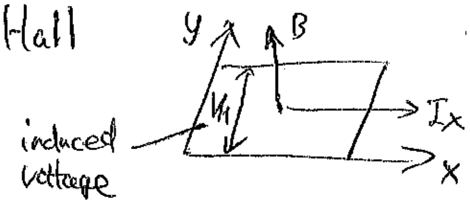
1 Quantum Hall

Magnetic Scale

Cyclotron Freq.	$\omega_B = \frac{eB}{M}$	
Magnetic Length	$l_B = \sqrt{\frac{\hbar}{eB}}$	
Quantum Flux	$\Phi_0 = \frac{2\pi\hbar}{e}$	
Hall Resistivity	$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$	
Hall Conductivity	$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu$	$\nu \in \mathbb{Z}$ $\nu \in \mathbb{Q}$

About: Quantum hall effect is the quantization of an emergent macroscopic property in a dirty system of many particles.

Classical Hall



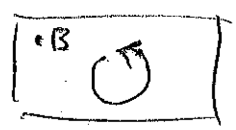
has classical motion

$$m\frac{d\vec{v}}{dt} = -e\vec{v}\times\vec{B}$$

Given $\vec{B} = (0,0,B)$, $\vec{v} = (\dot{x},\dot{y},0)$

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}0 & B\\ -B & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}.$$

cyclotron motion



$$x(t) = X - R\sin(\omega_B t + \phi)$$
$$y(t) = Y + R\cos(\omega_B t + \phi)$$

Drude Model (still Classical) to explore behavior of ρ_{xy} ρ_{xx}

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - e\vec{v}\times\vec{B} - \frac{m\vec{v}}{\underbrace{\tau}_{\text{scattering time}}}$$

friction due to impurity/collision

at equilibrium,

$$\vec{v} + \frac{e\tau}{m}\vec{v}\times\vec{B} = -\frac{e\tau}{m}\vec{E}$$

take again $\vec{v} = (v_x, v_y, 0)$, $\vec{B} = (0,0,B)$ then

$$\begin{pmatrix}1 & \omega_B\tau\\ -\omega_B\tau & 1\end{pmatrix}\vec{v} = -\frac{e\tau}{m}\vec{E}$$

For $\vec{J} = -ne V$

$$\begin{pmatrix}1 & \omega_B\tau\\ -\omega_B\tau & 1\end{pmatrix}\vec{J} = \frac{e^2 n\tau}{m}\vec{E}$$

Now we have $\vec{J} = \sigma\vec{E}$ w/ $\sigma = \begin{pmatrix}\sigma_{xx} & \sigma_{xy}\\ -\sigma_{xy} & \sigma_{yx}\end{pmatrix}$ conductivity tensor

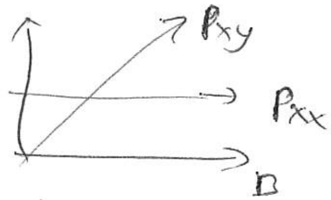
$$= \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{pmatrix}1 & -\omega_B\tau\\ \omega_B\tau & 1\end{pmatrix} \qquad \sigma_{DC} = \frac{ne^2\tau}{m}$$

When $B=0$, $\sigma = \begin{pmatrix}\sigma_{DC} & 0\\ 0 & \sigma_{DC}\end{pmatrix} \Rightarrow$ off diag. describes Hall effect.

Resistivity $\rho = \frac{1}{\sigma} = \frac{1}{\sigma_{DC}}\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}$ w/ $\rho_{xx} = \frac{m}{ne^2\tau}$ $\rho_{xy} = \frac{B}{ne}$

note that ρ_{xy} indept of scattering time!

we have picture of



In sense of actual measurement, we have resistance.

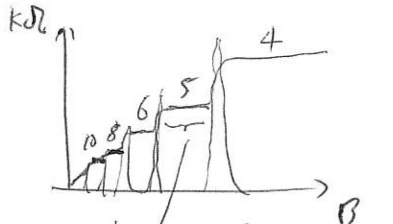
$$R_{xy} = \frac{V_y}{I_x} = \frac{L E_y}{L J_x} = \frac{E_y}{J_x} = -\rho_{xy}$$

Also define Hall coefficient.

$$R_H = -\frac{E_y}{J_x B} = \frac{\rho_{xy}}{B} \overset{\text{Drude Model}}{=} \frac{\omega_B}{B\sigma_{DC}} = \frac{1}{ne}$$

so depends on micro info of material.

Quantum Hall Effect

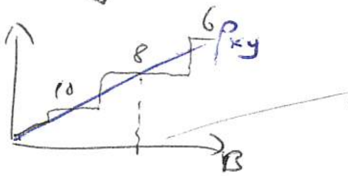


plateaux of B field

$$\rho_{xy} = \left(\frac{2\pi\hbar}{e^2}\right)\frac{1}{\nu} \qquad \nu \in \mathbb{Z}$$

quantum of resistivity / Von Klitzing const

Overlay classical result i.e. from Drude Model w/ quantum effect.



center of plateaux implies.

