§3.5 向量组的秩

1. 极大线性无关组

定义. 向量组 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 中, 若存在 r 个向量 $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ir}$ 线性无关, 再加进任一个向量 α_j $(j=1,\cdots,s)$ 就线性相关, 则称 $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ir}$ 是向量组 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 的一个极大线性无关组.

注 (1) $\{0\}$ 无极大线性无关组.

(2) $\{\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ir}\} \sim \{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ 等价 (HW)

定理. 若 $\{\alpha_{i1}, \alpha_{i2}, \cdots \alpha_{ir}\}$ 与 $\{\alpha_{j1}, \alpha_{j2}, \cdots, \alpha_{jt}\}$ 都是 $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ 的极大线性无关组, 则 $r=t$.

Pf. 向量组 $\alpha_{j1}, \alpha_{j2}, \cdots, \alpha_{jt}$ 可由向量组 $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ir}$ 线性表出. (由注(2))
而 $\alpha_{j1}, \alpha_{j2}, \cdots, \alpha_{jt}$ 线性无关
$\Rightarrow t \le r$

同理, $r \le t$. $\therefore r=t$. #　　两个极大线性无关组有相同个数的向量.

F, 求极大线性无关组的方法　　n 维向量 $\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_s}$.

$$A=[\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_s}] \xrightarrow[\text{行变换}]{\text{初等}} (RREF).\ R=[\vec{\beta_1}, \vec{\beta_2}, \cdots, \vec{\beta_s}]$$

设 $\mathrm{rank}(A)=\mathrm{rank}(R)=r$, 主元所在列 (pivot columns): $j_1, j_2, \cdots, j_r$ 列.

(1) $\vec{\beta}_{j1}, \vec{\beta}_{j2}, \cdots, \vec{\beta}_{jr}$ 为 $\vec{\beta_1}, \cdots, \vec{\beta_s}$ 极大线性无关组.
$\vec{\alpha}_{j1}, \vec{\alpha}_{j2}, \cdots, \vec{\alpha}_{jr}$ 为 $\vec{\alpha_1}, \cdots, \vec{\alpha_s}$ ---------

(2) $\vec{\beta_j} = x_1\vec{\beta}_{j1} + x_2\vec{\beta}_{j2} + \cdots + x_r\vec{\beta}_{jr} \iff \vec{\alpha_j} = x_1\vec{\alpha}_{j1} + x_2\vec{\alpha}_{j2} + \cdots + x_r\vec{\alpha}_{jr}$, $j=1,2,\cdots,s$.

2. 向量组的秩

定义. 向量组 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 的极大线性无关组中所含向量个数 r 称为这个向量组的秩.
记作 $r(\alpha_1, \alpha_2, \cdots, \alpha_s) = r$. 规定 $r(0)=0$.

定理. 设 $\beta_1, \beta_2, \cdots, \beta_t$ 可由 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 线性表出, 则 $r(\beta_1, \beta_2, \cdots, \beta_t) \le r(\alpha_1, \alpha_2, \cdots, \alpha_s)$

Pf. 比较极大线性无关组向量个数. (思路)
$\beta_1, \cdots, \beta_t$ 的极大线性无关组 可由 $\alpha_1, \cdots, \alpha_s$ 的极大… 线性表出
`- - - - - - - - - - - -` 线性无关
$\Rightarrow r(\beta_1, \cdots, \beta_t) \le r(\alpha_1, \cdots, \alpha_s)$.

推论: 若 $\{\alpha_1, \cdots, \alpha_s\} \sim \{\beta_1, \cdots, \beta_t\}$, 则 $r(\alpha_1, \cdots, \alpha_s) = r(\beta_1, \cdots, \beta_t)$.

