4.1 Symmetry in Quantum

$\mathscr{S}$ unitary operators (Sym. Op) and its inf. tranf. $\mathscr{S} = 1 - i\frac{\varepsilon}{\hbar}G$ $\quad \varepsilon \ll 1$

Prop If H invariant under $\mathscr{S}$ ($\mathscr{S}^\dagger H \mathscr{S} = H$), then $[G, H] = 0$ thus $\frac{dG}{dt} = 0$. So G constant of motion.

Remark G can be translation, rotation.

Prop If $[G, H] = 0$, given $|g'\rangle$ eigket of G at t_0, $|g', t_0; t\rangle$ also eigket of G at $t > 0$.

Pf b/c $[G, U] = 0$ also.

Degeneracies

Given $[H, \mathscr{S}] = 0$. Let $|n\rangle$ be energy eigket w/ eigval E_n.

Suppose $|n\rangle$ and $\mathscr{S}|n\rangle$ are two different states, then they're degenerate.

Rotation

Prop Given $[D(R), H] = 0$, then $[\vec{J}, H] = 0$, $[\vec{J}^2, H] = 0$.

This yields good quantum number n, l, m. We can then write sim eigket $|n; j, m\rangle$

Thus all states of form $D(R)|n; j, m\rangle$ are all degenerate.

Remark
- $D(R)|n; j, m\rangle = \sum_{m'} |n; j, m'\rangle D^{(j)}_{m'm}(R)$ as $2j+1$ lin. comb
- since rotation fixed j but not m, there're $2j+1$ degen.
- clearly $[J_\pm, H] = 0$, states due to $J_\pm$ are also degen.

Discrete symmetry – Parity

Parity transform coord. s.t LH $\Longleftrightarrow$ RH

Let π unitary be the parity op. acting on state ket. $|\alpha\rangle \rightarrow \pi|\alpha\rangle$

s.t $\langle\alpha|\pi^\dagger \vec{x} \pi|\alpha\rangle = -\langle\alpha|\vec{x}|\alpha\rangle$.

Position op

By construction, we have $\vec{x}\pi = -\pi\vec{x}$ thus anticommute $\boxed{\{\pi, \vec{x}\} = 0}$ $\Rightarrow$ $\pi|\vec{x}\rangle = |-\vec{x}\rangle$

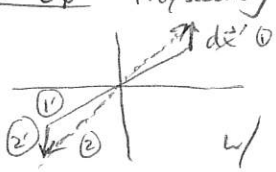
claim $\pi|\vec{x}'\rangle = e^{i\delta}|-\vec{x}'\rangle$

obs. $\vec{x}(\pi|\vec{x}'\rangle) = -\pi\vec{x}|\vec{x}'\rangle = -\vec{x}'(\pi|\vec{x}'\rangle)$ thus $\pi|\vec{x}'\rangle \sim |-\vec{x}'\rangle$

so $\pi|\vec{x}'\rangle \propto |-\vec{x}'\rangle$ by some phase factor ie $\pi|\vec{x}'\rangle = e^{i\delta}|-\vec{x}'\rangle$ $\quad \boxed{\pi|\vec{x}'\rangle = |-\vec{x}'\rangle}$

Remark
- by convention $e^{i\delta} = 1$, then
- $\boxed{\pi^2|\vec{x}'\rangle = |\vec{x}'\rangle \text{ thus } \pi^2 = 1 \text{ w/ } \pm 1 \text{ eigval.}}$

Momentum Op Physically



thus $\pi\,\mathscr{T}(d\vec{x}') = \mathscr{T}(-d\vec{x}')\,\pi$

w/ inf. tranf. $\pi\left(1 - \frac{i\vec{p}\cdot d\vec{x}'}{\hbar}\right) = \left(1 + \frac{i\vec{p}\cdot d\vec{x}'}{\hbar}\right)\pi$ so $\boxed{\{\pi, \vec{p}\} = 0}$ $\Rightarrow$ $\pi|\vec{p}\rangle = |-\vec{p}\rangle$

then $\pi P|p'\rangle = p'\pi|p'\rangle$, using anticommutation, then $P(\pi|p'\rangle) = -p'(\pi|p'\rangle)$

we have $\boxed{\pi|p'\rangle = |-p'\rangle}$

Ang Mom $\vec{J}$

For $\vec{L}$, $\boxed{[\Pi, \vec{L}] = 0}$.

