

On the Role of Mobility for Multimessage Gossip

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Abstract—We consider information dissemination in a large n -user wireless network in which k users wish to share a unique message with all other users. Each of the n users only has knowledge of its own contents and state information; this corresponds to a one-sided push-only scenario. The goal is to disseminate all messages efficiently, hopefully achieving an order-optimal spreading rate over unicast wireless random networks. First, we show that a random-push strategy—where a user sends its own or a received packet at random—is order-wise suboptimal in a random geometric graph: specifically, $\Omega(\sqrt{n})$ times slower than optimal spreading. It is known that this gap can be closed if each user has “full” mobility, since this effectively creates a complete graph. We instead consider velocity-constrained mobility where at each time slot the user moves locally using a discrete random walk with velocity $v(n)$ that is much lower than full mobility. We propose a simple two-stage dissemination strategy that alternates between individual message flooding (“self promotion”) and random gossiping. We prove that this scheme achieves a close to optimal spreading rate (within only a logarithmic gap) as long as the velocity is at least $v(n) = \omega(\sqrt{\log n/k})$. The key insight is that the mixing property introduced by the partial mobility helps users to spread in space within a relatively short period compared to the optimal spreading time, which macroscopically mimics message dissemination over a complete graph.

Index Terms—Gossip algorithms, information dissemination, mobility, wireless random networks.

I. INTRODUCTION

In wireless ad hoc or social networks, a variety of scenarios require agents to share their individual information or resources with each other for mutual benefits. A partial list includes file sharing and rumor spreading [2]–[5], distributed computation and parameter estimation [6]–[10], and scheduling and control [11], [12]. Due to the huge centralization overhead and unpredictable dynamics in large networks, it is usually more practical to disseminate information and exchange messages in a decentralized and asynchronous manner to combat unpredictable topology changes and the lack of global state

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information. This motivates the exploration of dissemination strategies that are inherently simple, distributed, and asynchronous while achieving optimal spreading rates.

A. Motivation and Related Work

Among distributed asynchronous algorithms, gossip algorithms are a class of protocols which propagate messages according to rumor-style rules, initially proposed in [13]. Specifically, suppose that there are $k \leq n$ distinct pieces of messages that need to be flooded to all n users in the network. Each agent in each round attempts to communicate with one of its neighbors in a random fashion to disseminate a limited number of messages. There are two types of push-based strategies on selecting which message to be sent in each round: 1) one-sided protocols that are based only on the disseminator’s own current state; and 2) two-sided protocols based on current states of both the sender and the receiver. Encouragingly, a simple uncoded one-sided push-only gossip algorithm with random message selection and peer selection is sufficient for efficient dissemination in some cases like a static complete graph, which achieves a spreading time of $\Theta(k \log n)$,¹ within only a logarithmic gap with respect to the optimal lower limit $\Theta(k)$ [14]–[16]. This type of one-sided gossiping has the advantages of being easily implementable and inherently distributed.

Since each user can receive at most one message in any single slot, it is desirable for a protocol to achieve close to the fastest possible spreading time $\Theta(k)$ (e.g., within a $\text{polylog}(n)$ factor). It has been pointed out, however, that the spreading rate of one-sided random gossip algorithms is frequently constrained by the network geometry, e.g., the conductance of the graph [15], [17]. For instance, for one-sided rumor-style all-to-all spreading (i.e., $k = n$), the completion time T is much lower in a complete graph ($T = O(n \log n)$) than in a ring ($T = \Omega(n^2)$). Intuitively, since each user can only communicate with its nearest neighbors, the geometric constraints in these graphs limit the location distribution of all copies of each message during the evolution process, which largely limits how fast the information can flow across the network. In fact, for message spreading over static wireless networks, one-sided uncoded push-based random gossiping can be quite inefficient: specifically up to $\Omega\left(\sqrt{\frac{n}{\text{poly}(\log n)}}\right)$ times slower than the optimal lower limit $\Theta(k)$ (i.e., a polynomial factor away from the lower bound), as will be shown in Theorem 2.

¹The standard notion $f(n) = \omega(g(n))$ means $\lim_{n \rightarrow \infty} g(n)/f(n) = 0$; $f(n) = o(g(n))$ means $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$; $f(n) = \Omega(g(n))$ means \exists a constant c such that $f(n) \geq cg(n)$; $f(n) = O(g(n))$ means \exists a constant c such that $f(n) \leq cg(n)$; $f(n) = \Theta(g(n))$ means \exists constants c_1 and c_2 such that $c_1g(n) \leq f(n) \leq c_2g(n)$.

Although one-sided random gossiping is not efficient for static wireless networks, it may potentially achieve better performance if each user has some degree of mobility—an intrinsic feature of many wireless and social networks. For instance, full mobility² changes the geometric graph with transmission range $O\left(\sqrt{\frac{\log n}{n}}\right)$ to a complete graph in the unicast scenario. Since random gossiping achieves a spreading time of $\Theta(n \log(n))$ for *all-to-all* spreading over a complete graph [15], [16], this allows near-optimal spreading time to be achieved within a logarithmic factor from the fundamental lower limit $\Theta(n)$. However, how much benefit can be obtained from more realistic mobility—which may be significantly lower than idealized best-case full mobility—is not clear. Most existing results on uncoded random gossiping center on evolutions associated with static homogeneous graph structure or a fixed adjacency matrix, which cannot be readily extended for dynamic topology changes. To the best of our knowledge, the first work to analyze gossiping with mobility was [18], which focused on *energy-efficient* distributed averaging instead of *time-efficient* message propagation. Another line of work by Clementi *et al.* investigate the speed limit for information flooding over Markovian evolving graphs (see, e.g., [19]–[21]), but they did not study the spreading rate under multimesage gossip. Recently, Pettarin *et al.* [22] explored the information spreading over sparse mobile networks with no connected components of size $\Omega(\log n)$, which does not account for the dense (interference-limited) network model we consider in this paper.

For a broad class of graphs that include both static and dynamic graphs, the lower limit on the spreading time can be achieved through random linear coding where a random combination of all messages is transmitted instead of a specific message [23], or by employing a two-sided protocol which always disseminates an innovative message if possible [16]. Specifically, through a unified framework based on dual-space analysis, recent work [24] demonstrated that the optimal all-to-all spreading time $\Theta(n)$ can be achieved for a large class of graphs [24] including complete graphs, geometric graphs, and the results hold in these network models even when the topology is allowed to change dynamically at each time. However, performing random network coding incurs very large computation overhead for each user, and is not always feasible in practice. On the other hand, two-sided protocols inherently require additional feedback that increases communication overhead. Also, the state information of the target may sometimes be unobtainable due to privacy or security concerns. Furthermore, if there are $k \ll \sqrt{n}$ messages that need to be disseminated over a static uncoordinated unicast wireless network or a random geometric graph with transmission radius $\Theta\left(\sqrt{\frac{1}{n} \text{poly}(\log n)}\right)$, neither network coding nor two-sided protocols can approach the lower limit of spreading time $\Theta(k)$. This arises due to

²By full mobility, we mean that the location of the mobile is distributed independently and uniformly random over the entire network over consecutive time steps (i.e., the velocity of the mobile can be “arbitrarily large”). This is sometimes also referred to in literature as the i.i.d. mobility model. In this paper, we consider nodes with “velocity-limited” mobility capability.

the fact that the diameter of the underlying graph with transmission range $\Theta\left(\sqrt{\frac{\text{poly}(\log n)}{n}}\right)$ scales as $\Omega\left(\sqrt{\frac{n}{\text{poly}(\log n)}}\right)$, and hence, each message may need to be relayed through $\Omega\left(\sqrt{\frac{n}{\text{poly}(\log n)}}\right)$ hops in order to reach the node farthest to the source.

