(4) Scaling hypothesis

Consider free energy

$$f(t,h) = \min\left[\frac{t}{2}m^2 + um^4 - hm\right] = \begin{cases} -\frac{t^2}{16u} & (h=0 \quad t<0) \\ -\frac{3}{4^{4/3}}\frac{h^{4/3}}{u^{1/3}} & (h\neq 0 \quad t=0) \end{cases}$$

<u>Claim</u> Singularity in free energy can be written as

$$f(t,h) = |t|^2 g_f\left(\frac{h}{|t|^\Delta}\right) \qquad x \equiv \frac{h}{|t|^\Delta} \leftarrow \text{gap expo.}$$

- For $h \to 0$ $x \to 0$ then $g_f(x) \to g_f(0) \sim \frac{1}{u}$ finite then $f \sim \frac{|t|^2}{u}$ recovered $h=0$ limit.
- For $t \to 0$ $x \to \infty$ then $g_f(\infty)$ finite b/c analy. now $g_f(x) \sim \frac{x^{4/3}}{u^{1/3}}$; $f \sim \frac{t^2}{u^{1/3}}\left(\frac{h}{t^\Delta}\right)^{4/3}$; required $t^2 t^{-\frac{4\Delta}{3}} = t^0$; $\Delta = \frac{3}{2}$

<u>Homogeneity</u> required sing. form of free energy remains homogeneous beyond saddle approx.

ie $f_{sing}(t,h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^\Delta}\right)$ w/ α, Δ depend on critical pt.

<u>Consider</u>

$$(S)\ E_{sing} \sim \frac{\partial f}{\partial t} \sim |t|^{1-\alpha}\left[(2-\alpha)g_f\left(\frac{h}{|t|^\Delta}\right) - \frac{\Delta h}{|t|^\Delta} g_f'\left(\frac{h}{|t|^\Delta}\right)\right] \equiv |t|^{1-\alpha} g_E\left(\frac{h}{|t|^\Delta}\right)$$

$$C_{sing} \sim \frac{\partial^2 f}{\partial t^2} \sim |t|^{-\alpha} g_c\left(\frac{h}{|t|^\Delta}\right)$$

<u>Remark</u>:
- derivatives of a hom. func is a hom. func.
- as $h \to 0$ $C_{sing} \sim |t|^{-\alpha} g_c(0)$ recovered.

<u>Remarks</u> on uniqueness

Claim to have more general form:

$$C_\pm(t,h) = |t|^{-\alpha_\pm} g_\pm\left(\frac{h}{|t|^{\Delta_\pm}}\right) \qquad \begin{matrix}(+) \text{ for } t>0 \\ (-) \text{ for } t<0\end{matrix} \quad \& \quad C_\pm \text{ match at } t=0, \text{ except in coexistence line } (h=0 \ \ t<0)$$

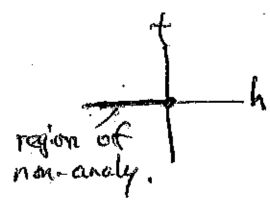
rule out by following:

- By assumption C analy. within point $t=0$ h finite.
- thus allows power law expansion; with leading power term

$$C_\pm = |t|^{-\alpha_\pm}\left[A_\pm\left(\frac{h}{|t|^{\Delta_\pm}}\right)^{p_\pm} + B_\pm\left(\frac{h}{|t|^{\Delta_\pm}}\right)^{q_\pm} + \cdots\right]$$

- O.T.O.H by Taylor expansion about $t=0$,

$$C \simeq \underbrace{A(h)}_{h \text{ vary only b/c } t=0} + tB(h) + O(t^2)$$



- matching two series, we arrived at $\alpha_+ = \alpha_- = \alpha$ and $\Delta_+ = \Delta_- = \Delta$. Thus unique!