B/c $\vec{L} = \vec{x} \times \vec{p} = \varepsilon_{ijk} x_i p_j$; $[\Pi, x_i p_j] = \Pi x_i p_j - x_i p_j \Pi = -x_i(\Pi p_j) - x_i p_j \Pi = 0$

Now in 3x3 rep.

$R^{(Parity)} = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$ clearly $R^P R^{rot} = R^{rot} R^P \xrightarrow[\text{For } \vec{J} \text{ in op. space}]{\text{by postulate}} \Pi D(R) = D(R) \Pi$

using inf. transf, $D(R) = 1 - \frac{\vec{J} \cdot \hat{n} \varepsilon}{\hbar}$ then

$$\boxed{[\Pi, \vec{J}] = 0}$$

since $\vec{J} = \vec{L} + \vec{S} \Rightarrow [\Pi, \vec{S}] = 0$

Remark①. $\vec{x}, \vec{J}$ transf. same way as vector / spherical tensor (rank 1) under rotation

- $\vec{x}, \vec{p}$ (odd parity) are vector ; $\vec{J}$ (even parity) is pseudovector.

Remark②. $\vec{S} \cdot \vec{x}$, $\vec{S} \cdot \vec{L}$, $\vec{x} \cdot \vec{p}$ as scalar under rot.

- $\Pi^{-1} \vec{S} \cdot \vec{x} \Pi = -\vec{S} \cdot \vec{x}$ (pseudoscalar) note $\Pi S_i X_i \Pi = -S_i X_i$
- $\Pi^{-1} \vec{L} \cdot \vec{S} \Pi = \vec{L} \cdot \vec{S}$ (scalar).

$[\Pi, \vec{L}] = 0$

$[\Pi, \vec{J}] = 0$

$R^P = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$

$R^{rot} = D(5)$

vector (odd parity)

pseudo-vec (even parity)

Scalar (even parity)

pseudo-scal (odd-parity)

Wavefunc. Under Parity

$\langle \vec{x} | \Pi | \alpha \rangle = \pm \langle \vec{x} | \alpha \rangle$ or $\langle \vec{x} | \Pi | \alpha \rangle = \langle -\vec{x} | \alpha \rangle$

thus $\boxed{\psi(-\vec{x}) = \pm \psi(\vec{x})}$ even / odd parity $\Leftrightarrow |\alpha\rangle$

ex momentum eigket is not a parity ket

$$e^{+i\vec{p} \cdot \vec{x}} \neq \pm e^{-i\vec{p} \cdot \vec{x}}$$

$\Leftrightarrow \psi(-\vec{x}) = \pm \psi(\vec{x})$

c. example.

non-ket $e^{i\vec{p} \cdot \vec{x}} \neq \pm e^{-i\vec{p} \cdot \vec{x}}$

ex eigket of L^2, L_z under parity

$\langle \vec{x} | \alpha, l m \rangle = R_\alpha(r) Y_l^m(\theta, \phi)$

for $\vec{x}' \to -\vec{x}'$ then
$$r \to r$$
$$\theta \to \pi - \theta$$
$$\phi \to \pi + \phi$$

since $\sin(\pi - \theta) = \sin\theta$ and $\left(\frac{d}{d\cos\theta}\right)^{l-m} \to (-1)^{l-m} \left(\frac{d}{d\cos\theta}\right)^{l-m}$

thus $\boxed{Y_l^m \to (-1)^l Y_l^m}$

and $\boxed{\Pi | \alpha, l m \rangle = (-1)^l | \alpha, l, m \rangle}$

$Y_l^m \to (-1)^l Y_l^m$

$\Pi | \alpha, l, m \rangle = (-1)^l | \alpha, l, m \rangle$

~~ex~~ example hydrogen atom 2p, 2s states

$$Y_l^m = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

Thm If $[H,\Pi]=0$ $H|n\rangle = E_n|n\rangle$ where $|n\rangle$ non.degen
then $|n\rangle$ is a parity eigket.

