① Phase Transitions

In Canonical Ensemble (no ext. work)

$F = -k_B T \ln Z$ (yields description of therm. properties)

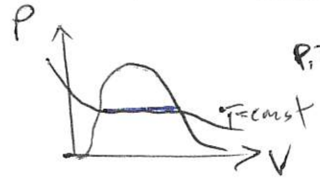
- $\boxed{\text{Phase transition emergence of new macro. state from another}}$
- Change of macro behavior char. by response func.

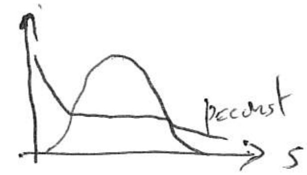
$C_{V,P} = T\left(\frac{\partial S}{\partial T}\right)_{T,P}$

$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ isoth. compressibility

$\chi_T = \frac{\partial M}{\partial B}\Big|_T$ or $\chi = \frac{\partial m}{\partial B}\Big|_T$ susceptibility.

Under phase trans,



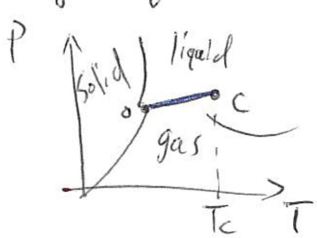


assoc. drastic change in response func. ie K_T, C_V

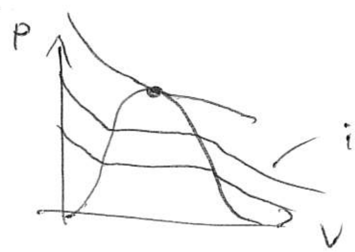
- w/ $dF = -S\,dT - P\,dV$ $C = -T^2 \frac{\partial^2 F}{\partial T^2}$ $\Leftrightarrow$ sing. in F!

Claim
- Z analytic for finite particles
- In therm. limit, assoc. F to phase transition becomes non-analytic. $(N \to \infty)$

ex gas-liquid condensate



phase trans. along line OC

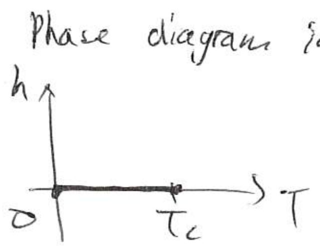


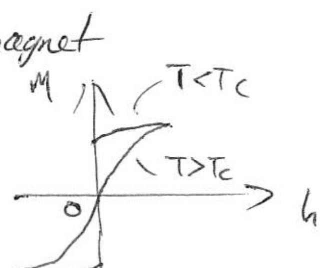
isotherm line

K_T diverges as $T \to T_c$ due to abrupt change in V while $\Delta P \to 0$

Remarks: F analytic in PT plane except branch cut along OC.

ex Phase diagram in magnet





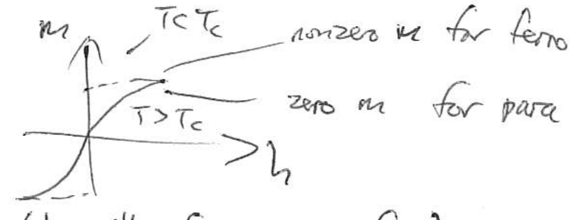
Critical behavior

- Critical exponent represents sing. behavior at critical pt./non-analyticity of therm. func.
- therm func = order parameter ie $M(T) = \frac{1}{V}\lim_{h \to 0} M(h, T)$ magnetization

Types of exp ν. (exponents universal via exp. obs)

magnetization.

$$m(T, h=0) \propto \begin{cases} 0 & T > T_c \\ |t|^{\beta} & T < T_c \end{cases}$$



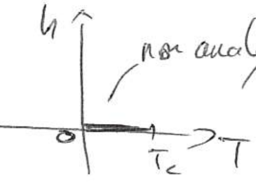
(transition from para to ferro)

$t = \frac{T_c - T}{T_c}$ reduced temp.

Remark. critical expo β indicates sing. behavior along coexistence line.

$m(T=T_c, h) \propto h^{1/\delta}$

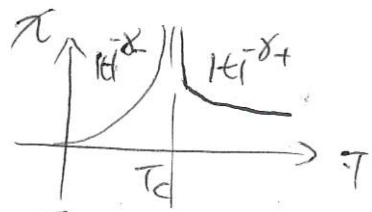
Th plane



Remark. critical expo δ occurs in Th plane.

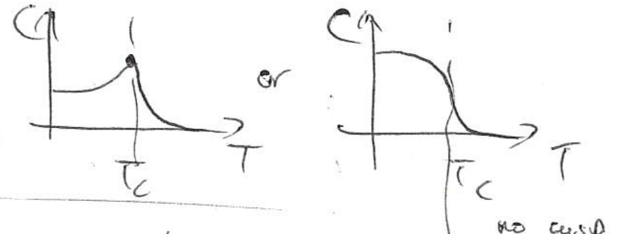
$\chi_T = \frac{\partial m}{\partial h}$

$\chi_{\pm}(T, h=0) \propto |t|^{-\gamma_{\pm}}$ most case $\gamma_+ = \gamma_- = \gamma$



Remark: infinite susceptibility near T_c in magnetic

$C(T, h=0) \propto |t|^{-\alpha}$ $\alpha > 0$ divergence
$\alpha < 0$ w/o cusp (maybe)



Remark. sing. in heat cap at T_c w/ zero field $h=0$

Long-range correlation.

