/m reaches a fixed threshold c
    - often, we choose c = 1.0

c is a **constant**



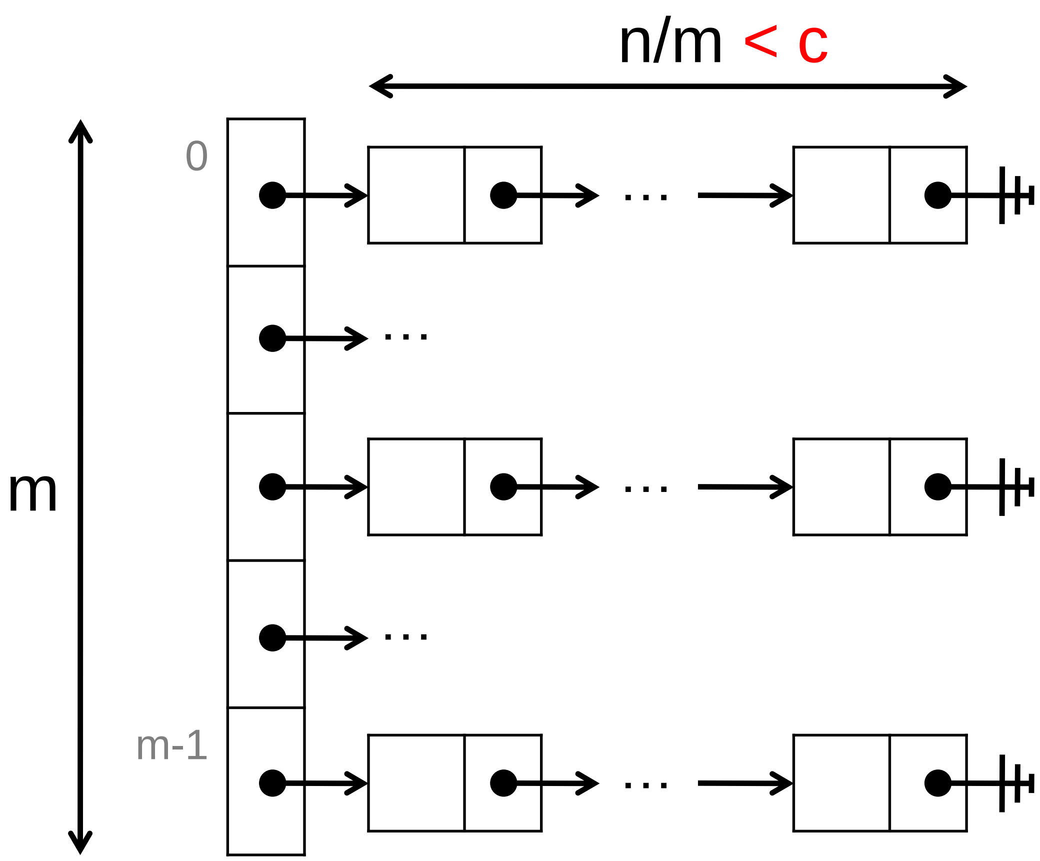
- When inserting, **double** the size of the table when n/m reaches c
- The cost of insert becomes **O(1) amortized**
  - like with unbounded arrays

Best Possible Layout

Why O(1) amortized?

