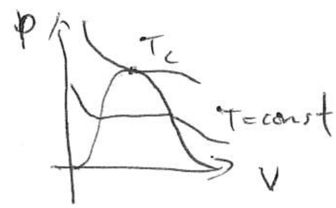
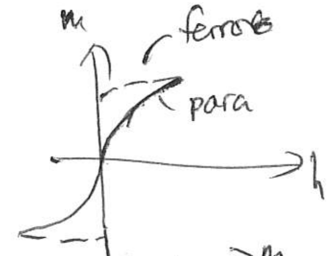
Phase Transition



$K_T = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_T$

$C = T\left(\frac{\partial S}{\partial T}\right)_{T,p}$.



$\chi = \frac{\partial m}{\partial h}$

(jumps on response func. during phase transition.

Recall Canonical ensemble.

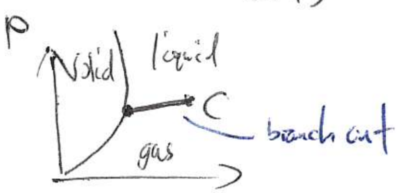
$$Z = e^{-\beta H}$$

$$\beta F = -\ln Z$$

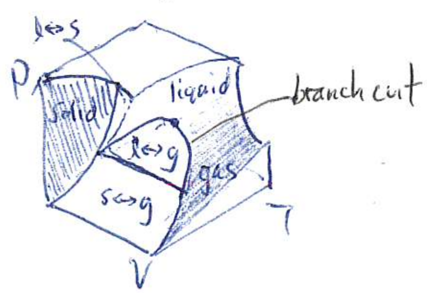
$$dF = -SdT - PdV$$

$$S \sim \left(\frac{\partial F}{\partial T}\right)_V \sim \frac{\partial F}{\partial T}$$

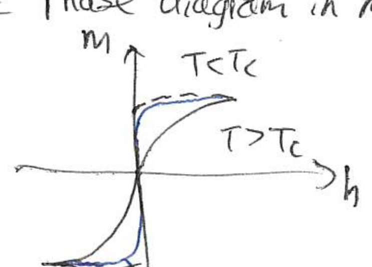
$$C = -T\left(\frac{\partial S}{\partial T}\right) \sim -T\frac{\partial^2 F}{\partial T^2}$$



where non-analy of F/Z given by jumps in response func.



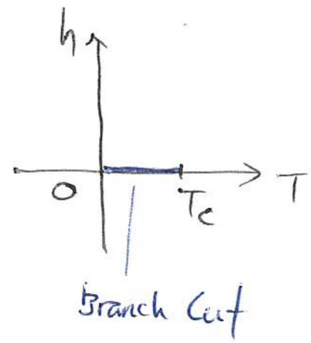
ex Phase diagram in magnet.



asymptotic curve approaching T_c

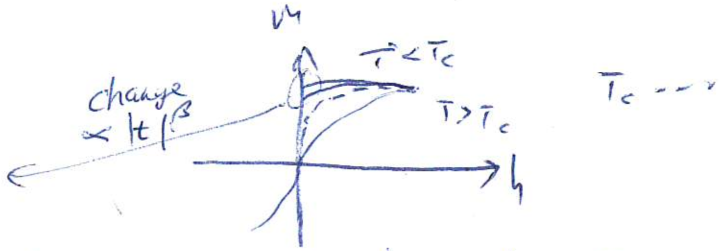
phase transition occurs at $T \to T_c$

$\Rightarrow$



Critical exponents.