$Z(h) = \mathrm{tr}\{\exp[-\beta H_0 + \beta h M]\}$ Gibbs ensemble in magnet

$\langle M \rangle = \frac{\partial \ln Z}{\partial(\beta h)}$, $\chi = \frac{\partial^2 \ln Z}{\partial(\beta h)^2} = \beta(\langle M^2 \rangle - \langle M \rangle^2) = \beta \int d^3r\, d^3r' (\langle m(\vec{r}) m(\vec{r}') \rangle - \langle m(\vec{r}) \rangle \langle m(\vec{r}') \rangle)$

w/ $\boxed{G(\vec{r} - \vec{r}') = \langle m(\vec{r}) m(\vec{r}') \rangle}$ the connected correlation func. where $\boxed{M = \int d^3r\, m(\vec{r})}$

from Wick's thm. $\boxed{\langle m(\vec{r}) m(\vec{r}') \rangle_c = G(\vec{r} - \vec{r}') - m^2}$

Centering r' at origin: $\boxed{\chi = \beta V \int d^3r \langle m(\vec{r}) m(0) \rangle_c}$

Remark. How local fluctuation affect other part of sys is measured by relating bulk response func. w/ connected correlation func.

- Such influence occurs over char. length ξ (correlation length)
- most case $G_c(\vec{r}) \equiv \langle m(\vec{r}) m(0) \rangle_c \propto \exp\left(-\frac{|\vec{r}|}{\xi}\right)$ thus $G_c \to 0$ as $|\vec{r}| \gg \xi$
- micro fluc. can be probed by scattering exp. i.e. critical opalescence in gas-liquid interface density fluctuation $G_c(\vec{r})$ w/ $\xi \sim$ λlight to scatter visible light

scaling

$\frac{k_B T \chi}{V} \sim \int d^3r\, G \sim \xi^3 G$ (*)

$\boxed{\xi_{\pm}(T, h=0) \propto |t|^{-\nu_{\pm}} \quad \text{w/ } \nu_{\pm} = \nu}$ (See ex. of estimate ν using (*))

Remark. fluc. has long wavelength i.e. $\xi \gg a$ (interparticle spacing)

(2) Mean Field Theory

Remark:
- Obs. long wave length near crt. pt. $\xi \gg a$.
- make sense to see sys. in mesoscopic scale
- ie magnetization field $\vec{m}(\vec{x})$: average elemental spins in area of $\vec{x}$.
- no variation of $\vec{x}$ within lattice spacing; $\vec{x}$ as cont' variable.
- Fourier transform valid within cutoff $\Lambda \sim 1/a$.

Corresponding probability for configs of $\vec{m}(\vec{x})$:

$$(*) \quad Z(T) = \mathrm{tr}\left[e^{-\beta H_{mic}}\right] = \int \underbrace{\mathcal{D}\vec{m}(\vec{x})}_{\text{overall allowed config}} \underbrace{W[\vec{m}(\vec{x})]}_{\text{weighted probability}}$$

Remark
- $W[\vec{m}(\vec{x})]$ is described in terms of "phenom" parameters $(\vec{m})$.
- $\vec{m}(\vec{x})$ is n-component order parameter field in dim d.
  ie $\vec{x} \in \mathbb{R}^d$, $\vec{m} \in \mathbb{R}^n$.

Landau-Ginzburg Hamiltonian

From $(*)$ gives effective Hamiltonian $\beta H = -\ln W[\vec{m}(\vec{x})]$

Construction axioms

I
- Sys w/ disconnected parts can be described as product of indep. prob.
- Σ over distributions of each parts $\to \int$ in continuum rep.

$$\beta H = \int d^d\vec{x}\, \Phi[\vec{m}(\vec{x}), \vec{x}]$$

II. now uniformity ~~depe~~ $\vec{x}$ independent.

III. Locality $\Rightarrow$ few derivatives ie short range interaction in van der Waals gas.

So

$$\boxed{\beta H = \int d^d\vec{x}\, \Phi[\vec{m}(\vec{x}), \nabla\vec{m}, \nabla^2\vec{m}, \dots]}$$

IV Analyticity w/ poly. expansion.

Remark
- Gaussian distrib. achieved via mesoscopic scale.
- this is in sense of CLT generalization.
- Φ smooth thus can be approximated via poly. expansion.

V Symmetries. ie absence of ext B field Ham invariant under rotation (R_n): $H[R_n\vec{m}(\vec{x})] = H[\vec{m}(\vec{x})]$
- even order of $\vec{m}$ are rot. invar. ie. $\vec{m}^2 = \vec{m}\cdot\vec{m}$; $m^4 = (m^2)^2$; $m^6 = (m^2)^3$.
- So as $(\nabla\vec{m})^2$ ie all direction in space are equivalent.
  else more terms are allowed ie $(\nabla^2\vec{m})^2$, $m^2(\nabla\vec{m})^2$.

Landau-Ginzburg Hamiltonian:

$$\boxed{\beta H = \beta F_0 + \int d^d\vec{x}\left[\frac{t}{2}m^2(\vec{x}) + u m^4(\vec{x}) + \frac{K}{2}(\nabla m)^2 + \cdots - \vec{h}\cdot\vec{m}(\vec{x})\right]}$$

Stability caveat
- probability from MFT config shouldn't diverge for $\vec{m} \gg 1$.
- thus LGH depends on set of phenom. parameter $\{t, u, K, \dots\}$

Saddle pt approximation

$$Z = \int \mathcal{D}\vec{m}(\vec{x}) \exp\left\{-\beta \overset{\text{LGH}}{H}[\vec{m}(\vec{x})]\right\}$$; Recalled: $H \sim \int d^d\vec{x}\left(\frac{t}{2}m^2(\vec{x}) + um^4 + vm^6 + \frac{K}{2}(\nabla m)^2 + \cdots - \vec{h}\cdot\vec{m}\right) + F_0$

Remark: • $K > 0$ for ie parallel magnetization & stability.

