A CALCULATION OF THE UNRAMIFIED INTERTWINING INTEGRAL OF $\mathrm{SU}(3)$

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Let k be a nonarchimedean local field whose residue character is not 2, and E/k be a quadratic unramified extension of k. We let $E = k(j)$ for $j \in \mathcal{O}_E^\times$ and say $j^2 = d$. Since in genera we normalize the absolute value on a nonarchimedean local field F by $\mathrm{Vol}(a\mathcal{O}) = |a|_F \mathrm{Vol}(\mathcal{O})$. Hence $|\cdot|_E \mid_k = |\cdot|_k^2$.

Let $|\cdot|'_E = |\cdot|_E^{1/2}$ be the absolute value on E whose restriction on k is $|\cdot|_k$. Since E is an unramified quadratic extension of k, the ramification index $|E^\times|' = |k^\times|$. Let $q_k = \#(\mathcal{O}_k/\mathfrak{p}_k) = |\varpi|_k$ and $q_E = \#(\mathcal{O}_E/\mathfrak{p}_E) = |\varpi|_E$ where $\varpi = \varpi_E = \varpi_k$ is an uniformizer of k and E. We have $q_E = q_k^2$.

Let $G = SU(3)/k$, with respect to

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Fix a Borel subgroup B which is subgroup of upper triangular matrices and a maximal torus T as the diagonal ones. Hence

$$T(k) = \left\{ \begin{pmatrix} z & & \\ & \bar{z}/z & \\ & & \bar{z}^{-1} \end{pmatrix} \mid z \in E^\times \right\}$$

and

$$N(k) = \left\{ \begin{pmatrix} 1 & x & \frac{x\bar{x}}{2} + jt \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \mid x \in E,\ t \in k \right\}.$$

Take $\mu \in X_{un}(T(k))$ an unramified character. By abuse of notation we regard μ as an unramified character of $k^\times$, i.e.

$$\mu\left(\begin{pmatrix} z & & \\ & \bar{z}/z & \\ & & \bar{z}^{-1} \end{pmatrix}\right) = \mu(z).$$

An unramified character μ is always of the form $\mu(z) = |z|_E^s$ for some $s \in \mathbb{C}$, in particular, it is a function of $|z|_E$.

Let $f_0 \in I(\mu)^K$ be the normalized spherical vector in the unramified principal series representation, i.e.

$$f_0(bk) = \delta_B^{1/2}\mu(b) = |b_{11}|\mu(b) = |b_{11}|_E \mu(b_{11}).$$

The main result in these notes is

Proposition 1.

$$\int_{\bar{N}} f_0(n)dn = \frac{(1 + q_k^{-1}\mu(\varpi))(1 - q_k^{-2}\mu(\varpi))}{1 - \mu(\varpi)^2}.$$

Proof. According to the Iwazawa decomposition $G = BK$, we can write

$$\begin{pmatrix} 1 & & \\ x & 1 & \\ \frac{x\bar{x}}{2} + jt & -\bar{x} & 1 \end{pmatrix} = \begin{pmatrix} z & * & * \\ & \bar{z}/z & * \\ & & \bar{z}^{-1} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ k_1 & k_2 & k_3 \end{pmatrix}.$$

Since $\max\{|k_1|, |k_2|, |k_3|\} = 1$, we have

$$\begin{aligned} |\bar{z}^{-1}|_E &= \max(|\bar{z}^{-1}k_1|_E, |\bar{z}^{-1}k_2|_E, |\bar{z}^{-1}k_3|_E) \\ &= \max(|\frac{x\bar{x}}{2} + jt|_E, |t|_E, 1) \\ &= \max(|x|_E^2, |t|_E, 1). \end{aligned}$$

Here we used the fact that $|\frac{x\bar{x}}{2}+jt|_E=\max(|\frac{x\bar{x}}{2}|_E,|jt|_E)$ and $|2|_E=1$. Hence $|z|_E$ is a function of x and t:

$$|z|_E=\max(|x|_E^2,|t|_E,1)^{-1}.$$

We split the integral

$$\int_{\bar{N}} f_0(n)dn=\int_E\int_k f_0\left(\begin{pmatrix}1 & & \\ x & 1 & \\ \frac{x\bar{x}}{2}+jt & -\bar{x} & 1\end{pmatrix}\right)\mathrm{d}t\mathrm{d}x=\int_E\int_k |z|_E\mu(z)\mathrm{d}t\mathrm{d}x$$

into the following five parts:

