§3.3 非齐次线性方程组 $A\vec{x}=\vec{b}$ 的解

<u>定理</u> $A\vec{x}=\vec{b}$ 有解 $\Longleftrightarrow$ $\vec{b}\in C(A)$

augmented matrix $[A \ \vec{b}] \xrightarrow{\text{Gauss}} [U \ \vec{c}] \xrightarrow{\text{Jordan}} [R \ \vec{d}]$

$A\vec{x}=\vec{b}$, $U\vec{x}=\vec{c}$ 及 $R\vec{x}=\vec{d}$ 同解.

若一行出现"$0=$非零数"的情形, 无解 ($\vec{b}$ 无法由 A 的列线性表出); 否则, 有解

<u>解的结构</u>

记 $A\vec{x}=\vec{b}$ 的解集为 W. 取定 $\vec{x}_p\in W$ (某个已知解, 称为特解, particular solution).

$A\vec{x}=\vec{0}$ 称为 $A\vec{x}=\vec{b}$ 的导出组 / 对应的齐次线性方程组, 其解集为 $N(A)$

(1) $\forall\ \vec{x}_1, \vec{x}_2\in W$, 有 $\vec{x}_1-\vec{x}_2\in N(A)$

Pf. $A(\vec{x}_1-\vec{x}_2)=A\vec{x}_1-A\vec{x}_2=\vec{b}-\vec{b}=\vec{0} \Rightarrow \vec{x}_1-\vec{x}_2\in N(A)$.

(2) $\forall\ \vec{x}\in W$, 则 $\exists\ \vec{\eta}\in N(A)$. s.t. $\vec{x}=\vec{x}_p+\vec{\eta}$.

Pf. $\vec{x}=\vec{x}_p+(\vec{x}-\vec{x}_p) \xlongequal{\text{记}\vec{\eta}=\vec{x}-\vec{x}_p} \vec{x}_p+\vec{\eta}$

由(1), 有 $\vec{\eta}\in N(A)$.

(3) $\forall\ \vec{\eta}\in N(A)$, 有 $\vec{x}_p+\vec{\eta}\in W$

Pf. $A(\vec{x}_p+\vec{\eta})=A\vec{x}_p+A\vec{\eta}=\vec{b}+\vec{0}=\vec{b} \Rightarrow \vec{x}_p+\vec{\eta}\in W$.

综合(2),(3), 有: $W=\{\vec{x}_p+\vec{\eta} \mid \forall\vec{\eta}\in N(A)\}$, 其中 $\vec{x}_p$ 为 $A\vec{x}=\vec{b}$ 的一个特解.

<u>定理</u> 数域 F 上的非齐次线性方程组 $A\vec{x}=\vec{b}$, 若 $\vec{b}\in C(A)$ (有解), 且已知 $\vec{s}_1, \vec{s}_2, \cdots, \vec{s}_{n-r}$ 是 $A\vec{x}=\vec{0}$ 的 special solutions, $\vec{x}_p$ 是 $A\vec{x}=\vec{b}$ 的某个 particular solution. 则 $A\vec{x}=\vec{b}$ 的通解为

$$\vec{x}=\underset{\substack{\uparrow\\ \text{particular solution}}}{\vec{x}_p}+\underbrace{c_1\vec{x}_1+c_2\vec{x}_2+\cdots+c_{n-r}\vec{x}_{n-r}}_{\in N(A)\ \uparrow\ A\vec{x}=\vec{0}\text{通解}}, \quad \forall\ c_1, c_2, \cdots, c_{n-r}\in F.$$

例 $\begin{cases} x_1+2x_2+3x_3+5x_4=b_1 \\ 2x_1+4x_2+8x_3+12x_4=b_2 \\ 3x_1+6x_2+7x_3+13x_4=b_3 \end{cases}$, $A\vec{x}=\vec{b}$

(1). Reduce $[A\ \vec{b}]$ to $[U\ \vec{c}]$.

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \\ \begin{array}{cccc} \vec{\alpha}_1 & \vec{\alpha}_2 & \vec{\alpha}_3 & \vec{\alpha}_4 \end{array} \end{array} \xrightarrow[(-3)r_1+r_3\to r_3]{(-2)r_1+r_2\to r_2} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & -2 & -2 & b_3-3b_1 \end{array}\right] \xrightarrow{r_2+r_3\to r_3} \begin{array}{c} \begin{array}{cc} U & \vec{c} \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 5 & b_1 \\ 0 & 0 & \boxed{2} & 2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_2+b_3-5b_1 \end{array}\right] \\ \begin{array}{cccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{array} \end{array}$$

(2) Find the condition on b_1, b_2, b_3 for $A\vec{x}=\vec{b}$ to have a solution.

