Propagator 2.6

Time evolution w/ time-independent Hamiltonian,

$$|\alpha, t_0 ; t\rangle = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right]|\alpha, t_0\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha, t_0\rangle \exp\left[-\frac{iE_{a'}(t-t_0)}{\hbar}\right]$$

then the wave-function

$$\psi(\vec{x}', t) = \sum_{a'} C_{a'}(t_0)\, U_{a'}(\vec{x}') \exp\left[-\frac{iE_{a'}(t-t_0)}{\hbar}\right] \; ; \; \psi(\vec{x}', t) \equiv \langle \vec{x}|\alpha, t_0 ; t\rangle$$

$$U_{a'}(\vec{x}') = \langle \vec{x}'|a'\rangle$$

$$C_{a'}(t_0) = \langle a'|\alpha, t_0\rangle = \int d^3x' \langle a'|\vec{x}\rangle\langle\vec{x}'|\alpha, t_0\rangle = \int d^3x'\, U^{*}_{a'}(\vec{x}')\,\psi(\vec{x}', t_0)$$

Substitute

$$\boxed{\psi(\vec{x}'', t) = \int d^3x'\, K(\vec{x}'', t ; \vec{x}', t_0)\, \psi(\vec{x}', t_0)}$$

$$\boxed{K(\vec{x}'', t ; \vec{x}', t_0) = \sum_{a'} \langle \vec{x}''|a'\rangle\langle a'|\vec{x}'\rangle \exp\left[-\frac{iE_{a'}(t-t_0)}{\hbar}\right] \text{ or } \langle\vec{x}''|\exp\left[-\frac{iH(t-t_0)}{\hbar}\right]|\vec{x}'\rangle}$$

is known as the propagator.

Remarks

If K & $\psi(\vec{x}', t_0)$ is given, Schrödinger's wave mech. is a causal theory (deterministic)
But when measurement intervenes, wave func. change abruptly and unpredictably.

Properties of $K(\vec{x}'', t ; \vec{x}', t_0)$

1. For $t > t_0$ K Satisfies SE ($\vec{x}''$, t as variable ; $\vec{x}'$, t_0 fixed)
2. $\lim_{t \to t_0} K(\vec{x}'', t ; \vec{x}', t_0) = \delta^3(\vec{x}'' - \vec{x}')$

Remarks

K function of $\vec{x}''$ is a func. (wavefunc.) of a particle at t that originally localized precisely at $\vec{x}'$ at earlier time t_0.

Can be summarized:

$$\boxed{\begin{aligned} &\hat{S}\, K(\vec{x}'', t ; \vec{x}', t_0) = -i\hbar\,\delta^3(\vec{x}'' - \vec{x}')\,\delta(t - t_0) \\ &K(\vec{x}'', t ; \vec{x}', t_0) = 0 \quad \text{for } t < t_0. \end{aligned}}$$

$\hat{S} \equiv \frac{-\hbar^2}{2m}\nabla^2 + V(x) - i\hbar\frac{\partial}{\partial t}$

ex

Form of propagator based on potential particle experiences.
For free particle (From 1.7.32)

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar}\right) \qquad \langle x''|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x''}{\hbar}\right)$$

$$K(\vec{x}'', t ; \vec{x}', t_0) = \langle x''|x'\rangle \exp\left[-\frac{iE_{a'}(t-t_0)}{\hbar}\right)$$

Substitute & complete square

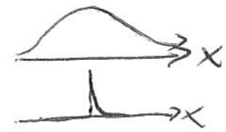
$$K(x'', t ; x', t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]$$

to address spreading of gaussian wave func.

Simplify $\frac{mx^2}{2\hbar t} \sim \frac{\Delta p \Delta x}{2\hbar}$ since $\Delta p \Delta x \geq \hbar$
$(x' = 0,\ t_0 = 0)$

$\Delta p \ll 1 \quad \Delta x \gg$

$\Delta p \gg 1 \quad \Delta x \ll 1$



Space-time integral of K at simplest sense, $x'' = x'$; $t_0 = 0$

$$G(t) = \int d^3x'\, K(x', t; x', 0) = \sum_{a'} \exp\left(\frac{-iE_{a'}t}{\hbar}\right)$$

Remarks:
- process as taking trace of $U(t,0)$
- trace is indep of rep.
- if $|a'\rangle$ as basis $U(t,0)$ diagonal
- Set $\beta = \frac{it}{\hbar}$ then has resemblance of partition func.

i.e. $\int_0^\infty e^{i(k-z_0)x}\, dx$

need to work out how to do complex integral

Laplace-Fourier trans.

$$\tilde{G}(E) = -\frac{i}{\hbar}\int_0^\infty dt\, G(t) \exp\left(\frac{iEt}{\hbar}\right);\ E \to E + i\varepsilon;\ \tilde{G}(E) = \sum_{a'} \frac{1}{E - E_{a'}}$$

exhibited simple poles of $\tilde{G}(E)$ in complex plane.

