3.1 Rotation and Angular Momentum

ex $R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Infinitesimal upon cyclic permutation $R_z(\varepsilon) = \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_x(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix}$ $R_y(\varepsilon) = \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\frac{\varepsilon^2}{2} \end{pmatrix}$

Remark. Infinitesimal rotation commutes by ignoring $\mathcal{O}(\varepsilon^2)$; keep $\mathcal{O}(\varepsilon^2)$ one has commutation rel.

(*) $R_x(\varepsilon)R_y(\varepsilon) - R_y(\varepsilon)R_x(\varepsilon) = R_z(\varepsilon^2) - \mathbb{1}$

Introduce Rotation Operator $D(R)$

$|\alpha\rangle_R = D(R)|\alpha\rangle$

rotated ket

Recall infinitesimal operator $U(\varepsilon) = 1 - iG\varepsilon$ here $G \to \frac{J_k}{\hbar}$ $\varepsilon \to d\phi$

so, $D(\hat{n}, \phi) = 1 - \frac{i\vec{J}\cdot\hat{n}}{\hbar}d\phi$ (rotate about $\hat{n}$ by $d\phi$ $\vec{J}$ angular mom.)

successive inf rot.

$D(\hat{n}, \phi) = \exp\left(-\frac{iJ_z\phi}{\hbar}\right)$ (here $\hat{n} = \hat{z}$)

axioms $R \in SO(3)$ postulate $D(R)$ has same group properties as R:

yields from (*)

$\left(1 - \frac{iJ_x\varepsilon}{\hbar} - \frac{J_x^2\varepsilon^2}{2\hbar^2}\right)\left(1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{2\hbar^2}\right) - \left(1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{2\hbar^2}\right)\left(1 - \frac{iJ_x\varepsilon}{\hbar} - \frac{J_x^2\varepsilon^2}{2\hbar^2}\right) = 1 - \frac{iJ_z\varepsilon^2}{\hbar} - 1$

then $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$ in general. (J_k rotational generator about k-axis)

Spin $\frac{1}{2}$

- From Pauli Matrix & by Exp. $[S_i, S_j] = i\hbar\varepsilon_{ijk}S_z$
- note $\vec{J} = \vec{L} + \vec{S}$ w/ $\vec{L} = 0$ $D_z(\phi) = \exp\left(-\frac{iS_z\phi}{\hbar}\right)$

Consider $\langle S_x\rangle = {}_R\langle\alpha|S_x|\alpha\rangle_R = \langle\alpha|D_z^\dagger(\phi)S_xD_z(\phi)|\alpha\rangle$

$\Rightarrow \exp\left(\frac{iS_z\phi}{\hbar}\right)S_x\exp\left(-\frac{iS_z\phi}{\hbar}\right) = S_x\cos\phi - S_y\sin\phi$ solved by BH lemma or using expression $S_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)$

thus

$\langle S_x\rangle = \langle S_x\rangle\cos\phi - \langle S_y\rangle\sin\phi$

$\langle S_y\rangle = \langle S_y\rangle\cos\phi + \langle S_x\rangle\sin\phi$

$\langle S_z\rangle = \langle S_z\rangle$ ie $[S_z, D_z(\phi)] = 0$

obs. $\Longrightarrow \langle S_k\rangle = \sum_l R_{kl}\langle S_l\rangle$ ie spin op. rotates as vector.

in general $\langle J_k\rangle = \sum_l R_{kl}\langle J_l\rangle$

ex Rotate a general ket in spin sys.

$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle \Rightarrow \exp\left(-\frac{iS_z\phi}{\hbar}\right)|\alpha\rangle = e^{-i\phi/2}|+\rangle\langle+|\alpha\rangle + e^{i\phi/2}|-\rangle\langle-|\alpha\rangle$

obs. 1.) $D_z(2\pi)|\alpha\rangle = -|\alpha\rangle$, $D_z(4\pi)|\alpha\rangle = |\alpha\rangle$ (ie mobius strip)

2.) Sign vanished in $\langle\vec{S}\rangle$ ie $(-1)^2\langle\alpha|\vec{S}|\alpha\rangle$ (will neg sign be observable)

ex Spin Precession Revisit

$H = -\frac{e}{m_ec}\vec{S}\cdot\vec{B} = \omega S_z$ then time-evo op. $U(t,0) = \exp\left(-\frac{iHt}{\hbar}\right) = \exp\left(-\frac{iS_z\omega t}{\hbar}\right) = D_z(\omega t)$

so $U(t,0) = D_z(\theta)$ is precisely a rotation operator!

Pauli Formalism / Matrix Rep of Spin $\frac{1}{2}$

We define $\chi_+ = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \chi_- = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle = \begin{pmatrix} \langle+|\alpha\rangle \\ \langle-|\alpha\rangle \end{pmatrix}$$

define two comp. spinor as $\chi = \begin{pmatrix} \langle+|\alpha\rangle \\ \langle-|\alpha\rangle \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+\chi_+ + c_-\chi_-$

Pauli Matrices $\langle\pm| S_k |+\rangle = \frac{\hbar}{2}\sigma_{k_{\pm,+}}$ $\qquad \langle\pm|S_k|-\rangle = \frac{\hbar}{2}\sigma_{\pm,-}$

$$\langle S_k\rangle = \langle\alpha| S_k |\alpha\rangle = \sum_{a'=\pm}\sum_{a''=\pm} \langle\alpha|a'\rangle\langle a'|S_k|a''\rangle\langle a''|\alpha\rangle$$

thus $\langle S_k\rangle = \frac{\hbar}{2}\chi^\dagger\sigma_k\chi$

Properties of Pauli Matrix

$\sigma_i^2 = 1$, $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$

$\det(\sigma_i) = 1$, $\mathrm{tr}(\sigma_i) = 0$

$\vec{\sigma}\cdot\vec{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$

Claim $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

Corollary $(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2$ if $\vec{a}$ real

Pf $a_i\sigma_i b_j\sigma_j = a_ib_j\left(\frac{1}{2}\{\sigma_i,\sigma_j\} + \frac{1}{2}[\sigma_i,\sigma_j]\right)$
$= a_ib_j\delta_{ij} + i\sigma_k\varepsilon_{ijk}a_ib_j$

using the claim, we have $(\vec{\sigma}\cdot\hat{n})^n = \begin{cases} 1 & n \text{ even} \\ \vec{\sigma}\cdot\hat{n} & n \text{ odd} \end{cases}$

then $\exp\left(-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right) = 1\cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2} = \begin{pmatrix} \cos\frac{\phi}{2} - in_z\sin\frac{\phi}{2} & (-in_x - n_y)\sin\frac{\phi}{2} \\ (-in_x + n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + in_z\sin\frac{\phi}{2} \end{pmatrix}$

$$\begin{array}{ccc} & \text{Operator Space} & \\ |\alpha\rangle & \longrightarrow & \mathcal{D}(R)|\alpha\rangle \\ \downarrow & & \downarrow \\ \chi & \longrightarrow & M_R\chi \\ & \text{Matrix Space} & \end{array}$$

i.e. $\chi \to \exp\left(-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)\chi$ $\quad$ • σ remains unchange under rot.

