(7) Permutation sym.

Axiom: In QM, particles are indistinguishable.

Can arrange particles in

$|k'\rangle|k''\rangle$ or $|k''\rangle|k'\rangle$

$p_1 \quad p_2 \qquad p_1 \quad p_2$

or more generally $c_1|k'\rangle|k''\rangle + c_2|k''\rangle|k'\rangle$

Claim: a observable upon measuring the lin. comb. yields all same eig.val.
This outcome is coined exchange degeneracy.

Def permutation op P_{12} s.t. $P_{12}|k'\rangle|k''\rangle = |k''\rangle|k'\rangle$

Prop. $P_{12}^2 = 1$, $P_{12}P_{12}^{-1} = 1 = P_{12}^{-1}P_{12}$

Def Sym and Antisym. $\frac{|k'\rangle|k''\rangle + |k''\rangle|k'\rangle}{2}$, $\frac{|k'\rangle|k''\rangle - |k''\rangle|k'\rangle}{2}$

then P_{12} has correspond. eig. val ± 1.

Ops in particle # ordering.

$A_1|a'\rangle|a''\rangle = a'|a'\rangle|a''\rangle$ or $A_1|a''\rangle|a'\rangle = a''|a''\rangle|a'\rangle$

$A_2|a'\rangle|a''\rangle = a''|a'\rangle|a''\rangle$ etc.

Consider

$P_{12}A_1|a'\rangle|a''\rangle = P_{12}A_1P_{12}^{-1}|a''\rangle|a'\rangle = a'|a''\rangle|a'\rangle = A_2|a''\rangle|a'\rangle$

so $P_{12}A_1P_{12}^{-1} = A_2$ thus P_{12} change particle lable of observable.

$P_{12}A_1P_{12}^{-1} = A_2$

$\Downarrow$

Now for Ham. of two identical particle.

$H = \frac{\vec{P}_1^2}{2m} + \frac{\vec{P}_2^2}{2m} + V(|\vec{x}_1 - \vec{x}_2|) + V(\vec{x}_1) + V(\vec{x}_2)$

clearly from above, $P_{12}HP_{12}^{-1} = H$ and P_{12} has eig.val ± 1 with previous constructed eig. state.

$P_{12}HP_{12}^{-1} = H$

So $[P_{12}, H] = 0$
share eigenstate
eig.val $P_{12} = \pm 1 \Rightarrow$ odd/even eigenstate

$|k'k''\rangle = \frac{1}{\sqrt{2}}(|k'\rangle|k''\rangle + |k''\rangle|k'\rangle)$ & $|k'k''\rangle = \frac{1}{\sqrt{2}}(|k'\rangle|k''\rangle - |k''\rangle|k'\rangle)$

Def symmetrizer $S_{12} = \frac{1}{2}(1+P_{12})$ and antisymmetrizer $A_{12} = \frac{1}{2}(1-P_{12})$

Thus for any $c_1|k'\rangle|k''\rangle + c_2|k''\rangle|k'\rangle$ we have $\begin{Bmatrix} S_{12} \\ A_{12} \end{Bmatrix}(c_1|k'\rangle|k''\rangle + c_2|k''\rangle|k'\rangle) = \frac{c_1 \pm c_2}{2}(|k'\rangle|k''\rangle \pm |k''\rangle|k'\rangle)$

w/ resulting sym or antisym kets.

* Physical important in test of double α-decay of Conserved Vector Current CVC test

<u>Symmetrization Postulate</u> (What sys. like in nature?)

Sys. of N identical particles under interchange of any pairs are either

Boson — Satisfy boson Stat. or i.e. $P_{ij}\,|N \text{ bosons}\rangle = +\,|N \text{ bosons}\rangle$

Fermion — " fermion Stat. $P_{ij}\,|N \text{ fermions}\rangle = -\,|N \text{ fermions}\rangle$

← interchange of i^{th}, j^{th} particles

Also Boson – Integer spin

Fermion — half-integer

Fermion satisfies Pauli exclusion principle.

Boson demonstrates Bose-Einstein Condensation at low-temp (unique to boson not fermion)

Two electron system.