定义. $A\in M_{mn}(F)$, $A=\underbrace{[\vec{\alpha_1}\ \vec{\alpha_2}\cdots\vec{\alpha_n}]}_{\text{按列分块}}=\underbrace{\begin{bmatrix}\vec{\beta_1}^T\\ \vec{\beta_2}^T\\ \vdots\\ \vec{\beta_m}^T\end{bmatrix}}_{\text{按行分块}}$, $\vec{\alpha_j}$: $m\times1$ 第 j 列. $\vec{\beta_i}^T$: $1\times n$ 第 i 行

称 $r(\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n})$ 是 A 的列秩, $r(\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_m})$ 为 A 的行秩.

例. $A\longrightarrow(RREF)$. R.

A 的 pivot columns 为 $\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}$ 的极大线性无关组, $rank(A)=$ 主元的个数 $=A$ 的列秩

定理. 初等行变换不改变矩阵的行秩与列秩.

Pf. ① 列秩不变(已证)

初等行变换不改变了列之间的线性相关性.

② 行秩不变.

$A\in M_{m,n}(F)$. $A=\begin{bmatrix}\vec{\beta_1}^T\\ \vec{\beta_2}^T\\ \vdots\\ \vec{\beta_m}^T\end{bmatrix}$, 只需证作一次初等行变换, 行秩不变.

1°. $r_i\Leftrightarrow r_j$. $\{\vec{\beta_1},\cdots,\vec{\beta_j},\cdots,\vec{\beta_i},\cdots,\vec{\beta_m}\}\sim\{\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_m}\}$.

2°. λr_i, $(\lambda\neq0)$. $\{\vec{\beta_1},\cdots,\lambda\vec{\beta_i},\cdots,\vec{\beta_m}\}\sim\{\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_m}\}$.

3°. $kr_i+r_j\to r_j$: $\{\vec{\beta_1},\cdots,\vec{\beta_i},\cdots,k\vec{\beta_i}+\vec{\beta_j},\cdots,\vec{\beta_m}\}\sim\{\vec{\beta_1},\cdots,\vec{\beta_m}\}$.

无论是何种初等行变换, 新的行向量组与原行向量组等价. ∴ 行秩不变. #.

注. 初等列变换也不改变矩阵的行秩与列秩

(对 A 初等列变换 $\Leftrightarrow$ 对 A^T 行变换 ∴ 不改变 A^T 行秩与列秩

而 A^T 的行秩 (列秩) $=$ A 的列秩 (行秩))

定理. 矩阵的行秩 $=$ 主元的个数

Pf. $A\xrightarrow[\text{行变换}]{\text{消元}}$ 阶梯形 U. 设 $rank(A)=$ 主元个数 $=r$.

U 的前 r 行为 U 的行向量的极大线性无关组.

A 的行秩 $=U$ 的行秩 $=r=$ 主元个数. #.

注. 矩阵的行秩 $=$ 列秩 $=$ 主元个数, 统称为矩阵的秩.

推论. $A\in M_{m,n}(F)$. $P\in M_m(F)$. $Q\in M_n(F)$, P,Q 可逆, 则 $rank(PAQ)=rank(A)$

初等行/列变换

§3.6 基.维数与坐标

1. 基本概念

定义. 设V是线性空间，若V中有n个线性无关的向量$\alpha_1,\alpha_2,\cdots,\alpha_n$，且$V$中任何向量都可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出. 则称$\alpha_1,\alpha_2,\cdots,\alpha_n$为$V$的一组基，而称$V$是$n$维的，记作 $\dim V=n$

注. (1) $V=\mathrm{span}\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$. (3) 若有两组基. 其包含向量个数相同. 故可定义维数.

(2) V中,任意$m(m>n)$个向量都线性相关 (任取$\beta_1,\cdots,\beta_m$ 都可由$\alpha_1,\cdots,\alpha_n$ 线性表出, $m>n$)

(4) 一般向量组$\beta_1,\beta_2,\cdots,\beta_m$的极大线性无关组$\beta_{j_1},\cdots,\beta_{j_r}$为$V:=\mathrm{span}\{\beta_1,\beta_2,\cdots,\beta_m\}$一组基. 而$\dim V=r$.

