6 Scattering Theory

Given $H = H_0 + V(\vec{r})$ $\quad H_0 = \frac{\vec{p}^2}{2m}$ $\quad E_k = \frac{\hbar^2 \vec{k}^2}{2m}$ $\quad$ s.t $\quad H_0|\vec{k}\rangle = E_k|\vec{k}\rangle$

time. indep. scattering potential

plane wave eig. vec

Remark • Incoming particle see $V(\vec{r})$ as perturbation.

• analysis using time-dep. perturb. theory in interaction pic.

Recall $|\alpha, t; t_0\rangle_I = U_I(t; t_0)|\alpha, t_0; t_0\rangle_I$

where U_I satisfies • $U_I(t_0; t_0) = 1$

• $i\hbar \frac{\partial}{\partial t} U_I = V_I(t) U_I \quad w/ \quad V_I = e^{\frac{iH_0 t}{\hbar}} V e^{\frac{-iH_0 t}{\hbar}}$

thus

$$U_I(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t') U_I(t', t_0) dt'$$

and transition amplitude:

$$\langle n|U_I(t,t_0)|i\rangle = \delta_{ni} - \frac{i}{\hbar}\sum_m \langle n|V|m\rangle \int_{t_0}^{t} e^{i\omega_{nm}t'} \langle m|U_I(t';t_0)|i\rangle dt'$$

Remark: • Scattering states are continuum but $|m\rangle$ is discrete

• resolved inconsistency by considering scattering status in a "big box" w/ sides "L".

thus coord. rep. $\quad \langle \vec{x}|\vec{k}\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{x}}\Big|_{L\to\infty} \quad \& \quad \langle \vec{k}'|\vec{k}\rangle = \delta_{\vec{k}\vec{k}'}$

Obtain from recursive relation yielding 1^{st} order

$$\langle n|U_I(t,t_0)|i\rangle = \delta_{ni} - \frac{i}{\hbar}\langle n|V|i\rangle \int_{t_0}^{t} e^{i\omega_{ni}t'} dt'$$

Remark: • initial and final states exist asymptotically i.e $t, t_0 \to \infty, -\infty$

• transition rate from 1^{st} order amp "emerges" as Fermi's golden rule as $t \to \infty$

Now as $t_0 \to -\infty$ def T matrix s.t:

$$\langle n|U_I(t, \underset{-\infty}{t_0})|i\rangle = \delta_{ni} - \frac{i}{\hbar} T_{ni} \int_{t_0=-\infty}^{t} e^{i\omega_{ni}t' + \varepsilon t'} dt' \quad \text{where } T_{ni} \in \mathbb{C}$$

note: $\varepsilon > 0$, $\varepsilon \ll 1$ s.t $e^{\varepsilon t} \sim 1$ as $t \to \infty$

and $e^{\varepsilon t'} \to 0$ as $t_0 \to -\infty$.

Def Scattering Matrix

$$S_{ni} = \lim_{t\to\infty}\left[\lim_{\varepsilon\to 0}\langle n|U_I(t,-\infty)|i\rangle\right] = \underbrace{\delta_{ni}}_{\text{if final = initial states}} - \underbrace{\frac{i}{\hbar}T_{ni}\int_{-\infty}^{\infty} e^{i\omega_{ni}t'}dt'}_{\text{where scattering occurs \& governing by T matrix}} = \delta_{ni} - 2\pi i\,\delta(E_n - E_i)\,T_{ni}$$

Interpretation of cross-section v.s Transition rates.