$[S^2_{tot}, H] = 0$

$\Rightarrow \psi = \phi(\vec{x}_1,\vec{x}_2)\,\chi(m_{s_1}, m_{s_2})$

From $\langle \vec{x}_1, m_{s_1}; \vec{x}_2, m_{s_2} | P_{12} | \alpha\rangle$

$\Rightarrow \langle \vec{x}_1, m_{s_1}; \vec{x}_2, m_{s_2} | \alpha\rangle = -\langle \vec{x}_2, m_{s_2}; \vec{x}_1, m_{s_1} | \alpha\rangle$

So $P_{12} = P_{12}^{(space)} P_{12}^{(spin)}$

If $[S^2_{tot}, H] = 0$ then $\psi = \phi(\vec{x}_1,\vec{x}_2)\chi$ w/

$$\chi(m_{s_1}, m_{s_2}) = \begin{cases} \text{triplet}, & \chi_{++} \\ & \frac{1}{\sqrt{2}}(\chi_{+-} + \chi_{-+}) \\ & \chi_{--} \\ \text{singlet} & \frac{1}{\sqrt{2}}(\chi_{+-} - \chi_{-+}) \end{cases}$$

note $m_{s_1} = \frac{1}{2}$ $m_{s_2} = -\frac{1}{2}$

$P_{12}^{spin} = \frac{1}{2}\left(1 + \frac{4}{\hbar^2}\vec{S}_1\cdot\vec{S}_2\right)$ $\vec{S}_1\cdot\vec{S}_2 = \begin{cases} \frac{\hbar^2}{4} & \text{tripl} \\ -\frac{3\hbar^2}{4} & \text{singl} \end{cases}$

Since Fermion, then

$$\langle \vec{x}_1, m_{s_1}; \vec{x}_2, m_{s_2} | P_{12} | \alpha\rangle = -\langle \vec{x}_1, m_{s_1}; \vec{x}_2, m_{s_2} | \alpha\rangle$$

$$\langle \vec{x}_2, m_{s_2}; \vec{x}_1, m_{s_1} | \alpha\rangle \Rightarrow \boxed{\langle \vec{x}_1, m_{s_1}; \vec{x}_2, m_{s_2} | \alpha\rangle = -\langle \vec{x}_2, m_{s_2}; \vec{x}_1, m_{s_1} | \alpha\rangle} \quad (*)$$

Thus $P_{12} = P_{12}^{(space)} P_{12}^{(spin)}$ w/ $P_{12}^{spin} = \frac{1}{2}\left(1 + \frac{4}{\hbar^2}\vec{S}_1\cdot\vec{S}_2\right)$ $\vec{S}_1\cdot\vec{S}_2 = \begin{cases} \frac{\hbar^2}{4} & \text{triplet} \\ -\frac{3\hbar^2}{4} & \text{singlet} \end{cases}$

So $|\alpha\rangle \to P_{12}|\alpha\rangle \Rightarrow \phi(\vec{x}_1,\vec{x}_2) \to \phi(\vec{x}_2,\vec{x}_1)$ & $\chi(m_{s_1}, m_{s_2}) \to \chi(m_{s_2}, m_{s_1})$

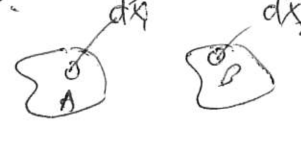
- So wave func. must be antisym from (*)
  If ϕ sym then χ anti and vice versa.
- ϕ gives prob interpretation w/ $|\phi(\vec{x}_1,\vec{x}_2)|^2 d^3x_1\, d^3x_2$
- when neglect interaction: $H\psi = \left[-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\vec{x}_1) + V(\vec{x}_2)\right]\psi = E\psi$

s.t. space. wave func. $\phi(\vec{x}_1,\vec{x}_2) = \frac{1}{\sqrt{2}}\left[\omega_A(\vec{x}_1)\omega_B(\vec{x}_2) \pm \omega_A(\vec{x}_2)\omega_B(\vec{x}_1)\right]$ $\pm$ ⟷ spin singlet / spin triplet

So Prob. $\frac{1}{2}\left\{|\omega_A(\vec{x}_1)|^2|\omega_B(\vec{x}_2)|^2 + |\omega_A(\vec{x}_2)|^2|\omega_B(\vec{x}_1)|^2 \pm \underbrace{2\mathrm{Re}\left[\omega_A(\vec{x}_1)\omega_B(\vec{x}_2)\omega_A^*(\vec{x}_2)\omega_B^*(\vec{x}_1)\right]}_{\text{exchange density}}\right\} d^3x_1\, d^3x_2$

- Remark. ex. den nonvanished when spin is singlet.
  - when particle wide well separated s.t. no overlap on wave func.
    only $|\omega_A(\vec{x}_1)|^2|\omega_B(\vec{x}_2)|^2$ non-vanished → classical.

d^3x_1 d^3x_2



7.1 Helium

$$H=\frac{\vec p_1^{\,2}}{2m}+\frac{\vec p_2^{\,2}}{2m}-\frac{2e^2}{r_1}-\frac{2e^2}{r_2}+\frac{e^2}{r_{12}}$$

Ignored $\frac{e^2}{r_{12}}$ then for electron in ground state another in excited state

$$\phi(\vec x_1,\vec x_2)=\frac{1}{\sqrt{2}}\left[\psi_{100}(\vec x_1)\psi_{n\ell m}(\vec x_2)\pm\psi_{100}(\vec x_2)\psi_{n\ell m}(\vec x_1)\right]$$

