Chapter 5. 行列式

§5.1 几何向量的数量积、向量积与混合积

1. 数量积

定义：两个几何向量$\vec{\alpha}$与$\vec{\beta}$的数量积/点积/内积（记作$\vec{\alpha}\cdot\vec{\beta}$）为一个数：

$$\vec{\alpha}\cdot\vec{\beta} := \|\vec{\alpha}\|\,\|\vec{\beta}\|\cos\langle\vec{\alpha},\vec{\beta}\rangle$$

其中$\langle\vec{\alpha},\vec{\beta}\rangle$为$\vec{\alpha}$与$\vec{\beta}$的夹角（不大于$\pi$）。若$\vec{\alpha}$或$\vec{\beta}$为$\vec{0}$，规定$\vec{\alpha}\cdot\vec{\beta}=0$.

Remark: $\|\vec{\alpha}\|^2 = \vec{\alpha}\cdot\vec{\alpha}$ 长度

$\cos\langle\vec{\alpha},\vec{\beta}\rangle = \dfrac{\vec{\alpha}\cdot\vec{\beta}}{\|\vec{\alpha}\|\,\|\vec{\beta}\|}$ 夹角

性质：
(1) $\vec{\alpha}\cdot\vec{\beta} = \vec{\beta}\cdot\vec{\alpha}$ 对称性
(2) $(k\vec{\alpha})\cdot\vec{\beta} = k(\vec{\alpha}\cdot\vec{\beta})$ 线性
(3) $(\vec{\alpha}+\vec{\beta})\cdot\vec{\gamma} = \vec{\alpha}\cdot\vec{\gamma}+\vec{\beta}\cdot\vec{\gamma}$
$\Rightarrow$ 对第2个向量也是线性的
(4) $\vec{\alpha}\cdot\vec{\alpha}\geq 0$，等号成立$\Leftrightarrow\vec{\alpha}=\vec{0}$ 正定性

定理：$\vec{\alpha}\perp\vec{\beta}\Leftrightarrow\vec{\alpha}\cdot\vec{\beta}=0$

用坐标计算数量积

定理：$\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_i y_j = [x_1\ x_2\ \cdots\ x_m]\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn}\end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n\end{bmatrix} = \vec{x}^T B\vec{y}$

定义. 矩阵 $A=\begin{bmatrix} \vec{e_1}\cdot\vec{e_1} & \vec{e_1}\cdot\vec{e_2} & \vec{e_1}\cdot\vec{e_3} \\ \vec{e_2}\cdot\vec{e_1} & \vec{e_2}\cdot\vec{e_2} & \vec{e_2}\cdot\vec{e_3} \\ \vec{e_3}\cdot\vec{e_1} & \vec{e_3}\cdot\vec{e_2} & \vec{e_3}\cdot\vec{e_3}\end{bmatrix}$ 称为仿射坐标系$\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$的**度量矩阵**.

定理. 给定仿射坐标系$\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$，$A$为度量矩阵，向量$\vec{\alpha}=x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3}$，$\vec{\beta}=y_1\vec{e_1}+y_2\vec{e_2}+y_3\vec{e_3}$，则内积$\vec{\alpha}\cdot\vec{\beta}=\vec{x}^T A\vec{y}$，其中$\vec{x}^T=[x_1\ x_2\ x_3]$，$\vec{y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3\end{bmatrix}$

Pf: $\vec{\alpha}\cdot\vec{\beta}=\left(\sum_{i=1}^{3}x_i\vec{e_i}\right)\cdot\left(\sum_{j=1}^{3}y_j\vec{e_j}\right)=\sum_{i=1}^{3}x_i\left[\vec{e_i}\cdot\left(\sum_{j=1}^{3}y_j\vec{e_j}\right)\right]=\sum_{i=1}^{3}x_i\left(\sum_{j=1}^{3}y_j\,\vec{e_i}\cdot\vec{e_j}\right)$

$=\sum_{i=1}^{3}\sum_{j=1}^{3}(\vec{e_i}\cdot\vec{e_j})x_iy_j=\vec{x}^TA\vec{y}$. #

特别地，若 $\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$ 为右手直角坐标系，则

$\vec{e_1}\cdot\vec{e_1}=\vec{e_2}\cdot\vec{e_2}=\vec{e_3}\cdot\vec{e_3}=1$, $\quad \vec{e_i}\cdot\vec{e_j}=0$ if $i\neq j$.

