Online Forecasting of Total-Variation-bounded Sequences

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Introduction and Objective

◇ Nonparametric Online Forecasting model

1. Fix action time intervals $1, 2, \ldots, n$
2. The player declares a forecasting strategy $\mathcal{A}_i : \mathbb{R}^{i-1} \to \mathbb{R}$ for $i = 1, \ldots, n$.
3. An adversary chooses a sequence $\theta_{1:n} = [\theta_1, \theta_2, \ldots, \theta_n] \in \mathbb{R}^n$.
4. For every time point $i = 1, \ldots, n$:
   - (a) We play $x_i = \mathcal{A}_i(y_1, \ldots, y_{i-1})$.
   - (b) We receive a feedback $y_i = \theta_i + Z_i$, where Z_i is a zero-mean, independent subgaussian noise.
5. At the end, the player suffers a cumulative error $\sum_{i=1}^n (x_i - \theta_i)^2$.

◇ Assumptions

1. Knowledge of σ^2 of sub-gaussian noise.
2. Ground truth sequence $\theta_{1:n} \in TV(C_n)$, where $\mathrm{TV}(C_n) := \{\theta_{1:n} \in \mathbb{R}^n \,|\, \|D\theta_{1:n}\|_1 \le C_n\}$ and D is the discrete difference operator. C_n is not required to be known a priori.
3. $|\theta_1| \le U$

◇ Questions of interest

1. What is the optimal Total Squared Error (TSE) for any method?
2. How to design a minimax policy that is locally adaptive to the non-uniform trends found in TV class?

Performance of existing policies

Theorem 1 (A lowerbound on TSE) *Assume that $\min\{U, C_n\} > 2\pi\sigma$ and $n > 3$, there is a universal constant c such that*

$$\inf_{x_{1:n}} \sup_{\theta_{1:n} \in \mathrm{TV}(C_n)} \mathbb{E}\left[\sum_{t=1}^n \left(x_t(y_{1:t-1}) - \theta_t\right)^2\right] \ge c(U^2 + C_n^2 + \sigma^2 \log n + n^{1/3} C_n^{2/3} \sigma^{4/3}).$$

◇ TSE for existing policies grows as $O(\sqrt{n})$

- Restarting OGD [1,2], Moving Averages
- Adaptive Optimistic Mirror Descent [3]

