7 Identical Particles in Hilbert Space

			state func	Prob
Config	$\overset{1}{\cdot}$ $\vec{x}_1$	$\overset{2}{\cdot}$ $\vec{x}_2$	$\psi(\vec{x}_1,\vec{x}_2)$	$\lvert\psi(\vec{x}_1,\vec{x}_2)\rvert^2$
	position			
exchange	$\overset{2}{\cdot}$ $\vec{x}_1$	$\overset{1}{\cdot}$ $\vec{x}_2$	$\psi(\vec{x}_2,\vec{x}_1)$	$\lvert\psi(\vec{x}_2,\vec{x}_1)\rvert^2$

thus we have

(*) $|\psi(1,2)\rangle = |\psi(2,1)\rangle$ or $|\psi(1,2)\rangle = -|\psi(2,1)\rangle$

N id.particles has $N!$ permutation forming permutation group S_N

From observation (*)

Parity permutation $(-1)^P = \begin{cases} 1 & \text{if } P \text{ even} \\ -1 & \text{if } P \text{ odd} \end{cases}$

Remark:
- action in permuting N id.particles yields S_N rep. in Hilbert space
- Yield two id.particles in nature.

1) Boson $P|\psi(1,\dots,N)\rangle = +|\psi(1,\dots,N)\rangle$ fully symmetric

2) Fermions $P|\psi(1,\dots,N)\rangle = (-1)^P|\psi(1,\dots,N)\rangle$ " anti-sym.

Caveat
- Ham of id.particles must be itself symmetric. (ie exchange loc of id.particles won't alter its energy)
- specified stat. of particles (Boson/Fermions) is needed to give correct eig. stat of H.

ex Free particle has Ham.

$$H = \sum_{\alpha=1}^{N} H_\alpha = \sum_{\alpha=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla_\alpha^2\right)$$ clearly it has product Hilbert space

$|\vec{k}_1,\dots,\vec{k}_N\rangle_\otimes = |\vec{k}_1\rangle\cdots|\vec{k}_N\rangle$ w/ which

$$H|\vec{k}_1,\dots,\vec{k}_N\rangle_\otimes = \sum_{\alpha=1}^{N}\frac{\hbar^2 k_\alpha^2}{2m}|\vec{k}_1,\dots,\vec{k}_N\rangle_\otimes$$ and w/ coord rep.

$$\langle\vec{x}_1,\dots,\vec{x}_N|\vec{k}_1,\dots,\vec{k}_N\rangle_\otimes = \frac{1}{V^{N/2}}\exp\left(i\sum_{\alpha=1}^{N}\vec{k}_\alpha\cdot\vec{x}_\alpha\right)$$

Fermionic subspace

$$|\vec{k}_1, \ldots, \vec{k}_N\rangle_- = \frac{1}{\sqrt{N_-}} \sum_P (-1)^P P |\vec{k}_1, \ldots, \vec{k}_N\rangle_\otimes$$

$N_- = N!$ for $N!$ diff. permutation - ie $\left(\frac{|\vec{k}_1, \vec{k}_2\rangle - |\vec{k}_2, \vec{k}_1\rangle}{2}\right)$

Bosonic

$$|\vec{k}_1, \ldots, \vec{k}_N\rangle = \frac{1}{\sqrt{N_+}} \sum_P P |\vec{k}_1, \ldots, \vec{k}_N\rangle_\otimes \qquad N_+ = N! \prod_{\vec{k}} n_{\vec{k}}!$$

hee $n_{\vec{k}}$ is # time a particular particle may repeated. Note $\sum_{\vec{k}} n_{\vec{k}} = N$

ex given $|\alpha\rangle$ $|\beta\rangle$ states, $n_\alpha = 2$ $n_\beta = 1$ then $N_+ = 3!\, 2!\, 1!$

$$|\alpha\alpha\beta\rangle_+ = \frac{1}{\sqrt{3}}\left(|\alpha\rangle|\alpha\rangle|\beta\rangle + |\alpha\rangle|\beta\rangle|\alpha\rangle + |\beta\rangle|\alpha\rangle|\alpha\rangle\right)$$