$\chi^\dagger\sigma_k\chi \to \sum_\ell R_{k\ell}\chi^\dagger\sigma_\ell\chi$ $\quad$ • $\chi^\dagger\vec{\sigma}\chi$ rot. as vector

Recalled $\boxed{\langle S_k\rangle = \frac{\hbar}{2}\chi^\dagger\sigma_k\chi}$

Eig. val Eigvec of $\vec{\sigma}\cdot\hat{n}$

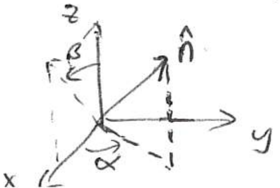
i.e. $\vec{\sigma}\cdot\hat{n}\chi = \chi$ then $\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \frac{\hbar}{2}|\vec{S}\cdot\hat{n};+\rangle$

Remark. • $\vec{\sigma}\cdot\hat{n}\,\chi = \chi$ implies spinor align in $\hat{n}$ direction of $\vec{\sigma}\cdot\hat{n}$ ops.

• $\exp\left(-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)$ rotates σ_z to $\hat{n}$ direction

• to find χ, all we need is to rotate $|+\rangle$ or χ_+ to align w/ $\hat{n}$

• for $\hat{n}$ in (β, α), we do $\exp\left(-\frac{i\sigma_3\alpha}{2}\right)\exp\left(-\frac{i\sigma_2\beta}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi$!



SO(3), SU(2)

R in $SO(3)$
3x3 orth. matrix

- R 9 entries
- $RR^T = R^TR = 1$ RR^T then is sym $\Rightarrow$ 6 indep. eqn thus 6 indep entries.
- 9−6 constraints leave 3 free parameters.
- $\det(RR^T) = 1 \Rightarrow \det(R) = \pm 1$ (take +1)

$\boxed{SO(3) \text{ has axioms: } RR^T = 1,\ \det(R) = 1,\ \dim(R) = 3}$

SU(2) · unitary metric preserved magnitude of complex vector

· Unitary Unimodular group

$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \text{ s.t. } U^\dagger U = 1,\ \det(U) = 1 \text{ ie } |a|^2 + |b|^2 = 1$$

ex 2X2 rep $\mathcal{D}_{\hat{n}}(\phi) = \exp\left(-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)$ is unimodular

Pf · Check $D_{\hat{n}}(\phi)$ satisfies entries $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

· identify $Re(a)$, $Im(a)$
$Re(b)$, $Im(b)$

clearly $|a|^2 + |b|^2 = 1$

Remark general form of $U(a,b)$ rep. a rotation.

Claim $\{U(a,b)\}$ forms a group

closure · $U(a_1,b_1)U(a_2,b_2) = U(\underbrace{a_1a_2 - b_1b_2^*}_{\tilde{a}}, \underbrace{a_1b_2 + a_2^*b_1}_{\tilde{b}})$ w/ $|\tilde{a}|^2 + |\tilde{b}|^2 = 1$

inverse · $U^{-1}(a,b) = U(a^*, -b)$ thus $SU(2)$ ie $n \times n$ unitary matrix w/ $\det(U) = 1$

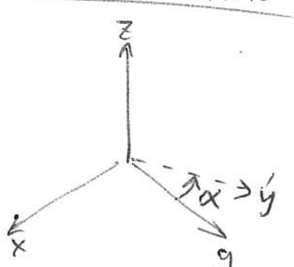
Unitary Unimodular Matrix $U(2)$

$U(2)$ has four parameters

Claim $U(2) = e^{i\gamma}U(a,b)$ w/ γ real

Remark $SU(2) \leq U(2)$ subgroup of $U(2)$.

Euler Rotations



$$R(\alpha,\beta,\gamma) \equiv R_{z'}(\gamma)R_{y'}(\beta)R_z(\alpha)$$

note. $R_{y'}(\beta) = R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha)$

$R_{z'}(\gamma) = R_{y'}(\beta)R_z(\gamma)R_{y'}^{-1}(\beta)$

$$\boxed{R(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma)}$$

In QM $R(\alpha,\beta,\gamma) \to \mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha)\mathcal{D}_y(\beta)\mathcal{D}_z(\gamma) \in SU(2)$

$= \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$

$\mathcal{D}(\alpha,\beta,\gamma)$

$$= \begin{pmatrix} e^{-i\frac{(\alpha+\gamma)}{2}}\cos\left(\frac{\beta}{2}\right) & -e^{-i\frac{(\alpha-\gamma)}{2}}\sin\left(\frac{\beta}{2}\right) \\ e^{i\frac{(\alpha-\gamma)}{2}}\sin\left(\frac{\beta}{2}\right) & e^{i\frac{(\alpha+\gamma)}{2}}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}$$

is $j = \frac{1}{2}$ irreducible rep. of $D(\alpha,\beta,\gamma)$
see (§3.5) denote $\mathcal{D}^{(1/2)}_{m'm}(\alpha,\beta,\gamma)$

3.4 Density operator

Def Ensemble: collection of identically prepared phys. sys characterized by same state ket $|\alpha\rangle$.

Ensemble of completely random spin orientation can be chara. by probability weight
ie composed of 50% $|+\rangle$ $(w_+ = 0.5)$ and 50% of $|-\rangle$ $(w_- = 0.5)$

Def Completely random ensemble: unpolarized. ie directly out of oven w/ no prefer spin dir.
Def Pure ensemble: polarized by single spin pointing in a definite direction.
Def Mix ensemble: 70% chara. by $|\alpha\rangle$ and 30% by $|\beta\rangle$.