Our Policy

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ARROWS: inputs - observed $y$ values, $\delta \in (0,1]$, $\sigma^2$, time horizon $n$
a hyper-parameter $\beta > 24$
1. Initialize $t_h = 1$, $newBin = 1$, $y_0 = 0$
2. For $t = 1$ to $n$:
   (a) if $newBin == 1$, predict $x_t^{t_h} = y_{t-1}$, else predict
       $x_t^{t_h} = \bar{y}_{t_h:t-1}$
   (b) set $newBin = 0$, observe $y_t$ and suffer loss $(x_t^{t_h} - \theta_t)^2$
   (c) Let $\hat{y} = pad_0(y_{t_h}, \ldots, y_t)$ and $k$ be the padded length.
   (d) Let $\hat{\alpha}(t_h : t) = T(H\hat{y})$
   (e) Restart Rule: If $\frac{1}{\sqrt{k}} \sum_{l=0}^{\log_2(k)-1} 2^{l/2} \|\hat{\alpha}(t_h : t)[l]\|_1 > \frac{\sigma}{\sqrt{k}}$
       i. set $newBin = 1$
       ii. set $t_h = t + 1$
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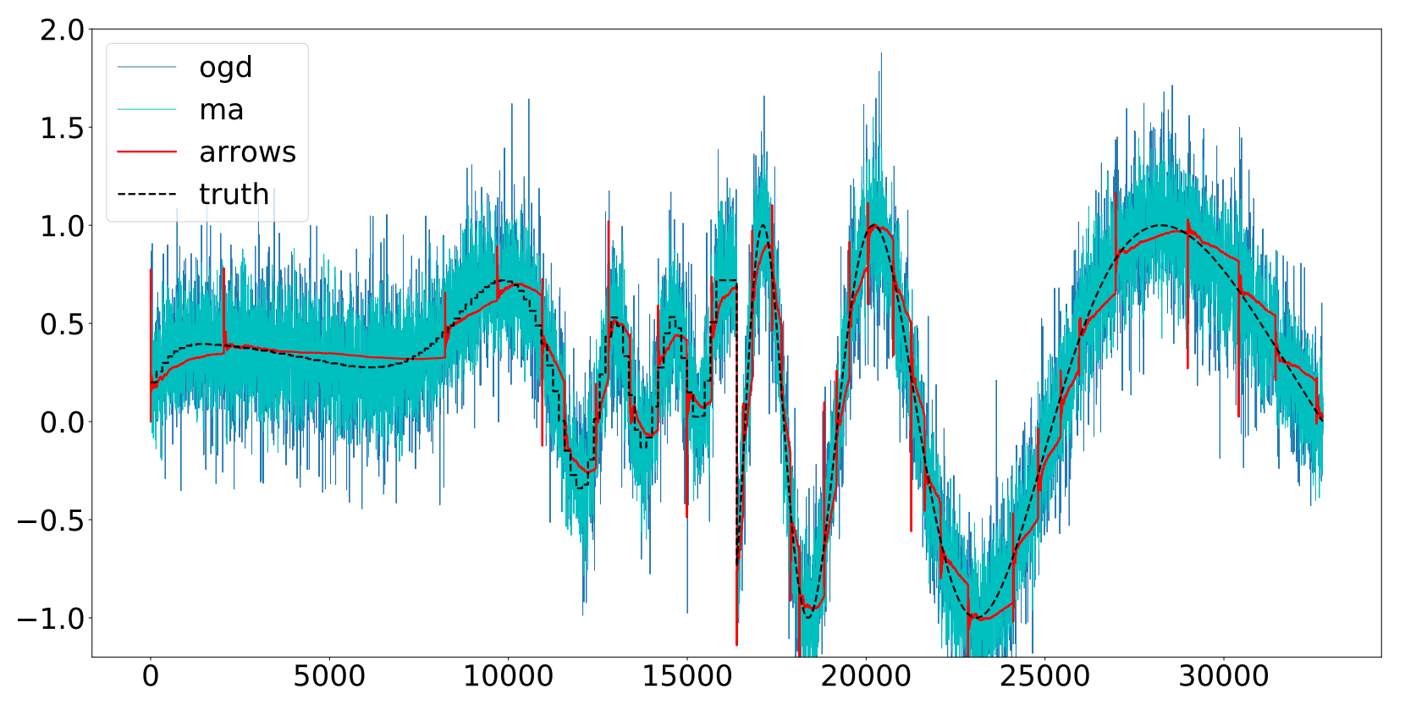
Theorem 2 (TSE of Arrows) *Let the feedback be $y_t = \theta_t + Z_t$, $t = 1, \ldots, n$ and Z_t be independent, σ-subgaussian random variables. If $\beta = 24 + \frac{8\log(8/\delta)}{\log(n)}$, then with probability at least $1 - \delta$,* Arrows *achieves a dynamic regret of $\tilde{O}(n^{1/3}\|D\theta_{1:n}\|_1^{2/3}\sigma^{4/3} + |\theta_1|^2 + \|D\theta_{1:n}\|_2^2 + \sigma^2)$ where $\tilde{O}$ hides a logarithmic factor in n and $1/\delta$.*

