66 Square Well

$$V = \begin{cases} -V_0 & r<R \\ 0 & r>R \end{cases} \quad V_0>0 \qquad k=\sqrt{\frac{2mE}{\hbar^2}},\quad E>0$$
(attractive)

$r>R \quad R_\ell = e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right]$

$r<R \quad R_\ell = A_\ell j_\ell(\alpha r)$, $\quad \alpha=\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

Governing Eqn.

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(rR_\ell) + \left[V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right]R_\ell = ER_\ell$$

low energy $k\ll1$, $\ell=0$ dominate

$\ell=0$, $j_0=\frac{\sin kr}{kr} \qquad n_0=-\frac{\cos kr}{kr} \qquad r>R$

$R^> = B\frac{\sin(kr+\delta_0)}{kr}$

$R^< = A\frac{\sin(\alpha r)}{\cancel{k}r}$

matching B.C $\qquad A\sin\alpha R = B\sin(kR+\delta_0)$

derivative of w.f cont', $\quad \frac{Bk\cos(kr+\delta_0)}{r}\Big|_R - \cancel{\frac{B\sin(kr+\delta_0)}{r^2}\Big|_R} = \frac{A\alpha\cos\alpha r}{r}\Big|_R - \cancel{\frac{A\sin\alpha r}{r^2}\Big|_R}$

then
$$A\alpha\cos\alpha R = Bk\cos(kR+\delta_0)$$

$\tan(kR+\delta_0)=\frac{k}{\alpha}\tan\alpha R \quad$ or $\quad \frac{\tan(kR)+\tan\delta_0}{1-\tan(kR)\tan\delta_0} = \frac{k}{\alpha}\tan\alpha R$

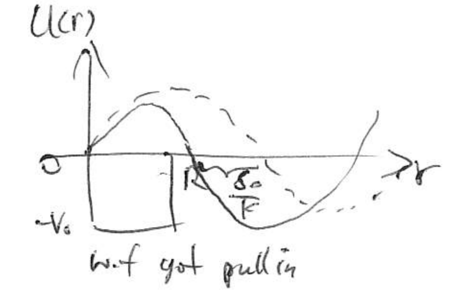
low energy $kR\ll1 \Rightarrow \tan\delta_0 \simeq \frac{k}{\alpha}\tan\alpha R$

since $\alpha>0, k>0 \Rightarrow \delta_0>0$

$\alpha=\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

attr $\nearrow \Rightarrow V_0\nearrow\ \alpha\nearrow$

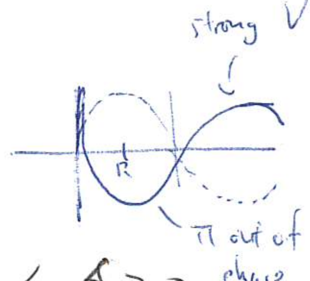
$\Rightarrow$ sin oscill rapidly (inside)



- low energy $\alpha\to\bar{\alpha}=\sqrt{\frac{2mV_0}{\hbar^2}}$
- very attractive potential $\delta_0\to\frac{\pi}{2}$

thus max. S-wave σ achieved

$$\sigma_{tot}=\frac{4\pi}{k^2}\sin^2\delta_0$$



- a more attractive well force $\delta_0\simeq\pi$ w/ $\sigma\to0$

Punch line : Extrem low energy results in perfect transmission ~~also~~ a/w/ strong attractive potential.

• Scattering at low energy ($k \simeq 0$).

$$\frac{d^2u}{dr^2} + \left(k^2 - \frac{2mV}{\hbar^2} - \frac{l(l+1)}{r^2}\right)u = 0$$

for $k=0$ $r>R$ s.t. $V=0$ $\frac{1}{r^2} \sim 0$, we have $\frac{d^2u}{dr^2} = 0$ for low ~~easy~~ zero energy approx.

thus $u(r) = C(r-a)$, for $k \sim 0$ $\lambda \sim$ infinite long.

O.T.O.H $r > R$

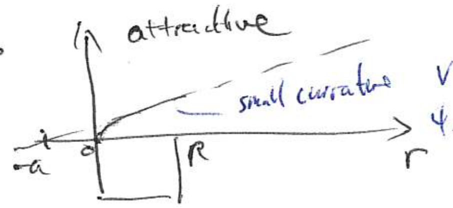
$u(r) \sim \frac{\sin(kr+\delta_0)}{kr}$ $r \gg 1$ $k \ll 1 \Rightarrow kr \sim$ const

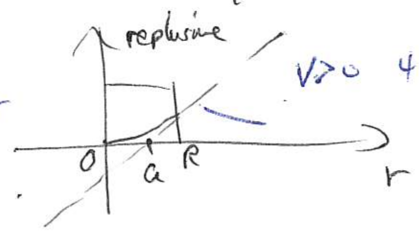
so $u(r) \sim \lim_{k\to 0} \sin(kr + \delta_0) \sim k(r + \frac{\delta_0}{k})$ as $C(r-a)$

