Visually Deriving the Wigner Rotation by Spacetime Diagrams*

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Abstract

The Wigner Rotation describes the paradoxical phenomenon in special relativity, where a composition of several Lorentz boosts could give a pure rotation. Despite having important applications, as well as being an extremely surprising paradox, it is rarely taught in undergraduate physics, and its geometric meaning is rarely mentioned anywhere.

This paper gives an illustrated derivation of Wigner rotation using the geometry of spacetime diagrams, assuming no background in the reader of the geometric style of reasoning about special relativity. This allows Wigner rotation to be easily derived and its geometric meaning understood, allowing it to be taught in an introductory course on special relativity. The Thomas precession formula becomes an easy corollary. Illustrative fables and further readings are attached at the end.

1 Introduction

1.1 The Wigner rotation

Imagine a space station at rest, with two reference frames made of steel, sitting parallel to each other. A rocket picks one up and flies once around the station. The frame on the rocket is mounted on a 3-gimbal mount, the same kind of stand used by gyroscopes, to keep it non-rotating whenever the rocket is rotating. Thus, while the rocket could rotate and boost, the frame itself only undergoes boosts. When the rocket returns, the two frames are compared, and astonishingly, the frame on the rocket has rotated relative to the frame on the station.

This is Wigner rotation: a rotation made of a composition of boosts.

1.2 History

The Wigner rotation was derived by Émile Borel in 1913 [Mal13], forgotten, rediscovered by Llewellyn Thomas in 1926 to explain the fine structure of atoms (thus also called the "Thomas precession"), and finally rederived by Eugene Wigner in 1939, who got his name attached to it. [Wal99]

1.3 Conventions in the paper

$c = 1$, so the lightcones make an angle of 45° with the t-axis in the spacetime diagrams.

The coordinates are written in the (t, x, y, z) order. t axis is always pointing up.

We will assume special relativity and Lorentz transforms are correct, and simply translate some of it into a geometric language. We could justify them by assuming some axioms of spacetime geometry, but we won't.

The sign convention is $(+, -, -, -)$, so that the Minkowski metric on $\mathbb{R}^{1+3}$ is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

and the dot product is

$$(t, x, y, z) \cdot (t', x', y', z') = tt' - xx' - yy' - zz' \tag{2}$$

We write the absolute norm of a 4-vector $V = (t, x, y, z)$ as $||V|| = \sqrt{|t^2 - x^2 - y^2 - z^2|}$. The absolute norm is not the norm, but it's convenient when we don't want to deal with sign problems.

A unit vector is a vector with absolute norm 1. Two vectors U, V are orthogonal, that is, $U \perp V$, iff $U \cdot V = 0$.

A reference frame is defined by its basis vectors e_t, e_x, e_y, e_z. The basis is orthonormal, meaning they are unit vectors that are pairwise orthogonal. Also, e_t is timelike, and points to the future. e_x, e_y, e_z are spacelike.

To "see" is to measure. We won't deal with the visual distortion effects of relativity.

We assume all inertial observers make their origins coincide, since translations are pretty trivial, and we'd rather not deal with them.

While the results in the paper apply to $1 + 3$ dimensional spacetime, for ease of plotting, we consider only $1 + 1$ or $1 + 2$ dimensions. One unfortunate side effect is that, confusingly, "3-vector" has only 1 or 2 components, and "4-vector" has only 2 or 3 components.

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2 Geometry of spacetime

For a more leisurely treatment, see [BJ04].

2.1 Symmetries

A geometric space is best understood by what symmetries it has, and the geometric concepts are the concepts unchanged by any symmetry operation on the space. This viewpoint is promoted since Klein's Erlangen Program in the 19th century.

Applied to spacetime, whose symmetries are translations, rotations, and boosts, it means that any geometric concept on spacetime must be unchanged by translations, rotations, and boosts. We call this "Lorentz-invariance".

Spacetime also have space-reflection and time-reflection symmetries, and all results in the paper still apply even when we allow these reflection symmetries, but for simplicity we will not consider them. Technically, it means we only consider proper orthochronous Lorentz transformations.

2.2 Physics and geometry

By special relativity, the physically meaningful concepts are the same for all inertial observers, and the reference frames of inertial observers are related by Lorentz-transforms, that is, the symmetries of spacetime. Thus, any physically meaningful concept must be a geometric concept on spacetime.