- Setup
  - dictionary contains n entries
  - table has capacity m
  - $n/m < c$

c is a constant



- After inserting a new entry,
  - either $(n+1)/m < c$
  - or $(n+1)/m \geq c$

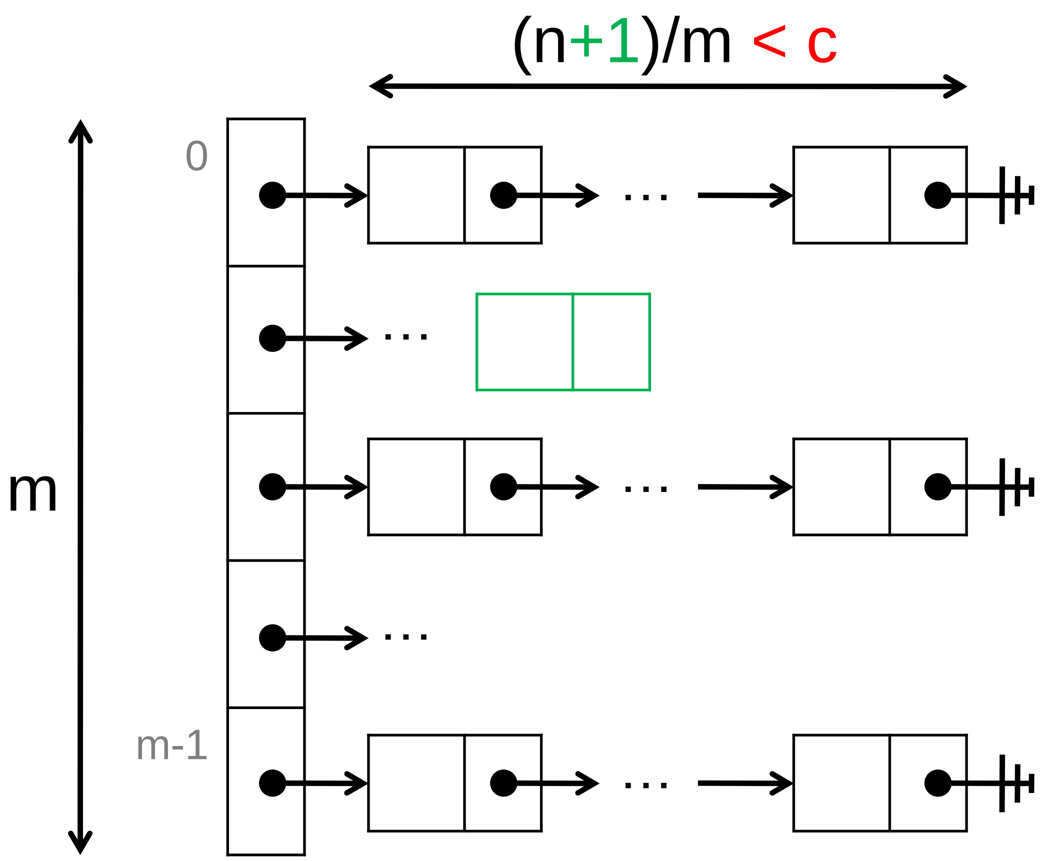
Resize the table

Best Possible Layout

Why O(1) amortized?

- Case (n+1)/m < c
  - go to the right bucket
  - check if it contains an entry with this key
    - examine about n/m nodes
    - that's at most c nodes

    c is a constant
  - insert or update the entry



This insert costs O(1)

Since (n+1)/m < c, the next lookup also costs O(1)

Best Possible Layout

Why O(1) amortized?