Density Operator

Ensemble of eigstates of ops A can be mixed ie. w_1 of $|\alpha^{(1)}\rangle$ w_2 of $|\alpha^{(2)}\rangle$
but subjects to constrains:
$$\sum_i w_i = 1$$

Quest for average measured value of A in large num. of measurement.
Consider ensemble average
$$\boxed{[A] = \sum_i w_i \langle\alpha^i|A|\alpha^i\rangle}$$
insert general basis
$$[A] = \sum_{b'b''}\left(\sum_i w_i \langle b''|\alpha^i\rangle\langle\alpha^i|b'\rangle\right)\langle b'|A|b''\rangle$$
def. density operator
$$\boxed{\rho = \sum_i w_i |\alpha^i\rangle\langle\alpha^i|}$$ has matrix rep. $\langle b''|\rho|b'\rangle = \sum_i w_i \langle b''|\alpha^i\rangle\langle\alpha^i|b'\rangle$

then
$$\boxed{[A] = \sum_{b'b''}\langle b''|\rho|b'\rangle\langle b'|A|b''\rangle = \operatorname{tr}(\rho A)}$$

Remark • $\operatorname{tr}(\rho A)$ indep. of rep.; can evaluate using any convenient basis.
• ρ hermitian, $\operatorname{tr}(\rho) = 1$.

ex. spin $\frac{1}{2}$ $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so 8 parameters
But $\rho^\dagger = \rho \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ so $a, d \in \mathbb{R}$ w/ $c_1 = b_1$ $c_2 = -b_2 \Rightarrow 4$ parameters
$\operatorname{tr}(\rho) = 1 \Rightarrow a + d = 1$ thus $4 - 1 = 3$ parameters

def. of pure ensemble $\quad w_i = 1$ for some $i = n$
$\quad w_i = 0$ else

thus
$$\rho = |\alpha^n\rangle\langle\alpha^n|$$
and $\rho^2 = \rho$ is idempotent

claim • If pure ensemble then $\operatorname{tr}(\rho^2) = \operatorname{tr}(\rho) = 1$
• eigval of pure ensemble are zero or one.

Remark. $\rho(\rho - 1) = 0$ then for $\rho|a'\rangle = \rho'|a'\rangle$
$$\sum_{a'}\rho|a'\rangle\langle a'|\rho - \sum_{a'}\rho|a'\rangle\langle a'| = \rho'(\rho' - 1)$$
thus eig. val are either 0 or 1.

thus diagnalized ρ in matrix
$$\rho = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0, & & \\ & & & 1, 0 & \\ 0 & & & & \ddots\ 0 \end{pmatrix}$$
Since $\operatorname{tr}(\rho) = 1$

3.4 Density Matrix

ex. Spin $\frac{1}{2}$ w/ $|+\rangle$ ensembles $\rho = |+\rangle\langle+| = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

ex unpolarized beam ie. 50% $|+\rangle$ & $|-\rangle$ $\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

ex pure if $|S_x;+\rangle$ or $|S_x;-\rangle$ then $\rho = \frac{1}{2}\begin{pmatrix}\frac{1}{2} & \pm\frac{1}{2}\\ \pm\frac{1}{2} & \frac{1}{2}\end{pmatrix}$

and unpolarized in $|S_x;\pm\rangle$ then $\rho = \sum \omega_i |\alpha^i\rangle\langle\alpha^i| = \begin{pmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}$

ex consider $[\vec{S}]$ $[\vec{S}] = tr(\rho\vec{S})$

(for unpolarized beam) $= tr(\rho S_x) + tr(\rho S_y) + tr(\rho S_z)$

$[\vec{S}] = 0$ b/c $tr(S_i) = 0$ $\rho = \frac{1}{2}\cdot\mathbb{1}$.

ex partially polarized : $\omega_{+,z} = 0.75$ $\omega_{+,x} = 0.25$

$$\rho = \frac{3}{4}\begin{pmatrix}1&0\\0&0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}7/8 & 1/8\\ 1/8 & 1/8\end{pmatrix}$$

$$[S_x] = tr(\rho S_x) = \frac{\hbar}{8} \quad [S_y] = 0 \text{ (why?)} \quad [S_z] = \frac{3\hbar}{8}$$

Time evol. of ensemble

Recalled $i\hbar\frac{\partial}{\partial t}|\alpha^i, t_0; t\rangle = H|\alpha^i, t_0; t\rangle$

$$\frac{\partial\rho}{\partial t} = \sum_i \omega_i \frac{\partial|\alpha^i, t_0; t\rangle}{\partial t}\langle\alpha^i, t_0; t| + \omega_i |\alpha^i, t_0; t\rangle \frac{\partial}{\partial t}\langle\alpha^i, t_0; t|$$

$$\boxed{\frac{\partial\rho}{\partial t} = -[\rho, H]} \quad (*)$$

Remark:
- Recalled $\frac{dA}{dt} = \frac{1}{i\hbar}[A,H] + \frac{\partial A}{\partial t}$ but ρ not dyn. obs.
- (*) analogue to Liouville's thm $\frac{\partial\rho_{ce}}{\partial t} = -\{\rho_{ce}, H\}$
- Classical analogue of $[A]$. $A_{ave} = \dfrac{\int\rho_{ce} A(q,p)\, d\Gamma}{\int\rho_{ce}\, d\Gamma}$ recalled in stat. mech. $d\Gamma = \prod_{i=1}^{N} d^3p_i\, d^3q_i$

Density Matrix in Continuum

$[A] = \int d^3x'' d^3x' \langle x''|\rho|x'\rangle\langle x'|A|x''\rangle$ rewrite $\langle x''|\rho|x'\rangle = \sum_i \omega_i \psi_i(x'')\psi_i^\dagger(x')$

diagonal elem. ie $x'' = x'$ $\Rightarrow$ $\sum_i \omega_i |\psi_i(x')|^2$ is the weighted sum of prob. density.

Connection to stat. mech.

- Spin $\frac{1}{2}$ $\rho = \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$
- In pure ensemble $\rho = \begin{pmatrix}0&&0\\&\ddots 1 \ddots&\\0&&0\end{pmatrix}$
- for completely rand. ensemble

$$\rho = \frac{1}{N}\begin{pmatrix}1&&0\\&\ddots&\\0&&1\end{pmatrix}$$

ie density of all ket equally populate!

Remarks

diag entries of ρ ~ counting of possible state.

prompt definition of $\boxed{\sigma = -tr(\rho\ln\rho)}$

- diag. ρ w/ chosen basis. then

$\sigma = -\sum_k \rho_{kk}\ln\rho_{kk}$; $0 \le \rho_{kk} \le 1$ $\sigma \ge 0$ positive semi def.