In ground state, we have

$$\underbrace{\psi_{100}(\vec x_1)\,\psi_{100}(\vec x_2)}_{sym}\,\chi_{singlet}=\frac{Z^3}{\pi a_0^3}e^{-\frac{Z(r_1+r_2)}{a_0}}\chi \qquad w/\ Z=2$$

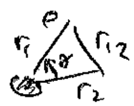
w/ ground-state energy

$$E=2\times\left(-\frac{(Ze)^2}{2a_0}\right)=-108.8\,eV \qquad Z=2.$$

O.T.O.H by 1st order perturbation

$$\Delta_{(1s)^2}=\left\langle\frac{e^2}{r_{12}}\right\rangle_{(1s)^2}=\iint\frac{Z^6}{\pi^2a_0^6}e^{-\frac{2Z(r_1+r_2)}{a_0}}\frac{e^2}{r_{12}}\,d^3x_1\,d^3x_2$$

w/ config.



$$\frac{1}{r_{12}}=\frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}}=\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma)$$

and $$P_\ell(\cos\gamma)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_\ell^{m*}(\theta_1,\phi_1)Y_\ell^m(\theta_2,\phi_2)$$

note

$$\int Y_\ell^m(\theta_i,\phi_i)\,d\Omega_i=\frac{1}{\sqrt{4\pi}}(4\pi)\delta_{\ell 0}\delta_{m0}$$

then $$\int_0^\infty\left[\int_0^{r_1}\frac{1}{r_1}e^{-\frac{2Z(r_1+r_2)}{a_0}}r_2^2dr_2+\int_{r_1}^\infty\frac{1}{r_2}e^{-\frac{2Z(r_1+r_2)}{a_0}}r_2^2dr_2\right]r_1^2dr_1$$ note $\ell=0$

$$=\frac{5}{128}\frac{a_0^5}{Z^5}$$

thus $$\Delta_{(1s)^2}=\left(\frac{Z^6e^2}{\pi^2a_0^6}\right)(4\pi)^2\left(\frac{5}{128}\right)\left(\frac{a_0^5}{Z^5}\right)=\frac{5e^2}{4a_0}$$

$$E_{est}=\left(-8+\frac{5}{2}\right)\frac{e^2}{2a_0}\sim-74.8\,eV$$ where $E_{exp}\simeq-78.8\,eV$

By variational method, take trail wave func. $\langle\vec x_1,\vec x_2|\tilde 0\rangle=\frac{Z_{eff}^3}{\pi a_0^3}e^{-\frac{Z_{eff}(r_1+r_2)}{a_0}}$ then

$$\bar H=\langle\tilde 0|\frac{p_1^2}{2m}+\frac{p_2^2}{2m}|\tilde 0\rangle-\langle\tilde 0|\frac{Ze^2}{r_1}+\frac{Ze^2}{r_2}|\tilde 0\rangle+\langle\tilde 0|\frac{e^2}{r_{12}}|\tilde 0\rangle=\left(2\frac{Z_{eff}^2}{2}-2ZZ_{eff}+\frac{5}{8}Z_{eff}\right)\left(\frac{e^2}{a_0}\right)$$

$$\frac{\partial\bar H}{\partial Z_{eff}}=0\Rightarrow Z_{eff}=2-\frac{5}{16}\Rightarrow E_{est}\simeq-77.5\,eV.$$

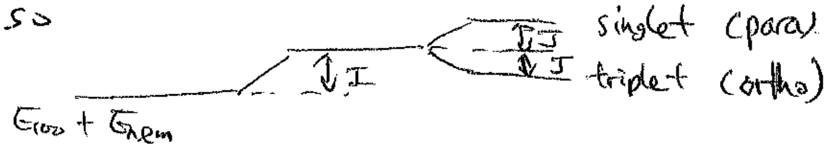
Consider an electron on excited state

$$E = E_{100} + E_{n\ell m} + \Delta E \quad \text{where} \quad \Delta E = \left\langle \frac{e^2}{r_{12}} \right\rangle = I \pm J \qquad \pm \begin{cases} \text{spin singlet} \\ \text{triplet} \end{cases}$$

$$I = \int d^3x_1 \int d^3x_2 \, |\psi_{100}(\vec{x}_1)|^2 \, |\psi_{n\ell m}(\vec{x}_2)|^2 \frac{e^2}{r_{12}}$$

$$J = \int d^3x_1 \int d^3x_2 \, \psi_{100}(\vec{x}_1) \psi_{n\ell m}(\vec{x}_2) \frac{e^2}{r_{12}} \psi^*_{100}(\vec{x}_2) \psi^*_{n\ell m}(\vec{x}_1) \qquad \text{note } I, J > 0$$

so



energy split of $(1s)(n\ell)$ in He

7.2 Multiple particles

$P_{ij}|k\rangle|k'\rangle\cdots|k^i\rangle|k^{i+1}\rangle\cdots|k^j\rangle\cdots = |k\rangle|k'\rangle\cdots|k^j\rangle|k^{i+1}\rangle\cdots|k^i\rangle\cdots$ note $P_{ij}^2=1$

