11. Prove that $H(p)$ is concave in p, i.e., for $0 \le \lambda \le 1$ and $\bar{\lambda} = 1 - \lambda$,

$$\lambda H(p_1) + \bar{\lambda} H(p_2) \le H(\lambda p_1 + \bar{\lambda} p_2).$$

Proof

$$\begin{aligned}
&\lambda H(p_1) + \bar{\lambda} H(p_2) - H(\lambda p_1 + \bar{\lambda} p_2) \\
&= -\lambda \sum p_{1i} \log p_{1i} - \bar{\lambda} \sum p_{2i} \log p_{2i} + \sum (\lambda p_{1i} + \bar{\lambda} p_{2i}) \log(\lambda p_{1i} + \bar{\lambda} p_{2i}) \\
&= -\lambda \sum p_{1i} \log \frac{p_{1i}}{\lambda p_{1i} + \bar{\lambda} p_{2i}} - \bar{\lambda} \sum p_{2i} \log \frac{p_{2i}}{\lambda p_{1i} + \bar{\lambda} p_{2i}} \\
&= -\lambda D(p_1 \| (\lambda p_1 + \bar{\lambda} p_2)) - \bar{\lambda} D(p_2 \| (\lambda p_1 + \bar{\lambda} p_2)) \\
&\le 0
\end{aligned}$$

So

$$\lambda H(p_1) + \bar{\lambda} H(p_2) \le H(\lambda p_1 + \bar{\lambda} p_2)$$

12. Let $(X, Y) \sim p(x, y) = p(x)p(y|x)$.
   a) Prove that for fixed $p(x)$, $I(X;Y)$ is a convex functional of $p(y|x)$.
   b) Prove that for fixed $p(y|x)$, $I(X;Y)$ is a concave functional of $p(x)$.

Proof

a)

$$\begin{aligned}
&I(X;Y) \\
&= H(X) - H(X|Y) \\
&=
\end{aligned}$$

15. Let X be a function of Y. Prove that $H(X) \le H(Y)$. Interpret this result.

Proof

Since X is a function of Y, we have $H(X|Y) = 0$.

So

$$H(X) = H(X|Y) + I(X;Y) = I(X;Y) \le H(Y)$$

This means further processing of information does not increase information.

16. Prove that for any $n \ge 2$,

$$H(X_1, X_2, \cdots, X_n) \ge \sum_{i=1}^{n} H(X_i | X_j, j \ne i).$$

Proof

$$H(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} H(X_i | X_1, \cdots X_{i-1}) \ge \sum_{i=1}^{n} H(X_i | X_1, \cdots, X_{i-1}, X_{i+1}, \cdots X_n)$$

17. Prove that

$$H(X_1, X_2) + H(X_2, X_3) + H(X_1, X_3) \geq 2H(X_1, X_2, X_3).$$

Hint: Sum the identities

$$H(X_1, X_2, X_3) = H(X_j, j \neq i) + H(X_i | X_j, j \neq i)$$

for $i = 1, 2, 3$ and apply the result in Problem 16.

Proof

$$\begin{aligned}
&H(X_1, X_2) + H(X_2, X_3) + H(X_1, X_3) \\
&= H(X_1) + H(X_2 \mid X_1) + H(X_2) + H(X_3 \mid X_2) + H(X_3) + H(X_1 \mid X_3) \\
&= H(X_1) + H(X_2 \mid X_1) + H(X_3) + H(X_2) + H(X_3 \mid X_2) + H(X_1 \mid X_3) \text{ (re - arrange)} \\
&\geq H(X_1) + H(X_2 \mid X_1) + H(X_3 \mid X_1 X_2) + H(X_2) + H(X_3 \mid X_2) + H(X_1 \mid X_2 X_3) \\
&= 2H(X_1 X_2 X_3)
\end{aligned}$$