- If completely rand. $\rho_{kk} = \frac{1}{N}$, $\sigma = \ln N$
- If pure $\rho_{kk} = 1$ for a k else 0 thus $\sigma = 0$

obs $\boxed{\sigma \text{ measures ordering}}$. $\Rightarrow$ entropy $\boxed{S = k\sigma}$

<u>Assumption</u> "Nature likes max. δ under constraint s.t $\langle H\rangle$ is fixed."

at thermal equilibrium. $\frac{\partial \rho}{\partial t}=0 \Rightarrow [\rho, H]=0$ (available for energy ket) / ρ_{kk} now fraction of eigket w/ eig E_k.

<u>Apply constraints</u> • fixed $H \Rightarrow U \equiv [H] = tr(\rho H)$

• $tr(\rho) = 1$

• then $\delta[H] = 0$ & $\delta(tr(\rho)) = 0$

note $\delta[H] = \sum_k \delta\rho_{kk} E_k = 0$ & $\delta(tr(\rho)) = \sum_k \delta\rho_{kk} = 0$

• Apply lagrange multiplier.

$$\delta\delta = 0 \Rightarrow \delta\delta = -\left(\sum_k \delta\rho_{kk}\ln\rho_{kk} + \delta\rho_{kk}\right)$$

then

$$\sum_k \delta\rho_{kk}(\ln\rho_{kk}+1) + \beta\sum_k \delta\rho_{kk}E_k + \gamma\sum_k \delta\rho_{kk} = 0$$

s.t $\sum_k \delta\rho_{kk}[\ln\rho_{kk} + 1 + \beta E_k + \gamma] = 0$

thus $\rho_{kk} = \exp(-\beta E_k - \gamma - 1)$

Now $\sum_k \rho_{kk} = 1 \Rightarrow \sum_l^N e^{-\beta E_l} = e^{\gamma+1}$ thus $\underbrace{\rho_{kk} = \frac{e^{-\beta E_k}}{\sum_l^N e^{-\beta E_l}}}_{\text{cannonical ensemble}}$ for distinct E_k

Remark. If w/o $\delta H = 0$, then $\ln\rho_{kk} + 1 + \gamma = 0$

$\rho_{kk} = \exp(-1-\gamma)$ w/ $tr(\rho) = 1 \Rightarrow e^{-(1+\gamma)} = \frac{1}{N}$

thus $\rho_{kk} = \frac{1}{N}$ (ie at $\beta \to 0$ high temp)

In stat mech, $\underbrace{Z}_{\text{partition func.}} = \sum_k^N e^{-\beta E_k} = tr(e^{-\beta H})$ then $\rho = \frac{e^{-\beta H}}{Z}$

note. $[A] = tr(\rho A) = \frac{tr(e^{-\beta H}A)}{Z} = \frac{\sum_k A_k \exp(-\beta E_k)}{Z}$

for $A = U$, then $U = -\frac{\partial}{\partial\beta}\ln Z$ where $\beta = \frac{1}{k_B T}$

<u>ex</u> $H = \vec{\mu}\cdot\vec{S} = \frac{e\hbar B}{2m_e c}S_z = \omega S_z$ So $E_\pm = \pm\frac{\hbar\omega}{2}$

$\rho = \frac{e^{-\beta H}}{Z}$ at therm. equilibrium

<u>Method 1</u> $e^{-\beta\omega S_z} = \cos\frac{\beta\omega\hbar}{2} - i\sigma_z \sin\frac{\beta\omega\hbar}{2} = \begin{pmatrix} e^{-\frac{\beta\hbar\omega}{2}} & 0 \\ 0 & e^{\frac{\beta\hbar\omega}{2}} \end{pmatrix}$

<u>" 2</u> H diag, $\rho_{kk} = \frac{e^{-\beta E_k}}{Z}$ thus $\rho = \begin{pmatrix} e^{-\frac{\beta\hbar\omega}{2}} & 0 \\ 0 & e^{\frac{\beta\hbar\omega}{2}} \end{pmatrix} / Z$

• $[S_x] = tr(\rho S_x)$ so $[S_x] = [S_y] = 0$

$[S_z] = -\frac{\hbar}{2}\tanh\left(\frac{\beta\hbar\omega}{2}\right)$

• magnetic moment $\mu = \frac{e}{m_e c}S_z$ w/ $[\mu] = \chi B$

then $[\mu] = \frac{e}{m_e c}[S_z] = \frac{|e|\hbar}{2m_e c}\tanh\left(\frac{\beta\hbar\omega}{2}\right)$ thus $\underbrace{\chi = \frac{|e|\hbar}{2m_e c B}\tanh\left(\frac{\beta\hbar\omega}{2}\right)}_{\text{the Brillouin's formula.}}$

3.5 Eig. val & Eig. state of Angular Mom.

General Ang. Mom. $J^2 = J_x^2 + J_y^2 + J_z^2$ clearly $[J^2, J_k] = 0$

- w.l.o.g J_z is selected to share eig.ket w/ J^2
- 1st convention: Let J^2 and J_z's eig.val be a, b.

ie $J^2|a,b\rangle = a|a,b\rangle$

$J_z|a,b\rangle = b|a,b\rangle$

- analogue to spin $\frac{1}{2}$, $\boxed{J_\pm \equiv J_x \pm iJ_y \text{ thus } [J_+, J_-] = 2\hbar J_z,\ [J_z, J_\pm] = \pm\hbar J_\pm,\ [J^2, J_\pm] = 0}$
- w/ above, we have $\underline{J_z(J_\pm|a,b\rangle) = (b\pm\hbar)(J_\pm|a,b\rangle)}$ $J_\pm|a,b\rangle$ sim. ket of J^2 & J_z

So $J_\pm$ steps up (down) eig.val b of J_z by $\hbar$

- note $[J_z, J_\pm] = \pm\hbar J_\pm$ and $[x_i, \mathcal{J}(\vec{l})] = l_i \mathcal{J}(\vec{l})$

$[N, a^+] = a^+$; $[N, a] = -a$ all same functionality.

- Naturally, $J_\pm|a,b\rangle = C_\pm|a, b\pm\hbar\rangle$ w.r.t J^2 & J_z.

