$$\vec{\beta_3}=\vec{\alpha_3}-\frac{\vec{\beta_1}^T\vec{\alpha_3}}{\vec{\beta_1}^T\vec{\beta_1}}\vec{\beta_1}-\frac{\vec{\beta_2}^T\vec{\alpha_3}}{\vec{\beta_2}^T\vec{\beta_2}}\vec{\beta_2}\neq\vec{0},\quad \text{否则 } \vec{\alpha_1},\vec{\alpha_2},\vec{\alpha_3} \text{ 线性相关.}$$

⋮

一般地，$$\vec{\beta_k}=\vec{\alpha_k}-\frac{\vec{\beta_1}^T\vec{\alpha_k}}{\vec{\beta_1}^T\vec{\beta_1}}\vec{\beta_1}-\cdots-\frac{\vec{\beta_{k-1}}^T\vec{\alpha_k}}{\vec{\beta_{k-1}}^T\vec{\beta_{k-1}}}\vec{\beta_{k-1}}$$

即从 $\vec{\alpha_k}$ 中分别减去其在 $\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_{k-1}}$ 方向上的投影.

⋮

可得 $\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_n}$ 正交

Step 2. 单位化

$$\vec{q_i}=\vec{\beta_i}/\|\vec{\beta_i}\|,\quad i=1,2,\cdots,n.$$

4. QR分解

定理. 设 $A\in M_{m,n}(\mathbb{R})$, $\mathrm{rank}(A)=n$, 则 A 可分解为 $A=QR$, 其中 $Q\in M_{m,n}$ 满足 $Q^TQ=I_n$ (Q 的列向量组标准正交), $R\in M_n$ 为主对角线元素为正的上三角阵, $A=QR$ 称为 A 的 QR 分解.

Pf. 设 $A=[\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}]$.

$\mathrm{rank}(A)=n \Rightarrow \vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}$ 线性无关

可由 Gram-Schmidt 正交化为标准正交向量组 $\vec{q_1},\vec{q_2},\cdots,\vec{q_n}$.

Step 1. $$\vec{\beta_k}=\vec{\alpha_k}-\frac{\vec{\beta_1}^T\vec{\alpha_k}}{\vec{\beta_1}^T\vec{\beta_1}}\vec{\beta_1}-\cdots-\frac{\vec{\beta_{k-1}}^T\vec{\alpha_k}}{\vec{\beta_{k-1}}^T\vec{\beta_{k-1}}}\vec{\beta_{k-1}}$$

$$\Rightarrow \vec{\alpha_j}=\sum_{i=1}^{j-1}\underbrace{\frac{\vec{\beta_i}^T\vec{\alpha_j}}{\vec{\beta_i}^T\vec{\beta_i}}}_{\text{常数}}\underbrace{\vec{\beta_i}}_{j\text{ 从前的向量}}+\vec{\beta_j}$$

$$\Rightarrow [\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}]=[\vec{\beta_1}\ \vec{\beta_2}\cdots\vec{\beta_n}]\begin{bmatrix}1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1\end{bmatrix}$$

Step 2. $$\vec{q_i}=\frac{1}{\|\vec{\beta_i}\|}\vec{\beta_i}\Rightarrow \vec{\beta_i}=\|\vec{\beta_i}\|\vec{q_i},\quad i=1,2,\cdots,n$$

$$\Rightarrow [\vec{\beta_1}\ \vec{\beta_2}\cdots\vec{\beta_n}]=[\vec{q_1}\ \vec{q_2}\cdots\vec{q_n}]\begin{bmatrix}\|\vec{\beta_1}\| & & \\ & \|\vec{\beta_2}\| & \\ & & \ddots \\ & & & \|\vec{\beta_n}\|\end{bmatrix}$$

综合 Steps 1, 2, 有:

$$A=[\vec{q_1}\ \vec{q_2}\cdots\vec{q_n}]\begin{bmatrix}\|\vec{\beta_1}\| & & \\ & \|\vec{\beta_2}\| & \\ & & \ddots \\ & & & \|\vec{\beta_n}\|\end{bmatrix}\begin{bmatrix}1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1\end{bmatrix}=\underbrace{[\vec{q_1}\cdots\vec{q_n}]}_{Q}\underbrace{\begin{bmatrix}\|\vec{\beta_1}\| & & ** \\ & \|\vec{\beta_2}\| & \\ & & \ddots \\ & & & \|\vec{\beta_n}\|\end{bmatrix}}_{R}\ \#$$

例. $A=\begin{bmatrix}1&2\\1&3\end{bmatrix}$. 作A的QR分解.

解. 令 $\vec{\alpha}_1=\begin{bmatrix}1\\1\end{bmatrix}$. $\vec{\alpha}_2=\begin{bmatrix}2\\3\end{bmatrix}$. 施密特正交化:

$\vec{\beta}_1=\vec{\alpha}_1=\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \vec{\alpha}_1=\vec{\beta}_1$

$\vec{\beta}_2=\vec{\alpha}_2-\frac{\vec{\beta}_1^T\vec{\alpha}_2}{\vec{\beta}_1^T\vec{\beta}_1}\vec{\beta}_1=\vec{\alpha}_2-\frac{5}{2}\vec{\beta}_1=\frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} \Rightarrow \vec{\alpha}_2=\frac{5}{2}\vec{\beta}_1+\vec{\beta}_2$.