Saddle pt approx.

$$Z \approx Z_{sp} = e^{-\beta F_0}\int d\vec{m}\, \exp\left[-V\left(\frac{t}{2}m^2 + um^4 + \cdots - \vec{h}\cdot\vec{m}\right)\right]$$

where

$$\beta F_{sp} = -\ln Z_{sp} \approx \beta F_0 + V \min\{\Psi(\vec{m})\}$$

with

$$\Psi(\vec{m}) \equiv \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + \cdots - \vec{h}\cdot\vec{m}$$

min Ψ by $\Psi'(\vec{m}) = 0 \Rightarrow \exists\ \bar{m}$ s.t. $\Psi'(\bar{m}) = t\bar{m} + 4u\bar{m}^3 + \cdots - h = 0$.

Remark. • $\Psi(\vec{m})$ analytic but βF_{sp} might not

• But in therm limit $V \to \infty$ and for finite V integral analytic.

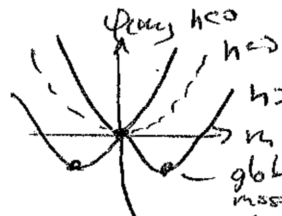
• around crt. pt $|\vec{m}|$ small, thus only lowest powers necessary.

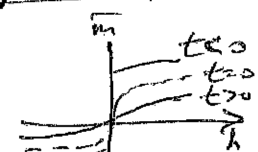
Condition of parameters: t, u, K.

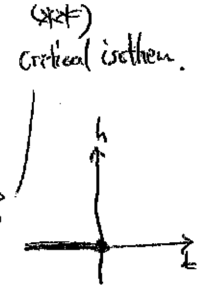
Given $\Psi(\vec{m}) = \frac{t}{2}\vec{m}^2 + u\vec{m}^4 + \cdots - \vec{h}\cdot\vec{m}$

$t > 0 \quad \bar{m} = \frac{h}{t}$

$t < 0$, need $u > 0$ (stability)







Remark. • $(t, u, K, \ldots)$ are analytic of temp.

• expansion around crt. pt:

$$t(T,\ldots) = a_0 + a_1(T - T_c) + O(T-T_c)^2 \qquad - \boxed{t = \frac{T - T_c}{T_c}} \Rightarrow a_0 = 0 \quad a_1 = a > 0$$

$$u(T,\ldots) = u + u_1(T-T_c) + O(T-T_c)^2$$

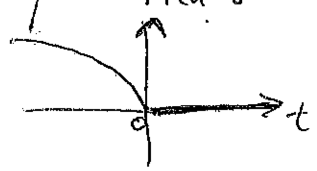
$$K(T,\ldots) = K + K_1(T-T_c) + O(T-T_c)^2$$

$\Big\}\ u, K > 0$ b/c stability near crt. pt.

Thus $(a, u, K) > 0$

ex. near crt pt. $h = 0$ (see graphs above) then $\Psi' = t\bar{m} + 4u\bar{m}^3$

thus $\bar{m}(h=0) = \begin{cases} 0 & t > 0 \\ \sqrt{\frac{-t}{4u}} \text{ or } \sqrt{\frac{a}{4u}}(T_c - T)^{1/2} & t < 0 \end{cases} \Rightarrow \beta = \frac{1}{2}$

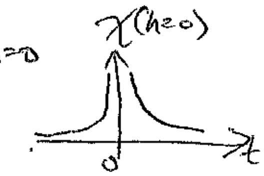


• along critical isotherm (**) $t=0$, $\Psi' = \cancel{t\bar{m}} + 4u\bar{m}^3 - h = 0$

so $\bar{m}(t=0) = \left(\frac{h}{4u}\right)^{1/3} \Rightarrow \delta = 3$

• magnetization by ext. field then $t\bar{m} + 4u\bar{m}^3 = h$ longitudinal susceptibility $\chi_\ell = \frac{\partial \bar{m}}{\partial h}\Big|_{h=0}$

$$\chi_\ell^{-1} = \frac{\partial h}{\partial \bar{m}}\Big|_{h=0} = t + 12u\bar{m}^2 = \begin{cases} t & t > 0 \quad h = 0 \\ -2t & t < 0 \quad h = 0 \end{cases}$$



thus $\chi \sim A|t|^{-\gamma} \Rightarrow \gamma = 1$

• Free energy $\beta F = \beta F_0 + V\Psi(\bar{m}) = \beta F_0 + V\begin{cases} 0 & t > 0 \\ -t^2/16u & t < 0 \end{cases}$ for $t = a(T - T_c)$ $\frac{\partial}{\partial T} \sim a\frac{\partial}{\partial t}$

then

$$C(h=0) = -T\frac{\partial^2 F}{\partial T^2} \approx -T_c a^2 \frac{\partial^2}{\partial t^2}(k_B T_c \beta F) = C_0 + V k_B a^2 T_c^2 \times \begin{cases} 0 & t > 0 \\ 1/8u & t < 0 \end{cases}$$ here $\alpha = 0$.

2. Spontaneous Sym. Breaking & Goldstone Modes.

<u>Remark</u> • At zero field, H_{micro} rotational invariant but low-temp phase is not.

• Spontaneous symmetry breaking occurs when direction of $\vec{M}$ specified.
The corresponding long-range order indicated majority of spins in sys. now oriented in $\vec{M}$.

• global sym. still persist by rotating all local spins together slowly s.t no change in energy occurs i.e $\vec{m}(\vec{x}) \longmapsto R(\vec{x})\vec{m}(\vec{x})$

• $R(\vec{x})$ long wavelength variation w/ little energy

• the low energy excitation is called Goldstone mode.