(5) V中任何向量都可由基$\alpha_1,\alpha_2,\cdots,\alpha_n$唯一线性表出 ($\alpha_1,\cdots,\alpha_n$线性无关)

定义. 设V是n维线性空间, $\alpha_1,\alpha_2,\cdots,\alpha_n$是一组基. 则$\forall\beta\in V$, $\exists|$ 数组$x_1,x_2,\cdots,x_n\in F$, s.t.

$$\beta=x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n \qquad \beta=(\alpha_1,\alpha_2,\cdots,\alpha_n)\vec{x},\quad \vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

称$(x_1,x_2,\cdots,x_n)^T$是β在基$\alpha_1,\alpha_2,\cdots,\alpha_n$下的坐标

例. 向量空间$\mathbb{R}^n$, "standard basis"

记 $\vec{e_i}=[0,\cdots,0,\underset{\text{第}i\text{个}}{1},0,\cdots,0]^T$ $i=1,\cdots,n$, 则 $\vec{e_1},\vec{e_2},\cdots,\vec{e_n}$ 线性无关

$\forall\vec{\alpha}=[a_1,a_2,\cdots,a_n]^T\in\mathbb{R}^n$, 有$\vec{\alpha}=a_1\vec{e_1}+a_2\vec{e_2}+\cdots+a_n\vec{e_n}$

$\Rightarrow \vec{e_1},\vec{e_2},\cdots,\vec{e_n}$是$\mathbb{R}^n$一组基. $\dim\mathbb{R}^n=n$, $\vec{\alpha}$在其下坐标: $(a_1,a_2,\cdots,a_n)^T$

定理. $\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}\in\mathbb{R}^n$ 构成$\mathbb{R}^n$的一组基 $\Leftrightarrow$ 方阵$A=[\vec{\alpha_1}\ \vec{\alpha_2}\cdots\vec{\alpha_n}]$可逆.

Pf. "$\Rightarrow$" 由题设, $\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}$ 线性无关, 则A可逆.

"$\Leftarrow$". A可逆,则$\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}$线性无关.

而 $\forall\vec{\beta}\in\mathbb{R}^n$, $A\vec{x}=\vec{\beta}$有解, 即$\vec{\beta}$可由$\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}$ 线性表出 $\Rightarrow \vec{\alpha_1},\cdots,\vec{\alpha_n}$为$F^n$一组基.

注. $\mathbb{R}^n$有无穷多组基.

例. $P_n[x]=\{f(x)\mid f(x)=\sum_{i=0}^{n}a_ix^i,\ a_i\in F,\ i=0,1,\cdots,n\}$

$1,x,x^2,\cdots,x^n$ 线性无关

$\forall f(x)\in P_n[x]$. 可由$1,x,\cdots,x^n$ 线性表出 $\Rightarrow 1,x,x^2,\cdots,x^n$ 为基. $\dim P_n[x]=n+1$

若$f(x)=\sum_{i=0}^{n}b_ix^i$, 则其在基 $1,x,\cdots,x^n$ 下坐标: $(b_0,b_1,\cdots,b_n)^T$.

例. $M_{m,n}(F)$, 数域 F

$$\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \\ k_{m1} & \cdots & k_{mn} \end{bmatrix}$$

定义 $E_{ij} \in M_{m,n}(\mathbb{R})$: (i,j) 元为 1, 其余为 0.

则 $E_{ij}, i=1,\cdots,m, j=1,\cdots,n$ 线性无关. $\because \sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij}E_{ij} = O \Rightarrow k_{ij}=0, i=1,\cdots,m, j=1,\cdots,n$

$\forall A=(a_{ij}) \in M_{m,n}(\mathbb{R})$, 有 $A=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}$, 可由 E_{ij} 线性表出

$\Rightarrow E_{ij}, i=1,\cdots,m, j=1,\cdots,n$ 为基, $\dim M_{m,n}(F)=mn$.

