Noether thm

Alternatives

Taylor expansion.

$L' = L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q$

$L' = L + \frac{dF(q,t)}{dt}$

subtraction,

$\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q = \frac{dF}{dt}$

(1) δS least action invariant under

$$L' = L + \frac{dF(q,t)}{dt}$$

(2) Given $q' \to q + \delta q$ we have $L(q+\delta q, \dot q + \delta \dot q, t)$

$$\delta L = L(q+\delta q, \dot q + \delta \dot q, t) - L(q, \dot q, t)$$

Also

$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q$ since $\delta \dot q = \frac{d}{dt}(\delta q)$, using chain rule we have

$\delta L = \frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q}\delta q}_{\text{as } F(q,t)}\Big)$ thus $L' = L + \frac{dF}{dt}$

(3) Define Noether current J,

for $\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q = \frac{dF}{dt} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q - F\right) = 0$

$\boxed{J = \frac{\partial L}{\partial \dot q}\delta q - F}$ is constant.

ex $L = \sum_i \frac{1}{2} m \dot x_i \dot x_i$ take $x_i' = x_i + a_i$ then $\delta x_i = a_i$ clearly $\dot x_i' = \dot x_i$

and $\delta L = 0 = \frac{dF}{dt} \Rightarrow F = c$ constant

and $J = \frac{\partial L}{\partial \dot q}\delta q - c = \text{constant}$ or $\sum_i m \dot x_i = \text{const}!$

ex Infinitesimal rotation

$L = \frac{1}{2} m(\dot x^2 + \dot y^2)$ $\quad x' = x\cos\theta - y\sin\theta \sim x' = x - y\theta$

$\quad y' = x\sin\theta + y\cos\theta \sim y' = x\theta + y$

Inspecting $\delta L \Rightarrow \delta L = 0 \Rightarrow F = C$

thus $J = \sum_i \frac{\partial L}{\partial \dot q}\delta q_i - F \Rightarrow J = m\dot x\delta x + m\dot y\delta y - F \Rightarrow m\dot x y - m\dot y x = \text{const}$

or $P_x y - P_y x = \text{const}$. where $\vec L = \vec r \times \vec p$.

④ Lagrange multiplier

Given non-integrable constraints: $\sum_{\delta=1}^{n} a_{j\delta}\,\delta q_\delta = 0$ $\quad j=1,\dots,k$ constraints.

& holonomic constraints $\quad f_j(q_1,\dots,q_n,t)=C_j\ ,\ \delta f_j = \sum_\delta \underbrace{\frac{\partial f_j}{\partial q_\delta}}_{a_{j\delta}}\delta q_\delta$

we have then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_\delta} - \frac{\partial L}{\partial q_\delta} = \sum_{j=1}^{K}\lambda_j a_{j\delta}$$

Remarks: (i) λ_j depends on # of constraint eqns.

(ii) $Q_\delta^c = \sum_{j=1}^{K}\lambda_j a_{j\delta}$ generalized force of constraint

⑤ Small oscillation.

$T=\sum_i \frac{1}{2}m\dot x_i^2 \quad V=V(\{q_\alpha\})$ where $X_i = X_i(\{q_\alpha\},t) \quad \alpha,\beta=1,\dots,N$

let $q_{0\alpha}$ be local min, then $q_\alpha = q_{0\alpha}+\eta_\alpha\ ,\ V(q_\alpha) = V(q_{0\alpha}) + \frac{1}{2}\sum_{\alpha\beta}\frac{\partial^2 V}{\partial q_\alpha\partial q_\beta}\Big|_0 \eta_\alpha\eta_\beta$ $\quad$ 1st der → 0

thus $\quad L = \frac{1}{2}\sum_\beta M_{\alpha\beta}\dot\eta_\beta^2 - \frac{1}{2}\sum_\beta V_{\alpha\beta}\eta_\beta^2$ $\quad \boxed{M_{\alpha\beta} = \sum_i m\frac{\partial x_i}{\partial q_\alpha}\frac{\partial x_i}{\partial q_\beta}\Big|_0}$ $\boxed{V_{\alpha\beta} = \frac{\partial^2 V}{\partial q_\alpha\partial q_\beta}\Big|_0}$

