
Question1 (1 points)

Find the least squares estimates of θ and ϕ in terms of y_1, y_2 and y_3 for the following model

$$\begin{aligned} Y_1 &= \theta + \varepsilon_1 \\ Y_2 &= 2\theta - \phi + \varepsilon_2 \qquad \text{and } \mathbb{E}[\varepsilon_i] = 0 \text{ for } i = 1, 2, 3. \\ Y_3 &= \theta + 2\phi + \varepsilon_3 \end{aligned}$$

Question2 (4 points)

Suppose that $\mathbf{Y} \sim \text{Normal}\left(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}\right)$, where $\mathbf{X}$ is $n \times (k+1)$ of rank $k+1$, that is, $\mathbf{X}$ has linearly independent columns. Let $\mathbf{P} = \mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}$ denote the projection/hat matrix.

(a) (1 point) Show both $\mathbf{P}$ and $(\mathbf{I} - \mathbf{P})$ are symmetric and idempotent.

(b) (1 point) Show the residual and fitted value are orthogonal, that is,

$$\hat{\mathbf{e}}^{\mathrm{T}}\hat{\mathbf{y}} = 0 \qquad \text{where } \hat{\mathbf{y}} = \mathbf{P}\mathbf{y} \text{ and } \hat{\mathbf{e}} = (\mathbf{I} - \mathbf{P})\,\mathbf{y}.$$

(c) (1 point) Find the covariance $\mathrm{Cov}\left[\hat{e}_i, \hat{e}_j \mid \mathbf{X}\right]$, where $i \neq j$, in terms of elements of $\mathbf{P}$.

(d) (1 point) Find the variance $\mathrm{Var}\left[\hat{\sigma}_u^2 \mid \mathbf{X}\right]$, where $\hat{\sigma}_u^2 = \dfrac{1}{n-k-1}\hat{\mathbf{e}}^{\mathrm{T}}\hat{\mathbf{e}}$.

Question3 (1 points)

Show the F-statistic for significance of linear regression with k predictors can be written as

$$F = \frac{R^2/k}{\left(1 - R^2\right)/\left(n - k - 1\right)}$$

Suppose that $n = 20$, $k = 4$, and $R^2 = 0.9$. Find the probability of observing datasets as extreme as the given one if the model assumptions and hypothesis are true. What conclusion would you draw about the relationship between the response and the four predictors?

Question4 (1 points)

Suppose that we scale the predictors so that $x_{ij} = k_j w_{ij}$ for all i, j. By expressing $\mathbf{X}$ in terms of a new data matrix $\mathbf{W}$, show that $\hat{\mathbf{Y}}$ remains unchanged under this change of scale.

Question5 (3 points)

Suppose we have observations of Y for $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

(a) (2 points) Consider the model $\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 x$ while the true mean is given by

$$\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 x + \beta_2 x^2$$

Determine whether LSE $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased. If not, determine the bias.

(b) (1 point) Consider the model $\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 x + \beta_2 x^2$ while the true mean is

$$\mathbb{E}[Y \mid X] = \beta_0 + \beta_1 x$$

Determine whether LSE $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are unbiased. If not, determine the bias.

(c) (2 points (bonus)) What is the variance $\mathrm{Var}\left[\hat{\beta}_1 \mid \mathbf{X}\right]$ in each of the above two cases?

(d) (1 point (bonus)) Compare the variance $\mathrm{Var}\left[\hat{\beta}_1 \mid \mathbf{X}\right]$ with the corresponding variance when the correct model is used in each of the above two cases.