Recall

$\psi(x,t)$

$\langle x' | \alpha, t_0; t\rangle$ $\qquad$ $\langle x', t | \alpha, t_0\rangle$ Heisenberg pic.

$|\alpha, t_0; t\rangle$ moves w/ time

$\langle x', t|$ moves oppositely w/ time.

Keyword "State ket"

$\boxed{K(x'', t; x', t_0) = \langle x'', t | x', t_0\rangle}$ in Heisenberg pic.

$\langle x'', t | x', t_0\rangle$ as prob. amplitude of particle prepare at t_0, x' to be found at t, x''

has expression "~~trandat~~ transition amplitude"

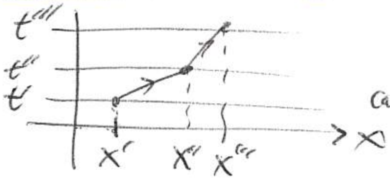
refer to time evolution.

- View as unitary trans. connecting two sets of base kets at diff. times.
- write $\langle x'', t'' | x', t'\rangle$ for symmetric beauty.

Use property $\boxed{\int |x'', t''\rangle\langle x'', t''| = 1}$ $\longrightarrow$ $\langle x''', t''' | x', t'\rangle = \int \langle x''', t''' | x'', t''\rangle \langle x'', t'' | x', t'\rangle\, d^3x''$

w/ $t''' > t'' > t'$

has mental pic



can divide as many subinterval as wished.

Path Integral. init. pt (x_1, t_1); final. pt (x_N, t_N) time interval in $N-1$ parts: $t_j - t_{j-1} = \frac{t_N - t_1}{N-1}$

So $\langle x_N, t_N | x_1, t_1\rangle = \int dx_{N-1}\, dx_{N-2} \cdots dx_2\, \langle x_N t_N | x_{N-1} t_{N-1}\rangle \langle x_{N-1} t_{N-1} | x_{N-2} t_{N-2}\rangle \cdots \langle x_2 t_2 | x_1 t_1\rangle$

Dirac: $\exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, L_{cl}(x,\dot{x})\right]$ corresponds to $\langle x_2 t_2 | x_1 t_1\rangle$ how?

First, there exists classical path s.t.

$$S(n,n-1) \equiv \int_{t_{n-1}}^{t_n} dt\, L_{cl}(x,\dot{x})$$ — $\int$ over time interval

Second, back to Dirac's statement above,

$$\prod_{n=2}^{N} \exp\left[\frac{i}{\hbar}S(n,n-1)\right] = \exp\left[\frac{i}{\hbar}S(N,1)\right]$$ for a definite path!

Third $\langle x_N, t_N | x_1, t_1\rangle \sim \sum_{\text{all path}} \exp\left[\frac{i}{\hbar}S(N,1)\right]$; $\hbar \to 0$ back to cl. path

sum over all path

recall $K(x'',t;x',t_0) = \langle x'',t | x',t_0\rangle$
and for free particle $K(x'',t;x',t_0) \sim \exp\left(\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right)$

Now (Feynman's approach)

Ansatz, $\boxed{\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \frac{1}{\omega\Delta t}\exp\left[\frac{i}{\hbar}S(n,n-1)\right]}$; $\frac{1}{\omega\Delta t}$ weight factor due to normalization of eig. kets must have dim. 1/length.