度量矩阵 $A=\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}=I_3$.

$\vec{\alpha}\cdot\vec{\beta}=\vec{x}^TI_3\vec{y}=\vec{x}^T\vec{y}=x_1y_1+x_2y_2+x_3y_3$

几何向量的数量积　　　　坐标向量的标准内积

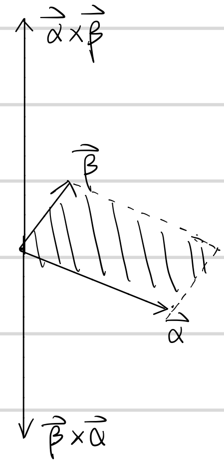
长度 $\|\vec{\alpha}\|=\sqrt{x_1^2+x_2^2+x_3^2}$.

2. 向量积

定义. 两个几何向量 $\vec{\alpha}$ 与 $\vec{\beta}$ 的向量积/叉积/外积（记作 $\vec{\alpha}\times\vec{\beta}$）仍是一个几何向量.

- 大小：以 $\vec{\alpha},\vec{\beta}$ 为边的平行四边行的面积，即 $\|\vec{\alpha}\times\vec{\beta}\|=\|\vec{\alpha}\|\cdot\|\vec{\beta}\|\sin\langle\vec{\alpha},\vec{\beta}\rangle$.
- 方向：$\vec{\alpha}\times\vec{\beta}$ 与 $\vec{\alpha},\vec{\beta}$ 都垂直，且 $\vec{\alpha},\vec{\beta},\vec{\alpha}\times\vec{\beta}$ 符合右手系

若 $\vec{\alpha}$ 或 $\vec{\beta}$ 为 $\vec{0}$，则 $\vec{\alpha}\times\vec{\beta}:=\vec{0}$



定理. $\vec{\alpha}\times\vec{\beta}=\vec{0}\iff\vec{\alpha}$ 与 $\vec{\beta}$ 共线 $(\vec{\alpha}/\!/\vec{\beta})$

性质 (1) $\vec{\alpha}\times\vec{\beta}=-\vec{\beta}\times\vec{\alpha}$，交换顺序，变号！（面积不变，方向相反）

$\left.\begin{array}{l}(2)\ (k\vec{\alpha})\times\vec{\beta}=k(\vec{\alpha}\times\vec{\beta})\\(3)\ (\vec{\alpha}+\vec{\beta})\times\vec{\gamma}=\vec{\alpha}\times\vec{\gamma}+\vec{\beta}\times\vec{\gamma}\end{array}\right\}$ 线性性

Remark: 结合(1), (2)和(3)，知对第2个向量也有线性性

用坐标计算向量积

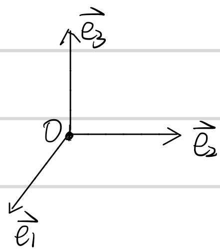
取仿射坐标系 $\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$　　　$\vec{e_1}\times\vec{e_1}=\vec{e_2}\times\vec{e_2}=\vec{e_3}\times\vec{e_3}=\vec{0}$

设 $\vec{\alpha}=x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3}$，$\vec{\beta}=y_1\vec{e_1}+y_2\vec{e_2}+y_3\vec{e_3}$

则 $\vec{\alpha}\times\vec{\beta}=(x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3})\times(y_1\vec{e_1}+y_2\vec{e_2}+y_3\vec{e_3})$

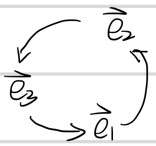
$$=x_1y_2(\vec{e_1}\times\vec{e_2})+x_1y_3(\vec{e_1}\times\vec{e_3})+x_2y_1(\vec{e_2}\times\vec{e_1})+\underline{x_2y_3(\vec{e_2}\times\vec{e_3})}+x_3y_1(\vec{e_3}\times\vec{e_1})+\underline{x_3y_2(\vec{e_3}\times\vec{e_2})}$$

$$=(x_2y_3-x_3y_2)\vec{e_2}\times\vec{e_3}+(x_3y_1-x_1y_3)\vec{e_3}\times\vec{e_1}+(x_1y_2-x_2y_1)\vec{e_1}\times\vec{e_2}.$$