In general,

$$|\{\vec{k}\}\rangle_\eta = \frac{1}{\sqrt{N_\eta}} \sum_P \eta^P P |\{\vec{k}\}\rangle \quad w/ \quad \eta = \begin{cases} +1 & \text{boson} \\ -1 & \text{fermion} \end{cases} \quad \text{and} \quad \sum_{\vec{k}} \underbrace{n_{\vec{k}}}_{\text{occ. \#}} = N$$

Remark (1) fermion $|\{\vec{k}\}\rangle_- = 0$ unless $n_{\vec{k}} = 0$ or 1

(2) boson by normalization.

$$_+\langle\{\vec{k}\}|\{\vec{k}\}\rangle_+ = \frac{1}{N_+} \sum_{P,P'} \langle P\{\vec{k}\}| P'\{k\}\rangle$$

$$1 = \frac{N!}{N_+} \sum_{P'} \langle\{\vec{k}\}| P'\{k\}\rangle$$

for identical permuted set $|\{\vec{k}\}\rangle$ s.t. $\langle\{\vec{k}\}|P\{\vec{k}\}\rangle \neq 0$. $P|\{\vec{k}\}\rangle = \prod_{\vec{k}} n_{\vec{k}}!$

thus $\quad N_+ = N! \prod_{\vec{k}} n_{\vec{k}}!$

Canonical formulation.

density matrix (non-interacting id. particles)

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle_\eta = \sum_{\{\vec{k}\}}' \sum_{P,P'} \eta^P \eta^{P'} \langle\{\vec{x}'\}|P'\{\vec{k}\}\rangle \rho(\{\vec{k}\}) \langle P\{\vec{k}\}|\{\vec{x}\}\rangle \frac{1}{N_\eta}$$

$$\rho = \exp\left[-\beta \sum_\alpha \frac{\hbar^2 k_\alpha^2}{2m}\right] / Z_N$$

Remarks: • $\sum'_{\{\vec{k}\}}$ sum over distinct particle states.

• To remove overcounting in the case of boson, divide over-count factor $\frac{N!}{\prod_k n_{\vec{k}}!}$

$$\sum'_{\{\vec{k}\}} = \sum_{\{\vec{k}\}} \frac{\prod_k n_{\vec{k}}!}{N!}$$

thus $$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle = \sum_{\{\vec{k}\}} \frac{\prod_k n_{\vec{k}}!}{N!} \underbrace{\frac{1}{N!\,\prod_k n_{\vec{k}}!}}_{N_\eta} \sum_{P,P'} \frac{\eta^P \eta^{P'}}{Z_N} \exp\left(-\beta \sum_{\alpha=1}^N \frac{\hbar^2 k_\alpha^2}{2m}\right) \langle\{\vec{x}'\}|P'\{\vec{k}\}\rangle\langle P\{\vec{k}\}|\{\vec{x}\}\rangle$$

In large volume limit,

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle = \frac{1}{Z_N (N!)^2} \sum_{P,P'} \eta^P \eta^{P'} \int \prod_{\alpha=1}^N \frac{V d^3 k_\alpha}{(2\pi)^3} \exp\left(-\beta \frac{\hbar^2 k_\alpha^2}{2m}\right) \frac{\exp\left[-i\sum_{\alpha=1}^N (\vec{k}_{P\alpha}\cdot\vec{x}_\alpha - \vec{k}_{P'\alpha}\cdot\vec{x}'_\alpha)\right]}{V^N}$$