Critical Expo. Analysis

- $m(t,h) \sim \frac{\partial f}{\partial h} \sim |t|^{2-\alpha-\Delta} g_m\left(\frac{h}{|t|^\Delta}\right)$; $x \to 0$ $g_m(0) = \text{const}$

  thus, $m(t, h=0) \sim |t|^{2-\alpha-\Delta}$ w/ $\beta = 2-\alpha-\Delta$.

- O.T.O.H, $x \to \infty$ $g_m(x) \sim x^p$ (for some highest power p)

  then, $m(t=0, h) \sim |t|^{2-\alpha-\Delta}\left(\frac{h}{|t|^\Delta}\right)^p$ w/ $p\Delta = 2-\alpha-\Delta$ (t indep)

  thus, $m(t=0, h) \sim h^{\frac{2-\alpha-\Delta}{\Delta}}$ w/ $\delta = \frac{\Delta}{\beta}$.

- $\chi(t,h) \sim \frac{\partial m}{\partial h} \sim |t|^{2-\alpha-2\Delta} g_\chi\left(\frac{h}{|t|^\Delta}\right) \Rightarrow \chi(t, h=0) \sim |t|^{2-\alpha-2\Delta}$

  w/ $\gamma = 2\Delta - 2 - \alpha$

Mic's trick
Scale $\frac{h}{|t|^\Delta} \sim 1$
thus $t = h^{\frac{1}{\Delta}}$
$m(t=0,h) \sim h^{\frac{2-\alpha-\Delta}{\Delta}} \underbrace{g_m(1)}_{\text{const.}}$

Homogeneity Assumption

- critical quantities Q (i.e. C, S etc) are homogeneous w/ same expo. on analytic area.
- due to therm. derivatives Δ presents in all therm quantities
- we can obtain all crt. expo. using only two indep. param. α, Δ
- together there provide a list of expo. identities.

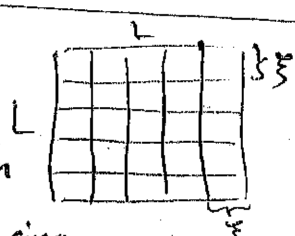
  i.e $\alpha + 2\beta + \gamma = 2$ ⟵ using $\beta = 2-\alpha-\Delta$

  $\gamma = \beta(\delta - 1)$ $\delta = \frac{\Delta}{\beta}$

  $\gamma = 2\Delta - 2 + \alpha$

Divergence of correlation length.

Given sys size L & divided by ξ



in dim d

Postulate: $\xi(t,h) \sim |t|^{\nu} g\left(\frac{h}{|t|^\Delta}\right)$ ⟵ recall Gaussian

Claim ξ most fundamental param. responsible for sing.

Remark sys can divided by unit of ξ^d, thus extensiveness of partition func.

$$\ln Z = \underbrace{\left(\frac{L}{\xi}\right)^d}_{\#\text{ of units}} \underbrace{g_s}_{\text{non sing func.}} + \cdots + \left(\frac{L}{a}\right)^d \underbrace{g_a}_{\text{micro scale}} \quad \text{then } f_{sing}(t,h) \sim \frac{\ln Z}{L^d} \sim \xi^{-d}$$

thus $f_{sing}(t,h) \sim |t|^{d\nu} g_f\left(\frac{h}{|t|^\Delta}\right)$ from 1st postulate $f_{sing} \sim |t|^{2-\alpha} g$

thus, we obtain Josephson's Identity

$$2 - \alpha = d\nu \qquad (*)$$

Remarks. Identities involve dim d are called hyperscaling relations.

Remarks. for $d=3$ $\alpha = 0.11$ $\nu = 0.63$ relation $(*)$ holds

- At saddle pt. $\alpha = 0$ $\nu = \frac{1}{2}$ $d=3$ breakdown which required $d > 4$ to hold.
- Any valid theory must be consistent w/ $(*)$ at low dim.

4i

Critical Correlation func. and Self-similarity

- Consider η - charac. decay of correlation func. at criticality.