EL, $\quad \sum_\beta (M_{\alpha\beta}\ddot\eta_\beta + V_{\alpha\beta}\eta_\beta)=0$, take trial soln. $\eta_\alpha = \mathrm{Re}\,Z_\alpha \quad Z_\alpha = a_\alpha e^{i\omega t}$

then $\sum_\beta (V_{\alpha\beta}-\omega^2 M_{\alpha\beta})a_\beta = 0 \ \Rightarrow\ \det(\underbrace{V-\omega^2 M}_{N\times N}) = 0$ $\quad$ N eig.val $\omega_1^2,\dots,\omega_N^2$; N eig.vec $\vec a^{(1)},\dots,\vec a^{(N)}$

Consider

$V\vec a = \lambda_s M\vec a \quad \lambda_s \equiv \omega_s^2$ $\quad$ take $A=(\vec a^{(1)}\ \vec a^{(2)}\dots\vec a^{(N)}) \quad \Lambda \equiv \lambda_\alpha\delta_{\alpha\beta}$

then we have

$VA = MA\Lambda$ $\quad$ from definition $V^+=V$ & $M^+=M$

after some algebra, we have (hint by writing in form $A^+VA = A^+MA\Lambda$)

$\boxed{A^+MA = \mathbb{1}}$ & $\boxed{A^+VA=\Lambda}$

General Soln

$\boxed{\vec\eta(t) = \mathrm{Re}\sum_{s=1}^{N} C_s\vec a^{(s)}e^{-i\omega_s t}}$ $\quad$ w/ initial cond. $\eta_\alpha(0)$ & $\dot\eta_\alpha(0)$

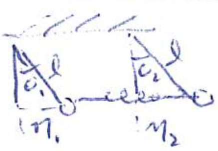
Note $\eta_\alpha(0) = \sum_{s'}\mathrm{Re}\,C_{s'}a_\alpha^{(s')}$ $\quad$ using completeness relation $A^+MA=\mathbb{1}$ or $\sum_{\alpha\beta}a_\alpha^s M_{\alpha\beta}a_\beta^{s'} = 1$

we get

$$\boxed{\begin{aligned}\mathrm{Re}\,C_s &= \sum_{\alpha\beta}M_{\alpha\beta}a_\beta^s\eta_\alpha(0)\\ \text{and}\quad \mathrm{Im}\,C_s &= \frac{1}{\omega_s}\sum_{\alpha\beta}M_{\alpha\beta}a_\beta^s\dot\eta_\alpha(0)\end{aligned}}$$

Recall s spans $1,\dots,N$ for N eigvec.

ex



$T = \frac{1}{2}ml^2\dot\theta_1^2 + \frac{1}{2}ml\dot\theta_2^2$

$V = mgl(1-\cos\theta_1) + mgl(1-\cos\theta_2) + \frac{1}{2}k(\eta_2-\eta_1)^2$

$\theta_{1,2}\ll 1 \Rightarrow \eta_{1,2} = l\theta_{1,2}$ also $\cos\theta \approx 1 + 0 + \frac{\theta^2}{2}$, $V = \frac{mg}{2l}(\eta_1^2+\eta_2^2) + \frac{k}{2}(\eta_2-\eta_1)^2$

Thus $\quad L = \frac{1}{2}m(\dot\eta_1^2+\dot\eta_2^2) - \frac{mg}{2l}(\eta_1^2+\eta_2^2) - \frac{1}{2}k(\eta_2-\eta_1)^2$

