2. 矩阵A的四个基本子空间

矩阵 $A \in M_{m,n}(\mathbb{R})$， $A = \begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_m^T \end{bmatrix} = [\vec{\beta}_1 \ \vec{\beta}_2 \cdots \vec{\beta}_n]$

(1) 列空间 (column space)

$C(A) = \operatorname{span}\{\vec{\beta}_1, \vec{\beta}_2, \cdots, \vec{\beta}_n\} = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$.

列向量的线性组合

(2) 化零空间 (nullspace)

$N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$

列的何种线性组合可得到 $\vec{0}$？

(3) 行空间 (row space)

$C(A^T) = \operatorname{span}\{\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_m\} = \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} = \{(\vec{x}^TA)^T \mid \vec{x} \in \mathbb{R}^m\} \subseteq \mathbb{R}^m$

行向量的线性组合（仍写成列的形式）

(4) the left nullspace

$N(A^T) = \{\vec{x} \in \mathbb{R}^m \mid A^T\vec{x} = \vec{0}\} = \{\vec{x} \in \mathbb{R}^m \mid \vec{x}^TA = \vec{0}^T\} \subseteq \mathbb{R}^m$

行向量的何种线性组合可得到 $\vec{0}^T$？

$$A \xrightarrow[\text{变换}]{\text{初等行}} R = \begin{bmatrix} 0 \cdots 0 & \boxed{1} & * \cdots * & 0 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & & \boxed{1} & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & & & & \boxed{1} & * \cdots * & 0 & * \cdots * \\ & & & & & & & \vdots & \\ & & & & & & & \boxed{1} & * \cdots * \\ & & & & & & & & \end{bmatrix}$$

$\operatorname{rank}(A) = r$.

r pivot columns. $n-r$ free columns.

special solutions of $A\vec{x} = \vec{0}$ ($R\vec{x} = \vec{0}$): $\vec{s}_1, \vec{s}_2, \cdots, \vec{s}_{n-r}$

$A = \begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_m^T \end{bmatrix} = [\vec{\beta}_1 \ \vec{\beta}_2 \cdots \vec{\beta}_n]$. $R = \begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} = [\vec{c}_1 \ \vec{c}_2 \cdots \vec{c}_n]$

(1) Column space.

$C(A)$ 和 $C(R)$ 可能不同

例: $A=\begin{bmatrix}1&2&3\\2&4&6\\3&7&10\end{bmatrix} \longrightarrow R=\begin{bmatrix}\boxed{1}&0&1\\&\boxed{1}&1\\&&\end{bmatrix}$, $C(A)\neq C(R)$

① $C(R)$

$\begin{cases} R\text{的pivot columns为}\ \vec{e_1},\vec{e_2},\cdots,\vec{e_r}\in F^n,\ \text{线性无关} \\ \text{任何free column都可由pivot columns线性表出} \end{cases}$

$\begin{pmatrix} R\vec{s_i}=\vec{0},\ \vec{s_i}\ \text{中第}i\text{个自由变量为}1,\ \text{其余自由变量为}0. \\ \vec{s_i}\ \text{给出了第}i\text{个free column由pivot columns线性表出的方式}.\ (i=1,2,\cdots,n-r). \end{pmatrix}$

$\Rightarrow$ R的pivot columns为$C(R)$的一组基. $\dim C(R)=r=\mathrm{rank}(A)$

② $C(A)$

初等行变换不改变列向量之间的线性相关性.

$$x_1\vec{\beta_1}+x_2\vec{\beta_2}+\cdots+x_n\vec{\beta_n}=\vec{0} \iff x_1\vec{c_1}+x_2\vec{c_2}+\cdots+x_n\vec{c_n}=\vec{0}$$

A的pivot columns为$C(A)$一组基. $\dim C(A)=r=\mathrm{rank}(A)$

(2) Nullspace

$N(A)=N(R)=\mathrm{span}\{\vec{s_1},\vec{s_2},\cdots,\vec{s_{n-r}}\}$.