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} = \frac{B}{ne} \Rightarrow B = \frac{n}{\nu}\Phi_0$$

But how to justify this?

Remark More plateaux emerge as disorder in sample decreases

Landau Level

Ignore spin due to following.

Zeeman splitting due to spin B-field interaction s.t

$\Delta = 2\mu_B B$ $\quad \mu_B = \frac{e\hbar}{2m}$ (Bohr magneton)

(diff. due to flipping of up to down spin. or vice versa.

Recalled gauge transf. $\vec{A} \to \vec{A} + \nabla\alpha$ then $L \to L + \dot{\alpha}$ ← total derivative

so EOM gauge inv.

Quantization

Lagrangian $\quad L = \frac{1}{2}m\dot{\vec{x}}^2 - e(\dot{\vec{x}} \times \vec{B}) \cdot \vec{x}$

$$L = \frac{1}{2}m\dot{\vec{x}}^2 - e\dot{\vec{x}} \cdot \vec{A}$$

Can. mom. $\quad \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} - e\vec{A}$

$$H = \dot{\vec{x}}\frac{\partial L}{\partial \dot{\vec{x}}} - L = \frac{1}{2}m\dot{\vec{x}}^2$$

$$H = \frac{1}{2}m(\vec{p} + e\vec{A})^2$$

Define $\quad \vec{\Pi} = \vec{p} + e\vec{A}$

Remark
- $\vec{p}$ not gauge inv. (unphysical)
- $\dot{\vec{x}}$ gauge inv. (physical)
- but $\dot{\vec{x}}$ no canonical poisson structure

i.e $\{m\dot{x}_i, m\dot{x}_j\} = -e\,\varepsilon_{ijk}B_k$

note x, p has canonical relations.

$\{x_i, p_i\} = \delta_{ij}$ $\quad \{x_i, x_j\} = \{p_i, p_j\} = 0$

$\{\Pi_i, \Pi_j\} = -e\,\varepsilon_{ijk}B_k$

Also

$[x_i, p_j] = i\hbar\delta_{ij}$ $\quad [x_i, x_j] = [p_i, p_j] = 0$

$[\Pi_i, \Pi_j] = -ie\hbar B$

H has form of SHO, we done this b4 in QM w/ quantization.

$$a = \frac{1}{\sqrt{2e\hbar B}}(\Pi_x - i\Pi_y) \qquad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\Pi_x + i\Pi_y)$$

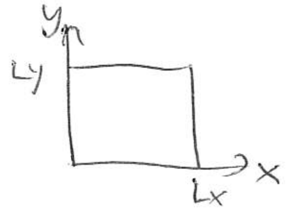
and $\quad [a, a^\dagger] = 1 \Rightarrow H = \frac{1}{2m}\vec{\Pi}\cdot\vec{\Pi} = \hbar\omega_B(a^\dagger a + \frac{1}{2})$

where $\quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$E_n = \hbar\omega_B(n + \tfrac{1}{2}) \quad n \in \mathbb{N}$$

Ways of Counting Degeneracy

① (Regularize wf in a box and count num. of states can fit into)



$\hat{y}$ periodic so $k = \frac{2\pi}{L_y}$

$\hat{x}$ not trivial

Heuristically, recalled num. of states:

$$N = \frac{V_{k_F}}{V_k \text{ per state}} = \frac{\frac{1}{8}\frac{4}{3}\pi k^3}{\pi^3/L^3} = \frac{Vk^3}{6\pi^2}$$

(V is vol. here)

we have for $x = -kl_B^2 \Rightarrow k = -\frac{x}{l_B^2}$ so varying x from $-L_x$ to 0 $(k>0)$

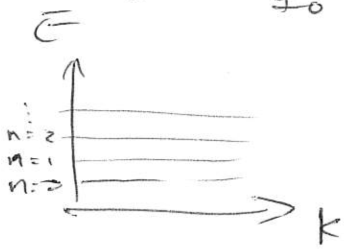
then

$$\mathcal{N} = \underbrace{\frac{L_y}{2\pi}}_{\text{taken unit vol.}} \int_{\underbrace{-\frac{L_x}{l_B^2}}_{\text{counting } \# \, k_x \text{ in } (-L_x,\, 0]}}^{0} dk_y = \frac{L_x L_y}{2\pi l_B^2} = \frac{BA}{\Phi_0}$$

$A - L_x L_y$

~~gap~~ ground state degen?

$$\mathcal{N} = \frac{BA}{\Phi_0}$$



Note
- E depends on n corresponding to w.f. $\psi_{n,k}$
- By comparing $\frac{L_x L_y}{2\pi l_B^2} = \frac{BA}{\Phi_0} \Rightarrow \Phi_0 = B(2\pi l_B^2)$ thus as if B-flux thru area of $2\pi l_B^2$.