Since all the physical concepts are already geometric, what work is left to do? Define the concepts in a coordinate-free language!

The usual definition of physical concepts uses a coordinate frame, but this is inconvenient for geometric reasoning. This is similar to the situation where a circle can be defined as $\{(x, y) : (x-x_0)^2+(y-y_0)^2 = r^2\}$ in a certain coordinate system, or as "the set of points having equal distance to a point" with no reference to any coordinate system. The second definition allows us to avoid algebra and proceed by geometric arguments.

We develop some of the geometry of spacetime, a bit more than enough for the purpose of this paper, so that any reader unfamiliar with this geometric view can be brought up to speed.

2.3 Timelike, lightlike, spacelike

The points are events in spacetime. Two points are timelike separated iff it's possible to send a massive particle from one to the other. Two points are lightlike separated iff it's possible to send a light from one to the other. Otherwise they are spacelike separated. Due to the choice of $c = 1$, the lightlike separations all make a 45° angle with the t-axis when drawn in a coordinate system.

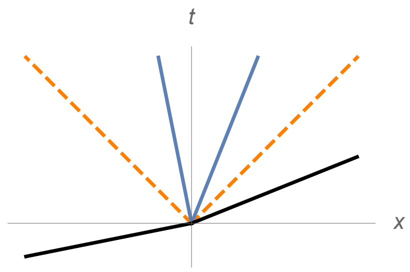


Figure 1: Three kinds of separations in $1+1$ dimensional spacetime. The dashed lines are lightlike, the blue lines are timelike, and the black lines are spacelike.

Similarly, a 4-vector is timelike, lightlike, or spacelike, based on how its head and tail are separated. These are shown in Figure 1.

2.4 Unit timelike vector

Any unit timelike vector e_t is timelike, so it can be traversed by a massive particle, or indeed, a clock, if the clock is small enough to be thought of as a particle. It is of unit length, meaning that the clock would measure unit time while it follows e_t.

Fix a reference frame, then the unit timelike vectors are $e_t = (t, x, y), ||e_t|| = 1$, that is, $t^2 - x^2 - y^2 = 1$. We want the timelike vectors to point to the future, so $t > 0$, then, we see that the unit timelike vectors have their heads on the positive sheet of hyperbola:

$$\mathbb{H} = \{t^2 - x^2 - y^2 = 1, t > 0\} \tag{3}$$

Figure 2 shows the case for $1 + 1$ dimensions.

2.5 3- and 4-velocity

We give a careful, geometry-flavored definition of 3- and 4-velocity, because the standard textbooks don't make the geometry clear. If you understand Figure 3, you can skip this section.

Consider a massive particle moving at constant speed, seen by a stationary observer. Its worldline is a straight line. Its 4-velocity is its 4-displacement during 1 unit of its proper time. Its 3-velocity is its 3-displacement during 1 unit of the stationary observer's time.

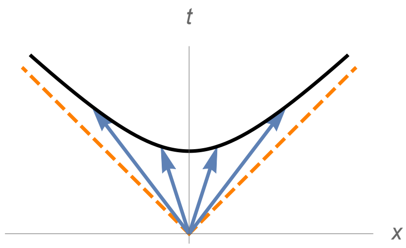


Figure 2: The hyperbola of unit timelike vectors. A clock takes unit time to follow any of the unit timelike vectors.

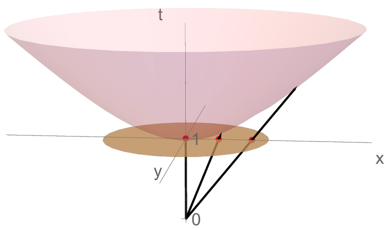


Figure 3: The vectors are 4-velocities. Their intersections with the disk are their corresponding 3-velocities. Γ maps a 3-velocity v to its corresponding 4-velocity U, and O, v, U are collinear.

Massive particles move slower than light, so the set of all possible 3-velocities is the open disk of radius $c = 1$:

$$\mathbb{D} = \{(v_x, v_y) \in \mathbb{R}^2 : v_x^2 + v_y^2 < 1\} \tag{4}$$

The 4-velocity is time-like, so its norm is the proper time experienced by a particle moving along the 4-velocity vector, which is 1 by definition. So it is a unit time-like vector, so the hyperboloid $\mathbb{H}$ is the set of all possible 4-velocities.