特别地，若 $\{O;\vec{e_1},\vec{e_2},\vec{e_3}\}$ 是右手直角坐标系

$$\begin{cases}\vec{e_1}\times\vec{e_2}=\vec{e_3},\quad \vec{e_2}\times\vec{e_3}=\vec{e_1},\quad \vec{e_3}\times\vec{e_1}=\vec{e_2}\\ \vec{e_2}\times\vec{e_1}=-\vec{e_3},\quad \vec{e_3}\times\vec{e_2}=-\vec{e_1},\quad \vec{e_1}\times\vec{e_3}=-\vec{e_2}\end{cases}$$

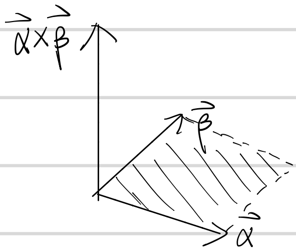


当两个向量的次序与箭头方向一致（相反）时，其向量积即为箭头所指的第3个向量（第3个向量的反向量）

$\vec{\alpha}\times\vec{\beta}=(x_2y_3-x_3y_2)\vec{e_1}+(x_3y_1-x_1y_3)\vec{e_2}+(x_1y_2-x_2y_1)\vec{e_3}$

坐标：$(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1)$

大小：$\|\vec{\alpha}\times\vec{\beta}\|=\sqrt{(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2+(x_1y_2-x_2y_1)^2}$　以 $\vec{\alpha},\vec{\beta}$ 为邻边的平行四边形的面积.



2D：若取 $x_3=y_3=0$，则

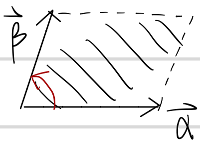
$\vec{\alpha}\times\vec{\beta}=(0,0,x_1y_2-x_2y_1)$　　(*)

$\|\vec{\alpha}\times\vec{\beta}\|=|x_1y_2-x_2y_1|$　　面积

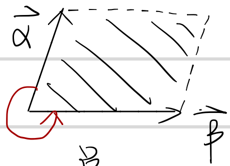
二阶行列式：$\begin{array}{c}\vec{\alpha}\\ \vec{\beta}\end{array}\begin{vmatrix}x_1 & x_2\\ y_1 & y_2\end{vmatrix}:=x_1y_2-x_2y_1$ 为以 $\vec{\alpha}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$，$\vec{\beta}=\begin{bmatrix}y_1\\y_2\end{bmatrix}$ 为邻边的平行四边形的"有向"面积.

次对角线　　主对角线

"有向"：$\begin{cases}\vec{\alpha}\text{ 沿逆时针旋转到 }\vec{\beta}\text{ 的角度小于 }180^\circ\text{，则 }\vec{\alpha}\times\vec{\beta}\text{ 指向 z 轴正向，由(*)}\Rightarrow x_1y_2-x_2y_1>0.\\ \text{------ 大于 ------ 负 ------}\Rightarrow x_1y_2-x_2y_1<0\end{cases}$



正

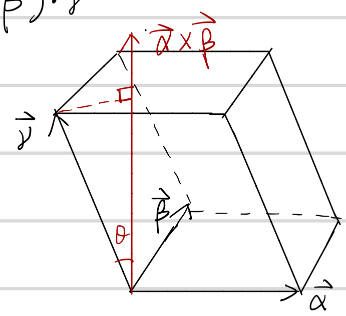


负

3. 混合积

有序

定义：三个几何向量 $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 的混合积是一个实数，记为 $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) := (\vec{\alpha} \times \vec{\beta}) \cdot \vec{\gamma}$



几何意义：$(\vec{\alpha} \times \vec{\beta}) \cdot \vec{\gamma} = \|\vec{\alpha} \times \vec{\beta}\| \, \|\vec{\gamma}\| \cos\langle \vec{\alpha}\times\vec{\beta}, \vec{\gamma} \rangle$.

$\|\vec{\alpha} \times \vec{\beta}\|$：平行六面体的底面积 $\left.\right\}$ $\Rightarrow |(\vec{\alpha}\times\vec{\beta})\cdot\vec{\gamma}| =$ 体积 V.