Quest for w.f

Given gauge potential $\vec{A}$ required that $\nabla \times \vec{A} = B\hat{z}$.
We have two typical choice

- Landau Gauge $\vec{A} = xB\hat{y}$ here $\vec{B}$ inv. under rotation & translation. by requirement
But $\vec{A}$ is not transl. inv.

then $H = \frac{1}{2m}\left[P_x^2 + (P_y + eBx)^2\right]$ — $H = \frac{1}{2m}(\vec{P} + e\vec{A})^2$

$(\vec{P} + e\vec{A}) = (P_x, P_y + eBx)$

↳ eig. state of P_y has assoc. good Q.num.

So plane wave in P_y direction.

Ansatz, $\psi_k(x, y) = e^{iky} f_k(x)$

& $H\psi_k = \frac{1}{2m}(P_x^2 + (\hbar k + eBx)^2)\psi_k = H_k \psi_k(x, y)$

rewrite in SHO,

$$H_k = \frac{1}{2m}P_x^2 + \frac{eB}{2m}\left(x + \frac{\hbar k}{eB}\right)^2$$

char. length scale gov. quantum phens. In magnetic field

$$H_k = \frac{1}{2m}P_x^2 + \frac{1}{2}m\omega_B^2(x + kl_B^2)^2$$

harmonic oscillator centers at $-kl_B^2$

Since P_x, X remains canonical, so $E_n = \hbar\omega_B(n + \frac{1}{2})$
So explicit w.f depends on Q.num $n \in \mathbb{N}$ & $k \in \mathbb{R}$

$$\psi_{n,k}(x, y) \sim e^{iky} H_n(x + kl_B^2) e^{-\frac{(x + kl_B)^2}{2l_B^2}}$$

Turning on E-field $H = \frac{1}{2m}(P_x^2 + (P_y + eBx)^2) - eEx$
completing square yields.

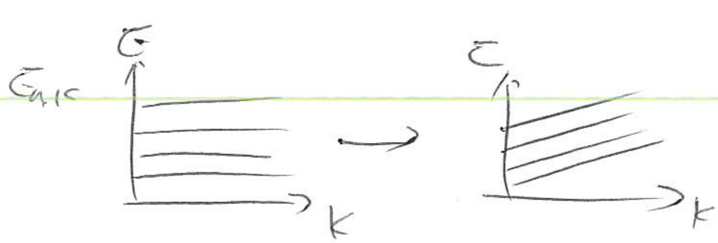
$$H = \frac{1}{2}m\omega_B^2\left[x + kl_B^2 - \frac{eE}{m\omega_B^2}\right]^2 + kl_B^2 eE - \frac{1}{2}m\frac{e^2E^2}{m^2\omega_B^2}$$

$$E_{n,k} = \hbar\omega_B(n + \tfrac{1}{2}) + \underbrace{eE\left(kl_B^2 - \frac{eE}{m\omega_B^2}\right)}_{\text{potential}} + \underbrace{\frac{m}{2}\frac{E^2}{B^2}}_{\text{kinetic}}$$

Previously we have $\psi(x, y) \sim e^{iky} H_n(x + kl_B^2) e^{-\frac{(x + kl_B^2)}{2l_B^2}} := \psi_{n,k}(x, y)$

now $\psi_{n,k}(x, y) \longrightarrow \psi_{n,k}\left(x - \frac{eE}{m\omega_B^2}, y\right)$

Eig. val



lifted as k increases.

Recalled k tagged in $\hat{y}$ direction.

energy is now k dependent $\Rightarrow$ states is drifted in $\hat{y}$ w/
group vel $V_y = \frac{1}{\hbar}\frac{\partial E_{n,k}}{\partial k} = \frac{eE l_B^2}{\hbar} = \frac{E}{B}$ w/ wavefunc localized at $-k l_B^2 - \frac{eE}{m\omega_B^2}$.

- Sym Gauge.

$\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B} = \frac{-yB}{2}\hat{x} - \frac{xB}{2}\hat{y}$ only preserved rot. sym $\Rightarrow$ ang. mom good q. num.

Remark convenient gauge in describing fractional QHall.

Recalled $\vec{\pi} = \vec{P} + e\vec{A}$ gauge invariant but not canonical.

Counting degeneracy for sym. gauge.

Introduce $\tilde{\pi} = \vec{P} - e\vec{A}$ not gauge inv.

w/ commutation relations: $[\tilde{\pi}_i, \tilde{\pi}_j] = ie\hbar\varepsilon_{ijk}B_k$ (*)

$[\pi_x, \tilde{\pi}_x] = 2ie\hbar\frac{\partial A_x}{\partial x}$ $\quad [\pi_x, \tilde{\pi}_y] = [\pi_y, \tilde{\pi}_x] = ie\hbar\left(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\right)$ $\quad [\pi_y, \tilde{\pi}_y] = 2ie\hbar\frac{\partial A_y}{\partial y}$.

not necessary zero. But for sym gauge, all ~~to~~ zeros. $[\pi_i, \tilde{\pi}_j] = 0$.

w/ (*) we have new sets of lowering and raising op.