Pf ① consider $\frac{1}{2}(1\pm\Pi)|n\rangle$, $\Pi(1\pm\Pi)|n\rangle = \pm(1\pm\Pi)|n\rangle$ thus parity ket w/ eigval ± 1.
Clearly it's also energy eigket w/ E_n. Since $|n\rangle$ nondegen, then $\frac{1}{2}(1\pm\Pi)|n\rangle = |n\rangle$

Pf ② $H\Pi|n\rangle = \Pi H|n\rangle = E_n(\Pi|n\rangle) \Rightarrow \Pi|n\rangle = \pm|n\rangle$.

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ex $\boxed{\text{SHO } |0\rangle \text{ being even b/c being Gaussian}}$
But $|1\rangle$ odd b/c $a^\dagger \sim x - ip$
$\Pi a^\dagger = -(x-ip)$
$\Pi|1\rangle = \Pi a^\dagger|0\rangle = -a^\dagger|0\rangle = -|1\rangle$ thus odd.
Generally $\boxed{\Pi|n\rangle = (-1)^n|n\rangle}$

Counter Example

① hydrogen atom. E_n depends on quantum num. n $E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2}$
thus degen in 2p, 2s states. Note $[H,\Pi]=0$.
but $|\alpha\rangle = C_p|2p\rangle + C_s|2s\rangle$ is not parity ket. (why?)
b/c $2s \Leftrightarrow l=0$ $2p \Leftrightarrow l=1$
$\psi_{nlm} = R_{nl}(r)Y_l^m(\theta,\phi)$, $\Pi|n,l,m\rangle = (-1)^l|l,m\rangle$ thus $\Pi|\alpha\rangle \neq \pm|\alpha\rangle$.

② free particle $[H,\Pi]=0$ but $|\vec{p}'\rangle$, $|-\vec{p}'\rangle$ degen.
in wave func, $\Pi e^{i\vec{p}\cdot\vec{x}} \neq \pm e^{-i\vec{p}\cdot\vec{x}}$ but $\cos\frac{\vec{p}\cdot\vec{x}'}{\hbar}$, $\sin\frac{\vec{p}\cdot\vec{x}'}{\hbar}$ are parity ket ie. $|\vec{p}'\rangle \pm |-\vec{p}'\rangle$.

Parity Selection Rule

Prop. If $\Pi|\alpha\rangle = \varepsilon_\alpha|\alpha\rangle$ $\Pi|\beta\rangle = \varepsilon_\beta|\beta\rangle$ where $\varepsilon_{\alpha,\beta} = \pm 1$ eigval of Π
then $\langle\beta|\vec{x}|\alpha\rangle = 0$ unless $\varepsilon_\alpha\varepsilon_\beta = -1$. if $\varepsilon_\alpha\varepsilon_\beta \neq -1$ then $\langle\beta|\vec{x}|\alpha\rangle = 0$

Pf ① $\langle\beta|\{\Pi, x\}|\alpha\rangle = 0$, $(\varepsilon_\beta + \varepsilon_\alpha)\langle\beta|x|\alpha\rangle = 0$.
② $\langle\beta|x|\alpha\rangle = \langle\beta|\Pi^{-1}\Pi x\Pi^{-1}\Pi|\alpha\rangle = -\varepsilon_\alpha\varepsilon_\beta\langle\beta|x|\alpha\rangle$.

4.4 Time Reversal (Intro)

Physical Intuition

Classical

Time Reversal
- reverse dir. of momentum $\vec{P}$

$\vec{x}(t)$, $\vec{x}(-t)$ both soln to $m\ddot{\vec{x}} = -\nabla V(\vec{x})$

Maxwell Eqn

$$\nabla\cdot\vec{E} = 4\pi\rho$$
$$\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$
$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

& Lorentz Force $\vec{F} = e(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})$ all time reversal invariant. (in sense of expectation value)

b/c $t\to -t$, $\rho\to\rho$, $\vec{E}\to\vec{E}$; $\vec{j}\to-\vec{j}$, $\vec{B}\to-\vec{B}$; $\vec{v}\to-\vec{v}$

Wave Mech

$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla + V\right)\psi$; $\psi(\vec{x},t)$ soln $\psi(\vec{x},-t)$ not

* But $\psi^*(\vec{x},-t)$ soln (clear using energy eigket $\psi(\vec{x},t) = u_n(\vec{x})e^{-\frac{iE_nt}{\hbar}}$, $\psi^*(\vec{x},-t) = u_n^*(\vec{x})e^{-\frac{iE_nt}{\hbar}}$)

Remark time reversal is related to complex conjugate.