The definition of 4-velocity is observer-independent, but 3-velocity depends on the choice of observer, in order to define the $t = 1$ plane, so it is not.

From the definition of 3- and 4-velocities, we see a geometric relation between the two.

Let the particle have 3-velocity $v = (v_x, v_y)$, and 4-velocity $U = (U_t, U_x, U_y)$. Then the 4-displacement of the particle during 1 unit of the stationary observer's time is $(1, v_x, v_y)$, and the 4-displacement of the particle during 1 unit of its proper time is (U_t, U_x, U_y). Thus, $(0, 0, 0)$, $(1, v_x, v_y)$, and (U_t, U_x, U_y) are collinear, illustrated in Figure 3.

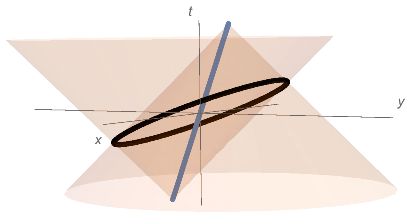


Figure 4: Given e_t, shine a lightcone from $-e_t$ forwards in time, and another lightcone from e_t backwards in time. Their intersection is $S(e_t)$.

2.6 Perpendicularity

$U \perp V$ is defined by $U \cdot V = 0$. If in coordinate system, $U = (t, x, y)$, $V = (t', x', y')$, then it's defined by $tt' - xx' - yy' = 0$, but when one of U, V is timelike, there's a more pictorial definition.

Given any unit-length timelike vector e_t, we define its **spacecircle**[1] as

$$S(e_t) = \{e : ||e|| = 1, e \perp e_t\} \tag{5}$$

One way to imagine a spacecircle $S(e_t)$ is to imagine it as the edge of a saucer with radius 1, and the saucer is flying at 4-velocity e_t. Then, in the saucer's frame, $e_t = (1, 0, 0)$ and $S(e_t) = \{(0, \cos\theta, \sin\theta) : 0 \leq \theta < 2\pi\}$.

A pictorial way to construct $S(e_t)$ from e_t is by shining a cone of light from $-e_t$ forwards in time, and shining another cone of light from e_t backwards in time. Their intersection is $S(e_t)$. This is illustrated in Figure 4. This procedure is observer-independent, and it works in one of the frames, namely the saucer's frame, so it works in any frame.

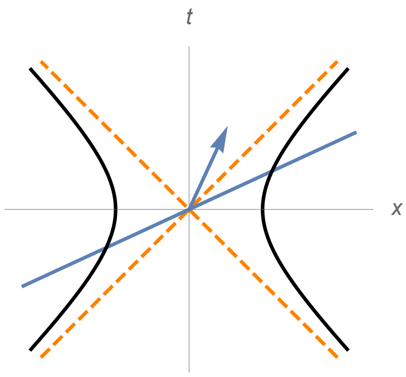
In a frame where e_t does not have coordinates $(1, 0, 0)$, $S(e_t)$ would appear tilted and elliptical, but that's only because our diagrams are Euclidean, so they don't faithfully picture the geometry of spacetime.

In $1 + 1$ dimensions, it's clear to see in the spacetime diagram that e_t and $S(e_t)$ are symmetric across the lightcone. This is true for $1 + 2$ dimensions too. Both cases are illustrated in Figure 5.

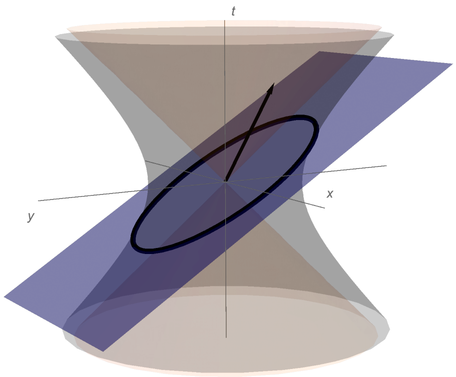
Also, any spacecircle is made of unit spacelike vectors, and in any reference frame, the set of all unit spacelike vectors makes a one-sheet hyperboloid

$$\mathbb{H}' = \{(t, x, y) : t^2 - x^2 - y^2 = -1\} \tag{6}$$

[1] We call it a spacecircle, because we'll plot it in 1+2 dimensions. In 1+3 dimensions, it might be more properly called a spacesphere.



