⑧ Natural units

$c = 1 \Rightarrow 1\,\text{sec} = 3\times10^{8}\,m$ since $E = mc^2$ $[E] = [M]$ in eV

$\hbar = 6.6\times10^{-16}\,eV\,s \simeq 2.0\times10^{-7}\,eV\,m$

$\hbar = 1 \Rightarrow 1m = \frac{1}{2\times10^{-7}\,eV}$ so $[L]$ in eV^{-1}

thus increasing energy decreasing length.

de Broglie wavelength

$\lambda_B = \frac{h}{p} \Rightarrow [P]$ in eV

implies probing small length scale needs high energy i.e particle accelerators like microscopes

Gravity

$F = \frac{GM_1M_2}{r^2}$ $\quad G_N \sim \frac{L^{d-1}}{MT^2}$

$d=4$ $\quad G_N = 6.7\times10^{-11}\frac{m^3}{kg\,s^2}$ $\quad$ (ie. in m $\quad G_N = 2.5\times10^{-70}\,m^2$)

In terms of time, length, mass

$L_{planck} = \sqrt{G_N} = 1.6\times10^{-35}\,m$

$M_{planck} = \frac{1}{\sqrt{G_N}} = 1.3\times10^{19}\,GeV$

$T_{planck} = \sqrt{G_N} = 5.4\times10^{-44}\,s$

So planck length scale is the natural distance at which quantum gravitational effect becomes significant.

Free Particles (Relativistic)

$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$ relativistic $E_p = \sqrt{p^2+m^2}$ $\quad H \approx m + \frac{p^2}{2m} - \frac{p^4}{8m^3} + \cdots$

then $i\frac{\partial}{\partial t}\langle \vec{x}|\psi(t)\rangle = \int \frac{d^3x' d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \underbrace{\langle \vec{x}'|E_p|\psi(t)\rangle}_{\text{non local due to H.O derivative}}$

Loss of covariance cost causality

Klein-Gordon Eqn.

$-\frac{\partial^2}{\partial t^2}|\psi(t)\rangle = H^2|\psi(t)\rangle$

$(\partial_t^2 + \nabla^2 + m^2)\psi(\vec{x},t) = 0$ Lorentz inv.

- m^2 term w/ $c\hbar$ is compton wavelength $\left(\frac{mc}{\hbar}\right)^2$
- def $\partial_\mu = (\partial_t, \nabla)$

KG.

$(\partial_\mu\partial^\mu + m^2)\Psi(\vec{x},t) = 0$ $\quad$ Free particle $\underline{\Psi(\vec{x},t) = N e^{-i(Et - \vec{p}\cdot\vec{x})}} = N e^{-ip^\mu x_\mu}$

equal w/ $-E^2 + p^2 + m^2 = 0 \Rightarrow E = \pm E_p$

KG Current $\partial_\mu j^\mu = 0$

$j^\mu = \frac{i}{2m}\left[\Psi^*\partial^\mu\Psi - \Psi(\partial^\mu\Psi)^*\right]$ $\quad$ check $\partial_\mu j^\mu = \left[\frac{i}{2m}N^2(-m^2+m^2)\right] = 0$

def.

$\rho = j^0 = \frac{i}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial t} - \left(\frac{\partial\Psi}{\partial t}\right)^*\Psi\right]$

so $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$ where ρ const for $j^0 = \frac{N^2E}{m}$

ρ either $+/-$ due to $\pm E_p$

EM interaction.

Def $A^\mu = (\Phi, -\vec{A})$ $\quad i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} - e\Phi$ $\quad p_j \to p_j - eA_j$

KG $\left[i\frac{\partial}{\partial t} - e\Phi\right]^2\Psi = \left\{\left[\frac{1}{i}\nabla - e\vec{A}\right]^2 + m^2\right\}\Psi$

Under Lorentz transf. of spinless particle

$\underset{\text{new}}{\Psi'(\vec{r}', t')} = \underset{\text{old}}{\Psi(\vec{r}, t)}$

$\vec{j} = \frac{1}{2m}\left[\Psi^*\left(\frac{1}{i}\nabla - e\vec{A}\right)\Psi + \Psi\left(-\frac{1}{i}\nabla - e\vec{A}\right)\Psi^*\right]$