After reordering $\beta = P\alpha$,

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle = \frac{1}{Z_N (N!)^2} \sum_{P,P'} \eta^P \eta^{P'} \prod_{\alpha=1}^N \left[\int \frac{d^3 \vec{k}_\alpha}{(2\pi)^3} e^{-i k_\alpha \cdot (\vec{x}_{P\alpha} - \vec{x}'_{P'^{-1}\alpha}) - \beta \frac{\hbar^2 k_\alpha^2}{2m}}\right]$$

we have $$\frac{1}{\lambda^3} \exp\left[-\frac{\pi}{\lambda^2}\left(\vec{x}_{P^{-1}\alpha} - \vec{x}'_{P'^{-1}\alpha}\right)^2\right]$$

set $Q = P'^{-1}P$ then

$$\langle\{\vec{x}'\}|\rho|\{\vec{x}\}\rangle = \frac{1}{Z_N \lambda^{3N} N!} \sum_Q \eta^Q \exp\left[-\frac{\pi}{\lambda^2} \sum_{\beta=1}^N (\vec{x}_\beta - \vec{x}'_{Q\beta})^2\right]$$

w/ $\mathrm{tr}(\rho) = 1 \Rightarrow \int \prod_{\alpha=1}^N d^3\vec{x}_\alpha \langle\{\vec{x}\}|\rho|\{\vec{x}\}\rangle = 1$ in cont'

then

$$Z_N^\pm \equiv Z_N = \frac{1}{N!\,\lambda^{3N}} \int \prod_{\alpha=1}^N d^3\vec{x}_\alpha \sum_Q \eta^Q \exp\left[-\frac{\pi}{\lambda^2} \sum_{\beta=1}^N (\vec{x}_\beta - \vec{x}_{Q\beta})^2\right]$$

which reduced to classical when $Q = 1$.

$$Z_N = \left(\frac{V}{\lambda^3}\right)^N \frac{1}{N!}$$

For lowest order approx. from exchange of two particles,

$$Z_N = \frac{1}{N!\,\lambda^{3N}} \int \prod_{\alpha=1}^{N} d^3\vec{x}_\alpha \left\{ 1 + \frac{N(N+1)}{2} \eta \exp\left[-\frac{2\pi}{\lambda^2} (\vec{x}_1 - \vec{x}_2)^2 \right] + \cdots \right\}$$

For $\alpha \geq 3$ $\int d^3\vec{x}_\alpha = V$ using $\vec{r}_{12} = \vec{x}_2 - \vec{x}_1$ then

$$(**) \qquad Z_N = \frac{1}{N!\,\lambda^{3N}} V^N \left[1 + \frac{N(N-1)}{2V} \eta \int d^3\vec{r}_{12}\, e^{-2\pi \vec{r}_{12}^{\,2}/\lambda^2} + \cdots \right]$$

$$Z_N = \frac{1}{N!} \left(\frac{V}{\lambda^3} \right)^N \left[1 + \frac{N(N-1)}{2V} \left(\sqrt{\frac{2\pi\lambda^2}{4\pi}} \right)^3 \eta + \cdots \right]$$

Free energy.

$$F = -k_B T \ln Z_N$$

which gives gas pressure:

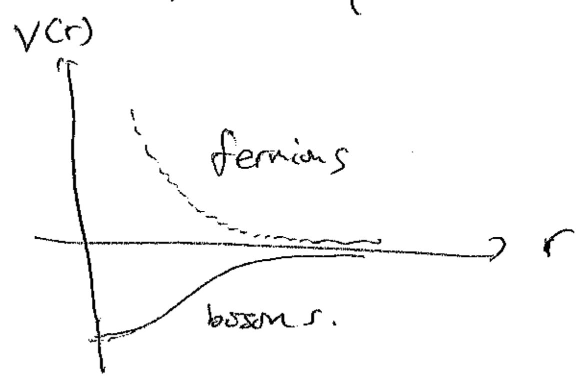
$$P = -\left.\frac{\partial F}{\partial V}\right|_T = n k_B T \left[1 - \frac{\eta \lambda^3}{2^{5/2}} n + \cdots \right] \qquad \text{w/ } B_2 = -\frac{\eta\lambda^3}{2^{5/2}}$$