Eigval of J^2 & J_z

Claim Given a, there exist a upper value J_+ that can raise on b, w/ $a \geq b^2$

Consider $J^2 - J_z^2 = J_x^2 + J_y^2 = \frac{J_+J_- + J_-J_+}{2} = \frac{J_+J_+^\dagger + J_+^\dagger J_+}{2}$

s.t $\boxed{\langle a,b|(J^2 - J_z^2)|a,b\rangle \geq 0}$ thus $a \geq b^2$ as claim! $\exists\, b_{max}$

So, $J_+|a, b_{max}\rangle = 0$ so as $J_-J_+|a, b_{max}\rangle = 0$

Now $J_-J_+ = J_x^2 + J_y^2 + i[J_x, J_y] = J^2 - J_z^2 - \hbar J_z$

$(J^2 - J_z^2 - \hbar J_z)|a, b_{max}\rangle = 0 \Rightarrow a - b_{max}^2 - \hbar b_{max} = 0$

$\Rightarrow a = b_{max}(b_{max} + \hbar)$

Since $a \geq b^2$, $b \geq -\sqrt{a}$ has low bound

So $J_-|a, b_{min}\rangle = 0$

Now $J_+J_- = J^2 - J_z^2 + \hbar J_z |a, b_{min}\rangle$ then $a = b_{min}(b_{min} - \hbar)$ together we have $b_{max} = -b_{min}$

so $-b_{max} \leq b \leq b_{max}$

Say $J_+^n|a, b_{min}\rangle \mapsto b_{min} + n\hbar = b_{max}$ so $b_{max} = \frac{n\hbar}{2}$

Let $j \equiv \frac{b_{max}}{\hbar} = \frac{n}{2}$

- j either integer or half
- max eigval of J_z is $j\hbar$
- $a = \hbar^2 j(j+1)$
- $b \equiv m\hbar$ s.t $m = \underbrace{-j, -j+1, \ldots, 0, \ldots, j-1, j}_{2j+1 \text{ states}}$

take convension $|a,b\rangle \to |j,m\rangle$ s.t

$$\boxed{\begin{aligned} J^2|j,m\rangle &= \hbar^2 j(j+1)|j,m\rangle \\ J_z|j,m\rangle &= m\hbar|j,m\rangle \end{aligned}}$$

Properties

$\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}$

Matrix elem. of J^2, $J_\pm$, J_z

$\langle j',m'|J^2|j,m\rangle = \hbar^2 j(j+1)\delta_{jj'}\delta_{mm'}$ ex. $j=1$ $m=-1,0,1$ $J^2 = 2\hbar\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $J_z=\begin{pmatrix}-\hbar&0&0\\0&0&0\\0&0&\hbar\end{pmatrix}$

$J_\pm$? Recalled $\boxed{J_\mp J_\pm = J^2 - J_z^2 \mp \hbar J_z}$ w/ $J_-J_+ = J_+^\dagger J_+$

thus $\langle j,m|J_+^\dagger J_+|j,m\rangle = \hbar^2[j(j+1)-m(m+1)]$

since $J_+|j,m\rangle = C_+|j,m+1\rangle$, $\boxed{J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm1\rangle}$

Its matrix elem.

$\langle j',m'|J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,\delta_{jj'}\delta_{m',m\pm1}$ Remark elem. of J_x, J_y using $J_\pm = J_x \pm iJ_y$

Matrix elem. of Rot operator using J_z, $J_\pm$

$D^j_{m'm}(R) = \langle j,m'|\exp\left(-\frac{\vec{J}\cdot\hat{n}\phi}{\hbar}\right)|j,m\rangle$

- obs. j fixed & $[J^2, D(R)]=0$ b/c $\vec{J}\cdot\hat{n}$

Dimensionality since $\# m$ is $2j+1$ $\mathcal{D}^j_{m'm}$ is $(2j+1)\times(2j+1)$ matrix referred as $(2j+1)$-dim. Irr. rep.

b/c $D(R)$ not chara. by j. It can be written as block matrix w/ each block for a given j $(2j+1)\times(2j+1)$. Where these $(2j+1)\times(2j+1)$ blocks cannot be reduced to smaller block.

Math Structure $\mathcal{D}^j_{m'm}(R)$ is a group • D unitary so is R b/c $D^\dagger(R) = D(R^{-1})$ where $RR^T=1$ w/ $R^T=R^{-1}$

Physic of $D(R)$: $D(R)\underbrace{|j,m\rangle}_{\text{origin}} \longrightarrow \underbrace{|j,m'\rangle}_{\text{rotated}}$

$D(R)|j,m\rangle = \sum_{m'}|j,m'\rangle \underbrace{D^j_{m'm}(R)}_{\text{matrix elem}}$. also amp. for rotated state to be found in $|j,m'\rangle$.

Arbitrary Rot. in term of Euler Angle.

$\mathcal{D}^j_{m'm}(\alpha,\beta,\gamma) = \langle j,m'|D_z(\alpha)D_y(\beta)D_z(\gamma)|j,m\rangle = e^{-i(m'\alpha+m\gamma)}d^j_{m'm}(\beta)$ $d^j_{m'm}(\beta) \equiv \langle j,m'|\exp\left(-\frac{iJ_y\beta}{\hbar}\right)|j,m\rangle$

ex $j=\frac{1}{2} \Rightarrow m=-\frac{1}{2},\frac{1}{2}$ & $J_y=S_y$ thus $d^{1/2} = \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}$

ex $j=1$ $m=-1,0,1$ $J_y = \frac{J_+-J_-}{2i}$ $J_y = \frac{\hbar}{2}\begin{pmatrix}0&-\sqrt{2}i&0\\ \sqrt{2}i&0&-\sqrt{2}i\\ 0&\sqrt{2}i&0\end{pmatrix}$.

$\exp\left(-\frac{iJ_y\beta}{\hbar}\right)$ in Taylor expansion. $\left(\frac{J_y}{\hbar}\right)^3 = \frac{J_y}{\hbar}$ thus $\exp\left(-\frac{iJ_y\beta}{\hbar}\right) = 1-\left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta) - i\left(\frac{J_y}{\hbar}\right)\sin\beta$

substitution yields

$$d^{(1)}(\beta) = \begin{pmatrix}\left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta)\\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta\\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta)\end{pmatrix}$$

3.6 Orbital Angular Mom.

$\vec{J} = \vec{L} + \vec{S}$ for $\vec{S} = 0$ or by definition $\vec{L} = \vec{x} \times \vec{p}$ we have $\boxed{[L_i, L_j] = i\varepsilon_{ijk}\hbar L_k}$

Consider $(1 - i\frac{\delta\phi}{\hbar}L_z)|x', y', z'\rangle = |x' - \delta\phi y', y' + \delta\phi x', z'\rangle$ note $L_z = xp_y - yp_x$

by mapping $\langle x', y', z'|\alpha\rangle \longmapsto \langle r, \theta, \phi|\alpha\rangle$

we have $\langle r, \theta, \phi|[1 - i\frac{\delta\phi}{\hbar}L_z]|\alpha\rangle = \langle r, \theta, \phi - \delta\phi|\alpha\rangle$ display a rotation about z by $\delta\phi$

$= \langle r, \theta, \phi|\alpha\rangle - \delta\phi\frac{\partial}{\partial\phi}\langle r, \theta, \phi|\alpha\rangle$

thus $\boxed{\langle \vec{x}'|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle \vec{x}'|\alpha\rangle}$

by writing x', y', z' in spherical coord.