Also $[P_{ij}, P_{kl}]\neq 0$ unless $i\neq j$ $k\neq l$ $i\neq k,l$ $j\neq k,l$

ex For $|k\rangle|k'\rangle|k''\rangle$ has total $3!$ lin. comb.

sym & antisym:

$$|kk'k''\rangle = \frac{1}{\sqrt{6}}\{|k\rangle|k'\rangle|k''\rangle \pm |k'\rangle|k\rangle|k''\rangle + |k'\rangle|k''\rangle|k\rangle \pm |k''\rangle|k'\rangle|k\rangle + |k''\rangle|k\rangle|k'\rangle \pm |k\rangle|k''\rangle|k'\rangle\}$$

Also $P_{123}(|k\rangle|k'\rangle|k''\rangle) = |k'\rangle|k''\rangle|k\rangle$ $\quad 1 \leftrightarrow 2\ 3$ then $2 \leftrightarrow 3$ where $\boxed{P_{123}=P_{12}P_{13}}$

now for N particles if it has N_1 same indices (ie $|k\rangle$) N_2 same indices (ie $|k'\rangle$)
then the normalized factor: $\boxed{\sqrt{\dfrac{N_1!\,N_2!\,N_n!}{N!}}}$ etc.

Second Quantization.

Define Fock space s.t $|n_1, n_2, \ldots, n_i, \ldots\rangle$

n_1 particles at k_1 state $\quad n_1$ occ. num.

Fock space
$|n_1, n_2, \ldots, n_j, \ldots\rangle$

Theory of many-particle.

def vaccum state $|0\rangle \equiv |0, 0, \ldots, 0, \ldots\rangle$

single particle state $|k_i\rangle \equiv |0, 0, \ldots, n_i=1, \ldots\rangle$

Vaccum
$|0, 0, \ldots, 0, \ldots\rangle$

Single
$|k_i\rangle = |0, \ldots, n_i=1, \ldots\rangle$

def ladder op. $a_i^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle \propto |n_1, n_2, \ldots, n_i+1, \ldots\rangle$

s.t creation op. on vaccum

$$a_i^\dagger|0\rangle = |k_i\rangle$$

Define Creation op.
$a_i^\dagger|0\rangle = |k_i\rangle$
normalization
$1=\langle k_i|k_i\rangle \Rightarrow a|k_i\rangle = |0\rangle$

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w/ normalization.

$$1 = \langle k_i|k_i\rangle = \langle 0|a_i a_i^\dagger|0\rangle = \langle 0|(a_i a_i^\dagger|0\rangle) = \langle 0|a_i|k_i\rangle$$

thus $a_i|k_i\rangle = |0\rangle$ acts as annihilation op.

w/

$$a_i|n_1, n_2, \ldots, n_i, \ldots\rangle \propto |n_1, n_2, \ldots, n_i-1, \ldots\rangle$$

$$a_i|k_j\rangle = \delta_{ij}|0\rangle$$

Introduce permutation sym.

$a_i^\dagger a_j^\dagger |0\rangle = \pm a_j^\dagger a_i^\dagger |0\rangle$

thus

Boson	$[a_i^\dagger, a_j^\dagger] = 0$	$[a_i, a_j] = 0$	$[a_i, a_j^\dagger] = \delta_{ij}$
fermion	$\{a_i^\dagger, a_j^\dagger\} = 0$	$\{a_i, a_j\} = 0$	$\{a_i, a_j^\dagger\} = \delta_{ij}$

Remark: built in Pauli principle. b/c $a_i^\dagger a_i^\dagger = 0$ for fermion.

$N = \sum_i a_i^\dagger a_i$

Def number ops. $N \equiv \sum_i a_i^\dagger a_i$

For Boson

$\langle n | a^\dagger a | n\rangle = n$ by def $\Rightarrow$ $\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$

$\langle n | a a^\dagger | n\rangle = n+1$ $\Rightarrow$ $\boxed{a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle}$

For Fermion

$\langle n | a a^\dagger | n\rangle = \langle n | 1 - a^\dagger a | n\rangle = 1-n \Rightarrow a^\dagger |n\rangle = \sqrt{1-n}\,|n+1\rangle$ where $n=0$.