$\begin{cases} \text{special solutions}\ \vec{s_1},\vec{s_2},\cdots,\vec{s_{n-r}}\ \text{线性无关}. \\ \text{任一解可写为}\ \vec{s_1},\vec{s_2},\cdots,\vec{s_{n-r}}\ \text{的线性组合} \end{cases}$

$\Rightarrow$ $\vec{s_1},\vec{s_2},\cdots,\vec{s_{n-r}}$ 为 $N(R)$ 及 $N(A)$ 的一组基. $\dim N(A)=\dim N(R)=n-r=n-\mathrm{rank}(A)$.

(3) Row space

初等行变换前后的行向量组等价 $\{\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_m}\}\sim\{\vec{r_1},\vec{r_2},\cdots,\vec{r_m}\}$

$C(A^T)=\mathrm{span}\{\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_m}\}=\mathrm{span}\{\vec{r_1},\vec{r_2},\cdots,\vec{r_m}\}=C(R^T)$

R的前r行线性无关, 后$m-r$行为零行

$\Rightarrow \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_r$ (R的前r行,写成列的形式) 为 $C(A^T)$ 及 $C(R^T)$ 一组基

$$\dim C(A^T) = \dim C(R^T) = r = \text{rank}(A)$$

注意: A的前r行(写成列的形式)未必是 $C(A^T)$ 一组基. 见(1)中例题.

(4) Left nullspace

$N(R^T)$ 与 $N(A^T)$ 未必相同.

① $N(R^T)$

$$\vec{y} \in N(R^T) \Leftrightarrow R^T\vec{y} = \vec{0} \Leftrightarrow \vec{y}^T R = \vec{0}^T \Leftrightarrow y_1\vec{r}_1 + y_2\vec{r}_2 + \cdots + y_m\vec{r}_m = \vec{0}$$

$$\xLeftrightarrow[\text{为零行}]{\text{R的后m-r行}} y_1\vec{r}_1 + y_2\vec{r}_2 + \cdots + y_r\vec{r}_r = \vec{0}, \quad y_{r+1}, \cdots, y_m \text{ 可任取}$$

$$\xLeftrightarrow[\text{无关}]{\vec{r}_1,\cdots,\vec{r}_r\text{线性}} y_1 = y_2 = \cdots = y_r = 0, \quad y_{r+1}, \cdots, y_m \text{ 可任取, 即 } \vec{y} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y_{r+1} \\ \vdots \\ y_m \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}0\\\vdots\\0\end{matrix}}\right\} r \\ \left.\vphantom{\begin{matrix}0\\\vdots\\0\end{matrix}}\right\} m-r \end{array}$$

$$\Rightarrow N(R^T) \text{一组基为: } \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (\text{前 } r \text{ 个分量为 } 0)$$

$$\dim N(R^T) = m - r = m - \text{rank}(A).$$

② $N(A^T)$

已知: $N(A) \subseteq \mathbb{R}^n$. $\dim N(A) = n - \text{rank}(A)$

应用到 A^T: $N(A^T) \subseteq \mathbb{R}^m$, $\dim N(A^T) = m - \text{rank}(A^T) = m - \text{rank}(A)$.

求 $N(A^T)$ 一组基:

1°. $\vec{x} \in N(A^T) \Leftrightarrow x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \cdots + x_r\vec{\alpha}_r = \vec{0}$ (*) 行的何种线性组合为 $\vec{0}^T$?

观察A, 找出 $m - \text{rank}(A)$ 个线性无关的向量满足(*)式.

2°. 将 $N(A^T)$ 看作 A^T 的 nullspace, 求基.

总结: ① $\dim C(A) = \dim C(A^T) = \text{rank}(A)$

② $\dim C(A) + \dim N(A) = n$

③ $C(A^T), N(A) \subseteq \mathbb{R}^n$.

$\dim C(A^T) = \text{rank}(A)$, $\dim N(A) = n - \text{rank}(A)$. (adding to n)

④ $C(A). N(A^T) \subseteq \mathbb{R}^m$

$\dim C(A) = \text{rank}(A)$. $\dim N(A^T) = m - \text{rank}(A)$. (adding to m)

例. P185 Example 3. incidence matrix 联接矩阵 graph

$$A = \begin{array}{c} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{c} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vec{\alpha}_3^T \\ \vec{\alpha}_4^T \\ \vec{\alpha}_5^T \end{array} \\ \begin{array}{cccc} \vec{\beta}_1 & \vec{\beta}_2 & \vec{\beta}_3 & \vec{\beta}_4 \end{array} \end{array} \xrightarrow{\text{rref}} R = \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$m = 5$, $n = 4$, $\text{rank}(A) = \text{rank}(R) = 3$.

pivot columns: 1, 2, 3列. free variable: x_4.