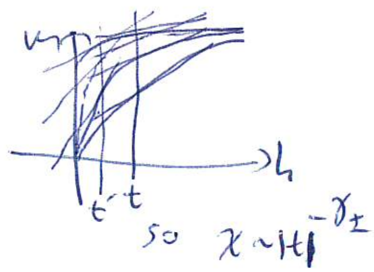
$$m(T, h=0) \propto \begin{cases} 0 & T > T_c \\ |t|^{\beta} & T < T_c \end{cases}$$

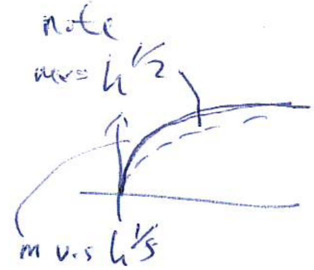


$$m(T=T_c, h) \propto h^{1/\delta}$$

One may ask also how at $T=T_c$ m varies w/ h

$$\chi_{\pm} = \frac{\partial m}{\partial h} \propto |t|^{-\gamma_{\pm}}$$ → slope





$$C(T, h=0) \propto |t|^{-\alpha} \quad \cancel{\text{near } T_c}$$

$$\xi_{\pm}(T, h=0) \propto |t|^{-\nu_{\pm}} \quad \text{w/ } \nu_{\pm} = \nu$$

MFT.

$$Z = tr(e^{-\beta H_{micro}}) \xrightarrow[\text{Graining}]{\text{Coarse}} \int Dm(\vec{x})\, W[m(\vec{x})]$$

$$e^{-\beta H_{micro}} \simeq W[m(\vec{x})]$$

$$\beta H \longrightarrow -\ln W[m(\vec{x})]$$

locality
uniformity

$$\beta H_{eff} \longrightarrow \int d^dx\, \Phi(\vec{m}(\vec{x}), \nabla\vec{m}, \nabla^2\vec{m}, \dots)$$

Sym.
Analyticity

<u>LGH</u> $$\beta H = \beta F_0 + \int d^dx \left(\frac{t}{2}m^2 + u m^4 + \frac{K}{2}(\nabla m)^2 - \vec{h}\cdot\vec{m}\right)$$

Saddle Point Approximation.

$$Z = \int Dm(\vec{x})\, e^{-\beta H(\vec{m})} = e^{-\beta F_0}\int Dm(\vec{x}) \exp\left\{V \underbrace{\left(\frac{t}{2}m^2 + u m^4 + \frac{K}{2}(\nabla m)^2 - \vec{h}\cdot\vec{m}\right)}_{\Phi(\vec{m})}\right\}$$

$\exists\, m^*$ s.t.

$$Z \simeq e^{-\beta F_0} \times \exp(V\Phi(m^*)) \quad \text{where } \Phi(m^*) \Rightarrow \Phi'(\bar{m}) = 0$$

then

$$\beta F = -\ln Z = \beta F_0 - V\Phi(m^*)$$

we use $\bar{m}$ or m^*

Phenom induced criteria of param. t, u etc.

ex

$$\Phi(\vec{m}) = \frac{t}{2}m^2 + um^4 + \dots - \vec{h}\cdot\vec{m}$$ $t = \alpha T$ or $\frac{T-T_c}{T_c}$

$$\Phi' = tm + 4um^3 - h = 0$$

then $t>0$ $\bar{m} = \frac{h}{t}$ if $h=0$ $\bar{m} = 0$

$t<0$ $u>0$ for stability.

note $t>0$, say $y(x) = x^2 + x^4 5 \leftarrow$ plays no role in stability

here u is chosen to ~~ignored~~ be ignored.

ex $\Phi(\vec{m}) = \frac{t}{2}m^2 + um^4 - hm$ magnetic

$$\Phi(m) = tm + 4um^3 - h$$

$$\bar{m}(t, h=0) \sim \bar{m} = \left(\frac{-t}{4u}\right)^{1/2} \sim |t|^{\beta} \quad w/ \ \boxed{\beta = \frac{1}{2}}$$

$$\bar{m}(t=0, h) \sim \bar{m} = \left(\frac{h}{4u}\right)^{1/3} \sim |t|^{1/\delta} \quad w/ \ \boxed{\delta = 3}$$

