
Question1 (2 points)

Let $Y_i = \beta x_i + \varepsilon_i$ for $i = 1, 2$, where

$$\varepsilon_1 \sim \text{Normal}\left(0, \sigma^2\right) \qquad \text{and} \qquad \varepsilon_2 \sim \text{Normal}\left(0, 2\sigma^2\right)$$

Suppose ε_1 and ε_2 are statistically independent. For $x_1 = 1$ and $x_2 = -1$, find the weighted least squares estimate of β and find the variance of your estimate in terms of σ^2.

Question2 (4 points)

Consider the AR(1) model

$$Y_t = 5 - 0.7Y_{t-1} + \varepsilon_t \qquad \text{where} \quad \sigma_\varepsilon^2 = \text{Var}\left[\varepsilon_t\right] = 2 \quad \text{for all } t.$$

(a) (1 point) Is this process stationary? Explain your answer.

(b) (1 point) What is the mean of this process?

(c) (1 point) What is the variance of this process?

(d) (1 point) What is the covariance of this process?

Question3 (3 points)

Let Y_t be a stationary AR(2) process

$$(Y_t - \mu) = \phi_1 \left(Y_{t-1} - \mu\right) + \phi_2 \left(Y_{t-2} - \mu\right) + \varepsilon_t$$

(a) (1 point) Show that the ACF of Y_t satisfies the following equation

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) \qquad \text{for} \quad k = 1, 2, \ldots$$

(b) (1 point) Show ϕ_1 and ϕ_2 solves the following system

$$\begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

(c) (1 point) Suppose that $\rho(1) = 0.4$ and $\rho(2) = 0.2$, find $\rho(3)$.

Question4 (1 points)

Suppose that $\mathbb{E}\left[\mathbf{Y}\right] = \boldsymbol{\theta}$, $\mathbf{A}\boldsymbol{\theta} = \mathbf{0}$, and $\text{Var}\left[\mathbf{Y}\right] = \sigma^2\mathbf{V}$, where $\mathbf{A}$ is an $m \times n$ matrix of rank m and $\mathbf{V}$ is a known $n \times n$ positive-definite matrix. Let $\hat{\boldsymbol{\theta}}$ be the generalised least squares estimate of $\boldsymbol{\theta}$; that is, $\hat{\boldsymbol{\theta}}$ minimises

$$\left(\mathbf{Y} - \boldsymbol{\theta}\right)^{\mathrm{T}} \mathbf{V}^{-1} \left(\mathbf{Y} - \boldsymbol{\theta}\right) \qquad \text{subject to} \qquad \mathbf{A}\boldsymbol{\theta} = \mathbf{0}$$

Show that

$$\mathbf{Y} - \hat{\boldsymbol{\theta}} = \mathbf{V}\mathbf{A}^{\mathrm{T}}\hat{\boldsymbol{\gamma}}$$

where $\hat{\boldsymbol{\gamma}}$ is the generalised least squares estimate of γ for the model

$$\mathbb{E}\left[\mathbf{Y}\right] = \mathbf{V}\mathbf{A}^{\mathrm{T}}\gamma \qquad \text{and} \qquad \text{Var}\left[\mathbf{Y}\right] = \sigma^2\mathbf{V}$$