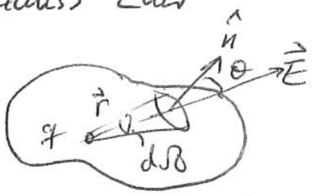
Gauss Law



$$\vec{E}\cdot\hat{n} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\hat{n} = \frac{q}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

integrate over surface

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{q}{\varepsilon_0} \text{ if charge with surface, else zero!}$$

Discrete charge distribution

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\sum_i q_i$$

Continuous charge distribution

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\vec{x})\,d^3x$$

$$\nabla\frac{1}{|\vec{x}-\vec{x}'|} = -\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$\nabla^2\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\,\delta(\vec{x}-\vec{x}')$$

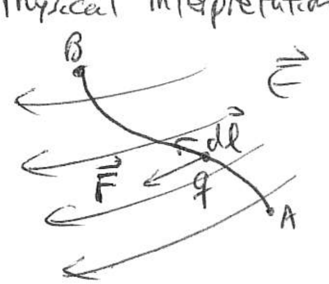
From divergence thm $\Rightarrow$ $\nabla\cdot\vec{E} = \rho/\varepsilon_0$

From Generalized Coulomb's Law

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3x' \quad \text{we have } \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

and $\vec{E} = -\nabla\Phi \qquad \nabla\times\vec{E} = 0$

Physical Interpretation of Scalar potential.



Work done on charge against action of $\vec{E}$

$$W = -\int_A^B \vec{F}\cdot d\vec{l} \quad \text{for } \vec{F} = q\vec{E}, \text{ we have}$$

$$W = q\Delta\Phi, \quad w/\ \Delta\Phi = \Phi_B - \Phi_A.$$

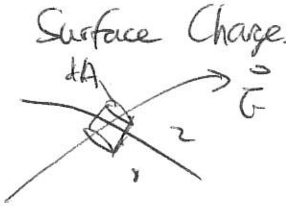
From this line integral, we also have

$$\int_A^B \vec{E}\cdot d\vec{l} = -(\Phi_B - \Phi_A) \quad \text{signify path independent}$$

and $\oint \vec{E}\cdot d\vec{l} = 0$ using Stoke's thm $\int\nabla\times\vec{E}\cdot d\vec{a} = \oint\vec{E}\cdot d\vec{l} = 0$

Give again $\nabla\times\vec{E} = 0$

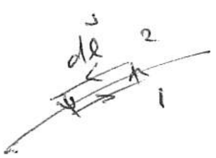
Surface Charges



Gauss law ← surface charge density.

$$\oint\vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0} \Rightarrow (\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}$$

show discontinuity of $\frac{\sigma}{\varepsilon_0}$ in normal component of $\vec{E}$ field crossing a surface.



$$\oint\vec{E}\cdot d\vec{l} = 0 \Rightarrow (\vec{E}_2 - \vec{E}_1)\cdot\hat{t} = 0$$

shows continuity of tangential component across the surface.

* Potential is continuous everywhere w/ surface or volume charge density (b/c $\vec{E}$ is bounded)

Remarks: w/ point charge, line charge or dipole layers, potential is not cont'!

Dipole layer

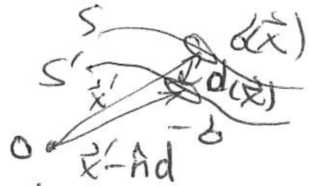
Recalled $\vec{\tau} = q\vec{r}\times\vec{E}$ or $\vec{p}\times\vec{E}$ (torque)

dipole moment $\vec{p} \equiv q\vec{r}$.