Another line of work has studied spreading scaling laws using more sophisticated nongossip schemes over static wireless networks (see, e.g., [25] and [26]). Recently, Resta and Santi [27] began investigating broadcast schemes for mobile networks with a *single static* source constantly propagating new data, while we focus on a different problem with multiple mobile sources each sharing distinct message. Besides, Resta and Santi [27] analyzed how to combat the adverse effect of mobility to ensure the same pipelined broadcasting as in static networks, whereas we are interested in how to take advantage of mobility to overcome the geometric constraints. In fact, with the help of mobility, simply performing random gossiping—which is simpler than most nongossip schemes and does not require additional overhead—is sufficient to achieve optimality.

Finally, we note that gossip algorithms have also been employed and analyzed for other scenarios like distributed averaging, where each node is willing to compute the average of all initial values given at all nodes in a decentralized manner (see, e.g., [6] and [8]). The objective of such distributed consensus is to minimize the total number of computations. It turns out that the convergence rates of both message sharing and distributed averaging are largely dependent on the eigenvalues or, more specifically, the mixing times of the graph matrices associated with the network geometry [8], [15].

B. Problem Definition and Main Modeling Assumptions

Suppose there are n users randomly located over a square of unit area. The task is to disseminate $k \leq n$ distinct messages (each contained in one user initially) among all users. The message spreading can be categorized into two types: 1) *single-message dissemination*: a single user (or $\Theta(1)$ users) wishes to flood its message to all other users; 2) *multimesage dissemination*: a large number k ($k \gg 1$) of users wish to spread individual messages to all other users. We note that distinct messages may not be injected into the network simultaneously. They may arrive in the network (possibly in batches) sequentially, but the arrival time information is unknown in the network.

Our objective is to design a gossip-style one-sided algorithm in the absence of coding, such that it can take advantage of the intrinsic feature of mobility to accelerate dissemination. Only the “push” operation is considered in this paper, i.e., a sender determines which message to transmit solely based on its own current state, and in particular not using the intended receiver’s state. We are interested in identifying the range of the degree of mobility within which our algorithm achieves near-optimal spreading time $O(k \text{poly}(\log n))$ for each message regardless of message arrival patterns. Specifically, our MOBILE PUSH protocol achieves a spreading time $O(k \log^2 n)$ as stated in Theorem 3 for the mobility that is significantly lower than the

idealized full mobility. As an aside, it has been shown in [16] and [23] that with high probability, the completion time for one-sided uncoded random gossip protocol over complete graphs is lower bounded by $\Omega(k \log n)$, which implies that, in general, the logarithmic gap from the universal lower limit $\Theta(k)$ cannot be closed with uncoded one-sided random gossiping.

Our basic network model is as follows. Initially, there are n users uniformly distributed over a unit square. We ignore edge effects so that every node can be viewed as homogeneous. Our models and analysis are mainly based on the context of wireless ad hoc networks, but one can easily apply them to other network scenarios that can be modeled as a random geometric graph of transmission radius $\Theta(\sqrt{\log n/n})$.

Physical-Layer Transmission Model: Each transmitter employs the same amount of power P , and the noise power density is assumed to be η . The path-loss model is used such that node j receives the signal from transmitter i with power $Pr_{ij}^{-\alpha}$, where r_{ij} denotes the Euclidean distance between i and j with $\alpha > 2$ being the path loss exponent. Denote by $\mathcal{T}(t)$ the set of transmitters at time t . We assume that a packet from transmitter i is successfully received by node j at time t if

$$\text{SINR}_{ij}(t) := \frac{Pr_{ij}^{-\alpha}}{\eta + \sum_{k \neq i, k \in \mathcal{T}(t)} Pr_{kj}^{-\alpha}} \geq \beta \quad (1)$$

where $\text{SINR}_{ij}(t)$ is the signal-to-interference-plus-noise ratio (SINR) at j at time t , and β the SINR threshold required for successful reception. For simplicity, we suppose only one fixed-size message or packet can be transmitted for each transmission pair in each time instance.

Suppose that each node can move with velocity $v(n)$ in this mobile network. We provide a precise description of the mobility pattern as follows.

Mobility Model: We use a mobility pattern similar to [28, Sec. VIII], which ensures that at steady state, each user lies in each subsquare with equal probability. Specifically, we divide the entire square into $m := 1/v^2(n)$ subsquares each of area $v^2(n)$ (where $v(n)$ denotes the velocity of the mobile nodes), which forms a $\sqrt{m} \times \sqrt{m}$ discrete torus. At each time instance, every node moves according to a *random walk* on the $\sqrt{m} \times \sqrt{m}$ discrete torus. More precisely, if a node resides in a subsquare $(i, j) \in \{1, \dots, \sqrt{m}\}^2$ at time t , it may choose to stay in (i, j) or move to any of the eight adjacent subsquares each with probability $1/9$ at time $t+1$. If a node is on the edge and is selected to move in an infeasible direction, then it stays in its current subsquare. The position inside the new subsquare is selected uniformly at random. See Fig. 1 for an illustration.

We note that when $v(n) = 1/3 = \Theta(1)$, the pattern reverts to the full mobility model. In this random-walk model, each node moves independently according to a uniform ergodic distribution. In fact, a variety of mobility patterns have been proposed to model mobile networks, including i.i.d. (full) mobility [29], random walk (discrete-time) model [30], [31], and Brownian motion (continuous-time) pattern [32]. For simplicity, we model it as a discrete-time random walk pattern, since it already captures intrinsic features of mobile networks like uncontrolled placement and movement of nodes.

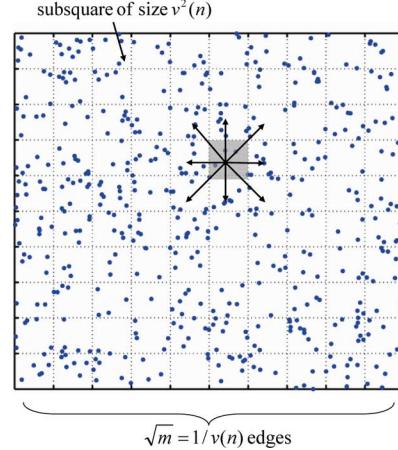


Fig. 1. Unit square is equally divided into $m = 1/v^2(n)$ subsquares. Each node can jump to one of its eight neighboring subsquares or stay in its current subsquare with equal probability $1/9$ at the beginning of each slot.