$\langle \vec{x}'|L_x|\alpha\rangle = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\langle \vec{x}'|\alpha\rangle$ $\langle \vec{x}'|L_y|\alpha\rangle = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\langle \vec{x}'|\alpha\rangle$

$\boxed{L^2 = L_x^2 + L_y^2 + L_z^2 \; ; \; L_\pm = L_x \pm iL_y \; , \; L^2 = L_z^2 + \frac{L_+L_- + L_-L_+}{2}}$

thus, $\langle \vec{x}'|L_\pm|\alpha\rangle = -i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\langle \vec{x}'|\alpha\rangle$

$\langle \vec{x}'|L^2|\alpha\rangle = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\langle \vec{x}'|\alpha\rangle$ Note L^2 only has θ, ϕ dependence.

Spherical Harmonic

Spinless particle wave func. $\langle \vec{x}'|n, l, m\rangle = R_{nl}(r)Y_l^m(\theta, \phi)$

- If H spherical sym. ie $H(r)$ then $[H, L_z] = [H, L^2] = 0$ $\Rightarrow$ compatible quantum numbers n, l, m.
- B/c $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_z$ we have by construction:
  - $L^2 \Longleftrightarrow$ eigval $\hbar^2 l(l+1)$
  - $L_z \Longleftrightarrow$ eigval $m\hbar$; $m = -l, -l+1, \ldots, 0, \ldots, l-1, l$

Consider Ang. term $\langle\hat{n}|l, m\rangle = Y_l^m(\theta, \phi) \overset{\text{or}}{=} Y_l^m(\hat{n})$ amp of state chara. by l, m in dir. $\hat{n}$

Since $L_z|l, m\rangle = m\hbar|l, m\rangle$

then $\underbrace{\langle\hat{n}|L_z|l, m\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle\hat{n}|l m\rangle}_{L.H.S} = \underbrace{m\hbar\langle\hat{n}|l, m\rangle}_{R.H.S}$ thus $-i\hbar\frac{\partial}{\partial\phi}Y_l^m(\theta, \phi) = m\hbar Y_l^m(\theta, \phi)$

$\Rightarrow \boxed{Y_l^m(\theta, \phi) \sim e^{im\phi}}$

Same reasoning, $L^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle$

then

$\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + l(l+1)\right]Y_l^m(\theta, \phi) = 0 \Rightarrow$ Soln of $P_l(\cos\theta)$

Other properties $\langle l', m'|l, m\rangle = \delta_{l'l}\delta_{m'm}$

$\int d\Omega\langle l', m'|\hat{n}\rangle\langle\hat{n}|l, m\rangle = \int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\, Y_{l'}^{m'*}(\theta, \phi)Y_l^m(\theta, \phi) = \delta_{l'l}\delta_{m'm}$ note here uses $\int d\Omega|\hat{n}\rangle\langle\hat{n}| = \ldots$

Quest for Explicit form Y_l^m

$L_+|l, l\rangle = 0 \Rightarrow -i\hbar e^{i\phi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right]\langle\hat{n}|l, l\rangle = 0 \Rightarrow \exists f(\theta)$ s.t $\frac{\partial f(\theta)}{\partial\theta} = l\cot\theta f(\theta)$ thus $f(\theta) = C_l\sin^l\theta$

and $\langle\hat{n}|l, l\rangle = Y_l^l(\theta, \phi) = C_l e^{il\phi}\sin^l\theta$ upon normalization $C_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}}$

take $\langle\hat{n}|l, l-1\rangle = \frac{\langle\hat{n}|L_-|l, l\rangle}{\sqrt{2l(l-l+1)}\hbar}$

$\vdots$

$\langle\hat{n}|l, m-1\rangle = \frac{\langle\hat{n}|L_-|l, m\rangle}{\sqrt{(l+m)(l-m+1)}\hbar}$

$\Bigg\}$ yields $Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(l+m)!}{4\pi\,(l-m)!}}\, e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}(\sin\theta)^{2l}$

Important equality

$Y_\ell^{-m}(\theta,\phi) = (-1)^m [Y_\ell^m(\theta,\phi)]^*$ For $m=0$ & by Legendre poly. $P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^n$

$Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$

Claim here ℓ must be integer

Pf by counter example if $\ell = m = \frac{1}{2}$, then $Y_{1/2}^{1/2}(\theta,\phi) = C_{1/2}\, e^{i\phi/2}\sqrt{\sin\theta}$

and $\langle \hat{n}|\frac{1}{2},-\frac{1}{2}\rangle = \frac{1}{\hbar}\langle \hat{n}|L_-|\frac{1}{2},\frac{1}{2}\rangle$

$Y_{\frac{1}{2}}^{-\frac{1}{2}}(\theta,\phi) = -C_{1/2}\, e^{-\frac{i\phi}{2}}\cot\theta\sqrt{\sin\theta}$ singular at $\theta = 0, \pi$

O.T.O.H,

$\langle \hat{n}|L_-|\frac{1}{2},-\frac{1}{2}\rangle = 0 \Rightarrow -i\hbar e^{-i\phi}(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi})\langle \hat{n}|\frac{1}{2},-\frac{1}{2}\rangle = 0$

so $Y_{\frac{1}{2}}^{-1/2} = C'_{1/2}\, e^{-\frac{i\phi}{2}}\sqrt{\sin\theta}$ $\neq$ above thus contradiction and ℓ must be integer.

3.8 Momentum Addition $\vec{J} = \vec{L} + \vec{S}$

Actual particle description accounts for position $\vec{x}$ and spin $|\pm\rangle$ (ie electron)

baseket $|\vec{x}', \pm\rangle = |\vec{x}'\rangle \otimes |\pm\rangle$

I) Remark
- operator in space $\vec{x}'$ commutes w/ that in spin $|\pm\rangle$
- $\vec{J} = \vec{L} + \vec{S}$ in math $\vec{J} = \vec{L}\otimes 1 + 1\otimes\vec{S}$ we have $[\vec{L}, \vec{S}] = 0$

in rot. operator space:

$$D(R) = D^{orb}(R)\otimes D^{spin}(R) = \exp\left(\frac{-i\vec{L}\cdot\hat{n}\phi}{\hbar}\right)\otimes\exp\left(-\frac{i\vec{S}\cdot\hat{n}\phi}{\hbar}\right)$$

- corresponding wavefunc of baseket:

$\langle \vec{x}', \pm|\alpha\rangle = \psi_\pm(\vec{x}')$ w/ vector form $\begin{pmatrix}\psi_+(\vec{x}') \\ \psi_-(\vec{x}')\end{pmatrix}$

- Prob density $|\psi_\pm(\vec{x}')|^2$

baseket notation in terms of quantum number.