At crt. pt $\xi \sim \infty$ all correlations decay as power of separation $|\vec{x}|$

i.e $G^c_{mm}(\vec{x}) \equiv \langle m(\vec{x}) m(\vec{0})\rangle - \langle m\rangle^2 \sim \frac{1}{|\vec{x}|^{d-2+\eta}}$

As for energy:

$$G^c_{\epsilon\epsilon}(\vec{x}) \equiv \langle H(\vec{x}) H(0)\rangle - \langle H\rangle^2 \sim \frac{1}{|\vec{x}|^{d-2+\eta'}} \quad \text{for some } \eta'$$

- Away from criticality ξ as cutoff, and response func. can obtained from integrating G^c.

ex recalled $\langle M\rangle = \frac{\partial \ln Z}{\partial(\beta h)} = \frac{\partial f}{\partial(\beta h)}$; $\langle M^2\rangle_c = \frac{\partial^2 \ln Z}{\partial(\beta h)^2} = \langle(\Delta M)^2\rangle = \frac{1}{\beta}\chi$

Note
$M \equiv \int d^d r\, m(\vec{r})$

$$\chi = \beta\langle(\Delta M)^2\rangle \cong \beta V \int d^d x\,(G(\vec{x}) - m^2)$$

thus $\chi \sim \int d^d x\, G^c_{mm}(\vec{x}) \sim \int^{\xi} \frac{d^d x}{|x|^{d-2+\eta}} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}$

but $\chi \sim |t|^{-\gamma} \Rightarrow \boxed{\gamma = (2-\eta)\nu}$ "Fisher's identity.

O.T.O.H. $C \sim \int d^d x\, G^c_{\epsilon\epsilon}(\vec{x}) \sim |t|^{-\nu(2-\eta')}$

$$\Rightarrow \boxed{\alpha = (2-\eta')\nu}$$

Remark

- Scaling inherits the dilation symmetry in critical sys., this renders scaling invariance:
  $G_{crt}(\lambda\vec{x}) = \lambda^p G_{crt}(\vec{x})$
- If dilation sym is included in LGH then it can describe crt. pt. behavior
- this is difficult except at $d=2$ which dilation sym. $\Rightarrow$ conformal invariance (conf. mapping)

Conceptual Renormalization Group

$\ast$ ξ the only important length near critical pt.

RG is a procedure to reduce dof to simplest and uncorrelated one at ξ based on self-similarity.

Steps outline

1. Coarse Grain reduce resolution in LGH by enlarging original coarse length scale i.e ba w/ $b>1$
2. Rescale restore original length by rescaling $\vec{x}_{new} = \frac{\vec{x}_{old}}{b}$
3. Renormalized remedies fluctuation variation by contrast factor ζ

$$m^{b}_i(\vec{x}) = \frac{1}{b^d}\int_{Cell(\vec{x})} d^d\vec{x}'\, m_i(\vec{x}') \longrightarrow \vec{m}_{new}(\vec{x}_{new}) = \frac{1}{\zeta b^d}\int_{Cell(\vec{x}_{new})} d^d\vec{x}'\, \vec{m}(\vec{x}')$$

Axioms $x < \xi$ renorm. config. statistically similiar to original s.t crt. Ham invariant under rot. & trans. Thus Ham still described by t, h.

new t', h'
old t, h
renorm $t_b\ h_b$ analy

$$\Rightarrow \quad t' = t_b(t,h) \overset{analy}{=} t_b(t,h) = A(b)t + B(b)h + \cdots$$
$$h' = h_b(t,h) \qquad h_b(t,h) = C(b)t + D(b)h + \cdots$$

expansion in this form b/c of analyticity.

s.t at crt. pt ($t=h=0$) so is ($t'=h'=0$).