Remark
- for symmetry operation: rotation, translation, even parity. inner product preserved.

  ie $|\alpha\rangle \to |\tilde{\alpha}\rangle$, $|\beta\rangle\to|\tilde{\beta}\rangle$ then $\langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle\alpha|U^\dagger U|\beta\rangle = \langle\alpha|\beta\rangle$

  since sym. op. unitary
- for time reversal op. impose weaker cond. $|\langle\tilde{\beta}|\tilde{\alpha}\rangle| = |\langle\beta|\alpha\rangle|$

  we have $\boxed{\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle}$

Introduce Antiunitary

Def $|\alpha\rangle\to|\tilde{\alpha}\rangle = \Theta|\alpha\rangle$ $|\beta\rangle\to|\tilde{\beta}\rangle = \Theta|\beta\rangle$ is antiunitary

if
- $\boxed{\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*}$
- $\Theta(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^*\Theta|\alpha\rangle + c_2^*\Theta|\beta\rangle$ antilinear.

Caveat
1. $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$
2. $\Theta = UK$, $|\tilde{\alpha}\rangle = \Theta|\alpha\rangle$

   K anti-linear.

   $|a'\rangle$ base-ket

   $K|a'\rangle = |a'\rangle$.
- 1, 2 self ~~constant~~ consistent

Claim $\Theta = UK$ s.t. $KC|\alpha\rangle = C^*K|\alpha\rangle$ $C\in\mathbb{C}$

unitary complex conjugate op

Say $|\alpha\rangle = \sum|a'\rangle\langle a'|\alpha\rangle \to |\tilde{\alpha}\rangle = \sum\langle a'|\alpha\rangle^* K|a'\rangle = \sum\langle a'|\alpha\rangle^*|a'\rangle$

- clearly antilinearity holds
- $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$

Pf $|\alpha\rangle \xrightarrow{UK} |\tilde{\alpha}\rangle = \sum_{a'}\langle a'|\alpha\rangle^* U|a'\rangle$ $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \sum_{a''}\sum_{a'}\langle a''|\beta\rangle\langle a''|U^\dagger U|a'\rangle\langle a'|\alpha\rangle^*$

$|\beta\rangle \xrightarrow{UK} |\tilde{\beta}\rangle = \sum_{a''}\langle a''|\beta\rangle^* U|a''\rangle$ $= \langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$

Time Reversal Op " Θ "

Consider $|\alpha\rangle \longrightarrow \Theta|\alpha\rangle$

ex Let $|\alpha\rangle = |\vec{p}'\rangle$ then $\Theta|\vec{p}'\rangle \sim |-\vec{p}'\rangle$

At $t=0$ sys has $|\alpha\rangle$. $t>0$ $|\alpha, t_0; \delta t\rangle = (1 - \frac{iH\delta t}{\hbar})|\alpha\rangle$ then

$$\Theta\left(1 + \frac{iH\delta t}{\hbar}\right)|\alpha\rangle = \left(1 - \frac{iH\delta t}{\hbar}\right)\Theta|\alpha\rangle$$ $|\alpha\rangle = |\vec{p}\rangle$

thus $-iH\Theta|\alpha\rangle = \Theta iH|\alpha\rangle \quad \forall |\alpha\rangle$

Claim Θ not unitary , else $-H\Theta = \Theta H$

Proof by contradiction

1) If unitary $H\Theta|n\rangle = -\Theta H|n\rangle = -E_n\Theta|n\rangle$ w/ neg. energy
but energy of say free particle ≥ 0 thus $-E_n$ unphysical.