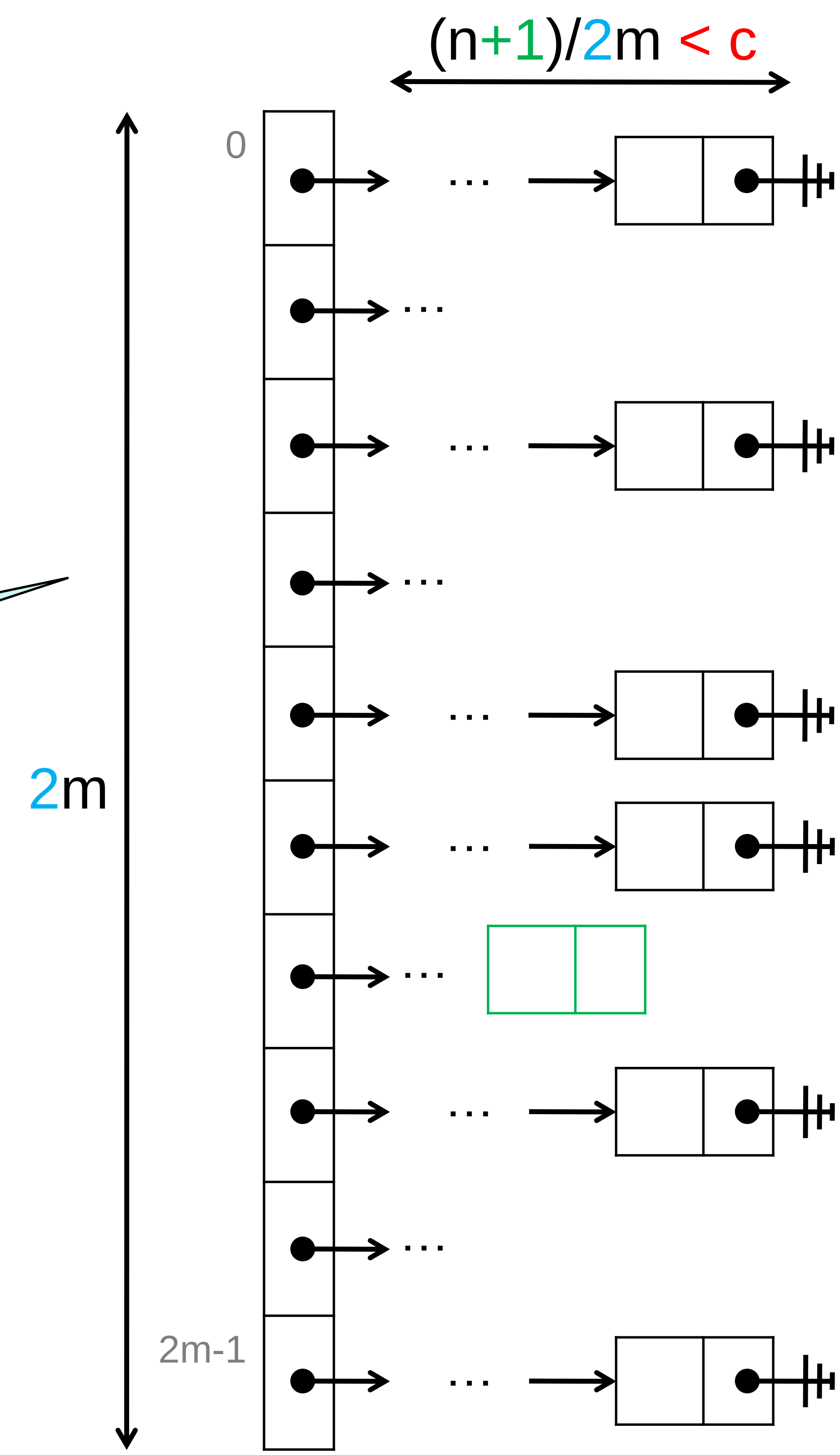
- Case $(n+1)/m \geq c$
  - double the table capacity to 2m
  - insert **all entries** into the new table
    - n times O(1)
    - that's O(n)

***If** we keep on being in the best possible layout*

This insert costs O(n)

  - The new load factor is $(n+1)/2m < c$
    - because $(n+1)/2m < 2n/2m = n/m < c$

Thus, the next lookup costs O(1)



Best Possible Layout

Why O(1) amortized?

- After inserting a new entry,
  - either $(n+1)/m < c$
    - costs O(1)
  - or $(n+1)/m \geq c$
    - costs O(n)
    - but the next n inserts will cost O(1)

This is cheap!

This is expensive!

