Matrix Rep

Operator X, write

$$\boxed{X = \sum_{a''a'} |a''\rangle\langle a''|X|a'\rangle\langle a'|}$$ General

so $\langle a''|X|a'\rangle$ (row, column) w/ $X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots \\ \vdots & & \ddots \end{pmatrix}$

this gives matrix form

ex Let $X = S_z$ w/ $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

$$S_z = |+\rangle\langle+|S_z|+\rangle\langle+| + |+\rangle\langle+|S_z|-\rangle\langle-| + |-\rangle\langle-|S_z|+\rangle\langle+| + |-\rangle\langle-|S_z|-\rangle\langle-|$$

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$$

$$S_z \doteq \langle a'|S_z|a''\rangle \to S_z \doteq \begin{pmatrix} \langle+|S_z|+\rangle & \langle+|S_z|-\rangle \\ \langle-|S_z|+\rangle & \langle-|S_z|-\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\langle a''|Z|a'\rangle = \langle a''|XY|a'\rangle = \sum_{a'''} \langle a''|X|a'''\rangle\langle a'''|Y|a'\rangle$

  $$Z_{a''a'} = X_{a''a'''}Y_{a'''a'}$$

- $|\gamma\rangle = X|\alpha\rangle = \sum_{a''} X|a''\rangle\langle a''|\alpha\rangle$

$\langle a'|\gamma\rangle = \sum_{a''} \langle a'|X|a''\rangle\langle a''|\alpha\rangle \sim \gamma_{a'} = X_{a'a''}\alpha_{a''}$

so we have $\gamma_{a'}$ or $|\gamma\rangle \doteq \begin{pmatrix} \langle a^{(1)}|\gamma\rangle \\ \langle a^{(2)}|\gamma\rangle \\ \vdots \end{pmatrix}$ & $\langle\gamma| = \left(\langle a^{(1)}|\gamma\rangle^*, \langle a^{(2)}|\gamma\rangle^*, \ldots\right)$

Now if $A|a'\rangle = a'|a'\rangle$
then $\boxed{A = \sum_{a'} a'|a'\rangle\langle a'|}$ special, or $A = \sum_{a'} a'\Lambda_a$ where $\Lambda_a \equiv |a'\rangle\langle a'|$

is diagonalized

b/c
$$A = \sum_{a''a'} |a''\rangle\langle a''|A|a'\rangle\langle a'|$$
$$= \sum_{a''a'} |a''\rangle\langle a'|\, a'\delta_{a'a''}$$

Given $|S_x \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$ $\quad |S_y \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$

using the general form we have:

ex $S_x = \frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right]$ and $S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$S_y = \frac{\hbar}{2}\left(-i|+\rangle\langle-| + i|-\rangle\langle+|\right)$ and $S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

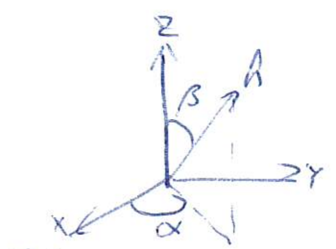
$S_\pm = S_x \pm iS_y \quad S_+ \equiv \hbar|+\rangle\langle-| \quad S_- \equiv \hbar|-\rangle\langle+|$

$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k$, $\{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij}$ $\quad S^2 \equiv S_x^2 + S_y^2 + S_z^2$

$[S^2, S_i] = 0$

$S_i^2 = \frac{\hbar^2}{4}\mathbb{1}$

$S^2 = \frac{3\hbar^2}{4}\cdot 1$

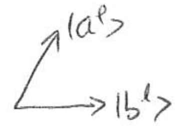


$(\vec{S}\cdot\hat{n})|\hat{n};+\rangle = \frac{\hbar}{2}|\hat{n};+\rangle$

$|\hat{n};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$

Change of basis

rotates $|a^l\rangle \to |b^l\rangle$



Claims ① $\boxed{|b^l\rangle = U|a^l\rangle}$

② $U^\dagger = U$ unitary b/c $\langle a^l|a^l\rangle = \langle b^l|b^l\rangle$

Ansatz $\boxed{U = \sum_k |b^k\rangle\langle a^k|}$ clearly satisfies ① & ②

Matrix Rep. $\boxed{U = \langle a^m|U|a^n\rangle = \langle a^m|b^n\rangle}$

as in Rotational Matrix

$$R = \begin{bmatrix} \hat{x}\cdot\hat{x}' & \hat{x}\cdot\hat{y}' & \hat{x}\cdot\hat{z}' \\ \hat{y}\cdot\hat{x}' & \hat{y}\cdot\hat{y}' & \hat{y}\cdot\hat{z}' \\ \hat{z}\cdot\hat{x}' & \hat{z}\cdot\hat{y}' & \hat{z}\cdot\hat{z}' \end{bmatrix}$$

Given $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$

Find Rep of $\langle b'|\alpha\rangle$ in term of $U^\dagger$

then $\langle b'|\alpha\rangle = \sum_{a''} \langle a'|U^\dagger|a''\rangle\langle a''|\alpha\rangle$ here we use simply $|b'\rangle = U|a'\rangle$

$(\text{new}) = (U^\dagger)(\text{old})$

Matrix relation between X' and X

$$X' = \langle b^m|X|b^n\rangle = \sum_{k,l} \underbrace{\langle b^m|a^k\rangle}_{U^\dagger}\underbrace{\langle a^k|X|a^l\rangle}_{X}\underbrace{\langle a^l|b^n\rangle}_{U}$$

we get similarity transformation $\boxed{X' = U^\dagger X U}$

Properties. • $tr(X) = \sum_{a'} \langle a'|X|a'\rangle$

can be written as: $= \sum_{a'}\sum_{b'}\sum_{b''} \langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle$

$= \sum_{b'} \langle b'|X|b'\rangle$

thus $\boxed{tr(X) = tr(X')}$

• $\boxed{tr(|a'\rangle\langle a''|) = \delta_{a'a''}}$

• $\boxed{tr(|b'\rangle\langle a'|) = \langle a'|b'\rangle}$

Remark trace of outer product is its inner product.

ie $tr\left(\begin{pmatrix} a \\ b \\ c\end{pmatrix}(i\ j\ k)\right)$ and $(a\ b\ c)\cdot(i\ j\ k)$

• The Schwarz inequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

• uncertainty principle $\langle\Delta A^2\rangle\langle\Delta B^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$

time evolution of state ket.