$\Rightarrow M_{\alpha\beta} = \frac{\partial L}{\partial\dot\eta_\alpha\partial\dot\eta_\beta} = \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix} \quad V_{\alpha\beta} = \frac{\partial^2 V}{\partial\eta_\alpha\partial\eta_\beta} = \begin{pmatrix}\frac{mg}{l}+k & -k \\ -k & \frac{mg}{l}+k\end{pmatrix}$

Solve eig.val

$\omega_1 = \sqrt{\frac{g}{l}} \quad a^{(1)} = \frac{1}{\sqrt{2m}}\begin{pmatrix}1\\1\end{pmatrix}$

$\omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}} \quad a^{(2)} = \frac{1}{\sqrt{2m}}\begin{pmatrix}1\\-1\end{pmatrix}$

Given initial cond. $\quad \eta_1(0) = \alpha \quad \eta_2(0) = 0$

$\dot\eta_1(0) = 0 \quad \dot\eta_2(0) = 0$

Given $x_i = x_i(\{q_\alpha\}, t)$

$$T = \frac{1}{2}\sum_i m_i \dot{x}_i^2 = \frac{1}{2}\sum_{\alpha,\beta} \underbrace{\sum_i m_i \frac{\partial x_i}{\partial q_\alpha}\frac{\partial x_i}{\partial q_\beta}}_{m_{\alpha\beta}} \dot{q}_\alpha \dot{q}_\beta$$

$$T = \frac{1}{2} m_{\alpha\beta}\dot{q}_\alpha \dot{q}_\beta$$

<u>Def</u> small oscillation :

For some local min $q_{0\alpha}$, $q_\alpha = q_{0\alpha} + \eta_\alpha$. This gives

- $\dot{q}_\alpha = \dot{\eta}_\alpha$
- $V(q_\alpha) = V(q_{0\alpha}) + \cancel{\sum_\beta \frac{\partial V}{\partial q_\beta}\Big|_0}^{\,0} \eta_\beta + \frac{1}{2}\sum_{\beta,\gamma} \frac{\partial^2 V}{\partial\eta_\beta \partial\eta_\gamma}\eta_\beta\eta_\gamma + O(\eta^3)$

Rewrite Lagrangian as.

$$L = \frac{1}{2}\dot{\eta}^T M \dot{\eta} - \frac{1}{2}\eta^T V \eta$$

Absorb $\frac{1}{2}$ into M & V, we have

$$M = \frac{\partial^2 L}{\partial\dot{\eta}_\alpha \partial\dot{\eta}_\beta} \quad \& \quad V = -\frac{\partial^2 L}{\partial\eta_\beta \partial\eta_\alpha}$$

<u>EOM</u>

$$M\ddot{\eta} = -V\eta \qquad \text{let} \quad \eta = \alpha e^{i\omega t}$$

then $\det(V - \omega^2 M)$ solve eigval & eigvec.

eigvec used to construct Modal Matrix A

$$A = (\vec{a}^{(1)}\ \vec{a}^{(2)}, \dots)$$

s.t

$$A^T M A = 1 \qquad A^T V A = \Lambda$$

define normalized eigenvec $\xi = A^{-1}\eta$ can rewrite L in diagonalized form :

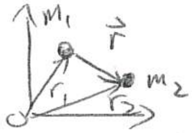
$$L = \frac{1}{2}\dot{\xi}^T\dot{\xi} - \frac{1}{2}\xi^T\Lambda\xi$$

<u>Ex</u> For 2x2 then

$$\vec{\xi}(t) = \xi_1(0)\begin{pmatrix}1\\0\end{pmatrix}e^{i\omega_1 t} + \xi_2(0)\begin{pmatrix}0\\1\end{pmatrix}e^{i\omega_2 t}$$

Then $\vec{\eta}(0) = A\vec{\xi}(0)$ is used to define initial cond that yield a given normal mode.