(a) e_t and $S(e_t)$ are symmetric across the light cone. $S(e_t)$ is the intersection of l with the hyperbola $t^2 - x^2 = -1$.



(b) The same situation but in $1+2$ dimensions. $S(e_t)$ is the intersection of l with the hyperboloid $t^2 - x^2 - y^2 = -1$.

Figure 5: Symmetry between e_t and $S(e_t)$

Thus, in any frame, if e_t is known, $S(e_t)$ can be constructed by drawing the plane that's symmetric to e_t across the lightcone, and its intersection with the hyperboloid $\mathbb{H}'$ is $S(e_t)$. Figure 5 illustrates both $1+1$ and $1+2$ dimensional cases.

2.7 Infinitesimal boosts

Consider $1+1$-dimensional spacetime, and consider two frames: the rest frame $\{e_t, e_x\}$, and a frame $\{e'_t, e'_x\}$ that's boosted by v in the $+x$ direction. The transform is

$$\begin{bmatrix} e'_t \\ e'_x \end{bmatrix} = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix} \begin{bmatrix} e_t \\ e_x \end{bmatrix} \tag{7}$$

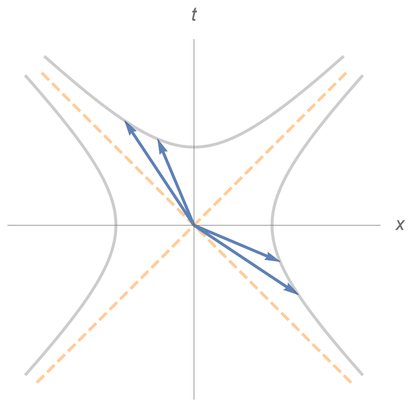


Figure 6: An infinitesimal boost of a frame e_t, e_x to $e_t + de_t, e_x + de_x$. Note that $de_x \parallel e_t$, and $de_t \parallel e_x$.

where $\gamma = 1/\sqrt{1-v^2}$. For infinitesimal v, to first order,

$$\begin{bmatrix} e'_t \\ e'_x \end{bmatrix} = \begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix} \begin{bmatrix} e_t \\ e_x \end{bmatrix} \tag{8}$$

So, for an infinitesimal boost in the x-direction, e_x is changed by $de_x = e_x - e'_x = ve_t$, thus $de_x \parallel e_t$. This works in general for $1+3$-dimensional spacetime, so we obtain the following rule: Under an infinitesimal boost, de_x, de_y, de_z are all parallel to e_t. Also, $de_t \parallel span\{e_x, e_y, e_z\}$, with its direction determined by the direction of the boost.

This allows us to perform an infinitesimal boost on any frame pictorially. Consider Figure 6. The frame defined by e_t, e_x looks tilted in the spacetime diagram, still $de_x \parallel e_t$ under an infinitesimal boost.

2.8 Boosting a spacecircle

Given a flying saucer e_t, and its spacecircle $S(e_t)$, if the saucer accelerates a tiny bit, without rotating, then $S(e_t)$ is boosted infinitesimally to $S(e'_t)$.

To find where any given $e \in S(e_t)$ goes after the boost, a pictorial method is:

First, draw $S(e_t)$ and $S(e'_t)$, by the intersection method in Figure 5. Then, for any $e \in S(e_t)$, draw a line parallel to e_t. The intersection with $S(e'_t)$ is where e is boosted to. This is illustrated in Figure 7.

3 Deriving the Wigner rotation

Fix a reference frame, and consider a particle that starts at rest, undergoes a series of pure boosts, and returns to

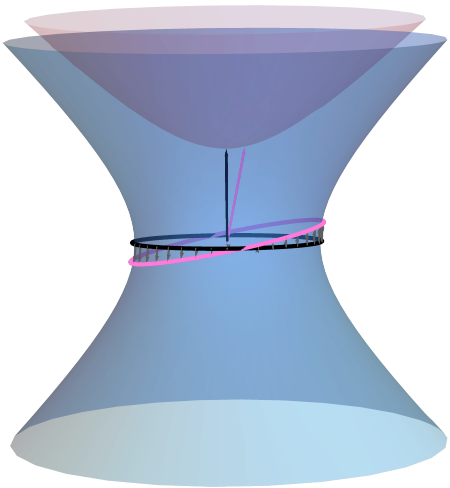


Figure 7: An infinitesimal boost of a frame. Its 4-velocity changes from U_1 to U_2, and its spacecircle changes from S_1 to S_2. The gray lines are the trajectories of the points on the spacecircle during the boost. Notice how they are all parallel to U_1.