Now build op in terms of $a_i^\dagger a_i$

Let op K additive s.t it has eig. val $\sum_i k_i n_i$ where $N_i = a_i^\dagger a_i$

we postulate $K = \sum_i k_i a_i^\dagger a_i$ ex $H_0 = \sum_i \frac{p_i^2}{2m}$ $H_0 = \sum_k \frac{\hbar^2 k^2}{2m} a_k^\dagger a_k$

Op K additive

$\Rightarrow K|\alpha\rangle = \sum_i k_i n_i |\alpha\rangle$

n_i assoc. $N_i = a_i^\dagger a_i$

$K = \sum_i k_i a_i^\dagger a_i$

What if change of basis?

Change of basis

mm. basis $|k_i\rangle = \sum_j |l_j\rangle\langle l_j | k_i\rangle$

where $|k_i\rangle = a_i^\dagger |0\rangle$ assume $\exists\ b_i^\dagger$ s.t

$|l_i\rangle = b_i^\dagger |0\rangle$

then $a_i^\dagger |0\rangle = \sum_j b_j^\dagger |0\rangle \langle l_j | k_i\rangle \Rightarrow a_i^\dagger = \sum_j b_j^\dagger \langle l_j | k_i\rangle$

$a_i = \sum_j \langle k_i | l_j\rangle b_j$

def $a_i^\dagger |0\rangle = |k_i\rangle$

$b_j^\dagger |0\rangle = |l_j\rangle$

write $|k_i\rangle$ in terms of $|l_j\rangle$ s.t.

$a_i^\dagger = \sum_j b_j^\dagger \langle l_j | k_i\rangle$

recall $K = \sum_i k_i a_i^\dagger a_i$

$K = \sum_{mn} b_m^\dagger b_n \langle l_m | K | l_n\rangle$

using above result and under change of basis.

$K = \sum_{mn} b_m^\dagger b_n \langle l_m | K | l_n\rangle$ ex $H_0 = \sum_{x x'} b_x^\dagger b_{x'} \langle x | \frac{p^2}{2m} | x'\rangle$

7.3 Many Particle Interaction

Let V_{ij} be sym matrix, eigval of two particle interaction between states $|k_i\rangle$ $|k_j\rangle$

The 2^{nd} quantized interaction op.

$$\mathcal{U} = \underbrace{\frac{1}{2}\sum_{i\neq j} V_{ij} N_i N_j}_{\text{all particle interaction}} + \underbrace{\frac{1}{2}\sum_i V_{ii} N_i (N_i - 1)}_{\frac{N_i(N_i-1)}{2} \text{ way of self interaction.}}$$

rewrite as

$$\mathcal{U} = \frac{1}{2}\sum_{ij} V_{ij} \underbrace{(N_i N_j - N_i \delta_{ij})}_{T_{ij}} \qquad T_{ij} = a_i^\dagger \underbrace{a_i a_j^\dagger}_{= \delta_{ij} \pm a_j^\dagger a_i} a_j - a_i^\dagger a_i \delta_{ij} = \pm a_i^\dagger a_j^\dagger a_j a_i$$

thus

$$\boxed{\mathcal{U} = \frac{1}{2}\sum_{ij} V_{ij} a_i^\dagger a_j^\dagger a_j a_i}$$

change of basis:

$$\mathcal{U} = \frac{1}{2}\sum_{mnpq} \langle mn | V | pq \rangle b_m^\dagger b_n^\dagger b_p b_q \quad \text{w/ } \langle mn | U | pq \rangle = \sum_{ij} V_{ij} \langle l_m | k_i \rangle \langle k_i | l_p \rangle \langle l_n | k_j \rangle \langle k_j | l_q \rangle$$

Ex Degenerate gas.

$$H = H_{ee} + H_b + H_{ee\text{-}b}$$

electron interaction. $H_{ee} = \sum_i \frac{\vec{P}_i^2}{2m} + \frac{1}{2} e^2 \sum_i \sum_{j\neq i} \frac{e^{-\mu|\vec{x}_i - \vec{x}_j|}}{|\vec{x}_i - \vec{x}_j|}$ — screening potential.

positive charge bkg. $H_b = \frac{1}{2} e^2 \int d^3x' d^3x'' \rho(x') \rho(x'') \frac{e^{-\mu|\vec{x}' - \vec{x}''|}}{|\vec{x}' - \vec{x}''|}$, assume uniform $\rho = \frac{N}{V}$

$$= \frac{1}{2} e^2 \frac{N^2}{V} \frac{4\pi}{\mu^2} \quad \text{diverg. } \mu \to 0.$$