Given $A|a'\rangle = a'|a'\rangle$

$|\alpha, t_0\rangle = \sum C_{a'}(t_0)|a'\rangle$ where $|C_{a'}(t_0)| \neq |C_{a'}(t)|$ in general

$|\alpha, t_0; t\rangle = \sum C_{a'}(t)|a'\rangle$ but $\sum_{a'}|C_{a'}(t_0)|^2 = \sum_{a'}|C_{a'}(t)|^2 = 1$

Motivation, In quest of $C_{a'}(t)$ or $|a, t_0\rangle \longrightarrow |a, t_0; t\rangle$

Define: • State ket $|\alpha\rangle$ or $|\alpha, t_0; t\rangle$

• eigenket $A|a'\rangle = a'|a'\rangle$

• baseket: vector basis that spans vector space

$\left(\{|a'\rangle\} \mid A|a'\rangle = a'|a'\rangle \text{ and } \forall|\alpha\rangle \in V \quad |\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle\right)$

1st Claim

$|\alpha, t_0; t\rangle = U(t, t_0)|\alpha, t_0\rangle$

axioms (1) $U^\dagger U = 1$ b/c $\langle\alpha, t_0; t|\alpha, t_0; t\rangle = 1$

(2) Composition $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$

(3) Infinitesimal $|\alpha, t; t+dt\rangle = U(t+dt, t)|\alpha, t\rangle$

where $\lim_{dt\to 0} U(t+dt, t) = 1$

Corollary: $U(t+dt, t) = 1 - i\Omega dt$ note from (1) U complex (2) 1st order dt

By construction $\Omega = \frac{H}{\hbar}$ (note: $H = E = \hbar\omega$) & Ωdt unitless

Remarks: easily check corollary satisfies all axioms.

thus $U(t+dt, t_0) = U(t+dt, t)U(t, t_0) = (1 - i\Omega dt)U(t, t_0)$

we get SE $\frac{\partial U}{\partial t} = -\frac{iH}{\hbar}U$

2nd Claim $U(t, t_0) = \exp\left[-\frac{iHt}{\hbar}\right]$ pf by expansion or Infinitesimal compounding.

Continue questing of $C_{a'}(t)$ ... (take $t_0 = 0$ for simplification)

Given $[A, H] = 0$ then for $A|a'\rangle = a'|a'\rangle$ we have $H|a'\rangle = E_{a'}|a'\rangle$

and Matrix Rep. $\exp\left(-\frac{iHt}{\hbar}\right) = \sum_{a'}|a'\rangle\exp\left(-\frac{iE_{a'}t}{\hbar}\right)\langle a'|$

Write $|\alpha, t_0\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle = \sum_{a'} C_{a'}|a'\rangle$

then $|\alpha, t_0; t\rangle = \exp\left(\frac{-iHt}{\hbar}\right)|\alpha, t_0\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle\exp\left(-\frac{iE_{a'}t}{\hbar}\right)$ (note $t_0 = 0$)

so $C_{a'}(t) = C_{a'}(0)\exp\left(-\frac{iE_{a'}t}{\hbar}\right)$

Remarks • $|\alpha, t_0; t\rangle$ no longer sim. eig.ket of A, H at later time

• If $|\alpha\rangle = |a'\rangle$ taken as base-ket

then $|\alpha, t\rangle = |a'\rangle\exp\left(-\frac{iE_{a'}t}{\hbar}\right) \Rightarrow$ remains as sim. eig.ket of A, H later time.

expectation value time dependency.

take energy eigket, $|a', t_0=0; t\rangle = U(t,0)|a'\rangle$

then $\langle B\rangle = \langle a'|U^{+}BU|a'\rangle = \langle a'|B|a'\rangle$

thus time independent w/ $|a'\rangle$, energy eigstate as stationary state.

O.T.O.H w/ superposition of energy state

ie $|\alpha, t_0=0\rangle = \sum_{a'} C_{a'}|a'\rangle$

$$\langle B\rangle = \langle \alpha, t_0|U^{+}BU|\alpha, t_0\rangle = \sum_{a'a''} C_{a'}^{*}C_{a''}\langle a'|B|a''\rangle \exp\left[-\frac{i(E_{a''}-E_{a'})t}{\hbar}\right]$$

Remark $\boxed{\langle B\rangle \text{ oscillating w/ time}}$ in the N. Bohr's frequency $\boxed{\omega_{a''a'} = \frac{(E_{a''}-E_{a'})}{\hbar}}$

ex Spin Precession $\vec{B} = B\hat{z}$ then $H = \vec{\mu}\cdot B = -\frac{eB}{m_e c}$ $(e<0)$

$H|\pm\rangle = E_{\pm}|\pm\rangle$ where $E_{\pm} = \mp\frac{e\hbar B}{2m_e c} \Rightarrow \omega = \frac{|e|B}{m_e c}$ (N. Bohr's freq)

So, $H = \omega S_z$ & $U(t,0) = \exp\left(-\frac{i\omega S_z t}{\hbar}\right)$

Given $|\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ Probability of finding $S_x\pm$ state at later time t:

$|\langle S_x\pm|\alpha, t_0=0; t\rangle|^2 = \cos^2\frac{\omega t}{2}$ (for S_x+) and $\sin^2\frac{\omega t}{2}$ (for S_x-).

