6.2 Phase Shifts and Partial Waves

Free particles $[H_0, L_z] = 0 \quad [H_0, L^2] = 0$

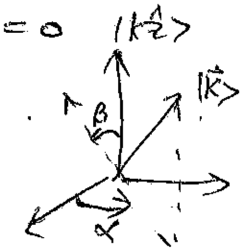
Quantum state is $|\vec{k}\rangle$ or $|E, l, m\rangle$ "sphr. wave state"

Quest for $\langle \vec{k}|E, l, m\rangle$ and $\langle \vec{x}|E, l, m\rangle$

Ansatz $$\langle \vec{k}|E\, l, m\rangle = g_{lE}(k)\, Y_l^m(\hat{k})$$

Pf $L_z|k\hat{z}\rangle = 0$ so $m = 0$

$$\langle E', l', m'|E, l, m\rangle = \delta_{ll'}\delta_{mm'}\delta(E - E') \quad \text{and} \quad \langle E', l', m'|k\hat{z}\rangle = 0 \quad |k\hat{z}\rangle$$



then $$|k\hat{z}\rangle = \sum_{l'} \int dE'\, |E', l', m' = 0\rangle\langle E', l', m' = 0|k\hat{z}\rangle$$

O.T.O.H $$|\vec{k}\rangle = \mathcal{D}(\alpha = \phi, \beta = \theta, \gamma = 0)|k\hat{z}\rangle$$

n+ $$\langle E, l, m|\vec{k}\rangle = \sum_{l'} \int dE' \underbrace{\langle E, l, m|\mathcal{D}(\alpha, \beta, 0)|E', l', 0\rangle}_{\mathcal{D}^{(l)}_{m0}\,\delta_{ll'}(E - E')} \langle E', l', 0|k\hat{z}\rangle$$

$$\langle E, l, m|\vec{k}\rangle = \underbrace{\mathcal{D}^{(l)}_{m0}(\alpha, \beta, 0)}_{\sim Y_l^m(\hat{k})} \underbrace{\langle E, l, 0|k\hat{z}\rangle}_{\text{indept. of } k \text{ orient.}}$$

so $\langle \vec{k}|E, l, m\rangle \propto g_{lE}(k)\, Y_l^m(\hat{k})$ Quest for $g_{lE}(k)$

note $(H_0 - E)|E, l, m\rangle = 0$

$$\langle \vec{k}|(H_0 - E)|E, l, m\rangle = \left(\frac{\hbar^2 k^2}{2m} - E\right)\langle \vec{k}|E, l, m\rangle = 0 \Rightarrow g_{lE}(k) = N\,\delta\left(\frac{\hbar^2 k^2}{2m} - E\right)$$

normalization w/ $$\int_0^{\pi}\int_0^{2\pi} Y_l^m\, Y_{l'}^{m'*}\, d\Omega = \delta_{ll'}\delta_{mm'}$$

we have $$g_{lE}(k) = \frac{\hbar}{\sqrt{mk}}\,\delta\left(\frac{\hbar^2 k^2}{2m} - E\right)$$

$$\langle \vec{k}|E, l, m\rangle = \frac{\hbar}{\sqrt{mk}}\,\delta\left(\frac{\hbar^2 k^2}{2m} - E\right) Y_l^m(\hat{k})$$

Remark: plane wave ∞ dim thus has all possible l

In position space $\langle \vec{x} | E, l, m \rangle$

$$\psi(\vec{r}) = C_l j_l(kr) Y_l^m(\hat{r}) \qquad j_l \text{ b/c } r \to 0 \ \psi \to 0$$

note

$$\langle \vec{x} | \vec{k} \rangle = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} = \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) i^l j_l(kr) P_l(\hat{k}\cdot\hat{r})$$

w/ addition thm,

$$\sum_m Y_l^m(\hat{r}) Y_l^{m*}(\hat{k}) = \frac{2l+1}{4\pi} P_l(\hat{k}\cdot\hat{r})$$

$$\langle \vec{x} | \vec{k} \rangle = \sum_l \frac{(2l+1)}{4\pi} P_l(\hat{k}\cdot\hat{r}) \frac{\hbar}{\sqrt{mk}} C_l j_l(kr)$$

after matching, $\quad C_l = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}$

$$\langle \vec{x} | E, l, m \rangle = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}} j_l(kr) Y_l^m(\hat{r})$$

For $V(r) \neq 0$ sphr. sym. transition op. T $\quad [T, L^2] = 0 \quad [T, \vec{L}] = 0$.