$\chi = \frac{\partial \bar{m}}{\partial h}\Big|_{h=0}$ under saddle pt. $h = tm + 4um^3$

$$\frac{\partial \bar{m}}{\partial h} = \frac{1}{t + 12u\bar{m}^2} \quad w/ \ \bar{m}(h=0) = \left(\frac{-t}{4u}\right)^{1/2}$$

$$\chi \sim |t|^{-1} \sim |t|^{-\gamma} \quad w/ \ \boxed{\gamma = 1} \quad \text{for } t<0$$

free energy $\beta F = \beta F_0 + V\Phi(\bar{m}) = \beta F_0 + V \begin{cases} 0 & t>0 \\ \frac{-t^2}{16u} & t<0 \end{cases}$

since $\frac{\partial}{\partial T} \sim a\frac{\partial}{\partial t}$

$$C(h=0) = -T\frac{\partial^2 F}{\partial T^2} \approx -T_c a^2 \frac{\partial^2}{\partial t^2}(k_B T_c \beta F) = C_0 + V k_B a^2 T_c^2 \begin{cases} 0 & t>0 \\ \frac{1}{8u} & t<0 \end{cases}$$

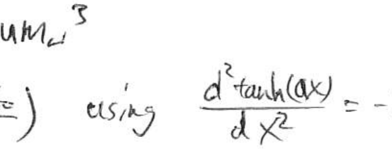
$$C \sim |t|^{-\alpha} \sim \frac{1}{8u} \quad w/ \ \boxed{\alpha = 0.}$$

$t \sim a(T - T_c)$

Domain Wall. 1D

$m(x \to -\infty) = -\bar{m}$

$m(x \to \infty) = \bar{m}$



from Hoff, $\int dx \left[\frac{K}{2}\left|\frac{dm}{dx}\right|^2 + \frac{t}{2}m^2 + um^4\right]$ since $\frac{\partial}{\partial_\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}$

EL eqn $\Rightarrow$ $K\frac{d^2 m_w}{dx^2} = tm_w + 4um_w^3$

Ansatz $m_w = \bar{m}\tanh\left(\frac{x - x_0}{\omega}\right)$ using $\frac{d^2\tanh(ax)}{dx^2} = -2a^2\tanh(ax)\left[1 - \tanh^2(ax)\right]$

this solves $w = \sqrt{\frac{2K}{-t}} \sim |t|^{-1/2}$ & $\bar{m} = \sqrt{\frac{-t}{4u}}$.

1st order transition $\quad m = \frac{\partial F}{\partial h} \to$ zero discontinuously.

2nd order transition $\quad m = \frac{\partial F}{\partial h}$ continuous but $\chi \sim \frac{\partial^2 F}{\partial h^2}$ discontinuous.

Gaussian Integrals.

$$I = \int_{-\infty}^{\infty} d\phi \; e^{-\frac{K}{2}\phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}}\, e^{\frac{h^2}{2K}}$$

cs $\quad e^{-\frac{K}{2}(\phi - \frac{h}{K})^2} \, e^{\frac{h^2}{2K}}$

$$\langle \phi \rangle = \frac{d(I)}{dh} = \frac{d}{dh}\left(\sqrt{\frac{2\pi}{K}}\, e^{\frac{h^2}{2K}}\right) = \sqrt{\frac{2\pi}{K}}\, e^{\frac{h^2}{2K}}\left(\frac{h}{K}\right)$$

$$\langle \phi \rangle = \frac{h}{K} \qquad \langle \phi^2 \rangle = \frac{h^2}{K^2} + \frac{1}{K}$$

$$\langle e^{-ik\phi} \rangle = \exp\left[-\frac{ikh}{K} - \frac{k^2}{2K}\right]$$