$\rho = \frac{1}{2m}\left[\Psi^*\left(i\frac{\partial}{\partial t} - e\Phi\right)\Psi + \Psi\left(-i\frac{\partial}{\partial t} - e\Phi\right)\Psi^*\right]$

3-1 Neg. Energy and Antiparticle

Positive Energy

at rest $\psi = e^{-imt}$ boost frame, particle at $\vec{v}$, $\vec{p} = \gamma m \vec{v}$ $E_p = \gamma m c^2$

so $\psi'(\vec{r}', t') = e^{-imt} = e^{i(\vec{p}\cdot\vec{x}' - E_p t')}$ Comparing rest frame w/ boosted frame

we have $\rho = \frac{E_p}{m} \Rightarrow j = \frac{p}{m} = \underbrace{\frac{p}{E_p}}_{\vec{v}_{rel}} \rho(\vec{r}', t)$

Negative Energy.

at rest $\psi = e^{imt}$ boost frame, particle at $\vec{v}$ $\vec{p} = \gamma m \vec{v}$ $E_p = \gamma m c^2$

$\psi'(\vec{r}', t) = e^{imt} = e^{-i(\vec{p}\cdot\vec{x}' - E_p t')}$ thus particle of energy $-E_p$ & momentum $-\vec{p}$

$$\rho = -\frac{E_p}{m} \Rightarrow j = -\frac{p}{m} = \frac{p}{E_p}\rho(\vec{r}', t')$$

Remark antiparticle ~~flot~~ flow in opposite direction w/ opp. current.

note $\left[i\frac{\partial}{\partial t} - e\Phi\right]^2 \Phi = \left(\left[\frac{1}{i}\nabla - e\vec{A}\right]^2 + m^2\right)\Phi$

c.c $\left[i\frac{\partial}{\partial t} + e\Phi\right]^2 \Phi^* = \left(\left[\frac{1}{i}\nabla + e\vec{A}\right]^2 + m^2\right)\Phi^*$ Φ^* soln of K.G of opp sign charge!

charge conjugate

$$\rho_c = -\frac{1}{2m}\left[\Phi\left(i\frac{\partial}{\partial t} + e\Phi\right)\Phi^* + \Phi^*\left(-i\frac{\partial}{\partial t} + e\Phi\right)\Phi\right] = -\rho$$

current "

$$\vec{j}_c = -\vec{j}$$

First Order KG

Def $D_\mu = \partial_\mu + ieA_\mu$, $[D_\mu D^\mu + m^2]\Phi(\vec{x}, t) = 0$

$D_\mu D^\mu = D_t^2 - D^2$ then $[D_\mu D^\mu + m^2]\psi = 0 \Rightarrow (m^2 + D_t^2)\psi = D^2\psi$

rewrite as $m\left(1 - \frac{i}{m}D_t\right)\left(1 + \frac{i}{m}D_t\right)\psi = \frac{D^2}{m}\psi$

def $\phi = \frac{1}{2}\left[\Phi + \frac{i}{m}D_t\Phi\right]$

$\chi = \frac{1}{2}\left[\Phi - \frac{i}{m}D_t\Phi\right]$

then
$$iD_t\phi = -\frac{1}{2m}D^2(\phi + \chi) + m\phi$$
$$iD_t\chi = \frac{1}{2m}D^2(\phi + \chi) - m\chi$$

define $\Upsilon = \begin{bmatrix}\phi\\ \chi\end{bmatrix}$ $\quad \tau_i = \cancel{\psi}\sigma_i$ (Pauli Matrix)

KG $\quad iD_t\Upsilon = \left[-\frac{1}{2m}D^2(\tau_3+i\tau_2)+m\tau_3\right]\Upsilon$

$j^\mu = \frac{i}{2m}\left[\Psi^* D_t\Psi - (D_t\Psi)^*\Psi\right]$

$\rho = \frac{i}{2m}\left[\Psi^* D_t\Psi - (D_t\Psi)^*\Psi\right]$

w/ $\Psi = \phi+\chi \qquad D_t\Psi = \frac{m}{i}(\phi-\chi)$

So $\rho = \phi^*\phi - \chi^*\chi = \Upsilon^\dagger\tau_3\Upsilon$ prob. charge density b/w particles & antiparticles.

normalization. $\int d^3r\,\Upsilon^\dagger\tau_3\Upsilon = \pm 1 \Rightarrow \langle\Upsilon|\Upsilon'\rangle = \int d^3r\,\Upsilon^\dagger\tau_3\Upsilon'$

