子空间的直和

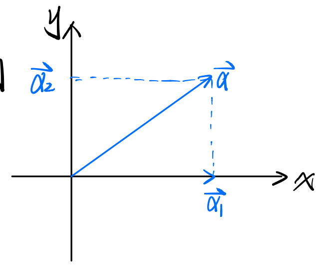
Def 设 W_1 和 W_2 是线性空间 V 的两个子空间. 若 W_1+W_2 中任何向量 α 的分解式

$$\alpha=\alpha_1+\alpha_2,\quad \alpha_1\in W_1,\ \alpha_2\in W_2$$

是唯一的, 则称 W_1+W_2 是 W_1 与 W_2 的直和, 记为 $W_1\oplus W_2$.

例. $\mathbb{R}^2=\left\{\begin{bmatrix}x\\y\end{bmatrix}\mid x,y\in\mathbb{R}\right\}$.

$W_1=\left\{\begin{bmatrix}x\\0\end{bmatrix}\mid x\in\mathbb{R}\right\}$ 及 $W_2=\left\{\begin{bmatrix}0\\y\end{bmatrix}\mid y\in\mathbb{R}\right\}$ 为 $\mathbb{R}^2$ 子空间



$W_1+W_2=\mathbb{R}^2$

$\forall\vec{\alpha}\in\mathbb{R}^2$. 分解式 $\vec{\alpha}=\vec{\alpha_1}+\vec{\alpha_2}$, $\vec{\alpha_1}\in W_1$, $\vec{\alpha_2}\in W_2$ 唯一

$\therefore\ \mathbb{R}^2=W_1\oplus W_2$.

Thm W_1+W_2 是直和

$\Longleftrightarrow$ 零向量的分解式唯一. 若 $\theta=\alpha_1+\alpha_2$, $\alpha_1\in W_1$, $\alpha_2\in W_2$, 则必有 $\alpha_1=\alpha_2=\theta$.

$\Longleftrightarrow$ $W_1\cap W_2=\{\theta\}$.

$\Longleftrightarrow$ $\dim W_1+\dim W_2=\dim(W_1+W_2)$

Thm 设 W_1 是线性空间 V 的子空间, 则 $\exists$ V 的子空间 W_2. s.t. $V=W_1\oplus W_2$.

例. 设 $W_1=\mathrm{span}\{\vec{\alpha_1},\vec{\alpha_2}\}$. $W_2=\mathrm{span}\{\vec{\beta_1},\vec{\beta_2}\}$. 其中

$\vec{\alpha_1}=[1,2,1,0]^T$, $\quad\vec{\alpha_2}=[-1,1,1,1]^T$

$\vec{\beta_1}=[2,-1,0,1]^T$, $\quad\vec{\beta_2}=[1,-1,3,7]^T$

求 W_1+W_2, $W_1\cap W_2$ 的基与维数.

解. (1) $W_1+W_2=\mathrm{span}\{\vec{\alpha_1},\vec{\alpha_2}\}+\mathrm{span}\{\vec{\beta_1},\vec{\beta_2}\}=\mathrm{span}\{\vec{\alpha_1},\vec{\alpha_2},\vec{\beta_1},\vec{\beta_2}\}$

$\vec{\alpha_1},\vec{\alpha_2},\vec{\beta_1},\vec{\beta_2}$ 的极大线性无关组即为 W_1+W_2 的基

$$A=\begin{array}{c}\begin{bmatrix}1 & -1 & 2 & 1\\ 2 & 1 & -1 & -1\\ 1 & 1 & 0 & 3\\ 0 & 1 & 1 & 7\end{bmatrix}\\ \begin{matrix}\vec{\alpha_1} & \vec{\alpha_2} & \vec{\beta_1} & \vec{\beta_2}\end{matrix}\end{array} \xrightarrow{Gauss} U=\begin{bmatrix}\boxed{1} & -1 & 2 & 1\\ & \boxed{1} & -1 & 1\\ & & \boxed{1} & 3\\ & & & \end{bmatrix}$$

pivot columns：1,2,3 列． 对应A的1,2,3列：$\vec{\alpha_1}, \vec{\alpha_2}, \vec{\beta_1}$

$\vec{\alpha_1}, \vec{\alpha_2}, \vec{\beta_1}$ 为 $\vec{\alpha_1}, \vec{\alpha_2}, \vec{\beta_1}, \vec{\beta_2}$ 的极大线性无关组，也为 W_1+W_2 的基．