N variable

$$I_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\sum_{ij} \frac{1}{2} K_{ij}\phi_i\phi_j + \sum_j h_j\phi_j\right] = \left(\frac{(2\pi)^N}{\det K}\right)^{1/2} \exp\left[\sum_{ij} \frac{K_{ij}^{-1} h_i h_j}{2}\right]$$

$$\langle e^{-i\sum_j k_j\phi_j} \rangle = \exp\left[-i\sum_{ij} K_{ij}^{-1} h_i k_j - \sum_{ij} \frac{K_{ij}^{-1} k_i k_j}{2}\right]$$

$$\langle \phi_i \rangle_c = \sum_j K_{ij}^{-1} h_j \qquad \langle \phi_i \phi_j \rangle_c = K_{ij}^{-1}$$

In F.M $\quad I_N = \prod_{q=1}^{N} \int_{-\infty}^{\infty} d\tilde{\phi}_q \exp\left[-\frac{K_q}{2}\tilde{\phi}_q^2 + \tilde{h}_q\tilde{\phi}_q\right]$

FT $\quad \phi(\vec{x}) = \frac{1}{\sqrt{V}}\sum_q e^{i\vec{q}\cdot\vec{x}}\,\tilde{\phi}_q$

Fluctuation.

$$\vec{m}(\vec{x}) = [\bar{m} + \phi_\ell(\vec{x})]\hat{e}_\ell + \phi_t(\vec{x})\hat{e}_t \qquad \phi_t\hat{e}_t = \sum_{\alpha=2}^{n} \phi_{t,\alpha}(\vec{x})\hat{e}_\alpha$$

$$P(\vec{m}(\vec{x})) \propto \exp\left\{-\int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4\right]\right\}$$

$$\beta H = -\ln P = V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \frac{K}{2}\int d^dx\left[(\nabla\phi_\ell)^2 + \xi_\ell^{-2}\phi_\ell^2\right] + \frac{K}{2}\int d^dx\left[(\nabla\phi_t)^2 + \xi_t^{-2}\phi_t^2\right]$$

Stiffness const, $\xi_\ell^{-2} = \frac{t + 12u\bar{m}^2}{K} = \begin{cases} t & t>0 \\ -2t & t<0 \end{cases} \qquad \xi_t^{-2} = \frac{t+4u\bar{m}^2}{K} = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$

In FM

$$P(\{\phi_{\ell,\vec{q}}; \phi_{t,\vec{q}}\}) \propto \prod_{\vec{q}} \exp\left\{-\frac{K}{2}(q^2 + \xi_\ell^{-2})|\phi_{\ell,\vec{q}}|^2\right\} \exp\left\{-\frac{K}{2}(q^2+\xi_t^{-2})|\phi_{t,\vec{q}}|^2\right\}.$$

using I_N, $\langle \phi_{\alpha,\vec{q}}\phi_{\beta,\vec{q}'}\rangle = \frac{\delta_{\alpha\beta}\delta_{\vec{q},-\vec{q}'}}{K(q^2+\xi_\alpha^{-2})} \qquad \alpha,\beta = \ell, t$

In limiting Case

Gaussian functional Integral: $\int \mathcal{D}\phi(x) \exp\left[-\int d^dx\, d^dx'\, \frac{K(\vec{x},\vec{x}')}{2}\phi(\vec{x})\phi(\vec{x}') + \int d^dx\, h(\vec{x})\phi(\vec{x})\right]$

$$\propto (\det K)^{-1/2} \exp\left[\int d^dx\, d^dx'\, \frac{K^{-1}(\vec{x},\vec{x}')}{2} h(\vec{x})h(\vec{x}')\right] \quad (*)$$

$\langle\phi(\vec{x})\rangle_c = \int d^dx' K^{-1}(\vec{x},\vec{x}')h(\vec{x}') \qquad \boxed{\langle \phi(\vec{x})\phi(\vec{x}')\rangle_c = K^{-1}(\vec{x},\vec{x}')}$

w/ $\int d^dx' K(\vec{x},\vec{x}')K^{-1}(\vec{x}',\vec{x}'') = \delta^d(\vec{x}-\vec{x}'')$ ← inverse of kernel K^{-1} must satisfy

by setting (*) w/ $h=0$

Small fluc.