⑥ Two body central force



$\vec r = \vec r_2 - \vec r_1 \quad CM = \vec R = \dfrac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2}$, then

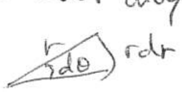
$\vec r_1 = \vec R - \dfrac{m_2}{m_1+m_2}\vec r \quad \vec r_2 = \vec R + \dfrac{m_1}{m_1+m_2}\vec r$, $\quad T = \frac{1}{2}M\dot{\vec R}^2 + \frac{1}{2}\mu\dot{\vec r}^2$ w/ $\mu = \dfrac{m_1 m_2}{m_1+m_2}$

Central force satisfies
- $\vec F(\vec r) = f(r)\hat r$
- $\vec F = -\dfrac{dV(r)}{dr}$

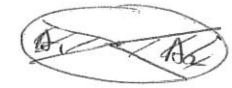
thus $V(r) = \int_r^\infty f(r)\,dr$; $L = \frac{1}{2}M\dot{\vec R}^2 + \frac{1}{2}\mu\dot{\vec r}^2 - V(r)$

Remarks
- $\dot{\vec R}$ = constant $\Rightarrow$ conservation of total linear momentum
- no external torque $\Rightarrow$ conserved angular momentum $\vec J = \vec r \times \vec p$
- no explicit time dependent $\Rightarrow$ conserved energy.

- Consider only $L = \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) - V(r)$; $P_\theta = \dfrac{\partial L}{\partial \dot\theta} = \mu r^2\dot\theta = \text{const}$



$\dfrac{dA}{dt} = \dfrac{r^2\dot\theta}{2} = \text{const} \Rightarrow$ Kepler's 2^{nd} law



Let $A_{1,2}$ be areas swept by a planet. if $A_1 = A_2$ then $T_1 = T_2$

Now Consider its EL,

$\mu\ddot r - \mu r\dot\theta^2 = f(r)$, $P_\theta = \mu r^2\dot\theta = l \Rightarrow \mu\ddot r - \dfrac{l^2}{\mu r^3} = f(r)$ or $\underbrace{\mu\ddot r = -\dfrac{\partial}{\partial r}\left(V(r) + \dfrac{l^2}{2\mu r^2}\right)}_{(*)}$

or from Hamiltonian, we have effective potential

$$\boxed{V_{eff} = V(r) + \frac{l^2}{2\mu r^2}}$$

Remark can shown easily that H = const (Hint: multiply (*) by $\dot r$ and invoke $\frac{1}{2}M\dot{\vec R}^2$ = const)

Consider System where $\vec R = 0$, then we have

$\frac{1}{2}\mu\dot r^2 + \dfrac{l^2}{2\mu r^2} + V = E$ constant, take $\mu = m$

then at $t=0$ $r=r_0$,
$$\boxed{t = \sqrt{\frac{m}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{E - V - \frac{l^2}{2mr^2}}}}$$

Also $mr^2\dot\theta = l$,
$$\boxed{\theta = l\int_0^t \frac{dt}{mr^2(t)} + \theta_0}$$

Kelper's third law

$dA = \frac{1}{2}r^2 d\theta \rightarrow \dfrac{dA}{dt} = \frac{1}{2}r^2\dot\theta \qquad \dot\theta = \dfrac{l}{mr^2}$

$= \dfrac{1}{2}\dfrac{l}{m}$

$\rightarrow A = \dfrac{l}{2m}\tau$ where $A = \pi ab \sim \pi a^{3/2}$

thus $\pi a^{3/2} \sim \dfrac{l}{2m}\tau \Rightarrow \tau^2 \propto a^3$

$E = \frac{1}{2}m\dot r^2 + \dfrac{l^2}{2mr^2} + V(r)$ w/ $\dot r = \dfrac{dr}{d\theta}\dot\theta$

