例 $A\in M_{m,n}(\mathbb{R})$, $\|A\|_F := (\text{trace}(A^TA))^{1/2}$. Frobenius 范数.

(1). $\text{trace}(A^TA)\geq 0$, $\|A\|_F=0 \Leftrightarrow A=0$.

Pf. 设 $A=[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]=(a_{ij})_{m\times n}$

A^TA: $n\times n$, $(A^TA)_{jj} = \vec{a}_j^T\vec{a}_j = \|\vec{a}_j\|^2$, $j=1,2,\cdots,n$.

$\therefore \text{trace}(A^TA) = \sum_{j=1}^n (A^TA)_{jj} = \sum_{j=1}^n \|\vec{a}_j\|^2 = \sum_{j=1}^n\sum_{i=1}^m (a_{ij})^2 \geq 0$.

$\|A\|_F=0 \Leftrightarrow \text{trace}(A^TA)=0 \Leftrightarrow a_{ij}=0,\ \forall i,j \Leftrightarrow A=0$.

(2) $\forall B\in M_{n,l}(\mathbb{R})$, 证明: $\|AB\|_F\leq \|A\|\,\|B\|_F$, $\|AB\|_F\leq\|A\|_F\,\|B\|$

其中 $\|A\| = \max_{\vec{x}\neq\vec{0}}\frac{\|A\vec{x}\|}{\|\vec{x}\|} = \max_{\|\vec{x}\|=1}\|A\vec{x}\|$ 谱范数.

Pf. ① $AB=[A\vec{b}_1\ A\vec{b}_2 \cdots A\vec{b}_l]$

$\|AB\|_F^2 = \sum_{j=1}^l \|A\vec{b}_j\|^2 \leq \sum_{j=1}^l \|A\|^2\|\vec{b}_j\|^2 = \|A\|^2\left(\sum_{j=1}^l\|\vec{b}_j\|^2\right) = \|A\|^2\|B\|_F^2$

$\therefore \|AB\|_F\leq\|A\|\,\|B\|_F$.

② $\|A\|_F = \left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2\right)^{1/2} = \|A^T\|_F$.

$\|B\| = B$的最大奇异值 $= B^T$的最大奇异值 $=\|B^T\|$.

$\|AB\|_F = \|B^TA^T\|_F \leq \|B^T\|\,\|A^T\|_F = \|B\|\,\|A\|_F$

(3) 若C是n阶半正定矩阵, 证明 $\|C\|_F\leq\text{trace}(C)$.

Pf. Method 1 C为半正定. $\therefore \exists$矩阵A, s.t. $C=A^TA$.

$\|C\|_F = \|A^TA\|_F \leq \|A\|\,\|A\|_F$

$$\left.\begin{array}{l}\text{而 } \|A\| = A\text{的最大奇异值}\\ \|A\|_F^2 = \text{trace}(A^TA) = A^TA\text{特征值之和} = A\text{的所有奇异值的平方和}\end{array}\right\}\Rightarrow \|A\|\leq\|A\|_F$$

$\therefore \|C\|_F\leq\|A\|_F^2 = \text{trace}(A^TA) = \text{trace}(C)$.

Method 2 设C的特征值为$\lambda_1,\lambda_2,\cdots,\lambda_n\geq 0$ (半正定). 则C^2特征值为$\lambda_1^2,\lambda_2^2,\cdots,\lambda_n^2$

$$\begin{cases}\|C\|_F^2 = \text{trace}(C^TC) = \text{trace}(C^2) = \lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2\\ \text{trace}(C)^2 = (\lambda_1+\lambda_2+\cdots+\lambda_n)^2\end{cases}$$

$\Rightarrow \|C\|_F\leq\text{trace}(C)$. #

例 设A为实方阵, 证明 A^TA与AA^T相似.

Pf. 取A的奇异值分解 $A=U\Sigma V^T$, Σ为对称方阵 (A为方阵)

$$\begin{cases} A^TA = V\Sigma^T U^T U \Sigma V^T = V(\Sigma^T\Sigma)V^T = V\Sigma^2 V^T, & V^T=V^{-1} \\ AA^T = U\Sigma V^T V \Sigma^T U^T = U(\Sigma\Sigma^T)U^T = U\Sigma^2U^T, & U^T = V^{-1}\end{cases}$$

都相似于Σ^2.

$\therefore \Sigma^2 = U^TAA^TU$

$\therefore A^TA = V(U^TAA^TU)V^T = (UV^T)^{-1}(AA^T)\underbrace{(UV^T)}_{可逆}$. 即 A^TA与AA^T相似. #

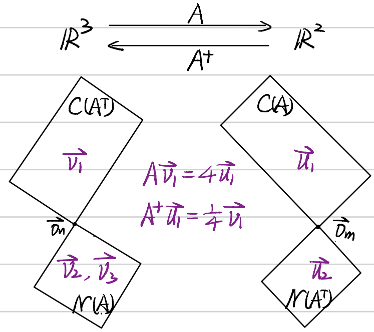
例. 设$A\in M_{2,3}$有如下奇异值分解

$$A=U\Sigma V^T = [\vec{u_1}\ \ \vec{u_2}]\begin{bmatrix}4&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\vec{v_1}^T\\ \vec{v_2}^T\\ \vec{v_3}^T\end{bmatrix}$$

(1) 求$N(A)$的一组标准正交基.