<u>ex</u> (superfluidity)

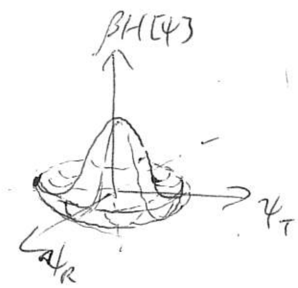
possesses macroscopic occupation of single quantum ground state in area of $\vec{x}$:

$$\psi(\vec{x}) \equiv \psi_R(\vec{x}) + i\psi_I(\vec{x}) = |\psi(\vec{x})| e^{i\theta(\vec{x})}$$

also as order parameter — not observable.

<u>Remark</u> in superfluidity $n=2$ thus $\vec{m}(\vec{x}) = (\psi_R(\vec{x}), \psi_I(\vec{x}))$ s.t

$$\beta H = \beta F_0 + \int d^dx \left[\frac{K}{2}|\nabla\psi|^2 + \frac{t}{2}|\psi|^2 + u|\psi|^4 + \cdots \right]$$



Now for slowly varying phase $\psi(\vec{x}) = \bar{\psi} e^{i\theta(\vec{x})}$ w/ $|\bar{\psi}| \ll 1$

$$\beta H = \beta H_0 + \frac{\bar{K}}{2}\int d^dx\, (\nabla\theta)^2 \quad \cancel{w/\ \bar{K} = K\bar{\psi}^2}$$

• rotation sym $\Rightarrow \theta \longmapsto \theta + \theta_0$ w/ no change in energy
thus energy density is $\theta(\vec{x})$ dependence.

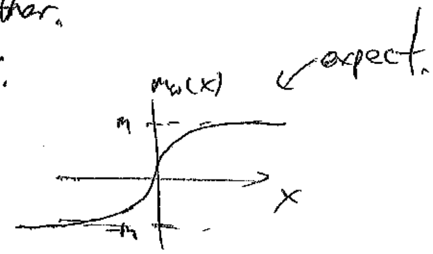
• incorporation of both normal and superfluid phases $\Rightarrow \bar{K} \propto \bar{\psi}^2 \propto t$
thus phase vanished slowly as $t \to 0$.

• this allows fourier mode $\theta(\vec{x}) = \frac{1}{\sqrt{V}} \sum_q e^{i\vec{q}\cdot\vec{x}} \theta_q$ s.t

$$\beta H = \beta H_0 + \frac{\bar{K}}{2} \sum_q q^2 |\theta(\vec{q})|^2$$ where q small at long wavelength $q = \frac{2\pi}{\lambda}$

Discrete Symmetry Breaking w/ Domain Walls

- One component scalar field has two possible magnetization values.
- But these two states can't can't deform into one another.
- these two states are separated by sharp domain walls.



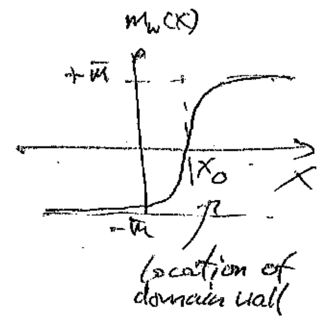
At $t<0$, $h=0$, transition occurs s.t.

$m(x\to-\infty)=-\bar{m}$ and $m(x\to+\infty)=+\bar{m}$

Consider $\int dx\left[\frac{K}{2}\left|\frac{dm}{dx}\right|^2+\frac{t}{2}m^2+um^4\right]$ — H_{eff}

EL eqn $\Rightarrow$ $K\frac{d^2 m_w}{dx^2}=tm_w(x)+4um_w(x)^3$ where energy minimized at $m=m_w$

Let $K=1$ for simplicity.



Using $\frac{d^2\tanh(ax)}{dx^2}=-2a^2\tanh(ax)\left[1-\tanh^2(ax)\right]$ we get

$$m_w(x)=\bar{m}\tanh\left[\frac{x-x_0}{\omega}\right] \quad w/ \quad w=\sqrt{\frac{2K}{-t}} \quad \bar{m}=\sqrt{\frac{-t}{4u}}$$

width of domain

Remark.
- Width diverges as $(T_c-T)^{-1/2}$

Free energy required to form domain wall

$$\beta F_w=\beta F[m_w(x)]-\beta F[\bar{m}]=\int d^dx\left[\frac{K}{2}\left(\frac{dm_w}{dx}\right)^2+\frac{t}{2}(m_w^2-\bar{m}^2)+u(m_w^4-\bar{m}^4)\right]$$

energy under uniform magnetization

sub $m_w(x)$ thus

$$\beta F_w=-\frac{t}{2}\bar{m}^2\int d^dx\cosh^{-4}\left(\frac{x-x_0}{w}\right)=-\frac{2}{3}t\bar{m}^2wA$$

cross-sectional area of sys. normal to x-direction

Remark
- Approaching phase trans. vanished as ; $\beta F_w\propto(T_c-T)^{3/2}$

Order of transitions

1st order transitions : order parameter $m=\frac{\partial F}{\partial h}$ goes to zero discontinuously
(or first derivative of free energy w.r.t ext. field is discontinuous)

2nd order transitions : continuous in m but discontinuous in $\frac{\partial^2 F}{\partial h^2}$

(3.) Scattering & Fluctuations

- Microscopic fluctuations at length scale $\xi \sim \lambda$ can be probed by scattering experiment.
- For elastic scattering

$\vec{q} = |\vec{k}_i - \vec{k}_s| \quad q = 2k\sin\frac{\theta}{2} \sim 2k\sin\theta$

