§2.4 矩阵运算

1. 加法与数量乘法

定义 设$A=(a_{ij})_{m\times n}$，$B=(b_{ij})_{m\times n}$，则矩阵$C=(C_{ij})_{m\times n}=(a_{ij}+b_{ij})_{m\times n}$称为矩阵A与B的和，记为$C=A+B$. matrix addition

定义 设$A=(a_{ij})_{m\times n}$，k是一个常数，则矩阵$(ka_{ij})_{m\times n}$称为数k与矩阵A的数乘，记作kA.

注：$-A:=(-1)A=(-a_{ij})_{m\times n}$ 称为A的负矩阵

$A-B:=A+(-B)$ 矩阵减法

性质 设A, B, C是三个同型矩阵，k, l是常数，则

(1) $(A+B)+C=A+(B+C)$ 结合律

(2) $A+B=B+A$ 交换律

(3) $A+0=0+A$ 其中0是与A同型的零矩阵

(4) $A+(-A)=0$

(5) $1A=A$ (6) $k(lA)=(kl)A$

(7) $k(A+B)=kA+kB$ (8) $(k+l)A=kA+lA$.

注：$M_{m,n}(F)$ 线性空间.

2. 矩阵乘法 Matrix multiplication

$A=(a_{ij})_{m\times n}$，$B=(b_{ij})_{n\times t}=[\vec{b_1}\ \vec{b_2}\ \cdots\ \vec{b_t}]$

$AB=A[\vec{b_1}\ \vec{b_2}\ \cdots\ \vec{b_t}]=[A\vec{b_1}\ A\vec{b_2}\ \cdots\ A\vec{b_t}]$ $m\times t$

AB第j列：$A\vec{b_j}$ (列向量)

AB的(i,j)元$=A\vec{b_j}$的第i个分量$=$A的第i行乘$\vec{b_j}$

$=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}=\sum_{k=1}^{n}a_{ik}b_{kj}$

$$
\begin{array}{c}
\\ \\ 行 \\ \\ \\
\end{array}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1t} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2t} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nt} \end{bmatrix}
=
\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1t} \\ c_{21} & c_{22} & \cdots & c_{2t} \\ \vdots & & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mt} \end{bmatrix}
$$

列

$m\times n$ $\quad$ $n\times t$ $\quad$ $m\times t$

例 (1) $A=[a_1\ a_2\ a_3]$, $B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. $\quad A: 1\times 3$, $B: 3\times 1$ 可乘

$AB=a_1b_1+a_2b_2+a_3b_3$ $\quad$, $BA=\begin{bmatrix} b_1a_1 & b_1a_2 & b_1a_3 \\ b_2a_1 & b_2a_2 & b_2a_3 \\ b_3a_1 & b_3a_2 & b_3a_3 \end{bmatrix}$ $\quad 3\times 3$

1×1

$AB\neq BA$，不同型

(2) $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$A: 2\times 2$, $B: 2\times 2$. $\Rightarrow AB: 2\times 2$, $BA: 2\times 2$.

$AB=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $BA=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\quad AB\neq BA$（即使同型）

注：① 矩阵乘法不满足交换律！一般地，$AB\neq BA$

② $AB=B \not\Rightarrow A=I$，不满足消去律

③ $A\neq O$, $B\neq O$，但 $BA=O$. 两个非零矩阵的乘积可能为零矩阵.

④ $AB=O \not\Rightarrow A=O$ 或 $B=O$.

定理 设 O, I, A, B, C 等矩阵在下面的乘法和加法运算中都能进行，则

(1) $OA=O$, $AO=O$. $\qquad$ (2) $IA=A$, $AI=A$.

(3) $A(BC)=(AB)C$ 结合律 $\qquad$ (4) $A(B+C)=AB+AC$ 左分配律

$(B+C)A=BA+CA$ 右分配律

Pf：(3) 为使乘法有意义，设 $A=(a_{ij})_{m\times s}$, $B=(b_{ij})_{s\times t}$, $C=(c_{ij})_{t\times n}$

$BC: s\times n$, $A(BC): m\times n$ } 同型矩阵，需验证对应元素相等.

$AB: m\times t$, $(AB)C: m\times n$

$$(A(BC))_{ij} = \sum_{k=1}^{s} \underset{A\text{的}i\text{行}}{a_{ik}} \underset{BC\text{的}j\text{列}}{(BC)_{kj}} = \sum_{k=1}^{s} a_{ik}\Big(\sum_{l=1}^{t} \underset{B\text{的}k\text{行}}{b_{kl}}\ \underset{C\text{的}j\text{列}}{c_{lj}}\Big) = \sum_{k=1}^{s}\sum_{l=1}^{t} a_{ik}b_{kl}c_{lj}$$

$$((AB)C)_{ij} = \sum_{l=1}^{t} \underset{AB\text{的}i\text{行}}{(AB)_{il}}\ \underset{C\text{的}j\text{列}}{c_{lj}} = \sum_{l=1}^{t}\Big(\sum_{k=1}^{s} \underset{A\text{的}i\text{行}}{a_{ik}}\ \underset{B\text{的}l\text{列}}{b_{kl}}\Big)c_{lj} = \sum_{k=1}^{s}\sum_{l=1}^{t} a_{ik}b_{kl}c_{lj}.$$

