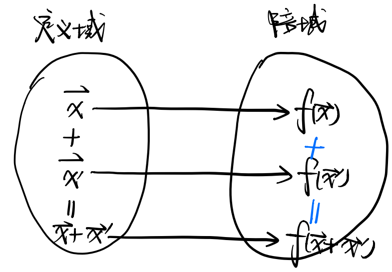
§1.3 线性映射



<u>定义</u>（线性映射）映射 $f: \mathbb{R}^n \to \mathbb{R}^m$，若满足

(1) $\forall\, \vec{x}, \vec{x}' \in \mathbb{R}^n$，都有 $f(\vec{x}+\vec{x}') = f(\vec{x}) + f(\vec{x}')$，

(2) $\forall\, \vec{x} \in \mathbb{R}^n$，$\forall\, k \in \mathbb{R}$，都有 $f(k\vec{x}) = kf(\vec{x})$，

则称 f 为 $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的线性映射。特别地，若 $m=n$，称 f 为 $\mathbb{R}^n$ 上的线性变换。

<u>性质</u> (1) $f(\vec{0}_n) = \vec{0}_m$: $f(\vec{0}_n) = f(0\vec{\alpha}) = 0f(\vec{\alpha}) = \vec{0}_m$.

(2) $f(-\vec{\alpha}) = -f(\vec{\alpha})$: $f(-\vec{\alpha}) = f((-1)\vec{\alpha}) = (-1)f(\vec{\alpha}) = -f(\vec{\alpha})$

(3) $f(k_1\vec{x}_1 + k_2\vec{x}_2 + \cdots + k_l\vec{x}_l) = k_1 f(\vec{x}_1) + k_2 f(\vec{x}_2) + \cdots + k_l f(\vec{x}_l)$ 保持线性运算

<u>例</u>: $f: \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{array}{r} \text{平时} \to \\ \text{期中} \to \\ \text{期末} \to \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto 0.2x_1 + 0.2x_2 + 0.6x_3 \text{ 总成绩}$$

验证: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$, $\vec{x}+\vec{x}' = \begin{bmatrix} x_1+x_1' \\ x_2+x_2' \\ x_3+x_3' \end{bmatrix}$

$$f(\vec{x}+\vec{x}') = 0.2(x_1+x_1') + 0.2(x_2+x_2') + 0.6(x_3+x_3')$$

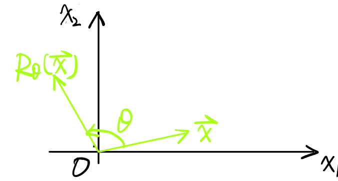
$$= (0.2x_1 + 0.2x_2 + 0.6x_3) + (0.2x_1' + 0.2x_2' + 0.6x_3') = f(\vec{x}) + f(\vec{x}')$$

$$f(k\vec{x}) = kf(\vec{x})$$

<u>例</u>: $\mathbb{R}^n$ 上的恒同变换 $I = id: \mathbb{R}^n \to \mathbb{R}^n$ 是线性变换.

$$\vec{x} \mapsto \vec{x}$$

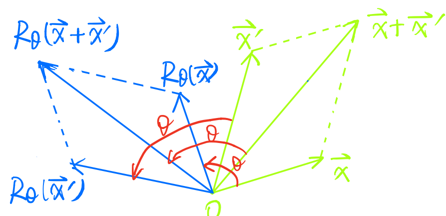
平面向量构成的线性空间$\mathbb{R}^2$上的线性变换 (直角坐标系)

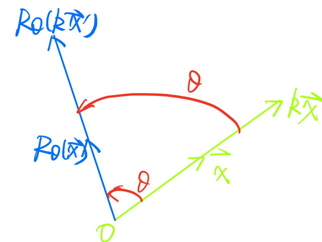


1. 旋转变换R_θ ($\theta\in\mathbb{R}$)

$\mathbb{R}^2\to\mathbb{R}^2$, $R_\theta(\vec{x})$: 将向量$\vec{x}$绕原点逆时针旋转θ角.