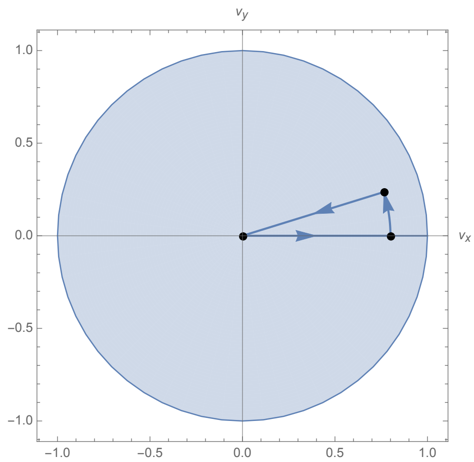


Figure 8: The 3-velocity traces around an infinitesimal sector of radius v and angle $d\theta$ in $\mathbb{D}$. The whole motion consists of three boosts, the second one is infinitesimal. The first and the third are straight boosts.

rest (but not necessarily returning to original position). Then, its 3-velocity v traces out a closed path in $\mathbb{D}$, the space of 3-velocities.

Let $e_{t'}$ be the particle's 4-velocity. We will follow the path of $e_{x'} \in S(e_{t'})$ during the motion, and find out that it would rotate by a certain angle $d\theta$ during the motion. Since $e_{x'}$ determines the "starting point" of the spacecircle, it means that the whole spacecircle would also have rotated by $d\theta$ during the motion. This is the Wigner rotation angle.

3.1 A pure boost

In $1+2$ dimensions, fix a stationary reference frame, and consider a particle accelerating by a series of boosts in the x-direction, its 3-velocity changes from $(0,0)$ to $(v,0)$, its 4-velocity from $U_1 = (1,0,0)$ to $U_2 = (\gamma, v\gamma, 0)$, where $\gamma = 1/\sqrt{1-v^2}$. During this process, $e_{t'}$ moves from U_1 to U_2, and $S(e_{t'})$ moves from $S_1 = S(U_1)$ to $S_2 = S(U_2)$.

Consider the trajectory of any $e' \in S(e'_t)$. The curve $e'(t)$ has tangent de'/dt, and so it must be parallel to $e_{t'}(t)$. Since the particle only accelerates in the x direction, $e_{t'}$ is in the $t-x$ plane at all times, so the trajectory of $e_{t'}(t)$ must lie in a plane parallel to the $t-x$ plane. Some trajectories are shown in Figure 9a, both shown in the full (t,x,y) space, as well as a projection onto (x,y) plane viewed from above.

Since $U_2 = (\gamma, v\gamma, 0)$, by symmetry across the light cone, $(e_{x'})_2 = (v\gamma, \gamma, 0)$, and when projected to the (x,y) plane, $proj((e_{x'})_2) = (\gamma, 0)$. Thus, the projected S_2 is an ellipse with semi long axis γ.

3.2 Three pure boosts

Now let the particle undergo three boosts, such that its 3-velocity traces around an infinitesimal sector of radius v and angle $d\theta$ in $\mathbb{D}$, as shown in Figure 8.

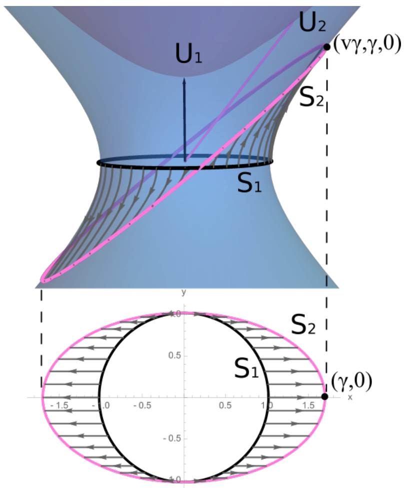
During the first boost, $e_{t'}$ moves from U_1 to U_2, and $S(e_{t'})$ moves from S_1 to S_2. This process is shown in Figure 9a.

During the second boost, $e_{t'}$ moves from U_2 to U_3. This boost is infinitesimal, so any $e \in S_2$ moved by an infinitesimal amount de that is parallel to U_2.