(2) 写出 $A\vec{x}=\vec{u_1}$ 的通解.

(3) 求 $A\vec{x}=\vec{u_1}$ 的长度最短的解.



解. (1) $A=U\Sigma V^T \Rightarrow AV=U\Sigma$

$\Rightarrow A\vec{v_1}=4\vec{u_1}$, $A\vec{v_2}=A\vec{v_3}=\vec{0}$

V为正交阵 $\therefore \vec{v_2},\vec{v_3}$为$N(A)$一组标准正交基.

$\mathrm{rank}(A)=1$

(2) $\because A\vec{v_1}=4\vec{u_1}$, $\therefore \frac14\vec{v_1}$为$A\vec{x}=\vec{u_1}$的一个特解 (particular solution).

通解: $\frac14\vec{v_1}+\underbrace{c_1\vec{v_2}+c_2\vec{v_3}}_{N(A)中任意向量}$ $\forall c_1,c_2\in\mathbb{R}$

(3) $A\vec{x}=\vec{u_1}$长度最短的解 $= A^+\vec{u_1} = \frac14\vec{v_1}$.

or. $\|\frac14\vec{v_1}+c_1\vec{v_2}+c_2\vec{v_3}\|^2 = (\frac14\vec{v_1}+c_1\vec{v_2}+c_2\vec{v_3})\cdot(\frac14\vec{v_1}+c_1\vec{v_2}+c_2\vec{v_3})$

$=\|\frac14\vec{v_1}\|^2+\|c_1\vec{v_2}+c_2\vec{v_3}\|^2$ $\because \vec{v_1}\perp(c_1\vec{v_2}+c_2\vec{v_3})$

$\geq \|\frac14\vec{v_1}\|^2$.

例. $A\in M_3(\mathbb{R})$ 且 $A=A^T$. 若$1,1,-2$是A的特征值, 且$[1,1,-1]^T$是属于-2的特征向量. 求A.

解. A为实对称阵 必可相似对角化.

$\therefore$ 特征值1的几何重数为2, $V_1\oplus V_2=\mathbb{R}^3$

A为实对称阵, 必有 $V_1\perp V_2$ (不同特征值的特征向量正交).

$\therefore V_1 = V_2^{\perp}$

$\swarrow$ 记为$\vec{x_3}$

设 $\vec{x}=[\alpha\ \beta\ \gamma]^T\in V_1$，则有 $\vec{x}\perp\begin{bmatrix}1\\1\\-1\end{bmatrix}$，即 $\alpha+\beta-\gamma=0$.

任取V_1一组基 $\vec{x_1}=[0,1,1]^T$，$\vec{x_2}=[1,0,1]^T$ (属于1的2个线性无关的特征向量)

则有 $A[\vec{x_1}\ \vec{x_2}\ \vec{x_3}]=[\vec{x_1}\ \vec{x_2}\ \vec{x_3}]\begin{bmatrix}1&&\\&1&\\&&-2\end{bmatrix}$

求得 $A=\begin{bmatrix}0&-1&1\\-1&0&1\\1&1&0\end{bmatrix}$

例. $A,B\in M_n(\mathbb{R})$ 且 $AB=BA$. 令 $C=\begin{bmatrix}A\\B\end{bmatrix}$. 证明: $\operatorname{rank}(A)+\operatorname{rank}(B)\geq\operatorname{rank}(AB)+\operatorname{rank}(C)$.

Pf. $\dim N(A)=n-\operatorname{rank}(A)$. $\dim N(B)=n-\operatorname{rank}(B)$.

$\dim N(AB)=n-\operatorname{rank}(AB)$. $\dim N(C)=n-\operatorname{rank}(C)$.

只需证明 $\dim N(A)+\dim N(B)\leq\dim N(AB)+\dim N(C)$. (*)

$\because N(C)=N(A)\cap N(B)$

$\therefore \dim N(C)=\dim N(A)+\dim N(B)-\dim(N(A)+N(B))$

(*)式变为 $\dim(N(A)+N(B))\leq\dim N(AB)$. (**)

$\left.\begin{array}{l}\text{而 } N(A)\subset N(BA)=N(AB)\\ N(B)\subset N(AB)\end{array}\right\}\Rightarrow N(A)+N(B)\subset N(AB)$

$\therefore$ (**)式成立. #.

例. 设$\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}$为$\mathbb{R}^n$一组标准正交基. 且 $(\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_n})=(\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n})Q$, Q为n阶方阵.

证明: $\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_n}$也为$\mathbb{R}^n$一组标准正交基 $\Leftrightarrow$ Q为正交阵.