$\therefore (A(BC))_{ij} = ((AB)C)_{ij}$, $i=1,\cdots,m$, $j=1,\cdots,n$

$\therefore A(BC) = (AB)C$. #

<u>定义</u>. 设A是方阵，则A的方幂定义为 $A^k = \overbrace{AA\cdots A}^{k\text{个}}$, $k=1,2,3,\cdots$

<u>性质</u>: $(A^p)(A^q) = A^{p+q}$, $(A^p)^q = A^{pq}$

注: $(AB)^k$ 与 A^kB^k 一般不相等.

3. 分块矩阵 (Block Matrix)

<u>定义</u> 将矩阵A用纵线和横线分成若干小块，每个小块称为A的子块(blocks)，分成子块的矩阵称为分块矩阵.

$$A = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

A_{ij}: 第i行，j列子块

$$= \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = [\vec{\beta}_1\ \vec{\beta}_2\ \vec{\beta}_3\ \vec{\beta}_4], \quad \text{where } \vec{\beta}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ a_{4j} \end{bmatrix}, \ j=1,2,3,4$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \hdashline a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \\ \hdashline a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vec{\alpha}_3^T \\ \vec{\alpha}_4^T \end{bmatrix}, \text{ where } \vec{\alpha}_i^T = [a_{i1}\ a_{i2}\ a_{i3}\ a_{i4}],\ i=1,2,3,4$$

(1) 加法

设A,B是同型矩阵,且采用相同的划分方法分块

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{bmatrix} \quad (*), \qquad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ B_{21} & B_{22} & \cdots & B_{2s} \\ \vdots & \vdots & & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} \end{bmatrix}$$

其中 A_{ij}, B_{ij} 是同型矩阵 $(i=1,\cdots,r,\ j=1,\cdots,s)$,则

$$A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1s}+B_{1s} \\ A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2s}+B_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1}+B_{r1} & A_{r2}+B_{r2} & \cdots & A_{rs}+B_{rs} \end{bmatrix}$$

(2) 数乘

A的划分如(*), $k \in F$, 则 $kA = \begin{bmatrix} kA_{11} & kA_{12} & \cdots & kA_{1s} \\ kA_{21} & kA_{22} & \cdots & kA_{2s} \\ \vdots & \vdots & & \vdots \\ kA_{r1} & kA_{r2} & \cdots & kA_{rs} \end{bmatrix}$

(3) 矩阵乘法

$$A = \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_r \\ \\ \end{array} \!\!\begin{array}{c} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{bmatrix} \\ \begin{matrix} l_1 & l_2 & \cdots & l_s \end{matrix} \end{array}, \qquad B = \begin{array}{c} l_1 \\ l_2 \\ \vdots \\ l_s \\ \\ \end{array} \!\!\begin{array}{c} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{st} \end{bmatrix} \\ \begin{matrix} n_1 & n_2 & \cdots & n_t \end{matrix} \end{array}$$

A的列划分与B的行划分一致!

A_{ik} 是 $m_i \times l_k$ 矩阵 $(i=1,\cdots,r,\ k=1,\cdots,s)$, B_{kj} 是 $l_k \times n_j$ 矩阵 $(k=1,\cdots,s,\ j=1,\cdots,t)$,

则有:

$$AB=\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rt} \end{bmatrix}$$

A在前

其中 $C_{ij}=A_{i1}B_{1j}+A_{i2}B_{2j}+\cdots+A_{is}B_{sj}=\sum_{k=1}^{s}A_{ik}B_{kj}$, $i=1,\cdots,r$, $j=1,\cdots,t$

例 $A=\underbrace{\left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline -1 & 2 & 1 \\ 1 & 1 & 0 \end{array}\right]}_{2\quad\ 1}=\begin{bmatrix} I & O \\ A_{21} & A_{22} \end{bmatrix}$, $B=\begin{array}{c} 2\{ \\ \\ 1\{ \end{array}\left[\begin{array}{cc:cc} 1 & 0 & 3 & 2 \\ -1 & 2 & 0 & 1 \\ \hdashline 1 & 0 & 4 & 1 \end{array}\right]=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

$$AB=\begin{bmatrix} I & O \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}=\begin{bmatrix} B_{11} & B_{12} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}=\left[\begin{array}{cc:cc} 1 & 0 & 3 & 2 \\ -1 & 2 & 0 & 1 \\ \hdashline -2 & 4 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{array}\right]$$

定理 用初等矩阵右乘给定的矩阵，其结果是对给定的矩阵做相应的初等列变换

Pf. 设 $A\in M_{m,n}$ ，$\vec\beta_j$ 为A的第j列，$j=1,\cdots,n$

①

$$AE_i(\lambda)=[\vec\beta_1\ \vec\beta_2\cdots\vec\beta_n]\begin{array}{c}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}\begin{array}{l} \\ \\ \\ \leftarrow i行 \\ \\ \\ \\ \end{array} \\ \uparrow \\ i列 \end{array}=[\vec\beta_1\cdots\vec\beta_{i-1}\ \ \lambda\vec\beta_i\ \ \vec\beta_{i+1}\cdots\vec\beta_n]$$