Consider scatter event $|i\rangle \neq |n\rangle$,

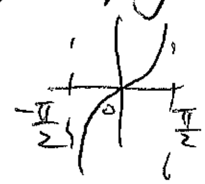
$$\langle n|U_I(t,t_0)|i\rangle = -\frac{i}{\hbar}T_{ni}\int_{-\infty}^{t} e^{i\omega_{ni}t' + \varepsilon t'}dt' = -\frac{i}{\hbar}T_{ni}\frac{e^{i\omega_{ni}t + \varepsilon t}}{i\omega_{ni} + \varepsilon}$$

thus transition rates,

$$\omega(i\to n) = \frac{d}{dt}\left|\langle n|U_I(t,-\infty)|i\rangle\right|^2 = \frac{1}{\hbar^2}|T_{ni}|^2\frac{2\varepsilon e^{2\varepsilon t}}{\omega_{ni}^2 + \varepsilon^2}$$

Analysis

- 1^{st} $\varepsilon\to 0$ $(t<\infty)$ then $t\to\infty$
- for $\omega_{ni}\neq 0$ $\omega\to 0$ else $\omega\to\infty$ $\Rightarrow \delta(\omega_{ni})$ emerging!
- note $\displaystyle\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2+\varepsilon^2} = \frac{1}{\varepsilon}\arctan\left(\frac{\omega}{\varepsilon}\right)\Big|_{-\infty}^{\infty} = \frac{\pi}{\varepsilon}$



- thus $\displaystyle\lim_{\varepsilon\to 0}\frac{e^{2\varepsilon t}}{\omega^2+\varepsilon^2} \sim \frac{1}{\omega^2+\varepsilon^2}$ rewrite $\displaystyle\int_{-\infty}^{\infty}\frac{\varepsilon\, d\omega}{\omega^2+\varepsilon^2} = \pi\int\delta(\omega)\,d\omega$

~~or simply~~ thus $\displaystyle\lim_{\varepsilon\to 0}\frac{\varepsilon e^{2\varepsilon t}}{\omega^2+\varepsilon^2} = \pi\,\delta(\omega)$ or simply $\displaystyle\lim_{a\to 0}\frac{a}{x^2+a^2} = \pi\,\delta(x)$

We now have :

$$\omega(i\to n) = \frac{2\pi}{\hbar}|T_{ni}|^2\delta(E_n - E_i)$$ form of Fermi's Golden rule!

Density of final states $\rho(E_n) = \frac{\Delta n}{\Delta E_n}$

For elastic scattering take $|i\rangle = |\vec{k}\rangle$ $|n\rangle = |\vec{k}'\rangle$ s.t. $|\vec{k}| = |\vec{k}'| = k$
ie diff. direction but same mom. magnitude.

Recall particle in the box:

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}|\vec{n}|\right)^2 \quad \text{s.t.} \quad \Delta E_n = \frac{\hbar^2}{m}\left(\frac{2\pi}{L}\right)^2 |\vec{n}|\,\Delta|\vec{n}| \quad \text{w/ } \vec{n} = (n_x, n_y, n_z) \quad n_i \in \mathbb{N}$$

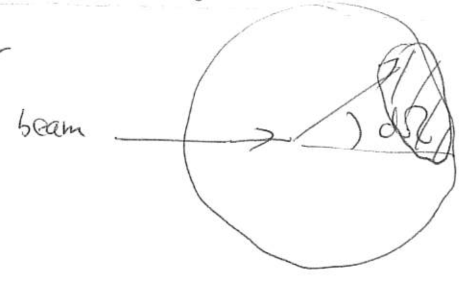
states within spherical shell of radius $|\vec{n}|$ (here L is large s.t. $|\vec{n}| \sim$ con't) Recalled $k = \frac{2\pi}{L}|\vec{n}|$

$$\Delta n = 4\pi |\vec{n}|^2 \Delta|\vec{n}| \frac{d\Omega}{4\pi} \qquad \text{thus } \rho(E_n) = \frac{\Delta n}{\Delta E_n} = \frac{mk}{\hbar^2}\left(\frac{L}{2\pi}\right)^3 d\Omega$$

And accounting for all final states

$$\omega(i \to n) = \sum_n \int \rho(E_n)|C_n|^2 dE_n = \frac{mkL^3}{(2\pi)^2\hbar^3}|T_{ni}|^2 d\Omega$$