By Wigner-Eckart thm.

$\langle E', l', m' | T | E, l, m \rangle = T_l(E) \delta_{l'l} \delta_{m'm}$ diag. elem. of T depends on E & l not m.

Take $\vec{k}$ in $\hat{z}$ dir.

since $L_z | k\hat{z} \rangle = 0 \quad m = 0$ then $Y_l^m = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \delta_{m0}$

take θ b/w $\vec{k}'$ and $\vec{k}$, $\quad Y_l^0(\hat{k}') = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$

Thus,

$$f(\vec{k}', \vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2} L^3 \langle \vec{k}' | T | \vec{k} \rangle = -\frac{4\pi^3}{k} \sum_{l,m} T_l(E) Y_l^m(\hat{k}') Y_l^{m*}(\hat{k})$$

$$= 4\pi \sum_{l,m} f_l(k) \frac{2l+1}{4\pi} \delta_{m0} P_l(\cos\theta) \quad \text{w/} \quad f_l(k) \equiv \frac{-\pi T_l(E)}{k}$$

So

$$f(\vec{k}, \vec{k}') = f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta)$$

For spherical sym and $V(r) \neq 0$

$$f(\vec{k}', \vec{k}) = f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta)$$

Recalled $\langle \vec{x} | \psi \rangle = \frac{1}{(2\pi)^{3/2}} \left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k}', \vec{k}) \right]$

sub. $\langle \vec{x} | \psi^+ \rangle = \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) \frac{P_l}{2ik} \left[[1 + 2ikf_l(k)] \frac{e^{ikr}}{r} - \frac{e^{-i(kr - l\pi)}}{r} \right]$

Absent of Scatter.

Plane wave = $\sum$ (sphr. out. wave) + (sphr. in. wave)

i.e. $\quad C_l \frac{e^{ikr}}{r} - \frac{e^{-i(kr-l\pi)}}{r} \quad$ w/ $C_l = 1 \longmapsto 1 + 2ikf_l(k)$

$\qquad$ w/o scatterer $\qquad$ w/ scatterer

$1 + 2ikf_l(k) = e^{2i\delta_l}$

Recalled

$\rho = |\psi|^2$ and $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$ conserv. of prob. $\nabla \cdot \vec{J} = 0$

Gauss's thm: $\oint \vec{j} \cdot d\vec{s} = 0$ so influx = outflux $\forall l$

Let $S_l(k) = 1 + 2ikf_l(k)$ its magnitude $|S_l| = 1$

Remark
- Out. wave changes only by phase for $r \gg 1$ "unitary relation"
- In QFT $S_l(k)$ the l^{th} diag. elem. of S matrix, S unitary for prob. conserv.
- By convention, take $S_l = e^{i2\delta_l}$

thus $f_l = \frac{S_l - 1}{2ik} = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l} \sin\delta_l}{k} = \frac{1}{k\cot\delta_l - ik}$

s.t.

$$\boxed{f(\theta) = \sum_{l=0} (2l+1) \left(\frac{e^{2i\delta_l} - 1}{2ik} \right) P_l(\cos\theta)}$$

or

$$\boxed{f(\theta) = \frac{1}{k} \sum_{l=0} (2l+1) e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta)}$$

w/

$$\boxed{\sigma_{tot} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2\delta_l}$$

note $\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \frac{2}{2l+1} \delta_{l'l}$

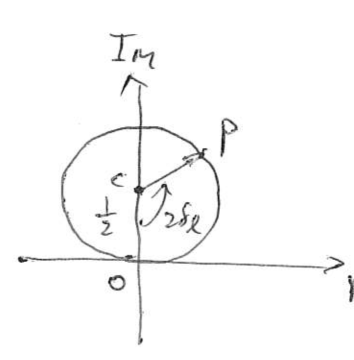
Recalled

$\text{Im} f(\theta = 0) = \frac{k}{4\pi} \sigma_{tot}$

Previous result, show

$\text{Im} f(\theta = 0) = \sum_l \frac{(2l+1)}{k} \sin^2\delta_l$

Consider $kf_l = \frac{i - ie^{2i\delta_l}}{2} = \underbrace{\frac{i}{2}}_{\substack{\text{center at} \\ \frac{1}{2} \text{ on Im}}} + \underbrace{\frac{1}{2} e^{-i\frac{\pi}{2} + 2i\delta_l}}_{\text{circle of radius } \frac{1}{2}}$