- Using Fermi golden rule, scattering amplitude:

$$A(\vec{q}) \propto \langle \underbrace{\vec{k}_s \otimes f}_{\text{final state}} | \underbrace{U}_{\text{scatter. potential}} | \underbrace{\vec{k}_i \otimes i}_{\text{init. state}} \rangle \propto \underbrace{\delta(\vec{q})}_{\text{scattering from individual element}} \int d^d\vec{x}\, e^{i\vec{q}\cdot\vec{x}} \underbrace{\rho(\vec{x})}_{\text{global density of scatters}}$$

Remark
- $\rho(\vec{x})$ depends on nature of probe (incident wave)
  - light scattering sense atomic density
  - electron scattering sense charge density
  - neutron scattering sense magnetization density

  } in sense of time average config. (nature of Fermi Golden rule)

thus scattering intensity is

$$S(\vec{q}) \propto \langle |A(\vec{q})|^2 \rangle \propto \langle |\rho(\vec{q})|^2 \rangle \qquad \text{note: } \langle\ \rangle \text{ either thermal or time average.}$$

Remark: Study long-wavelength fluc. at small angles or small k.

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ex Scattering by magnetization, LGH gives probability distribution

$$\mathcal{P}[\vec{m}(\vec{x})] \propto \exp\left\{-\int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4\right]\right\}$$

- Given most probable $\vec{m}(\vec{x}) = \bar{m}\,\hat{e}_1$ in direction $\hat{e}_1$, fluctuation around the config.:

$$\vec{m}(\vec{x}) = [\bar{m} + \underbrace{\phi_\ell(\vec{x})}_{\text{long. fluc.}}]\hat{e}_1 + \underbrace{\sum_{\alpha=2}^{n} \phi_{t,\alpha}(\vec{x})\hat{e}_\alpha}_{\text{trans fluc. in } n-1 \text{ direction}}$$

- taken $\mathcal{O}(m^2)$:

$$(\nabla m)^2 = (\nabla\phi_\ell)^2 + (\nabla\phi_t)^2 \ ; \quad m^2 = \bar{m}^2 + 2\bar{m}\phi_\ell + \phi_\ell^2 + \phi_t^2$$

$$m^4 = \bar{m}^4 + 4\bar{m}^3\phi_\ell + 6\bar{m}^2\phi_\ell^2 + 2\bar{m}^2\phi_t^2 + \mathcal{O}(\phi_\ell^3, \phi_t^3)$$

thus

$$\beta H = -\ln P = V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \int d^dx \left[\frac{K}{2}(\nabla\phi_\ell)^2 + \frac{t + 12u\bar{m}^2}{2}\phi_\ell^2\right] + \int d^dx \left[\frac{K}{2}(\nabla\phi_t)^2 + \frac{t + 4u\bar{m}^2}{2}\phi_t^2\right] + \cdots$$

In Kardar, $\frac{1}{\xi_\ell^2} \equiv \frac{t + 12u\bar{m}^2}{K} \qquad \frac{1}{\xi_t^2} \equiv \frac{t + 4u\bar{m}^2}{K}$

By EL eqn,

$$\frac{\partial^2 \Psi(m)}{\partial\phi_\ell^2} = t + 12u\bar{m}^2 = \frac{K}{\xi_\ell^2} = \begin{cases} t & t > 0 \\ -2t & t < 0 \end{cases}$$

$$\frac{\partial^2 \Psi(m)}{\partial\phi_t^2} = t + 4u\bar{m}^2 = \frac{K}{\xi_t^2} = \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} \quad \text{(Goldstone mode)}$$

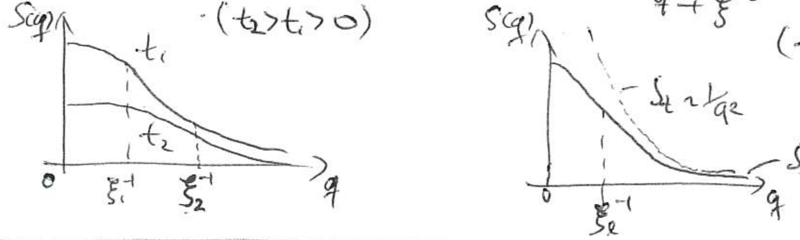
- using Fourier modes $\boxed{\phi(\vec{x}) = \sum_{\vec{q}} \phi_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} / \sqrt{V}}$

now,

$$\mathcal{P}(\{\phi_{\ell,\vec{q}}\, ; \phi_{t,\vec{q}}\}) \propto \prod_{\vec{q}} \exp\left\{-\frac{K}{2}(q^2 + \xi_\ell^{-2})|\phi_{\ell,\vec{q}}|^2\right\} \cdot \exp\left\{-\frac{K}{2}(q^2 + \xi_t^{-2})|\phi_{t,\vec{q}}|^2\right\}$$

Gives $\langle \phi_{\alpha,\vec{q}}\, \phi_{\beta,\vec{q}'} \rangle = \frac{\delta_{\alpha\beta}\,\delta_{\vec{q},\vec{q}'}}{K(q^2 + \xi_\alpha^{-2})}$ two pt correlation $\alpha, \beta = \ell, t$.