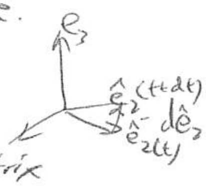
rewrite $d\theta = \dfrac{dr/r^2}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}}$ take $V = -\dfrac{k}{r}$ (kelper problem) and $u = \dfrac{1}{r}$

then
$\theta = \theta_0 + \displaystyle\int_{r_0}^{r}\frac{dr/r^2}{\sqrt{\frac{2mE}{l^2} + \frac{2mk}{l^2}\frac{1}{r} - \frac{1}{r^2}}} \Rightarrow \theta = \theta_0 - \int\frac{du}{\sqrt{a + bu + cu^2}} \rightarrow \dfrac{1}{r} = C[1 + \varepsilon\cos(\theta - \theta_0)]$

$\varepsilon = 0$ circle
$\varepsilon > 1$ hyperbola
$\varepsilon = 1$ parabola
$\varepsilon < 1$ ellipse

(7) Rigid body

$\vec{v} = v_i^0 \hat{e}_i^0 = v_i \hat{e}_i$ where $\hat{e}_i^0$ inertial frame.



axioms
- $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$
- $d\hat{e}_i \cdot \hat{e}_i = 0$

$\frac{d\vec{v}}{dt} = \dot{v}_i^0 \hat{e}_i^0 = \dot{v}_i \hat{e}_i + v_i \frac{d\hat{e}_i}{dt}$; prompt $\frac{d\hat{e}_i}{dt} = ?$

note $d\hat{e}_i = d\Omega_{ij} \hat{e}_j$ $\quad d\Omega_{ij}$ is infinitesimal rotation matrix

from axioms we know/learn: $d\Omega_{ii} = 0$ $\quad d\Omega_{ij} = d\Omega_{ji}$ $\Rightarrow$ define $d\Omega_{12} = d\Omega_3$ $\quad d\Omega_{23} = d\Omega_1$ $\quad d\Omega_{31} = d\Omega_1$

thus $\frac{d\hat{e}}{dt} = \frac{d\Omega}{dt} \times \hat{e} = \vec{\omega} \times \hat{e}$ $\Rightarrow$ generalized $\frac{d\vec{a}}{dt} = \vec{\omega} \times \vec{a}$

so $\boxed{\left(\frac{d\vec{v}}{dt}\right)_{int} = \left(\frac{d\vec{v}}{dt}\right)_{body} + \vec{\omega} \times \vec{v}}$ try $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\dot{\vec{r}}_{body} + \vec{\omega} \times \vec{r}_{body}\right) = (\ddot{\vec{r}})_{body} + \dot{\vec{r}} \frac{d\hat{e}_i}{dt} + \vec{\omega} \times \frac{d}{dt}(r_i \hat{e}_i)$

then $\boxed{\frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} + 2\vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r})}$

Now $\left(\frac{d\vec{r}}{dt}\right)_{int} = \left(\frac{d\vec{r}}{dt}\right)_{body} + \vec{\omega} \times \vec{r}$ rigid body $\Rightarrow \left(\frac{d\vec{r}}{dt}\right)_{body} = 0$

~~Thus~~ $\cancel{T = \frac{1}{2}\sum_p m_p v_p^2 \Rightarrow \boxed{T = \frac{1}{2}\sum_p m_p\left[\omega^2 r_p^2 - (\vec{\omega}\cdot\vec{r}_p)^2\right]}}$ b/c $\cancel{T = \frac{1}{2}\sum_p m_p\left[\omega^2 \vec{r}_p - (\vec{\omega}\cdot\vec{r}_p)\vec{\omega}\right]\cdot\vec{r}_p}$

~~Define~~ $\cancel{T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j \Rightarrow \boxed{I_{ij} = \sum_p m_p\left(r_p^2 \delta_{ij} - x_{p_i} x_{p_j}\right)}}$ $\cancel{(\vec{\omega}\cdot\vec{r}_p)^2 = (\omega_i r_{p_i})(\omega_j r_{p_j})}$ ~~note~~

$T = \frac{1}{2}\sum_p m_p V_p^2$ $\quad$ w/ $V_p^2 = (\vec{\omega} \times \vec{r}_p)\cdot(\vec{\omega} \times \vec{r}_p)$