$$K\int d^dx\left[(\nabla\phi)^2 + \xi^{-2}\phi^2\right] = \int d^dx\, d^dx'\, \phi(\vec{x}')\delta^d(\vec{x}-\vec{x}')(-\nabla^2+\xi^{-2})\phi(\vec{x}) \propto \ln\det K$$

thus kernel, $K(\vec{x},\vec{x}') = K\delta^d(\vec{x}-\vec{x}')(-\nabla^2+\xi^{-2})$ & $K\int d^dx'' \delta^d(\vec{x}-\vec{x}'')(-\nabla^2+\xi^{-2})K^{-1}(\vec{x}''-\vec{x}') = \delta^d(\vec{x}-\vec{x}')$

since $K(-\nabla^2+\xi^{-2})K^{-1}(\vec{x}) = \delta^d(\vec{x})$ & $(\nabla^2 - \xi^{-2})I_d(\vec{x}) = \delta^d(\vec{x})$

thus $\boxed{K^{-1}(\vec{x}) = -\frac{I_d(\vec{x})}{K} = \langle\phi(\vec{x})\phi(0)\rangle_c}$

FM: $K(\vec{q}) = K(q^2+\xi^{-2})$

O.T.O.H.

in FM: $\ln\det K = \sum_{\vec{q}} \ln K(\vec{q}) = V\int\frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi^{-2})\right]$ by setting $h=0$ of (*)

so free energy

$$\beta f = -\frac{\ln Z}{V} = \frac{t\bar{m}^2}{2} + u\bar{m}^4 + \ln\det K \quad ; \quad \ln\det K = \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi_\ell^{-2})\right] + \frac{n-1}{2}\int\frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi_t^{-2})\right]$$

$$\text{w/ } C_{sing} \propto -\frac{\partial^2(\beta f)}{\partial t^2} = \begin{cases} 0 + \frac{n}{2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2+t)^2} & t>0 \\ \frac{1}{8u} + 2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2-2t)^2} & t<0 \end{cases}$$

note Correction term (length)$^{4-d}$

$$C_F \sim \frac{1}{K^2}\begin{cases} a^{4-d} & d>4 \\ \xi^{4-d} & d<4 \end{cases}$$

mean field: $\langle \phi_\alpha(\vec{x}) \rangle = \langle M_\alpha(\vec{x}) - \bar{M}_\alpha \rangle = 0$

Connect func. $G^c_{\alpha\beta}(\vec{x},\vec{x}') = \langle \phi_{\alpha,\vec{x}} \phi_{\beta,\vec{x}'} \rangle = \frac{1}{V}\sum_{\vec{q},\vec{q}'} e^{i\vec{q}\cdot\vec{x} + i\vec{q}'\cdot\vec{x}'} \langle \phi_{\alpha,\vec{q}} \phi_{\beta,\vec{q}'} \rangle = \frac{\delta_{\alpha\beta}}{K} I_d(\vec{x}-\vec{x}', \xi_\alpha)$

in cont' $I_d(\vec{x},\xi) = -\int \frac{d^d q}{(2\pi)^d} \frac{e^{i\vec{q}\cdot\vec{x}}}{q^2+\xi^{-2}}$ Solve Hemholtz $(\nabla^2 - \xi^{-2}) I_d(\vec{x}) = \delta^d(\vec{x})$

I_d can also be:

$I_d \sim \frac{\exp(-\frac{X}{\xi})}{X^p}$

this yields $\frac{p(p+1)}{X^2} + \frac{2p}{X\xi} + \frac{1}{\xi^2} - \frac{p(d-1)}{X^2} - \frac{(d-1)}{X\xi} = \frac{1}{\xi^2}$