Recalled $\langle A\rangle = \langle\alpha|A|\alpha\rangle = \sum_{a'a''}\langle\alpha|a'\rangle\langle a'|A|a''\rangle\langle a''|\alpha\rangle = \sum_{a'} a'|\langle a'|\alpha\rangle|^2$

so $\langle S_x\rangle = \left(\frac{\hbar}{2}\right)\cos^2\left(\frac{\omega t}{2}\right) + \left(-\frac{\hbar}{2}\right)\sin^2\left(\frac{\omega t}{2}\right) = \frac{\hbar}{2}\cos\omega t$ //

Correlation amplitude

$$\boxed{C(t) \equiv \langle\alpha|\alpha, t_0; t\rangle = \langle\alpha|U(t,0)|\alpha\rangle}$$

to measure how state ket $|\alpha\rangle$ at later time is similar to $|\alpha\rangle$ at $t=0$.

ex if $|\alpha\rangle = |a'\rangle$ engket then

$C(t) = \langle a'|a', t_0=0; t\rangle = \exp\left(-\frac{iE_{a'}t}{\hbar}\right)$ has amplitude 1.

ex if $|\alpha\rangle = \sum C_{a'}|a'\rangle$,

$C(t) = \sum_{a'}|C_{a'}|^2\exp\left(-\frac{iE_{a'}t}{\hbar}\right)$

Remark: strong cancellation if $t\gg1$ so correlation amplitude decreases w/ time.

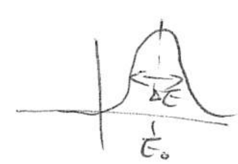
In quasi-continuous spec.

$\sum_{a'} \to \int dE\,\rho(E)$ and $C_{a'} \to g(E)\Big|_{E=E_{a'}}$

then $C(t) = \int dE\,|g(E)|^2\rho(E)\exp\left(-\frac{iEt}{\hbar}\right)$ w/ $\int dE\,|g(E)|^2\rho(E) = 1$

In real life, $|g(E)|^2\rho(E)$ peaked $\sim E_0$ w/ width ΔE so

$C(t) = \exp\left(-\frac{iE_0t}{\hbar}\right)\int dE\,|g(E)|^2\rho(E)\exp\left[-\frac{i(E-E_0)t}{\hbar}\right]$



Remark. rapid oscillation if $t\gg1$ unless $|E-E_0|$ small

- If $|E-E_0|$ much narrow than $\Delta E \Rightarrow \frac{\hbar}{t}$ or $t\gg1$ thus strong cancellation $C(t)\sim1$
- define characteristic time s.t $C(t)$ starts different from 1 be

$t \sim \frac{\hbar}{\Delta E}$ or $\Delta E\,\Delta t \sim \hbar$

Heisenberg Picture v.s Schrödinger Picture

$\langle A \rangle = \langle \alpha | U^\dagger A U | \alpha \rangle$

HP $|\alpha\rangle$ stationary ; $A_H(t) = U^\dagger A U$ evolves w/ time

SP $|\alpha\rangle$ evolves w/ time $|\alpha, t_0; t\rangle = U|\alpha, t_0\rangle$; A stationary

From HP $\boxed{\frac{dA_H}{dt} = \frac{1}{i\hbar}[A_H, H]}$ Heisenberg equation of motion
Note H time Indep.

Ehrenfest's Thm

Free Particle $H = \frac{\vec{p}^2}{2m}$ w/ $P = (P_x, P_y, P_z)$ as operators.

Now $\boxed{\frac{dX_i}{dt} = \frac{1}{i\hbar}[X_i, H] = \frac{P_i}{m}}$ b/c $[X_i, H] = U^\dagger X_i U U^\dagger H U - U^\dagger H U U^\dagger X_i U = U^\dagger [X_i, H] U = \frac{i\hbar U^\dagger P_i U}{m} = \frac{(P_i)_H}{m}$

recovered classical result!

$X(t) = X(0) + \frac{P_i(0)}{m} t$

Interesting observations

$[X_i(0), X_j(0)] = 0$

$[X_i(t), X_i(0)] = [\frac{P_i(0)}{m} t, X_i(0)] = \frac{-i\hbar t}{m} \neq 0$ for $t \neq 0$ using result $\frac{dX_i}{dt} = \frac{P_i}{m}$

then $\langle (\Delta X_i)^2 \rangle_t \langle (\Delta X_i)^2 \rangle_{t=0} \geq \frac{1}{4} \left| \langle [X_i(t), X_i(0)] \rangle \right|^2 = \frac{\hbar^2 t^2}{4m}$

Remarks if a particle well localized at $t=0$
its position becomes more uncertain over time.

Now w/ Potential

$H = \frac{\vec{p}^2}{2m} + V(\vec{x})$

$\frac{dP_i}{dt} = \frac{1}{i\hbar}[P_i, H] = -\frac{\partial V(\vec{x})}{\partial X_i}$

$\frac{dX_i}{dt} = \frac{1}{i\hbar}[X_i, H] = \frac{P_i}{m}$

$\frac{d}{dt}\left(\frac{dX_i}{dt}\right) = \frac{1}{i\hbar}\left[\frac{dX_i}{dt}, H\right] = \frac{1}{i\hbar}\left[\frac{P_i}{m}, H\right] = \frac{1}{i\hbar m}[P_i, H]$

so $m\frac{d^2 X_i}{dt^2} = -\nabla V(x)$ note operators in HP

Since $|\alpha\rangle$ stationary for state ket

then $m\frac{d^2\langle x \rangle}{dt} = -\langle \nabla V(x) \rangle$ w/ $\hbar$ disappear

Remarks:
- Result indep. of HP or SP
- Center of wavepacket moves like classical particle subjected to $V(\vec{x})$

Given baseket $|a'\rangle$

$\boxed{A|a'\rangle = a'|a'\rangle}$ in SP ; A time indep $\Rightarrow$ $|a'\rangle$ baseket time indep.