$= \frac{1}{2}\sum_p m_p\left[\omega^2 r_p^2 - (\vec{\omega}\cdot r_p)^2\right]$

$= \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$ $\quad$ w/ $I_{ij} = \sum_p m_p (r_p^2 \delta_{ij} - x_{p_i} x_{p_j}) \to \int d^3x\, \rho(x)(\delta_{ij} x^2 - x_i x_j)$

$\vec{L} = \sum_p m_p \vec{r}_p \times \vec{v}_p = \sum_p m_p\left[r_p^2 \vec{\omega} - (\vec{\omega}\cdot\vec{r}_p)\vec{r}_p\right] = \sum_{i=1}^{3} I_{ij}\omega_j$

so $\boxed{T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_i L_i\omega_i = \frac{1}{2}\vec{L}\cdot\vec{\omega}}$

Since I sym $\Rightarrow \exists\, A$ s.t. $\Lambda = A^T I A$ ~~with A~~

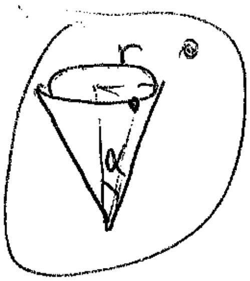
and $\Lambda \hat{e} = \lambda \hat{e} \Rightarrow I A\hat{e} = \lambda A\hat{e}$ where $A\hat{e} = \vec{a}$ the corresponding principle axes

$\Lambda = \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix}$ $\quad I_i = \lambda_i$

$T = \frac{1}{2}\omega^T I \omega = \frac{1}{2}\xi^T \Lambda \xi$ $\quad$ w/ $\xi = A^T\omega$

thus $T = \frac{1}{2}\sum_i I_s \xi_s^2$ & $L = I\vec{\omega} \Rightarrow L_{new} = A^T L = \Lambda A^T \vec{\omega}$

or $L_{new} = \Lambda \xi$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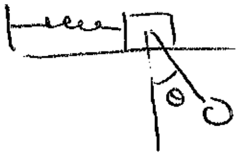
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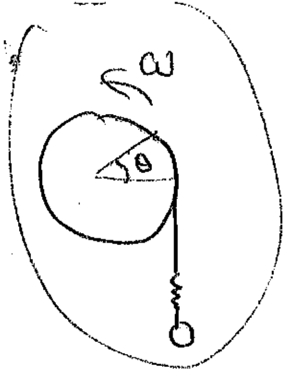
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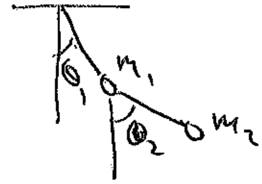
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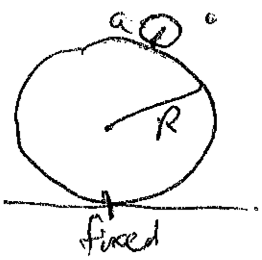
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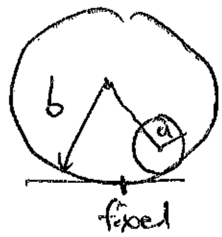
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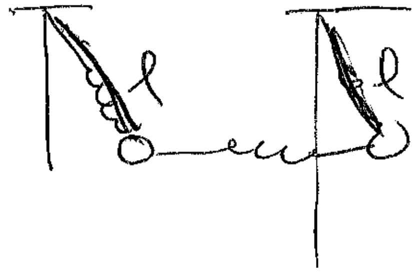
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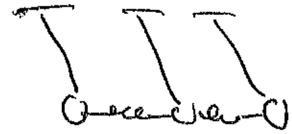