• Rot sym $\Rightarrow h \mapsto -h \quad t \mapsto t$ s.t $B = C = 0$

s.t $t_b(t,h) = A(b)t + \cdots$

$h_b(t,h) = D(b)h + \cdots$, where $A(b), D(b)$ depends on rescaling factor. "b"

when rescaled in sequence b_1 then b_2 the net effect is $b_1 b_2 \Rightarrow$ semi-group

• By semi-group,

$t(b_1 b_2) \sim A(b_1)A(b_2)t \sim A(b_1 b_2)t \Rightarrow \exists\, y_i$ s.t $A(b) = b^{y_i}$

so $\boxed{\begin{array}{l} t' = t_b = b^{y_t} t + \cdots \\ h' = h_b = b^{y_h} h + \cdots \end{array}}$ &

<u>Advantage</u>

• If $\beta H_{old} \Leftrightarrow \xi_{old}$ (large) then RG w/ $\beta H_{new} \Leftrightarrow \xi_{new}$ smaller

• thus βH_{new} appears less critical w/ parameters move away from origin i.e $y_t, y_h > 0$

<u>RG consequence</u>

① free energy : partition func conserved i.e $W'([m'])$ sum of old $W([m])$

thus $Z = \int Dm\, W([m]) = \int Dm'\, W([m']) = Z'$ s.t $\ln Z = \ln Z'$

and $V f(t,h) = V' f(t',h')$ then $f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$

<u>Remarks</u> • same Ham only t, h change.

• by homogeneity, we choose $b^{y_t} t = 1$ s.t

$f(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^{\Delta}}\right)$ w/ $2 - \alpha = \frac{d}{y_t}$ & $\Delta = \frac{y_h}{y_t}$

② Correlation length $\xi' = \xi/b$ b/c $\frac{\xi(t,h)}{b} = \xi(b^{y_t} t, b^{y_h} h)$ s.t

$\xi = t^{-1/y_t} \xi\left(1, \frac{h}{t^{y_h/y_t}}\right) \sim t^{-\nu}$ so $\nu = \frac{1}{y_t}$ & $2 - \alpha = d\nu$

③ w/ $\frac{V}{b^d} = V'$ $\frac{\partial}{\partial h} = \frac{\partial}{\partial h'} \frac{dh'}{dh}$; $\ln Z = \ln Z'$

magnetization :

$f = t^{d/y_t} g_f\left(\frac{h}{t^{y_h/y_t}}\right)$ then

$m(t,h) = -\frac{1}{V} \frac{\partial \ln Z(t,h)}{\partial h} = -\frac{1}{b^d V'} \frac{\partial \ln Z(t',h')}{b^{-y_h} \partial h'}$

$m(t,h) = b^{y_h - d} m(b^{y_t} t, b^{y_h} h)$

now scale w/ unity $b^{y_t} t = 1 \Rightarrow b = t^{-1/y_t}$

$m(t,h) = t^{\frac{-(y_h - d)}{y_t}} m\left(\frac{h}{t^{y_h/y_t}}\right)$ thus $\beta = \frac{y_h - d}{y_t}$

one can guess also for any quantity

$X(t,h) = b^{y_X} X(b^{y_t} t, b^{y_h} h)$

4.i <u>Formal Renormalization Group</u> (Formal procedure for effect of dilation op in Ham.)

Consider general Ham

$$\beta H = \int d^dx \left[\frac{t}{2} m^2 + u m^4 + v m^6 + \cdots + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots \right]$$

then RG. $m'(\vec{x}') = \frac{1}{\zeta b^d} \int_{\text{cell}(b\vec{x}')} d^d\vec{x}\, m(\vec{x})$ where $\vec{x}' = \frac{\vec{x}}{b}$

rot sym. so

$$\beta H'[m'(\vec{x}')] \equiv \ln P'[m'(\vec{x}')] = f_b + \int d^dx' \left[\frac{t'}{2} m'^2 + u' m'^4 + v' m'^6 + \cdots + \frac{K'}{2}(\nabla m')^2 + \frac{L'}{2}(\nabla^2 m')^2 \cdots \right]$$

let S be parameter space, $S = \{t, u, v, \cdots, \frac{K}{2}, \frac{L}{2}, \cdots\}$

$S' = R_b S$ s.t $t' = t_b(t, u, v, \cdots)$ $u' = u_b(t, u, v, \cdots)$

effect of dilation.