Analogue dipole layers

define dipole layers distrib.

strength: $D(\vec{x}) \equiv \lim_{\substack{d\to 0\\ \sigma\to\infty}} \sigma(\vec{x})\, d(\vec{x})$; $k = \frac{1}{4\pi\varepsilon_0}$



Potential due to two close surf.

$$\Phi(\vec{x}) = k\int_S \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}da' - k\int_{S'} \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'+\hat{n}d|}da''$$

for d small, apply Taylor expansion in 3D

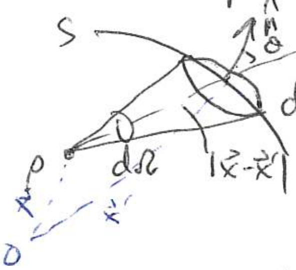
i.e. $\frac{1}{|\vec{x}+\vec{a}|} = \frac{1}{x} + \vec{a}\cdot\nabla\left(\frac{1}{x}\right) + \cdots$ w/ $|a| \ll |x|$

thus, $\Phi(\vec{x}) = \frac{-1}{4\pi\varepsilon_0}\int_S D(\vec{x}')\,\hat{n}\cdot\nabla'\frac{1}{|\vec{x}-\vec{x}'|}da'$

Remark. For point dipole w/ dipole moment

$\vec{p} = \hat{n}D\,da'$ $\quad \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$

Geom. interpretation.



$$\underbrace{\hat{n}\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}_{\text{Recalled } \vec{E}=-\nabla\Phi \,\sim\, \nabla'\frac{1}{|\vec{x}-\vec{x}'|} \text{ pointing in direction of } \vec{E}} da' = -\frac{\cos\theta\, da'}{|\vec{x}-\vec{x}'|^2} = -d\Omega$$

so $\Phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\int_S D(\vec{x}')\,d\Omega$

Motivation, investigate $\delta\Phi$ between two layer when obs. pt is placed closer in between layers. See Jump in potential $\Delta\Phi = \frac{D(\vec{x})}{\varepsilon_0}$

Remark If bring obs point closer

$D \to$ uniform.

so $\Phi_2 - \Phi_1 = D/\varepsilon_0$

discontinuity in crossing from inner to outer layer w/ $\frac{-D}{2\varepsilon_0}$ inner $\frac{D}{2\varepsilon_0}$ outer

Physically means potential drop occuring inside dipole layer.

Green's Identity and physically interpretation.

for $\vec{x}$ within volume $\partial V'$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{R}d^3x' + \frac{1}{4\pi}\oint_S\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'$$

If $\vec{x}$ lies outside volume, $\Phi(\vec{x})$ and LHS 1st term $\Rightarrow 0$ (write $\rho(\vec{x})$ as $\sigma(\vec{x})\,\delta(\vec{x}-\vec{x}')$)

2nd term has $\frac{\partial\Phi}{\partial n'} = \frac{\sigma}{\varepsilon_0}$ and $\varepsilon_0\Phi = D(\vec{x})$ consistent w/ dipole layer, cancelled to zero.

Thus, discontinuity in e-field & potential across surface render zero potential & zero field outside the volume!!

Uniqueness of poisson eqn. $\nabla^2\Phi = -\rho/\varepsilon_0$ inside volume subject to Dirichlet / Neumann

use Green's 1st identity let $\Phi_{1,2}$ satisfy poisson eqn. define $U = \Phi_2 - \Phi_1$ where $U=0$ on ∂ Dirichlet, $\frac{\partial U}{\partial n} = 0$ in ∂ Neu.

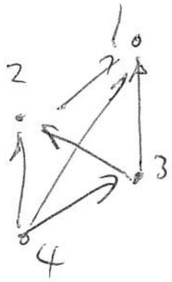
results in $\int|\nabla U|^2 d^3x = 0 \Rightarrow \nabla U = 0$ i.e. U = const

thus in Dirichlet, $U=0$ Φ unique! Also implies maxima located on bndy!