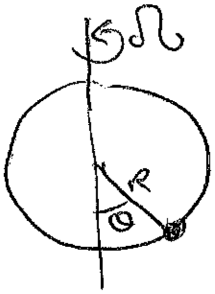
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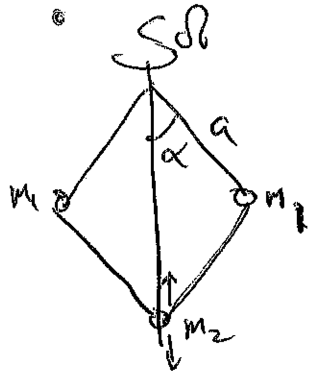
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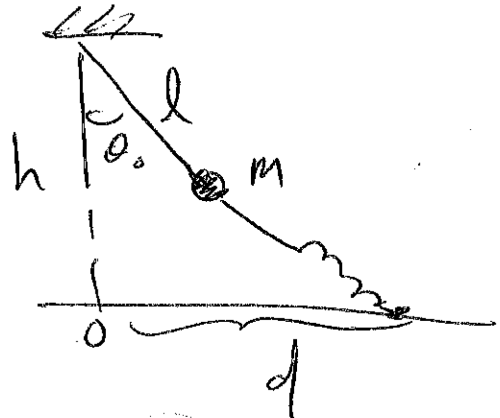
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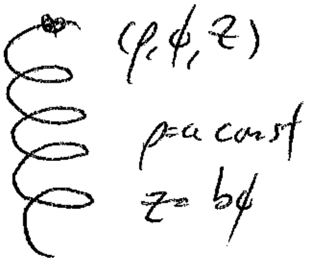
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ρ = a const
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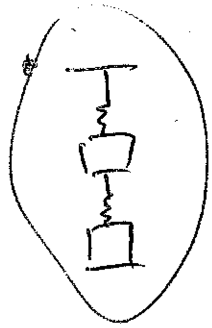

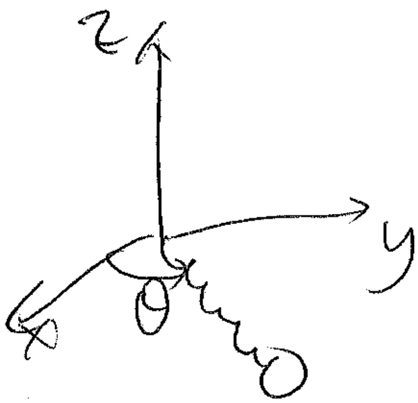
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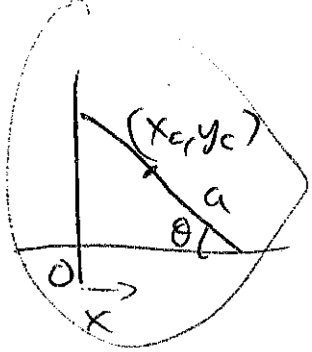
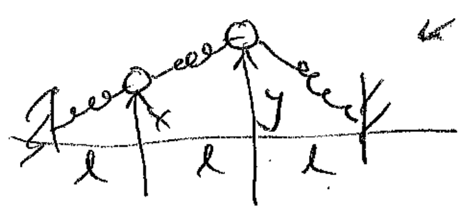
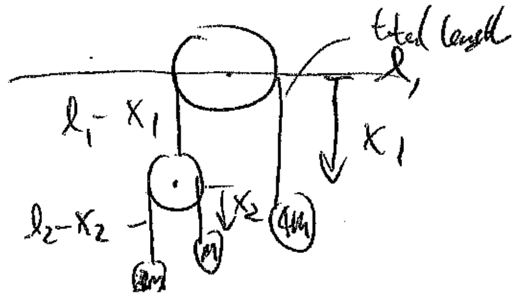
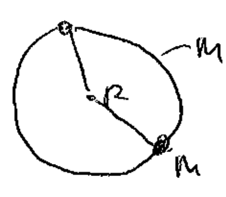
x
θ


(xc, yc)
a
θ
O
x


x
y
l
l
l


total length
l1
l1 − x1
x1
l2 − x2
x2
4m
m
3m


m
R
m