$A\vec{x}=\vec{b}$ 有解 $\Leftrightarrow b_2+b_3-5b_1=0$. (*)

(*) 也是 $\vec{c}\in C(U)$ 的条件, 也即 $\vec{b}\in C(A)$ 的条件. (初等行变换不改变列之间的关系)

(3) Describe the column space of A. Which plane in $\mathbb{R}^3$?

① $C(A)=S(\vec{\alpha}_1,\vec{\alpha}_2,\vec{\alpha}_3,\vec{\alpha}_4)$

$\begin{cases} \text{pivot columns: } \vec{u}_1, \vec{u}_3 \text{ 线性无关} \Rightarrow \vec{\alpha}_1, \vec{\alpha}_3 \text{ 线性无关} \\ \text{free columns: } \vec{u}_2 \text{和} \vec{u}_4 \text{都可写成} \vec{u}_1, \vec{u}_3 \text{的线性组合} \Rightarrow \vec{\alpha}_2, \vec{\alpha}_4 \text{可由} \vec{\alpha}_1, \vec{\alpha}_3 \text{线性表出} \end{cases}$

$\therefore C(A)=S(\vec{\alpha}_1,\vec{\alpha}_3)$ 两个不共线的3维向量张成一个平面.

② $C(A)$包含所有满足条件 $b_2+b_3-5b_1=0$ 的向量 $\vec{b}$.

$\therefore y+z-5x=0$ 即为$C(A)$所在的平面方程.

(4) Find the complete solution of $A\vec{x}=\vec{b}$.

$$[U\ \vec{c}] \xrightarrow{\text{Jordan}} [R\ \vec{d}] = \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2 & 4b_1-\frac{3}{2}b_2 \\ 0 & 0 & \boxed{1} & 1 & \frac{1}{2}b_2-b_1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot variables: x_1, x_3. free variables: x_2, x_4. rank of $A=2$.

$R\vec{x}=\vec{d}$: $\begin{cases} x_1 = 4b_1-\frac{3}{2}b_2-2x_2-2x_4 \\ x_3 = \frac{1}{2}b_2-b_1-x_4 \end{cases}$

取 $x_2=c_1$, $x_4=c_2$, $c_1, c_2\in F$, 则 $\begin{cases} x_1 = 4b_1-\frac{3}{2}b_2-2c_1-2c_2 \\ x_2 = c_1 \\ x_3 = \frac{1}{2}b_2-b_1-c_2 \\ x_4 = c_2 \end{cases}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 4b_1-\frac{3}{2}b_2 \\ 0 \\ \frac{1}{2}b_2-b_1 \\ 0 \end{bmatrix}}_{\vec{x}_p} + c_1 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{s}_1} + c_2 \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\vec{s}_2}, \quad c_1, c_2 \in F$$

① $\vec{x}_p$: particular solution of $A\vec{x}=\vec{b}$

取 free variables $x_2=x_4=0$，而 pivot variables 由 $\vec{d}$ 给出.

② $\vec{s}_1, \vec{s}_2$: special solutions of $A\vec{x}=\vec{0}$

取 free variables $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$，而 pivot variables 由R给出(取负号)

$N(A)=\{c_1\vec{s}_1+c_2\vec{s}_2 \mid c_1,c_2\in F\}$

The complete solution to $A\vec{x}=\vec{b}$: $\vec{x}=\vec{x}_p+c_1\vec{s}_1+c_2\vec{s}_2$, $\forall c_1,c_2\in F$.

$A\vec{x}=\vec{b}$. $A\in M_{m,n}$, $[A\ \vec{b}]\to[R\ \vec{d}]$，对A的秩r做分类讨论:

(1) $r=m=n$ A是可逆的方阵

$A\vec{x}=\vec{b}$ 有唯一解 $\vec{x}_p=A^{-1}\vec{b}$ (particular solution)

$$[R\ \ \vec{d}]=\begin{bmatrix} \boxed{1} & & & d_1 \\ & \boxed{1} & & d_2 \\ & & \ddots & \vdots \\ & & \boxed{1} & d_n \end{bmatrix}, \quad R=I, \ \vec{d}=A^{-1}\vec{b}$$

无自由变量，无special solutions. $N(A)=\{\vec{0}\}$.

The complete solution of $A\vec{x}=\vec{b}$: $\vec{x}=\vec{x}_p+\vec{0}=A^{-1}\vec{b}$.