O.T.O.H

electron - bkg interaction $H_{ee\text{-}b} = -e^2 \sum_i \int d^3x' \rho(x') \frac{e^{-\mu|\vec{x}' - \vec{x}_i|}}{|\vec{x}' - \vec{x}_i|} = -\frac{e^2 N^2}{V} \frac{4\pi}{\mu^2}$

thus

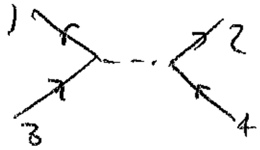
$$H = \underbrace{-\frac{1}{2} \frac{e^2 N^2}{V} \frac{4\pi}{\mu^2}}_{\text{num,}} + \underbrace{\sum_i \frac{\vec{P}_i^2}{2m}}_{\text{single body}} + \underbrace{\frac{1}{2} e^2 \sum_i \sum_{j\neq i} \frac{e^{-\mu|\vec{x}_i - \vec{x}_j|}}{|\vec{x}_i - \vec{x}_j|}}_{\text{two body}}$$

Define $K \equiv \sum_i \frac{P_i^2}{2m}$ w/ state $|\vec{k}\lambda\rangle$ $\lambda = \pm 1$ spin

s.t $\langle \vec{k}'\lambda' | \vec{P} | \vec{k}\lambda \rangle = \hbar \vec{k} \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'}$

$$K = \sum_{\vec{k}\lambda} \frac{\hbar^2 \vec{k}^2}{2m} a_{\vec{k}\lambda}^\dagger a_{\vec{k}\lambda}$$

$$U = \frac{1}{2} e^2 \sum_i \sum_{j \neq i} \frac{e^{-\mu|\vec{x}_i - \vec{x}_j|}}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{2} \sum_{\vec{k}_1\lambda_1} \sum_{\vec{k}_2\lambda_2} \sum_{\vec{k}_3\lambda_3} \sum_{\vec{k}_4\lambda_4} \langle \vec{k}_1\lambda_1 \vec{k}_2\lambda_2 | V | \vec{k}_3\lambda_3 \vec{k}_4\lambda_4 \rangle a^\dagger_{\vec{k}_1\lambda_1} a^\dagger_{\vec{k}_2\lambda_2} a_{\vec{k}_4\lambda_4} a_{\vec{k}_3\lambda_3}$$



where $\langle \vec{k}_1\lambda_1 \vec{k}_2\lambda_2 | V | \vec{k}_4\lambda_4 \vec{k}_3\lambda_3 \rangle = \int d^3x' d^3x'' V(\vec{x}', \vec{x}'') \langle \vec{k}_1\lambda_1 | \vec{x}' \rangle \langle \vec{x}' | \vec{k}_3\lambda_3 \rangle \langle \vec{k}_2\lambda_4 | \vec{x}'' \rangle \langle \vec{x}'' | \vec{k}_4\lambda_4 \rangle$

$$= \frac{e^2}{V} \delta_{\lambda_1\lambda_4} \delta_{\lambda_2\lambda_3} \delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4} \frac{4\pi}{\vec{q}^2 + \mu^2} \quad \text{w/ } \vec{x} = \vec{x}'' \quad \vec{y} = \vec{x}' - \vec{x}''$$

$$\vec{q} = \vec{k}_1 - \vec{k}_3$$

so $$U = \frac{e^2}{2V} \sum_{\vec{k}_1\lambda_1} \sum_{\vec{k}_2\lambda_2} \sum_{\vec{k}_3\lambda_3} \sum_{\vec{k}_4\lambda_4} \delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4} \frac{4\pi}{\vec{q}^2 + \mu^2} a^\dagger_{\vec{k}_1\lambda_1} a^\dagger_{\vec{k}_2\lambda_2} a_{\vec{k}_4\lambda_2} a_{\vec{k}_3\lambda_1}$$

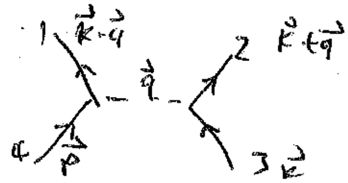
for $\vec{k}_3 = \vec{k}$ $\vec{k}_4 = \vec{p}$ $\vec{q} = 0$ $\Rightarrow$ $\vec{k}_1 = \vec{k}_3 = \vec{k}$, $\vec{k}_2 = \vec{p}$ then we have

$$\frac{e^2}{2V} \sum_{\vec{k}\vec{p}} \sum_{\lambda_1\lambda_2} \frac{4\pi}{\mu^2} a^\dagger_{\vec{k}\lambda_1} a^\dagger_{\vec{p}\lambda_2} a_{\vec{p}\lambda_2} a_{\vec{k}\lambda_1} = \underbrace{\frac{e^2}{2V} \frac{4\pi}{\mu^2}}_{\text{canceled num.}} (N^2 - N)$$

after using $[\,,\,]$ for fermion.

vanished for $\mu L \gg 1$

Remark thus $\vec{q} = 0$ term has no contribution.



