define translation. $\boxed{\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}'+d\vec{x}'\rangle}$

can write $\mathcal{J}(d\vec{x}')|\alpha\rangle = \int d^3x' |\vec{x}'+d\vec{x}'\rangle\langle \vec{x}'|\alpha\rangle$

$= \int d^3x' |\vec{x}'\rangle\langle \vec{x}'-d\vec{x}'|\alpha\rangle$ ⟲ by change of variable

thus wave function of the translated state:

$$\langle \vec{x}'|\mathcal{J}(d\vec{x}')|\alpha\rangle = \langle \vec{x}'-d\vec{x}'|\alpha\rangle$$

v.s

original state wave function

$$\langle \vec{x}'|\alpha\rangle$$

Structures of $\mathcal{J}(d\vec{x}')$

axioms

1. Unitary

i.e. $|\alpha'\rangle = \mathcal{J}(d\vec{x}')|\alpha\rangle$

then $\langle\alpha'|\alpha'\rangle = \langle\alpha|\alpha\rangle = 1$

$\Rightarrow \quad \mathcal{J}^\dagger(d\vec{x}')\mathcal{J}(d\vec{x}') = 1$

2. Composition.

$$\mathcal{J}(d\vec{x}'')\mathcal{J}(d\vec{x}') = \mathcal{J}(d\vec{x}'+d\vec{x}'')$$

3. Inverse

$$\mathcal{J}^{-1}(d\vec{x}') = \mathcal{J}(-d\vec{x}')$$

4. Identity

$$\lim_{d\vec{x}'\to 0}\mathcal{J}(d\vec{x}') = \underline{1}$$

⟲ conjectures that $\boxed{\mathcal{J}(d\vec{x}') = ~~\mathbb{1}~~ 1 - iK\cdot d\vec{x}'}$ where K hermitian operators

which satisfies 1, 2, 3, 4

5. $\boxed{[\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}'}$ "operator identity

states $-i\vec{x}\vec{K}\cdot d\vec{x}' + i\vec{K}\cdot d\vec{x}'\vec{x} = d\vec{x}'$

$-ix_iK_jdx_j + iK_jdx_jx_i = dx_i$

$[x_i, K_j]dx_j = i\,dx_i$

$[x_i, K_j] = i\left(\frac{dx_i}{dx_j}\right) = \delta_{ij}$

yields ⟶ $\boxed{[x_i, k_j] = i\delta_{ij}}$

⟲ $[x_i, p_j] = i\delta_{ij}\hbar$

thus $\langle\Delta x^2\rangle\langle\Delta p^2\rangle \geq \frac{\hbar^2}{4}$

de Broglie $p = \frac{h}{\lambda}$ $~~\hbar = \frac{h}{2\pi}~~$ $\hbar = \frac{h}{2\pi}$

$\Rightarrow k = \frac{2\pi}{\lambda}$

Continuous Spectrum

Position Eigenkets & Measurement

$$\boxed{X|x'\rangle = x'|x'\rangle \quad \& \quad \langle x'|x''\rangle = \delta(x'-x'')}$$

Given $|\alpha\rangle$ we have $\boxed{|\alpha\rangle = \int dx' |x'\rangle\langle x'|\alpha\rangle}$

where the probability : $\langle\alpha|\alpha\rangle = \int dx' |\langle x'|\alpha\rangle|^2$

Prob $= |\langle x'|\alpha\rangle|^2$ of measuring particle within $(x'-\frac{\Delta}{2}, x'+\frac{\Delta}{2})$.

and $\boxed{\int_{-\infty}^{\infty} dx' |\langle x'|\alpha\rangle|^2 = 1}$

as $|\alpha\rangle \xrightarrow{\text{in}} |a'\rangle$ w/ prob $|\langle a'|\alpha\rangle|^2$ & $\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1$

Generalized to 3D.

$$|\alpha\rangle = \int d^3x' |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle \quad \text{w/ } |\vec{x}'\rangle = |x', y', z'\rangle$$

$$X|\vec{x}'\rangle = x'|\vec{x}'\rangle$$
$$Y|\vec{x}'\rangle = y'|\vec{x}'\rangle$$
$$Z|\vec{x}'\rangle = z'|\vec{x}'\rangle$$

$$\boxed{[X_i, X_j] = 0}$$

In practice, detector clicks when ~~a~~ a particle is seen within $(x'-\frac{\Delta}{2}, x'+\frac{\Delta}{2})$. Thus state ket changes suddenly as