(2) $r=m<n$. A is short and wide. full row rank 行满秩

方程# < 未知数#，$A\vec{x}=\vec{b}$ is underdetermined.

$$[R\ \ \vec{d}]=\begin{bmatrix} \boxed{1} & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & d_1 \\ & & & & \boxed{1} & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & d_2 \\ & & & & & & & & \boxed{1} & * & \cdots & * & 0 & * & \cdots & * & d_3 \\ & & & & & & & & & & & & \vdots & \vdots & & \vdots & \vdots \\ & & & & & & & & & & & & \boxed{1} & * & \cdots & * & d_m \end{bmatrix}$$

每一行都有主元

无zero rows

有n-r个自由变量

$\forall \vec{b} \in F^m$，一定有解，即$\vec{b} \in C(A) \Rightarrow C(A) = F^m$

$N(A)$有$n-m$个special solutions，$A\vec{x}=\vec{b}$有无穷多解.

A的m行线性无关 $\Rightarrow A^T$的列线性无关 $\Rightarrow N(A^T)=\{\vec{0}\}$. 书上例2.

(3) $r=n<m$. A is tall and thin. full column rank.

$A\vec{x}=\vec{b}$ is overdetermined.

$$[R \ \ \vec{d}] = \begin{bmatrix} \boxed{1} & & & & d_1 \\ & \boxed{1} & & & d_2 \\ & & \ddots & & \vdots \\ & & & \boxed{1} & d_n \\ & & & & \vdots \\ & & & & d_m \end{bmatrix}$$

每列都有主元，每列都是pivot colums.

$A\vec{x}=\vec{b}$可能无解

no free variables. no special solutions. $N(A)=\{\vec{0}\}$

若$A\vec{x}=\vec{b}$有解，则只有唯一解.

A的n个列线性无关. 书上例1

(4) $r<m$ 且 $r<n$. not full rank

$A\vec{x}=\vec{b}$无解或有无穷多解 ($n-r$个free variables)

Rank One Matrix

<u>定理</u> 设$\mathrm{rank}(A)=1$，则A可写成某个非零列向量$\vec{\beta}$和某个非零行向量$\vec{\alpha}^T$的乘积

即 $A=\vec{\beta}\vec{\alpha}^T$ (分解不唯一)

例. $A=\begin{bmatrix} 0 & 4 & 8 \\ 0 & 8 & 16 \\ 0 & 12 & 24 \end{bmatrix} \longrightarrow U=\begin{bmatrix} 0 & \boxed{4} & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\mathrm{rank}(A)=\mathrm{rank}(U)=1$

pivot row：第一行

pivot column：第二列.

(1) every row is a multiple of the pivot row

取$\vec{\alpha}^T=[0 \ \ 4 \ \ 8]$. 则 $A=\begin{bmatrix} \vec{\alpha}^T \\ 2\vec{\alpha}^T \\ 3\vec{\alpha}^T \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \vec{\alpha}^T \xlongequal{\text{记}\vec{\beta}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \vec{\beta}\vec{\alpha}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [0 \ \ 4 \ \ 8]$

(2) every column is a multiple of the pivot column.

取 $\vec{\beta}=\begin{bmatrix}4\\8\\12\end{bmatrix}$，则 $A=[0\vec{\beta}\ \ \vec{\beta}\ \ 2\vec{\beta}]=\vec{\beta}\underbrace{[0\ \ 1\ \ 2]}_{\vec{\alpha}^T}=\vec{\beta}\vec{\alpha}^T=\begin{bmatrix}4\\8\\12\end{bmatrix}[0\ \ 1\ \ 2]$

例. $\begin{bmatrix}1&3&10\\2&6&20\\3&9&30\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}[1\ \ 3\ \ 10]$, $\begin{bmatrix}0&0\\3&5\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}[3\ \ 5]$

设 rank(A)=1. 且 $A=\vec{\beta}\vec{\alpha}^T$，则

(1) $A\vec{x}=\vec{0}$ is easy to understand:

$$\vec{\beta}(\vec{\alpha}^T\vec{x})=\vec{0}\text{，而 }\vec{\beta}\neq\vec{0}\Leftrightarrow\overbrace{\vec{\alpha}^T\vec{x}}^{\text{数}}=0.$$

∴ $\vec{\alpha}$ (in the row space) $\perp$ $\vec{x}$ (in the nullspace)

(2) $A^n=(\vec{\beta}\vec{\alpha}^T)^n=\vec{\beta}\underbrace{(\vec{\alpha}^T\vec{\beta})^{n-1}}_{\text{数}}\vec{\alpha}^T$ 易求秩1矩阵的幂.