C. Contributions and Organization

The main contributions of this paper include the following.

- 1) *Single-message dissemination over mobile networks.* We derive an upper bound on the single-message ($k = \Theta(1)$) spreading time using push-only random gossiping (called RANDOM PUSH) in mobile networks. A gain of $\Omega(v(n)\sqrt{n}/\log^2 n)$ in the spreading rate can be obtained compared with static networks, which is, however, still limited by the underlying geometry unless there is full mobility.
- 2) *Multimessage dissemination over static networks.* We develop a lower bound on the multimessage spreading time under RANDOM PUSH protocol over static networks. It turns out that there may exist a gap as large as $\Omega(\frac{\sqrt{n}}{\text{poly}(\log n)})$ between its spreading time and the optimal lower limit $\Theta(k)$. The key intuition is that the copies of each message M_i tend to cluster around the source i at all time instances, which results in capacity loss. This inherently constrains how fast the information can flow across the network.
- 3) *Multimessage dissemination over mobile networks.* We design a *one-sided uncoded* message-selection strategy called MOBILE PUSH that accelerates multimessage spreading ($k = \omega(\log n)$) with mobility. An upper bound on the spreading time is derived, which is the main result of this paper. Once $v(n) = \omega(\sqrt{\frac{\log n}{k}})$ (which is still significantly smaller than full mobility), the near-optimal spreading time $O(k \log^2 n)$ can be achieved with high probability. The underlying intuition is that if the mixing time arising from the mobility model is smaller than the optimal spreading time, the mixing property approximately *uniformizes* the location of all copies of each message, which allows the evolution to mimic the propagation over a complete graph.

The remainder of this paper is organized as follows. In Section II, we describe our unicast physical-layer transmission strategy and two types of message selection strategies, including RANDOM PUSH and MOBILE PUSH. Our main

theorems are stated in Section II as well, with proof ideas illustrated in Section III. Detailed derivation of auxiliary lemmas are deferred to the Appendix.

II. STRATEGIES AND MAIN RESULTS

The strategies and main results of this study are outlined in this section, where only the unicast scenario is considered. The dissemination protocols for wireless networks are a class of scheduling algorithms that can be decoupled into 1) physical-layer *transmission* strategies (link scheduling) and 2) message selection strategies (message scheduling).

One physical-layer transmission strategy and two message selection strategies are described separately, along with the order-wise performance bounds.

A. Strategies

1) *Physical-Layer Transmission Strategy:* In order to achieve efficient spreading, it is natural to resort to a decentralized transmission strategy that supports the order-wise largest number [i.e., $\Theta(n)$] of concurrent successful transmissions per time instance. The following strategy is a candidate that achieves this objective with local communication.

UNICAST Physical-Layer Transmission Strategy:

- At each time slot, each node i is designated as a sender independently with constant probability θ , and a potential receiver otherwise. Here, $\theta < 0.5$ is independent of n and k .
- Every sender i attempts to transmit one message to its *nearest* potential receiver $j(i)$.

This simple “link” scheduling strategy, when combined with appropriate push-based message selection strategies, leads to the near-optimal performance in this paper. We note that the authors in [29], by adopting a slightly different strategy in which θn nodes are randomly designated as senders (as opposed to link-by-link random selection as in our paper), have shown that the success probability for each unicast pair is a constant. Using the same proof as for [29, Th. III-5], we can see (which we omit here) that there exists a constant c such that

$$\mathbb{P}(\text{SINR}_{i,j(i)}(t) > \beta) \geq c \quad (2)$$

holds for our strategy. Here, c is a constant irrespective of n , but may depend on other salient parameters P , α and η . That said, $\Theta(n)$ concurrent transmissions can be successful, which is order-optimal. For ease of analysis and exposition, we further assume that physical-layer success events are *temporally independent* for simplicity of analysis and exposition. Indeed, even accounting for the correlation yields the same scaling results, detailed in Remark 1.

Remark 1: In fact, the physical-layer success events are correlated across different time slots due to our mobility model and transmission strategy. However, we observe that our analysis framework would only require that the transmission success probability at time $t + 1$ is always a constant irrespective of n

given the node locations at time t . To address this concern, we show in Lemma 1 that for any $m < n/(32 \log n)$, the number of nodes N_{A_i} residing in each subsquare A_i is bounded within $[\frac{n}{6m}, \frac{7n}{3m}]$ with probability at least $1 - 2n^{-3}$. Conditional on this high-probability event that $N_{A_i} \in [\frac{n}{6m}, \frac{7n}{3m}]$ with all nodes in each subsquare uniformly located, we can use the same proof as [29, Th. III-5] to show that $\mathbb{P}(\text{SINR}_{i,j(i)}(t) > \beta) \geq c$ holds for some constant c .

Although this physical-layer transmission strategy supports $\Theta(n)$ concurrent local transmissions, it does not tell us how to take advantage of these resources to allow efficient propagation. This will be specified by the message-selection strategy, which potentially determines how each message is propagated and forwarded over the entire network.

2) *Message Selection Strategy:* We now turn to the objective of designing a one-sided message-selection strategy (only based on the transmitter’s current state) that is efficient in the absence of network coding. We are interested in a decentralized strategy in which no user has prior information on the number of distinct messages existing in the network. One common strategy within this class is:

RANDOM PUSH Message Selection Strategy:

- In every time slot: each sender i randomly selects one of the messages it possesses for transmission.

This is a simple gossip algorithm solely based on random message selection, which is surprisingly efficient in many cases like a complete graph. It will be shown later, however, that this simple strategy is inefficient in a static unicast wireless network or a random geometric graph with transmission range $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$.

In order to take advantage of the mobility, we propose the following alternating strategy within this class:

MOBILE PUSH Message Selection Strategy:

- Denote by M_i the message that source i wants to spread, i.e., its own message.
- In every odd time slot: for each sender i , if it has an individual message M_i , then i selects M_i for transmission; otherwise i randomly selects one of the messages it possesses for transmission.
- In every even time slot: each sender i randomly selects one of the messages it has received for transmission.

In the above strategy, each sender alternates between random gossiping and self-promotion. This alternating operation is crucial if we do not know *a priori* the number of distinct messages. Basically, random gossiping enables rapid spreading by taking advantage of all available throughput and provides a nondegenerate approach that ensures an approximately “uniform” evolution for all distinct messages. On the other hand, individual message flooding step plays the role of self-advocating, which guarantees that a sufficiently large number of copies of each message can be forwarded with the assistance of mobility (which is not

true in static networks). This is critical at the initial stage of the evolution.

B. Main Results (Without Proof)

Now we proceed to state our main theorems, each of which characterizes the performance for one distinct scenario. Detailed analysis is deferred to Section III.

1) Single-Message Dissemination in Mobile Networks With RANDOM PUSH: The first theorem states the limited benefits of mobility on the spreading rate for single-message spreading when RANDOM PUSH is employed. We note that MOBILE PUSH reverts to RANDOM PUSH for single-message dissemination and, hence, has the same spreading time.