Now consider

$U^\dagger A U U^\dagger |a'\rangle = a' U^\dagger |a'\rangle$, define $\boxed{|a',t\rangle_H = U^\dagger |a'\rangle}$ we have

$A_H |a',t\rangle_H = a' |a',t\rangle_H$ thus baseket in HP evolves w/ time in fashion of $U^\dagger |a'\rangle$

using $(i\hbar \frac{\partial U}{\partial t} = HU)^\dagger$, we have $i\hbar \frac{\partial}{\partial t}|a',t\rangle_H = -H|a',t\rangle_H$ "a wrong-sign SE",

Remark : • note that eigval unchange w/ time due to unitary operator preserves size.

• $\boxed{U^\dagger \text{ rotates } |a'\rangle \text{ in opposite direction}}$ in HP v.s SP.

Writing now the matrix rep.

$A_H(t) = \sum_{a'} |a',t\rangle_H \langle a',t|_H a' = U^\dagger (\sum_{a'} |a'\rangle \langle a'| a') U$

thus $A_H(t) = U^\dagger A_S U$ as expected.

If stateket as superposition of baseket ;

SP $C_{a'}(t) = \langle a'|\alpha,t\rangle = \underbrace{\langle a'|}_{\text{base}} \underbrace{U|\alpha\rangle}_{\text{stateket}}$

HP $C_{a'}(t) = \langle a'|U|\alpha\rangle = {}_H\langle \underbrace{a',t}_{\text{base}}|\alpha\rangle$

Remark

Continuous Spec. Wave func $\langle x'|\alpha\rangle$ can be regard as

SP $\langle \underbrace{\vec{x}'}_{\text{stationary}}|\underbrace{\alpha}_{\text{moving}}\rangle$ or HP $\langle \underbrace{\vec{x}'}_{\text{moving}}|\underbrace{\alpha}_{\text{stationary}}\rangle$

Probability amplitude as transition amplitude

Given at $t=0$ $|\alpha\rangle = |a'\rangle$ where $A|a'\rangle = a'|a'\rangle$

Probability amplitude of finding system in eigenstate $|b'\rangle$ of a observable B at later time.

so

$U|a'\rangle \longrightarrow \langle b'|U|a'\rangle = P(b'|a,t)$

In SP $\underbrace{\langle b'|}_{\text{base bra}} \cdot \underbrace{(U|a'\rangle)}_{\text{state ket}}$ HP $\underbrace{(\langle b'|U)}_{\text{base bra}} \cdot \underbrace{|a'\rangle}_{\text{stateket}}$

Remark transition amp from $|a'\rangle$ to $|b'\rangle$

Harmonic Oscillator

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

define ① $a, a^\dagger$ s.t. $\boxed{[a, a^\dagger] = 1}$

② $N \equiv a^\dagger a$ number operator

③ $H = \hbar\omega(N + \frac{1}{2})$ thus N, H share energy eigenstate $|n\rangle$ w/ $N|n\rangle = n|n\rangle$

④ $H|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$

From ③, we deduce $a, a^\dagger$: $a^\dagger a = \frac{H}{\hbar\omega} - \frac{1}{2} = \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2}\right) - \frac{1}{2}$

so $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ $\Rightarrow$ $x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$ $p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$

⑤ $\boxed{[N, a] = -a \quad [N, a^\dagger] = a^\dagger}$ $n = \langle n|N|n\rangle \geq 0$ (positivity requirement)

⑥ $a|n\rangle = \sqrt{n}|n-1\rangle$ $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

eigenket $|1\rangle = a^\dagger|0\rangle$ $|2\rangle = \frac{a^\dagger}{\sqrt{2}}|1\rangle = \frac{(a^\dagger)^2}{\sqrt{2}}|0\rangle$ ...

$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$

starts w/ ground state eig. ket

$a|0\rangle = 0|0\rangle \Rightarrow \langle x'|a|0\rangle = 0$

$\langle x'|a|0\rangle = \langle x'|x'|0\rangle + \frac{i}{2m}\langle x'|p|0\rangle = 0$ recall $\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x'|\alpha\rangle$

is an ODE

$\left(x' + x_0^2\frac{d}{dx}\right)\langle x'|0\rangle$ where $x_0 = \sqrt{\frac{\hbar}{2m}}$

solve ODE and normalized yields

$$\langle x'|0\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}}\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$$

energy eigenfunc for excited state.

$$\langle x'|1\rangle = \langle x'|a^\dagger|0\rangle = \frac{1}{\sqrt{2}x_0}\left(x' - x_0^2\frac{d}{dx'}\right)\langle x'|0\rangle$$

$$\langle x'|2\rangle = \frac{1}{\sqrt{2}}\langle x'|(a^\dagger)^2|0\rangle = \frac{1}{\sqrt{2}}\int dx''\langle x'|a^\dagger|x''\rangle\langle x''|a^\dagger|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\left(\frac{1}{\sqrt{2}x_0}\right)^2\left(x' - x_0^2\frac{d}{dx'}\right)^2\langle x'|0\rangle$$

$$\langle x'|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}x_0}\right)^n\left(x' - x_0^2\frac{d}{dx'}\right)^n\langle x'|0\rangle$$

$$\langle x'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\left(\frac{1}{x_0^{n+1/2}}\right)\left(x' - x_0^2\frac{d}{dx'}\right)^n\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$$

uncertainty principle.