For Δt small, valid to take cl. path be straight line.

$$S(n,n-1) = \int_{t_{n-1}}^{t_n} dt\left[\frac{m\dot{x}^2}{2} - V(x)\right] = \Delta t\left\{\frac{m}{2}\left[\frac{x_n - x_{n-1}}{\Delta t}\right]^2 - V\left(\frac{x_n + x_{n-1}}{2}\right)\right\}$$

For $V=0$, $\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \left[\frac{1}{\omega(\Delta t)}\right]\exp\left[\frac{im(x_n - x_{n-1})^2}{2\hbar\Delta t}\right]$

If $t_n = t_{n-1}$, orthonormality $\langle x_n, t_n | x_{n-1}, t_n\rangle = \delta(x_n - x_{n-1})$

So above expression must be reconciled as $t_n \to t_{n-1}$ ie $\Delta t \to 0$

Since $\int_{-\infty}^{\infty} dy \exp\left(\frac{imy^2}{2\hbar\Delta t}\right) = \sqrt{\frac{2\pi i\hbar\Delta t}{m}}$

$$\lim_{\Delta t\to 0}\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left(\frac{imy^2}{2\hbar\Delta t}\right) = \delta(y)$$ thus $\boxed{\frac{1}{\omega(\Delta t)} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}}$

Defining $\int_{x_1}^{x_N} D[x(t)] = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N-1}{2}}\int dx_{N-1}\cdots dx_2$

$$\langle x_N, t_N | x_1, t_1\rangle = \int_{x_1}^{x_N} D[x]\exp\left[\frac{i}{\hbar}\int_{t_1}^{t_N} dt\, L_{cl}(x,\dot{x})\right]$$

It left to show such formation is well defined. ie $\langle x_N, t_N | x_1, t_1\rangle$ satisfies SE.

let $y = x_n - x_{n-1}$ $\Delta t \to 0$ or $\Delta t \ll 1$

$$\delta(y) = \frac{1}{\omega\Delta t}\exp\left(\frac{imy^2}{2\hbar\Delta t}\right)$$

thus, $\langle x_N, t_N | x_1, t_1\rangle = \int_{x_1} D[x]\exp\left[\frac{i}{\hbar}S(N,1)\right]$

$$\int_{-\infty}^{\infty}\delta(y)dy = \frac{1}{\omega\Delta t}\int_{-\infty}^{\infty}dy\exp\left(\frac{imy^2}{2\hbar\Delta t}\right) = \frac{1}{\omega\Delta t}\sqrt{\frac{2\pi i\hbar\Delta t}{m}}$$

thus $\frac{1}{\omega\Delta t} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \Rightarrow \langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left(\frac{im(x_n - x_{n-1})^2}{2\hbar(t_n - t_{n-1})}\right)$

2.7 Constant potential & Gauge invariant.

Remarks
- Observable effects such at time evolution of $\langle x \rangle$, $\langle S \rangle$ always depend on energy diff. "review 2-1.47"
- gravity in cl is said to be a pure geometry theory due to mass cancelation
- not true in QM, mass remain:

$$\left[-\left(\frac{\hbar^2}{2m}\right)\nabla^2 + m\Phi_{grav}\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Gauge Trans. in EM

$$\vec{E} = -\nabla\phi \qquad \vec{B} = \nabla\times\vec{A} \qquad H = \frac{1}{2m}\left(\vec{P} - \frac{e\vec{A}}{c}\right)^2 + e\phi$$

$$\frac{dx_i}{dt} = \frac{[x_i, H]}{i\hbar} = \frac{P_i - eA_i/c}{m}$$

P_i canonical mom

$\Pi = \frac{P_i - eA_i/c}{m}$ mech. mom

$$\boxed{[\Pi_i, \Pi_j] = i\hbar\frac{e}{c}\varepsilon_{ijk}B_k}$$

Now $H = \frac{\Pi^2}{2m} + e\phi$ Lorentz force (QM) $m\ddot{x} = \dot{\Pi} = e\left[\vec{E} + \frac{1}{2c}\left(\vec{v}\times\vec{B} - \vec{B}\times\vec{v}\right)\right]$

Study Schrödinger's wave eqn. w/ ϕ & $\vec{A}$
- take the complex conjugate of SG
- yields $\frac{\partial\rho}{\partial t} + \nabla'\cdot\vec{j} = 0 \quad (\nabla = j)$
- $\rho = |\psi|^2$; $\vec{j} = \frac{\hbar}{m}\operatorname{Im}(\psi^*\nabla\psi) - \frac{e}{mc}\vec{A}|\psi|^2$

$$\vec{j} = \left(\frac{\rho}{m}\right)\left(\nabla S - \frac{e\vec{A}}{c}\right)$$

- $\int d^3x\, \vec{j} = \frac{\langle \Pi \rangle}{m}$

2.7 Gauge trans.

$\vec{A} \to \vec{A} + \nabla\Lambda$

$\phi \to \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$

$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$\vec{B} = \nabla \times \vec{A}$

ex take $\vec{B} = B\hat{z}$

1. $A_x = -\frac{By}{2}$ $A_y = \frac{Bx}{2}$ $A_z = 0$
2. $A_x = -By$ $A_y = 0$ $A_z = 0$