Theorem 1: Assume that the velocity obeys $v(n) > \sqrt{\frac{32 \log n}{n}}$, and that the number of distinct messages obeys $k = \Theta(1)$. RANDOM PUSH message selection strategy is assumed to be employed in the unicast scenario. Denote by $T_{\text{sp}}^{\text{uc}}(i)$ the time taken for all users to receive message M_i after M_i is injected into the network, then with probability at least $1 - n^{-2}$ we have

$$\forall i, \quad T_{\text{sp}}^{\text{uc}}(i) = O\left(\frac{\log n}{v(n)}\right) \quad \text{and} \quad T_{\text{sp}}^{\text{uc}}(i) = \Omega\left(\frac{1}{v(n)}\right). \quad (3)$$

Since the single-message flooding time is $\Omega(\sqrt{n}/\log n)$ under RANDOM PUSH over static wireless networks or random geometric graphs of radius $\Theta(\sqrt{\log n/n})$ [17], the gain in dissemination rate due to mobility is $\Omega(v(n)\sqrt{n}/\log^2 n)$. When the mobility is large enough (e.g., $v(n) = \omega\left(\sqrt{\frac{\log n}{n}}\right)$), it plays the role of increasing the transmission radius, thus resulting in the speedup. It can be easily verified, however, that the universal lower bound on the spreading time is $\Theta(\log n)$, which can only be achieved in the presence of full mobility. To summarize, while the speedup $\Omega(v(n)\sqrt{n}/\log^2 n)$ can be achieved in the regime $\sqrt{\frac{32 \log n}{n}} \leq v(n) \leq \Theta(1)$, RANDOM PUSH cannot achieve near-optimal spreading time $O(\log n)$ for single-message dissemination unless full mobility is present.

2) Multimessage Dissemination in Static Networks With RANDOM PUSH: Now we turn to multimessage spreading over static networks with uncoded random gossiping. Our analysis is developed for the regime where there are k distinct messages that satisfies $k = \omega(\text{polylog}(n))$, which subsumes most multispread cases of interest. For its complement regime where $k = O(\text{polylog}(n))$, an apparent lower bound $\Omega(\sqrt{n}/\text{polylog}(n))$ on the spreading time can be obtained by observing that the diameter of the underlying graph with transmission radius $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$ is at least $\Omega(\sqrt{n}/\text{polylog}(n))$. This immediately indicates a gap $\Omega(\sqrt{n}/\text{polylog}(n))$ between the spreading time and the lower limit $k = O(\text{polylog}(n))$.

The spreading time in the regime $k = \omega(\log(n))$ is formally stated in Theorem 2, which implies that simple RANDOM GOSSIP is inefficient in static wireless networks, under a message injection scenario where users start message dissemination sequentially. The setting is as follows: $(k-1)$ of the sources inject their messages into the network at some

time prior to the k th source. At a future time when each user in the network has at least $w = \omega(\text{poly}(\log n))$ messages, the k th message (denoted by M^*) is injected into the network. This pattern occurs, for example, when a new message is injected into the network much later than other messages, and hence, all other messages have been spread to a large number of users. We will show that without mobility, the spreading time under MOBILE PUSH in these scenarios is at least of the same order as that under RANDOM PUSH,³ which is a polynomial factor away from the universal lower limit $\Theta(k)$. In fact, the individual message flooding operation of MOBILE PUSH does not accelerate spreading since each source has only $O(\text{poly}(\log n))$ potential neighbors to communicate.

The main objective of analyzing the above scenario is to uncover the fact that uncoded one-sided random gossiping fails to achieve near-optimal spreading for a large number of message injection scenarios over static networks. This is in contrast to mobile networks, where protocols like MOBILE PUSH with the assistance of mobility is robust to all initial message injection patterns and can always achieve near-optimal spreading, as will be shown later.

Theorem 2: Assume that a new message M^* arrives in a static network later than other $k-1$ messages, and suppose that M^* is first injected into the network from a state such that each node has received at least $w = \omega(\log n)$ distinct messages. Denote by T^* the time until every user receives M^* using RANDOM PUSH; then for any constant $\epsilon > 0$ we have

$$T^* > w^{1-\epsilon} \sqrt{\frac{n}{128 \log n}} \quad (4)$$

with probability exceeding $1 - n^{-2}$.

Remark 2: Our main goal is to characterize the spreading inefficiency when each node has received a few messages, which becomes most significant when each has received $\Theta(k)$ messages. In contrast, when only a constant number of messages are available at each user, the evolution can be fairly fast since the piece selection has not yet become a bottleneck. Hence, we consider $w = \omega(\text{polylog}(n))$, which captures most of the spreading-inefficient regime ($\omega(\text{polylog}(n)) \leq w \leq \Theta(k)$). The spreading can be quite slow for various message-injection process over static networks, but can always be completed within $O(k \log^2 n)$ with the assistance of mobility $v(n) = \omega\left(\sqrt{\log n/k}\right)$ regardless of the message-injection process, as will be shown in Theorem 3.

Theorem 2 implies that if M^* is injected into the network when each user contains $\omega\left(\frac{k^{1+2\epsilon}}{\sqrt{n} \text{poly}(\log n)}\right)$ messages for any $\epsilon > 0$, then RANDOM PUSH is unable to approach the fastest possible spreading time $\Theta(k)$. In particular, if the message is first transmitted when each user contains $\Omega(k/\text{poly}(\log n))$ messages, then at least $\Omega(k^{1-\epsilon}\sqrt{n}/\text{poly}(\log n))$ time slots are required to complete spreading. Since ϵ can be arbitrarily small, there may exist a gap as large as $\Omega\left(\frac{\sqrt{n}}{\text{poly}(\log n)}\right)$ between the lower limit $\Theta(k)$ and the spreading time using RANDOM

³We note that this section is devoted to showing the spreading inefficiency under two uncoded one-sided push-only protocols. It has recently been shown in [24] that a network coding approach can allow the optimal spreading time $\Theta(k)$ to be achieved over static wireless networks or random geometric graphs.

PUSH. The reason is that as each user receives many distinct messages, a bottleneck of spreading rate arises due to the low piece-selection probability assigned for each message. A number of transmissions are wasted due to the blindness of the one-sided message selection, which results in capacity loss and hence largely constrain how efficient information can flow across the network. The copies of each message tend to cluster around the source—the density of the copies decays rapidly with the distance to the source. Such inefficiency becomes more severe as the evolution proceeds, because each user will assign an increasingly smaller piece-selection probability for each message.

3) *Multimessage Dissemination in Mobile Networks With MOBILE PUSH:* Although the near-optimal spreading time $O(\text{polylog}(n))$ for single-message dissemination can only be achieved when there is near-full mobility $v(n) = \Omega(1/\text{polylog}(n))$, a limited degree of velocity turns out to be remarkably helpful in the multimessage case as stated in the following theorem.