$\|\vec{\gamma}\| \, |\cos\langle \vec{\alpha}\times\vec{\beta}, \vec{\gamma} \rangle|$：高　　绝对值

θ

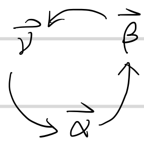
当 $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 成右手系：$\vec{\alpha}\times\vec{\beta}$ 与 $\vec{\gamma}$ 在底面的同一侧，夹角小于 $\frac{\pi}{2}$，$\sim > 0 \Rightarrow (\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = V$.

- - - - - - 左 - - - ：- - - - - - - - - 两侧，- - 大于 - -，$\sim < 0 \Rightarrow (\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = -V$.

总结：$(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ 为以 $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 为棱的平行六面体的"有向"体积.

Remarks: ① 若 $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 为右(左)手系，则 $\vec{\beta}, \vec{\gamma}, \vec{\alpha}$ 及 $\vec{\gamma}, \vec{\alpha}, \vec{\beta}$ 也是右(左)手系

$(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = (\vec{\beta}, \vec{\gamma}, \vec{\alpha}) = (\vec{\gamma}, \vec{\alpha}, \vec{\beta})$　循环性(圈上同向)



② $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = (\vec{\beta}, \vec{\gamma}, \vec{\alpha}) = (\vec{\beta}\times\vec{\gamma})\cdot\vec{\alpha} = \vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma})$

混合积可理解为相邻两个向量先作向量积，再与第三个向量作数量积.

定理：$\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 共面 $\Leftrightarrow (\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = 0$.

性质 (1) 交换有序向量组 $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 中任意两个向量的位置，混合积 $(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ 改变符号

$(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = -(\vec{\beta}, \vec{\alpha}, \vec{\gamma}) = -(\vec{\gamma}, \vec{\beta}, \vec{\alpha}) = -(\vec{\alpha}, \vec{\gamma}, \vec{\beta})$　(改变左、右手系)

交错性

(2) 混合积对第三个向量有线性性. $\begin{cases} (\vec{\alpha}, \vec{\beta}, k\vec{\gamma}) = k(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) \\ (\vec{\alpha}, \vec{\beta}, \vec{\gamma}_1 + \vec{\gamma}_2) = (\vec{\alpha}, \vec{\beta}, \vec{\gamma}_1) + (\vec{\alpha}, \vec{\beta}, \vec{\gamma}_2) \end{cases}$

注：① $(\vec{\beta}, \vec{\gamma}, \vec{\alpha}) = -(\vec{\alpha}, \vec{\gamma}, \vec{\beta}) = (\vec{\alpha}, \vec{\beta}, \vec{\gamma})$　循环性.

奇数次对换改变混合积的符号，偶数次对换不改变混合积符号.

② 结合(1),(2)，可知混合积对第一个和第二个向量都是线性的.　多线性.

例. $(\vec{\alpha}+\vec{\beta})\times\vec{\gamma} = \vec{\alpha}\times\vec{\gamma} + \vec{\beta}\times\vec{\gamma}$.　叉乘的线性性.

Pf. Claim: $\forall \vec{\eta}$, 有 $[(\vec{\alpha}+\vec{\beta})\times\vec{\gamma} - \vec{\alpha}\times\vec{\gamma} - \vec{\beta}\times\vec{\gamma}]\cdot\vec{\eta} = 0$

$LHS = (\vec{\alpha}+\vec{\beta}, \vec{\gamma}, \vec{\eta}) - (\vec{\alpha}, \vec{\gamma}, \vec{\eta}) - (\vec{\beta}, \vec{\gamma}, \vec{\eta}) = 0$.

$\therefore [(\vec{\alpha}+\vec{\beta})\times\vec{\gamma} - \vec{\alpha}\times\vec{\gamma} - \vec{\beta}\times\vec{\gamma}] \perp \vec{\eta}$, $\forall \vec{\eta}$

$\therefore (\vec{\alpha}+\vec{\beta})\times\vec{\gamma} - \vec{\alpha}\times\vec{\gamma} - \vec{\beta}\times\vec{\gamma} = \vec{0}$　#.