Consider



- beam scattered into $d\Omega$ w/ $p = \hbar k$ w/ speed $v = \frac{\hbar k}{m}$
- time for particles to cross the box : $\frac{L}{v}$
- thus flux $= \underbrace{\frac{1}{L^2}}_{1/\text{area}} \Big/ \underbrace{\left(\frac{L}{v}\right)}_{\text{time/particle}}$ = # particle/area/sec

or in term of prob. flux
recall $\vec{j} = \frac{\hbar}{m}\,\text{Im}(\psi^* \nabla \psi)$ here $\psi = \langle \vec{x}|\vec{k}\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{x}}$

$$\vec{j}(\vec{x}, t) = \frac{\hbar}{m}\frac{\vec{k}}{L^3} = \frac{v}{L^3}$$

Now define cross section $d\sigma = \frac{\text{trans. rate}}{\text{flux}}$ = effective area of scattered particle

$\Rightarrow$ $$\frac{d\sigma}{d\Omega} = \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 |T_{ni}|^2$$

state from $i \to n$ per particle in given area.

or eff. area that has event of state change $i \to n$ per particle

Remark: What's T_{ni} in terms of $V(\vec{r})$?

note $d\sigma = \frac{\omega(i \to n)}{|\vec{j}|}$

Quest for T matrix

Sub (*) $\langle n|U_I(t,-\infty)|i\rangle = \delta_{ni} + \frac{1}{\hbar} T_{ni} \frac{e^{i\omega_{ni}t+\varepsilon t}}{-\omega_{ni}+i\varepsilon}$ into original form, so we get

$$\langle n|U_I(t,-\infty)|i\rangle = \delta_{ni} + \frac{1}{\hbar} V_{ni} \frac{e^{i\omega_{ni}t+\varepsilon t}}{-\omega_{ni}+i\varepsilon} - \frac{i}{\hbar}\frac{1}{\hbar}\sum_m V_{nm} \frac{T_{mi}}{-\omega_{mi}+i\varepsilon}\int_{-\infty}^{t} e^{i\omega_{nm}t'+i\omega_{mi}t'+\varepsilon t'}\,dt'$$

where $\omega_{ni} = \omega_{nm}+\omega_{mi}$, comparing to (*) we get

(**) $$T_{ni} = V_{ni} + \sum_m V_{nm} \frac{T_{mi}}{E_i - E_m + i\hbar\varepsilon}$$

Remark: • this is sys. of inhomo. linear eqn

- solve T_{ni} in terms of V_{nm} still hard! —— not important

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What's next? Quest for $|\psi\rangle$ Alternative 1.

Assume $\exists\, |\psi^+\rangle$ w/ $|\psi^+\rangle = \sum |j\rangle\langle j|\psi^+\rangle$

Ansatz $T_{ni} = \langle n|V|\psi^+\rangle = \sum_i \langle n|V|i\rangle\langle i|\psi^+\rangle$

s.t (**):

$$\langle n|V|\psi^+\rangle = \langle n|V|i\rangle + \sum_m \langle n|V|m\rangle \frac{\langle m|V|\psi^+\rangle}{E_i - E_m + i\hbar\varepsilon} \quad \forall\, |n\rangle$$

from which, $$\boxed{|\psi^+\rangle = |i\rangle + \frac{1}{E_i - H_0 + i\hbar\varepsilon} V|\psi^+\rangle}$$ (Lippmann-Schwinger Eqn)

Now $$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 |\langle n|V|\psi^+\rangle|^2}$$

O.T.O.H for T operator, claim $\exists\, |\psi^+\rangle$ s.t $T|i\rangle = V|\psi^+\rangle$

This leads to $$T = V + V\frac{1}{E_i - H_0 + i\hbar\varepsilon} T$$

For weak V, including H.O.T;

$$T = V + V\frac{1}{E_i - H_0 + i\hbar\varepsilon} T + V\frac{1}{E_i - H_0 + i\hbar\varepsilon} V \frac{1}{E_i - H_0 + i\hbar\varepsilon} V + \cdots$$