$\dim(W_1+W_2)=3$．

(2) 设 $\vec{\alpha}\in W_1\cap W_2$，则 $\vec{\alpha}\in W_1$ 且 $\vec{\alpha}\in W_2$．

设 $\vec{\alpha}=x_1\vec{\alpha_1}+x_2\vec{\alpha_2}=-y_1\vec{\beta_1}-y_2\vec{\beta_2}$ 则

$$x_1\vec{\alpha_1}+x_2\vec{\alpha_2}+y_1\vec{\beta_1}+y_2\vec{\beta_2}=\vec{0} \quad (*) \quad \text{线性方程组}$$

求解 $(*)$，得 x_1,x_2,y_1,y_2 关系．

$A=[\vec{\alpha_1}\ \vec{\alpha_2}\ \vec{\beta_1}\ \vec{\beta_2}]\longrightarrow U$ （同(1)）

取 y_2 为自由变量．解得： $x_1=y_2$，$x_2=-4y_2$，$y_1=-3y_2$．

$\therefore\ \vec{\alpha}=x_1\vec{\alpha_1}+x_2\vec{\alpha_2}=y_2\vec{\alpha_1}-4y_2\vec{\alpha_2}=y_2(\vec{\alpha_1}-4\vec{\alpha_2})=y_2[5,-2,-3,-4]^T$

或：$\vec{\alpha}=-y_1\vec{\beta_1}-y_2\vec{\beta_2}=3y_2\vec{\beta_1}-y_2\vec{\beta_2}=y_2(3\vec{\beta_1}-\vec{\beta_2})=y_2[5,-2,-3,-4]^T$

$\therefore\ [5,-2,-3,-4]^T$ 为 $W_1\cap W_2$ 一组基．

$\dim(W_1\cap W_2)=1$．

注：可验证 $\dim W_1+\dim W_2=\dim(W_1+W_2)+\dim(W_1\cap W_2)$．

例 证明: $\text{rank}(AB) \le \text{rank}(B)$, $\text{rank}(AB) \le \text{rank}(A)$

Pf. 设 $A=(a_{ij})_{m\times n}$, $B=\begin{bmatrix} \vec{b_1}^T \\ \vec{b_2}^T \\ \vdots \\ \vec{b_n}^T \end{bmatrix}$, 则

$$AB=\begin{bmatrix} a_{11}\vec{b_1}^T + a_{12}\vec{b_2}^T + \cdots + a_{1n}\vec{b_n}^T \\ a_{21}\vec{b_1}^T + a_{22}\vec{b_2}^T + \cdots + a_{2n}\vec{b_n}^T \\ \vdots \\ a_{m1}\vec{b_1}^T + a_{m2}\vec{b_2}^T + \cdots + a_{mn}\vec{b_n}^T \end{bmatrix} \overset{\text{记为}}{=\!=} \begin{bmatrix} \vec{\alpha_1}^T \\ \vec{\alpha_2}^T \\ \vdots \\ \vec{\alpha_m}^T \end{bmatrix}$$

every row of AB is a combination of the rows of B

$\therefore$ AB的行向量组 $\vec{\alpha_1}^T, \vec{\alpha_2}^T, \cdots, \vec{\alpha_m}^T$ 可由B的行向量组 $\vec{b_1}^T, \vec{b_2}^T, \cdots \vec{b_n}^T$ 线性表出

$\therefore \text{rank}(AB) = \underbrace{r(\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_m})}_{\text{行秩}} \le r(\vec{b_1}, \vec{b_2}, \cdots, \vec{b_n}) = \text{rank}(B)$.

each column of AB is a combination of the columns of A.

$\Rightarrow \text{rank}(AB) \le \text{rank}(A)$. #

例 $A\in M_{n,m}$, $B\in M_{m,n}$, $n<m$, $AB=I_n$

求证: B的列向量组线性无关

Pf. $\text{rank}(AB)=\text{rank}(I_n)=n$

$\left.\begin{array}{l} \Rightarrow \text{rank}(B) \ge \text{rank}(AB) = n \\ \text{而 } \text{rank}(B) \le n \quad (\text{B只有n列}) \end{array}\right\} \Rightarrow \text{rank}(B)=n$, $\therefore$ ······. #

例 $A\in M_{m,n}$, $B\in M_{n,s}$. 若 $AB=0$. 证明: $\text{rank}(A)+\text{rank}(B)\le n$