验证:





$$R_\theta(\vec{x}+\vec{x}') = R_\theta(\vec{x})+R_\theta(\vec{x}') \qquad R_\theta(k\vec{x}) = kR_\theta(\vec{x})$$

表达式: $R_\theta\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = R_\theta\left(x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}\right) = x_1R_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+x_2R_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

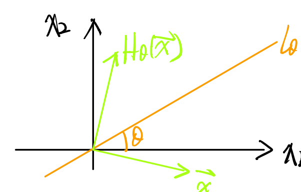
$$R_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix},\quad R_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta+\frac{\pi}{2})\\\sin(\theta+\frac{\pi}{2})\end{bmatrix}$$

$$\Rightarrow R_\theta\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}+x_2\begin{bmatrix}\cos(\theta+\frac{\pi}{2})\\\sin(\theta+\frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}x_1\cos\theta-x_2\sin\theta\\x_1\sin\theta+x_2\cos\theta\end{bmatrix}$$

注: R_θ仅由其在$\begin{bmatrix}1\\0\end{bmatrix}$和$\begin{bmatrix}0\\1\end{bmatrix}$($\mathbb{R}^2$的标准坐标向量)上的取值决定.

2. 反射变换

给定直线l_θ: $x_2\cos\theta-x_1\sin\theta=0$, 其中$\theta$是$l_\theta$与$x_1$坐标轴的夹角.

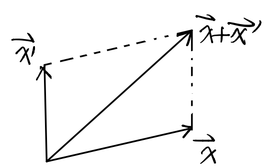


H_θ: $\mathbb{R}^2\to\mathbb{R}^2$. 向量关于$l_\theta$的反射.

H_θ是线性映射(几何作图验证)

$$H_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos2\theta\\\sin2\theta\end{bmatrix}.\quad H_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta-(\frac{\pi}{2}-\theta))\\\sin(\theta-(\frac{\pi}{2}-\theta))\end{bmatrix}$$

$$\Rightarrow H_\theta\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = H_\theta\left(x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}\right) = x_1H_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+x_2H_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}x_1\cos2\theta+x_2\sin2\theta\\x_1\sin2\theta-x_2\cos2\theta\end{bmatrix}$$



例: 映射L: $\mathbb{R}^2\to\mathbb{R}$, 取长度 $\begin{bmatrix}x_1\\x_2\end{bmatrix}\longrightarrow\sqrt{x_1^2+x_2^2}$

$L(\vec{x}+\vec{x}')\neq L(\vec{x})+L(\vec{x}')$. 不是线性映射

§1.4 线性映射的表示矩阵

线性映射 $f: \mathbb{R}^n \to \mathbb{R}^m$.

设 $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underset{\uparrow \atop \vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + x_2 \underset{\uparrow \atop \vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}} + \cdots + x_n \underset{\uparrow \atop \vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$ 线性表示.

则 $f(\vec{x}) = f(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) = x_1 f(\vec{e}_1) + x_2 f(\vec{e}_2) + \cdots + x_n f(\vec{e}_n)$ (*)

<u>定义</u> $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$ 称为 $\mathbb{R}^n$ 的标准坐标向量组. $\vec{e}_i$: 第 i 个标准坐标向量.

f 由 $f(\vec{e}_1), f(\vec{e}_2), \cdots, f(\vec{e}_n)$ 所唯一决定. $f(\vec{e}_1), \cdots, f(\vec{e}_n)$ 可任意取值吗?

<u>命题</u> $\forall\, \vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n \in \mathbb{R}^m$, $\exists!$ 线性映射 $f: \mathbb{R}^n \to \mathbb{R}^m$, s.t. $f(\vec{e}_j) = \vec{a}_j$. $j = 1, \cdots, n$

Pf: (1) "存在性". 定义映射 $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n$$

其满足 $f(\vec{e}_i) = \vec{a}_i$, $i = 1, \cdots, n$

下面验证其是线性映射: 设 $\vec{x}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$, $k \in \mathbb{R}$.