Alt. argument. $\exists\, |\psi^+\rangle$ s.t $T|i\rangle = V|\psi^+\rangle$ let $G_0 = \frac{1}{E_i - H_0 + i\hbar\varepsilon}$ so LS eqn $|\psi^+\rangle = |i\rangle + G_0 V|\psi^+\rangle$

so $T|i\rangle = V|i\rangle + VG_0V|\psi^+\rangle$

$= V|i\rangle + VG_0V|i\rangle + VG_0VG_0V|\psi^+\rangle$

$\Rightarrow T = V + VG_0V + (VG_0)^2V + (VG_0)^3V + \cdots$ allow expansion of T op.

Scattering Amplitude

Alternatives: Consider $H_0|\psi^0\rangle = E|\psi^0\rangle$

Assume $\exists\, |\psi^+\rangle$ s.t. $H|\psi^+\rangle = E|\psi^+\rangle$ where $H = H_0 + V$

By construction $|\psi^+\rangle = \dfrac{V}{E-H_0}|\psi^+\rangle \xrightarrow{\text{remove pathology}} |\psi^+\rangle = |\psi^0\rangle + \dfrac{1}{E-H_0+i\varepsilon}V|\psi^+\rangle$

Anyway, including scattering from future to past, then

$$|\psi^\pm\rangle = |i\rangle + \frac{1}{E-H_0\pm i\varepsilon}V|\psi^\pm\rangle$$

thus

$$\langle\vec{x}|\psi^\pm\rangle = \langle\vec{x}|i\rangle + \int d^3x' \langle\vec{x}|\frac{1}{E-H_0\pm i\varepsilon}|\vec{x}'\rangle\langle\vec{x}'|V|\psi^\pm\rangle$$

w/ Green func.

$$\boxed{G_\pm(\vec{x},\vec{x}') \equiv \frac{\hbar^2}{2m}\langle\vec{x}|\frac{1}{E-H_0\pm i\varepsilon}|\vec{x}'\rangle}$$

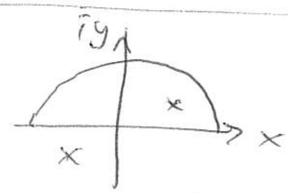
Solving $G_\pm$.

$$G_\pm(\vec{x},\vec{x}') = \frac{\hbar^2}{2m}\int d^3\vec{k}'\int d^3\vec{k}''\langle\vec{x}|\vec{k}'\rangle\langle\vec{k}'|\frac{1}{E-H_0\pm i\varepsilon}|\vec{k}''\rangle\langle\vec{k}''|\vec{x}'\rangle$$

$$= \frac{\hbar^2}{2m}\frac{1}{(2\pi)^3}\int d^3\vec{k}'\int d^3\vec{k}''\, e^{i\vec{k}'\cdot\vec{x}-i\vec{k}''\cdot\vec{x}'}\,\delta(\vec{k}'-\vec{k}'')\frac{1}{\frac{\hbar^2\vec{k}^2}{2m}-\frac{\hbar^2\vec{k}'^2}{2m}\pm i\varepsilon}$$

$$= \frac{1}{8\pi^2}\frac{1}{i|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty}k'dk'\left[\frac{e^{-ik'|\vec{x}-\vec{x}'|}-e^{ik'|\vec{x}-\vec{x}'|}}{k^2-k'^2\pm i\varepsilon}\right]$$

Contour Analysis



Consider G_+: $k^2-k'^2+i\varepsilon = 0 \Rightarrow k' = \pm(k+i\varepsilon)$

where

$$(-1)\int\frac{dk'\,k\,e^{ik'|\vec{x}-\vec{x}'|}}{k^2-k'^2+i\varepsilon} = \oint dk'\frac{k'e^{ik'|\vec{x}-\vec{x}'|}}{(k'-(k+i\varepsilon))(k'+(k+i\varepsilon))}$$

here $k' = x+iy \Rightarrow \operatorname{Im}k' = -y$

$$= (-1)(2\pi i)\left.\frac{k'e^{ik'|\vec{x}-\vec{x}'|}}{k'+k+i\varepsilon}\right|_{k'=k+i\varepsilon,\ \varepsilon\to 0}$$

$$= -\pi i\, e^{ik|\vec{x}-\vec{x}'|}$$

By sym. same result for $e^{-ik|\vec{x}-\vec{x}'|}$ term.