Remark
- $\delta_l \ll 1$ then f_l lies near bottom of circle $\Rightarrow f_{l=0}$ real
- $\delta_l \sim \frac{\pi}{2}$ $kf_l \sim \text{Im}$ & $|kf_l|$ is max, thus resonance from l^{th} partial wave
- Result $\sigma^{(l)}_{max} = 4\pi \lambda\!\!\!^{-2} (2l+1)$ $\quad \lambda\!\!\!^{-} = \frac{\lambda}{2\pi} \quad \lambda = \frac{2\pi}{k}$

Finding Phase Shifts

- $V \to 0$ $\quad r > R \quad \Rightarrow$ free sphr. wave
- w/o $\rho \to 0$ s.t. $\psi \to 0$ constraint, we have

$$\psi \sim j_\ell(kr) P_\ell(\cos\theta) + n_\ell(kr) P_\ell(\cos\theta) \quad \text{or} \quad h_\ell^{(1)} P_\ell + h_\ell^{(2)} P_\ell$$

recall sphr. wave expansion.

$$\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} = \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1)\, i^\ell j_\ell(kr) P_\ell(\cos\theta)$$

where

$$A_\ell(r) = \begin{cases} j_\ell(kr) & r < 0 \quad \rho \to 0 \\ C_\ell^{(1)} h_\ell^{(1)}(kr) + C_\ell^{(2)} h_\ell^{(2)}(kr) & r > R \end{cases}$$

$C_\ell^{(1,2)}$ choose to satisfy B.C

for $r \gg 1$,

$$h_\ell^{(1)} \to \frac{e^{i(kr - \frac{\ell\pi}{2})}}{ikr} \qquad h_\ell^{(2)} \to -\frac{e^{-i(kr-\frac{\ell\pi}{2})}}{ikr}$$

B.C $\quad C_\ell^{(1)} = \frac{1}{2} e^{2i\delta_\ell} \quad C_\ell^{(2)} = \frac{1}{2} \quad$ so

$$A_\ell = \frac{1}{2} e^{2i\delta_\ell} (j_\ell + i n_\ell) + \frac{1}{2}(j_\ell - i n_\ell)$$

$$A_\ell = e^{i\delta_\ell} \left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right]$$

log derivative, for $r = R$

$(r > R)$ $\quad \beta_\ell = \frac{r}{A_\ell} \frac{d(A_\ell)}{dr}\Big|_{r=R}$ then written in terms of δ_ℓ, $\tan\delta_\ell = \dfrac{kR j_\ell'(kR) - \beta_\ell j_\ell(kR)}{kR n_\ell'(kR) - \beta_\ell n_\ell(kR)}$

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now for $r < R$, since $V(r)$ sphr. sym.

we have

$$u_\ell'' + \left(k^2 - \frac{2m}{\hbar^2} V - \frac{\ell(\ell+1)}{r^2}\right) u_\ell = 0 \quad \text{w/} \quad u_\ell = r A_\ell(r)$$

so $\quad R(r) = C\, j_\ell(\alpha r) \quad$ w/ $\quad \alpha = \dfrac{\sqrt{2m(E-V)}}{\hbar}$

ex Hard sphere (low energy)

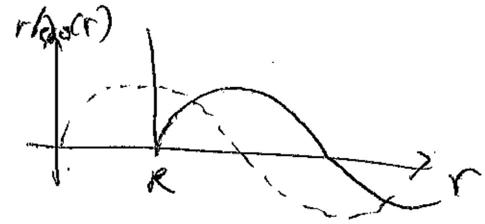
$r > R$ $\quad A_\ell(r) = [\cos\delta_\ell j_\ell(kr) - \sin\delta_\ell n_\ell(kr)] e^{i\delta_\ell}$