Theorem 3: Assume that the velocity obeys $v(n) = \omega\left(\sqrt{\frac{\log n}{k}}\right)$, where the number of distinct messages obeys $k = \omega(\text{poly}(\log n))$. MOBILE PUSH message selection strategy is employed along with unicast transmission strategy. Let $T_{\text{mp}}^{\text{uc}}(i)$ be the time taken for all users to receive message M_i after M_i is first injected into the network; then with probability at least $1 - n^{-2}$, we have

$$\forall i, \quad T_{\text{mp}}^{\text{uc}}(i) = O(k \log^2 n). \quad (5)$$

Since each node can receive at most one message in each time slot, the spreading time is lower bounded by $\Theta(k)$ for any graph. Thus, our strategy with limited velocity spreads the information essentially as fast as possible. Intuitively, this is due to the fact that the velocity (even with restricted magnitude) helps *uniformize* the locations of all copies of each message, which significantly increases the conductance of the underlying graph in each slot. Although the velocity is significantly smaller than full mobility (which simply results in a complete graph), the relatively low mixing time helps to approximately achieve the same objective of uniformization. On the other hand, the low spreading rates in static networks arise from the fact that the copies of each message tend to cluster around the source at any time instant, which decreases the number of flows going toward new users without this message.

Remark 3: Note that there is a $O(\log^2 n)$ gap between this spreading time and the lower limit $\Theta(k)$. We conjecture that $\Theta(k \log n)$ is the exact order of the spreading time, where the logarithmic gap arises from the blindness of peer and piece selection. A gap of $\Theta(\log n)$ was shown to be indispensable for complete graphs when one-sided random push is used [23]. Since the mobility model simply mimics the evolution in complete graphs, a logarithmic gap appears to be unavoidable when using our algorithm. Nevertheless, we conjecture that with a finer tuning of the concentration of measure techniques, the current gap $O(\log^2 n)$ can be narrowed to $\Theta(\log n)$. See Remark 5.

III. PROOFS AND DISCUSSIONS OF MAIN RESULTS

The proofs of Theorem 1–3 are provided in this section. Before continuing, we would like to state some preliminaries regarding the mixing time of a random walk on a 2-D grid, some related concentration results, and a formal definition of conductance.

A. Preliminaries

1) *Mixing Time:* Define the probability of a typical node moving to subsquare A_i at time t as $\pi_i(t)$ starting from any subsquare, and denote by π_i the steady-state probability of a node residing in subsquare A_i . Define the *mixing time* of our random walk mobility model as $T_{\text{mix}}(\epsilon) := \min\{t : |\pi_i(t) - \pi_i| \leq \epsilon, \forall i\}$, which characterizes the time until the underlying Markov chain is close to its stationary distribution. It is well known that the mixing time of a random walk on a grid satisfies (e.g., see [33, Corollary 2.3] and [33, Appendix C]):

$$T_{\text{mix}}(\epsilon) \leq \hat{c}m \left(\log \frac{1}{\epsilon} + \log n \right) \quad (6)$$

for some constant \hat{c} . We take $\epsilon = n^{-10}$ throughout this paper, so $T_{\text{mix}}(\epsilon) \leq c_0 m \log n$ holds with $c_0 = 10\hat{c}$. After $c_0 m \log n$ amount of time slots, all the nodes will reside in any subsquare almost uniformly likely. In fact, n^{-10} is very conservative and a much larger ϵ suffices for our purpose, but this gives us a good sense of the sharpness of the mixing time order. See [34, Sec. 6] for detailed characterization of the mixing time of random walks on graphs.

2) *Concentration Results:* The following concentration result is also useful for our analysis.

Lemma 1: Assume that b nodes are thrown independently into m subsquares. Suppose for any subsquare A_i , the probability q_{A_i} of each node being thrown to A_i is bounded as

$$\left| q_{A_i} - \frac{1}{m} \right| \leq \frac{1}{3m}. \quad (7)$$

Then, for any constant ϵ , the number of nodes $N_{A_i}(t)$ falling in any subsquare A_i ($1 \leq i \leq m < n$) at any time $t \in [1, n^2]$ satisfies

a) if $b = \Theta(m \log n)$ and $b > 32m \log n$, then

$$\mathbb{P}\left(\forall(i, t) : \frac{b}{6m} \leq N_{A(i)}(t) \leq \frac{7b}{3m}\right) \geq 1 - \frac{2}{n^3};$$

b) if $b = \omega(m \log n)$, then

$$\mathbb{P}\left(\forall(i, t) : \frac{(\frac{2}{3} - \epsilon)b}{m} \leq N_{A(i)}(t) \leq \frac{(\frac{4}{3} + \epsilon)b}{m}\right) \geq 1 - \frac{2}{n^3}.$$

Proof: See Appendix A. ■

This implies that the number of nodes residing in each subsquare at each time of interest will be reasonably close to the true mean. This concentration result follows from standard Chernoff bounds [35, Appendix A] and forms the basis for our analysis.

3) *Conductance:* Conductance is an isoperimetric measure that characterizes the expansion property of the underlying

graph. Consider an irreducible reversible transition matrix P with its states represented by V ($|V| = n$). Assume that the stationary distribution is uniform over all states. In spectral graph theory, the conductance associated with P is [34]

$$\Phi(P) = \inf_{B \subset V, |B| \leq \frac{n}{2}} \frac{\sum_{i \in B, j \in B^c} P_{ij}}{|B|} \quad (8)$$

which characterizes how easy the probability flow can cross from one subset of nodes to its complement. If the transition matrix P is chosen such that

$$P_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } j \in \text{neighbor}(i) \\ 0, & \text{else} \end{cases} \quad (9)$$

where d_i denotes the degree of vertex i , then the conductance associated with random geometric graph with radius $r(n)$ obeys $\Phi(P) = \Theta(r(n))$ [33], where $r(n)$ is the transmission radius.

B. Single-Message Dissemination in Mobile Networks

We only briefly sketch the proof for Theorem 1 in this paper, since the approach is somewhat standard (see [15]). Lemma 1 implies that with high probability, the number of nodes residing in each subsquare will exhibit sharp concentration around the mean n/m once $n > 32m \log n$. For each message M_i , denote by $N_i(t)$ the number of users containing M_i at time t . The spreading process is divided into two phases: $1 \leq N_i(t) \leq n/2$ and $n/2 < N_i(t) \leq n$.