Discrete

$$W_i = q_i \Phi(\vec{x}_i)$$

$$W_i = k q_i \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$



$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

$$\text{or } W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j \\ i\neq j}} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \quad \text{or} \quad \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j \\ i\neq j}} q_i \Phi_j(\vec{x}_i)$$

Continuous

$$W = \frac{1}{8\pi\varepsilon_0} \iint \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x\, d^3x'$$

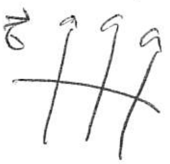
$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3x$$

rewrite

$$W = -\frac{\varepsilon_0}{2} \int \Phi(\vec{x}) \nabla^2 \Phi \, d^3x$$

$$\boxed{W = \frac{\varepsilon_0}{2} \int |\vec{E}|^2 d^3x} \quad \text{w/} \quad \boxed{w = \frac{\varepsilon_0 |\vec{E}|^2}{2}} \text{ as energy density.}$$

ex force per unit area on surf. of a conductor w/ $\sigma(\vec{x})$ charge density.



$$w = \frac{\varepsilon_0 |\vec{E}|^2}{2} \simeq \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2$$ from jump in $\vec{E}$

$$\Delta W = w \Delta A \Delta x$$
$$= \left(\frac{\sigma^2}{2\varepsilon_0} \Delta A\right) \Delta x$$

thus Force/Area $\sim \frac{\sigma^2}{2\varepsilon_0}$!!

Capacitance Matrix.

conductors (V_1, Q_1) (V_2, Q_2)

$(V_i, Q_i) \ldots (V_j, Q_j)$

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

C_{ii} capacitance

C_{ij}, $i \neq j$ inductance

recalled in capacitor

$U = V dQ$ w/ $V = \frac{Q}{C}$

$U = \frac{1}{2} Q V$

analogue

$$\text{So } W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i$$
$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j$$

① $\circ 1$

ex
$$\Phi_1(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{x}-\vec{x}_1|}$$

② $\circ 1$

$2\circ$
$$W_2 = q_2\Phi_1(\vec{x}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{x}_2-\vec{x}_1|}$$

③ $1\circ$

$2\circ$
$$W_3 = q_3\Phi_1(\vec{x}_3) + q_3\Phi_2(\vec{x}_3)$$

$3\circ$

④
$$W_{tot} = W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{|\vec{x}_2-\vec{x}_1|} + \frac{q_3 q_1}{|\vec{x}_3-\vec{x}_1|} + \frac{q_3 q_2}{|\vec{x}_3-\vec{x}_2|}\right)$$

Green Function

Satisfies $\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

~~ex free sp~~

space	G
free space	$\frac{1}{\lvert\vec{x}-\vec{x}'\rvert}$
General	$\frac{1}{\lvert\vec{x}-\vec{x}'\rvert} + F(\vec{x},\vec{x}')$ s.t. $\nabla'^2 F(\vec{x},\vec{x}') = 0$.

this extra freedom allows ~~one to~~ customization to make chosen type of bndy conditions possible!!

~~free space | G~~

from Green's identity.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \right] da'$$

Dirichlet, demand: $G_D(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on S

Neumann bndy:

Consider $\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x},\vec{x}')$.

Gauss's thm $\oint_S \frac{\partial G}{\partial n'} da' = -4\pi \Rightarrow \frac{\partial G_N}{\partial n'}(\vec{x},\vec{x}') = -\frac{4\pi}{S}$ $\quad x'$ on S

~~Now~~, thus. $\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} da \simeq \langle \Phi \rangle_S$. — vanished by setting two surface one finite one infinite.

$G(\vec{x},\vec{x}') = G(\vec{x}',\vec{x})$ symmetric for Dirichlet bndy cond.