$A\vec{x} = \vec{0} \Leftrightarrow R\vec{x} = \vec{0}$: $\begin{cases} x_1 = x_4 \\ x_2 = x_4 \\ x_3 = x_4 \end{cases}$, 取 $x_4 = c$. 则 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\forall c \in \mathbb{R}$

special solution: $\vec{s}_1 = [1, 1, 1, 1]^T$

(1) Column space $C(A)$.

basis: $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$, $\dim C(A) = \text{rank}(A) = 3$.

(2) Nullspace $N(A)$.

basis: $\vec{s}_1$. $\dim N(A) = n - \text{rank}(A) = 4 - 3 = 1$

(3) Row space $C(A^T)$.

basis: R的前3行(写成列的形式). $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\dim C(A^T) = \text{rank}(A) = 3$.

观察发现: $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_4$ 线性无关, 为 $C(A^T)$ 另一组基.

(4) Left nullspace $N(A^T)$

$\dim N(A^T) = m - \mathrm{rank}(A) = 5-3 = 2.$

考虑 $A^T\vec{y}=\vec{0}$，即 $\vec{y}^TA=\vec{0}^T$，行的何种线性组合为 $\vec{0}^T$？

观察：$\vec{\alpha_3} = \vec{\alpha_2} - \vec{\alpha_1} \Rightarrow \vec{y_1} = [1,-1,1,0,0]^T \in N(A^T)$

$\vec{\alpha_3} = \vec{\alpha_4} - \vec{\alpha_5} \Rightarrow \vec{y_2} = [0,0,-1,1,-1]^T \in N(A^T)$

$\vec{y_1}$ 与 $\vec{y_2}$ 线性无关，为 $N(A^T)$ 一组基.

注：也可将 A^T 化为 RREF，从而求 $C(A^T)$ 及 $N(A^T)$.

例.

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 0 & 2 & 2 & 6 \end{bmatrix} \begin{matrix} \vec{\alpha_1}^T \\ \vec{\alpha_2}^T \\ \vec{\alpha_3}^T \end{matrix} \xrightarrow{RREF} R = \begin{bmatrix} \boxed{1} & 0 & 1 & 2 \\ & \boxed{1} & 1 & 3 \\ & & & \end{bmatrix}$$

$A \in M_{m,n}$. $m=3$, $n=4$. $\mathrm{rank}(A) = \mathrm{rank}(R) = 2$.

pivot variables: x_1, x_2, free variable: x_3, x_4

pivot columns: 1, 2 列. free columns: 3, 4 列.

$$R\vec{x}=\vec{0}: \begin{cases} x_1 = -x_3 - 2x_4 \\ x_2 = -x_3 - 3x_4 \end{cases}, \text{取 } x_3 = c_1,\ x_4 = c_2: \begin{cases} x_1 = -c_1 - 2c_2 \\ x_2 = -c_1 - 3c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad \forall c_1, c_2 \in \mathbb{R}.$$

special solutions.

(1) Column space $C(A) \subseteq \mathbb{R}^3$.

basis: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ (A 的 pivot columns)

$\dim C(A) = \mathrm{rank}(A) = 2.$

(2) Nullspace $N(A) \subseteq \mathbb{R}^4$

basis: $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$, (special solutions)

$\dim N(A) = n - \text{rank}(A) = 4-2 = 2$.

(3) Row space $C(A^T) \subseteq \mathbb{R}^4$.

basis: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ (R的非零行,写成列向量).

$\dim C(A^T) = \text{rank}(A) = 2$.

观察: A的前2行线性无关. $\therefore \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ 也为 $C(A^T)$ 一组基.

事实上, A的任意2行都线性无关, 写成列向量后也构成 $C(A^T)$ 的基.

(4) Left nullspace $N(A^T) \subseteq \mathbb{R}^3$.

$\dim N(A^T) = m - \text{rank}(A) = 3-2 = 1$

考虑 $A^T\vec{y} = \vec{0}$, 即 $\vec{y}^T A = \vec{0}^T$, 行的何种线性组合为 $\vec{0}^T$?