thus

$$G_\pm(\vec{x},\vec{x}') = -\frac{1}{4\pi}\frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

which is soln of Helmholtz eqn.

$$(\nabla^2+k^2)\,G_\pm(\vec{x},\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$$

Remark: $\vec{x}\neq\vec{x}'$ solves $H_0G_\pm = E_0G_\pm$.

Thus we have explicit form

$$\langle \vec{x}|\psi^\pm\rangle = \underbrace{\langle \vec{x}|i\rangle}_{\text{Incident wave}} - \underbrace{\frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}\langle \vec{x}'|V|\psi^\pm\rangle}_{\text{effect of scattering}}$$

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ex Consider V local ie $V=V(\vec{x}')$ s.t. $\langle \vec{x}'|V|\vec{x}''\rangle = V(\vec{x}')\,\delta^{(3)}(\vec{x}'-\vec{x}'')$

thus $\langle \vec{x}'|V|\psi^\pm\rangle = \int d^3x'' \langle \vec{x}'|V|\vec{x}''\rangle\langle \vec{x}''|\psi^\pm\rangle = V(\vec{x}')\langle \vec{x}'|\psi^\pm\rangle$

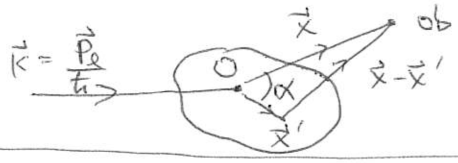
which simplifies

$$\langle \vec{x}|\psi^\pm\rangle = \langle \vec{x}|i\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} V(\vec{x}')\langle \vec{x}'|\psi^\pm\rangle$$

<u>Remark - Physics</u>

- $\vec{x}$ obs. pt
- limit region to a nonvanishing contribution for finite range potential

$|\vec{x}| \gg |\vec{x}'|$ — • Since obs. by detector far away from scatter, interesting to learn effect of scatter outside range of $V(\vec{r})$



Analysis ① Let $r = |\vec{x}|$ $\quad r' = |\vec{x}'|$ for $r \gg r'$

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 - 2rr'\cos\alpha + r'^2} \approx r - \hat{r}\cdot\vec{x}'$$

Since $k|\vec{x}-\vec{x}'| = kr - \underbrace{k\hat{r}}_{\vec{k}'}\cdot\vec{x}'$ thus $e^{\pm ik|\vec{x}-\vec{x}'|} \approx e^{\pm ikr} e^{\mp i\vec{k}'\cdot\vec{x}'}$

② $r \gg r'$, $\frac{1}{|\vec{x}-\vec{x}'|} \sim \frac{1}{r}$

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take $|i\rangle = |\vec{k}\rangle$ eig. state of free particle then

$$\langle \vec{x}|\psi^{(+)}\rangle \xrightarrow{r\gg 1} \langle \vec{x}|\vec{k}\rangle - \frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{e^{ikr}}{r}\int d^3x' e^{-i\vec{k}'\cdot\vec{x}'} V(\vec{x}')\langle \vec{x}'|\psi^+\rangle$$

$$= \frac{1}{L^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k}',\vec{k})\right]$$

where $f(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}L^3\int d^3x' \frac{e^{-i\vec{k}'\cdot\vec{x}'}}{L^{3/2}} V(\vec{x}')\langle \vec{x}'|\psi^+\rangle = -\frac{mL^3}{2\pi\hbar^2}\langle \vec{k}'|V|\psi^+\rangle$

$\int d^3x' d^3x'' \langle \vec{k}'|x'\rangle\langle x'|V|x''\rangle\langle x''|\psi\rangle$