Using ~~Bg~~ Green's thm: setting $\phi = G(\vec{x},\vec{y})$ $\psi = G(\vec{x}',\vec{y})$ w/ $\vec{y}$ integration variable

Some math techniques

Divergence thm.

$$\int_V \nabla\cdot\vec{A}\,d^3x = \int_S \vec{A}\cdot\hat{n}\,da$$

$$\int_V \nabla\times\vec{A}\,d^3x = \int_S \hat{n}\times\vec{A}\,da$$

$$\int_V \nabla\psi\,d^3x = \int_S \psi\hat{n}\,da$$

Stokes's thm

$$\int_S (\nabla\times\vec{A})\cdot\hat{n}\,da = \oint_C \vec{A}\cdot d\vec{l}$$

$$\int_S \hat{n}\times\nabla\psi\,da = \oint_C \psi\,d\vec{l}$$

Delta function

$$\int f(x)\,\delta'(x-a)\,dx = -f'(a)$$

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \qquad \text{Given that } f(x_i)=0$$

$$-\nabla_x\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

also $\nabla\times\nabla f = 0$ $\quad \varepsilon_{ijk}\nabla_i\nabla_j f\, e_k = 0$

b/c ε_{ijk} antisymmetric !

For $\frac{1}{|\vec{x}+\vec{a}|}$ $\quad |a| \ll |\vec{x}|$, Taylor series expansion in 3D.

$$\frac{1}{|\vec{x}+\vec{a}|} = \frac{1}{x} + \vec{a}\cdot\nabla\left(\frac{1}{x}\right) + \cdots$$

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

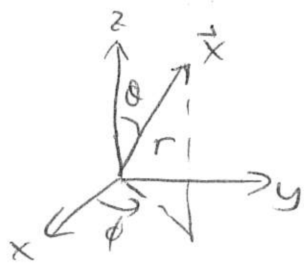
Green's identity

1st identity $\quad \nabla\cdot(\phi\nabla\psi) \Rightarrow \int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\,da$

2nd identity w/ $\nabla\cdot(\phi\nabla\psi)$ $\nabla\cdot(\psi\nabla\phi) \Rightarrow \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]da$

Laplace eqn in Spherical Coord.

$$\mathcal{L}_{r\theta\phi}\Phi = 0$$



$$\Phi = \frac{U(r)}{r} P(\theta) Q(\phi)$$

$$Q = e^{\pm i m \phi}$$

$$U = A r^{l+1} + B r^{-l}$$

P in the form of Legendre Polynomials: has following properties

$$P_0(x) = 1 \quad P_1(x) = x \quad P_{2n+1}(0) = 0 \quad P_l(1) = 1 \quad \forall\, l$$

orthogonality $\int_{-1}^{1} P_{l'}(x) P_l(x)\,dx = \frac{2}{2l+1}\delta_{l'l}$

if $f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$ then $A_l = \frac{2l+1}{2}\int_{-1}^{1} f(x) P_l(x)\,dx$

For BVP w/ Azimuthal Sym. $(m=0)$

$$\boxed{\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)}$$

ex $V(\theta)$ on surf. of sphere w/ radius a.

Potential inside sphere =

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

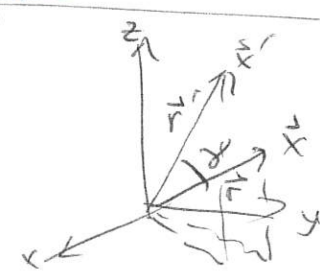
B.C $\Phi(a,\theta) = V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) \Rightarrow A_l = \frac{2l+1}{2a^l}\int_{-1}^{1} V(\theta) P_l(\cos\theta)\, d(\cos\theta)$

On symmetry axis ($\vec{x}$ on z) so $z = r$ $\theta = 0$ ($m=0$ still)

$$\boxed{\Phi(r) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]}$$

Expansion of $\frac{1}{|\vec{x}-\vec{x}'|}$ due to pt. charge at $\vec{x}'$

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)} \quad (3.38)$$

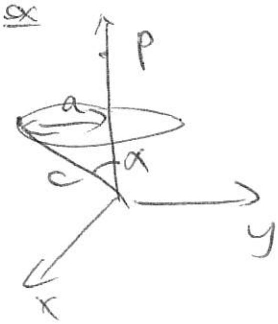


If $\vec{x}, \vec{x}'$ both on z but $\vec{x} \neq \vec{x}'$ then

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l \quad \text{w/ } \frac{1}{|\vec{x}-\vec{x}'|} \rightarrow \frac{1}{|r-r'|}$$