$b = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \quad b^+ = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y)$ w/ $[b, b^+] = 1$

but since sym. gauge yields $[\pi_i, \tilde{\pi}_j] = 0$, we can define ground state
$|0,0\rangle$ w/ $a|0,0\rangle = 0 \quad b|0,0\rangle = 0$
w/ general Hilbert space $|n,m\rangle = \frac{a^{+n}b^{+m}}{\sqrt{n!\,m!}}|0,0\rangle$

Corresponding to energy eigval.

$E_n = \hbar\omega_B(n + \frac{1}{2})$ n dependence only.

Lowest Landau Level

Consider lowering op. a

$$a = \frac{1}{\sqrt{2e\hbar B}}(\Pi_x - i\Pi_y) = \frac{1}{\sqrt{2e\hbar B}}\left(P_x - iP_y + e(A_x - iA_y)\right) = \frac{1}{\sqrt{2e\hbar B}}\left(-i\hbar(\partial_x - i\partial_y) + \frac{eB}{2}(-y - ix)\right)$$

observation suggesting definitions:

$$z = x - iy \qquad \partial = \tfrac{1}{2}(\partial_x + i\partial_y) \qquad \text{s.t.} \quad \partial z = \bar\partial \bar z = 1$$

$$\bar z = x + iy \qquad \bar\partial = \tfrac{1}{2}(\partial_x - i\partial_y) \qquad \partial \bar z = \bar\partial z = 0$$

so $a = -i\sqrt{2}\left(l_B \bar\partial + \frac{z}{4l_B}\right)$ since $\frac{\bar\partial \psi_{LL}}{\partial \bar z} = -\frac{z}{4l_B^2}\psi_{LL}$ $\quad \bar\partial \bar z = \partial z = 1$

$a^\dagger = -i\sqrt{2}\left(l_B \partial - \frac{\bar z}{4l_B}\right)$

$$\ln \psi_{LL} = -\frac{z\bar z}{4l_B^2} + g(z)$$

$$\psi_{LL} = f(z) = e^{-|z|^2/4l_B^2}$$

thus $\psi_{LL}(z,\bar z) = f(z)e^{-|z|^2/4l_B^2}$ where $f(z)$ holomorphic.

Similarly $b = -i\sqrt{2}\left(l_B\partial + \frac{\bar z}{4l_B}\right)$. $b^\dagger = -i\sqrt{2}\left(l_B\bar\partial - \frac{z}{4l_B}\right)$

try $\frac{\partial \psi_{LL,m=0}}{\partial z} = -\frac{\bar z}{4l_B^2}\psi_{LL,m=0} \Rightarrow \psi_{LL,m=0} \sim e^{-|z|^2/4l_B^2}$

now $\psi_{LL,m} = f(z)e^{-|z|^2/4l_B^2}$ $\quad \psi_{LL,m=0} \sim e^{-|z|^2/4l_B^2} \Rightarrow f(z) \sim$ order of m?

try $\psi_{LL,m=1} = b^\dagger \psi_{LL,m=0} \sim i\sqrt{2}\,\frac{z}{2l_B}e^{-\frac{z\bar z}{4l_B^2}}$ $\qquad b^{\dagger 2}\psi_{LL,m=0} = (i\sqrt{2})^2\left(\frac{z}{2l_B}\right)^2 e^{-\frac{z\bar z}{4l_B^2}}$

thus $\underline{\psi_{LL,m} \sim (i\sqrt{2})^m\left(\frac{z}{2l_B}\right)^m e^{-\frac{z\bar z}{4l_B^2}}}$

$\psi_{LL,m}$ also eig. state of ang. mom.!!

Check $\vec J = \vec x \times \vec p = -i\hbar(x\partial_y - y\partial_x)$

$$z\partial = \tfrac{1}{2}(x\partial_x + ix\partial_y - iy\partial_x + y\partial_y)$$

$$\bar z\bar\partial = \tfrac{1}{2}(x\partial_x - ix\partial_y + iy\partial_x + y\partial_y)$$

so $z\partial - \bar z\bar\partial = i(x\partial_y - y\partial_x)$ and $\vec J = -\hbar(z\partial - \bar z\bar\partial)$

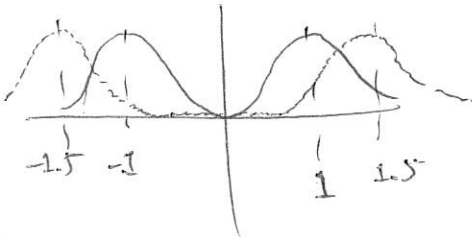
then $J\psi_{LL,m} \sim J(i\sqrt{2})^m\left(\frac{z}{2l_B}\right)^m e^{-z\bar z/4l_B^2} = \hbar m\psi_{LL,m}$

$$J\psi_{LL,m} = \hbar m\,\psi_{LL,m} \quad \text{as stated}$$

Recalled $\psi_{LLL,m} \sim \left(\frac{z}{2l_B}\right)^m e^{-z\bar{z}/4l_B^2}$

draw in wolfram,

$x^2 e^{-x^2}$



note

- $\psi_{LLL,m}$ forms concentric rings
- as ang. mom. m increases, radius of ring increases
- radius of the ring?