Remark. attractive potential $\frac{\delta_0}{k} > 0$ $\frac{\delta_0}{k} = -a \Rightarrow a < 0$

repulsive " $\frac{\delta_0}{k} < 0$ $a = -\frac{\delta_0}{k} \Rightarrow a > 0$

so

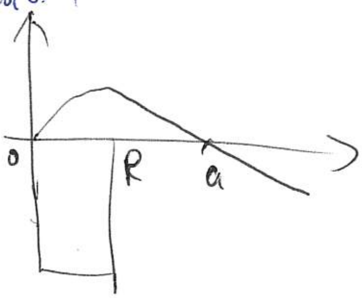




Consider $\frac{u'}{u} = k\cot\left(k(r+\frac{\delta_0}{k})\right)_{k\to 0} \simeq \frac{1}{r-a}$ thus for $r=0$ $\lim_{k\to 0} k\cot\delta_0 \sim -\frac{1}{a}$

or $\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0$
$\sin^2\delta_0 = \frac{\tan^2\delta_0}{1+\tan^2\delta_0} = \frac{1}{\cot^2\delta_0 + 1}$

$\sigma_{tot} \simeq \sigma_0 = 4\pi \left|\frac{1}{k\cot\delta_0 - ik}\right|^2_{k\to 0} = 4\pi a^2$ so a scattering length!



sign change from increasing attraction
for $E \sim 0$ $r \gg R$
bound state w.f. $\sim e^{-kr} \simeq 1 - kr$ thus sign flipped!
here $k \simeq 0$

~~$k = \frac{\sqrt{2m(E-V)}}{\hbar}$~~

~~$k \ll 1$ b/c $r \gg R$~~ $r > R$ V small ~~s.t. $E \sim 0$~~

~~s.t. k imaginary. & $e^{ikr} \sim e^{-|k|r} \sim 1 - |k|r$ neg slope~~

from ~~qualitat~~ qualitatively analysis we have sign flipped

$u(r) \sim (a-r) \sim e^{-kr} \sim 1 - kr$

Binding energy w/ infs. $E>0$

bound state w/ infinitesimal neg E bound state

For $E \sim 0$ ②

$r > R \quad u \sim e^{-kr} \quad k = \sqrt{\frac{2m|E|}{\hbar^2}}$

$r < R \quad u \sim \sin \alpha r \quad \alpha = \sqrt{\frac{2m(E+|V_0|)}{\hbar^2}}$

matching B.C

$$\frac{u'}{u}\Big|_{r=R_<} = \frac{u'}{u}\Big|_{r=R_>} \Rightarrow \frac{\alpha \cos \alpha r}{\sin \alpha r}\Big|_{r=R} = -\frac{k e^{-kr}}{e^{-kr}}\Big|_{r=R}$$

$$\frac{1}{r-a}\Big|_{r=R} = -k$$

$u'' + (\underbrace{k^2 - \frac{2m}{\hbar^2}V}_{\alpha^2})u = 0$

for $R \ll a$, $\quad k \simeq \frac{1}{a}$ //

$u(r) \sim \sin(\alpha r) \quad r \gg 1$

$\alpha r \to \text{const.}$

Binding energy: $B_e (=) E_{BE} = -E_{\text{bound state}} = \frac{\hbar^2 k^2}{2m} \simeq \frac{\hbar^2}{2ma^2}$.

$u'(r) \sim \alpha \cos(\alpha r)$

O.T.O.H we have

$u(r) \sim e^{-kr}$

$u'(r) \sim -k e^{-kr}$

at R we have

$\tan \alpha R \sim \frac{\alpha}{k} \Rightarrow$ cond.

O.T.O.H we have

$$\tan \alpha R = -\sqrt{\frac{E+|V_0|}{E}} \quad \text{when } |E| \to 0 \quad \tan(\alpha) \Rightarrow \infty.$$

$\Rightarrow \quad \alpha R = \frac{3\pi}{2}$

$$\sqrt{\frac{2m|V_0|}{\hbar^2}}R = \frac{3\pi}{2} \Rightarrow \frac{2m|V_0|^2}{\hbar^2}R^2 = \frac{9\pi^2}{4}$$ is the cond. for 0 energy bound state.

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ex. deuteron $^2H \quad 2 = 1_p + 1_n \quad$ Given $E_{BE} = 2.22\,\text{MeV} \quad a = 5.4 \times 10^{-13}\,\text{cm}$

$$E_{BE} = \frac{\hbar^2}{2\mu a^2} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{M_N}{2}$$

$$E_{BE} = \frac{\hbar^2}{M_N a^2} \sim 1.4\,\text{MeV}$$

//

note $R_>$ scattering event

$R_<$ related to binding

Why $E_{BE} = -E_{BS}$. $\quad u_{\text{scatt}} = \sin\left(\sqrt{\frac{2m E_{sc}}{\hbar^2}}\,r + \delta_0\right)$

$$\frac{u'}{u}\Big|_{R>} = \sqrt{\frac{2mE_{sc}}{\hbar^2}}\underbrace{\cot\left(\frac{2mE_{sc}}{\hbar^2}R + \delta_0\right)}_{\simeq \cot \delta_0} \qquad \frac{u'}{u}\Big|_{R<} = \sqrt{\frac{2m(E_b+|V_0|)}{\hbar^2}}\underbrace{\cot\left(\sqrt{\frac{2m(E_b+|V_0|)}{\hbar^2}}R\right)}_{-\sqrt{\frac{|E_b|}{E_b+|V_0|}}}$$

thus

$$\sqrt{E_{sc}}\cot\delta_0 = -\sqrt{E_b}$$

$$E_{sc} \simeq -E_b \quad //$$