---

ex



(i) for $r > c$

$$\Phi(z=r) = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}} P_l(\cos\alpha)$$

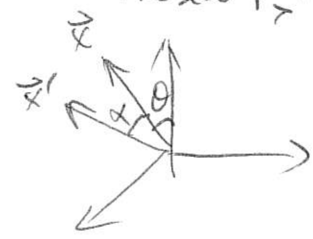
(ii) $r < c$

$$\Phi(z=r) = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{c^{l+1}} P_l(\cos\alpha)$$

q is total charge dist. uniformly on ring.

(iii) Now any pt in space is:

$$\Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta)$$



General soln for BVP w/o azimuthal is

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\phi)$$

Important relations

orthogonality $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, Y^*_{\ell' m'}(\theta,\phi) Y_{\ell m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m' m}$

completeness

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$$

$$Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi) \qquad Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta) e^{im\phi}$$

$$\int_{-1}^{1} P^m_{\ell'}(x) P^m_{\ell}(x) dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell'\ell} \quad ; \quad Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) \; ; \; P_1(\cos\theta) = \cos\theta$$

if $g(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi)$, $A_{\ell m} = \int d\Omega \, Y^*_{\ell m}(\theta,\phi) g(\theta,\phi)$

Additional thm for SH $Y_{\ell m}(\theta,\phi)$

Can expand $P_\ell(\cos\gamma)$ as:

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi) Y_{\ell m}(\theta,\phi)$$

then

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) \quad (*)$$

For Completeness, Green's function for exterior problem w/ spherical bdry $r=a$ is

$$G(\vec{x},\vec{x}') = 4\pi \sum_{\ell,m}\frac{1}{2\ell+1}\underbrace{\left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1}\right]}_{I} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) \quad (**)$$

$$I = \begin{cases} \frac{1}{r'^{\ell+1}}\left[r^{\ell} - \frac{a^{2\ell+1}}{r^{\ell+1}}\right], & r<r' \\ \left[r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right]\frac{1}{r^{\ell+1}}, & r>r' \end{cases}$$

Green's func. for spherical shell bounded by $r=a$, $r=b$

$$G(\vec{x},\vec{x}') = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]}\left(r_<^{\ell} - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)$$

for $b>a$

Remarks: If $a\to 0$, $b\to\infty$ we recover $(*)$

If $b\to\infty$ we recover $(**)$ exterior

If $a\to 0$ we recover interior

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$$

Remark: See (P.122) Ex 3.10

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)$$

Cylindrical Coord

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho) Q(\phi) Z(z)$$

take $Z'' = k^2 Z$ $\quad Q'' = -\nu^2 Q$

we have $Z(z) = e^{\pm kz}$

$Q(\phi) = e^{\pm i\nu\phi}$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

or $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) + \left(k^2 - \frac{\nu^2}{\rho^2}\right)f$

(solns: Bessel func.

If polar then $z=0 \Rightarrow k=0$ then soln form is of $f = A_\nu \rho^\nu + B_\nu \frac{1}{\rho^\nu}$

Bessel func.

If ν integer $\nu = m$ $\quad J_{-m}(x) = (-1)^m J_m(x)$ $\quad J_\nu(x)$ Bessel func. of 1st kind.

Its linearly indep. counter part,

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

Neumann func./ Bessel func. of 2nd kind.

$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$

$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$

Hankel func/ Bessel func. of 3rd kind.