① $f(\vec{x} + \vec{x}') = (x_1 + x_1')\vec{a}_1 + (x_2 + x_2')\vec{a}_2 + \cdots + (x_n + x_n')\vec{a}_n$

$= (x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n) + (x_1'\vec{a}_1 + x_2'\vec{a}_2 + \cdots + x_n'\vec{a}_n) = f(\vec{x}) + f(\vec{x}')$

② $f(k\vec{x}) = kx_1\vec{a}_1 + \cdots + kx_n\vec{a}_n = k(x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) = kf(\vec{x})$

(2) "唯一性".

假设 $f, g: \mathbb{R}^n \to \mathbb{R}^m$ 是两个线性映射, 且 $f(\vec{e}_i) = g(\vec{e}_i)$, $i = 1, \cdots, n$.

则 $\forall \vec{x} \in \mathbb{R}^n$, 有 $f(\vec{x}) = x_1 f(\vec{e}_1) + x_2 f(\vec{e}_2) + \cdots + x_n f(\vec{e}_n)$

$= x_1 g(\vec{e}_1) + x_2 g(\vec{e}_2) + \cdots + x_n g(\vec{e}_n) = g(\vec{x})$

$\therefore f = g$. #

线性映射 $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. 设 $f(\vec{e_j}) = \vec{a_j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, $j = 1, 2, \cdots, n$

则 $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto f(\vec{x}) = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$ (※)

$\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ 是线性映射} \Longleftrightarrow f \text{ 可写成(※)的形式} \\ m\text{个输出, 每个均为 } \underbrace{x_1, x_2, \cdots, x_n}_{n\text{个输入}} \text{ 的线性函数 (常数项为0的一次函数)} \end{cases}$

<u>定义</u> 矩形的数表 $A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \vec{r_1}^T \\ \vec{r_2}^T \\ \vdots \\ \vec{r_m}^T \end{matrix} \\ \begin{matrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{matrix} \end{array}$ 称为 $m \times n$ 矩阵.

A 的 (i,j) 元: a_{ij}. i 行, j 列

$A = [\vec{a_1}\ \vec{a_2} \cdots \vec{a_n}] = \begin{bmatrix} \vec{r_1}^T \\ \vec{r_2}^T \\ \vdots \\ \vec{r_m}^T \end{bmatrix} = [a_{ij}]_{m \times n}$

$\begin{cases} \text{列向量 } \vec{a_j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} & m \times 1 \text{ 矩阵} \\ \text{行向量 } \vec{r_i}^T = [a_{i1}\ a_{i2} \cdots a_{in}] & 1 \times n \text{ 矩阵} \end{cases}$

两个 $m \times n$ 矩阵相等: 每个元素都相等. 数域 F 上的全体 $m \times n$ 矩阵组成的集合记作 $M_{m,n}(F)$

<u>定义</u> 设线性映射 $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. $\vec{e_i}$, $i = 1, \cdots, n$ 为 $\mathbb{R}^n$ 的标准坐标向量. 若 $\vec{a_i} = f(\vec{e_i})$, $i = 1, \cdots, n$

则称矩阵 $A = [\vec{a_1}\ \vec{a_2} \cdots \vec{a_n}]$ 为线性映射 f 在标准坐标向量下的表示矩阵.

<u>定义</u> (矩阵与向量的乘积) A: $m \times n$ 矩阵. $\vec{x}$: n 维列向量

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \leftarrow m\text{维列向量}$$

$$= \underbrace{x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}}_{A\text{的列向量的线性组合}} = \begin{bmatrix} \vec{r_1}^T\vec{x} \\ \vec{r_2}^T\vec{x} \\ \vdots \\ \vec{r_m}^T\vec{x} \end{bmatrix}$$

$\vec{r_i}^T\vec{x} = a_{i1}x_1 + \cdots + a_{in}x_n$ 内积

$= \vec{r_i} \cdot \vec{x}$