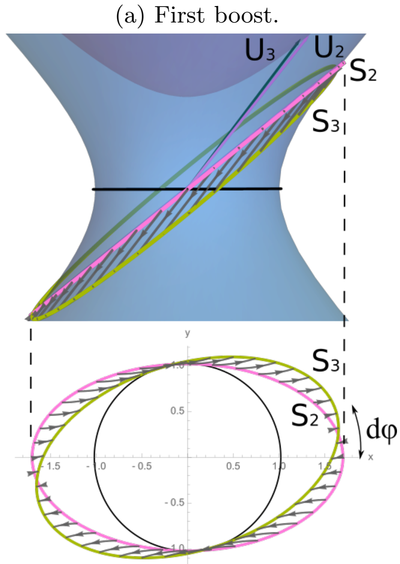
U_2 is parallel to the (t,x) plane, so de is parallel to the (t,x) plane too. So, when projected to (x,y) plane, de is parallel to the x axis. This process is shown in Figure 9b.

During the third boost, $e_{t'}$ moves from U_3 to U_1. The third boost is similar to the first boost, but reversed, and rotated by an angle $d\phi$. This process is shown in Figure 9c.

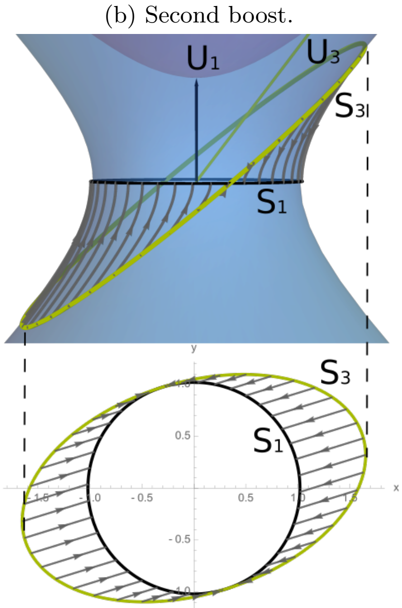
Now follow the trajectory of $e_{x'}$ during the motion, projected onto the (x,y) plane. This is shown in Figure 10. It's a simple exercise to show that the distance between points 2 and 3 in the figure is second-order in $d\phi$, and so can be ignored. Then, the arc from 1 to 4 has



(a) First boost.



(b) Second boost.



(c) Third boost.

Figure 9

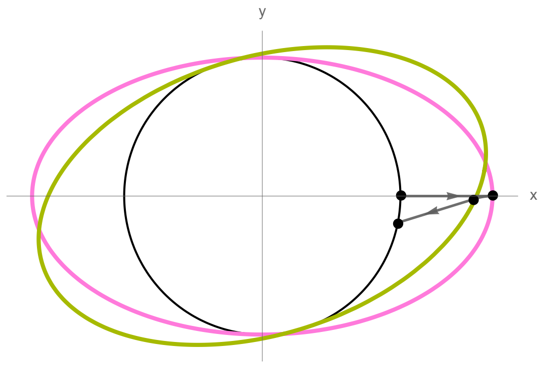


Figure 10: The trajectory of $e_{x'}$ during the motion, projected onto the (x, y) plane. The circle and the ellipses are S_1, S_2, and S_3. $d\theta$ is the Wigner rotation angle. Triangle $\Delta O24$ is enlarged for clarity.

length

$$d\theta = \widehat{1-4} = d\phi(\gamma - 1)$$

Notice what has been proved here. The particle has returned to rest, but its $e_{x'}$ vector has not returned to the original position, instead, it had rotated by $d\theta$ clockwise. This is Wigner rotation.