By comparing (**), we obtain the classical potential $\mathcal{V}(\vec{r})$:

$$f(\vec{r}) = e^{-\beta \mathcal{V}(\vec{r})} - 1 = \eta e^{-2\pi \vec{r}^{\,2}/\lambda^2}$$

$$\mathcal{V}(\vec{r}) = -k_B T \ln \left[1 + \eta e^{-2\pi \vec{r}^{\,2}/\lambda^2} \right] \approx -k_B T \eta e^{-2\pi \vec{r}^{\,2}/\lambda^2}$$

If plot.



this mimicks quantum correlation of the force nature at high temp.
i.e bosons attractive
fermions repulsive.

Grand Canonical Formulation

Z_N^* is difficult, but in Grand Canonical:

$$Q_\eta(T,\mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{n_k\}}' \exp\left[-\beta \sum_{\vec{k}} \mathcal{E}(\vec{k}) n_{\vec{k}}\right]$$

note $N = \sum_{\vec{k}} n_{\vec{k}}$

$n_k = 0, 1$ fermi
$n_k = 0, 1, 2, \ldots$

w/ $\sum'_{\{n_k\}} = \sum^{N}_{\{n_k\}}$ one rewrite $\sum_{N=0}^{\infty}\sum_{\{n_k\}}^{N} = \sum_{\{n_k\}}$

n_k here account for degen. of particle w/ energy $\mathcal{E}(\vec{k})$

so $Q = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{\mu_s\}} e^{-\beta H_N(\mu_s)}$

Now take $\mu_s \to n_k$ then

$Q = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum'_{\{n_k\}} e^{-\beta \sum_k \mathcal{E}(k) n_k}$

what "$\sum'$" means

thus

$$Q_\eta = \sum_{\{n_k\}} \prod_k \exp\left[-\beta(\mathcal{E}-\mu) n_{\vec{k}}\right]$$

$$\ln Q_\eta = -\eta \sum_{\vec{k}} \ln\left[1 - \eta \exp(\beta\mu - \beta\mathcal{E}(\vec{k}))\right]$$

($\eta = -1$ fermions $\eta = +1$ bosons)

note $N = \sum_k n_k$ $\quad \sum_{N=0}^{\infty}\sum_{\{n_k\}} e^{\beta\sum_k (\mu-\mathcal{E}) n_k}$

$= \prod_k \sum_{\{n_k\}} e^{-\beta(\mathcal{E}-\mu) n_k}$

joint prob

$$P_\eta(\{n(\vec{k})\}) = \frac{1}{Q_\eta} \prod_k \exp\left[-\beta(\mathcal{E}(\vec{k}) - \mu) n_{\vec{k}}\right]$$

key properties

occ. # $\boxed{\langle n_{\vec{k}} \rangle_\eta = -\frac{\partial \ln Q_\eta}{\partial(\beta\mathcal{E})} = \frac{1}{z^{-1} e^{\beta\mathcal{E}} - \eta}}$; $z = e^{\beta\mu}$, $\mathcal{E} = \mathcal{E}(\vec{k})$

$$N_\eta = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle_\eta = \sum_{\vec{k}} \frac{1}{z^{-1} e^{\beta\mathcal{E}} - \eta}$$

$$E_\eta = \sum_{k} \mathcal{E}(\vec{k}) \langle n_{\vec{k}} \rangle_\eta = \sum_{\vec{k}} \frac{\mathcal{E}(\vec{k})}{z^{-1} e^{\beta\mathcal{E}} - \eta}$$

Non-relativistic gas

Absence of B field account for spin degeneracy $g = 2s+1$

non-relativistic gas $\mathcal{E} = \frac{\hbar^2 k^2}{2m}$ w/ $\boxed{\sum_{\vec{k}} \longrightarrow V \int \frac{d^3k}{(2\pi)^3}}$ $\Rightarrow$ new expressions for P_η, n_η, $\mathcal{E}_\eta$

and w/ change of variable $x = \frac{\beta\hbar^2 k^2}{2m}$

$$\beta P_\eta = \frac{g}{\lambda^3} \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{z^{-1} e^x - \eta}$$