From above $\langle \psi_{LLL,m} | r | \psi_{LLL,m} \rangle = 0$ due to sym.

need instead

$$\langle \psi_{LLL,m} | r^2 | \psi_{LLL,m} \rangle \sim 2m l_B^2 \Rightarrow r \sim \sqrt{2m}\, l_B$$

didn't quite get what it claims.

now in disc shape region πR^2, no. of state.

$$\mathcal{N} = \frac{\pi R^2}{2\pi l_B^2} = \frac{A}{2\pi l_B^2} = \frac{eBA}{2\pi\hbar} = \frac{BA}{\Phi_0} \quad \text{as in Landau gauge.}$$

Connection to Classical physics

$x(t) = X - R\sin(\omega_B t + \phi) \quad y(t) = Y + R\cos(\omega_B t + \phi)$

rewrite as $X = x - \frac{\dot{y}}{\omega_B} = x - \frac{\pi_y}{m\omega_B} \qquad Y = y + \frac{\dot{x}}{\omega_B} = y + \frac{\pi_x}{m\omega_B}$

Recalled $H = \frac{1}{2m}\vec{\pi}\cdot\vec{\pi}$ so $[X, H] = [x - \frac{\pi_y}{m\omega_B}, \pi_x\pi_x + \pi_y\pi_y] = 2i\hbar\,[\pi_x - \pi_x] = 0$

note $[\pi_x, \pi_y] = -ie\hbar B, \quad [x, \pi_x] = [x, p_x] = i\hbar$

So $i\hbar\dot{X} = [X, H] = 0$ & $i\hbar\dot{Y} = [Y, H] = 0 \Rightarrow$ const. motion.

For sym gauge $\vec{A} = -\frac{yB}{2}\hat{x} + \frac{xB}{2}\hat{y}$

$X = x - \frac{\pi_y}{eB} = \frac{1}{eB}(2eA_y - \pi_y) = -\frac{\tilde{\pi}_y}{eB}$

$Y = y + \frac{\pi_x}{eB} = \frac{\tilde{\pi}_x}{eB}$

Recalled $\pi_x = p_x + eA_x \qquad 2eA_x = \pi_x - \tilde{\pi}_x$

$\tilde{\pi}_x = p_x - eA_x$

Remarks in sym. gauge, the alternative mom $\tilde{\pi}_{x,y}$ can be seen as center of orbits $[X, Y]$

Check $[X, Y] = [x, \frac{\pi_x}{m\omega_B}] + [-\frac{\pi_y}{m\omega_B}, y] + [-\frac{\pi_y}{m\omega_B}, \frac{\pi_x}{m\omega_B}] = \frac{2i\hbar}{eB} - \frac{i\hbar}{eB} = i l_B^2$

From Heisenberg principle $\Delta X \Delta Y \sim 2\pi l_B^2$

a naive semi-classical counting of states $\mathcal{N} = \frac{A}{\Delta X \Delta Y} \simeq \frac{eBA}{2\pi\hbar}$ as before!

Berry Phase and Berry Connection

Given general Hamiltonian $H(\underbrace{x^\alpha}_{\text{DoF evolves over time i.e. position, spin}}; \overbrace{\lambda^i}^{\text{due to external apparatus}})$

For given λ, assume sys. sitting on ground state $|\psi\rangle$
By varying λ slowly, H changes so as $|\psi\rangle \to |\psi(\lambda(t))\rangle$

Adiabatic Thm in QM.

If sys. placed in non-degen $|\psi\rangle$ energy state;
If λ varying slowly in a closed path, then H changes but $|\psi\rangle$ remains the same.
caveat: How slow depends on energy gap between current states and the nearest states.
so $|\psi\rangle \to e^{i\gamma}|\psi\rangle$ — phase diff.

<u>Remark</u> $e^{i\gamma}$ has two contributions ① dynamical phase $e^{-\frac{iEt}{\hbar}}$
② Berry phase.

Computing Berry Phase

$i\hbar \frac{\partial|\psi\rangle}{\partial t} = H(\lambda(t))|\psi\rangle$ for some ref. state $|n(\lambda)\rangle$ — ground state $\Leftrightarrow \lambda$

Ground state $|\psi(t)\rangle$ obeys adiabatic thm thus.

$$|\psi(t)\rangle = U(t)|n(\lambda)\rangle$$
time dep. phase

so for $t=0$ s.t. $|\psi(0)\rangle = U(0)|n(\lambda(0))\rangle \Rightarrow |\psi(0)\rangle = |n(\lambda(0))\rangle$
w/ $U(0) = 1$.