观察得: $-2\vec{\alpha_1} + 2\vec{\alpha_2} - \vec{\alpha_3} = \vec{0}$

$\therefore \vec{y} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \in N(A^T)$. 线性无关. 为 $N(A^T)$ 一组基.

注: 也可将 A^T 做初等行变换, 得RREF, 从而求 $C(A^T)$ 及 $N(A^T)$.

<u>定理</u>. $A\vec{x} = \vec{b}$ 有解 $\Leftrightarrow \vec{b} \in C(A) \Leftrightarrow \text{rank}(A) = \text{rank}([A \ \vec{b}])$.

Rank One Matrix

$$A=\begin{bmatrix}2&3&7&8\\2a&3a&7a&8a\\2b&3b&7b&8b\end{bmatrix}=\begin{bmatrix}1\\a\\b\end{bmatrix}\begin{bmatrix}2&3&7&8\end{bmatrix}$$

rank(A) = 1 = 行秩 = 列秩

$\begin{bmatrix}1\\a\\b\end{bmatrix}$: basis of the column space

$\begin{bmatrix}2&3&7&8\end{bmatrix}$: basis of the row space.

Rank Two Matrix

$$A=\begin{bmatrix}1&0&3\\1&1&7\\4&2&20\end{bmatrix}\xrightarrow{\text{行}} R=\begin{bmatrix}\boxed{1}&0&3\\0&\boxed{1}&4\\0&0&0\end{bmatrix}$$

$\vec{u}_1$, $\vec{u}_2$: the first two columns of A

row rank = column rank = rank(A) = 2

相当于：$EA=R$. E: 一系列初等矩阵乘积

$$\Rightarrow A=E^{-1}R=\begin{array}{c}\begin{bmatrix}1&0&0\\1&1&0\\4&2&1\end{bmatrix}\\ \vec{u}_1\quad\vec{u}_2\quad\vec{u}_3\end{array}\begin{bmatrix}\boxed{1}&0&3\\0&\boxed{1}&4\\0&0&0\end{bmatrix}\begin{array}{c}\vec{v}_1^T\\\vec{v}_2^T\\\vec{v}_3^T\end{array}$$

$$\begin{cases}\vec{u}_1,\vec{u}_2: \text{ pivot columns of } A. \text{ basis of } C(A).\\ \vec{v}_1,\vec{v}_2: \text{ nonzero rows of } R. \text{ basis of } C(A^T)\end{cases}$$

$$\Rightarrow A=[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]\begin{bmatrix}\vec{v}_1^T\\\vec{v}_2^T\\\vec{0}^T\end{bmatrix}=\vec{u}_1\vec{v}_1^T+\vec{u}_2\vec{v}_2^T=\text{Rank One}+\text{Rank One}$$

Rank r matrix

<u>Thm</u>. Every rank r matrix is a sum of r rank one matrices.

Pf. $A=[\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_n]\longrightarrow R=[\vec{r}_1\ \vec{r}_2\ \cdots\ \vec{r}_n]$. rank(A) = r.

pivot columns of A: $\vec{\alpha}_{j_1},\vec{\alpha}_{j_2},\cdots,\vec{\alpha}_{j_r}$

pivot columns of R: $\vec{r}_{j_1}=\vec{e}_1,\ \vec{r}_{j_2}=\vec{e}_2,\ \cdots,\ \vec{r}_{j_r}=\vec{e}_r$

$EA = R \Rightarrow A = E^{-1}R$. 记 $E^{-1} = [\vec{u}_1\ \vec{u}_2\ \cdots\ \vec{u}_m]$, $R = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \\ \vec{0}^T \\ \vdots \\ \vec{0}^T \end{bmatrix}$ $\}$ r个非零行

则 $\vec{\alpha}_{j_1} = E^{-1}\vec{r}_{j_1} = E^{-1}\vec{e}_1 = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_m]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{u}_1$.