◇ Runtime of Arrows is $O(n \log n)$

Experimental results



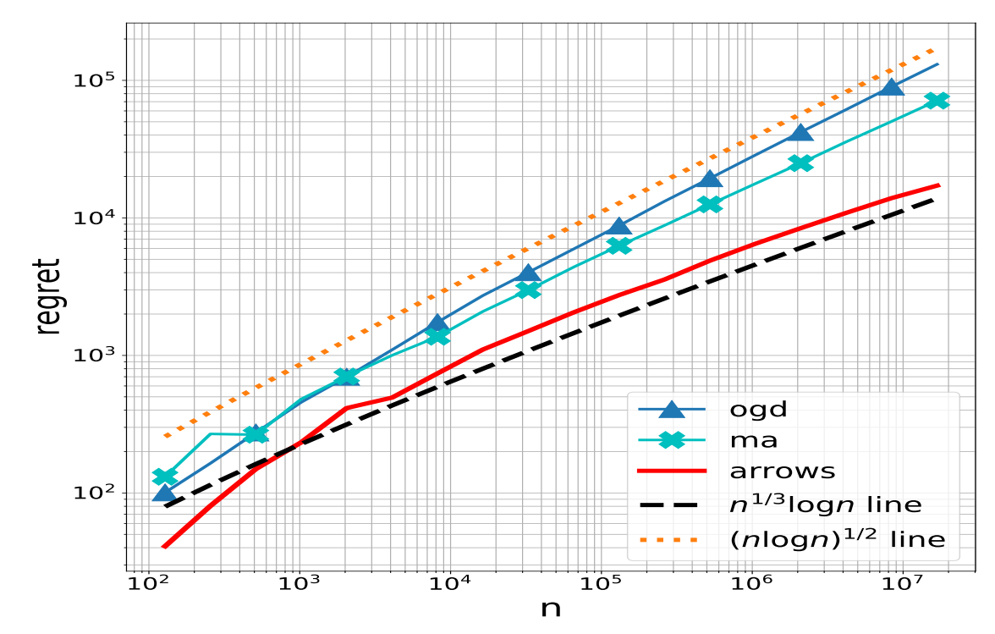
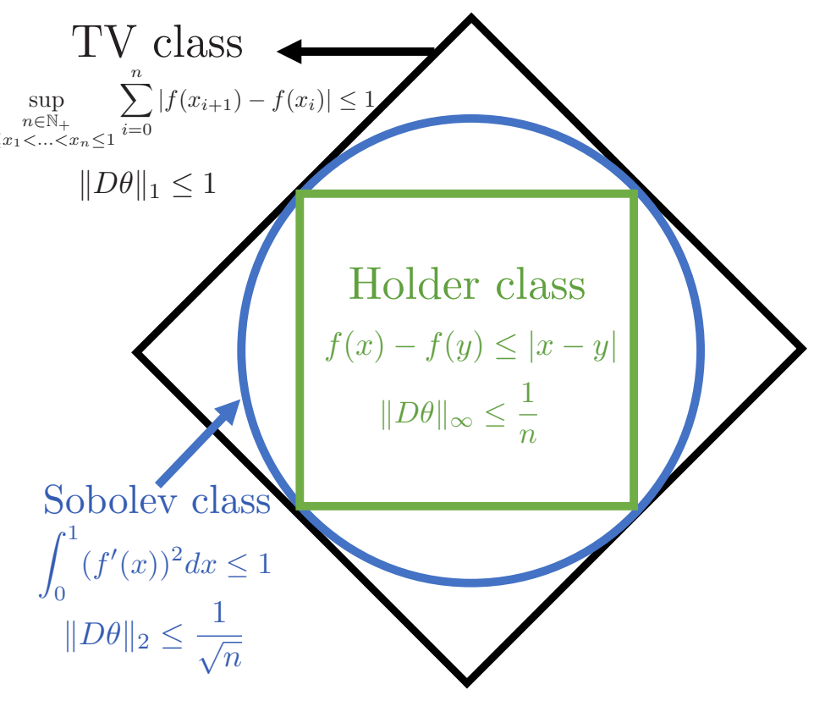


Figure description: The figure shows the results on a function with heterogeneous smoothness. The top panel illustrates that Arrows is locally adaptive to heterogeneous smoothness of the ground truth. Red peaks in the top left figure signifies restarts. During the initial and final duration, the signal varies smoothly and Arrows chooses a larger window size for online averaging. In the middle, signal varies rather abruptly. So Arrows chooses a smaller window size. OGD and MA can't adapt to non-uniform smoothness and has a suboptimal $\tilde{O}(\sqrt{n})$ TSE while Arrows attains the $\tilde{O}(n^{1/3})$ minimax TSE!