Quest for Berry Phase

Recall $|\psi(t)\rangle = U(t)|n(\lambda(t))\rangle$ then $|\dot\psi\rangle = \dot U|n\rangle + U|\dot n\rangle$ note $[H, U(t)] = 0$

$\langle\psi|\dot\psi\rangle = \langle n|U^\dagger \dot U|n\rangle + \langle n|\dot n\rangle = 0$ b/c $H|n\rangle = 0$

then $U^\dagger \dot U = -\langle n|\dot n\rangle$ w/ $\langle n|\dot n\rangle = \langle n|\partial_{\lambda_i}|n\rangle\dot\lambda$

where we define Berry connection $A^i(t) = -i\langle n|\frac{\partial}{\partial\lambda_i}|n\rangle$

From which we have $e^{i\gamma} = U(t) = \exp\left(-i\oint_C A^i(t)\,d\lambda_i\right)$ $i = 1, 2, \ldots, d$.

Berry phase γ can be computed taken a closed path.

Recalled Gauge transf. $A_\mu \to A'_\mu = A_\mu + \partial_\mu \omega$

- $A_\mu(x)$ is one-form over Minkowski space
- $\mathcal{A}^i(x)$ is one-form over parameter space λ_i

Now for any func. $\omega(\lambda)$ for every choice of λ we required that ω goes back to a fixed pt after a closed path.

It can have now $|n'(\lambda)\rangle = e^{i\omega(\lambda)}|n(\lambda)\rangle$ s.t

the Berry connection is now:

$$\mathcal{A}'_i = -i\langle n'|\frac{\partial}{\partial\lambda_i}|n'\rangle \qquad \text{where } \frac{\partial}{\partial\lambda_i}|n'\rangle = -i\frac{\partial\omega}{\partial\lambda}|n\rangle + e^{i\omega(\lambda)}\frac{\partial}{\partial\lambda_i}|n\rangle$$

As gauge transf, $\mathcal{A}_i \to \mathcal{A}_i + \frac{\partial\omega}{\partial\lambda^i}$ so $\oint \partial_i\omega\, d\lambda^i = 0$

$\mathcal{A}_i$ kinda gauge inv.

In high dim. stokes $F_{ij} = \varepsilon_{ijk}\varepsilon_{k\ell m}\partial_\ell A_m$

Consider $$\int_S F_{ij}\,ds^{ij} = \varepsilon_{ijk}\int_S(\varepsilon_{k\ell m}\partial_\ell A_m)\,ds^{ij} = \oint_C A_k\,d\lambda^k$$

so $$e^{i\gamma} = \exp\left(-i\oint_C \vec{A}(\lambda)\cdot d\vec{\lambda}\right) = \exp\left(-i\int F_{ij}\,ds^{ij}\right)$$

as in EM. $F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$ we have connection curvature $F_{ij}(\lambda) = \frac{\partial A_i}{\partial\lambda^j} - \frac{\partial A_j}{\partial\lambda^i}$

4-vector Field Strength Tensor in EM (review)

$$\left.\begin{aligned} A^\alpha &= (\Phi, \vec{A}) \\ \partial^\alpha &= \left(\frac{\partial}{\partial x^0}, -\nabla\right) \\ \partial_\alpha &= \left(\frac{\partial}{\partial x^0}, \nabla\right) \end{aligned}\right\} \partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla\cdot\vec{A} \qquad \Box = \partial_\alpha\partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2$$

$$J^\alpha = (c\rho, \vec{J}) \Rightarrow \partial_\alpha J^\alpha = \frac{\partial \rho}{\partial x^0} + \nabla\cdot\vec{J} = 0$$

Recall $\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \Rightarrow \Box A^\alpha = \frac{4\pi}{c}\vec{J}$

Lorentz cond. $\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \nabla\cdot\vec{A} = 0 \Rightarrow \partial_\alpha A^\alpha = 0$

$$\left.\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\Phi \\ \vec{B} &= \nabla\times\vec{A} \end{aligned}\right\} \Rightarrow F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Now $\tilde{F}^{\alpha\beta}$ dual of $F^{\alpha\beta}$, $\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ w/ $\vec{B}\to-\vec{E}$, $\vec{E}\to\vec{B}$

So $$\boxed{\left.\begin{aligned} \nabla\cdot\vec{E} &= 4\pi\rho \\ \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &= \frac{4\pi}{c}\vec{J} \end{aligned}\right\} \Rightarrow \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta} \quad \boxed{\left.\begin{aligned} &\nabla\cdot\vec{B} \\ &\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \end{aligned}\right\} \Rightarrow \partial_\alpha\tilde{F}^{\alpha\beta} = 0}$$

Also $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

Example. Spin in Mag. Field

$$H = \vec{B}\cdot\vec{\sigma} + B \quad \text{s.t.} \quad H|\uparrow\rangle = 2B|\uparrow\rangle \ \& \ H|\downarrow\rangle = 0|\downarrow\rangle$$