As an aside, if we denote by p_{lj} the probability that l successfully transmit to j in the next time slot conditional on the event that there are $\Theta(n/m)$ users residing in each subsquare, then if $m < \frac{n}{32 \log n}$, one has

$$p_{lj} = \begin{cases} \Theta\left(\frac{m}{n}\right), & \text{if } l \text{ and } j \text{ can move to the same} \\ & \text{subsquare in the next time slot} \\ 0, & \text{else.} \end{cases}$$

Concentration results imply that for any given time t and any user l , there are $\Theta\left(\frac{n}{m}\right)$ users that can lie within the same subsquare as l with high probability. On the other hand, for a geometric random graph with $r(n) = \sqrt{1/m} = v(n)$, the transition matrix defined in (9) satisfies $P_{lj} = \Theta\left(\frac{m}{n}\right)$ for all j inside the transmission range of l (where there are with high probability $\Theta\left(\frac{n}{m}\right)$ users inside the transmission range). Therefore, if we define the conductance related to this mobility model as $\Phi(n) = \inf_{B \subset V, |B| \leq \frac{n}{2}} \frac{\sum_{i \in B, j \in B^c} p_{lj}}{|B|}$, then this is order-wise equivalent to the conductance of the geometric random graph with $r(n) = v(n)$, and hence $\Phi(n) = \Theta(r(n)) = \Theta(v(n))$.

1) *Phase 1*: Look at the beginning of each slot, all senders containing M_i may transmit it to any nodes in the nine subsquares equally likely with constant probability by the end of this slot. Using the same argument as [15], one can see that the expected increment of $N_i(t)$ by the end of this slot can be lower bounded by the number of nodes $N_i(t)$ times the conductance related to the mobility model $\Phi(n) = \Theta(v(n))$ defined above. We can thus conclude that before $N(t) = n/2$

$$\mathbb{E}(N_i(t+1) - N_i(t) | N_i(t)) \geq b_1 N_i(t) \Phi(n) = \tilde{b}_1 N_i(t) v(n)$$

holds for some constant b_1 and \tilde{b}_1 . Following the same martingale-based proof technique used for single-message dissemination in [15, Th. 3.1], we can prove that for any $\epsilon > 0$, the time $T_{i1}(\epsilon)$ by which $N_i(t) \geq n/2$ holds with probability at least $1 - \epsilon$ can be bounded by

$$T_{i1}(\epsilon) = O\left(\frac{\log n + \log \epsilon^{-1}}{\Phi(n)}\right) = O\left(\frac{\log n + \log \epsilon^{-1}}{v(n)}\right). \quad (10)$$

Take $\epsilon = n^{-3}$; then $T_{i1}(\epsilon)$ is bounded by $O\left(\frac{\log n}{v(n)}\right)$ with probability at least $1 - n^{-3}$.

2) *Phase 2*: This phase starts at T_{i1} and ends when $N_i(t) = n$. Since the roles of j and l are symmetric, the probability of j having l as the nearest neighbor is equal to the probability of l having j as the nearest neighbor. This further yields $p_{lj} = p_{jl}$ by observing that the transmission success probability is the same for each designated pair. Therefore, we can see

$$\begin{aligned} & \mathbb{E}(N_i(t+1) - N_i(t) | N_i(t)) \\ & \geq b_2 \sum_{l \in \mathcal{N}_i(t), j \notin \mathcal{N}_i(t)} p_{lj} \end{aligned} \quad (11)$$

$$\begin{aligned} & = b_2 (n - N_i(t)) \frac{\sum_{j \notin \mathcal{N}_i(t), l \in \mathcal{N}_i(t)} p_{jl}}{n - N_i(t)} \\ & \geq b_2 (n - N_i(t)) \Phi(n). \end{aligned} \quad (12)$$

Denote by T_{i2} the duration of Phase 2. We can follow the same machinery in [17] to see

$$T_{i2} = O\left(\frac{\log n}{\Phi(n)}\right) = O\left(\frac{\log n}{v(n)}\right) \quad (13)$$

with probability exceeding $1 - n^{-3}$.

By combining the duration of Phases 1 and 2 and applying the union bound over all distinct messages, we can see that $T_{sp}^{uc}(i) = O\left(\frac{\log n}{v(n)}\right)$ holds for all distinct messages with high probability. When $v(n) > \sqrt{\frac{32 \log n}{n}}$, at any time instance, each node can only transmit a message to nodes at a distance at most $O(v(n))$, and hence, it will take at least $\Omega(1/v(n))$ time instances for M_i to be relayed to the node farthest from i at time 0, which is a universal lower bound on the spreading time. Therefore, $T_{sp}^{uc}(i)$ is only a logarithmic factor away from the fundamental lower bound $\Omega\left(\frac{1}{v(n)}\right)$. It can be seen that the bottleneck of this upper bound lies in the conductance of the underlying random network. When $v(n) = \omega\left(\sqrt{\frac{\log n}{n}}\right)$, mobility accelerates spreading by increasing the conductance. The mixing time duration is much larger than the spreading time, which implies that the copies of each message is still spatially constrained in a fixed region (typically clustering around the source) without being spread out over the entire square. We note that with full mobility, the spreading time in single-message dissemination case achieves the universal lower bound $\Theta(\log n)$, which is much smaller than that with a limited degree of velocity.

C. Multimessage Spreading in Static Networks With RANDOM PUSH

The proof idea of Theorem 2 is sketched in this subsection.

1) *Lower Bound on the Spreading Time*: To begin our analysis, we partition the entire unit square as follows:

- The unit square is divided into a set of nonoverlapping **tiles** $\{B_j\}$ each of side length $\sqrt{32 \log n / n}$ as illustrated in Fig. 2 (Note that this is a different partition from sub-squares $\{A_j\}$ resulting from the mobility model).
- The above partition also allows us to slice the network area into **vertical strips** each of width $\sqrt{32 \log n / n}$ and length 1. Label the vertical strips as $\{V_l\}$ ($1 \leq l \leq \sqrt{n / (32 \log n)}$) in increasing order from left to right, and denote by $N_{V_l}(t)$ and $\mathcal{N}_{V_l}(t)$ the number and the set of nodes in V_l that contains M^* by time t .
- The vertical strips are further grouped into **vertical blocks** $\{V_1^b\}$ each containing $\log n$ strips, i.e., $V_1^b = \{V_l : (j-1)\log n + 1 \leq l \leq j\log n\}$.

Remark 4: Since each tile has an area of $32 \log n / n$, concentration results (see Lemma 1) imply that there are $\Theta(\log n)$ nodes residing in each tile with high probability. Since each sender only attempts to transmit to its nearest receiver, then with high probability the communication process occurs only to nodes within the same tile or in adjacent tiles.

Without loss of generality, we assume that the source of M^* resides in the *leftmost* vertical strip V_1 . We aim at counting the time taken for M^* to cross each vertical block horizontally. In order to decouple the counting for different vertical blocks, we construct a new spreading process \mathcal{G}^* as follows.

Spreading in Process \mathcal{G}^* :

- At $t = 0$, distribute M^* to all nodes residing in vertical strip V_1 .
- Each node adopts RANDOM PUSH as the message selection strategy
- Define $T_l^b = \min \{t : N_{V_l^b}(t) > 0\}$ as the first time that M^* reaches vertical block V_l^b . For all $l \geq 2$, distribute M^* to all nodes residing in either V_{l-1}^b or the leftmost strip of V_l^b at time $t = T_l^b$.