Alternatives: L^2, S^2, L_z, S_z $|n, l, m; S, m_s\rangle$

J^2, J_z, L^2, S^2 $|n, j, m_j; l, s\rangle$

II) Additions of two spins $\vec{S}_1 + \vec{S}_2$ $\vec{S} = \vec{S}_1 + \vec{S}_2 \rightarrow \vec{S}_1\otimes 1 + 1\otimes\vec{S}_2$

- Properties of commutation relation.

$[S_{1x}, S_{2y}] = 0$ $[S_{1x}, S_{1y}] = i\hbar S_{1z}$ $[S_{2x}, S_{2y}] = i\hbar S_{2z}$ generally $[S_x, S_y] = i\hbar S_z$

- Corresponding eigval

$S^2 = S(S+1)\hbar^2$ $S_z = m\hbar$ $S_{1z} = m_1\hbar$ $S_{2z} = m_2\hbar$

- baseket in term of quantum num.

Alternatives: i) S^2, S_z $|S, m\rangle = |S=1, m=\pm1, 0\rangle$ and $|S=0, m=0\rangle$

ii) S_{1z}, S_{2z} $|m_1, m_2\rangle$ w/ $m_1, m_2 = \pm\frac{1}{2}$ or $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

ex $S=1$, $m=\pm1, 0$

$|S=1, m=1\rangle = |++\rangle$ ($m=1 \Leftrightarrow \uparrow\uparrow$)

$|S=1, m=0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$ ($m=0 \Leftrightarrow \uparrow\downarrow$ or $\downarrow\uparrow$) triplet (sym)

$|S=1, m=-1\rangle = |--\rangle$

$|S=0, m=0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ singlet (antisym)

3.8 ex $|S=1, m=1\rangle \longrightarrow |S=1, m=-1\rangle$ using S_-

$S_- = S_{1-} + S_{2-} = (S_{1x} - iS_{1y}) + (S_{2x} - iS_{2y})$

$S_-|S=1, m=1\rangle = (S_{1-} + S_{2-})|++\rangle$ S_{1-} affects 1st entry of $|++\rangle$

$\sqrt{2}|S=1, m=0\rangle = |-+\rangle + |+-\rangle \Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$

the CG coeff which yield relation between $|S, m\rangle$ and $|m_{1z}, m_{2z}\rangle$

(III) Ang. mom. Add. $\vec{J} = \vec{J}_1 + \vec{J}_2 \to \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$

w/ $\boxed{\begin{array}{ll} [J_{1i}, J_{1j}] = i\hbar \varepsilon_{ijk} J_{1k} \quad \text{etc} & \\ [J_{1i}, J_{2j}] = 0 & \text{in general } [J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \end{array}}$

- infinitesimal rot.

$$\left(1 - \frac{i\vec{J}_1\cdot\hat{n}\,\delta\phi}{\hbar}\right) \otimes \left(1 - \frac{i\vec{J}_2\cdot\hat{n}\,\delta\phi}{\hbar}\right) = 1 - \frac{i(\vec{J}_1\otimes 1 + 1\otimes\vec{J}_2)\cdot\hat{n}\,\delta\phi}{\hbar}$$

from which we define $\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2$

- finite angle version.

$$\boxed{D(R) = D_1(R) \otimes D_2(R) = \exp\left(\frac{-i\vec{J}_1\cdot\hat{n}\phi}{\hbar}\right) \otimes \exp\left(-\frac{i\vec{J}_2\cdot\hat{n}\phi}{\hbar}\right)}$$

- baseket notation:

Alt 1. $J_1^2, J_2^2, J_{1z}, J_{2z}$ $|j_1, j_2; m_1, m_2\rangle$

Alt 2. J^2, J_1^2, J_2^2, J_z $|j_1, j_2; j, m\rangle$

note $J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$

shows although $\boxed{\begin{array}{ll} [J^2, J_z] = 0 & \\ [J^2, J_{1z}] \neq 0 & [J^2, J_{2z}] \neq 0 \end{array}}$

- relation between Alt.

$$|j_1 j_2; j\, m\rangle = \sum_{m_1, m_2} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m\rangle}_{\text{CG coeff mapping A2 to A1.}}$$

- inherited constraint 1

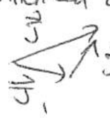
$\boxed{m = m_1 + m_2}$

Pf $J_z = J_{1z} + J_{2z}$ then $(J_z - J_{1z} - J_{2z})|j_1 j_2; j m\rangle = 0$

$$\langle j_1 j_2; m_1 m_2|(\cdot\cdot)|j_1 j_2; j\, m\rangle = (m - m_1 - m_2)\underbrace{\langle j_1 j_2; m_1 m_2|j_1 j_2; j\, m\rangle}_{CG} = 0$$

If CG $\neq 0$ then $m_1 + m_2 = m$

- inherited constraint 2



triangle ineq. $|\vec{J}| \leq |\vec{J}_1| + |\vec{J}_2|$ ~~claim if $|j_1 - j_2| \leq j \leq j_1 + j_2$ then~~

claim if $|j_1 - j_2| \leq j \leq j_1 + j_2$ then $\dim |j_1 j_2; m_1 m_2\rangle = \dim |j_1 j_2; j m\rangle$

For $(m_1 m_2)$ we have dim $N_1 = (2j_1 + 1)(2j_2 + 1)$

For (j, m), we have $N_2 = \sum_{j=j_1-j_2}^{j_1+j_2} \underbrace{(2j+1)}_{\# m} = N_1$ (w.l.o.g let $j_1 \geq j_2$)

CG is a matrix

$\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle$ where $m_1 = -j_1, \dots, j_1$ $m_2 = -j_2, \dots, j_2$

- Claim CG is unitary

$(CG)^*(CG) = 1$ (cus completeness relation)

- since CG is real thus CG is orthogonal.

Recursion Rel. of CG

Using $J_\pm |j_1 j_2; j m\rangle = (J_{1\pm} + J_{2\pm})\sum_{m_1, m_2} |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle$

after multiply by

$\langle j_1 j_2; m_1 m_2|$ one obtains recursion.

$$\sqrt{(j\mp m)(j\pm m+1)}\, \langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m\pm 1\rangle$$
$$= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\, \langle j_1 j_2; m_1 \mp 1, m_2 | j_1 j_2; j m\rangle$$
$$+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}\, \langle j_1 j_2; m_1, m_2 \mp 1 | j_1 j_2; j m\rangle$$

3.11 Tensor Operators

Def of Vector Ops.

