Dirac Eqn. Spin 1/2

Axiom: spin as internal angular momentum, so has transf. prop of angular momentum.

transf. prop.
- rotation as vector
- parity as pseudo-vec
- Lorentz transform

Recall $\vec{a}\times\vec{b} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\hat{e}_i a_j b_k = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

For NxN matrix A

$$\det A = \sum_{i_1,i_2,\cdots i_N=1}^{N} \varepsilon_{i_1 i_2 \cdots i_N} a_{1i_1} a_{2i_2} a_{3i_3} \cdots a_{Ni_N}$$

so angular momentum $\vec{L} = \vec{r}\times\vec{p}$ has above form ie second-rank tensor.

ex EM-field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H_3 & -H_2 \\ -E_2 & -H_3 & 0 & H_1 \\ -E_3 & H_2 & -H_1 & 0 \end{pmatrix}$$

Recall Lorentz boost, in x

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

so $F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$ s.t

$$H_\parallel' = H_\parallel \quad E_\parallel' = E_\parallel \qquad \vec{H}_\perp' = \gamma(\vec{H}_\perp - \vec{v}\times\vec{E}) \quad \vec{E}_\perp' = \gamma(\vec{E}_\perp + \vec{v}\times\vec{H})$$

under parity

$\vec{E}\to-\vec{E}$ but $\vec{H}\to\vec{H}$ so as $\vec{L}\to\vec{L}$ thus

$\vec{L}\sim\vec{H}$ but $? \sim \vec{E}$

in orbital mom. $-c\vec{p}t - \vec{r}E/c \sim \vec{E}$

Spin ?

$\vec{S}\sim\vec{L}$ so $\vec{S}\sim\vec{H}$

Conjecture $i\vec{\alpha}\sim\vec{E}$ s.t Lorentz transf. hold w/

$\sigma_\parallel' = \sigma_\parallel \quad i\alpha_\parallel' = i\alpha_\parallel$

$$\vec{\sigma}_\perp' = \gamma(\vec{\sigma}_\perp - \vec{v}\times i\vec{\alpha}) \quad i\vec{\alpha}_\perp' = \gamma(i\vec{\alpha}_\perp + \vec{v}\times\vec{\sigma}) \qquad \sigma^{\mu\nu} = \begin{pmatrix} 0 & i\alpha_1 & i\alpha_2 & i\alpha_3 \\ -i\alpha_1 & 0 & \sigma_3 & -\sigma_2 \\ -i\alpha_2 & -\sigma_3 & 0 & \sigma_1 \\ -i\alpha_3 & \sigma_2 & -\sigma_1 & 0 \end{pmatrix}$$

By construction taking $\vec{v}$ in $\hat{z}$ then

$$\sigma_x' = \gamma(\sigma_x + iv\alpha_y)$$

$$\sigma_y' = \gamma(\sigma_y - iv\alpha_x)$$

$\sigma_i^2 = 1$ then $\sigma_x'^2 = 1 \Rightarrow \sigma_x\alpha_y + \alpha_y\sigma_x = 0 \quad \alpha_y^2 = 1.$

Generalization: $\{\sigma_i, \alpha_j\} = 0 \quad i \neq j$, $\alpha_i^2 = 1$

Now $\sigma_y'\sigma_z' = i\sigma_x' \Rightarrow \alpha_x\sigma_z = -i\alpha_y$

$\sigma_z'\sigma_y' = -i\sigma_x' \Rightarrow \sigma_z\alpha_x = i\alpha_y$

Generalization: $\alpha_i\sigma_j = i\varepsilon_{ijk}\alpha_k$, $[\alpha_i, \sigma_j] = 2i\varepsilon_{ijk}\alpha_k$

Now $\sigma_x'\sigma_y' = i\sigma_z \Rightarrow \alpha_y\alpha_x = -i\sigma_z$

$\sigma_y'\sigma_x' = -i\sigma_z \Rightarrow \alpha_x\alpha_y = i\sigma_z$

$[\alpha_i, \alpha_j] = 2i\varepsilon_{ijk}\sigma_k$, $\{\alpha_i, \alpha_j\} = 0$

Summarize

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

$$[\alpha_i, \alpha_j] = 2i\varepsilon_{ijk}\sigma_k$$

$$[\alpha_i, \sigma_j] = 2i\varepsilon_{ijk}\alpha_k$$

$$\{\alpha_i, \sigma_j\} = 0 \quad i \neq j$$

Quest for Rep. α

Recall parity $\sigma \to \sigma$ but $\alpha \to -\alpha$ $\quad \alpha, \sigma$ diff. behavior.

Def pairity op "β" on spin, we have $\beta^2 = +1$ or $-1 \Rightarrow \beta = \pm 1$ or $\beta = \pm i$

take $\beta^2 = 1$

σ inv under parity $\Rightarrow [\sigma, \beta] = 0$ or $\beta^{-1}\sigma\beta = \sigma$ but $\beta^{-1}\alpha\beta = -\alpha$

Since $\det(\beta^{-1}\sigma\beta) = \det(\sigma) = 1 \qquad \det(\alpha) = \det(\beta^{-1}\alpha\beta) = \det(-\alpha) = (-1)^N \det\alpha$

so N even. But $N \neq 2$ b/c if so α lincomb of σ. since $[\beta, \sigma] = 0$

$\Rightarrow [\beta, \alpha] = 0$ contradiction to $\beta^{-1}\alpha\beta = -\alpha$

thus $\dim(\alpha, \beta) = 4$ at least.