Scattering Intensity in Lorentzian form $S(\vec{q}) \propto \frac{1}{q^2+\xi^{-2}}$ due to magnetic fluctuations



$(t<0)$

Correlation functions

mean $\langle \phi_\alpha(\vec{x})\rangle = \langle m_\alpha(\vec{x}) - \bar{m}_\alpha\rangle = 0$ but connected correlation func.

$$G^c_{\alpha,\beta}(\vec{x},\vec{x}') \equiv \langle (m_{\alpha,\vec{x}} - \bar{m}_\alpha)(m_{\beta,\vec{x}'} - \bar{m}_\beta)\rangle = \langle \phi_{\alpha,\vec{x}}\, \phi_{\beta,\vec{x}'}\rangle = \frac{1}{V}\sum_{\vec{q},\vec{q}'} e^{i\vec{q}\cdot\vec{x} + i\vec{q}'\cdot\vec{x}'}\langle \phi_{\alpha,\vec{q}}\, \phi_{\beta,\vec{q}'}\rangle$$

$$= \frac{\delta_{\alpha,\beta}}{V}\sum_{\vec{q}} \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{K(q^2+\xi_\alpha^{-2})} \equiv -\frac{\delta_{\alpha\beta}}{K} I_d(\vec{x}-\vec{x}', \xi_\alpha) \qquad (***)$$

in continuum, $I_d(\vec{x},\xi) = -\int \frac{d^dq}{(2\pi)^d}\frac{e^{i\vec{q}\cdot\vec{x}}}{q^2+\xi^{-2}}$

techniques:

$$\nabla^2 I_d(x) = \int \frac{d^dq}{(2\pi)^d}\frac{q^2 e^{i\vec{q}\cdot\vec{x}}}{q^2+\xi^{-2}} = \int \frac{d^dq}{(2\pi)^d}\left(1 - \frac{\xi^{-2}}{q^2+\xi^{-2}}\right)e^{i\vec{q}\cdot\vec{x}} = \delta^d(\vec{x}) + \frac{I_d(\vec{x})}{\xi^2}$$

thus solve "in spherical sym."

$$(*) \quad \frac{d^2 I_d}{dx^2} + \frac{d-1}{x}\frac{dI_d}{dx} = \frac{I_d}{\xi^2} + \delta^d(\vec{x}) \qquad \text{Helmholtz eqn.}$$

soln $I_d(\vec{x}) \propto \frac{\exp(-x/\xi)}{x^p}$ solves $\underbrace{\frac{d^2 I_d}{dx^2} = \left(\frac{p(p+1)}{x^2} + \frac{2p}{x\xi} + \frac{1}{\xi^2}\right) I_d}_{(**)}$ where $\frac{dI_d}{dx} = -\left(\frac{p}{x} + \frac{1}{\xi}\right) I_d$

If $x \neq 0$, forcing $(*)$ & $(**)$ agree upon each other.

then

$$\frac{p(p+1)}{x^2} + \frac{2p}{x\xi} + \frac{1}{\xi^2} - \frac{p(d-1)}{x^2} - \frac{(d-1)}{x\xi} = \frac{1}{\xi^2}$$

- for $(x \ll \xi)$ $\frac{1}{x^2}$ dominates thus $p(p+1) = p(d-1) \Rightarrow p = d-2$ recovers coulomb interaction.

proper normalization

$$I_d(x) \simeq C_d(x) = \frac{x^{2-d}}{(2-d)S_d}$$

- for $(x \gg \xi)$ $\frac{1}{x\xi}$ dominates thus $p = \frac{d-1}{2}$ for $x \simeq \xi$

$$I_d(x) \simeq \frac{\xi^{(3-d)/2}}{(2-d)S_d\, x^{(d-1)/2}}\exp(-x/\xi) \qquad \text{note } \xi^{\frac{(3-d)}{2} - \left(\frac{d-1}{2}\right)} \simeq x^{2-d}$$

Char. length near crt. pt. long. correlation length

$$\xi_\ell = \begin{cases} t^{-1/2}/\sqrt{K} & t>0 \\ (-2t)^{-1/2}/\sqrt{K} & t<0 \end{cases} \qquad \text{from } (***) \qquad \text{at sing. } \xi_\pm \simeq \underbrace{\xi_0}_{\sqrt{K}} B_\pm |t|^{-\nu_\pm} \quad \nu_\pm = 1/2 \text{ univ.}$$

$$\xi_t = \begin{cases} \xi_\ell & t>0 \\ \infty & t<0 \quad (\text{w/o fluc.}) \end{cases}$$

Suceptibility from $I_d(x)$ $\sim T_c$ correlation decays as $x^{-(d-2)}$ (actually $x^{-(d-2+\eta)}$)

then from 2nd cumulant, for $t<0$ & $x \ll \xi$

$$\chi_\ell \propto \int d^dx\, G^c_\ell(\vec{x}) \propto \int_0^{\xi_\ell}\frac{d^dx}{x^{d-2}} \propto \xi_\ell^2 \simeq A_\pm t^{-1}$$

But for $T<T_c$ not upper cutoff,

$$\chi_t \propto \int d^dx\, G^c_\ell(\vec{x}) \propto \int_0^L \frac{d^dx}{x^{d-2}} \propto L^2 \qquad L \text{ sys. size}$$

3. Lower critical dimension

ex superfluid w/ local order parameter $\psi(x) = |\psi(x)| e^{i\theta(x)}$ thus

$$P[\theta(x)] \propto \exp\left[-\frac{K}{2}\int d^d x (\nabla\theta)^2\right] \stackrel{F.T.}{=} \exp\left[-\frac{K}{2}\sum_q q^2 |\theta(\vec{q})|^2\right] \propto \prod_q P(\theta_q)$$

Remark
- each mode $\theta_{\vec{q}}$ indept. random variable w/ zero mean in gaussian distrb.
- $\langle \theta_{\vec{q}} \theta_{\vec{q}'} \rangle = \dfrac{\delta_{\vec{q},-\vec{q}'}}{K q^2}$

In real space $\langle \theta_{\vec{x}} \theta_{\vec{x}'} \rangle = \frac{1}{V}\sum_{\vec{q},\vec{q}'} e^{i\vec{q}\cdot\vec{x} + i\vec{q}'\cdot\vec{x}'} \langle \theta_{\vec{q}} \theta_{\vec{q}'} \rangle = \frac{1}{V}\sum_{\vec{q}} \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{K q^2}$