Roots of J_ν $\quad J_\nu(x_{\nu n}) = 0 \quad n = 1, 2, 3 \ldots$

normalized integral

$$\int_0^a \rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n}$$

Fourier Bessel Series

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})}\int_0^a \rho f(\rho) J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) d\rho$$

$x \ll 1$ $\quad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$

$$N_\nu(x) \to \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

$x \gg 1$ $\quad J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$

$N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$

Modified Bessel If $k^2 \to -k^2$ then $Z(z) = e^{\pm ikz}$ and

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R = 0$$

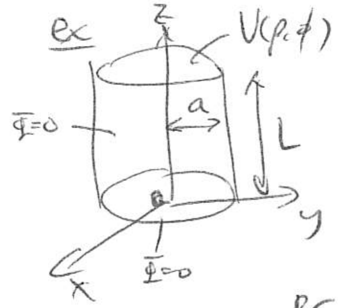
has soln: $I_\nu(x)$, $K_\nu(x)$

$x \ll 1$, $I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$ $\quad K_\nu(x) \to \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + 0.5772\ldots\right] & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$

$x \gg 1$, $I_\nu(x) \to \frac{1}{\sqrt{2\pi x}} e^x\left[1 + O\left(\frac{1}{x}\right)\right]$

$K_\nu(x) \to \sqrt{\frac{\pi}{2x}} e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right]$

ex



$Q(\phi) = A\sin m\phi + B\cos m\phi$

$Z(z) = \sinh kz$

$R(\rho) = CJ_m(k\rho) + DN_m(k\rho)$

Find potential inside cylinder.

BC · inside $D=0$ · $\rho = a$ $J_m(ka)=0 \Rightarrow k_{mn} = \frac{x_{mn}}{a}$ — roots

thus $V(\rho,\phi) = \sum_{m,n} \sinh(k_{mn}L) J_m(k_{mn}\rho)(A_{mn}\sin m\phi + B_{mn}\cos m\phi)$

valid for $0 \le \rho \le a$

$$\text{w/ } A_{mn} = \frac{2\,\mathrm{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho,\phi) J_m(k_{mn}\rho)\sin m\phi$$

$$B_{mn} = \frac{2\,\mathrm{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho,\phi) J_m(k_{mn}\rho)\cos m\phi$$

Now for $a \to \infty$ $z \ge 0$ w/ $\Phi \to 0$ when $z \to \infty$

Instead we have $\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, e^{-kz} J_m(k\rho)\left[A_m(k)\sin m\phi + B_m(k)\cos m\phi\right]$

B.C If $z=0$ $V(\rho,\phi)$ for whole plane,

$$V(\rho,\phi) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho)\left[A_m(k)\sin m\phi + B_m(k)\cos m\phi\right]$$

using $\int_0^{\infty} x J_m(kx) J_m(k'x)\, dx = \frac{1}{k}\delta(k'-k)$

then
$$\left.\begin{matrix} A_m(k) \\ B_m(k) \end{matrix}\right\} = \frac{k}{\pi}\int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\phi\, V(\rho,\phi) J_m(k\rho) \left\{\begin{matrix} \sin m\phi \\ \cos m\phi \end{matrix}\right.$$

Remark: Poisson Problem ~ Green's thm / second identity
Laplace Problem ~ BVP using seperation of variables.

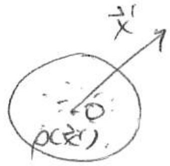
Remarks $J_\nu''(x) + \frac{1}{x}J_\nu'(x) + J_\nu(x) = 0$ for $\nu=0$

this ODE can use to solve $\int_0^a x J_0(x)\, dx$

note w/ $J_0(x) = \frac{\sin x}{x}$ $J_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \Rightarrow J_0'(x) = -J_1(x)$

Multiple Expansion

Motivation



$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\, q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

multiple.