$X \ll \xi$ $\frac{1}{X^2}$ dominate $\Rightarrow p = d-1$ $\quad I_d(X) \sim \frac{X^{2-d}}{(2-d)S_d}$

$X \gg \xi$ $\frac{1}{\xi X}$ dominate $\Rightarrow p = \frac{d-1}{2}$ $\quad I_d(X) \sim \frac{\xi^{(\frac{3-d}{2})}}{(2-d)S_d X^{\frac{d-1}{2}}} \exp(-\frac{X}{\xi})$ where $X \sim \xi$

Recalled. $\xi_\ell = \begin{cases} t^{-1/2}/\sqrt{K} & t>0 \\ (-2t)^{-1/2}/\sqrt{K} & t<0 \end{cases}$ $\qquad \xi_t = \begin{cases} \xi_\ell & t>0 \\ \infty & t<0 \end{cases}$

for $t<0$ & $x \ll \xi$ $\quad \chi_\ell \propto \int d^d x\, G^c_\ell(\vec{x}) \propto \xi_\ell^2 \approx A t^{-1}$

$t<0$ & $x \ll \xi$ $\quad \chi_t \propto \int d^d x\, G^c_\ell(\vec{x}) \propto \int_0^L \frac{d^d x}{x^{d-2}} \propto L^2$ sys size.

Lower Critical Dimension

ex supfluid $n=1$ w/ $\psi(x) = |\psi(x)| e^{i\theta(x)}$

$P[\theta(x)] \propto \exp\left(-\frac{\bar{K}}{2}\int d^d x (\nabla\theta)^2\right] \overset{FM}{=} \exp\left[-\frac{\bar{K}}{2}\sum_q q^2 |\theta(\vec{q})|^2 \propto \prod_q P(\theta_q)\right)$

has $\langle \theta_{\vec{q}} \theta_{\vec{q}'} \rangle = \frac{\delta_{\vec{q},-\vec{q}'}}{\bar{K} q^2}$ $\qquad \langle \theta_{\vec{x}} \theta_{\vec{x}'} \rangle = \frac{1}{V}\sum_q \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{\bar{K} q^2}$

cont' $\langle \theta_{\vec{x}} \theta_{\vec{x}'} \rangle = \frac{-C_d(\vec{x}-\vec{x}')}{\bar{K}}$ where $C_d(\vec{x}) = -\int \frac{d^d q}{(2\pi)^d} \frac{e^{i\vec{q}\cdot\vec{x}}}{q^2}$ solves $\nabla^2 C_d(\vec{x}) = \delta^d(\vec{x})$

use Gauss thm, $C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + C_0$

So long dist

$\lim_{x\to\infty} C_d(x) = \begin{cases} C_0 & d>2 \\ \frac{x^{2-d}}{(2-d)S_d} & d<2 \\ \frac{\ln x}{2\pi} & d=2 \end{cases}$

finite phase fluc.

large fluc. long range order in phase is destroyed.

Scaling hypothesis

fundametal parameter ξ

Claim $\xi(t,h) \sim |t|^{-\nu} g_\xi\left(\frac{h}{|t|^\Delta}\right)$ also claim $f(t,h) \sim |t|^{2-\alpha} g_f\left(\frac{h}{|t|^\Delta}\right)$ (free energy ↓ f)

system size L

then $\ln Z \sim \left(\frac{L}{\xi}\right)^d g_s + \cdots$

$$f(t,h) \sim \frac{\ln Z}{L^d} \sim \xi^{-d}$$

Joshephson's identity $2-\alpha = d\nu$

Green's func/Correlation func.

$$G^c_{mm}(\vec{x}) \sim \frac{1}{|\vec{x}|^{d-2+\eta}}$$

$$\chi \sim \int d^d x\, G^c_{mm}(\vec{x}) \sim \int^{\xi} d^d x \frac{1}{|x|^{2-d+\eta}} \sim \xi^{2-\eta}$$

also $\chi \sim |t|^{-\gamma} \Rightarrow \gamma = (2-\eta)\nu$ Fisher's identity