At $r=R$ $\quad A_\ell(R) = 0$, so $\tan\delta_\ell = \frac{j_\ell(kR)}{n_\ell(kR)}$

consider s wave ($\ell = 0$)

$\tan\delta_0 = -\tan kR \Rightarrow \quad \delta_0 = -kR$ thus $A_0(r) \sim \frac{1}{kr}\sin(kr+\delta_0)$

shifted by R



at low energy $kR \ll 1 \approx kr \ll 1$

thus $j_\ell(kr) \approx \frac{(kr)^\ell}{(2\ell+1)!!}$ $\quad n_\ell(kr) \approx -\frac{(2\ell-1)!!}{(kr)^{\ell+1}}$ $\quad$ so $\tan\delta_\ell = \frac{-(kR)^{2\ell+1}}{(2\ell-1)!!(2\ell+1)!!}$

Recalled

$$f(\theta) = \frac{1}{k}\sum_{\ell=0} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell P_\ell(\cos\theta)$$

low energy s-wave $f(\theta) \sim \frac{1}{k} e^{i\delta_0}\sin\delta_0$, $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\sin^2\delta_0}{k^2} = R^2$, $\sigma_{tot} = 4\pi R^2$ //

* high energy limit w/ Eikonal Approx.

Eikonal Approx.

Assumptions
- $V(\vec{x})$ varies slowly over distance of length scale λ
- wave travels at high energy i.e $E \gg |V|$
- from these we have semi-classical (why?)

· wavefunc. $\psi \sim e^{\frac{iS(x)}{\hbar}}$ and Ham. Jacobi eqn. $\frac{(\nabla S)^2}{2m} + V = \frac{\hbar^2 k^2}{2m}$

rewrite

$$\frac{S}{\hbar} = \int_{-\infty}^{z} \left[k^2 - \frac{2m}{\hbar^2} V(\sqrt{b^2+z'^2})\right]^{1/2} dz' + \text{const.} \quad \text{s.t. } \frac{S}{\hbar} \to kz \text{ as } V \to 0.$$

thus

$$\frac{S}{\hbar} \cong kz - \frac{m}{\hbar^2 k}\int_{-\infty}^{z} V(\sqrt{b^2+z'^2})\, dz' \quad \text{b/c } E \gg V \quad k^2 \gg \frac{2mV}{\hbar^2}$$

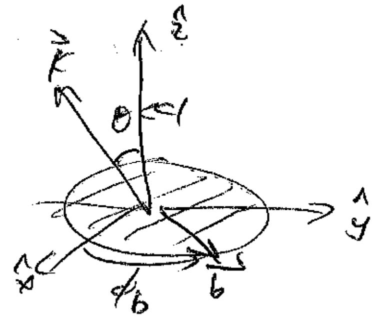
$$\sqrt{k^2 - \frac{2mV}{\hbar^2}} = k\sqrt{1 - \frac{2mV}{\hbar^2 k^2}} \sim k - \frac{mV}{\hbar^2 k}$$

Recall $f(\vec{k},\vec{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}L^3\int d^3x' \frac{e^{-i\vec{k}'\cdot\vec{x}'}}{L^{3/2}} V(\vec{x}') \underbrace{\langle \vec{x}' | \psi^+ \rangle}_{\text{use semi-class approx}}$

small deflection

note. $kz = \vec{k}\cdot\vec{x}' = \vec{k}\cdot(\vec{b}+z\hat{z})$ w/ $\vec{k} \parallel \hat{z}$ & $\vec{k}\perp\vec{b}$ then $(\vec{k}-\vec{k}')\cdot\hat{z} \sim \mathcal{O}(\theta^2)$

so $(\vec{k}-\vec{k}')\cdot\vec{x}' \sim \vec{k}'\cdot\vec{b}$



$\vec{k}' = k\sin\theta\,\hat{x} + k\cos\theta\,\hat{z}$

$\vec{b} = b\cos\phi_b\,\hat{x} + b\sin\phi_b\,\hat{y}$

thus $\vec{k}'\cdot\vec{b} \simeq kb\theta\cos\phi_b$

$$f(\vec{k},\vec{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int_0^\infty b\,db \underbrace{\int_0^{2\pi} d\phi_b\, e^{-ikb\theta\cos\phi_b}}_{2\pi J_0(kb\theta)} \underbrace{\int_{-\infty}^{\infty} dz\, V\exp\left[-\frac{im}{\hbar^2 k}\int_{-\infty}^{z} V\,dz'\right]}_{I}$$