compare w/ $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$ and use spherical harmonic exp. of $\frac{1}{|\vec{x}-\vec{x}'|}$

we have $q_{lm} = \int Y^*_{lm}(\theta',\phi')\, r'^{l}\rho(\vec{x}')\,d^3x'$ and $q_{l,-m} = (-1)^m q^*_{lm}$ from Y_{lm} & Y^*_{lm}

$q_{00} \sim \int \rho(\vec{x}')\,d^3x'$

dipole $\vec{p} = \int \vec{x}'\rho(\vec{x}')\,d^3x'$

~~Cartesian~~

Quadrupole $Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\,\rho(\vec{x}')\,d^3x'$ $(Q_{ij} = Q_{ji})$ Some nice property

$(tr(Q_{ij}) = 0)$

then under multipole expansion,

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j}Q_{ij}\frac{x_ix_j}{r^5} + \cdots\right]$$

dipole field

$$\vec{E}(\vec{x}) = -\nabla_x\left(\frac{\vec{p}\cdot\vec{x}}{r^3}\right) = \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{4\pi\varepsilon_0|\vec{x}-\vec{x}_0|^3}$$; $\vec{x}_0$ location of dipole $\vec{p}$

need modification

$\left(\nabla_j\left(\frac{r_k}{r^3}\right) = \frac{\delta_{jk}}{r^3} - \frac{3r_jr_k}{r^5} \quad E_j = -p_k\nabla_j\left(\frac{r_k}{r^3}\right)\right)$

Cartesian Multiple moment vs q_{lm}

$$Q_{ijk\cdots} = \int \rho(x_i, x_j, x_k, \ldots)\, x_ix_jx_k\cdots dV$$

Motivation: Q_{ijk} are independent of origin (translation invariant) if all H.O.T vanished except the leading term.

ie take $x_i' = x_i + X_i$

Dipole field Modification

$$\int_{r<R} d^3x\,\vec{E}(\vec{x}) = -\frac{\vec{p}}{3\varepsilon_0}$$

O.T.O.H

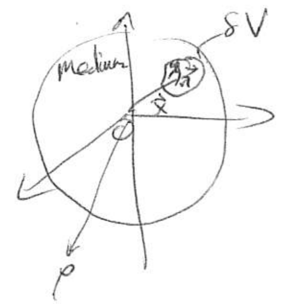
$$\int_{r<R} d^3x\,\vec{E}(\vec{x}) = \frac{4\pi}{3}R^3\vec{E}(0)$$

Thus dipole field must account for dipole within ~~volume~~ radius of observation

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right]$$

no contribution if away from $\vec{x}_0$

Ponderable Materials,



let a denote a species w/ $\vec{p}_a$ thin inside δV_x

$$\sum_{\vec{p}_a \in \delta V_x} \vec{p}_a = \delta N_a \langle \vec{p}_a \rangle$$

δN_a: tot. # of such species

define density $\delta n_a = \frac{\delta N_a}{\delta V_x}$

then

$$\sum_{\vec{p}_a \in \delta V} \vec{p}_a = \delta V_x \delta n_a \langle \vec{p}_a \rangle$$

electric polarization (dipole moment/volume)

$$\vec{p}_{tot,x} = \delta V_x \sum_a \delta n_a \langle \vec{p}_a \rangle = \delta V_x \vec{P}(\vec{x})$$

Let ρ be external, $\quad \Phi = \Phi_\rho + \Phi_{\text{from } \vec{p}_a}$; $\Phi_{\text{pt dipole}} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{1}{4\pi\varepsilon_0} \vec{P}(\vec{x}')\cdot\nabla' \frac{1}{|\vec{x}-\vec{x}'|}$

then in cont' spec,

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left\{ \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} - \vec{P}\cdot\nabla \frac{1}{|\vec{x}-\vec{x}'|} \right\}$$

Macroscopic field $\nabla\cdot\vec{E} = \frac{\rho_{tot}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left[\rho - \nabla\cdot\vec{P}\right] \Rightarrow \boxed{\nabla\cdot\vec{D} = \rho \quad \text{w/ } \vec{D} = \varepsilon_0\vec{E} + \vec{P}}$