It can be verified using a coupling approach that \mathcal{G}^* evolves stochastically faster than the true process. By enforcing mandatory dissemination at every $t = T_l^b$, we enable separate counting for spreading time in different blocks—the spreading in V_{l+1}^b after T_{l+1}^b is independent of what has happened in V_l^b . Roughly speaking, since there are $\sqrt{\frac{n}{32 \log^3 n}}$ blocks, the spreading time over the entire region is $\Theta\left(\sqrt{\frac{n}{\text{poly}(\log n)}}\right)$ times the spreading time over a typical block.

We perform a single-block analysis in the following lemma, and characterize the rate of propagation across different strips over a typical block in \mathcal{G}^* . By Property 3 of \mathcal{G}^* , the time taken to cross each typical block is equivalent to the crossing time in V_1^b . Specifically, we demonstrate that the time taken for a message to

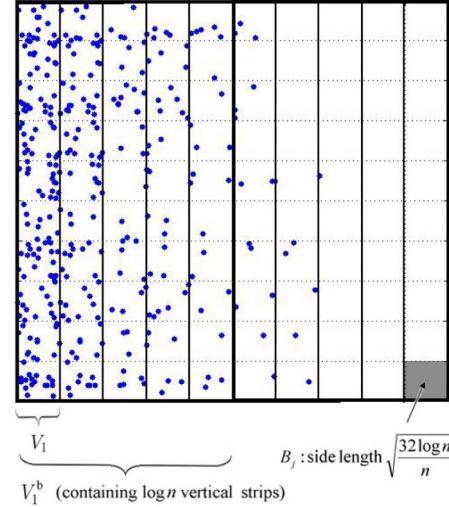


Fig. 2. Plot illustrates that the number $N_{V_l}(t_X)$ of nodes containing M^* in vertical strip V_l by time t_X is decaying rapidly with geometric rate.

cross a single block is at least $\Omega(w^{1-\epsilon})$ for any positive ϵ . Since the crossing time for each block in \mathcal{G}^* is statistically equivalent, this single-block analysis further allows us to lower bound the crossing time for the entire region.

Lemma 2: Consider the spreading of M^* over V_1^b in the original process \mathcal{G} . Suppose each node contains at least $w = \omega(\text{poly}(\log n))$ messages initially. Define $t_X := w^{1-\epsilon}$, and define $l^* = \min \{l : N_{V_l}(t_X) = O(w^\epsilon \log n)\}$. Then, with probability at least $1 - 3n^{-3}$, we have

- $l^* \leq \frac{1}{2} \log n$;
- $\forall s (1 \leq s < l^*)$, there exists a constant c_{31} such that

$$N_{V_s}(t_X) \leq \left(\frac{\log n}{w^\epsilon}\right)^{s-1} (c_{31} \sqrt{n} \log n). \quad (14)$$

$$\text{c)} \quad N_{V_{\frac{1}{2} \log n}}(t_X) \leq \log^2 n.$$

Sketch of Proof of Lemma 2: The proof makes use of the fixed-point type of argument. The detailed derivation is deferred to Appendix B. ■

The key observation from the above lemma is that the number of nodes in V_s containing M^* is decaying rapidly as s increases, which is illustrated in Fig. 2. We also observe that $N_{V_l}(t_X)$ decreases to $O(\log^2 n)$ before $V_{\frac{1}{2} \log n}$.

While Lemma 2 determines the number of copies of M^* inside $V_1 \sim V_{\frac{1}{2} \log n}$ by time t_X , it does not indicate whether M^* has crossed the block V_1^b or not by t_X . In order to characterize the crossing time, we still need to examine the evolution in strips $V_{\frac{1}{2} \log n + 1} \sim V_{\log n}$. Since communication occurs only between adjacent strips or within the same strip, all copies lying to the right of $V_{\frac{1}{2} \log n}$ must be relayed via a path that starts from V_1 and passes through $V_{\frac{1}{2} \log n}$. That said, all copies in $V_{\frac{1}{2} \log n + 1} \sim V_{\log n}$ by time t_X must have been forwarded (possibly in a multihop manner) via some nodes having received M^* by t_X . If we denote by $\mathcal{N}_{V_{\log n/2}}^*(t_X)$ the set of nodes in $V_{\log n/2}$ having received M^* by t_X in \mathcal{G}^* , then we can construct a process $\bar{\mathcal{G}}$ in which all nodes in $\mathcal{N}_{V_{\log n/2}}^*(t_X)$ receive

M^* from the very beginning ($t = 0$), and hence, the evolution in $\bar{\mathcal{G}}$ can be stochastically faster than \mathcal{G}^* by time t_X .

Spreading in Process $\bar{\mathcal{G}}$:

- 1) Initialize (a): at $t = 0$, for all $v \in \mathcal{N}_{V_{\log n/2}}^*(t_X)$, distribute M^* to all nodes residing in the same tile as v
- 2) Initialize (b): at $t = 0$, if v_1 and v_2 are two nodes in $\mathcal{N}_{V_{\log n/2}}^*(t_X)$ such that v_1 and v_2 are less than $\log n$ tiles away from each other, then distribute M^* to all nodes in all tiles between v_1 and v_2 in $V_{\log n/2}$. After this step, tiles that contain M^* forms a set of *nonoverlapping* substrips.
- 3) By time $t_X = w^{1-\epsilon}$, the evolution to the left of $V_{\log n/2}$ occurs exactly the same as in \mathcal{G}^* .
- 4) At the first time slot in which any node in the above substrips selects M^* for transmission, distribute M^* to all nodes in all tiles adjacent to any of these substrips. In other words, we expand all substrips outwards by one tile.
- 5) Repeat from 4) but consider the new set of substrips after expansion.

By our construction of $\bar{\mathcal{G}}$, the evolution to the left of $V_{\frac{1}{2}\log n}$ stays completely the same as that in \mathcal{G}^* , and hence, there is no *successful* transmission of M^* between nodes in $V_1 \sim V_{\frac{1}{2}\log n-1}$ and those in $V_{\log n/2}$ but not contained in $\mathcal{N}_{V_{\log n/2}}^*(t_X)$. Therefore, in our new process $\bar{\mathcal{G}}$, the evolution to the left of $V_{\frac{1}{2}\log n}$ by time t_X is decoupled with that to the right of $V_{\frac{1}{2}\log n}$ by time t_X .

Our objective is to examine how likely $T_2^* = \min\{t : M^* \text{ reaches } V_{\log n} \text{ in } \bar{\mathcal{G}}\}$ is smaller than t_X . It can be observed that any two substrips would never merge before T_2^* since they are initially spaced at least $\log n$ tiles from each other. This allows us to treat them separately. Specifically, the following lemma provides a lower bound on T_2^* by studying the process $\bar{\mathcal{G}}$.

Lemma 3: Suppose $t_X = w^{1-\epsilon}$ and each node contains at least w distinct messages since $t = 0$. Then, we have

$$\mathbb{P}(T_2^* \leq t_X) \leq \frac{4}{n^3}. \quad (15)$$

Proof: See Appendix C. ■

This lemma indicates that M^* is unable to cross V_1^b by time $t_X = w^{1-\epsilon}$ in $\bar{\mathcal{G}}$. Since $\bar{\mathcal{G}}$ is stochastically faster than the original process, the time taken for M^* to cross a vertical block in the original process exceeds t_X with high probability. In other words, the number of nodes having received M^* by t_X vanishes within no more than $O(\log n)$ further strips.