Assuming we still have the best possible layout …

- Just like with unbounded array
  - many cheap operations can pay for the rare expensive ones
- Thus, insert has **O(1) amortized cost**
  - because lookup depends on what was inserted in the table, it has cost O(1)

Best Possible Layout

- Assuming chains always have the same length and the table is self-resizing
  - **insert** costs **O(1) amortized**
    - *amortized* because some insertions trigger a table resize
  - **lookup** costs **O(1)**
    - lookup never triggers a resize
- *But is this a reasonable assumption to make?*

Most insertions cost O(1), but a few cost O(n)

Lookups always cost O(1)

Without this assumption, both lookup and insert cost O(n) in the worst case

Best Possible Layout

- What does it take to be in this ideal case?
  - The indices associated with the keys in the table need to be **uniformly distributed** over [0,m)
  - This happens when the keys are chosen at **random** over the integers

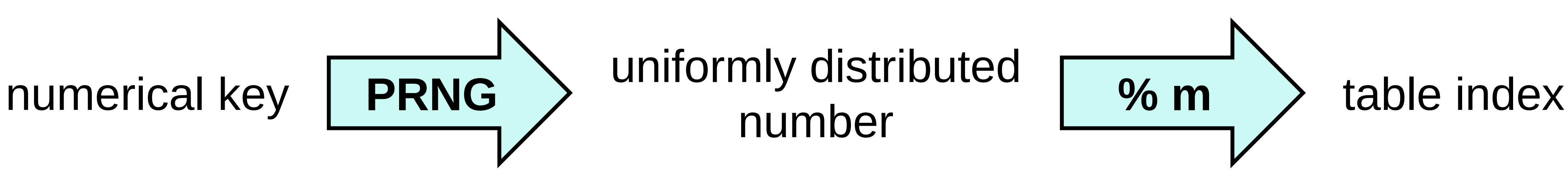
- Is this typical?
  - Keys are rarely random
    - e.g., if we take first digit of zip code (instead of last)
      - many students from Pennsylvania: lots of 1
      - many students from the West Coast: lots of 9 (mapped to 4, modulo 5)
  - We shouldn't count on it

- Making this assumption is not reasonable

Best Possible Layout

- Can we *arrange* so that we **always** end up in this ideal case?
  - ➢ unless we are really, really unlucky
  - We want the indices associated to keys to be scattered
    - ➢ be **uniformly distributed** over the table indices
    - ➢ bear little relation to the key itself

- Run the key through a **pseudo-random number generator**
  - *"random number generator":* result *appears* random
    - ❑ uniformly distributed
    - ❑ (apparently) unrelated to input
  - *"pseudo":* always returns the same result for a given key
    - ❑ deterministic



Best Possible Layout

- *Arrange so that we **always** end up in the ideal case*
  - *unless we are really, really unlucky*
  - by running the key through a pseudo-random number generator

numerical key $\xrightarrow{\textbf{PRNG}}$ uniformly distributed number $\xrightarrow{\textbf{\% m}}$ table index

- Then, lookup has O(1) **average case complexity**
  - because it will almost always be in the ideal case
    - but if we are really, really unlucky
      - all keys may end up in the same bucket
      - the worst-case complexity remains is O(n)
- And insert has O(1) **average and amortized** complexity

Hash Tables

$$\text{numerical key} \; \underbrace{\xrightarrow{\textbf{PRNG}}}_{\textbf{hash function}} \; \underbrace{\text{uniformly distributed number}}_{\textbf{hash value}} \; \xrightarrow{\textbf{\% m}} \; \underbrace{\text{table index}}_{\textbf{hash index}}$$

This is a **hash table**

- a PRNG an example of a **hash function**
  - a function that turns a key into a number on which to base the table index
- its result is a **hash value**
- it is then turned into a **hash index** in the range [0, m)

$$\text{numerical key} \; \xrightarrow{\textbf{hash function}} \; \text{hash value} \; \xrightarrow{\textbf{\% m}} \; \text{hash index}$$

Hash Table Complexity

- Complexity of **insert**, assuming
  - the dictionary contains *n* entries
  - the table has capacity *m*
  - and …

Output is **uniformly distributed** and **unrelated to input**

Average

	Bad hash function	***Good hash function***
No resizing	*O(n)*	*(Left as exercise)*
UBA-style resizing	*(Left as exercise)*	*O(1) <u>average</u> and <u>amortized</u>*