Recalled $\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 |\langle n|V|\psi^+\rangle|^2}$

thus $\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k}',\vec{k})|^2}$ $\quad$ & $\quad f(\vec{k},\vec{k}') = -\frac{mL^3}{2\pi\hbar^2}\langle \vec{k}'|V|\psi^+\rangle$

Optical Thm

Relate Imaginary part of forward scattering amplitude

$$f(\theta=0) = f(\vec{k},\vec{k})$$

to total cross section $\sigma_{tot} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$ as

$$\mathrm{Im} f(\theta=0) = \frac{k\,\sigma_{tot}}{4\pi}$$

Pf. Recall $f(\vec{k},\vec{k}) = -\frac{mL^3}{2\pi\hbar^2}\langle \vec{k}|V|\psi^+\rangle$ using LG eqn

where $\langle \vec{k}|V|\psi^+\rangle = \underbrace{\langle\psi^+|V|\psi^+\rangle}_{\text{real}} - \langle\psi^+|V\frac{1}{E-H_0-i\varepsilon}V|\psi^+\rangle$ (*)

Remark: • want Im on $f(\vec{k},\vec{k})$

• 2^{nd} term of (*) has sing. on $E=H_0$ as $\varepsilon\to0$. So need Cauchy principle value.

Tool CPV

$$I \equiv \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx$$ sing. on $\mathbb{R}$ at x_0

Partitioning I into: $I = P(I) + \int_C \frac{f(z)}{z-z_0}dz$

where  $P(I) \equiv \int_{-\infty}^{x_0-\delta}\frac{f(x)}{x-x_0}dx + \int_{x_0+\delta}^{\infty}\frac{f(x)}{x-x_0}dx$ w/ $\delta \ll 1$

Consider 2^{nd} $\int$ where def $z = z_0+\delta e^{i\theta}$ w/ $z_0=x_0$ & $dz = i\delta e^{i\theta}d\theta$

thus $$\int_C \frac{f(z)}{z-z_0}dz = \lim_{\delta\to0}\int_{-\pi}^{0}\frac{f(z)}{\delta e^{i\theta}}\, i\delta e^{i\theta}d\theta = i\int_{-\pi}^{0} f(z_0)\,d\theta = i\pi f(x_0)$$ //

Now Consider $$\lim_{\varepsilon\to0}\frac{1}{E-H_0-i\varepsilon} = \lim_{\varepsilon\to0}\int_{-\infty}^{+\infty}\frac{\delta(E-E')}{E'-H_0-i\varepsilon}dE' = i\pi\,\delta(E-H_0)$$

Thus $$\mathrm{Im}\, f(\vec{k},\vec{k}) = -\frac{mL^3}{2\pi\hbar^2}\mathrm{Im}\langle\vec{k}|V|\psi^+\rangle = \frac{mL^3}{2\hbar^2}\langle\vec{k}|T^\dagger\delta(E-H_0)T|\vec{k}\rangle$$ using $T|\vec{k}\rangle = V|\psi^+\rangle$

$$= \frac{mL^3}{2\hbar^2}\int d^3k'\langle\vec{k}|T^\dagger\delta(E-H_0)|\vec{k}'\rangle\langle\vec{k}'|T|\vec{k}\rangle$$

$$= \frac{mL^3}{2\hbar^2}\int d^3k'\int d^3k''\langle\vec{k}|T^\dagger|\vec{k}''\rangle\langle\vec{k}''|\delta(E-H_0)|\vec{k}'\rangle\langle\vec{k}'|T|\vec{k}\rangle$$

$$= \frac{mL^3}{2\hbar^2}\int d^3k'\,|\langle\vec{k}'|T|\vec{k}\rangle|^2\delta\left(E-\frac{\hbar^2k'^2}{2m}\right)$$

$$= \frac{2\pi^2\hbar^2}{m(2\pi)^3}\int d^3k'\,|f(\vec{k}',\vec{k})|^2\delta\left(E-\frac{\hbar^2k'^2}{2m}\right) = \frac{\hbar^2}{4\pi m}\frac{mk}{\hbar^2}\int d\Omega_k\frac{d\sigma}{d\Omega_k}$$