用坐标计算混合积

仿射坐标系 $\{O; \vec{e_1}, \vec{e_2}, \vec{e_3}\}$,

$\vec{\alpha} = a_{11}\vec{e_1} + a_{12}\vec{e_2} + a_{13}\vec{e_3}$, $\vec{\beta} = a_{21}\vec{e_1} + a_{22}\vec{e_2} + a_{23}\vec{e_3}$, $\vec{\gamma} = a_{31}\vec{e_1} + a_{32}\vec{e_2} + a_{33}\vec{e_3}$

$$
\begin{aligned}
(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) &= \left(\sum_{i=1}^{3} a_{1i}\vec{e_i}, \sum_{j=1}^{3} a_{2j}\vec{e_j}, \sum_{k=1}^{3} a_{3k}\vec{e_k}\right) \\
&= \sum_{i=1}^{3} a_{1i}\left(\vec{e_i}, \sum_{j=1}^{3} a_{2j}\vec{e_j}, \sum_{k=1}^{3} a_{3k}\vec{e_k}\right) = \sum_{i=1}^{3} a_{1i}\left[\sum_{j=1}^{3} a_{2j}\left(\sum_{k=1}^{3} a_{3k}(\vec{e_i}, \vec{e_j}, \vec{e_k})\right)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{1i}a_{2j}a_{3k}(\vec{e_i}, \vec{e_j}, \vec{e_k})
\end{aligned}
$$

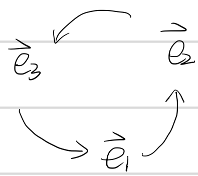
注意到: $(\vec{e_i}, \vec{e_j}, \vec{e_k}) \neq 0 \Leftrightarrow i, j, k$ 互不相同　　$3!$ 种取法

$$
\begin{aligned}
(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = {} & a_{11}a_{22}a_{23}(\vec{e_1}, \vec{e_2}, \vec{e_3}) + a_{12}a_{23}a_{31}(\vec{e_2}, \vec{e_3}, \vec{e_1}) + a_{13}a_{21}a_{32}(\vec{e_3}, \vec{e_1}, \vec{e_2}) \\
& + a_{13}a_{22}a_{31}(\vec{e_3}, \vec{e_2}, \vec{e_1}) + a_{12}a_{21}a_{33}(\vec{e_2}, \vec{e_1}, \vec{e_3}) + a_{11}a_{23}a_{32}(\vec{e_1}, \vec{e_3}, \vec{e_2})
\end{aligned}
$$

前一行: $(\vec{e_1}, \vec{e_2}, \vec{e_3}) = (\vec{e_2}, \vec{e_3}, \vec{e_1}) = (\vec{e_3}, \vec{e_1}, \vec{e_2}) = 1$

后一行: $(\vec{e_3}, \vec{e_2}, \vec{e_1}) = (\vec{e_2}, \vec{e_1}, \vec{e_3}) = (\vec{e_1}, \vec{e_3}, \vec{e_2}) = -1$

} 右手直角坐标系



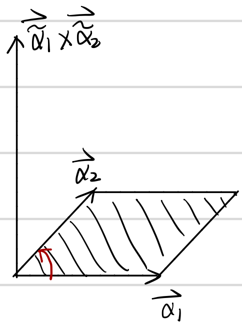
$(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = a_{11}a_{22}a_{23} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$　　3个正项

$\quad - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$.　　3个负项

三阶行列式: $\begin{array}{c} \vec{\alpha} \\ \vec{\beta} \\ \vec{\gamma} \end{array}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix} = (\vec{\alpha}, \vec{\beta}, \vec{\gamma})$　以 $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ 为棱的平行六面体的"有向"体积

划线记忆法只适用于3阶行列式!