$\vec{\alpha}_{j_2} = E^{-1}\vec{r}_{j_2} = E^{-1}\vec{e}_2 = \vec{u}_2, \ \cdots\cdots, \ \vec{\alpha}_{j_r} = \vec{u}_r$

$$\therefore A = E^{-1}R = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n]\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_r^T \\ \vec{0}^T \\ \vdots \\ \vec{0}^T \end{bmatrix} = \vec{u}_1\vec{v}_1^T + \vec{u}_2\vec{v}_2^T + \cdots + \vec{u}_r\vec{v}_r^T \quad (*)$$

where $\begin{cases} \vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r : \text{ basis of } C(A) \\ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r : \text{ basis of } C(A^T). \end{cases}$ $\vec{u}_i, \vec{v}_i \neq \vec{0}$, $i = 1, \cdots, r$. #

A is a sum of r rank one matrices. #

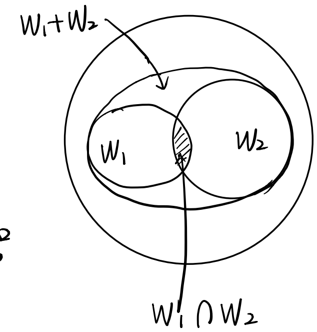
<u>定理</u> 设 $A \in M_{m,n}$ 且 $\text{rank}(A) = r$, 则 ∃ 列满秩矩阵 $U \in M_{m,r}$ 和行满秩矩阵 $V \in M_{r,n}$. s.t. $A = UV$. (A 的满秩分解).

Pf. (*)式 $\Rightarrow A = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_r]\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix} = UV$

其中 $U = [\vec{u}_1 \cdots \vec{u}_r] \in M_{m,r}$. $\text{rank}(U) = r$.

$V = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix} \in M_{r,n}$. $\text{rank}(V) = r$. #

§3.7 子空间的交与和



Def W_1和W_2是线性空间V的两个子空间.

$W_1\cap W_2 := \{\alpha \mid \alpha\in W_1 \text{ 且 } \alpha\in W_2\}$. 称为$W_1$与$W_2$的交.

$W_1+W_2 := \{\alpha=\alpha_1+\alpha_2 \mid \alpha_1\in W_1,\ \alpha_2\in W_2\}$ 称为W_1与W_2的和

Thm $W_1\cap W_2$ 与 W_1+W_2 仍是V的子空间.

Pf. $\forall \alpha,\beta\in W_1\cap W_2$, 则有

$$\left.\begin{aligned}&\alpha\in W_1,\ \beta\in W_1 \xRightarrow{W_1\text{是子空间}} \alpha+\beta\in W_1\\ &\alpha\in W_2,\ \beta\in W_2 \xRightarrow{W_2\text{是子空间}} \alpha+\beta\in W_2\end{aligned}\right\}\Rightarrow \alpha+\beta\in W_1\cap W_2 \quad \text{加法封闭}$$

$\forall \alpha\in W_1\cap W_2$. $\forall k\in F$, 有:

$$\left.\begin{aligned}&\alpha\in W_1 \Rightarrow k\alpha\in W_1\\ &\alpha\in W_2 \Rightarrow k\alpha\in W_2\end{aligned}\right\}\Rightarrow k\alpha\in W_1\cap W_2 \quad \text{数乘封闭}$$

$\therefore$ $W_1\cap W_2$为V的子空间. #

Thm. 设V是有限维线性空间, W是V的子空间, 则W的任何一组基可扩充为V的一组基.

Pf. 设 $\dim V=n$, $\dim W=m$. $\alpha_1,\alpha_2,\cdots,\alpha_m$ 为W一组基.

- 若 $m=n$, 则 $\alpha_1,\alpha_2,\cdots,\alpha_m$ 为V的一组基.
- 若 $m<n$, 则 $\exists\, \alpha_{m+1}\in V$, s.t. α_{m+1}不可由$\alpha_1,\alpha_2,\cdots,\alpha_m$线性表出

  否则, $\dim V=m<n$ 矛盾.

  $\therefore$ $\alpha_1,\alpha_2,\cdots,\alpha_{m+1}$ 线性无关.

  若 $m+1=n$, 得证。

  否则, 用相同的方法扩充 $\alpha_{m+2},\cdots,\alpha_n$, s.t. $\alpha_1,\cdots,\alpha_n$ 线性无关, 为V一组基. #

Thm (维数公式). W_1, W_2为有限维线性空间V的子空间, 则

$$\dim W_1+\dim W_2=\dim(W_1+W_2)+\dim(W_1\cap W_2)$$

Pf. 设 $\dim W_1 = r$, $\dim W_2 = s$, $\dim(W_1 \cap W_2) = t$.