Clearly $\langle x\rangle = \langle p\rangle = 0$ $\langle x\rangle = \frac{\hbar}{2m\omega}$ $\langle p\rangle = \frac{\hbar m\omega}{2}$ w/ some work.

note $\langle n|a^\dagger a^\dagger|n\rangle = \langle n|aa|n\rangle = 0$ ~~$\langle n|aa^\dagger|n\rangle$~~

thus $\langle\Delta x^2\rangle\langle\Delta p^2\rangle = \frac{\hbar^2}{4}$ for ground state.

$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = (n+\frac{1}{2})^2\hbar^2$ for excited state

Time-dependent wave eqn.

$\psi(\vec{x},t) = \langle x' | \alpha, t_0; t\rangle$ by def

Recalled $\frac{\partial \mathcal{U}}{\partial t} = -\frac{iH}{\hbar}\mathcal{U}$ where $|\alpha, t_0; t\rangle = \mathcal{U}|\alpha\rangle$

since $|x'\rangle$ time indep. in SP, then we have

$i\hbar \frac{\partial}{\partial t}\langle x' | \alpha, t_0; t\rangle = \langle x' | H | \alpha, t_0; t\rangle$

For $H = \frac{p^2}{2m} + V(\vec{x})$ note $V(\vec{x})$ local b/c $\langle x'' | V(x) | x'\rangle = V(x')\delta(x'-x'')$

we have

$$i\hbar \frac{\partial}{\partial t}\psi(x',t) = -\frac{\hbar^2}{2m}\nabla'^2\psi(x',t) + V(x')\psi(x',t)$$

Note $\langle x' | \alpha, t_0; t\rangle = \underbrace{\langle x' | a'\rangle}_{U_E(x')} \exp\left(-\frac{iE_{a'}t}{\hbar}\right)$ energy eig. func.

this yields time-indep. wave eqn.

$$-\left(\frac{\hbar^2}{2m}\right)\nabla'^2 U_E + V U_E = E U_E$$

Wave func Interpretation. — Probabilistic

Probability density $\rho(x',t) = |\psi(x',t)|^2 = |\langle x' | \alpha, t_0; t\rangle|^2$

Using SE and its complex conjugate, we have

$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$ where $\vec{j}(\vec{x},t) = \frac{\hbar}{m}\,\mathrm{Im}(\psi^* \nabla \psi)$ probability flux

By integrating over all space,

$$\int d^3x\, \vec{j} = \mathrm{Im}\left[\frac{i}{m}\int d^3x\, \psi^*(-i\hbar\nabla)\psi\right] = \frac{\langle p\rangle_t}{m}$$ — expectation value at time t.

Physical Significance of wave func:

Ansatz Let $\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}\,\underbrace{\exp\left[\frac{i\overbrace{S(\vec{x},t)}^{\text{phase}}}{\hbar}\right]}_{\text{phase term}}$

then $\vec{j} = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\nabla\psi) = \frac{\rho \nabla S}{m}$ after some cal. "asc"

Remark: • $\vec{j}$ is characterized by phase variation

• Increase phase variation intesifies the flux

ex plane wave $\psi \sim \exp\left(\underbrace{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}_{\text{phase term}} - \frac{iEt}{\hbar}\right)$ p, x equal.

we have $\nabla S = \vec{p}$ and $\frac{\nabla S}{m} \sim$ velocity "v"

thus $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \text{"}v\text{"}) = 0$

In Classical limit

take $\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}\, e^{\frac{i S(\vec{x},t)}{\hbar}}$

after subject to SE and for $\hbar \to 0$, we have

$$\boxed{\frac{1}{2m}|\nabla S|^2 + V + \frac{\partial S}{\partial t} = 0}$$

which is the Hamilton-Jacobi eqn in classical mech.

$S(\vec{x},t)$ — Hamilton's principal func.

At stationary states

$S(\vec{x},t) = W(\vec{x}) - Et$

Hamilton characteristic func.

Potential & Gauge Transf

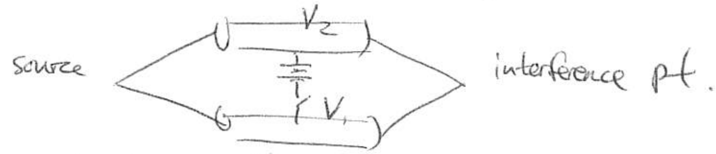
$V(x) \longrightarrow V(x) + V_0(t)$ diff potent'l at any instant time.

$|\alpha, t_0; t\rangle \Leftrightarrow V(\vec{x})$

$|\alpha, \tilde{t_0}; t\rangle \Leftrightarrow V(\vec{x}) + V_0$

then $|\alpha, \tilde{t_0}; t\rangle = \exp\left(-i\int_{t_0}^{t} dt' \frac{V_0(t')}{\hbar}\right) |\alpha, t_0; t\rangle$

ex.



thus $\phi_1 - \phi_2 = \frac{1}{\hbar}\int_{t_i}^{t_f} dt\,[V_2(t) - V_1(t)]$

- $\langle obs\rangle$ no diff. in expect. val
- result in observable effect namely $\sin(\phi_1-\phi_2)$, $\cos(\phi_1-\phi_2)$ induced by $(V_2-V_1)(t)$
- pure QM effect.

