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Question1 (1 points)

Use the law of total expectation to prove the law of total variance

$$\operatorname{Var}[X] = \mathbb{E}\big[\operatorname{Var}[X \mid Y]\big] + \operatorname{Var}\big[\mathbb{E}[X \mid Y]\big]$$

where X and Y denote random variables.

Question2 (1 points)

Let X have a negative binomial distribution with an unknown probability of success p

$$X \sim \mathrm{NB}(r = 3, p)$$

where r denotes the number of failures. Find the maximum likelihood estimate of p given a single observation of $x = 2$.

Question3 (2 points)

In 1000 tosses of a coin, 560 heads appeared. Is it reasonable to assume the coin is fair? Justify your answer.

Question4 (2 points)

Let $Y_1, \ldots, Y_n$ be i.i.d. Normal$(\mu, 1)$, A 95% confidence interval for μ is given by

$$\mathcal{I} = \left(\bar{y} - 1.96/\sqrt{n}, \bar{y} + 1.96/\sqrt{n}\right)$$

Let p denote probability that an additional independent observation, Y_{n+1}, fall in this interval $\mathcal{I}$. Is p greater than, less than, or equal to 0.95? Justify your answer.

Question5 (4 points)

Let X have a normal distribution $\mathrm{N}(\mu, \sigma^2)$. Suppose 10 independent realisations of X are obtained such that the sample mean is

$$\bar{x} = 10$$

(a) (2 points) Suppose $\sigma^2 = 1$, is it reasonable to suggest $\mu = 5$? Justify your answer.

(b) (2 points) Suppose σ^2 is unknown but the sample variance is known to be 4, is it reasonable to suggest $\mu = 5$? Justify your answer.

Question6 (2 points)

Let $X_1, X_3, \ldots, X_n$ be independent random variables with common mean μ and variances

$$\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2.$$

Show $\dfrac{1}{n(n-1)} \displaystyle\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$ is an unbiased estimator of $\operatorname{Var}\left[\bar{X}\right]$, where $\bar{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$.

Question7 (13 points)

Consider the simple linear regression, and the following estimators for β_0 and β_1.

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \qquad \text{and} \qquad \hat{\beta}_1 = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}$$

(a) (4 points) Given $\hat{\beta}_1$ is unbiased and consistent, show $\hat{\beta}_0$ is unbiased and consistent.

(b) (5 points) Derive the sampling distribution of $\hat{\beta}_1$.

(c) (4 points) Derive the sampling distribution of $\hat{\beta}_0$.