Now. $H = H_0 + H_1$

$$H_0 = \sum_{\vec{k}\lambda} \frac{\hbar^2 k^2}{2m} a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} \qquad H_1 = \frac{e^2}{2V} \sum'_{\vec{k}\vec{p}\vec{q}} \sum_{\lambda_1\lambda_2} \frac{4\pi}{q^2} a^\dagger_{\vec{k}+\vec{q},\lambda_1} a^\dagger_{\vec{p}-\vec{q},\lambda_2} a_{\vec{p}\lambda_2} a_{\vec{k}\lambda_1}$$

Use H to find ground state & 1st excited state.

Remark. Pauli's principle $\Rightarrow$ electrons fill available energy level up to $k_{max} = k_F$

so total # elec. $$N = \sum_{\vec{k}} \Theta(k_F - k)$$

Recall. $k = \frac{2\pi}{L} n$ $\quad n = V \frac{k^3}{(2\pi)^3}$ $\quad$ so $N = \sum_{\vec{k}} \Theta \to V \int \frac{d^3k}{(2\pi)^3} \Theta(k_F - k) \times 2$ # electron $\uparrow\downarrow$

$$N = \frac{V}{3\pi^2} k_F^3 \quad \text{so} \quad k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

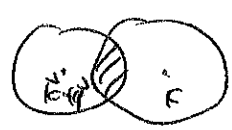
note $\frac{N}{V} = \frac{1}{\frac{4}{3}\pi r_0^3}$, $k_F = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_0}$

thus. $$E^{(0)} = \sum_{\vec{k}\lambda} \frac{\hbar^2 k^2}{2m} \Theta(k_F - k) = \sum_\lambda \frac{\hbar^2}{2m} \frac{V}{(2\pi)^3} \int_0^{k_F} d^3k\, k^2 = \frac{e^2}{2a_0} N \frac{3}{5} \left(\frac{9\pi}{4}\right)^{2/3} \frac{1}{r_s^2} \quad \text{w/ } r_s = \frac{r_0}{a_0} \quad a_0 = \frac{\hbar^2}{me}$$

$$E^{(1)} = \langle F | H_1 | F \rangle = \frac{e^2}{2V} \sum_{\vec{k}\vec{p}\vec{q}} \sum_\lambda \frac{4\pi}{q^2} \langle F | a^\dagger_{\vec{k}+\vec{q},\lambda_1} a^\dagger_{\vec{p}-\vec{q},\lambda_2} a_{\vec{p}\lambda_2} a_{\vec{k}\lambda_1} | F \rangle$$

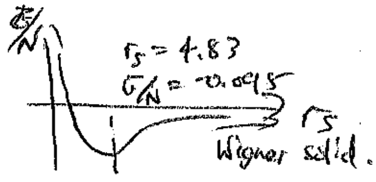
$|F\rangle$ assoc. w/ either 0, 1

$$= -\frac{e^2}{2V^2} \frac{V^2}{(2\pi)^6} \int d^3k \int d^3q \frac{4\pi}{q^2} \Theta(k_F - |\vec{k}+\vec{q}|) \Theta(k_F - k)$$



$$= -\frac{e^2}{2a_0} N \frac{3}{2\pi} \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s}$$

so $$\frac{E}{N} = \frac{E^{(0)} - E^{(1)}}{N} = \frac{e^2}{2a_0} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{3}{5} \frac{1}{r_s^2} - \frac{3}{2\pi} \frac{1}{r_s}\right)$$



7.4 Quantization of EM Field

$\nabla\cdot\vec{E}=0$ $\qquad \frac{1}{c}\frac{\partial\vec{B}}{\partial t}+\nabla\times\vec{E}=0$

$\nabla\cdot\vec{B}=0$ $\qquad \frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\nabla\times\vec{B}=0$

w/ Coulomb gauge $\nabla\cdot\vec{A}=0$ then $\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0$

Guess $\vec{A}(\vec{x},t)=\vec{A}(\vec{k})e^{\pm i\vec{k}\cdot\vec{x}\pm i\omega t}$ w/ $\omega=kc$

Remark $\nabla\cdot\vec{A}=0 \Rightarrow \vec{k}\cdot\vec{A}(\vec{k})=0 \Rightarrow \vec{k}\perp\vec{A}$ thus $\vec{A}$ referred as transverse gauge.