Self-similarity $G^c_{\epsilon\epsilon}(\vec{x}) \sim \frac{1}{|\vec{x}|^{d-2+\eta'}}$ for some $\tilde{\eta}$

s.t

$$C \sim \int d^d x\, G^c_{\epsilon\epsilon}(\vec{x}) \sim |t|^{-\nu(2-\tilde{\eta})}, \quad \alpha = (2-\tilde{\eta})\nu$$

Dilation sym. in critical sys.

$$G_{crt}(\lambda\vec{x}) = \lambda^p G_{crt}(\vec{x})$$

RG. Coarse Grain (further enlarge) $a \mapsto ba$ $b>1$

Rescale $\vec{x}_{new} = \frac{\vec{x}_{old}}{b}$

Renormalized ξ s.t

$$m_i(\vec{x}) = \frac{1}{b^d}\int_{cell(\vec{x})} d^d x'\, m_i(\vec{x}') \longrightarrow \vec{m}_{new}(\vec{x}_{new}) = \frac{1}{5b^d}\int_{Cell(\vec{x}_{new})} d^d x'\, \vec{m}(\vec{x}')$$

Caveats of RG

- $x<\xi$ renorm. config $\approx$ original w/ Ham rot & trans invariant
- Only need t, h to describe ham.
- (new) t', h' ; (old) t, h (renorm) t_b h_b then

$$\left.\begin{aligned} t' &= t_b(t,h) = b^{y_t} t + \cdots \\ h' &= h_b(t,h) = b^{y_h} h + \cdots \end{aligned}\right\} \text{ due to semi-group prop.}$$

$t=h=0$ fixed pt.

Result of RG. $\vec{x}' = \frac{\vec{x}}{b}$

① free energy. $\boxed{Z = Z'}$ thus $V f(t,h) = V' f(t',h')$

then $\boxed{f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)}$

homogeneity $b^{y_t} t = 1$,

$$f(t,h) = t^{2-\alpha} g_t\left(\frac{h}{t^{\Delta}}\right) \quad w/\ 2-\alpha = \frac{d}{y_t} \ \& \ \Delta = \frac{y_h}{y_t}$$

② Correlation length. $\xi' = \frac{\xi}{b}$.

$$\xi(t,h) = b\,\xi(b^{y_t} t, b^{y_h} h)$$

$$\xi(t,h) = t^{-1/y_t} \xi\left(\frac{h}{y^{y_h/y_t}}\right) \sim t^{-\nu} \qquad \nu = \frac{1}{y_t} \Rightarrow 2-\alpha = d\nu$$

③ magnetization.

$$V' = \frac{V}{b^d} \quad \frac{\partial}{\partial h} = \frac{\partial}{\partial h'} \frac{\partial h'}{\partial h} \quad \ln Z = \ln Z'$$

$$f = t^{d/y_t} g_f\left(\frac{h}{t^{y_h/y_t}}\right) \text{ then}$$

$$m(t,h) = \frac{-1}{V} \frac{\partial \ln Z(t,h)}{\partial h} = -\frac{1}{b^d V'} \frac{\partial \ln Z'(t',h')}{b^{-y_h} \partial h'}$$

thus,

$$m(t,h) = b^{y_h - d} m(b^{y_t} t, b^{y_h} h)$$

homogeneity $b^{y_t} t = 1 \Rightarrow b = t^{-1/y_t}$

now

$$m(t,h) = t^{-\frac{y_h - d}{y_t}} m\left(\frac{h}{t^{y_h/y_t}}\right) \quad \beta = \frac{y_h - d}{y_t}$$

④ For any quantity X

$$X(t,h) = b^{y_x} X(b^{y_t} t, b^{y_h} h)$$