Assignment 4

Chapter 3

9. (a) Prove that under the constraint that $X \to Y \to Z$ forms a Markov chain, $X \perp Y|Z$ and $X \perp Z$ imply $X \perp Y$.
   (b) Prove that the implication in a) continues to be valid without the Markov chain constraint.

10. (a) Show that $Y \perp Z|T$ does not imply $Y \perp Z|(X,T)$ by giving a counterexample.
    (b) Prove that $Y \perp Z|T$ implies $Y \perp Z|(X,T)$ conditioning on $X \to Y \to Z \to T$.

11. (a) Let $X \to Y \to (Z,T)$ form a Markov chain. Prove that $I(X;Z)+I(X;T) \leq I(X;Y)+I(Z;T)$.
    (b) Let $X \to Y \to Z \to T$ form a Markov chain. Determine which of the following inequalities always hold:
       i. $I(X;T)+I(Y;Z) \geq I(X;Z)+I(Y;T)$
       ii. $I(X;T)+I(Y;Z) \geq I(X;Y)+I(Z;T)$
       iii. $I(X;Y)+I(Z;T) \geq I(X;Z)+I(Y;T)$.

14. Prove that for random variables X, Y, Z, and T,

$$\left.\begin{array}{l} X \perp Z|Y \\ (X,Y) \perp T|Z \\ Y \perp Z|T \\ Y \perp Z|X \\ X \perp T \end{array}\right\} \Rightarrow Y \perp Z.$$

    Hint: Observe that $X \perp Z|Y$ and $(X,Y) \perp T|Z$ are equivalent to $X \to Y \to Z \to T$ and use an information diagram.

15. Prove that

$$\left.\begin{array}{l} X \perp Y \\ X \perp Y|(Z,T) \\ Z \perp T|X \\ Z \perp T|Y \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} Z \perp T \\ Z \perp T|(X,Y) \\ X \perp Y|Z \\ X \perp Y|T. \end{array}\right.$$

    (Studený [346].)

Chapter 4

5. Prove directly that the codeword lengths of a prefix code satisfy the Kraft inequality without using Theorem 4.4.

7. *Suffix codes* A code is a suffix code if no codeword is a suffix of any other codeword. Show that a suffix code is uniquely decodable.

9. *Random coding for prefix codes* Construct a binary prefix code with codeword lengths $l_1 \leq l_2 \leq \cdots \leq l_m$ as follows. For each $1 \leq k \leq m$, the codeword with length l_k is chosen independently from the set of all 2^{l_k} possible binary strings with length l_k according the uniform distribution. Let $P_m(good)$ be the probability that the code so constructed is a prefix code.

(a) Prove that $P_2(good) = (1 - 2^{-l_1})^+$, where

$$(x)^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(b) Prove by induction on m that

$$P_m(good) = \prod_{k=1}^{m} \left(1 - \sum_{j=1}^{k-1} s^{-l_j} \right)^+ .$$

(c) Observe that there exists a prefix code with codeword lengths $l_1, l_2, \cdots, l_m$ if and only if $P_m(good) > 0$. Show that $P_m(good) > 0$ is equivalent to the Kraft inequality.

By using this random coding method, one can derive the Kraft inequality without knowing the inequality ahead of time. (Ye and Yeung [395].)

Additional Problem

In Example 4.5, verify that $A_3 = 8$ and list the 8 sequences of 2 codewords with a total length of 3 code symbols.