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Q1. From $\hat{\beta}_w = (X^T W X)^{-1} X^T W y$ $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $W = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^2} \end{pmatrix}$

$$= \left[\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \left[\begin{pmatrix} \frac{3}{2}\frac{1}{\sigma^2} & \frac{1}{2}\frac{1}{\sigma^2} \\ \frac{1}{2}\frac{1}{\sigma^2} & \frac{3}{2}\frac{1}{\sigma^2} \end{pmatrix} \right]^{-1} \begin{pmatrix} \frac{1}{\sigma^2} & \frac{1}{2\sigma^2} \\ \frac{1}{\sigma^2} & -\frac{1}{2\sigma^2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ \frac{1}{2}y_1 - \frac{1}{2}y_2 \end{pmatrix}$$

$$Var[\hat{\beta}_w] = (X^T W X)^{-1} X^T \cdot W \, Var[\varepsilon] \cdot ((X^T W X)^{-1} X^T W)^T$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\sigma^2 & -\frac{1}{4}\sigma^2 \\ -\frac{1}{4}\sigma^2 & \frac{3}{4}\sigma^2 \end{pmatrix}$$

Q2 a) Yes. AR(1) process $Y_t = \phi Y_{t-1} + w_t$ $(1 - \phi B) Y_t = w_t$

characteristic polynomial $\phi(B) = (1 - \phi B)$

Then process is stationary if and only if $|\phi| < 1$

$\phi = -0.7$, so the pross is stationary

b) 2.94. With the satisfaction of $|\phi| < 1$ $E[Y_t] = E[Y_{t-1}] = \mu$

$\mu = 5 - 0.7\mu$ $\mu = 2.94$

c) 3.92 $Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t$ let $\tilde{Z}_t = Y_t - \mu$

$$\tilde{Z}_t = \phi \tilde{Z}_{t-1} + \varepsilon_t$$

Variance: $E[\tilde{Z}_t^2] = \phi^2 E[\tilde{Z}_{t-1}^2] + 2\phi E[\tilde{Z}_{t-1}\varepsilon_t] + E[\varepsilon_t^2]$

Assume σ_Z^2 as the variance of the stationary process

$\tilde{Z}_{t-1}$ and ε_t are independent and have null expectation

$$\sigma_Z^2 = \frac{2}{1 - (-0.7)^2} = 3.92$$

d) $Cov(Y_t, Y_{t-1}) = Cov(5 - 0.7Y_{t-1} + \varepsilon_t, Y_{t-1}) = -0.7\,Cov(Y_{t-1}, Y_{t-1}) + Cov(\varepsilon_t, Y_{t-1})$

$= -0.7\,Var(Y)$

$Var[Y_t] = Var[5 - 0.7Y_{t-1} + \varepsilon_t] = Var[\varepsilon_t] + (-0.7)^2 Var[Y_{t-1}] = Var[Y_{t-1}]$

for all t $Var[Y] = \frac{2}{1 - (0.7)^2} = 3.92$

$Cov[Y_t, Y_{t-1}] = -2.74$

Q3 $(Y_t - \mu) = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + \varepsilon_t$

a) Denote $\tilde{z}_t = Y_t - \mu$

Then $\tilde{z}_t = \phi_1 \tilde{z}_{t-1} + \phi_2 \tilde{z}_{t-2} + \varepsilon_t$ ①

auto covariance of order k $\gamma_k = E[(z_{t-k} - \mu)(z_t - \mu)]$

$= E[\tilde{z}_{t-k}(\phi \tilde{z}_{t-1} + \varepsilon_t]$

multiply ① by $\tilde{z}_{t-k}$ and taking expectation

$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}, \quad k \geq 1$ ②

$E[Y_t | Y_{t-1}, Y_{t-2}] = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2}$, dividing by ②

could obtain the relationship between autocorrelation coefficients

$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2), \quad k = 1, 2$ ③

b) Specify the equation ② for $k=1$ $\quad \gamma_{-1} = \gamma_1$

Then $\gamma_1 = \phi_1 + \gamma_0 + \phi_2 \gamma_1$

from ③ $k=1$ $\quad \rho(-1) = \rho(1) \quad \rho(1) = \phi_1 + \phi_2 \rho(1), \; \rho(1) = \frac{\phi_1}{1-\phi_2}$

from ③ $k=2$, plug in $\rho(1) = \frac{\phi_1}{1-\phi_2}$, gives

$\rho(2) = \phi_1 \rho(1) + \phi_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2$

c) From b) $\rho(1) = \frac{\phi_1}{1-\phi_2} = 0.4$ $\qquad \phi_1 = 0.381$

$\rho(2) = \frac{\phi_1^2}{1-\phi_2} + \phi_2 = 0.2 = 0.4\phi_1 + \phi_2$ $\qquad \phi_2 = 0.0476$

$\rho(3) = \phi_1 \rho(2) + \phi_2 (\rho_1) = 0.0952$

Q4 From $\mathbb{E}[Y] = VA^T \gamma$ and $\mathbb{E}[Y] = \theta$

$\hat{\gamma} = [(VA^T)^T \cdot (6^2 V)^{-1} \cdot VA^T]^{-1} \cdot (VA^T)^T (6^2 V)^{-1} Y$

$= (A^T)^{-1} \cdot V^{-1} \cdot Y$