Pf. 设 $A=[\vec{\alpha_1}\ \vec{\alpha_2}\ \cdots\ \vec{\alpha_n}]$, $Q=[\vec{q_1}\ \vec{q_2}\ \cdots\ \vec{q_n}]$. 则 $\vec{\beta_j}=A\vec{q_j}$. $j=1,2,\cdots,n$

$\vec{\beta_i}\cdot\vec{\beta_j}=(A\vec{q_i})^T(A\vec{q_j})=\vec{q_i}^TA^TA\vec{q_j}$

而$\vec{\alpha_1},\cdots,\vec{\alpha_n}$为标准正交基. $\therefore (A^TA)_{ij}=\vec{\alpha_i}\cdot\vec{\alpha_j}=\delta_{ij}$. $A^TA=I_n$.

$\therefore \vec{\beta_i}\cdot\vec{\beta_j}=\vec{q_i}\cdot\vec{q_j}$, $i,j=1,2,\cdots,n$

$\vec{\beta_1},\cdots,\vec{\beta_n}$是标准正交基 $\Leftrightarrow$ $\vec{\beta_i}\cdot\vec{\beta_j}=\delta_{ij},\ \forall i,j$ $\Leftrightarrow$ $\vec{q_i}\cdot\vec{q_j}=\delta_{ij},\ \forall i,j$ $\Leftrightarrow$ $Q^TQ=I$, 即Q为正交阵.

例 $A\in M_n(\mathbb{C})$. 证明:

(1) A一定相似于形如 $\begin{bmatrix}\lambda_1 & b_{12} & \cdots & b_{1n}\\ & b_{22} & \cdots & b_{2n}\\ & \vdots & & \vdots\\ & b_{n2} & \cdots & b_{nn}\end{bmatrix}$ 的矩阵.

(2) A一定与一个上三角阵相似

Pf. (1) 取A的一个特征值λ_1及对应的一个特征向量$\vec{x}_1$, 有

$$A\vec{x}_1=\lambda_1\vec{x}_1.$$

将$\vec{x}_1$扩充为$\mathbb{C}^n$一组基: $\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_n$

设 $A\vec{x}_j=b_{1j}\vec{x}_1+b_{2j}\vec{x}_2+\cdots+b_{nj}\vec{x}_n$, 则

$$A[\vec{x}_1\ \vec{x}_2\cdots\vec{x}_n]=[\vec{x}_1\ \vec{x}_2\cdots\vec{x}_n]\underbrace{\begin{bmatrix}\lambda_1 & b_{12} & \cdots & b_{1n}\\ & b_{22} & \cdots & b_{2n}\\ & \vdots & & \vdots\\ & b_{n2} & \cdots & b_{nn}\end{bmatrix}}_{B}, \text{ i.e. } AX=XB$$

而$\vec{x}_1,\cdots,\vec{x}_n$线性无关. $\therefore X$可逆. $\therefore X^{-1}AX=B$. 即A相似于B.

(2) 对n作数学归纳法.

① $n=1$时, 显然.

② 假设结论对$n-1$阶方阵成立.

③ $A\in M_n(\mathbb{C})$, n阶的情形.

由(1), $\exists$ 可逆矩阵X_1, s.t. $X_1^{-1}AX_1=\begin{bmatrix}\lambda_1 & b_{12} & \cdots & b_{1n}\\ & b_{22} & \cdots & b_{2n}\\ & \vdots & & \\ & b_{n2} & \cdots & b_{nn}\end{bmatrix}$

令 $C=\begin{bmatrix}b_{22} & \cdots & b_{2n}\\ \vdots & & \vdots\\ b_{n2} & \cdots & b_{nn}\end{bmatrix}\in M_{n-1}(\mathbb{C})$ 由归纳假设. $\exists$ $n-1$阶可逆矩阵X_2, s.t.

$$X_2^{-1}CX_2=\begin{bmatrix}c_1 & & *\\ & c_2 & \\ & & \ddots \\ & & & c_{n-1}\end{bmatrix}$$ (C相似于上三角阵).

令 $X=X_1\begin{bmatrix}1 & 0\\ 0 & X_2\end{bmatrix}$, 则$X$可逆, 且

$$X^{-1}AX=\begin{bmatrix}1 & \\ & X_2^{-1}\end{bmatrix}X_1^{-1}AX_1\begin{bmatrix}1 & \\ & X_2\end{bmatrix}=\begin{bmatrix}1 & \\ & X_2^{-1}\end{bmatrix}\begin{bmatrix}\lambda_1 & *\\ & C\end{bmatrix}\begin{bmatrix}1 & \\ & X_2\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_1 & **\\ & X_2^{-1}CX_2\end{bmatrix}=\begin{bmatrix}\lambda_1 & & & \tilde{*}\\ & c_1 & & \\ & & c_2 & \\ & & & \ddots \\ & & & & c_{n-1}\end{bmatrix}$$

#.