2) $H = \frac{p^2}{2m}$, w/ $\Theta^{-1}H\Theta = -H \Rightarrow \Theta^{-1}\frac{p^2}{2m}\Theta = -\frac{p^2}{2m}$ contradict by construction.

thus $\boxed{\Theta \text{ must be antiunitary}}$

then $\Theta iH = -i\Theta H = -iH\Theta \Rightarrow \boxed{[\Theta, H] = 0}$ fundamental prop of H under time reversal.
under inf. time translation.

Important rule of Θ

$$\langle\beta|\Theta|\alpha\rangle = \begin{cases} (\langle\beta|)\cdot(\Theta|\alpha\rangle) & ✓ \\ (\langle\beta|\Theta)\cdot|\alpha\rangle & \times \text{ (undefine)} \end{cases}$$ so Θ is left hand action only.

Now for $|\tilde{\alpha}\rangle = \Theta|\alpha\rangle$ $|\tilde{\beta}\rangle = \Theta|\beta\rangle$

Claim $\boxed{\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A^{\dagger}\Theta^{-1}|\tilde{\beta}\rangle}$

- $\langle\beta|A|\alpha\rangle \overset{T.R}{=} \langle\tilde{\alpha}|\Theta A^{\dagger}\Theta^{-1}|\tilde{\beta}\rangle$
- If A Herm. the parity of A ; $\Theta A\Theta^{-1} = \pm A$

Pf let $|\gamma\rangle = A^{\dagger}|\beta\rangle$ s.t $\langle\gamma| = \langle\beta|A$

now $\langle\beta|A|\alpha\rangle = \langle\gamma|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\gamma}\rangle = \langle\tilde{\alpha}|\Theta A^{\dagger}|\beta\rangle = \langle\tilde{\alpha}|\Theta A^{\dagger}\Theta^{-1}|\tilde{\beta}\rangle$ //

- thus by const.
$\Theta\vec{P}\Theta^{-1} = -\vec{P}$ or $\{\vec{P}, \Theta\} = 0$
$\Theta\vec{X}\Theta^{-1} = \vec{X}$ $\quad [\Theta, \vec{X}] = 0$

Corrollary If A Hermitian,
$\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A\Theta^{-1}|\tilde{\beta}\rangle$

Remark : Parity of A is defined by $\boxed{\Theta A\Theta^{-1} = \pm A}$ then $\langle\beta|A|\alpha\rangle = \pm\langle\tilde{\beta}|A|\tilde{\alpha}\rangle^*$

- then expectation value $\langle\alpha|A|\alpha\rangle = \pm\langle\tilde{\alpha}|A|\tilde{\alpha}\rangle$

ex mom. By construction, it's reasonable to expect
$\langle\alpha|\vec{P}|\alpha\rangle = -\langle\tilde{\alpha}|\vec{P}|\tilde{\alpha}\rangle$ or $\Theta\vec{P}\Theta^{-1} = -\vec{P}$

thus
$\vec{P}(\Theta|\vec{p}'\rangle) = -\Theta\vec{P}\Theta^{-1}\Theta|\vec{p}'\rangle = -\vec{p}'(\Theta|\vec{p}'\rangle)$ mom. eigket w/ eigval $-\vec{p}'$

ex position. $\Theta\vec{X}\Theta^{-1} = \vec{X}$ by construction.
then $\Theta|\vec{x}'\rangle = |\vec{x}'\rangle$
since $\langle\alpha|\vec{X}|\alpha\rangle = \langle\alpha|\Theta\vec{X}\Theta^{-1}|\alpha\rangle = \langle\tilde{\alpha}|\vec{X}|\tilde{\alpha}\rangle$

Commutation relation under time reversal

$[X_i, P_j] = i\hbar\delta_{ij}$

$\Theta[X_i, P_j]\Theta^{-1} = -i\hbar\delta_{ij}$

$\Theta \tilde{i} \Theta^{-1} = -\tilde{i}$

$\Theta \vec{J} \Theta^{-1} = -\vec{J}$

$[X_i, P_j] = i\hbar\delta_{ij}$ preserved under time reversal.