Since $\vec{A}\perp\vec{k}$ we write $\vec{A}$ in terms of $\hat{e}_{\vec{k}\lambda(\pm 1)}$ $\quad \hat{e}_{k\pm}=\mp\frac{1}{\sqrt{2}}\left(\hat{e}^{(1)}_{\vec{k}}\pm i\hat{e}^{(2)}_{\vec{k}}\right)$

Quantization suggested

$$\vec{A}(\vec{x},t)=\sum_{\vec{k}\lambda}\hat{e}_{\vec{k}\lambda}\vec{A}_{\vec{k}\lambda}(\vec{x},t)\ ;\ \vec{A}_{\vec{k}\lambda}(\vec{x},t)=A_{\vec{k}\lambda}e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}+A^{*}_{\vec{k}\lambda}e^{i(\omega_k t-\vec{k}\cdot\vec{x})}$$

thus $\vec{A}_{\vec{k}\lambda}$ real when quantize A^{*}_{kA} creat. op $A_{\vec{k}\lambda}$ annihilation op.

note. $\boxed{\hat{e}^{*}_{\vec{k}\lambda}\cdot\hat{e}_{\pm\vec{k}\lambda'}=\pm\delta_{\lambda\lambda'} \qquad \hat{e}^{*}_{\vec{k}\lambda}\times\hat{e}_{\pm\vec{k}\lambda'}=\pm i\lambda\delta_{\lambda\lambda'}\hat{k}}$

For $\mathcal{E}=\frac{1}{8\pi}\int_V\left[|\vec{E}(\vec{x},t)|^2+|\vec{B}(\vec{x},t)|^2\right]d^3x$

$$\mathcal{E}=\frac{V}{4\pi}\sum_{\vec{k},\lambda}\frac{\omega_k^2}{c^2}\left[\vec{A}^{*}_{\vec{k}\lambda}A_{\vec{k},\lambda}+A_{\vec{k},\lambda}A^{*}_{\vec{k},\lambda}\right]$$ claim to promote to op.

Photon & Energy Quantization

Claim Photon bosonic.

Recalled $D_z(\phi)=\exp\left(\frac{-iJ_z\phi}{\hbar}\right)$ yield spin quantum num "m"

For $\hat{e}^{(1)}_k\to\hat{e}^{(1)'}_{\vec{k}}=\cos\phi\,\hat{e}^{(1)}_k-\sin\phi\,\hat{e}^{(2)}_{\vec{k}}$
$\hat{e}^{(2)}_k\to\hat{e}^{(2)'}_{\vec{k}}=\sin\phi\,\hat{e}^{(1)}_k+\cos\phi\,\hat{e}^{(2)}_{\vec{k}}$
$\Rightarrow$ $\hat{e}_{k+}=-\frac{1}{\sqrt{2}}(e^{(1)}_k+ie^{(2)}_k)$, $\quad \hat{e}'_{k+}=e^{i\phi}\hat{e}_{k+}$ $\Rightarrow$ $\boxed{m=1 \text{ thus boson.}}$

Orientation/Rotation to show photon

$$H=\sum_{\vec{k}\lambda}\hbar\omega_k a^{\dagger}_{\lambda}(\vec{k})a_{\lambda}(\vec{k})=\sum_{\vec{k}\lambda}\frac{1}{2}\hbar\omega_k\left[a^{\dagger}_{\lambda}(\vec{k})a_{\lambda}(\vec{k})+a_{\lambda}(\vec{k})a^{\dagger}_{\lambda}(\vec{k})+1\right]$$

so $A=\frac{(4\pi\hbar c^2)^{1/2}}{\sqrt{V}}\frac{1}{\sqrt{2\omega_k}}a_{\lambda}$

w/ zero-point/vacuum energy

$$E_0=\frac{1}{2}\sum_{\vec{k}\lambda}\hbar\omega_k=\sum_{k}\hbar\omega_k=\hbar c\left(\frac{L}{2a}\right)^2\left(\frac{a}{2\pi}\right)\int d^3k\,k=\frac{3c\hbar L^2 a}{\pi^2}\frac{1}{\lambda^4}$$

Casimir Effect

Between plates, $E_{II} = \frac{\hbar c}{2}\left(\frac{L}{2\pi}\right)^2 \int d^2k \int_0^\infty dn_z \sqrt{k^2 + k_z^2}$ w/ $\int dn_z \to \sum_{n=0}^{\infty}$

$$= \frac{\hbar c}{2}\left(\frac{L}{2\pi}\right)^2 \int d^2k \left[k + 2\sum_{n=1}^{\infty} \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2} \right]$$

$$= \frac{3c\hbar L^2 a}{\pi^2} \frac{1}{\lambda^4} - \frac{1}{720} \frac{c\hbar L^2 \pi^3}{a^3}$$

thus

$$\frac{E_{II} - E_0}{L^2} = \frac{\Delta E}{L^2} = -\frac{1}{720} \frac{c\hbar L^2 \pi^3}{a^3}$$

Force between plate: $-\frac{\partial E}{\partial a} = F = -\frac{c\hbar\pi^2}{240a^4}$ thus an attractive force

This is Casimir effect.