Consider for I, $u = e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z} V dz'}$

then $I = \frac{i\hbar^2 k}{m} e^{U(z)}\Big|_{-\infty}^{\infty}$, define $\Delta(b) = \frac{-m}{2\hbar^2 k}\int_{-\infty}^{\infty} V\,dz$

thus, (*) $f(\vec{k},\vec{k}') = -ik\int_{-\infty}^{\infty} db\, b\, J_0(kb\theta)\left[e^{2i\Delta(b)} - 1\right]$ when $b \gg$ range of V then $V(\sqrt{b^2+z^2}) = 0$ so $e^{2i\Delta(b)} - 1 = 0$

Partial Wave Recall $k = \frac{2\pi}{\lambda}$ $k' = \frac{2\pi}{R}$ ← ie lattice size

high energy $\lambda \ll R$ so kr large w/ many partial wave ie l's

Consider angular mom. in semi-class.

$p = \hbar k \to l \approx kb$, $l_{max} = kR$

thus l has cutoff at l_{max}.

Comparing (*) w/ general expression $f(\theta)$, $\Rightarrow \delta_l \to \Delta(b)\big|_{b=\frac{l}{k}}$

and $\frac{l_{max}}{k} = R \ll \frac{l}{k}$ if $l \gg l_{max}$ $e^{2i\Delta(b)} - 1 \to 0$

from general $f(\theta)$,

$$f(\theta) = \sum_{l=0}(2l+1)\left(\frac{e^{2i\delta_l}-1}{2ik}\right)P_l(\cos\theta) \to k\int db\,\frac{2kb}{2ik}\left(e^{2i\Delta(b)} - 1\right)J_0(kb\theta)$$

agree w/ (*)

Eikonal ex of hard sphere scattering at high energy

Total cross-section $\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{l\approx kR}(2l+1)\sin^2\delta_l$ write $\sin^2\delta_l = \frac{\tan^2\delta_l}{1+\tan^2\delta_l}$

as $r\to R$, we want tails of $A_l(r)\to 0$ s.t. $\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}$ (still in hard sphere?)

w/ $j_l(kr)\sim\frac{1}{kr}\sin(kr-\frac{l\pi}{2})$ and $n_l(kr)\sim -\frac{1}{kr}\cos(kr-\frac{l\pi}{2})$

$\sin^2\delta_l \sim \sin^2(kR-\frac{\pi l}{2})$ thus

$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{kR}(2l+1)\sin^2(kR-\frac{\pi l}{2})$ note $\langle\sin^2(kR-\frac{\pi l}{2})\rangle\sim\frac{1}{2}$

$\sigma_{tot} = 2\pi R^2$

Quest for factor 2

Split $f(\theta) = f_{ref} + f_{shad}$ $f_{refl} = \frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)e^{2i\delta_l}P_l(\cos\theta)$

$f_{shad} = \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)$ $\in\mathbb{C}$ pure Im.

$\int|f_{ref}|^2 d\Omega = \frac{2\pi}{4k^2}\sum_{l=0}^{l_{max}}\int_{-1}^{1}(2l+1)^2P_l^2(\cos\theta)\,d\cos\theta$ recall $\int_{-1}^{1}P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$

$= \frac{\pi}{k^2}(kR)^2 = \pi R^2$

$f_{shad}\to\frac{i}{2k}\int_0^R b\,db\,J_0(kb\theta)$ note $J_{-n} = (-1)^n J_n(x)$, $-x^{-n}J_{n+1}(x) = \frac{d}{dx}[x^{-n}J_n(x)]$

thus $\frac{i}{2k}\int_0^R b\,db\,J_0(kb\theta) = \frac{iRJ_1(kR\theta)}{\theta}$

$\int|f_{shad}|^2 d\Omega = 2\pi R^2\int_0^\infty\frac{J_1(\xi)^2}{\xi}d\xi \simeq \pi R^2$

$\sigma_{tot} = \pi R^2 + \pi R^2$

note interference term $Re(f^*_{shad}f_{ref})\simeq 0$