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx''|x''\rangle\underbrace{\langle x''|\alpha\rangle} \xrightarrow{\text{measurement}} \int_{x'-\frac{\Delta}{2}}^{x'+\frac{\Delta}{2}} dx''|x''\rangle\langle x''|\alpha\rangle$$

also probability amplitude — $\psi_\alpha(x')$

$\langle x'|\alpha\rangle$ is a wave function corresponding to $|\alpha\rangle$

thus,

$$\langle\beta|\alpha\rangle = \int dx' \langle\beta|x'\rangle\langle x'|\alpha\rangle$$
$$= \int dx' \psi_\beta^*(x')\psi_\alpha(x') \implies$$

represents probability amplitude for $|\alpha\rangle$ to be found in $|\beta\rangle$

Position Space Wave Function

Given A (operator) w/ $A|a'\rangle = a'|a'\rangle$
and for any eigenket

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$\psi_\alpha(x') = \langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle = \sum_{a'} c_{a'} \underbrace{u_{a'}(x')}_{\text{eigenfunc. of operator } A}$$

$$\boxed{u_{a'}(x') = \langle x'|a'\rangle}$$

$$\langle\beta|A|\alpha\rangle = \int dx' \int dx'' \psi_\beta^*(x')\langle x'|A|x''\rangle \psi_\alpha(x'')$$

if $A \to f(x)$ then $\boxed{\langle\beta|f(x)|\alpha\rangle = \int dx' \psi_\beta^*(x') f(x') \psi_\alpha(x')}$ $(*)$

~~Mome~~ Momentum operator

$$\underbrace{\left(1 - \frac{ip\Delta x}{\hbar}\right)}_{\mathcal{J}(\Delta x)} |\alpha\rangle = \int dx' |x'\rangle\langle x' - \Delta x|\alpha\rangle$$

k or $\frac{p}{\hbar}$ as generator for translation.

Note
$\langle x' - \Delta x'|\alpha\rangle \equiv f_\alpha(x' - \Delta x')$
Taylor,
$f_\alpha(x' - \Delta x') = f_\alpha(x') - \frac{\partial f_\alpha(x')}{\partial x'}\Delta x'$
$\langle x' - \Delta x'|\alpha\rangle = \langle x'|\alpha\rangle - \Delta x' \frac{\partial \langle x'|\alpha\rangle}{\partial x'}$

$$\Rightarrow \quad p|\alpha\rangle = \int dx' |x'\rangle\left(-i\hbar\frac{\partial}{\partial x'}\right)\langle x'|\alpha\rangle$$

$$\langle\alpha|p|\alpha\rangle = \int dx' \psi_\alpha^*(x')\left(-i\hbar\frac{\partial}{\partial x'}\right)\psi_\alpha(x')$$

from $(*)$

$$\boxed{p = -i\hbar\frac{\partial}{\partial x}} \quad \text{or} \quad \boxed{\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x'|\alpha\rangle}$$

Matrix element of p in x-representation,

$$\boxed{\langle x'|p|x''\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x' - x'').}$$

Clearly, $\boxed{\langle x'|p^n|\alpha\rangle = (-i\hbar)^n\frac{\partial^n}{\partial x'^n}\langle x'|\alpha\rangle}$