~~here we define~~

$$n_\eta = \frac{g}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{1/2}}{z^{-1} e^x - \eta}$$

$$\beta\mathcal{E}_\eta = \frac{g}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{z^{-1} e^x - \eta}$$

we define $f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{dx\, x^{m-1}}{z^{-1} e^x - \eta}$ w/ $m! \equiv \Gamma(m+1) = \int_0^\infty dx\, x^m e^{-x}$; $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$ $\quad (\frac{3}{2})! = \frac{3}{2}\frac{\sqrt{\pi}}{2}$

these eqns take simple forms:

$$\beta P_\eta = \frac{g}{\lambda^3} f_{5/2}^\eta(z)$$

$$n_\eta = \frac{g}{\lambda^3} f_{3/2}^\eta(z)$$

$$\mathcal{E}_\eta = \frac{3}{2} P_\eta$$

~~Rem~~ Remark
photon energy $\mathcal{E} = \hbar c k$
phonon energy $\mathcal{E} = \hbar v k$

how does $f_m^\eta(z)$ behave? recall $z = e^{\beta\mu}$

At high temp $z \ll 1$, rewrite/recast

$$f_m^\eta(z) = \frac{1}{(m-1)!}\int_0^\infty dx\, x^{m-1} z e^{-x}(1-\eta z e^{-x})^{-1} = \frac{1}{(m-1)!}\int_0^\infty dx\, x^{n-1}\sum_{\alpha=1}^\infty (ze^{-x})^\alpha \eta^{\alpha-1}$$

$$f_m^\eta(z) = \sum_{\alpha=1}^\infty \eta^{\alpha-1}\frac{z^\alpha}{\alpha^m} = z + \eta\frac{z^2}{2^m} + \frac{z^3}{3^m} + \eta\frac{z^4}{4^m} + \cdots$$

Remark $n_\eta, P_\eta \ll 1$ as $z \to 0$ degenerate limit;

$$\frac{n_\eta\lambda^3}{g} = f_{3/2}^\eta(z) = z + \eta\frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \eta\frac{z^4}{4^{3/2}} + \cdots$$

$$\frac{\beta P_\eta\lambda^3}{g} = f_{5/2}^\eta(z) = z + \eta\frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \eta\frac{z^4}{4^{5/2}} + \cdots$$

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using recursive relation: $z = n_\eta\frac{\lambda^3}{g} - \eta\frac{z^2}{2^{3/2}} - \frac{z^3}{3^{3/2}} - \cdots = \left(\frac{n_\eta\lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta\lambda^3}{g}\right)^3 \cdots$

we have $\frac{\beta P_\eta\lambda^3}{g} = \frac{n_\eta\lambda^3}{g} - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta\lambda^3}{g}\right)^2 + \cdots$

recover virial expansion:

$$P_\eta = n_\eta k_B T\left[1 - \frac{\eta}{2^{5/2}}\left(\frac{n_\eta\lambda^3}{g}\right) + \left(\frac{1}{8} - \frac{2}{3^{5/2}}\right)\left(\frac{n_\eta\lambda^3}{g}\right)^2 + \cdots\right] \quad w/ \quad B_2 = \frac{-\eta\lambda^3}{2^{5/2} g}$$

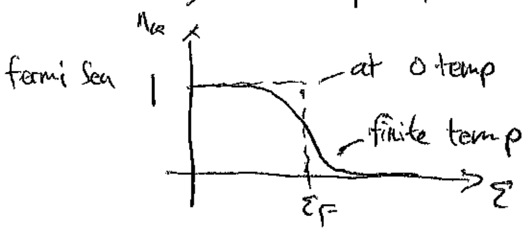
Remark. the dimless expansion parameter $\frac{n_\eta\lambda^3}{g}$ show quantum effect is important as $n_\eta\lambda^3 \geq g$
this is the quantum degenerate limit.