Pf. 设 $B=[\vec{\beta_1}\ \vec{\beta_2} \cdots \vec{\beta_s}]$

$AB=[A\vec{\beta_1}\ A\vec{\beta_2} \cdots A\vec{\beta_s}]=0 \Rightarrow A\vec{\beta_j}=\vec{0} \Rightarrow \vec{\beta_j}\in N(A)$. $j=1,2,\cdots,s$

$\therefore \text{rank}(B)=r(\vec{\beta_1}\ \vec{\beta_2} \cdots, \vec{\beta_s}) \le \dim N(A)$

而 $\dim N(A) = n - \text{rank}(A)$

$\therefore \text{rank}(A)+\text{rank}(B)\le n$. #

例 $A \in M_{m,n}(\mathbb{R})$. $\forall \vec{b} \in \mathbb{R}^m$，有 $A\vec{x}=\vec{b}$ 总有解.

证明：A 的行向量组线性无关.

Pf. $\forall \vec{b} \in \mathbb{R}^m$, $A\vec{x}=\vec{b}$ 有解，则 $\vec{b} \in C(A)$.

$\therefore \mathbb{R}^m \subseteq C(A)$

而 $C(A)$ 是 $\mathbb{R}^m$ 子空间. $C(A) \subseteq \mathbb{R}^m$

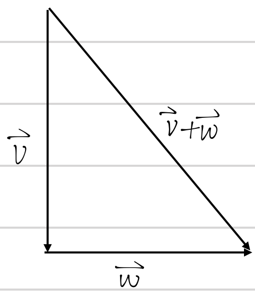
$\therefore C(A) = \mathbb{R}^m$. $\therefore \dim C(A) = \dim \mathbb{R}^m = m$

$\therefore \operatorname{rank}(A) = \dim C(A) = m$

$\therefore A$ 的行秩 $= m$. $\therefore A$ 的行线无关. #

第四章 正交性

§4.1 四个子空间的正交性



回顾：当 $\vec{v}\cdot\vec{w}=\vec{v}^T\vec{w}=0$ 时，称 $\vec{v}$ 和 $\vec{w}$ 正交。($\vec{v}\perp\vec{w}$)

$\|\vec{v}\|^2=\vec{v}\cdot\vec{v}$

$\|\vec{v}+\vec{w}\|^2=(\vec{v}+\vec{w})^T(\vec{v}+\vec{w})=\|\vec{v}\|^2+\overbrace{\vec{v}^T\vec{w}+\vec{w}^T\vec{v}}^{0}+\|\vec{w}\|^2$

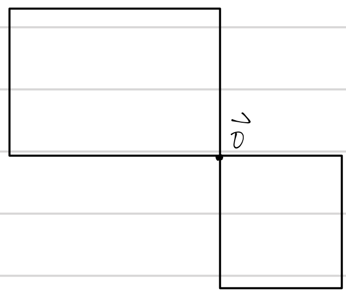
定义. 设 W_1, W_2 是 $\mathbb{R}^n$ 的两个子空间，若 $\forall\vec{\alpha}\in W_1$，$\forall\vec{\beta}\in W_2$，有 $\vec{\alpha}\perp\vec{\beta}$，则称 W_1 与 W_2 正交，记作 $W_1\perp W_2$.

例. (1) The floor of the room $\perp$ the line where two walls meet. $2+1=3$

(2) Two walls are not orthogonal! (相交于一条线, or. 维数相加 $2+2>3$)

$\mathbb{R}^3$ 中正交空间：线 $\perp$ 线 $(1+1=2<3)$；线 $\perp$ 面 $(1+2=3)$；$\{\vec{0}\}\perp W$

定理. 设 $W_1\perp W_2$，则 $W_1\cap W_2=\{\vec{0}\}$.



不会重叠

Pf. W_1, W_2 是子空间. $\therefore \vec{0}\in W_1$ 且 $\vec{0}\in W_2$.

$\forall\vec{\alpha}\in W_1\cap W_2$，即 $\vec{\alpha}\in W_1$ 且 $\vec{\alpha}\in W_2$

$\because W_1\perp W_2$. $\therefore\ \vec{\alpha}\cdot\vec{\alpha}=0 \Rightarrow \vec{\alpha}=\vec{0}$. #

注: $\dim W_1+\dim W_2=\dim(W_1+W_2)+\dim(W_1\cap W_2)$.

