

# CALCULATION ON THE UNRAMIFIED INTERTWINING INTEGRAL OF $SU(3)$

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Let  $k$  be a nonarchimedean local field whose residue character is not 2, and  $E/k$  be a quadratic unramified extension of  $k$ . We let  $E = k(j)$  for  $j \in \mathcal{O}_E^\times$  and say  $j^2 = d$ . Since in genera we normalize the absolute value on a nonarchimedean local field  $F$  by  $\text{Vol}(a\mathcal{O}) = |a|_F \text{Vol}(\mathcal{O})$ . Hence  $|\cdot|_E|_k = |\cdot|_k^2$ .

Let  $|\cdot|'_E = |\cdot|_E^{1/2}$  be the absolute value on  $E$  whose restriction on  $k$  is  $|\cdot|_k$ . Since  $E$  is an unramified quadratic extension of  $k$ , the ramification index  $|E^\times|' = |k^\times|$ . Let  $q_k = \#(\mathcal{O}_k/\mathfrak{p}_k) = |\varpi|_k$  and  $q_E = \#(\mathcal{O}_E/\mathfrak{p}_E) = |\varpi|_E$  where  $\varpi = \varpi_E = \varpi_k$  is a uniformizer of  $k$  and  $E$ . We have  $q_E = q_k^2$ .

Let  $G = SU(3)/k$ , with respect to

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Fix a Borel subgroup  $B$  which is subgroup of upper triangular matrices and a maximal torus  $T$  as the diagonal ones. Hence

$$T(k) = \left\{ \begin{pmatrix} z & & \\ & \bar{z}/z & \\ & & \bar{z}^{-1} \end{pmatrix} \mid z \in E^\times \right\}$$

and

$$N(k) = \left\{ \begin{pmatrix} 1 & x & \frac{x\bar{x}}{2} + jt \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \mid x \in E, t \in k \right\}.$$

Take  $\mu \in X_{un}(T(k))$  an unramified character. By abuse of notation we regard  $\mu$  as an unramified character of  $k^\times$ , i.e.

$$\mu \left( \begin{pmatrix} z & & \\ & \bar{z}/z & \\ & & \bar{z}^{-1} \end{pmatrix} \right) = \mu(z).$$

An unramified character  $\mu$  is always of the form  $\mu(z) = |z|_E^s$  for some  $s \in \mathbb{C}$ , in particular, it is a function of  $|z|_E$ .

Let  $f_0 \in I(\mu)^K$  be the normalized spherical vector in the unramified principal series representation, i.e.

$$f_0(bk) = \delta_B^{1/2} \mu(b) = |b_{11}| \mu(b) = |b_{11}|_E \mu(b_{11}).$$

The main result in these notes is

**Proposition 1.**

$$\int_N f_0(n) dn = \frac{(1 + q_k^{-1} \mu(\varpi))(1 - q_k^{-2} \mu(\varpi))}{1 - \mu(\varpi)^2}.$$

*Proof.* According to the Iwazawa decomposition  $G = BK$ , we can write

$$\begin{pmatrix} 1 & & \\ x & 1 & \\ \frac{x\bar{x}}{2} + jt & -\bar{x} & 1 \end{pmatrix} = \begin{pmatrix} z & * & * \\ & \bar{z}/z & * \\ & & \bar{z}^{-1} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ k_1 & k_2 & k_3 \end{pmatrix}.$$

Since  $\max\{|k_1|, |k_2|, |k_3|\} = 1$ , we have

$$\begin{aligned} |\bar{z}^{-1}|_E &= \max(|\bar{z}^{-1}k_1|_E, |\bar{z}^{-1}k_2|_E, |\bar{z}^{-1}k_3|_E) \\ &= \max(|\frac{x\bar{x}}{2} + jt|_E, |t|_E, 1) \\ &= \max(|x|_E^2, |t|_E, 1). \end{aligned}$$

Here we used the fact that  $|\frac{x\bar{x}}{2} + jt|_E = \max(|\frac{x\bar{x}}{2}|_E, |jt|_E)$  and  $|2|_E = 1$ . Hence  $|z|_E$  is a function of  $x$  and  $t$ :

$$|z|_E = \max(|x|_E^2, |t|_E, 1)^{-1}.$$

We split the integral

$$\int_{\bar{N}} f_0(n)dn = \int_E \int_k f_0 \left( \begin{pmatrix} 1 & & \\ x & 1 & \\ \frac{x\bar{x}}{2} + jt & -\bar{x} & 1 \end{pmatrix} \right) dt dx = \int_E \int_k |z|_E \mu(z) dt dx$$

into the following five parts:

(1)

$$\int_{|x|_E \leq 1} \int_{|t|_E \leq 1} 1 dt dx = 1;$$

(2)

$$\begin{aligned} & \int_{|x|_E > 1} \int_{|t|_E \leq 1} |x|_E^{-2} \mu(x^{-2}) dt dx \\ &= \frac{q_E - 1}{q_E} \sum_{n=1}^{\infty} q_E^{-n} \mu(\varpi)^{2n}; \end{aligned}$$

(3)

$$\begin{aligned} & \int_{|x|_E \leq 1} \int_{|t|_E > 1} |t|_E^{-1} \mu(t^{-1}) dt dx \\ &= \frac{q_k - 1}{q_k} \sum_{n=1}^{\infty} q_k^{-n} \mu(\varpi)^n; \end{aligned}$$

(4)

$$\begin{aligned} & \int_{|x|_E \geq |t|_k > 1} |x|_E^{-2} \mu(x^{-2}) dt dx \\ &= \sum_{n=1}^{+\infty} \sum_{m=1}^{2n} q_E^{-2n} \mu(\varpi)^{2n} q_E^n \frac{q_E - 1}{q_E} q_k^m \frac{q_k - 1}{q_k} \\ &= \frac{(q_E - 1)(q_k - 1)}{q_E q_k} \sum_n q_E^{-n} \mu(\varpi)^{2n} \sum_{m=1}^{2n} q_k^m \\ &= \frac{q_E - 1}{q_E} \sum_{n=1}^{\infty} \mu(\varpi)^{2n} - \frac{q_E - 1}{q_E} \sum_{n=1}^{\infty} q_E^{-n} \mu(\varpi)^{2n}; \end{aligned}$$

(5)

$$\begin{aligned} & \int_{|t|_k > |x|_E > 1} |t|_E^{-1} \mu(t)^{-1} dt dx \\ &= \sum_{n=1}^{+\infty} \sum_{m=2n+1}^{+\infty} q_E^{-m} \mu(\varpi)^m q_E^n \frac{q_E - 1}{q_E} q_k^m \frac{q_k - 1}{q_k} \\ &= \frac{(q_E - 1)(q_k - 1)}{q_E q_k} \sum_{n=1}^{\infty} q_E^n \sum_{m=2n+1}^{\infty} q_k^{-m} \mu(\varpi)^m \\ &= \frac{(q_E - 1)(q_k - 1)}{q_E q_k} \frac{q_k^{-1} \mu(\varpi)}{1 - q_k^{-1} \mu(\varpi)} \sum_{n=1}^{\infty} \mu(\varpi)^{2n}. \end{aligned}$$

Note that part 2 and some of part 4 cancelled. Denote  $\alpha = \mu(\varpi)$  and recall  $q_E = q_k^2$ . We have

$$\begin{aligned} (1) &= 1 \\ (2) + (4) &= \frac{(q_k^2 - 1)\alpha^2}{q_k^2(1 - \alpha^2)} \\ (3) &= \frac{(q_k - 1)\alpha}{q_k(q_k - \alpha)} \\ (5) &= \frac{(q_k^2 - 1)(q_k - 1)\alpha^3}{q_k^3(q_k - \alpha)(1 - \alpha^2)} \end{aligned}$$

Then

$$\begin{aligned} (1) + (2) + (4) &= \frac{q_k^2 - \alpha^2}{q_k^2(1 - \alpha^2)} \\ (3) + (5) &= \frac{\alpha(q_k + \alpha)(q_k - 1)}{q_k^3(1 - \alpha^2)} \end{aligned}$$

Hence the integral is

$$\frac{(1 + q_k^{-1}\alpha)(1 - q_k^{-2}\alpha)}{1 - \alpha^2}.$$

□

**Acknowledgement.** The key point of this computation is the observation that  $|z|_E$  can be realized as a very explicit function of the variables  $x$  and  $t$ :

$$|z|_E = \max(|x|_E^2, |t|_E, 1)^{-1}.$$

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