A 在基 $E_{ij}, i=1,\cdots,m, j=1,\cdots,n$ 下坐标: $(a_{ij}, i=1,\cdots,m, j=1,\cdots n)$ ← 与基的顺序有关

H.W. 写出 M_n 的下列子空间的基与维数:

(1) $V=\{n$ 阶上三角矩阵$\}$. $\dim V=\frac{1}{2}n^2+\frac{1}{2}n$

(2) $V=\{n$ 阶对角矩阵$\}$. $\dim V=n$

(3) $V=\{n$ 阶对称矩阵$\}$. $\dim V=\frac{1}{2}n^2+\frac{1}{2}n$

四个基本子空间

矩阵 $A \in M_{m,n}(\mathbb{R})$ $\quad A=\begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_m^T \end{bmatrix}=[\vec{\beta}_1\ \vec{\beta}_2 \cdots \vec{\beta}_n]$

(1) 列空间 (column space): $C(A)=\text{span}\{\vec{\beta}_1,\vec{\beta}_2,\cdots,\vec{\beta}_n\}=\{A\vec{x} \mid \vec{x}\in\mathbb{R}^n\} \subseteq \mathbb{R}^m$

列向量的线性组合

(2) 化零空间 (nullspace): $N(A)=\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\} \subseteq \mathbb{R}^n$

列向量的何种线性组合可得到 $\vec{0}$?

(3) 行空间 (row space): $C(A^T)=\text{span}\{\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_m\}=\{A^T\vec{x} \mid \vec{x}\in\mathbb{R}^m\} \subseteq \mathbb{R}^n$ $\quad (\vec{x}^TA)^T$

行向量的线性组合 (仍写成列的形式)

(4) the left nullspace: $N(A^T)=\{\vec{x}\in\mathbb{R}^m \mid A^T\vec{x}=\vec{0}\}=\{\vec{x}\in\mathbb{R}^m \mid \vec{x}^TA=\vec{0}^T\} \subseteq \mathbb{R}^m$

行向量的何种线性组合可得到 $\vec{0}^T$?

$A=(a_{ij}) \in M_{m,n} \longrightarrow$ Reduced row echelon form $R=\begin{bmatrix} 0\cdots 0 & \boxed{1} & * \cdots * & 0 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & & \boxed{1} & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & & & & \boxed{1} & * \cdots * & 0 & * \cdots * \\ & & & & & & & \vdots & \vdots \\ & & & & & & & \boxed{1} & * \cdots * \end{bmatrix}_{m\times n}$ } r 行

r pivot columns

rank$(A)=r$. pivot columns: $j_1, j_2, \cdots, j_r$. free columns: $k_1, k_2, \cdots, k_{n-r}$.

special solutions of $A\vec{x}=\vec{0}$ $(R\vec{x}=\vec{0})$: $\vec{s}_1, \vec{s}_2, \cdots, \vec{s}_{n-r}$. $\quad N(A)=N(R)$

记 $A=\begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_m^T \end{bmatrix}=[\vec{\beta}_1\ \vec{\beta}_2 \cdots \vec{\beta}_n]$, $R=\begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix}=[\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n]$

1. Column space.

(1) $C(R)$

$\left.\begin{array}{l} R\text{的 pivot columns 为 } \vec{e}_1, \vec{e}_2, \cdots, \vec{e}_r \in F^n\text{，线性无关} \\ R\text{的其余列(free columns)可由 pivot columns 线性表出} \end{array}\right\} \Rightarrow \begin{array}{l} R\text{的 pivot columns 为 } C(R)\text{一组基} \\ \dim C(R) = r = \operatorname{rank}(A) \end{array}$

$\left.\begin{array}{l} \text{special solutions}：R\vec{s}_i = \vec{0},\ i=1,2,\cdots,n-r\text{，列的组合}=\vec{0} \\ \text{而 } \vec{s}_i \text{ 中第 } i \text{ 个自由变量为1，其余自由变量为0.} \end{array}\right\} \Rightarrow \begin{array}{l} \vec{s}_i\text{ 给出了第 } i \text{ 个 free column 由 pivot columns} \\ \text{线性表出的方式. } i=1,2,\cdots,n-r. \end{array}$