若 $W_1\perp W_2$，则 $\dim W_1+\dim W_2=\dim(W_1+W_2)\leq\dim$ (whole space)

- $A\in M_{m\times n}(\mathbb{R})$，$A=\begin{bmatrix}\vec{\alpha}_1^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}=[\vec{\beta}_1,\vec{\beta}_2,\cdots,\vec{\beta}_n]$

① nullspace: $N(A)=\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\}$

② row space: $C(A^T)=\{A^T\vec{y}\mid\vec{y}\in\mathbb{R}^m\}=\mathrm{span}\{\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_m\}$ } $\mathbb{R}^n$ 子空间

③ left nullspace: $N(A^T)=\{\vec{y}\in\mathbb{R}^m\mid A^T\vec{y}=\vec{0}\}$

④ column space: $C(A)=\{A\vec{x}\mid\vec{x}\in\mathbb{R}^n\}$ } $\mathbb{R}^m$ 子空间

a row of A can not be in N(A)

定理. $N(A)\perp C(A^T)$，$N(A^T)\perp C(A)$

Pf. (1) $\forall\vec{x}\in N(A)$. 有:

$$\vec{0}=A\vec{x}=\begin{bmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\\ \vdots\\ \vec{\alpha}_m^T\end{bmatrix}\vec{x}=\begin{bmatrix}\vec{\alpha}_1^T\vec{x}\\ \vec{\alpha}_2^T\vec{x}\\ \vdots\\ \vec{\alpha}_m^T\vec{x}\end{bmatrix}\Rightarrow \vec{\alpha}_i\cdot\vec{x}=\vec{\alpha}_i^T\vec{x}=0,\ \text{即 } \vec{\alpha}_i\perp\vec{x}$$

每一行

$i=1,2,\cdots,m$

$\forall\vec{z}=\sum_{i=1}^m y_i\vec{\alpha}_i\in C(A^T)$，有 $\vec{x}\cdot\vec{z}=\sum_{i=1}^m y_i(\vec{x}\cdot\vec{\alpha}_i)=0$.

$\therefore N(A)\perp C(A^T)$.

注. 矩阵形式: $\forall\vec{x}\in N(A)$，$\forall A^T\vec{y}\in C(A^T)$，有 $\vec{x}\cdot(A^T\vec{y})=\vec{x}^T(A^T\vec{y})=(A\vec{x})^T\vec{y}=\vec{0}^T\vec{y}=0$

(2) 将 (1) 应用到 $B=A^T\in M_{n,m}$:

$$N(A^T)=N(B)\perp C(B^T)=C(A).\quad \#$$

注: $N(A)+C(A^\perp)=\mathbb{R}^n$, $N(A^\perp)+C(A)=\mathbb{R}^m$ $\left(N(A^\perp)+C(A)\subseteq\mathbb{R}^m,\ \dim(N(A^\perp)+C(A))=\dim N(A^\perp)+\dim C(A)=m\right)$

定义 设 W 是 $\mathbb{R}^n$ 子空间, 则 $W^\perp:=\{\vec{\alpha}\mid\vec{\alpha}\in\mathbb{R}^n,\ \vec{\alpha}\perp W\}$ 称为 W 的正交补 (orthogonal complete)

注. ① $W^\perp$ 是 $\mathbb{R}^n$ 子空间. ② $W\perp W^\perp\Rightarrow W\cap W^\perp=\{\vec{0}\}$

③ $W+W^\perp=\mathbb{R}^n$, $\dim(W)+\dim(W^\perp)=\dim(W+W^\perp)=n$ (一般两个子空间正交: $\leq n$)

定理 (1) $N(A)$ 与 $C(A^T)$ 在 $\mathbb{R}^n$ 中互为正交补

(2) $N(A^T)$ 与 $C(A)$ 在 $\mathbb{R}^m$ 中互为正交补.

Pf. (1) 先证 $N(A)=C(A^T)^\perp$.

① $\because N(A)\perp C(A^T)$. $\therefore N(A)\subseteq C(A^T)^\perp$

② $\forall\vec{\alpha}\in C(A^T)^\perp$, $\vec{\alpha}\perp C(A^T)=\mathrm{span}\{\vec{\alpha}_1,\cdots,\vec{\alpha}_m\}$.

则 $\vec{\alpha}\perp\vec{\alpha}_i,\ i=1,\cdots,m$. $\therefore A\vec{\alpha}=\vec{0}$

$\therefore\vec{\alpha}\in N(A)$. $\Rightarrow C(A^T)^\perp\subseteq N(A)$

综合①,② $\Rightarrow N(A)=C(A^T)^\perp$.