要证: $\dim(W_1 + W_2) = r + s - t$.

取 $W_1 \cap W_2$ 一组基 $\alpha_1, \alpha_2, \cdots, \alpha_t$

$\begin{cases} \text{将 } \alpha_1, \cdots, \alpha_t \text{ 扩充为 } W_1 \text{ 一组基}: \alpha_1, \cdots, \alpha_t, \beta_1, \beta_2, \cdots, \beta_{r-t} \\ \text{将 } \alpha_1, \cdots, \alpha_t \text{ 扩充为 } W_2 \text{ 一组基}: \alpha_1, \cdots, \alpha_t, \gamma_1, \gamma_2, \cdots, \gamma_{s-t}. \end{cases}$

下证: $\alpha_1, \alpha_2, \cdots, \alpha_t, \beta_1, \beta_2, \cdots, \beta_{r-t}, \gamma_1, \gamma_2, \cdots, \gamma_{s-t}$ 为 $W_1 + W_2$ 一组基.

① 设 $\underbrace{\sum_{i=1}^{t} x_i \alpha_i + \sum_{i=1}^{r-t} y_i \beta_i}_{\text{记为 } \delta} + \sum_{i=1}^{s-t} z_i \gamma_i = \theta$. $(*)$

$\left.\begin{array}{l} \delta = \sum_{i=1}^{t} x_i \alpha_i + \sum_{i=1}^{r-t} y_i \beta_i \in W_1 \\ \delta = -\sum_{i=1}^{s-t} z_i \gamma_i \in W_2 \end{array}\right\} \Rightarrow \delta \in W_1 \cap W_2$

设 $\delta = \sum_{i=1}^{t} c_i \alpha_i$, 代入$(*)$, 得:

$\sum_{i=1}^{t} c_i \alpha_i + \sum_{i=1}^{s-t} z_i \gamma_i = \theta$. 而 $\alpha_1, \cdots, \alpha_t, \gamma_1, \cdots, \gamma_{s-t}$ 线性无关

$\Rightarrow c_1 = \cdots = c_t = z_1 = \cdots = z_{s-t} = 0$.

$\left.\begin{array}{l} \text{代入}(*):\ \sum_{i=1}^{t} x_i \alpha_i + \sum_{i=1}^{r-t} y_i \beta_i = \theta \\ \alpha_1, \cdots, \alpha_t, \beta_1, \cdots, \beta_{r-t} \text{ 线性无关} \end{array}\right\} \Rightarrow x_1 = \cdots = x_t = y_1 = \cdots = y_{r-t} = 0.$

$\therefore$ 由$(*)$只能推出系数全为0.

$\therefore \alpha_1, \cdots, \alpha_t, \beta_1, \cdots, \beta_{r-t}, \gamma_1, \cdots, \gamma_{s-t}$ 线性无关.

② $\forall u \in W_1 + W_2$, 则 $\exists\, u_1 \in W_1, u_2 \in W_2$ s.t. $u = u_1 + u_2$.

设 $\begin{cases} u_1 = \sum_{i=1}^{t} x_i \alpha_i + \sum_{i=1}^{r-t} y_i \beta_i \\ u_2 = \sum_{i=1}^{t} \tilde{x}_i \alpha_i + \sum_{i=1}^{s-t} z_i \gamma_i \end{cases}$

$\Rightarrow u = \sum_{i=1}^{t} (x_i + \tilde{x}_i) \alpha_i + \sum_{i=1}^{r-t} y_i \beta_i + \sum_{i=1}^{s-t} z_i \gamma_i$

可由 $\alpha_1, \cdots, \alpha_t, \beta_1, \cdots, \beta_{r-t}, z_1, \cdots, z_{s-t}$ 线性表出.

由①, ②. $\alpha_1, \cdots, \alpha_t, \beta_1, \cdots, \beta_{r-t}, z_1, \cdots, z_{s-t}$ 为 $W_1 + W_2$ 一组基

$\therefore \dim(W_1 + W_2) = r + s - t = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$. #