(2) $C(A)$

$x_1\vec{c}_1 + \cdots + x_n\vec{c}_n = \vec{0} \Leftrightarrow x_1\vec{\beta}_1 + \cdots + x_n\vec{\beta}_n = \vec{0}$

$C(A) \neq C(R)$. 但 $A\vec{x} = \vec{0}$ 与 $R\vec{x} = \vec{0}$ 同解，初等行变换不改变列向量之间的线性关系.

A的 pivot columns 为 $C(A)$一组基. $\dim C(A) = r = \operatorname{rank}(A)$.

2. Nullspace

$\left.\begin{array}{l} N(A) = N(R) = \operatorname{span}\{\vec{s}_1, \vec{s}_2, \cdots, \vec{s}_{n-r}\} \\ \text{special solutions } \vec{s}_1, \vec{s}_2, \cdots, \vec{s}_{n-r} \text{ 线性无关} \end{array}\right\} \Rightarrow \begin{array}{l} \vec{s}_1, \vec{s}_2, \cdots, \vec{s}_{n-r}\text{ 为 } N(R)\text{ 及 } N(A)\text{ 的一组基} \\ \dim N(A) = \dim N(R) = n-r = n-\operatorname{rank}(R) \end{array}$

Counting theorem：$r + (n-r) = n$.

3. Row space.

初等行变换前后的行向量组等价. $\{\vec{\alpha}_1, \vec{\alpha}_2, \cdots \vec{\alpha}_m\} \sim \{\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m\}$

$\left.\begin{array}{l} C(A^T) = C(R^T) = \operatorname{span}\{\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m\} \\ \text{前 } r \text{ 行线性无关，后 } m-r \text{ 行为零行} \end{array}\right\} \Rightarrow \begin{array}{l} \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_r\text{ 为 } C(A^T)\text{ 及 } C(R^T)\text{一组基} \\ \dim C(A^T) = \dim C(R^T) = r = \operatorname{rank}(A) = \dim C(A) \end{array}$

4. Left nullspace

(1) $N(R^T)$

$\left.\begin{array}{l} \left.\begin{array}{l} \vec{y} \in N(R^T) \Rightarrow \vec{y}^T R = \vec{0}^T \Rightarrow y_1\vec{r}_1 + y_2\vec{r}_2 + \cdots + y_m\vec{r}_m = \vec{0} \\ \text{后 } m-r \text{ 行为零行. } \vec{r}_{r+1} = \cdots = \vec{r}_m = \vec{0} \end{array}\right\} \Rightarrow y_1\vec{r}_1 + y_2\vec{r}_2 + \cdots + y_r\vec{r}_r = \vec{0} \\ \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_r \text{ 线性无关} \end{array}\right\}$

$\Rightarrow y_1 = y_2 = \cdots = y_r = 0$，而 $y_{r+1}, \cdots, y_m$ 可任取. $\vec{y} = [0, \cdots, 0, y_{r+1}, \cdots, y_m]$

$\dim N(R^T) = m-r$.

(2) $N(A^T)$

已知：$N(A) \subseteq \mathbb{R}^n$，$\dim N(A) = n - \dim C(A) = n-\operatorname{rank}(A)$

应用到A^T：$N(A^T) \subseteq \mathbb{R}^m$，$\dim N(A^T) = m - \dim C(A^T) = m-r$.

总结：① $\dim C(A) = \dim C(A^T) = \operatorname{rank}(A)$. ② $\dim C(A) + \dim N(A) = n$

③ $C(A^T), N(A) \subseteq \mathbb{R}^n$. $\dim C(A^T) = r$，$\dim N(A) = n-r$. (adding to n)

④ $C(A), N(A^T) \subseteq \mathbb{R}^m$. $\dim C(A) = r$，$\dim N(A^T) = m-r$. (adding to m)