$A \to A - \nabla\left(\frac{Bxy}{2}\right)$ w/ $\Lambda = \frac{Bxy}{2}$

obs. that

$\frac{dP_x}{dt} = -\frac{\partial H}{\partial x}$; $\frac{\partial P_y}{\partial t} = -\frac{\partial H}{\partial y}$

P_x is const in (1) not (2)

so canonical mom $\vec{p}$ not gauge. inv But Π is

Now check $\langle x \rangle$

Assume $\tilde{A} = A + \nabla\Lambda \leftrightarrow |\tilde{\alpha}\rangle$

$A \leftrightarrow |\alpha\rangle$

required: $\langle\alpha|\vec{x}|\alpha\rangle = \langle\tilde{\alpha}|\vec{x}|\tilde{\alpha}\rangle$

$\langle\alpha|\Pi|\alpha\rangle = \langle\tilde{\alpha}|\Pi|\tilde{\alpha}\rangle$

Remarks

demands expectation values in QM behaves as in classical, similarly under ~~GT~~ GT.

$\langle x\rangle, \langle\Pi\rangle$ not to change

In additional, $\langle\alpha|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\alpha}\rangle$ (a)

need operator g st

$|\tilde{\alpha}\rangle = g|\alpha\rangle$ inv. if $g^\dagger x g = x$ (b)

and $g^\dagger\left(\vec{p} - \frac{e}{c}\vec{A} - \frac{e\nabla\Lambda}{c}\right)g = \vec{p} - \frac{e\vec{A}}{c}$ (c)

Ansatz, ~~$g = \exp\left[\frac{-ie\Lambda}{\hbar c}\right]$~~

$$\boxed{g = \exp\left[\frac{ie\Lambda}{\hbar c}\right]}$$

g unitary satisfies (a) (b) (c)

Another way is use SE

$H|\alpha;t\rangle = i\hbar\frac{\partial}{\partial t}|\alpha;t\rangle$ GT. $A \mapsto A + \nabla\Lambda$

$H'|\tilde{\alpha};t\rangle = i\hbar\frac{\partial}{\partial t}|\tilde{\alpha};t\rangle$

$H = \frac{(p - e\frac{A}{c})^2}{2m} + e\phi \leftrightarrow$ eigket $|\alpha;t\rangle$

$H' = \frac{(p - e\frac{A}{c} - e\nabla\Lambda/c)^2}{2m} + e\phi \leftrightarrow |\tilde{\alpha;t}\rangle$

If $|\tilde{\alpha},t\rangle = \exp\left(\frac{ie}{\hbar c}\Lambda\right)|\alpha,t\rangle$ then $g^\dagger H' g = H$

Also $\tilde{\psi} = \exp\left(\frac{ie\Lambda}{\hbar c}\right)\psi$

SHO & EM

Consider $H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2$ for $A = \frac{1}{2}\vec{r}\times\vec{B} = \frac{B}{2}(-y\hat{x} + x\hat{y})$

$$H = \frac{1}{2m}\left(P_x + \frac{eBy}{2c}\right)^2 + \frac{1}{2m}\left(P_y - \frac{eBx}{2c}\right)^2$$

Analog to SHO $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

Consider

$$\left[P_x + \frac{eBy}{2c},\ P_y - \frac{eBx}{2c}\right] = \frac{eB}{c}i\hbar$$

We can define

$$Q \equiv \left(P_x + \frac{Bey}{2c}\right)\sqrt{\frac{c}{eB}}$$

$$P \equiv \left(P_y - \frac{Bex}{2c}\right)\sqrt{\frac{c}{eB}}$$

s.t $[Q,P] = i\hbar$

And

$$\cancel{H = \frac{eB}{2mc}(P^2+Q^2)} \qquad \omega_0 \equiv \frac{eB}{mc}$$

$$H = \frac{\omega_0}{2}(Q^2 + P^2)$$

Now define $a = \frac{Q+iP}{\sqrt{2\hbar}}$ $a^\dagger = \frac{Q-iP}{\sqrt{2\hbar}}$ s.t $[a, a^\dagger] = 1$

thus $H = \hbar\omega_0\left[a^\dagger a + \frac{1}{2}\right]$ equip w/ $E_n = \hbar\omega_0\left(n + \frac{1}{2}\right)$

Also $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$