$$\cancel{= \frac{\hbar^2}{4\pi m}\frac{1}{\hbar^2k/m}k^2\int d\Omega_k\frac{d\sigma}{d\Omega_k} = \frac{k}{4\pi}\sigma_{tot}}$$ //

thus $\mathrm{Im} f(\vec{k},\vec{k}) = \frac{k}{4\pi}\sigma_{tot}$

Born Approximation

Objective: Approximate $f(\vec{k}',\vec{k}) = \frac{-mL^3}{2\pi\hbar^2}\langle \vec{k}'|V|\psi^+\rangle$ for elastic scattering.

Approach.

$|\phi\rangle = |\phi_0\rangle + G_0 V|\phi\rangle \qquad G_0 = \frac{1}{E - H_0 + i\epsilon}$

recursive then

$|\phi\rangle = |\phi_0\rangle + GV|\phi_0\rangle$ where $G = G_0 + G_0VG_0 + G_0VG_0VG_0 + \cdots$

Let $\boxed{T \equiv V + VG_0V + VG_0VG_0V + \cdots}$ then ~~$T = V + VGG$~~ $T = V + VGV$

and Prop 1 $\boxed{T|\phi_0\rangle = V|\phi\rangle}$

b/c $T|\phi_0\rangle = V|\phi_0\rangle + VGV|\phi_0\rangle$

side note

$G = G_0 \frac{1}{1 - VG_0} = G_0 \frac{1}{G_0^{-1} - V} G_0$

$= G_0 \frac{1}{E - H + i\epsilon} G_0$

$\shortparallel$

T

So $G = G_0 T G_0$

propagator in loop diag.

Remark neither $\langle \vec{x}|\psi^+\rangle$ nor T are known.

But using 1^{st} order $T = V + VG_0V + \cdots$

Assume · $|\psi^+\rangle = |\vec{k}\rangle$

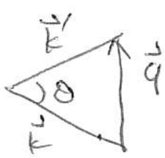
· elastic scattering $|\vec{k}'| = |\vec{k}|$

1^{st} Born Approx. $T \sim V$

$$f^{(1)}(\vec{k}',\vec{k}) = \frac{-mL^3}{2\pi\hbar^2}\langle \vec{k}'|V|\vec{k}\rangle = \frac{-m}{2\pi\hbar^2}\int d^3x'\, e^{i\vec{x}'\cdot(\vec{k}-\vec{k}')} V(\vec{x}')$$

Claim $f(\vec{k}',\vec{k}) = f(\theta)$ angle dependent for central potential $V(\vec{r})$

Elasticity



$q = |\vec{k}' - \vec{k}| = 2k\sin\frac{\theta}{2}$

then $f^{(1)}(\vec{k}',\vec{k}) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty r V(r) \sin(qr)\, dr$

thus $f^{(1)}(\vec{k}',\vec{k}) = f(\theta)$ Remark: · $f(\theta)$ depends on energy $\frac{\hbar^2k^2}{2m}$ & θ only

· $f(\theta)$ always real

· $\frac{d\sigma}{d\Omega}$ impartial to sign of V

· low energy $k \ll 1 \Rightarrow q \ll 1$ thus $\frac{\sin qr}{qr} \to 1$

· high energy $\sin(qr) \sim 0$ thus $f^{(1)}(\theta) \ll 1$

low energy $f^{(1)}(\theta) \simeq -\frac{2m}{4\pi\hbar^2}\int_0^\infty V(r)\, d^3x$

high energy $f^{(1)}(\theta) \ll 1$.