Pf $\Theta[X_i, P_j]\Theta^{-1}\Theta|\alpha\rangle = \Theta i\hbar\delta_{ij}|\alpha\rangle = -i\hbar\delta_{ij}\Theta|\alpha\rangle$

thus $\Theta[X_i, P_j]\Theta^{-1} = -i\hbar\delta_{ij} \Rightarrow [X_i, P_j] = -i\hbar\delta_{ij}$ ∎

Also intuitively $\Theta\vec{J}\Theta^{-1} = -\vec{J}$ then $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$ (Pf by same reasoning)

Wave func

W.f.

$\langle x'|\alpha\rangle = \psi(x')$

$\langle x'|\tilde{\alpha}\rangle = \psi^*(x')$

Spinless particle at $t=0$ $|\alpha\rangle$

$|\alpha\rangle = \int d^3x' |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$; $\Theta|\alpha\rangle = \int d^3x' \Theta|\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle^* = \int d^3x' |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle^*$

thus

$$\boxed{\begin{aligned} &\psi(\vec{x}) \to \psi^*(\vec{x}) \\ &Y_\ell^m(\theta,\phi) \to Y_\ell^{m*}(\theta,\phi) = (-1)^m Y_\ell^{-m}(\theta,\phi) \end{aligned}}$$

therefore $\boxed{\Theta|\ell, m\rangle = (-1)^m|\ell, -m\rangle}$

Physically $m > 0 \Rightarrow$ current flow C.C.W ; $m<0$ C.W.

$Y_\ell^m(\theta,\phi) \to Y_\ell^{m*}(\theta,\phi)$

$= (-1)^m Y_\ell^{-m}(\theta,\phi)$

$\Theta|\ell, m\rangle = (-1)^m|\ell, -m\rangle$

Thm If $[H, \Theta] = 0$ w/ $H|n\rangle = E_n|n\rangle$ & $|n\rangle$ nondegen.
then the corresponding wavefunc i.e. $\langle\vec{x}'|n\rangle$ is real.

Pf $H\Theta|n\rangle = \Theta H|n\rangle = E_n\Theta|n\rangle \Rightarrow \Theta|n\rangle = |n\rangle$

w/ wavefunc $\langle\vec{x}'|n\rangle \longleftrightarrow |n\rangle$

$\langle\vec{x}'|\Theta|n\rangle \longleftrightarrow \Theta|n\rangle$

$\langle\vec{x}'|n\rangle = \langle\vec{x}'|\Theta|n\rangle = \langle\vec{x}'|\tilde{n}\rangle = \langle\tilde{\vec{x}}'|\tilde{n}\rangle = \langle\vec{x}'|n\rangle^*$

thus $\langle\vec{x}'|n\rangle = \langle\vec{x}'|n\rangle^*$ real.

ex. Consider $\Theta|\alpha\rangle = \int d^3p' |-\vec{p}'\rangle\langle\vec{p}'|\alpha\rangle^*$

$$= \int d^3p' \int d^3p'' |-\vec{p}'\rangle \underbrace{\langle\vec{p}'|-\vec{p}''\rangle}_{\delta_{\vec{p}'(-\vec{p}'')}} \langle-\vec{p}''|\alpha\rangle^*$$

$\Theta|\alpha\rangle = \int d^3p'' |\vec{p}''\rangle\langle-\vec{p}''|\alpha\rangle^*$

$\langle\vec{p}'|\Theta|\alpha\rangle = \langle-\vec{p}'|\alpha\rangle^* = \phi^*(-\vec{p}')$ //

or $\Theta\vec{P}\Theta^{-1} = -\vec{P}$, $\vec{P}\Theta|\vec{p}'\rangle = -\Theta\vec{P}|\vec{p}'\rangle = -\vec{p}'\Theta|\vec{p}'\rangle \Rightarrow \Theta|\vec{p}'\rangle = |-\vec{p}'\rangle$

we have $\widetilde{|-\vec{p}'\rangle} = \Theta|-\vec{p}'\rangle = |\vec{p}'\rangle$

now $\langle\vec{p}'|\Theta|\alpha\rangle = \langle\vec{p}'|\tilde{\alpha}\rangle = \langle\widehat{-\vec{p}'}|\tilde{\alpha}\rangle = \langle-\vec{p}'|\alpha\rangle^*$

thus $\langle\vec{p}'|\Theta|\alpha\rangle = \phi^*(-\vec{p}')$ //