<u>Remark</u> if Ham specificed by S Ham-self-similar $\Rightarrow \exists S^*$ s.t $R_b S^* = S^*$

As correlation length rescaled under RG: $\xi(S) = b\xi(S') = b\xi(R_b S)$

under fixed pt S^*, $\xi(S^*) = b\xi(S^*) \Rightarrow \xi = 0$ or ∞.

<u>Remark</u>

$\xi^* = 0 \Rightarrow$ completely disorder $(T = \infty)$ or completely order $(T = 0)$

$\xi^* = \infty \Rightarrow T = T_c$ crt. pt

<u>Remark</u>

$t = h = 0$ is fixed pt.

Stability of fixed pt by linearization.

$$S_\alpha^* + \delta S_\alpha' = S_\alpha^* + \left.\frac{\partial S_\alpha'}{\partial S_\beta}\right|_{S^*} \delta S_\beta \quad \text{where } (R_b^L)_{\alpha\beta} \equiv \left.\frac{\partial S_\alpha'}{\partial S_\beta}\right|_{S^*}$$

w/ semigroup property:

$$R_b^L R_{b'}^L = R_{bb'}^L = R_{b'b}^L \Rightarrow \text{w/ eigvec of } O_i \quad (\text{ie } R_b^L O_i = \lambda(b) O_i)$$

eigval.

we have

$$R_b^L R_{b'}^L O_i = \lambda(b)_i \lambda(b')_i O_i$$

$$R_{bb'}^L O_i = \lambda(bb')_i O_i$$

$\Rightarrow$ since $\lambda(1) = 1$ then $\lambda(b)_i = b^{y_i}$ for some y_i const.

Ham in vicinity of S^* has $S = S^* + \Sigma_i g_i O_i$

RG Ham. has $S' = R_b^L(S) = S^* + \Sigma_i g_i b^{y_i} O_i$

<u>we have classification</u>

- $y_i > 0$ g_i increases O_i relevant op.
- $y_i < 0$ g_i decreases O_i irrelevant op.
- $y_i = 0$ g_i marginal H.O.T needed.

- near S^*, ξ diverges.

  $\xi(g_1, g_2, \dots) = b\,\xi(b^{y_1} g_1, b^{y_2} g_2, \dots)$
- Since irrev $O_i \Rightarrow y_i < 0$ thus for b large enough, all $O_{i,\text{irrev}}$ vanished which left $O_{i,\text{rev}}$
- Let O_i index by decreasing dim order, so reduce dim by scaling $b^{y_1} g_1 = 1$ s.t.

  $\xi(g_1, g_2, \dots) = g_1^{-1/y_1} f\left(\frac{g_2}{g_1^{y_2/y_1}}, \dots\right)$ w/ $\nu_1 = \frac{1}{y_1}$ for describing ξ divergence and

  gap exp. $\Delta_\alpha = \frac{y_\alpha}{y_1}$

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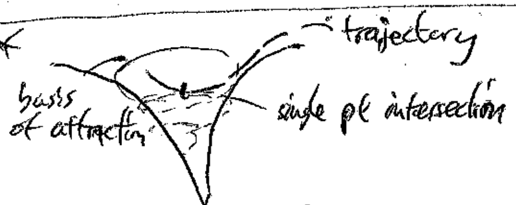
now if



$\Rightarrow$ basis of attraction has only co-dim one

So $\xi(g_1, g_2) = g_1^{-1/y_1} f\left(\frac{g_2}{g_1^{y_2/y_1}}\right)$ thus explains universality of two parameter t, h.