Amortized

Double the size of the table when load factor exceeds target

From good hash function

From UBA-style resizing

Hash Table Complexity

- Complexity of **lookup**, assuming
  - the dictionary contains *n* entries
  - the table has capacity *m*
  - and …

Average

Output is **uniformly distributed** and **unrelated to input**

	Bad hash function	***Good hash function***
No resizing	*O(n)*	*(Left as exercise)*
UBA-style resizing	*(Left as exercise)*	*O(1) <u>average</u>*

insert doubles the size of the table when load factor exceeds target

From good hash function and insert producing chains of about the same length

Pseudo-Random Number Generators

Linear Congruential Generators

- A common form of PRNG is

$$f(x) = a * x + c \bmod d$$

  - for appropriate constants *a*, *c* an *d*

- With 32-bit ints and handling overflow via modular arithmetic, we choose $d = 2^{32}$
  - *mod d* is automatic
- To get uniform distribution, we pick
  - a ≠ 0
  - a and d to be relative primes
  - c and d to be relative primes

- This is called a **linear congruential generator** (LCG)
  - Cost is O(1)

Linear Congruential Generators

$$f(x) = a * x + c \bmod d$$

- $a \neq 0$, and c and d relatively prime
- $d = 2^{32}$

- Implemented in the C0 **rand library**

  #use <rand>

  - a = 1664525
  - c = 1013904223

- Do it yourself?

```
int lgc(int x) {
  return 1664525 * x + 1013904223 ;
}
```

The rand library is a bit more general. It's interface is:

```
// typedef ___ rand_t;
rand_t init_rand (int seed);
int rand(rand_t gen):
```

Look it up!

Cryptographic Hash Functions

- Hash functions are used pervasively in cryptography
- Cryptographic hash functions have additional requirements
  - practically impossible to find x given h(x)
  - practically impossible to find x and a different y such that h(x) = h(y)
- Cryptographic hash functions are overkill for use in hash tables

Non-numerical Keys

Hashing Non-numerical Keys

- Simply transform the key into a number first (*cheaply*)

$$\text{key} \;\underbrace{\xrightarrow{\textbf{transformation}} \text{number} \xrightarrow{\textbf{PRNG}}}_{\textbf{hash function}}\; \text{hash value} \xrightarrow{\textbf{\% m}} \text{hash index}$$

- The whole transformation from key to hash value is called the hash function
  - often implemented as a single function

$$\text{key} \xrightarrow{\textbf{hash function}} \text{hash value} \xrightarrow{\textbf{\% m}} \text{hash index}$$

Dictionaries Summary

- We can use hash tables to implement efficient dictionaries
  - type of keys can be anything we want
  - O(1) average cost for **lookup**
  - O(1) average and amortized cost for **insert**

key ⇒ **hash function** ⇒ hash value ⇒ **% m** ⇒ hash index

  - Collision resolved via separate chaining or open addressing
    - Open addressing is more common in practice
      - faster
      - uses less space
- They are called **hash dictionaries**

Dictionaries Summary

- Complexity assuming
  - the dictionary contains *n* entries
  - the table has capacity *m*

	unsorted array with (key, value) data	***(key, value) array sorted by key***	***linked list with (key, value) data***	***Hash Tables***
lookup	*O(n)*	*O(log n)*	*O(n)*	*O(n)* *O(1) average**
insert	*O(1) amortized*	*O(n)*	*O(1)*	*O(n)* *O(1) average* and amortized***

**average = by using a good hash function*

***amortized = by resizing the table*

  - *The same analysis applies for open addressing hash tables*

What about Sets?

- A **set** can be understood as a special case of a dictionary
  - keys = entries
    - These are the elements of the set
  - **lookup** can simply return true or false
    - this now checks set membership

- A set implemented as a hash dictionary is called a **hash set**