Minimax rates for TSE

Class		Minimax rate for Forecasting	Minimax rate for Smoothing[4]	Minimax rate for Linear Forecasting
TV	$\|D\theta\|_1 \le C_n$	$n^{1/3}C_n^{2/3}\sigma^{4/3} + C_n^2 + \sigma^2$	$n^{1/3}C_n^{2/3}\sigma^{4/3} + \sigma^2$	$n^{1/2}C_n\sigma + C_n^2 + \sigma^2$
Sobolev	$\|D\theta\|_2 \le C_n'$	$n^{2/3}[C_n']^{2/3}\sigma^{4/3} + [C_n']^2 + \sigma^2$	$n^{2/3}[C_n']^{2/3}\sigma^{4/3} + \sigma^2$	$n^{2/3}[C_n']^{2/3}\sigma^{4/3} + [C_n']^2 + \sigma^2$
Holder	$\|D\theta\|_\infty \le L_n$	$nL_n^{2/3}\sigma^{4/3} + nL_n^2 + \sigma^2$	$nL_n^{2/3}\sigma^{4/3} + \sigma^2$	$nL_n^{2/3}\sigma^{4/3} + nL_n^2 + \sigma^2$
Minimax Algorithm		Arrows	Wavelet Smoothing[4] Trend Filtering[5]	Restarting OGD[1,2] Moving Averages


TV class
$\sup_{\substack{n\in\mathbb{N}_+\\0\le x_1<\ldots<x_n\le 1}} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| \le 1$
$\|D\theta\|_1 \le 1$
Holder class
$f(x) - f(y) \le |x - y|$
$\|D\theta\|_\infty \le \frac{1}{n}$
Sobolev class
$\int_0^1 (f'(x))^2 dx \le 1$
$\|D\theta\|_2 \le \frac{1}{\sqrt{n}}$

Canonical Scaling[a]		Forecasting	Smoothing	Linear Forecasting
TV	$C_n \asymp 1$	$n^{1/3}$	$n^{1/3}$	$n^{1/2}$
Sobolev	$C_n' \asymp 1/\sqrt{n}$	$n^{1/3}$	$n^{1/3}$	$n^{1/3}$
Holder	$L_n \asymp 1/n$	$n^{1/3}$	$n^{1/3}$	$n^{1/3}$

[a] The "canonical scaling" are obtained by discretizing functions in canonical function classes. Under the canonical scaling, Holder class ⊂ Sobolev class ⊂ TV class, as shown in the figure on the left.

1. *For compactness we hide the dependence of U and $\log n$ from all forecasting rates.*
2. Arrows is adaptively minimax over the described classes.
3. Linear forecasters are fundamentally limited in predicting TV bounded sequences
   - (a) Policies such as Restarting OGD/MA are unable to come up with a single window size that performs optimally through out the duration.

Adaptivity to unknown parameters

1. Arrows adapts optimally to the unknown TV of ground truth $\|D\theta_{1:n}\|_1$.
2. Adaptivity to time horizon is achievable by doubling trick.
3. σ if unknown can be robustly estimated via the MAD estimator due to the sparsity of wavelet coefficients in TV class.

References

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[4] David L Donoho and Iain M Johnstone. Minimax estimation via wavelet shrinkage. In *Annals of statistics*, 1998.

[5] Seung-Jean Kim, Kwangmoo Koh, Stephen Boyd, and Dimitry Gorinevsky. ℓ_1 trend filtering. In *SIAM Review*, 2009.