Possible rep

$\vec{\tau}$ Pauli matrix $\quad \vec{\sigma} = \begin{pmatrix} \vec{\tau} & 0 \\ 0 & \vec{\tau} \end{pmatrix} \quad \alpha = \begin{pmatrix} 0 & \vec{\tau} \\ \vec{\tau} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note $\mathrm{tr}(\sigma) = \mathrm{tr}(\alpha) = \mathrm{tr}(\beta) = 0$

Phys. Interpretation of $\sigma^{\mu\nu}$

$\vec{\sigma}$ op. generates rot. of coord. sys

$i\vec{\alpha}$ op. generates rot. of space about time axis (Lorentz transf.)

Recalled rot. op.

$\langle\alpha|O|\alpha\rangle$ for $|\beta\rangle = R|\alpha\rangle$, $\langle\beta|O|\beta\rangle = \langle\alpha|O|\alpha\rangle$

$\langle\alpha|R^{\dagger}OR|\alpha\rangle = \langle\alpha|O|\alpha\rangle_R \Rightarrow O' = R^{\dagger}OR$

so $\sigma' = D(\phi)\sigma D^{\dagger}(\phi)$ $\alpha' = D(\phi)\alpha D^{\dagger}(\phi)$ $\beta' = D(\phi)\beta D^{\dagger}(\phi)$

In Lorentz transf

$$\sigma' = \mathcal{L}_v \sigma \mathcal{L}_v^{-1} \quad \alpha' = \mathcal{L}_v \alpha \mathcal{L}_v^{-1} \quad \beta' = \mathcal{L}_v \beta \mathcal{L}_v^{-1} \quad w/ \ \mathcal{L}_v = e^{-i(i\vec{\alpha})\cdot\vec{\omega}/2} = e^{\vec{\alpha}\cdot\omega/2}$$

Check

Note $\tanh\omega = \frac{v}{c}$ recalled in SR ($\tanh\xi = \beta$)

Note $\mathcal{L}(\vec{\alpha}\cdot\vec{\omega})$ in $\hat{z} \parallel \sigma_{\parallel}$ and $[\alpha_i, \sigma_i] = 0$

so. $\sigma_{\parallel}' = \mathcal{L}\sigma_{\parallel}\mathcal{L}^{-1} = \mathcal{L}\mathcal{L}^{-1}\sigma_{\parallel} = \sigma_{\parallel}$

$\vec{\sigma}_{\perp}' = e^{\vec{\alpha}\cdot\vec{\omega}/2}\vec{\sigma}_{\perp}e^{-\vec{\alpha}\cdot\vec{\omega}/2} = e^{\vec{\alpha}\cdot\vec{\omega}}\sigma_{\perp}$ where $e^{\vec{\alpha}\cdot\vec{\omega}} = \cosh\omega + \vec{\alpha}\cdot\hat{\omega}\sinh\omega$

since $1 - \tanh^2 x = \operatorname{sech}^2 x \Rightarrow \cosh^2 x = \frac{1}{1-\tanh^2 x}$

so $\vec{\sigma}_{\perp}' = \cosh\omega\,[1 + \vec{\alpha}\cdot\hat{\omega}\tanh\omega]\,\vec{\sigma}_{\perp} = \frac{(\vec{\sigma}_{\perp} + \vec{\alpha}\cdot\vec{v})\vec{\sigma}_{\perp}}{\sqrt{1-v^2}}$

where $(\vec{\alpha}\cdot\vec{v})\vec{\sigma}_{\perp} = -i\frac{\vec{v}}{c}\times\vec{\alpha}$ b/c $(\alpha_i v_i)\sigma_{\perp k}$ & $\alpha_i\sigma_j = i\varepsilon_{ijk}\alpha_k$

transf holds in σ also in α, β

obs $\beta = \mathcal{L}_v\beta\mathcal{L}_v^{-1} = \gamma(\beta - \frac{\vec{v}}{c}\beta\vec{\alpha})$

see that β as time compoment of 4 vec where $\beta\alpha$ is spacelike

prompt to check $\beta'\vec{\alpha}_{\perp}' = \mathcal{L}_v\beta\vec{\alpha}_{\perp}\mathcal{L}_v^{-1} = \beta\vec{\alpha}_{\perp}$

& $\beta'\alpha_{\parallel}' = \mathcal{L}\beta\alpha_{\parallel}\mathcal{L}^{-1} = \gamma(\beta\alpha_{\parallel} - \frac{v}{c}\beta)$

thus β, $\beta\vec{\alpha}$ transf as x^{μ}

Def $\gamma^{\mu} = (\beta, \beta\vec{\alpha})$ ie $\gamma^0 = \beta$ $\gamma^i = \beta\alpha_i$ $i = 1,2,3$

so $\gamma = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}$

Prop of γ^μ

$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = \beta\alpha_i\beta\alpha_i = -\beta^2\alpha_i^2 = -1$$

$$\{\gamma^0, \gamma^\mu\} = 0$$

$$\gamma^i\gamma^j = \beta\alpha_i\beta\alpha_j = -\beta^2\alpha_i\alpha_j = \beta^2\alpha_j\alpha_i = -\gamma^j\gamma^i$$

$$\{\gamma^i, \gamma^j\} = 0 \quad \text{if } i \neq j$$

In general

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad \text{w/} \quad g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$