§5.2 n阶行列式



2阶行列式:
(2维, $\mathbb{R}^2$)

$$\begin{array}{c} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \end{array}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := \overset{\text{正}}{a_{11}a_{22}} - \overset{\text{负}}{a_{12}a_{21}}$$

, 记为 $\det(\vec{\alpha}_1, \vec{\alpha}_2)$, 2! 项相加

拓到3维向量 $\vec{\alpha}_1 = (a_{11}, a_{12}, 0)$, $\vec{\alpha}_2 = (a_{21}, a_{22}, 0)$

右手直角坐标系下. $\vec{\alpha}_1 \times \vec{\alpha}_2 = (0, 0, \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\det(\vec{\alpha}_1, \vec{\alpha}_2)})$

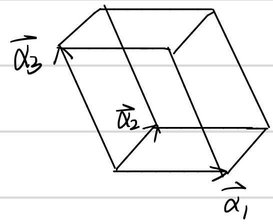
$\det(\vec{\alpha}_1, \vec{\alpha}_2)$ 为以 $\vec{\alpha}_1, \vec{\alpha}_2$ 为邻边的平行四边形的有向面积

3阶行列式:
(3维, $\mathbb{R}^3$)

$$\begin{array}{c} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vec{\alpha}_3^T \end{array}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left.\begin{array}{ll} a_{11}a_{22}a_{23} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} & \text{3个正项} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} & \text{3个负项} \end{array}\right\} 3!\text{项}$$

$\det(\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3) = (\vec{\alpha}_1 \times \vec{\alpha}_2) \cdot \vec{\alpha}_3$ 混合积

以 $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$ 为棱的平行六面体的有向体积.



2. n阶行列式 (Big formula)

(n维, $\mathbb{R}^n$)

$$\begin{array}{c} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_n^T \end{array}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

记为 $\det(\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n)$, 或 $\det(A)$. 或 $|A|$

看作以 $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ 为"棱"的"n维立体"的"n维有向体积".

n-dimensional box.

满足以下基本性质:

(1) $\mathbb{R}^n$ 的自然基 $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$ 决定的n维有向体积 $\det(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n) = 1$. i.e. $\underbrace{\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{vmatrix}}_{\text{单位阵}} = 1$

(2) 对换两行的位置, 行列式变号. $\det(\cdots, \vec{\alpha}_i, \cdots, \vec{\alpha}_j, \cdots) = -\det(\cdots, \vec{\alpha}_j, \cdots, \vec{\alpha}_i, \cdots)$

(3) 行列式关于每一行都有线性性.

"⋯"表示其余行不变.

$$\det(\cdots, x\vec{\alpha}_i + y\vec{\beta}_i, \cdots) = x\det(\cdots, \vec{\alpha}_i, \cdots) + y\det(\cdots, \vec{\beta}_i, \cdots)$$

推论: (4) 若有两行一样, 即∃ $1 \le i < j \le n$, s.t. $\vec{\alpha}_i = \vec{\alpha}_j$, 则 $\det(\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n) = 0$

$$\det(\cdots, \vec{\alpha}_i, \cdots, \vec{\alpha}_j, \cdots) = -\det(\cdots, \vec{\alpha}_j, \cdots, \vec{\alpha}_i, \cdots) = -\det(\cdots, \vec{\alpha}_i, \cdots, \vec{\alpha}_j, \cdots)$$

用基本性质推导行列式:

① $\det(\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n) = \det\left(\sum_{j_1=1}^{n} a_{1j_1}\vec{e}_{j_1}, \sum_{j_2=1}^{n} a_{2j_2}\vec{e}_{j_2}, \cdots, \sum_{j_n=1}^{n} a_{nj_n}\vec{e}_{j_n}\right)$

$= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_n=1}^{n} a_{1j_1}a_{2j_2}\cdots a_{nj_n}\det(\vec{e}_{j_1}, \vec{e}_{j_2}, \cdots, \vec{e}_{j_n})$ (性质3). 共 n^n 项.

若 $j_1, j_2, \cdots, j_n$ 中有某2个相同, 由推论, 有 $\det(\vec{e}_{j_1}, \vec{e}_{j_2}, \cdots, \vec{e}_{j_n}) = 0$.

$1 \le j_1, j_2, \cdots, j_n \le n$ 两两不同, 称为 $1, 2, \cdots, n$ 的一个排列. 记作 $(i_1, i_2, \cdots, i_n)$. 共 $n!$ 项.