3.3 A full cycle of boosts

Suppose that the particle is rotating around the origin at 3-speed v, then its 3-velocity follows the circular path in Figure 11. We can cut it into many small cycles, so that the Wigner rotation in each small cycle causes a Wigner rotation of angle

$$d\theta = d\phi(\gamma - 1) \tag{9}$$

and then, summing these rotations up, we have

$$\Delta\theta = 2\pi(\gamma - 1)$$

is the Wigner rotation angle during one revolution of the particle. The direction of Wigner rotation is clockwise, while the particle is moving counterclockwise, so to account for the sign, we have

$$\Delta\theta = -2\pi(\gamma - 1) \tag{10}$$

3.4 Thomas precession formula

The angular speed of Wigner rotation observed in the stationary frame is

$$\omega_T = |\Delta\theta/T| = 2\pi(\gamma - 1)/(2\pi R/v) = \frac{\gamma - 1}{v^2} va$$

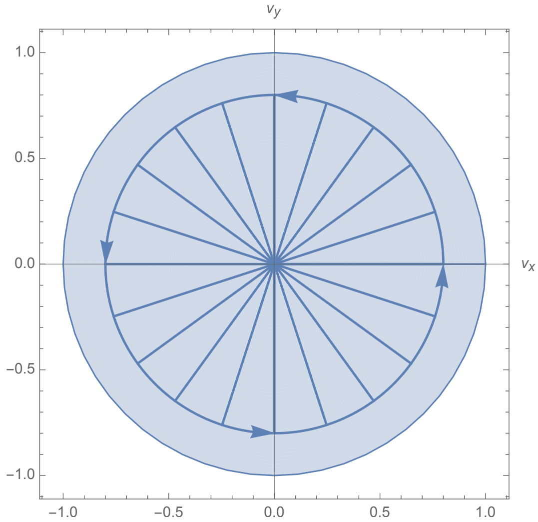


Figure 11: The path of the 3-velocity of the particle on a circular motion is dissected into infinitesimal sectors. Each sector gives a Wigner rotation by $d\theta = -d\phi(\gamma - 1)$, giving a total of $\Delta\theta = -2\pi(\gamma - 1)$.

where T is the orbit period, R is the radius of the particle's orbit, and $a = v^2/R$ is the particle's 3-acceleration.

To account for the direction of Wigner rotation, we have the 3-vector equation.

$$\omega_T = \frac{\gamma - 1}{v^2} a \times v$$

Simplifying by $\gamma = 1/\sqrt{1 - v^2}$,

$$\omega_T = \frac{\gamma^2}{\gamma + 1} a \times v \tag{11}$$

which is the Thomas precession formula.

4 Implications for pedagogy

The study of hyperbolic geometry of special relativity began in the 1910s by Vladimir Varičak, Gilbert N. Lewis, Émile Borel, among others, but it has never been mainstream. We checked all the relativity textbooks in the university library, and could not find a single reference to hyperbolic geometry.

The Wigner rotation is every bit as paradoxical and fascinating as the more popular phenomena of special relativity. Regarded as a paradox, it is arguably a more significant paradox than the "twin paradox" or "ladder and barn paradox", since it has a very practical significance: Electrons orbit the nucleus at relativistic speeds, and the Wigner rotation becomes significant, the effect of which shows up in the fine structure of atomic spectra, and is verified countless times.

Despite this, it is surprisingly obscure. We checked the whole library aisle on relativity, and Goldstein's *Classical Mechanics* is the only textbook we found that treats it in any detail. We suspect that this neglect is half because a purely algebraic treatment (compared to the hyperbolic-geometric treatment) is very arduous, and half due to a superstition that special relativity cannot deal with accelerating frames.

This highly pictorial method of deriving the Wigner rotation in the paper was discovered by the author during their attempt at a visual understanding this phenomenon, and to the best of our knowledge, is new. This new method allows Wigner rotation to be easily derived and understood, overcoming the barrier of formidable algebra. The Thomas precession formula, very useful in atomic physics, is derived as a simple corollary.

The author strongly recommend Wigner rotation, derived by the geometric method in the paper, to be incorporated in introductory courses on special relativity.

5 Sidenotes

5.1 A hint of hyperbolic geometry

Consider a particle moving circularly at small 3-velocity v, then after completing one cycle, its Wigner rotation angle is $|\Delta\theta| = 2\pi(\gamma - 1) \approx \pi v^2$.

This looks like the area of a circle of radius v, which is the area enclosed by the curve traced out by the particle's 3-velocity. As it turns out, it's possible to make this connection rigorous.

In 1+2 dimensional spacetime, the space of 4-velocities is $\mathbb{H} = \{t^2 - x^2 - y^2 = 1, t > 0\}$, then the Minkowski metric on it makes it into the hyperboloid model of hyperbolic plane of curvature -1. This induces a metric on the space of 3-velocities, $\mathbb{D}$, making it into the Beltrami-Klein disk model.