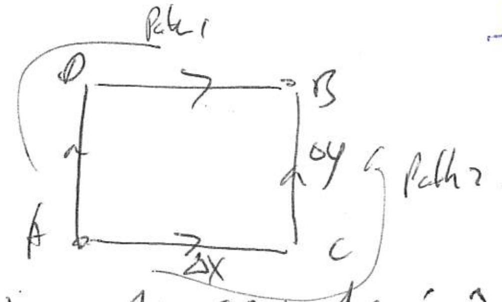
i.e. $\langle x'|p^2|\alpha\rangle = \int dx'' \langle x'|p|x''\rangle\langle x''|p|\alpha\rangle.$

translation

$\cancel{\mathcal{J}(\Delta\vec{x}) = \mathcal{J}\left(\frac{\Delta\vec{x}}{N}\right)^N = \mathcal{J}}$

$$\mathcal{J}(\Delta x\,\hat{x}) = \mathcal{J}^N\left(\frac{\Delta x}{N}\hat{x}\right) = \left(1 - i\frac{p\,\Delta x}{\hbar N}\right)^N \underset{N\to\infty}{=} \boxed{\exp\left(-i\frac{p\Delta x}{\hbar}\right)}$$

~~Now con~~ Now w/



$$\mathcal{J}(\Delta y'\hat{y})\,\mathcal{J}(\Delta x'\hat{x}) = \mathcal{J}(\Delta x'\hat{x})\,\mathcal{J}(\Delta y'\hat{y}) = \mathcal{J}(\Delta x'\hat{x} + \Delta y'\hat{y})$$

By investigate

$$[\mathcal{J}(\Delta y'\hat{y}), \mathcal{J}(\Delta x'\hat{x})] \approx \frac{(\Delta x')(\Delta y')}{\hbar^2}[p_x, p_y]$$

$$\rightarrow [p_i, p_j] = 0.$$

Collected Canonical ~~Re~~ Commutation :*

$$\boxed{\begin{aligned} [x_i, x_j] &= 0 \\ [p_i, p_j] &= 0 \\ [x_i, p_j] &= i\hbar\,\delta_{ij} \end{aligned}}$$

effect of $\mathcal{J}(d\vec{x}')$ on $|\vec{p}'\rangle$ momentum eigenket.

$$\mathcal{J}(d\vec{x}')|\vec{p}'\rangle = \left(1 - i\frac{\vec{P}\cdot d\vec{x}'}{\hbar}\right)|\vec{p}'\rangle = \left(1 - \frac{i\vec{p}'\cdot d\vec{x}'}{\hbar}\right)|\vec{p}'\rangle$$

v.s

$$\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle.$$

* $\cancel{[A+B, C] =}$

$[A, BC] = [A, B] + B[A, C]$ Product rule treat $[A, \cdot]$ as derivative.

$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ Jacobi Identity

Momentum Space Wave Function

in P-basis $P|p'\rangle = p'|p'\rangle$.

$\langle p'|p''\rangle = \delta(p'-p'')$ and. $\langle p'|\alpha\rangle = \phi_\alpha(p')$ momentum space wave func.

$|\alpha\rangle = \int dp' |p'\rangle\langle p'|\alpha\rangle$ w/ $\int dp' |\phi_\alpha(p')|^2 = 1$

~~Now~~ take

$$\underbrace{\langle x'|P|p'\rangle}_{\substack{" \\ p'\langle x'|p'\rangle}} = -i\hbar\frac{\partial}{\partial x}\langle x'|p'\rangle \Rightarrow \boxed{\langle x'|p'\rangle = N\exp\left(\frac{ip'x'}{\hbar}\right)}$$

$\Rightarrow p' = -i\hbar\frac{\partial}{\partial x}$

wave func. of ~~momentum eig. state~~ $|p'\rangle$

$N = \frac{1}{\sqrt{2\pi\hbar}}$ from $\langle x'|x''\rangle$ calculation.

thus,

$$\boxed{\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)}$$

Now.

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle$$

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$\boxed{\begin{aligned}\psi_\alpha(x') &= \int dp' \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)\phi_\alpha(p') \\ \phi_\alpha(p') &= \int dx' \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{ip'x'}{\hbar}\right)\psi_\alpha(x')\end{aligned}}$$

define fourier transform from x' to p' space

Gaussian Integrals:

$$\boxed{\begin{aligned}\int_0^\infty x^{2n} e^{-\alpha x^2} dx &= \frac{(2n-1)!!}{\alpha^n 2^{n+1}}\sqrt{\frac{\pi}{\alpha}} \\ \int_0^\infty x^{2n+1} e^{-\alpha x^2} dx &= \frac{n!}{2\alpha^{n+1}}\end{aligned}}$$

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-a)}\,dp.$$