Given arbitrary direction of $\vec{B} = \begin{pmatrix} B\sin\theta\cos\phi \\ B\sin\theta\sin\phi \\ B\cos\theta \end{pmatrix}$

s.t. $|\uparrow\rangle = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$ $|\downarrow\rangle = \begin{pmatrix} e^{-i\phi}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$ w/ parameter θ, ϕ

So $F_{\theta\phi} = \frac{\partial A_\phi}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi} = -\sin\theta$ where $A_\theta = -i\langle\downarrow|\partial_\theta|\downarrow\rangle = 0$, $A_\phi = -i\langle\downarrow|\partial_\phi|\downarrow\rangle = -\sin^2\left(\frac{\theta}{2}\right)$

claim spherical to cartesian

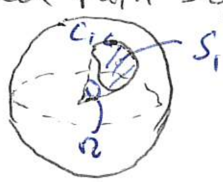
$$F_{ij} = -\epsilon_{ijk}\frac{B^k}{2|\vec{B}|^3} \quad \text{(magnetic monopole)}$$

For mag. monopole, if has charge $g = -\frac{1}{2}$ then from Gauss thm, over S^2

$$\int_{S^2} F_{ij}\, dS^{ij} = 4\pi g = -2\pi = \frac{\Omega_0}{2} \quad \text{w/ solid angle } \Omega_0 = 4\pi$$

now if a closed path suspend a solid angle Ω

$S' \equiv S^2 \setminus S_1$



then

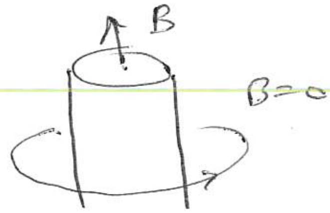
$$e^{i\gamma} = \exp\left(-i\int_{S_1} F_{ij}\, dS^{ij}\right) = \exp\left(-i\oint_{C_1} A_i\, d\lambda^i\right) = \exp\left(\frac{i\Omega}{2}\right)$$
$$= \exp\left(-i\int_{S'} F_{ij}\, dS^{ij}\right) = \exp\left(-i\oint_{-C_1} A_i\, d\lambda^i\right) = \exp\left(-\frac{i(4\pi-\Omega)}{2}\right) \quad (*)$$

In this case, we have phase diff. of 2π corresponding to $2g$ monopole charges. Quantization of monopole charge require $2g \in \mathbb{Z}$; take $C = 2g$ (chern number)

Thus we have over any (1) close surface $\int_S F_{ij}\, dS^{ij} = 2\pi C$

(2) phase difference in (*) also $\Leftrightarrow 2\pi C$ $\quad C \in \mathbb{Z}$.

Flux tube



Gauss thm

$$\oint \vec{A}\cdot d\vec{r} = \int \vec{B}\cdot d\vec{A} = \Phi \Rightarrow \vec{A} = \frac{\Phi}{2\pi r}\hat{e}_\phi$$

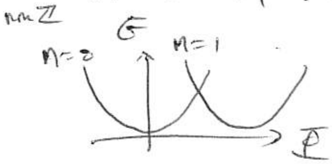
For particle outside solenoid, at radius r, Ham. is

$$H = \frac{1}{2m}(P_\phi + eA_\phi)^2 = \frac{1}{2mr^2}\left(-i\hbar\frac{\partial}{\partial\phi} + \frac{e\Phi}{2\pi}\right)^2$$

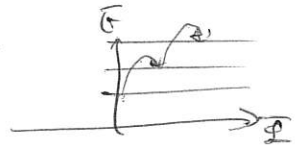
Ansatz $\psi = \frac{1}{\sqrt{2\pi r}}e^{in\phi}$ where $H\psi = E\psi$ w/ $\int |\psi|^2 dr = 1$

where $$E = \frac{1}{2mr^2}\left(n\hbar + \frac{e\Phi}{2\pi}\right)^2 = \frac{\hbar^2}{2mr^2}\left(n + \frac{\Phi}{\Phi_0}\right)^2$$

If Φ not integer multiple of Φ_0 then spectrum shifts

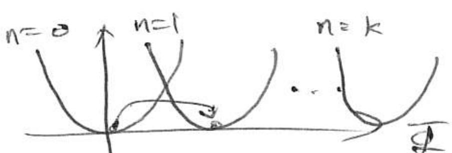


integer.



Remark

- switch off B initially so let particle in ground state
  By adiabatic thm, turning $\Phi = 0 \to \Phi = \Phi_0$ so
  all particle shifts from n to n+1 state

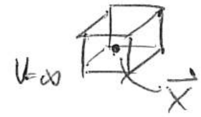


This phenomenon is called "spectral flow".