Vector $V_i \to \sum_j R_{ij} V_j$ (*)

Q oper $|\alpha\rangle \to D(R)|\alpha\rangle$

By construction we assume with (*)

$\langle\alpha|V_i|\alpha\rangle \to \langle\alpha|D^\dagger(R) V_i D(R)|\alpha\rangle = \sum_j R_{ij}\langle\alpha|V_j|\alpha\rangle$

$|\alpha\rangle$ arbitrary then we have $D^\dagger(R) V_i D(R) = \sum_j R_{ij} V_j$ (!)

Infinitesimal $D(R) = 1 - \frac{i\varepsilon \vec{J}\cdot\hat{n}}{\hbar}$ thus

$$D^\dagger(R) V_i D(R) = \left(1 + \frac{i\varepsilon \vec{J}\cdot\hat{n}}{\hbar}\right) V_i \left(1 - \frac{i\varepsilon \vec{J}\cdot\hat{n}}{\hbar}\right)$$
$$= V_i + \frac{\varepsilon}{i\hbar}[V_i, \vec{J}\cdot\hat{n}] + O(\varepsilon^2)$$
$$= \sum_j R_{ij}(\hat{n};\varepsilon) V_j$$

} follow (!) If set $\hat{n}$ along z-axis, we get $R(\hat{z};\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

w/ $R(\hat{z};\varepsilon)$ we have $\boxed{[V_i, J_j] = i\varepsilon_{ijk}\hbar V_k}$ general case of $[J_i, J_j]$

Generalizing $V_i \to \sum_j R_{ij} V_j$ to tensors

ie $V_{ij} \to \sum_{i'j'} R_{ii'} R_{jj'} V_{i'j'}$

So $T_{ijk\cdots} \to \sum_{i'j'k'\cdots} R_{ii'} R_{jj'} R_{kk'} \cdots \underbrace{T_{i'j'k'\cdots}}_{\#\text{ of indices is called rank}}$

ex rank 2 tensor is outer product of two vectors $T_{ij} = U_i V_j$ which is a matrix

---

Claim Cartesian tensor is reducible

W.l.o.g $U_i V_j = \frac{\vec{U}\cdot\vec{V}}{3}\delta_{ij} + \underbrace{\frac{(U_i V_j - U_j V_i)}{2}}_{\text{anti sym}} + \underbrace{\left(\underbrace{\frac{U_i V_j + U_j V_i}{2}}_{\text{sym}} - \frac{\vec{U}\cdot\vec{V}}{3}\delta_{ij}\right)}_{\text{traceless}}$

- 1st term invariant under rot thus has 1 parameter
- 2nd term antisym thus 3 parameters
- 3rd term traceless sym thus 6-1 = 5 param.
- total 3x3 = 1+3+5 (remind of young diagrams)
- here 1, 3, 5 $\iff$ $l = 0, 1, 2$
- this is rank 2 tensor in the simplest example of cartesian tensor reduction into irre. spherical tensors.

ex Spherical tensor of rank k quantum num. q

$T_q^{(k)} \equiv Y_{l=k}^{m=q}(V)$ For $k=1$ we have $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \to T_0^{(1)} = \sqrt{\frac{3}{4\pi}} V_z$

$$Y_1^{\pm1} = \mp\sqrt{\frac{3}{4\pi}}\frac{x \pm iy}{\sqrt{2}r} \to T_{\pm1}^{(1)} = \sqrt{\frac{3}{4\pi}}\left(\mp\frac{V_x \pm iV_y}{\sqrt{2}}\right)$$

Constructing Spherical Tensor

- $|\hat{n}'\rangle = D(R)|\hat{n}\rangle$; $Y_\ell^m(\hat{n}) = \langle \hat{n}|\ell, m\rangle$

By $D(R^{-1})|\ell, m\rangle = \sum_{m'} |\ell, m'\rangle D^{(\ell)}_{m'm}(R^{-1})$

$$Y_\ell^m(\hat{n}') = \sum_{m'} Y_\ell^{m'}(\hat{n}) \underline{D^{(\ell)}_{m'm}(R^{-1})}$$

By promote $Y_\ell^m(\hat{n}')$ to vector operator, (R.H.S)

$$D^\dagger(R)\, Y_\ell^m(\vec{V})\, D(R) = \sum_{m'} Y_\ell^{m'}(\vec{V}) \underline{D^{(\ell)*}_{mm'}(R)}$$

Let $T_q^{(k)} = Y_k^q(\vec{V})$ we have

$\boxed{\text{Def}}$ (*) $\quad D^\dagger(R)\, T_q^{(k)} D(R) = \sum_{q'=-k}^{k} D^{(k)*}_{qq'} T_{q'}^{(k)}$ or $D(R)\, T_q^{k} D^\dagger(R) = \sum_{q'=-k}^{k} D^{(k)}_{q'q}(R)\, T_{q'}^{(k)}$

- Alt definition, by infinitesimal form of (*), we have

$$[\vec{J}\cdot\hat{n}, T_q^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle kq'|\vec{J}\cdot\hat{n}|kq\rangle$$

for $\hat{n} = \hat{z}$,

$\boxed{\text{Def}}$ $\quad \boxed{[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}}$ and $\boxed{[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\; T_{q\pm 1}^{(k)}}$

Tools: Clebsch-Gordan Series:

- $$D^{(j_1)}_{m_1m_1'} \otimes D^{(j_2)}_{m_2m_2'}(R) = \sum_j \sum_m \sum_{m'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm'\rangle D^{(j)}_{mm'}(R)$$

CG orthogonality

- $$\sum_{m_1}\sum_{m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; j'm'\rangle = \delta_{jj'}\delta_{mm'}$$

- $$\sum_j \sum_m \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'}$$

Thm

Let $X_{q_1}^{(k_1)}$ $Z_{q_2}^{(k_2)}$ be irr. spherical tensors of rank k_1 & k_2, then

$$T_q^{(k)} = \sum_{q_1}\sum_{q_2} \langle k_1 k_2; q_1 q_2 | k_1 k_2; kq\rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

is irreducible spherical tensor of rank k.

---

m-selection rule: $\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = 0$ if $m' \neq q+m$ $\quad$ pf $\langle \alpha', j'm'|\left([J_z, T_q^{(k)}] - \hbar q T_q^{(k)}\right)|\alpha, jm\rangle = 0$