Assignment 5

Chapter 4

1. Construct a binary Huffman code for the distribution $\{0.25, 0.05, 0.1, 0.13, 0.2, 0.12, 0.08, 0.07\}$.

2. Construct a ternary Huffman code for the source distribution in Problem 1.

3. Show that a Huffman code is an optimal uniquely decodable code for a given source distribution.

4. Construct an optimal binary prefix code for the source distribution in Problem 1 such that all the codewords have even lengths.

6. Prove that if $p_1 > 0.4$, then the shortest codeword of a binary Huffman code has length equal to 1. Then prove that the redundancy of such a Huffman code is lower bounded by $1 - h_b(p_1)$. (Johnsen [192].)

10. Let X be a source random variable. Suppose a certain probability mass p_k in the distribution of X is given. Let

$$l_j = \begin{cases} \lceil -\log p_j \rceil & \text{if } j = k \\ \lceil -\log(p_j + x_j) \rceil & \text{if } j \neq k, \end{cases}$$

where

$$x_j = p_j \left(\frac{p_k - 2^{-\lceil -\log p_k \rceil}}{1 - p_k} \right)$$

for all $j \neq k$.

(a) Show that $1 \leq l_j \leq \lceil -\log p_j \rceil$ for all j.

(b) Show that $\{l_j\}$ satisfies the Kraft inequality.

(c) Obtain an upper bound on L_{Huff} in terms of $H(X)$ and p_k which is tighter than $H(X) + 1$. This shows that when partial knowledge about the source distribution in addition to the source entropy is available, tighter upper bounds on L_{Huff} can be obtained.

(Ye and Yeung [396].)

Chapter 5

1. Show that for any $\epsilon > 0$, $W^n_{[X]\epsilon}$ is nonempty for sufficiently large n.

2. *The source coding theorem with a general block code* In proving the converse of the source coding theorem, we assume that each codeword in $\mathcal{I}$ corresponds to a unique sequence in $\mathcal{X}^n$. More generally, a block code with block length n is defined by an encoding function $f : \mathcal{X}^n \to \mathcal{I}$ and a decoding function $g : \mathcal{I} \to \mathcal{X}^n$. Prove that $P_e \to 1$ as $n \to \infty$ even if we are allowed to use a general block code.

5. *Alternative definition of weak typicality* Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ be an i.i.d. sequence whose generic random variable X is distributed with $p(x)$. Let $q_{\mathbf{x}}$ be the empirical distribution of the sequence $\mathbf{x}$, i.e., $q_{\mathbf{x}}(x) = n^{-1}N(x; \mathbf{x})$ for all $x \in \mathcal{X}$, where $N(x; \mathbf{x})$ is the number of occurrence of x in $\mathbf{x}$.

(a) Show that for any $\mathbf{x} \in \mathcal{X}^n$,

$$-\frac{1}{n} \log p(\mathbf{x}) = D(q_{\mathbf{x}} \| p) + H(q_{\mathbf{x}}).$$

(b) Show that for any $\epsilon > 0$, the weakly typical set $W^n_{[X]\epsilon}$ with respect to $p(x)$ is the set of sequences $\mathbf{x} \in \mathcal{X}^n$ such that

$$|D(q_{\mathbf{x}} \| p) + H(q_{\mathbf{x}}) - H(p)| \leq \epsilon.$$

(c) Show that for sufficiently large n,

$$\Pr\{|D(q_{\mathbf{x}} \| p) + H(q_{\mathbf{x}}) - H(p)| \leq \epsilon\} > 1 - \epsilon.$$

(Ho and Yeung [167].)

9. *Universal source coding* Let $\mathcal{F} = \{\{X_k^{(s)}, k \geq 1\} : s \in \mathcal{S}\}$ be a family of i.i.d. information sources indexed by a finite set $\mathcal{S}$ with a common alphabet $\mathcal{X}$. Define

$$\bar{H} = \max_{s \in \mathcal{S}} H(X^{(s)})$$

where $X^{(s)}$ is the generic random variable for $\{X_k^{(s)}, k \geq 1\}$, and

$$A_\epsilon^n(\mathcal{S}) = \bigcup_{s \in \mathcal{S}} W^n_{[X^{(s)}]\epsilon},$$

where $\epsilon > 0$.

(a) Prove that for all $s \in \mathcal{S}$,

$$\Pr\{\mathbf{X}^{(s)} \in A_\epsilon^n(\mathcal{S})\} \to 1$$

as $n \to \infty$, where $\mathbf{X}^{(s)} = (X_1^{(s)}, X_2^{(s)}, \cdots, X_n^{(s)})$.

(b) Prove that for any $\epsilon' > \epsilon$,

$$|A_\epsilon^n(\mathcal{S})| \leq 2^{n(\bar{H} + \epsilon')}$$

for sufficiently large n.

(c) Suppose we know that an information source is in the family $\mathcal{F}$ but we do not know which one it is. Devise a compression scheme for the information source such that it is asymptotically optimal for every possible source in $\mathcal{F}$.