再证 $C(A^T)=N(A)^\perp$

① $C(A^T)\perp N(A)\Rightarrow C(A^T)\subseteq N(A)^\perp$

② $\forall\vec{\alpha}\in N(A)^\perp$, 则 $\vec{\alpha}\perp N(A)=\{\vec{x}\in\mathbb{R}^n\mid A\vec{x}=\vec{0}\}\Rightarrow$ 若 $A\vec{x}=\vec{0}$, 则有 $\vec{\alpha}^T\vec{x}=\vec{0}$.

若 $\vec{\alpha}\notin C(A^T)$, 令 $\hat{A}=\begin{bmatrix}A\\ \vec{\alpha}^T\end{bmatrix}$, 则 $N(\hat{A})=N(A)$

但 $\dim C(\hat{A}^T)=\dim C(A^T)+1$.

于是 $\dim C(\hat{A}^T)+\dim N(\hat{A})=\dim C(A^T)+\dim N(A)+1=n+1$. 矛盾.

$\therefore\vec{\alpha}\in C(A^T)$. 即 $N(A)^\perp\subseteq C(A^T)$

综合①,② $\Rightarrow C(A^T)=N(A)^\perp$.

(2) 将 (1) 中的 A 替换为 A^T. #

定理 若 W_1 与 W_2 为 $\mathbb{R}^n$ 子空间, $W_1\perp W_2$, 且 $\dim W_1+\dim W_2=n$, 则 W_1 与 W_2 互为 $\mathbb{R}^n$ 中正交补.

Pf. 只证 $W_1=W_2^\perp$, 由对称性, 易得 $W_2=W_1^\perp$.

① $W_1\perp W_2$, 则 $W_1\subseteq W_2^\perp$

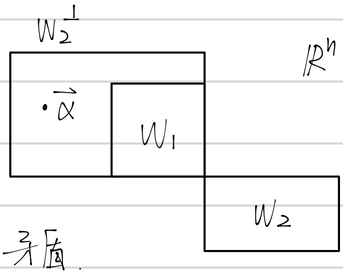
② 假设 $\exists\ \vec{\alpha} \in W_2^{\perp}$ 但 $\vec{\alpha} \notin W_1$

取 $W = \mathrm{span}\{W_1, \vec{\alpha}\}$，则：

$W \perp W_2$，且 $\dim W = \dim W_1 + 1$

但 $\dim W + \dim W_2 = \dim W_1 + \dim W_2 + 1 = n+1 > n$. 矛盾.

$\therefore\ \forall \vec{\alpha} \in W_2^{\perp}$，有 $\vec{\alpha} \in W_1$. 即 $W_2^{\perp} \subseteq W_1$



综合①,②，$W_1 = W_2^{\perp}$. #

Thm. $\mathbb{R}^n$中，$N(A) \perp C(A^T)$，且 $\dim N(A) + \dim C(A^T) = n \Rightarrow N(A)$与$C(A^T)$互为$\mathbb{R}^n$中的正交补

$\mathbb{R}^m$中，$N(A^T) \perp C(A)$，且 $\dim N(A^T) + \dim C(A) = m \Rightarrow N(A^T)$与$C(A)$ ··· $\mathbb{R}^m$ ······

$\forall \vec{x} \in \mathbb{R}^n = N(A) + C(A^T)$，有 $\vec{x} = \vec{x}_n + \vec{x}_r$，其中 $\vec{x}_n \in N(A)$，$\vec{x}_r \in C(A^T)$

$A\vec{x} = A\vec{x}_n + \underbrace{A\vec{x}_r}_{\text{this part is invertible.}} = \vec{0} + A\vec{x}_r = A\vec{x}_r \in C(A)$

(1) $\forall\ \vec{x}_n \in$ nullspace $\xrightarrow{映射A}$ $A\vec{x}_n = \vec{0}$ （多对一）

(2) $\forall\ \vec{x}_r \in$ row space $\xrightarrow{映射A}$ $A\vec{x}_r$ in column space

若 $\exists\ \vec{x}_r, \vec{x}_r' \in C(A^T)$. s.t. $A\vec{x}_r = A\vec{x}_r'$，则

$$\begin{cases} A(\vec{x}_r - \vec{x}_r') = \vec{0} \Rightarrow \vec{x}_r - \vec{x}_r' \in N(A) \\ 又\ \vec{x}_r - \vec{x}_r' \in C(A^T) \end{cases}$$

而 $N(A) \perp C(A^T)$. $\therefore\ \vec{x}_r = \vec{x}_r'$ （一一映射，可逆）

注. 每个秩r矩阵包含一个$r \times r$可逆子矩阵

(Example 4).

Figure 4.3