单位化: $\vec{q}_1=\frac{1}{\|\vec{\beta}_1\|}\vec{\beta}_1=\frac{1}{\sqrt{2}}\vec{\beta}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \vec{\beta}_1=\sqrt{2}\vec{q}_1$

$\vec{q}_2=\frac{1}{\|\vec{\beta}_2\|}\vec{\beta}_2=\frac{1}{\frac{\sqrt{2}}{2}}\vec{\beta}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} \Rightarrow \vec{\beta}_2=\frac{\sqrt{2}}{2}\vec{q}_2$

$$\Rightarrow[\vec{\alpha}_1\ \vec{\alpha}_2]=[\vec{\beta}_1\ \vec{\beta}_2]\begin{bmatrix}1&\frac{5}{2}\\0&1\end{bmatrix}=[\vec{q}_1\ \vec{q}_2]\underbrace{\begin{bmatrix}\sqrt{2}&\\&\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}1&\frac{5}{2}\\0&1\end{bmatrix}}=\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\sqrt{2}&\frac{5\sqrt{2}}{2}\\0&\frac{\sqrt{2}}{2}\end{bmatrix}}_{R}$$

Remarks.

$A=QR$, $Q^TQ=I$. R可逆上三角

(1) $Q^TA=\underbrace{Q^TQ}_{I}R=R \Rightarrow (R)_{ij}=\vec{q}_i\cdot\vec{\alpha}_j=0$ when $j\le i$ ($\vec{\alpha}_j\perp\vec{q}_i$). $\Rightarrow$ R为上三角

(2) normal equation: $A^TA\vec{x}=A^T\vec{b}$

$A^TA=R^T\underline{Q^TQ}R=R^TR$

$A^TA\vec{x}=A^T\vec{b} \Rightarrow R^TR\vec{x}=R^TQ^T\vec{b} \xrightarrow{R^T\text{可逆}} R\vec{x}=Q^T\vec{b}$, R为上三角,易求解.

$\Rightarrow \vec{x}=R^{-1}Q^T\vec{b}$. back substitution. fast.

Chapter 5. 行列式

n阶行列式：

$\mathbb{R}^n$上n个向量所构成的平行2n面体的有向体积.

§5.1 几何向量的数量积、向量积与混合积

1. 数量积

定义：两个几何向量$\vec{\alpha}$与$\vec{\beta}$的数量积/点积/内积（记作$\vec{\alpha}\cdot\vec{\beta}$）为一个数：

$$\vec{\alpha}\cdot\vec{\beta} := \|\vec{\alpha}\|\,\|\vec{\beta}\|\cos<\vec{\alpha},\vec{\beta}>$$

其中$<\vec{\alpha},\vec{\beta}>$为$\vec{\alpha}$与$\vec{\beta}$的夹角（不大于$\pi$）。若$\vec{\alpha}$或$\vec{\beta}$为$\vec{0}$，规定$\vec{\alpha}\cdot\vec{\beta}=0$.

Remark：$\|\vec{\alpha}\|^2 = \vec{\alpha}\cdot\vec{\alpha}$　　长度

$\cos<\vec{\alpha},\vec{\beta}> = \dfrac{\vec{\alpha}\cdot\vec{\beta}}{\|\vec{\alpha}\|\,\|\vec{\beta}\|}$　　夹角

性质：(1) $\vec{\alpha}\cdot\vec{\beta} = \vec{\beta}\cdot\vec{\alpha}$　　对称性

(2) $(k\vec{\alpha})\cdot\vec{\beta} = k(\vec{\alpha}\cdot\vec{\beta})$　　线性

(3) $(\vec{\alpha}+\vec{\beta})\cdot\vec{\gamma} = \vec{\alpha}\cdot\vec{\gamma}+\vec{\beta}\cdot\vec{\gamma}$

$\Rightarrow$ 对第2个向量也是线性的

(4) $\vec{\alpha}\cdot\vec{\alpha}\geq 0$，等号成立$\Leftrightarrow \vec{\alpha}=\vec{0}$　　正定性

定理：$\vec{\alpha}\perp\vec{\beta} \Leftrightarrow \vec{\alpha}\cdot\vec{\beta}=0$

用坐标计算数量积

定理：$\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_iy_j = [x_1\ x_2\ \cdots\ x_m]\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn}\end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n\end{bmatrix} = \vec{x}^TB\vec{y}$

定义：矩阵 $A=\begin{bmatrix} \vec{e_1}\cdot\vec{e_1} & \vec{e_1}\cdot\vec{e_2} & \vec{e_1}\cdot\vec{e_3} \\ \vec{e_2}\cdot\vec{e_1} & \vec{e_2}\cdot\vec{e_2} & \vec{e_2}\cdot\vec{e_3} \\ \vec{e_3}\cdot\vec{e_1} & \vec{e_3}\cdot\vec{e_2} & \vec{e_3}\cdot\vec{e_3}\end{bmatrix}$ 称为仿射坐标系$\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$的度量矩阵.

定理：给定仿射坐标系$\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$，A为度量矩阵，向量$\vec{\alpha}=x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3}$，$\vec{\beta}=y_1\vec{e_1}+y_2\vec{e_2}+y_3\vec{e_3}$，则内积$\vec{\alpha}\cdot\vec{\beta}=\vec{x}^TA\vec{y}$，其中$\vec{x}^T=[x_1\ x_2\ x_3]$，$\vec{y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3\end{bmatrix}$