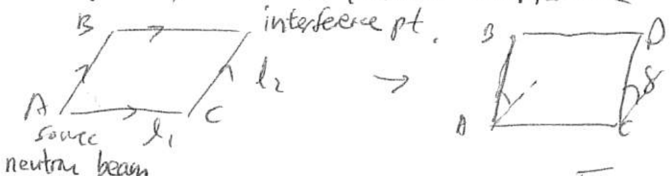
ex falling object ~~cm~~ $m\ddot{\vec{x}} = -m\nabla\Phi_{grav} = -mg\hat{z}$ & $\delta\int_{t_1}^{t_2} dt\left(\frac{m\dot{\vec{x}}^2}{2} - mgz\right) = 0$

~~cm~~ pure geom

QM $\left(-\frac{\hbar^2}{2m}\nabla^2 + m\Phi_{grav}\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$ & $\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[i\int_{t_{n-1}}^{t_n}\frac{(\frac{1}{2}m\dot{x}^2 - mgz)}{\hbar}\right]$

$\Rightarrow$ $m, \hbar$ can't be cancelled.

ex gravity-induced quantum interference



elevated by δ $\Delta V = m_n g l_2 \sin\delta$

T taken from BD or AC

G induced phase change : $\exp\left[-\frac{i m_n g l_2 \sin\delta\, T}{\hbar}\right]$ w/ δ dependence

introduced de Broglie : $p = \frac{\hbar}{\lambda} \Rightarrow mv = \frac{\hbar}{\lambda} \Rightarrow \frac{m l_1}{T} = \frac{\hbar}{\lambda} \Rightarrow T = \frac{m l_1 \lambda}{\hbar}$

then $\phi_{ABD} - \phi_{ACD} = -\frac{m_n^2 g l_1 l_2 \lambda \sin\delta}{\hbar^2}$

Remarks :
- w/ detectable magnitude
- ie $\lambda \simeq 1.42\,Å$ $l_1 l_2 = 10\,cm^2$ $\Rightarrow$ $\Delta\phi \sim 55.6$
- pure QM b/c as $\hbar \to 0 \Rightarrow$ fast oscillation $\Rightarrow$ cancellation $\Rightarrow \Delta\phi \sim 0$

$$e^{iA\lambda} B e^{-iA\lambda} = B + i\lambda[A,B] + \frac{i^2\lambda^2}{2!}[A,[A,B]] + \cdots$$

Baker-Hausdorff Lemma.

$$\exp\left(\frac{iHt}{\hbar}\right) x(0) \exp\left(\frac{-iHt}{\hbar}\right) = x(0) + \left(\frac{it}{\hbar}\right)[H, x(0)] + \left(\frac{i^2 t^2}{2!\hbar^2}\right)[H,[H,x(0)]] + \cdots$$

Gauge Transf. in EM.

$\vec{E} = -\nabla\phi \quad \vec{B} = \nabla\times\vec{A} \qquad H = \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + e\phi$

Consider $\frac{dx_i}{dt} = \frac{[x_i, H]}{i\hbar} = \frac{p_i - eA_i/c}{m}$

Thus define $\Pi_i = p_i - \frac{eA_i}{c}$

Remarks
- p_i canonical momentum
- Π_i mechanical momentum.

Properties
- easy to show $[\Pi_i, \Pi_j] = \frac{i\hbar e}{c}\varepsilon_{ijk}B_k$
- $H = \frac{\Pi^2}{2m} + e\phi$
- Lorentz force $m\frac{d^2x}{dt^2} = \frac{d\Pi}{dt} = e\left[\vec{E} + \frac{1}{2c}\left(\frac{d\vec{x}}{dt}\times\vec{B} - \vec{B}\times\frac{d\vec{x}}{dt}\right)\right]$

(*) Standard trick $X^\dagger Y X = X^\dagger[Y, X] + Y$

thus

$g^\dagger p g = g^\dagger[p, g] + p$

$= g^\dagger(-i\hbar)\nabla g + p$

$= p + \frac{e\nabla\Lambda}{c}$

SE w/ ϕ and $\vec{A}$.

Consider $\langle\vec{x}'|\Pi^2|\alpha, t_0; t\rangle$ using $\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x'|\alpha\rangle$

we have $\langle\vec{x}'|H|\alpha, t_0; t\rangle = \frac{1}{2m}\Pi\cdot\Pi\langle\vec{x}'|\alpha, t_0; t\rangle + e\phi(\vec{x}')\langle\vec{x}'|\alpha, t_0; t\rangle = i\hbar\frac{\partial}{\partial t}\langle\vec{x}'|\alpha, t_0; t\rangle$

where $\Pi \doteq -i\hbar\nabla' - \frac{eA(\vec{x}')}{c}$

define $\psi = \langle\vec{x}'|\alpha, t_0; t\rangle$ and $\rho = |\psi|^2$

we have $\frac{\partial\rho}{\partial t} + \nabla'\cdot\vec{j} = 0 \quad$ w/ $\vec{j} = \frac{\hbar}{m}\mathrm{Im}(\psi^*\nabla'\psi) - \frac{e}{mc}\vec{A}|\psi|^2$

notice that w/ A, $\nabla' \to \nabla' - \left(\frac{ie}{\hbar c}\right)\vec{A}$

Stoke's thm

$\int_S (\nabla\times A)\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{l}$

review 2.4

$\vec{j} = \left(\frac{\rho}{m}\right)\left(\nabla S - \frac{e\vec{A}}{c}\right)$

$\int d^3x'\,\vec{j} = \frac{\langle\Pi\rangle}{m}$

Gauge transf. $\quad \vec{A} \to \vec{A} + \nabla\Lambda \qquad$ then $\quad \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$\phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \qquad \vec{B} = \nabla\times\vec{A}$

Demonstrate invariant QM under gauge transf.

ex. $B = B\hat{z}$ then $A_x = \frac{-By}{2}$ $A_y = \frac{Bx}{2}$ $A_z = 0$ & $A_x = -By$, $A_y = 0$ $A_z = 0$.

$\Rightarrow \vec{A} \to \vec{A} - \nabla\left(\frac{Bxy}{2}\right)$ from $\vec{A}' - \vec{A}$

Remarks observed that $\frac{dp_x}{dt} = -\frac{\partial H}{\partial x}$, $\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} \Rightarrow \langle p\rangle$ not gauge invariant.