* Recall claims that $\exists\, T$ s.t. $V|\psi^+\rangle = T|\vec{k}\rangle$

Born approx. $T = V + VG_0V + \cdots$ 1^{st} approx $\Rightarrow$ $T|\vec{k}\rangle \doteq V|\vec{k}\rangle$.

ex $V(r) = \begin{cases} V_0 & r \le a \\ 0 & r > a \end{cases}$ s.t $f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{V_0 a^3}{(qa)^2}\left(\frac{\sin qa}{qa} - \cos qa\right)$

ex Yukawa potential $V(r) = \frac{V_0 e^{-\mu r}}{\mu r}$ $\frac{1}{\mu}$ range of potential

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2 q}\frac{V_0}{\mu}\int_0^\infty \sin(qr) e^{-\mu r} dr = -\frac{2m}{\hbar^2 q}\frac{V_0}{\mu} \operatorname{Im}\int_0^\infty e^{iqr-\mu r} dr = -\left(\frac{2mV_0}{\mu\hbar^2}\right)\frac{1}{q^2+\mu^2}$$

Now w/ $q^2 = 2k^2(1-\cos\theta)$

$$\frac{d\sigma}{d\Omega} \simeq |f^{(1)}(\theta)|^2 = \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta)+\mu^2]^2}$$

O.T.O.H if $\mu \to 0$ & $\frac{V_0}{\mu} = ZZ'e^2$ then for $p = \hbar k$ s.t $E_{KE} = \frac{p^2}{2m}$, we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\left(\frac{ZZ'e^2}{E_{KE}}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$ we have the coulomb scattering.

Validity of Born Approx.

Recall
$$\langle \vec{x}|\phi\rangle = \langle \vec{x}|k\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} V(\vec{x}')\langle \vec{x}'|\phi\rangle$$

Born $\Rightarrow$ $T \sim V$ thus $|\phi\rangle \sim |\vec{k}\rangle$

so $\langle \vec{x}'|\phi\rangle \sim \langle \vec{x}'|\vec{k}\rangle$ w/ $\vec{x} \sim 0$, so 2nd term expects ~ small

i.e.
$$\left|\frac{2m}{\hbar^2}\frac{1}{4\pi}\cancel{2\pi}\int d^3x' \frac{e^{ikr'}}{r'} V(\vec{x}') e^{i\vec{k}\cdot\vec{x}'}\right| \ll 1$$ Let $\vec{k}\cdot\vec{x}' = kx'\cos\theta$

$$= \frac{2m}{k\hbar^2}\left|\int dr' e^{ikr'} V(r') \sin kr'\right|$$

Low energy
$kr' \ll 1$: $\frac{2mk}{k\hbar^2}\left|\int_0^\infty V(r') r' dr'\right| \ll 1$

now for yukawa one gets $\frac{m|V_0|}{\hbar^2\mu^2} \ll 1$ but condition for yukawa requires $\frac{2m|V_0|}{\hbar^2\mu^2} \geq 2.7$

Born approx seems misleading at low energy.

take $\sin kr' = \frac{e^{ikr'} - e^{-ikr'}}{2i}$ s. h.o.t cancelled. oscill term

high energy
$|kr'| \gg 1$ $\Rightarrow r \gg \frac{1}{k}$

$$\frac{2m}{k\hbar^2}\left|\int_0^\infty dr' e^{ikr'} \frac{V_0 e^{-\mu r'}}{\mu r'} \sin kr'\right| \simeq \frac{2m}{k\hbar^2}\left|\frac{V_0}{\mu}\frac{1}{(-2i)}\int_0^\infty \frac{e^{-\mu r'}}{r'} dr'\right|$$ $\frac{e^{-\mu r}}{r} \approx \frac{1}{r} - \mu$

$\mu r \ll 1$ $\Rightarrow r \ll \frac{1}{\mu}$

$$\simeq \frac{2m}{k\hbar^2}\frac{V_0}{2\mu}\left|\int_{1/k}^{1/\mu} \frac{1}{r'} dr'\right| = \left|\frac{V_0 m}{\hbar^2 k\mu}\ln\left(\frac{k}{\mu}\right)\right| \ll 1$$ since $\frac{\hbar^2 k^2}{2m} \gg 1$

thus Born approx gets better at high energy.