共 $n!$ 项

$$\Rightarrow \det(\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}) = \sum_{(j_1,j_2,\cdots,j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n}\det(\vec{e_{j_1}},\vec{e_{j_2}},\cdots,\vec{e_{j_n}})$$

置换矩阵 $P = \begin{bmatrix} \vec{e_{j_1}}^T \\ \vec{e_{j_2}}^T \\ \vdots \\ \vec{e_{j_n}}^T \end{bmatrix}$

② 对排列 $(j_1,j_2,\cdots,j_n)$，求行列式 $\det(\vec{e_{j_1}},\vec{e_{j_2}},\cdots,\vec{e_{j_n}})$.

已知：每对换两行，行列式变号；$\det(\vec{e_1},\vec{e_2},\cdots,\vec{e_n})=1$.　(性质1,2)

若由 $(j_1,j_2,\cdots,j_n)$ 需经过 s 次对换变成 $(1,2,\cdots,n)$，则

$$(-1)^s\det(\vec{e_{j_1}},\vec{e_{j_2}},\cdots,\vec{e_{j_n}}) = \det(\vec{e_1},\vec{e_2},\cdots,\vec{e_n}) = 1$$

$\begin{cases} 1°. 若 (j_1,j_2,\cdots,j_n) 为偶排列，则 s 为偶数，此时 \det(\vec{e_{j_1}},\vec{e_{j_2}},\cdots,\vec{e_{j_n}})=1. \\ 2°. 若 \cdots\cdots 奇 \cdots, \cdots\cdots 奇 \cdots, \cdots \det(\vec{e_{j_1}},\vec{e_{j_2}},\cdots,\vec{e_{j_n}})=-1. \end{cases}$

$$\Rightarrow \det(\vec{\alpha_1},\vec{\alpha_2},\cdots,\vec{\alpha_n}) = \sum_{(j_1,j_2,\cdots,j_n)} \operatorname{sgn}(j_1,j_2,\cdots,j_n)\, a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

即书上的 $\det P$.

Big Formula

$$= \sum_{(j_1,j_2,\cdots,j_n)} (-1)^{\tau(j_1,j_2,\cdots,j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

1. 排列

定义. 由 n 个不同的自然数 $1,2,\cdots,n$ 按任何一种顺序排成一个有序数组 $(j_1,j_2,\cdots,j_n)$ 称为一个 n 元排列 (permutation)

注：1° n 元排列的总数是 $n!$　　2°. $(1,2,\cdots,n)$ 称为标准排列.

定义. 排列 $(j_1,j_2,\cdots,j_n)$ 中每出现一对 $p<q$, s.t. $j_p>j_q$，称 (j_p,j_q) 为该排列的一个逆序 (reverse order)

排列中逆序的总数称为这个排列的逆序数 (number of reverse order)，记为 $\tau(j_1,j_2,\cdots,j_n)$.

$\begin{cases} 若 \tau(j_1,j_2,\cdots,j_n) 为奇数，称 (j_1,j_2,\cdots,j_n) 为奇排列 \text{ (odd permutation)} \\ \cdots\cdots 偶 \cdots, \cdots\cdots 偶排列 \text{ (even permutation)} \end{cases}$

例：4元排列 $(3,1,4,2)$，逆序：$(3,1)$, $(3,2)$, $(4,2)$，共3个.

$\tau(3,1,4,2)=3$，$(3,1,4,2)$ 为奇排列.　　标准排列 $(1,2,\cdots,n)$ 是偶排列.

定义. 在 n 元排列 $(j_1,j_2,\cdots,j_n)$ 中，将两个数 j_p, j_q 的位置互换，其余数位置不变，称为这个排列的一次对换 (transposition).

定理 ① 任一个排列经过任一次对换，必改变奇偶性。

② $n!$ 个 n 元排列中，奇排列和偶排列各占一半.

定理 每个排列 $(j_1,j_2,\cdots,j_n)$ 都可经过有限次对换变成标准排列 $(1,2,\cdots,n)$，同个排列 $(j_1,j_2,\cdots,j_n)$ 变成标准排列所经过的对换次数 s 不唯一，但 s 的奇偶性是唯一的，且与排列的奇偶性相同。

奇偶性符号：$\operatorname{sgn}(j_1,j_2,\cdots,j_n)=(-1)^{\tau(j_1,j_2,\cdots,j_n)}=\begin{cases}1, & 若 (j_1,j_2,\cdots,j_n) 为偶排列\\ -1, & 若 (j_1,j_2,\cdots,j_n) 为奇排列\end{cases}$