分块矩阵乘法 用λ乘第i列

②

$$AE_{ij}(\mu)=[\vec\beta_1\ \vec\beta_2\cdots\vec\beta_n]\begin{array}{c}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & \mu & \cdots & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow i行 \\ \\ \leftarrow j行 \\ \\ \\ \end{array} \\ \uparrow\qquad\uparrow \\ i列\quad\ j列 \end{array}=[\vec\beta_1,\cdots,\vec\beta_i+\mu\vec\beta_j,\cdots,\vec\beta_n]$$

第j列的μ倍加到第i列

③ AE_{ij}：交换A的i列和j列.

矩阵相乘的不同解读

1、$A\in M_{m,n}$, $A=\begin{bmatrix}\vec{\alpha}_1^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}=[\vec{\beta}_1\ \vec{\beta}_2\cdots\vec{\beta}_n]$

$\vec{\alpha}_i^T$: 第i行

$\vec{\beta}_j$: 第j列

(1) $\vec{x}\in\mathbb{R}^n$

$$A\vec{x}=[\vec{\beta}_1\ \vec{\beta}_2\cdots\vec{\beta}_n]\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}=x_1\vec{\beta}_1+x_2\vec{\beta}_2+\cdots+x_n\vec{\beta}_n$$

$A\vec{x}$: A的列的线性组合

a combination of the columns of A.

$$=\begin{bmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}\vec{x}=\begin{bmatrix}\vec{\alpha}_1^T\vec{x}\\ \vec{\alpha}_2^T\vec{x}\\ \vdots\\ \vec{\alpha}_m^T\vec{x}\end{bmatrix}$$

A的每行与$\vec{x}$点积

(2) $\vec{x}\in\mathbb{R}^m$

$$\vec{x}^TA=[x_1\ x_2\cdots x_m]\begin{bmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}=x_1\vec{\alpha}_1^T+x_2\vec{\alpha}_2^T+\cdots+x_m\vec{\alpha}_m^T$$

$\vec{x}^TA$: A的行的线性组合

a combination of the rows of A.

$$=\vec{x}^T[\vec{\beta}_1\ \vec{\beta}_2\cdots\vec{\beta}_n]=[\vec{x}^T\vec{\beta}_1\ \ \vec{x}^T\vec{\beta}_2\ \cdots\ \vec{x}^T\vec{\beta}_n]$$

$\vec{x}$与A的每列点积

2、$A\in M_{m,n}$, $B\in M_{n,t}$

$$A=\begin{bmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}=[\vec{\beta}_1\ \vec{\beta}_2\cdots\vec{\beta}_n].\qquad B=\begin{bmatrix}\vec{r}_1^T\\ \vec{r}_2^T\\ \vdots\\ \vec{r}_n^T\end{bmatrix}=[\vec{C}_1\ \vec{C}_2\cdots\vec{C}_t]$$

(1) $AB=A[\vec{C}_1\ \vec{C}_2\cdots\vec{C}_t]=[A\vec{C}_1\ \ A\vec{C}_2\ \cdots\ A\vec{C}_t]$

each column of AB is a combination of the columns of A.

Matrix A times every column of B.

(2) $$AB=\begin{bmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}B=\begin{bmatrix}\vec{\alpha}_1^TB\\ \vec{\alpha}_2^TB\\ \vdots\\ \vec{\alpha}_m^TB\end{bmatrix}$$

every row of AB is a combination of the rows of B.

Every row of A times matrix B

(3) $$AB=[\vec{\beta}_1\ \vec{\beta}_2\cdots\vec{\beta}_n]\begin{bmatrix}\vec{r}_1^T\\ \vec{r}_2^T\\ \vdots\\ \vec{r}_n^T\end{bmatrix}=\vec{\beta}_1\vec{r}_1^T+\vec{\beta}_2\vec{r}_2^T+\cdots+\vec{\beta}_n\vec{r}_n^T$$

matrix

例. 若 $AB=O$，则 B 的每列都是 $A\vec{x}=\vec{0}$ 的解.

例. 已知 $AB=I$，$BC=I$，证明 $A=C$

Pf: $A=AI=A(BC)=(AB)C=IC=C$. #

例. 已知 $C=\begin{bmatrix}1\\-1\\2\end{bmatrix}$，$D=[2\ \ 0\ \ 1]$

(1) 计算 $A=CD$

(2) 计算 A^{100}

解: (1) $A=\begin{bmatrix}1\\-1\\2\end{bmatrix}[2\ \ 0\ \ 1]=\begin{bmatrix}2&0&1\\-2&0&-1\\4&0&2\end{bmatrix}$

(2) $A^{100}=(CD)^{100}=C(DC)^{99}D$ 结合律

而 $DC=4$.

$\therefore A^{100}=C\cdot 4^{99}D=4^{99}A=\cdots$