Behavior of degenerate fermi/boson gas

fermi gas

$$\langle n_k\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \begin{cases} 1 & w/\ \varepsilon < \mu \\ 0 & \text{else} \end{cases}$$ ~~given μ~~

given $\mu = \varepsilon(0) = \varepsilon_F \gg 1$



For idea gas in the box, $\varepsilon = \frac{\hbar^2 k^2}{2m}$ wrt # particle within fermi level (fermi wavenumber k_F)

$$N = gV\int^{k<k_F}\frac{d^3k}{(2\pi)^3} = \frac{gV}{6\pi^2}k_F^3 \Rightarrow k_F = \left(\frac{6\pi^2 n}{g}\right)^{1/3}$$

$$\varepsilon_F \neq \varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

how fermi sea behaves at low temp?

Consider $z \gg 1$ s.t

$$f_m^-(z) = \frac{1}{m!}\int_0^\infty dx\, x^m \frac{d}{dx}\left(\frac{-1}{z^{-1}e^x+1}\right) = \frac{(\ln z)^m}{m!}\sum_{\alpha=0}^{\infty}\frac{m!}{\alpha!(m-\alpha)!}(\ln z)^{-\alpha}\int_{-\infty}^{\infty} dt\, t^\alpha \frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

For $\frac{d}{dx}$ spike at ε_F, let ($e^t = z^{-1}e^x \Rightarrow x = \ln z + t$). Now upon expansion & anti-sym $t \to -t$ the integral:

$$\frac{1}{\alpha!}\int_{-\infty}^\infty dt\, t^\alpha \frac{d}{dt}\left(\frac{-1}{e^t+1}\right) = \begin{cases} 0 & \alpha \text{ odd} \\ \frac{2}{(\alpha-1)!}\int_0^\infty dt\, \frac{t^{\alpha-1}}{e^t+1} = 2 f_\alpha^-(1) & \alpha \text{ even}\end{cases}$$

so expanding $f_\alpha^-(1) \Rightarrow$ sommerfeld expansion.

$$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!}\left[1+\frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2}+\frac{7\pi^4}{360}\frac{m(m-1)(m-2)(m-3)}{(\ln z)^4}+\cdots\right]$$

In degenerate limit

$$\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{(3/2)!}\left[1+\frac{\pi^2}{6}\frac{3}{2}\frac{1}{2}(\ln z)^{-2}+\cdots\right] \gg 1$$

At lowest order

$$\lim_{T\to 0}\ln z = \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} = \beta\varepsilon_F \quad \text{which reproduce } \varepsilon_F{}^*$$

since $\mu = k_B T \ln z$,

Here $T_F \equiv \frac{\varepsilon_F}{k_B}$ Fermi temp. note μ change sign as $T \gg T_F$

Low temp expansion for fermi pressure.

$$\beta P = \frac{g}{\lambda^3} f_{5/2}^-(z) = \frac{g}{\lambda^3}\frac{(\ln z)^{5/2}}{(5/2)!}\left[1+\frac{\pi^2}{6}\frac{5}{2}\frac{3}{2}(\ln z)^{-2}+\cdots\right]$$

$$\beta P = \beta P_F\left[1+\frac{5}{12}\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2+\cdots\right]$$

$$P_F = \frac{2}{5} n \varepsilon_F$$

Remark fermi gas at zero temp has finite pressure & internal energy

$$\frac{E}{V} = \frac{3}{2}P = \frac{3}{5} n k_B T_F\left[1+\frac{5}{12}\pi^2\left(\frac{T}{T_F}\right)^2+\cdots\right]$$

$$C_V = \frac{dE}{dT} = \underbrace{\frac{\pi^2}{2} N k_B\left(\frac{T}{T_F}\right)}_{\text{lin. scaling: valid in all dim.}} + O\left(\frac{T}{T_F}\right)^3$$