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realization of neg. energy in the compact form.

free particle $\Upsilon \sim e^{-iEt+i\vec{p}\cdot\vec{x}}$

$$iD_t\Upsilon = \left[-\frac{1}{2m}D^2(\tau_3+i\tau_2)+m\tau_3\right]\Upsilon \Rightarrow -E\Upsilon = \begin{array}{cc} a & b \\ \end{array}\begin{bmatrix}\frac{p^2}{2m}+m & \frac{p^2}{2m}\\ -\frac{p^2}{2m} & -(\frac{p^2}{2m}+m)\end{bmatrix}\Upsilon$$

then $\lambda^2 - a^2 + b^2 = 0 \Rightarrow \lambda = \pm E_p$

eig.vec.

$E = +E_p \quad \Upsilon = \frac{1}{2\sqrt{mE_p}}\begin{pmatrix}E_p+m\\ m-E_p\end{pmatrix}e^{-iE_pt+i\vec{p}\cdot\vec{x}}$ $\qquad$ slow motion amplitude $\sim \begin{pmatrix}1\\ -\frac{v^2}{4}\end{pmatrix}$

$E = -E_p \quad \Upsilon = \frac{1}{2\sqrt{mE_p}}\begin{pmatrix}m-E_p\\ E_p+m\end{pmatrix}e^{iE_pt+i\vec{p}\cdot\vec{x}}$ $\qquad \sim \begin{pmatrix}-\frac{v^2}{4}\\ 1\end{pmatrix}$

Check $\quad \rho = \Upsilon^\dagger\tau_3\Upsilon = \pm 1 \Leftrightarrow \pm E_p$

Particle at rest for $+E_p$, $\Upsilon \sim \begin{pmatrix}2m\\0\end{pmatrix} \Rightarrow \phi$ only

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Current interpretation

$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k$

$\cancel{\frac{\partial\rho}{\partial t} \Rightarrow \frac{\partial\Upsilon}{\partial t} = -\frac{i}{2m}\left[-\frac{i}{2}\right.}$ $\quad \frac{\partial\Upsilon}{\partial t} = \left[-\frac{i}{2m}\nabla^2(\tau_3 - i\tau_2)+im\tau_3\right]\Upsilon$

$\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{j} \Rightarrow \vec{j} = \frac{1}{i2m}\left[\Upsilon^\dagger(1+\tau_1)\nabla\Upsilon - (\nabla\Upsilon^\dagger)(1+\tau_1)\Upsilon\right]$

free particle $\vec{j} = \frac{p}{m}\Upsilon^\dagger(1+\tau_1)\Upsilon = \frac{p}{E_p}$ $\quad$ need sign correction to match non compact KG.

def of sign e then.

c.c. $\left[(\partial_\mu - ieA_\mu)(\partial^\mu - ieA_\mu)+m^2\right]\Psi$ $\qquad$ w/ +e

$\left[(\partial_\mu + ieA_\mu)(\partial^\mu + ieA_\mu)+m^2\right]\Psi^*$ $\qquad$ w/ −e $\quad e<0$

8.2 Conjugate in Compact form

$$i D_t \Upsilon = \left[-\frac{1}{2m} D^2 (\tau_3 + i\tau_2) + m\tau_3 \right] \Upsilon$$

solve eigval eigvect

recast

$$i \frac{\partial \Upsilon}{\partial t} = H \Upsilon \qquad H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 (\tau_3 + i\tau_2) + e\Phi + m\tau_3$$

not hermitian

then obs. $\Upsilon_c = \tau_1 \Upsilon^*$

note $\tau_3(\tau_3 + i\tau_2) = (\tau_3 + i\tau_2)^+ \tau_3 = 1 + \tau_1$

thus $\tau_3 H = H^+ \tau_3$ or $H = \tau_3 H^+ \tau_3$

~~under charge conjugate~~

from $\langle \Upsilon | \Upsilon' \rangle = \int d^3r\, \Upsilon^+ \tau_3 \Upsilon'$