(1)

$$\int_{|x|_E\leq 1}\int_{|t|_E\leq 1}1\mathrm{d}t\mathrm{d}x=1;$$

(2)

$$\begin{aligned}&\int_{|x|_E>1}\int_{|t|_E\leq 1}|x|_E^{-2}\mu(x^{-2})\mathrm{d}t\mathrm{d}x\\ =&\frac{q_E-1}{q_E}\sum_{n=1}^{\infty}q_E^{-n}\mu(\varpi)^{2n};\end{aligned}$$

(3)

$$\begin{aligned}&\int_{|x|_E\leq 1}\int_{|t|_E>1}|t|_E^{-1}\mu(t^{-1})\mathrm{d}t\mathrm{d}x\\ =&\frac{q_k-1}{q_k}\sum_{n=1}^{\infty}q_k^{-n}\mu(\varpi)^{n};\end{aligned}$$

(4)

$$\begin{aligned}&\int_{|x|_E\geq|t|_k>1}|x|_E^{-2}\mu(x^{-2})\mathrm{d}t\mathrm{d}x\\ =&\sum_{n=1}^{+\infty}\sum_{m=1}^{2n}q_E^{-2n}\mu(\varpi)^{2n}q_E^{n}\frac{q_E-1}{q_E}q_k^{m}\frac{q_k-1}{q_k}\\ =&\frac{(q_E-1)(q_k-1)}{q_Eq_k}\sum_{n}q_E^{-n}\mu(\varpi)^{2n}\sum_{m=1}^{2n}q_k^{m}\\ =&\frac{q_E-1}{q_E}\sum_{n=1}^{\infty}\mu(\varpi)^{2n}-\frac{q_E-1}{q_E}\sum_{n=1}^{\infty}q_E^{-n}\mu(\varpi)^{2n};\end{aligned}$$

(5)

$$\begin{aligned}&\int_{|t|_k>|x|_E>1}|t|_E^{-1}\mu(t)^{-1}\mathrm{d}t\mathrm{d}x\\ =&\sum_{n=1}^{+\infty}\sum_{m=2n+1}^{+\infty}q_E^{-m}\mu(\varpi)^{m}q_E^{n}\frac{q_E-1}{q_E}q_k^{m}\frac{q_k-1}{q_k}\\ =&\frac{(q_E-1)(q_k-1)}{q_Eq_k}\sum_{n=1}^{\infty}q_E^{n}\sum_{m=2n+1}^{\infty}q_k^{-m}\mu(\varpi)^{m}\\ =&\frac{(q_E-1)(q_k-1)}{q_Eq_k}\frac{q_k^{-1}\mu(\varpi)}{1-q_k^{-1}\mu(\varpi)}\sum_{n=1}^{\infty}\mu(\varpi)^{2n}.\end{aligned}$$

Note that part 2 and some of part 4 cancelled. Denote $\alpha = \mu(\varpi)$ and recall $q_E = q_k^2$. We have

$$\begin{aligned}
(1) &= 1 \\
(2) + (4) &= \frac{(q_k^2 - 1)\alpha^2}{q_k^2(1-\alpha^2)} \\
(3) &= \frac{(q_k - 1)\alpha}{q_k(q_k - \alpha)} \\
(5) &= \frac{(q_k^2 - 1)(q_k - 1)\alpha^3}{q_k^3(q_k - \alpha)(1-\alpha^2)}
\end{aligned}$$

Then

$$\begin{aligned}
(1) + (2) + (4) &= \frac{q_k^2 - \alpha^2}{q_k^2(1-\alpha^2)} \\
(3) + (5) &= \frac{\alpha(q_k + \alpha)(q_k - 1)}{q_k^3(1-\alpha^2)}
\end{aligned}$$

Hence the integral is

$$\frac{(1 + q_k^{-1}\alpha)(1 - q_k^{-2}\alpha)}{1-\alpha^2}.$$

□

Acknowledgement. The key point of this computation is the observation that $|z|_E$ can be realized as a very explicit function of the variables x and t:

$$|z|_E = \max(|x|_E^2, |t|_E, 1)^{-1}.$$

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