With this metric, if a particle moves in such a way, so that its 3-velocity traces out a circuit l in $\mathbb{D}$, then the Wigner rotation experienced by the particle is the area in the circuit! This can be understood as the Gauss-Bonnet formula on the hyperbolic plane, which says that the angle-defect of a polygon is equal to the negative of the area of the polygon.

5.2 A few illustrative fables

5.2.1 Cosmic dark age

An alien race used to worship Rigel, Deneb, and Betelgeuse, and they prayed towards the three stars. In order to pray in the correct direction even without seeing them, they constructed the Sacred Tripods, which are steel tripods that are oriented, such that each leg points at one of the Sacred Stars. These Sacred Tripods are put in public places, and required to point exactly at the Sacred Stars.

Then a Cosmic Dark Age began and the stars winked out of existence, including the Sacred Stars, but the aliens did not lose their faith. Instead, they intensified their prayer in the hope of resurrecting the Sacred Stars.

In a region of space, there were two space stations A and B, at rest with each other. The Sacred Tripod on B was unstable and required yearly checking against the more stable Sacred Tripod at A. So station A would align its spare Sacred Tripod to its own one, put it on a gimbal-mount, and send it by rocket to station B. B would align its Tripod with the one sent, then let the Tripod go back. Years passed, and aliens on station B felt confident that its Tripod doesn't drift more than $1''$ per year.

One day, an asteroid field blocked the straight path, forcing the next shipment to make a big detour around the asteroid field. When it arrived, to their astonishment, it was found that the Tripod on B was clearly misaligned with the Tripod from A. What's going on?

Answer: Wigner rotation of the Tripod from A, caused by the detour.

5.2.2 Carrying a frame in the car

Instead of carrying a gimbal-mounted frame on a rocket, now we carry a plate in a car driven on a sphere. The plate is mounted on a central axis perpendicular to the ground, and can turn frictionlessly. Thus, just like in the previous story, the car can boost and rotate, but the plate only boosts.

We start at the North Pole and drive along the prime meridian to the equator, turn left by $90°$ to face east, drive to the $90°E$ meridian, turn left by $90°$ to face north, drive to the North Pole, then turn left by $90°$.

Now the car is back at the same position and direction, but the plate has turned right by $270°$, that is, turned left by $90°$.

By the angle defect formula for spheres, it's easy to see that if the sphere has radius R, then given any triangular path γ, driving around γ causes the plate to rotate by $\Omega(\gamma) = Area(\gamma)/R^2$. For a general path, it can be dissected into small triangles, so summing up, we get $\Omega(\gamma) = Area(\gamma)/R^2$ for a general, nontriangular path γ.

5.2.3 The Day the Earth Stood Still

One day, the Earth stopped turning. This annoyed the visitors to the Foucault Pendulum exhibition in the science museum, because they could not see it turn anymore. To save the museum, the curator mounted the pendulum on a disk affixed to a car and started driving around the axis of earth, as if the earth is still rotating. The museum-goers sat on the car and watched the pendulum rotate as it should, and were satisfied. After driving for a whole day, the car returned to the museum, having gone around the world once.

The pendulum behaves the same way as the mounted plate in the previous story, since it can also only be forced to undergo boosts, but not rotations. Its turning angle is thus $\Omega(\gamma) = Area(\gamma)/R^2$, where γ is the latitude line that the car followed. By the spherical cap formula, if the museum is at latitude θ, then $Area(\gamma) = 2\pi(1-\sin\theta)R^2$, thus $\Omega(\gamma) = 2\pi(1-\sin\theta)$, the formula for the turning angle of Foucault Pendulum.

5.3 Further reading

[Wal99] gives a thorough discussion of the early history of use of hyperbolic geometry in special relativity. [Ara97] derives the same result. [Mal13] derives the same result following Émile Borel's 1913 derivation of Wigner Rotation, and gives historical notes. [CA01] is a pedagogical paper that applies hyperbolic geometry to special relativity. [Kri09] is a pedagogical paper that uses geometric reasoning to study Foucault Pendulum and Wigner rotation in a unified way, and hints at their common geometric nature.

On the pure algebra side, [OV11] is a pedagogical paper that studies Wigner rotation in gory algebraic details, and [RS04] is a lengthy paper that gives a thorough treatment on Wigner rotation, that unfortunately does not have pictures.

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