H hermitian in following

$$\langle \Upsilon | (H | \Upsilon' \rangle) = (\langle \Upsilon | H^+) | \Upsilon'^+ \rangle \quad \text{or} \quad \langle \Upsilon | (H | \Upsilon' \rangle) = [\langle \Upsilon' | (H | \Upsilon \rangle)]^+$$

under charge conjugate $\varphi \to \chi^*$ $\chi \to \varphi^*$

b/c $\rho = |\varphi|^2 - |\chi|^2 = -\rho_c$ ⟵ note $\rho = \Upsilon^+ \tau_3 \Upsilon$ so $\int \rho\, d^3r = \pm 1$

so $\Upsilon_c = \tau_1 \Upsilon^*$

$\Rightarrow \int d^3r\, \Upsilon^+ \tau_3 \Upsilon = \pm 1$

claim Υ_c solves KG w/ opp. sign charge

$\langle \Upsilon | \Upsilon' \rangle$ $=$ $\int d^3r\, \Upsilon^+ \tau_3 \Upsilon'$

Pf

$$H^*(e) = \frac{1}{2m}(-\vec{p} - e\vec{A})^2 (\tau_3 + i\tau_2) + e\Phi + m\tau_3$$

$$\tau_1 H^*(e) \tau_1 = -H(-e) \qquad \text{b/c} \quad \{\tau_1, \tau_{2,3}\} = 0$$

$p^* = -p$

thus

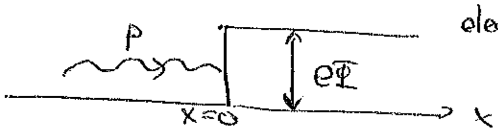
$$-i \frac{\partial \Upsilon^*}{\partial t} = H^*(e) \Upsilon^* = -\tau_1 H(-e) \tau_1 \Upsilon^*$$

by def

$$i \frac{\partial \Upsilon_c}{\partial t} = H(-e) \Upsilon_c$$

//

Klein's Paradox



electrostatic potential barrier.

For $x<0$ w.f $\psi(x) = a e^{ipx} + b e^{-ipx}$

For $x>0$ KG $(E_p - V)^2\psi(x) = -\frac{\partial^2\psi}{\partial x^2} + m^2\psi \Rightarrow \psi'' = -k^2\psi$

here $k^2 = (E_p - V)^2 - m^2$

w.f $\psi(x) = d e^{ikx}$ both w/ time dep. $e^{-iE_p t}$

B.C ψ, ψ' cont' at $x=0$ thus

$$\left.\begin{array}{l}\psi_<(0) = a+b \\ \psi_>(0) = d\end{array}\right\} a+b=d$$

$$\left.\begin{array}{l}\psi'_<(0) = ipa - ipb \\ \psi'_>(0) = ikd\end{array}\right\} pa-pb = kd$$

$\Rightarrow$ $b = \frac{p-k}{p+k}a$, $d = \frac{2p}{p+k}a$

Analysis. $k = \sqrt{(E_p - V + m)(E_p - V - m)}$

① $E_p > V+m$ part reflected part transmitted.

② $E_p - m < V < E_p + m$ ie $E_p - V + m > 0$ & $E_p - V - m < 0$

so $k = i\kappa$ w/ $\kappa = \sqrt{m^2 - (E_p - V)^2}$ so wave totally reflected at barrier

charge density $\rho(\vec{x},t) = \frac{1}{2m}\left[\psi^*(i\frac{\partial}{\partial t} - e\Phi)\psi + \psi(-i\frac{\partial}{\partial t} - e\Phi)\psi^*\right]$

$\rho_> = \frac{E_p - V}{m}|d|^2 e^{-2\kappa x}$

For $E_p > V$ $\rho > 0$ and decay to the right

$E_p < V$ $\rho < 0$ so has reflected + charge partl from wall find − charge partl w/in wall

③ if $V > E_p + m$, k real $\exists$ partl current on the right

$v_g = \frac{\partial E_p}{\partial k} = \frac{k}{E_p - V}$ $E_p - V < 0$ but wave moves to right so $k<0$

refl. coeff

$\frac{b}{a} = \frac{p-k}{p+k} > 1$ More wave reflected than incident!

$\rho = \frac{E_p - V}{m}|d|^2 < 0$ ie neg charge & current on right

explaination

Inc. wave created partl & antipartl at barrier w/ anti-partl on R.H.S w/ neg. current created partl on left. Thus net outgoing wave on left > inc. wave.
But total created current = inc. b/c charge conservation.