In cont' limit

$\sum_{\vec{q}} \to V\int \frac{d^d q}{(2\pi)^d}$: $\langle \theta_{\vec{x}} \theta_{\vec{x}'} \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{K q^2} \equiv -\frac{C_d(\vec{x}-\vec{x}')}{K}$, $C_d(\vec{x}) = -\int \frac{d^d q}{(2\pi)^d} \frac{e^{i\vec{q}\cdot\vec{x}}}{q^2}$ Coulomb potential

Ansatz. $\nabla^2 C_d(\vec{x}) = \delta^d(\vec{x})$

By Gauss thm, $1 = \int d^d x \, \delta^d(x) = \int d^d x \, \nabla^2 C_d(x) = \oint dS_d \cdot \nabla_d \stackrel{\text{sph. sym}}{=} \oint dS_d \left(\frac{dC_d}{dx}\right) = \frac{dC_d}{dx} S_d$

where $S_d = \dfrac{2\pi^{d/2}}{(\frac{d}{2}-1)!}$ thus $C_d(x) = \dfrac{x^{2-d}}{(2-d)S_d} + C_0$

Long dist. behavior of $C_d(x)$ ie $x \to \infty$

$$\lim_{x\to\infty} C_d(x) = \begin{cases} C_0 & d>2 \\ \dfrac{x^{2-d}}{(2-d)S_d} & d<2 \\ \dfrac{\ln x}{2\pi} & d=2 \end{cases}$$

Remark
- $d>2$, phase fluc finite
- $d\le 2$ " " large. Since periodic, thus long range order in phase destroyed.

Instead of phase fluc. Consider two point fluc.

$\langle \psi(\vec{x}) \psi^*(\vec{0}) \rangle = \bar{\psi}^2 \langle e^{i[\theta(\vec{x})-\theta(\vec{0})]} \rangle$; for any Gaussian variable $\langle e^{\alpha\theta} \rangle = e^{\frac{\alpha^2}{2}\langle\theta^2\rangle}$

transverse correlation

$$= \bar{\psi}^2 \exp\left[-\frac{1}{2}\langle [\theta(\vec{x})-\theta(0)]^2 \rangle\right] = \bar{\psi}^2 \exp\left[-\frac{x^{2-d}-a^{2-d}}{K(2-d)S_d}\right]$$

b/c $\langle [\theta(\vec{x})-\theta(\vec{x}')]^2 \rangle = 2\langle \theta(\vec{x})^2 \rangle - 2\langle \theta(\vec{x})\theta(\vec{x}') \rangle = \dfrac{2(|\vec{x}-\vec{x}'|^{2-d} - a^{2-d})}{K(2-d)S_d}$

thus $\lim_{x\to\infty} \langle \psi(\vec{x}) \psi^*(0) \rangle = \begin{cases} \bar{\psi}^2 & d>2 \quad \text{(order reduction)} \\ 0 & d\le 2 \quad \text{(complete order destruction)} \end{cases}$

Remark
- above is example of Mermin-Wagner thm
- thm stated that no spontaneous breaking of cont' sym in sys w/ short-range interaction in lower dim $d\le 2$.
- $d=2$ lower critical dim
- no Goldstone mode in discrete sym. ie $n=1$ thus long-range order possible down to $d=1$.

Gaussian Integrals

fluctuation.

$$I_1 = \int_{-\infty}^{\infty} d\phi\, e^{-\frac{K}{2}\phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}}\, e^{\frac{h^2}{2K}}$$ ← normalized form

w/ $$\int_{-\infty}^{\infty} d\phi\, \phi\, e^{-\frac{K}{2}\phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}}\, e^{\frac{h^2}{2K}}\, \frac{h}{K} \quad \text{etc}$$

s.t $\langle \phi \rangle = h/K \qquad \langle \phi^2 \rangle = \frac{h^2}{K^2} + \frac{1}{K}$ ← direct integration.

And $$\langle e^{-ik\phi} \rangle \equiv \exp\left[\sum_{\ell=1}^{\infty} \frac{(-ik)^\ell}{\ell!} \langle \phi^\ell \rangle_c\right] = \exp\left[-\frac{ikh}{K} - \frac{k^2}{2K}\right]$$

thus H.O.T are zeros by def (punchline)

N variables,

$$I_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\sum_{ij} \frac{1}{2} K_{ij}\phi_i\phi_j + \sum_i h_i\phi_i\right]$$; K diagonalized w/ eigvec, val $\hat{q}$, K_q

In coordinate sense, $\phi_i = \sum_q \tilde{\phi}_q \hat{q}_i$ i.e $\{\phi_i\}$ and $\{\tilde{\phi}_q\}$ are equivalent.

w/ unity Jacobian under unitary transform.

$$I_N = \prod_{q=1}^{N} \int_{-\infty}^{\infty} d\tilde{\phi}_q \exp\left[-\frac{K_q}{2}\tilde{\phi}_q^2 + \tilde{h}_q\tilde{\phi}_q\right] = \prod_{q=1}^{N} \sqrt{\frac{2\pi}{K_q}} \exp\left[\frac{\tilde{h}_q K_q^{-1} \tilde{h}_q}{2}\right]$$

or in terms of original coord.

$$I_N = \frac{\sqrt{(2\pi)^N}}{\sqrt{\det K}} \exp\left[\sum_{ij} \frac{K^{-1}_{ij} h_i h_j}{2}\right]$$

the joint characteristic func.