O.T.O.H Take $\tilde{A} = A + \nabla\Lambda \Leftrightarrow$ eskat $|\tilde{\alpha}\rangle$ & $A \Leftrightarrow |\alpha\rangle$; define $\tilde{\Pi} = p - \frac{e\tilde{A}}{c}$

Consider ops. g s.t. $|\tilde{\alpha}\rangle = g|\alpha\rangle$ required: $\langle\alpha|\vec{x}|\alpha\rangle = \langle\tilde{\alpha}|\vec{x}|\tilde{\alpha}\rangle$ or $g^\dagger\vec{x}g = \vec{x}$, $\langle\alpha|\Pi|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\Pi}|\tilde{\alpha}\rangle$ or $g^\dagger\tilde{\Pi} = \Pi$, $\langle\alpha|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\alpha}\rangle$

Ansatz $g = \exp\left[\frac{ie\Lambda(x)}{\hbar c}\right]$ satisfy all! So $\langle\Pi\rangle$ gauge invariant! See (*) So as H

~~So as H~~

2.6 Propagator

Mov Quest for position state wave function of a given initial state $|\alpha\rangle$ at later time such that it can be written as a convolution integral interms of Green's like func $K(x'', t''; x', t')$

Thm Given that $|\alpha\rangle$ can be written in terms of known eig. state $|a'\rangle$, of a known H. Then its position function can then be written w/ a convolution integral as stated above.

$$\psi(x'', t'') = \int d^3x' \, K(x'', t''; x', t') \, \psi(x', t')$$

~~Collor~~ Corollary it's easy to shown

- $K(x'', t; x', t_0) = \langle x''| \exp\left[-\frac{iH(t-t_0)}{\hbar}\right] |x'\rangle$

  or simply $K(x'', t''; x', t') = \langle x'', t'' | x', t'\rangle$

- Using free particle example, we can show that

$$K(x'', t''; x', t') = \sqrt{\frac{m}{2\pi i \hbar (t''-t')}} \exp\left[\frac{i\, m(x''-x')^2}{2\hbar (t''-t')}\right]$$

$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, e^{ik(x-x')}$

this generalized to

$$\langle x'', t'' | x', t'\rangle = \sqrt{\frac{m}{2\pi i \hbar (t''-t')}} \exp\left[\frac{i S(t'', t')}{\hbar}\right]$$

- More general

$$\langle x_N, t_N | x_1, t_1\rangle = \lim_{N\to\infty} \int d^3x_{N-1} \cdots d^3x_2 \, d^3x_1 \left(\frac{m}{2\pi i \hbar (t_N - t_1)}\right)^{\frac{N-1}{2}} \exp\left[\frac{i}{\hbar}\int L_{cl}(x, \dot{x})\, dt\right]$$

note that $1 = \int d^3x \, |x, t\rangle\langle x, t|$

or simply

$$\langle x_N, t_N | x_1, t_1\rangle = \int D[x(t)] \exp\left[\frac{iS}{\hbar}\right]$$

K/Momentum as spatial translation generator

mov. $\mathcal{J}(\Delta x) = (1 - ik\Delta x)$ w/ $k = \frac{p}{\hbar}$ s.t $\mathcal{J}(\Delta x)|x\rangle = |x + \Delta x\rangle$

Corollary
- N infinitesimal translation (i.e $\frac{\Delta x}{N}$, $N \to \infty$) gives

  $\mathcal{J}(\Delta x) = \exp\left(-\frac{ip}{\hbar}\Delta x\right)$

- Studying $\mathcal{J}(\Delta x)|\alpha\rangle$, we have $\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x}\langle x'|\alpha\rangle$
- from above, we have $\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip' \cdot x'}{\hbar}}$

Magnetic Monopole

$\vec{B} = \frac{e_m}{r^2}\hat{r}$ $\quad \nabla \times \vec{A} = \hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]$ $\quad A_\theta = A_r = 0$

$\vec{A} = \underbrace{\frac{e_m(1+\cos\theta)}{r\sin\theta}}_{A^{I}}\hat{\phi}$ $\quad$ Can also have $A^{II} = -\frac{e_m(1-\cos\theta)}{r\sin\theta}$

sing. at $\theta = \pi$ $\qquad$ sing. at $\theta = 0$

Pathology is no sing. free potential.

Pf knowing $\int \vec{B}\cdot d\vec{s} = 4\pi e_m$ if $\vec{A}$ sing. free, then $\int \nabla\cdot\vec{B}\, d^3x = 0$ so contradiction.

O.T.O.H under gauge transf.

$A \Leftrightarrow |\alpha\rangle$ $\quad \tilde{A} = A + \nabla\Lambda \Leftrightarrow |\tilde{\alpha}\rangle$ $\quad$ then $\exists\, g$ s.t. $|\tilde{\alpha}\rangle = g|\alpha\rangle$

ansatz, $\quad g = \exp\left(\frac{ie\Lambda}{\hbar c}\right)$

Now from the monopole,

$\nabla\Lambda = A_\phi^{II} - A_\phi^{I} = \frac{2e_m}{r\sin\theta}\hat{\phi}$ $\qquad$ know: $\nabla\Lambda = \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\phi}$

$\Lambda = -2e_m\phi$.