Relation between $\vec{E}$ & $\vec{D}$ or $\vec{P}$

For dielectric considered $\vec{P}\big|_{\vec{E}=0} = 0$, $P_i = \sum_{j=1}^{3} \left(\frac{\partial P_i}{\partial E_j}\right)_{\vec{E}=0} E_j + \cdots \simeq \varepsilon_0 \sum_j (\chi_e)_{ij} E_j$ — elec. susceptibility tensor

thus,

$$\vec{D} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} = \underbrace{\underbrace{\varepsilon_0(1+\chi_e)}_{\text{diele. constant tensor}}}_{\text{elec. permittivity}}\vec{E} = \varepsilon\vec{E}$$

thus $\boxed{\nabla\cdot\vec{E} = \rho/\varepsilon}$, $\boxed{\nabla\times\vec{E} = 0}$ holds

For isotropic Material $\left.\begin{aligned} \chi_e &= \chi_e \cdot \mathbb{1} \\ \epsilon &= \varepsilon \cdot \mathbb{1} \end{aligned}\right\}$ diagonalized tensor

bdry between two medium, tangential comp of $\vec{E}$ contin,

$$(\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0$$

normal comp. of $\vec{D}$ jump.

$$(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = \sigma$$

if isotropic $(\varepsilon_2\vec{E}_2 - \varepsilon_1\vec{E}_1)\cdot\hat{n} = \sigma$

Polarized Charge density

$$\sigma_{pol} = -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n}_{21} \qquad \text{where } \vec{P}_i = (\varepsilon_i - \varepsilon_0)\vec{E}_i$$

Electrostatic Energy in Dielectric Medium.

- material not necessary linear nor uniform
- need to account for small change $\delta\rho$ in macro. charge density ρ in all space.

Thus work done, $\delta W = \int \delta\rho(\vec{x})\Phi(\vec{x})d^3x$

due to material — due to $\rho(\vec{x})$

s.t. $\delta\rho = \nabla\cdot(\delta\vec{D})$

thus $\delta W = \int \vec{E}\cdot\delta\vec{D}\,d^3x$, total work $W = \int d^3x \int_0^D \vec{E}\cdot\delta\vec{D}$

If med. linear, $\vec{E}\cdot\delta\vec{D} = \frac{1}{2}\delta(\vec{E}\cdot\vec{D})$ thus

$$\boxed{W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,d^3x = \frac{1}{2}\int \varepsilon|\vec{E}|^2 d^3x}$$

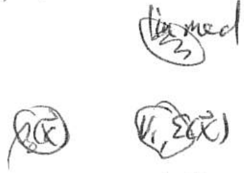
this recover $\frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$

Change in energy when lin. dielectric obj is placed in E-field of fixed source.

(1)

$W_0 = \frac{1}{2}\int \vec{E}_0\cdot\vec{D}_0\,d^3x$ w/ $\vec{D}_0 = \varepsilon_0\vec{E}_0$

(2) Now introduce obj w/ volume V_1, $\varepsilon(\vec{x})$. thus change $\vec{E}_0 \to \vec{E}$



so, $W_1 = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,d^3x$ $\vec{D} = \varepsilon\vec{E}$

Energy diff. $W = \frac{1}{2}\int(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0)d^3x = \frac{1}{2}\int(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0)d^3x = -\frac{1}{2}\int_{V_1}(\varepsilon_1-\varepsilon_0)\vec{E}\cdot\vec{E}_0\,d^3x$

$$\boxed{W = -\frac{1}{2}\int \vec{P}\cdot\vec{E}_0\,d^3x}$$

$\vec{P}$ polarization of dielectric placed in $\vec{E}_0$

w/ energy density

$$\boxed{w = \frac{-1}{2}\vec{P}\cdot\vec{E}_0}$$