(*) $$\langle e^{-i\sum_j k_j\phi_j} \rangle = \exp\left[-i\sum_{ij} K^{-1}_{ij} h_i k_j - \sum_{ij} \frac{K^{-1}_{ij} k_i k_j}{2}\right]$$

thus $\langle \phi_i \rangle_c = \sum_j K^{-1}_{ij} h_j$ & $\langle \phi_i \phi_j \rangle_c = K^{-1}_{ij}$

(*) can be generalized to

$$\langle e^A \rangle = \exp\left[\langle A \rangle_c + \frac{1}{2}\langle A^2 \rangle_c\right] \quad \text{for } A = \sum_i a_i\phi_i$$ — Gauss. distrib. variable (note in sense that $k \to i$)

Recall cumulant. $\tilde{p}(k) = \langle e^{-ikx} \rangle = \int dx\, p(x)\, e^{-ikx}$

and $\ln \tilde{p}(k) = \sum_{\ell=1}^{\infty} \frac{(-ik)^\ell}{\ell!} \langle x^\ell \rangle_c$ or $\tilde{p}(k) = \exp\left(\sum_{\ell=1}^{\infty} \frac{(-ik)^\ell}{\ell!} \langle x^\ell \rangle_c\right)$

3 Gaussian Integral Cont'

Gaussian functional Integral.

- limiting case of N variable gaussian is Gaussian func. Integrals.
  ie ϕ_i on i site of d-dim lattice as lattice spacing $\to 0$, $\phi_i \to \phi(\vec{x})$ & $K_{ij} \to K(\vec{x},\vec{x}')$
  thus cont' gen.

$$(*) \quad \int_{-\infty}^{\infty} D\phi(\vec{x}) \exp\left[-\int d^dx\, d^dx' \frac{K(\vec{x},\vec{x}')}{2}\phi(\vec{x})\phi(\vec{x}') + \int d^dx\, h(\vec{x})\phi(\vec{x})\right] \propto (\det K)^{-1/2} \exp\left[\int d^dx\, d^dx' \frac{K^{-1}(\vec{x},\vec{x}')}{2} h(\vec{x}) h(\vec{x}')\right]$$

w/ $\int d^dx'\, K(\vec{x},\vec{x}')\, K^{-1}(\vec{x}',\vec{x}'') = \delta^d(\vec{x}-\vec{x}'')$

and

$$\langle \phi(\vec{x}) \rangle_c = \int d^dx'\, K^{-1}(\vec{x},\vec{x}')\, h(\vec{x}') \qquad \langle \phi(\vec{x})\phi(\vec{x}')\rangle_c = K^{-1}(\vec{x},\vec{x}')$$

Small fluc. in LG Ham.

$$\int d^dx \left[(\nabla\phi)^2 + \frac{\phi^2}{\xi^2}\right] = \int d^dx\, d^dx'\, \phi(\vec{x}')\,\underline{\delta^d(\vec{x}-\vec{x}')\,(-\nabla^2+\xi^{-2})}\,\phi(\vec{x})$$

thus

$$K(\vec{x},\vec{x}') = K\,\delta^d(\vec{x}-\vec{x}')(-\nabla^2+\xi^{-2})$$

w/ inverse kernel.

$$K\int d^dx''\, \delta^d(\vec{x}-\vec{x}'')(-\nabla^2+\xi^{2})\, K^{-1}(\vec{x}''-\vec{x}') = \delta^d(\vec{x}'-\vec{x})$$

Satisfied diff. eqn $\quad K(-\nabla^2+\xi^{-2})\,K^{-1}(\vec{x}) = \delta^d(\vec{x})$

Recall $(\nabla^2-\xi^{-2})\, I_d(\vec{x}) = \delta^d(\vec{x}) \Rightarrow \quad K^{-1}(\vec{x}) = -\frac{I_d(\vec{x})}{K} = \langle \phi(\vec{x})\phi(0)\rangle_c$ //

Fluctuation Corrections to Saddle pt

Recalled.

$$Z \approx \exp\left[-V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right)\right] \int D\phi_\ell(\vec{x}) \exp\left\{-\frac{K}{2}\int d^dx \left[(\nabla\phi_\ell)^2 + \frac{\phi_\ell^2}{\xi_\ell^2}\right]\right\} \int D\phi_t(\vec{x}) \exp\left\{-\frac{K}{2}\int d^dx\left[(\nabla\phi_t)^2 + \frac{\phi_t^2}{\xi_t^2}\right]\right\}$$

w/ F.T. $\tilde{\phi}(\vec{q}) = \int d^dx \exp(-i\vec{q}\cdot\vec{x})\,\phi(\vec{x})/\sqrt{V}$, yield corresp. eig. val $K(\vec{q}) = K(q^2+\xi^{-2})$

w/ $\ln\det K = \sum_q \ln K(\vec{q}) = V\int \frac{d^dq}{(2\pi)^d} \ln\left[K(q^2+\xi^{-2})\right]$. since $\vec{h}=0$, from (*)

Free energy

$$\beta f = -\frac{\ln Z}{V} = \frac{t\bar{m}^2}{2} + u\bar{m}^4 + \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi_\ell^{-2})\right] + \frac{n-1}{2}\int\frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi_t^{-2})\right]$$

thus

$$C_{sing} \propto -\frac{\partial^2(\beta f)}{\partial^2 t} = \begin{cases} 0 + \frac{n}{2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2+t)^2} & t>0 \\ \frac{1}{8u} + 2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2-2t)^2} & t<0 \end{cases}$$

where $C_F = \frac{1}{K^2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+\xi^{-2})^2}$

Corr. term

Integral in $C_F \sim (\text{length})^{4-d}$

$d>4$ diverges w/ cutoff $\Lambda \sim 1/a$ $\Rightarrow C_F \sim \frac{1}{K^2}\begin{cases} a^{4-d} & d>4 \\ \xi^{4-d} & d<4\end{cases}$

$d<4$ convergent