We have $\psi^{II}(r,\theta,\phi) = \exp\left(\frac{ie\Lambda}{\hbar c}\right)\psi^{I}(r,\theta,\phi)$

Given $\theta = \frac{\pi}{2}$ at radius r,

By periodicity we have $\psi^{II,I}(r,\frac{\pi}{2},2\pi) = \psi^{II,I}(r,\frac{\pi}{2},0)$

This yields $\quad \exp\left(\frac{-2ie e_m 2\pi}{\hbar c}\right) = 1$

$$-\frac{4ee_m\pi}{\hbar c} = 2N\pi$$

$$e_m = \frac{N\hbar c}{2e}$$

this explains that electric charges are quantized

Summary. $\vec{B} = \frac{e_m}{r^2}\hat{r}$ (magnetic monopole) ~~under gauge transformation $A \to A + \nabla\Lambda$~~

, monopole yields $A_\phi^{I,II} = \frac{e_m(\cos\theta \pm 1)}{r\sin\theta}$, for $\psi^{II,I}(r,\theta,\phi) \Leftrightarrow A^{II,I}$

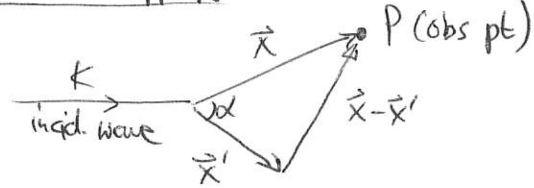
under gauge transform. $A \to A + \nabla\Lambda$ $\exists$, $g = \exp\left(\frac{ie\Lambda}{\hbar c}\right)$ s.t. $\psi^{II} = g\psi^{I}$.

Given radius r, $\theta = \frac{\pi}{2}$. By periodicity we obtain quantized e charges!

Note $A^{II} - A^{I} \to \Lambda = -2e_m\phi$ using $\nabla\Lambda = \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\phi}$

Scattering & Born Approx.

Scattering



$$G_\pm(\vec{x},\vec{x}') = -\frac{1}{4\pi}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

(General) position-basis states

$$\boxed{\langle\vec{x}|\psi\rangle = \langle\vec{x}|E\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} V(\vec{x}')\langle\vec{x}'|\psi\rangle}$$

Apply assumptions

consider $|\vec{x}| >> |\vec{x}'|$, take $|\vec{x}| = r$ $\quad |\vec{x}'| = r'$

thus $|\vec{x}-\vec{x}'| = \sqrt{r^2 - 2rr'\cos\alpha + r'^2} \approx r - \hat{r}\cdot\vec{x}'$ $\quad$ (note take $\vec{k}' = k\hat{r}$)

$$\frac{1}{|\vec{x}-\vec{x}'|} \sim \frac{1}{r}$$

then

$$\langle\vec{x}|\psi\rangle = \langle\vec{x}|\vec{E}\rangle - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x\, e^{-i\vec{k}'\cdot\vec{x}'} V(\vec{x}')\langle\vec{x}|\psi\rangle$$

using $\langle\vec{x}|\vec{k}\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{x}}$

$$\langle\vec{x}|\psi\rangle = \frac{1}{L^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k},\vec{k}')\right]$$

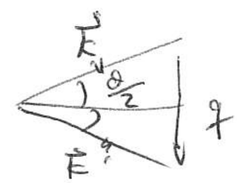
where

$$f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2} L^{3/2}\int d^3x\, e^{-i\vec{k}'\cdot\vec{x}'} V(\vec{x}')\langle\vec{x}|\psi\rangle$$

and diff-crossection $\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k},\vec{k}')|^2}$

Born Approx $\quad$ (i) $T \approx V \Rightarrow |\psi\rangle = |k\rangle$

(ii) spherical sym $V \Rightarrow q = |\vec{k}-\vec{k}'| = 2k\sin\frac{\theta}{2}$



then $\boxed{f^{(1)}(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int d^3x'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'} V(\vec{x}')}$

take $d^3x = r^2 dr\, d\cos\theta\, d\phi$ and $\vec{q}\cdot\vec{x} = qx\cos\theta$, we have

$$\boxed{f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin qr\, dr.}$$

ex Yukawa Potential $V(r) = \frac{V_0 e^{-\mu r}}{\mu r}$ w/ effective range $a \sim \frac{1}{\mu}$.

$$f^{(1)}(\theta) = \frac{-2m}{\hbar^2}\frac{V_0}{q\mu}\int_0^\infty e^{-\mu r}\sin qr\, dr$$ write $\sin qr = \frac{e^{iqr} - e^{-iqr}}{2i}$ and integrate.

we have $$\frac{d\sigma}{d\Omega} = |f'(\theta)|^2 = \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta)+\mu^2]^2}$$

Born Approx (at low energy) ie $ka \ll 1$

- $r' = |\vec{x} - \vec{x}'|$
- $V(\vec{x}') \sim V_0$ act in range a ie $|\vec{x} - \vec{x}'| \sim a$

thus the general

$$\langle \vec{x} | \psi \rangle = \langle \vec{x} | \vec{k} \rangle - \frac{2m}{\hbar^2} \int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi |\vec{x}-\vec{x}'|} V(\vec{x}') \langle \vec{x}' | \psi \rangle \quad w/ \ |\psi\rangle = |\vec{k}\rangle$$

has 2^{nd} term $\ll$ 1^{st} term.

by approximate integral,

$$\left| \frac{2m}{\hbar^2} \left(\frac{4}{3} \pi a^3 \right) \frac{e^{ikr'}}{4\pi a} V_0 \frac{e^{i\vec{k}\cdot\vec{x}'}}{L^{3/2}} \right| \ll \left| \frac{e^{i\vec{k}\cdot\vec{x}}}{L^{3/2}} \right|$$

exponential comp. ~ 1 b/c $ka \ll 1$, in order of magnitude, we have

$$\frac{m|V_0|}{\hbar^2} a^2 \ll 1$$

ex in Yukawa potential $a \sim \frac{1}{\mu}$ thus

$$\frac{m|V_0|}{\mu^2 \hbar^2} \ll 1 \quad //$$