Aharonov-Bohm Effect

Relation between $\mathcal{A}$ Berry connection & A vector potential.

take small box



so $\vec{A}(\vec{x}) \sim \text{const}$ for ~~its center~~ $\vec{x} \in \text{Box}$

If centered at $\vec{x}$ then Ham.

$$H = \frac{1}{2m}\left(-i\hbar\nabla + e\vec{A}(\vec{x})\right)^2 + V(\vec{x}-\vec{X})$$

Let $\vec{X} = \vec{X}_0$ s.t $\vec{A}(\vec{X}_0) = 0$. we have ground state w.f. $\underbrace{\psi(\vec{x}-\vec{X}_0)}_{\text{the initial gs wf}}$ localized at $\vec{X}_0$

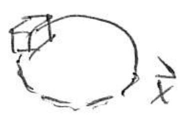
$$\vec{A}(\vec{X}_0) \to \vec{A}(\vec{X})$$

now $\psi(\vec{x}-\vec{X}_0) \longrightarrow \hat{U}\psi(\vec{x}-\vec{X}_0) = \psi(\vec{x}-\vec{X})$ where $\hat{U} = \exp\left(\frac{-ie}{\hbar}\int_{\vec{X}_0}^{\vec{X}} \vec{A}(\vec{x})\cdot d\vec{x}\right)$

so we have

$$H\psi(\vec{x}-\vec{X}) = \hat{U}H\psi(\vec{x}-\vec{X}_0) = \hat{U}E\psi(\vec{x}-\vec{X}_0) \quad \text{solves S.E.}$$

now close w/ a loop,



by def $\psi(\vec{x}-\vec{X}_0) = e^{i\gamma}\psi(\vec{x}-\vec{X}_0)$

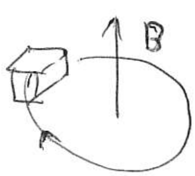
w/ $e^{i\gamma} = \exp\left(-i\oint_C \mathcal{A}_i dx^i\right)$

by construction, we also have

$$\hat{U}' = \exp\left(-\frac{ie}{\hbar}\oint_C \vec{A}(\vec{x})\cdot d\vec{x}\right)$$

$$\Rightarrow \boxed{\mathcal{A}(\vec{x}) = \frac{e}{\hbar}\vec{A}(\vec{x})}$$ ie $\mathcal{A}$ as vec. potential!

If



then $\oint_C \vec{A}\cdot d\vec{x} = \int_S \nabla\times\vec{A}\cdot d\vec{S} = \Phi$

thus we have for general charge q, a AB phase $e^{\frac{iq\Phi}{\hbar}}$

~~Remark~~ Remark

In AB exp. integer phase difference doesn't induce interference but only fractional one does!

Non-Abelian Berry Connection

Previous ground state is unique.

Now take g.s w.f N-fold degen $\forall \lambda$

<u>Remark</u> • pertrb will break degen.

• want change of H w/o breaking degen (need sym. protection for state)

• process of adiabatic thm only yields one of degen states diff by phase

Consider instead
$$i\frac{\partial|\psi\rangle}{\partial t} = H(\lambda(t))|\psi\rangle = 0$$
where g.s $E=0$ is assumed.

For each λ, we have N-dim basis $|n^a(\lambda)\rangle$ $\quad a=1,\ldots,N$

For unitary matrix $U(N)$, in Schrödinger pic,
$$|\psi_a(t)\rangle = U_{ab}(t)|n_b(\lambda(t))\rangle \qquad \text{w/ } U(t) \subset U(N)$$
again, $\quad |\dot{\psi}_a\rangle = \dot{U}_{ab}|n_b\rangle + U_{ab}|\dot{n}_b\rangle = 0$ from above S.E.

taken $\langle\psi_a|\dot{\psi}_a\rangle$, we have
$$U^{\dagger}_{ab}\dot{U}_{bb} = -\langle n_a|\dot{n}_b\rangle = -\langle n_a|\frac{\partial}{\partial\lambda^i}|n_b\rangle\dot{\lambda}^i$$
Define the non-Abelian Berry connection
$$\underset{N\times N \text{ matrix}}{(A_i)_{ba}} = -i\langle n_a|\frac{\partial}{\partial\lambda_i}|n_b\rangle \qquad \text{lives in Lie algebra } u(N)$$
Remark: Ambiguity for definition of A_i due to choice of basis.

ex one can pick $|n'_a(\lambda)\rangle = \Omega_{ab}|n_a(\lambda)\rangle$ w/ $\Omega \subset U(N)$

s.t
$$A'_i = \Omega A_i \Omega^{\dagger} + i\frac{\partial\Omega}{\partial\lambda^i}\Omega^{\dagger} \qquad (*)$$
can define again curvature of field strength
$$F_{ij} = \frac{\partial A_i}{\partial\lambda^j} - \frac{\partial A_j}{\partial\lambda^i} - i[A_i, A_j] \qquad \text{not like abelian case gauge inv.}$$
but transf. as $\quad F'_{ij} = \Omega F_{ij}\Omega^{\dagger}$

now expression of $U \neq \exp(-i\oint A_i d\lambda^i)$ $\quad$ matrix $\quad$ b/c $[A_i, A_j] \neq 0$

but instead
$$U = P\exp(-i\oint A_i d\lambda^i)$$
Berry holonomy is gauge inv. under $(*)$. P path ordering.