Since there are $\Theta(\sqrt{n/\text{poly}(\log n)})$ vertical blocks in total, and crossing each block takes at least $\Omega(w^{1-\epsilon})$ time slots, the time taken for M^* to cross all blocks can thus be bounded as

$$T^* = \Omega\left(w^{1-\epsilon} \sqrt{\frac{n}{\text{poly}(\log n)}}\right) \quad (16)$$

with high probability.

2) *Discussion:* Theorem 2 implies that if a message M^* is injected into the network when each user contains $\Omega(k/\text{poly}(\log n))$ messages, the spreading time for M^* is

$\Omega(k^{1-\epsilon} \sqrt{n/\text{poly}(\log n)})$ for arbitrarily small ϵ . That said, there exists a gap as large as $\Omega(\sqrt{n/\text{poly}(\log n)})$ from optimality. The tightness of this lower bound can be verified by deriving an *upper bound* using the conductance-based approach as follows.

We observe that the message selection probability for M^* is always lower bounded by $1/k$. Hence, we can couple a new process adopting a different message-selection strategy such that a transmitter containing M^* selects it for transmission with *state-independent* probability $1/k$ at each time. It can be verified that this process evolves stochastically slower than the original one. The conductance associated with the new evolution for M^* is $\Phi(n) = \frac{1}{k} \Theta(r(n)) = O\left(\frac{1}{k} \sqrt{\frac{\log n}{n}}\right)$. Applying similar analysis as in [17] yields

$$T_i = O\left(\frac{\text{poly}(\log n)}{\Phi(n)}\right) = O(k\sqrt{n}\text{poly}(\log n)) \quad (17)$$

with probability exceeding $1 - n^{-2}$, which is only a poly-logarithmic gap from the lower bound we derived.

The tightness of this upper bound implies that the propagation bottleneck is captured by the conductance-based measure—the copies of each message tend to cluster around the source at any time instead of spreading out (see Fig. 4). That said, only the nodes lying around the boundary are likely to forward the message to new users. Capacity loss occurs to the users inside the cluster since many transmissions occur to receivers who have already received the message and are thus wasted. This graph expansion bottleneck can be overcome with the assistance of mobility.

D. Multimessage Spreading in Mobile Networks With MOBILE PUSH

The proof of Theorem 3 is sketched in this section. We divide the entire evolution process into three phases. The duration of Phase 1 is chosen to allow each message to be forwarded to a sufficiently large number of users. After this initial phase (which acts to “seed” the network with a sufficient number of all the messages), random gossiping ensures the spread of all messages to all nodes.

1) *Phase 1:* This phase accounts for the first $c_6 \left(c_0 m \log n + \frac{m}{c_h \log n} \right) \log^2 n = \Theta(m \log^3 n)$ time slots, where c_0 , c_6 , and c_h are constants independent of m and n . At the end of this phase, each message will be contained in at least $32m \log n = \Theta(m \log n)$ nodes. The time intended for this phase largely exceeds the mixing time of the random walk mobility model, which enables these copies to “uniformly” spread out over space.

We are interested in counting how many nodes will contain a particular message M_i by the end of Phase 1. Instead of counting all potential multihop relaying of M_i , we only look at the set of nodes that receive M_i directly from source i in *odd* slots. This approach provides a crude lower bound on $N_i(t)$ at the end of Phase 1, but it suffices for our purpose.

Consider the following scenario: at time t_1 , node i attempts to transmit its message M_i to receiver j . Denote by $Z_i(t)$ ($1 \leq i \leq$

n) the subsquare position of node i , and define the relative coordinate $Z_{ij}(t) := Z_i(t) - Z_j(t)$. Clearly, $Z_{ij}(t)$ forms another 2-D random walk on a discrete torus. For notational convenience, we introduce the notation $\mathbb{P}_0(\cdot) \stackrel{\Delta}{=} \mathbb{P}(\cdot | Z_{ij}(0) = (0, 0))$ to denote the *conditional measure* given $Z_{ij}(0) = (0, 0)$. The following lemma characterizes the hitting time of this random walk to the boundary.

Lemma 4: Consider the symmetric random walk $Z_{ij}(t)$ defined above. Denote the set \mathcal{A}_{bd} of subsquares on the boundary as

$$\mathcal{A}_{\text{bd}} = \left\{ A_i \mid A_i = \left(\pm \frac{\sqrt{m}}{2}, j \right) \text{ or } A_i = \left(j, \pm \frac{\sqrt{m}}{2} \right), \forall j \right\}$$

and define the first hitting time to the boundary as $T_{\text{hit}} = \min \{t : Z_{ij}(t) \in \mathcal{A}_{\text{bd}}\}$; then, there is a constant c_h such that

$$\mathbb{P}_0 \left(T_{\text{hit}} < \frac{m}{c_h \log n} \right) \leq \frac{1}{n^4}. \quad (18)$$

Proof: See Appendix D. \blacksquare

Besides, the following lemma provides an upper bound on the expected number of time slots by time t during which the walk returns to $(0, 0)$.

Lemma 5: For the random walk $Z_{ij}(t)$ defined above, there exist constants c_3 and c_h such that for any $t < \frac{m}{c_h \log n}$:

$$\mathbb{E} \left(\sum_{k=1}^t \mathbb{1}(Z_{ij}(k) = (0, 0)) \mid Z_{ij}(0) = (0, 0) \right) \leq c_3 \log t. \quad (19)$$

Here, $\mathbb{1}(\cdot)$ denotes the indicator function.

Sketch of Proof of Lemma 5: Denote by \mathcal{H}_{bd} the event that $Z_{ij}(t)$ hits the boundary \mathcal{A}_{bd} (as defined in Lemma 4) before $t = m/(c_h \log n)$. Conditional on $Z_{ij}(0) = (0, 0)$, the probability $q_{ij}^0(t)$ of $Z_{ij}(t)$ returning to $(0, 0)$ at time t can then be bounded as

$$q_{ij}^0(t) \leq \mathbb{P}_0(\mathcal{H}_{\text{bd}}) + \mathbb{P}_0(Z_{ij}(t) = (0, 0) \wedge \overline{\mathcal{H}}_{\text{bd}}). \quad (20)$$

Now, observe that when restricted to the set of sample paths where $Z_{ij}(t)$ does not reach the boundary by t , we can couple the sample paths of $Z_{ij}(t)$ to the sample paths of a random walk $\tilde{Z}_{ij}(t)$ over an infinite plane before the corresponding hitting time to the boundary. Denote by $\tilde{\mathcal{H}}_{\text{bd}}$ the event that $\tilde{Z}_{ij}(t)$ hits \mathcal{A}_{bd} by $t = m/(c_h \log n)$, then

$$\begin{aligned} \mathbb{P}_0(Z_{ij}(t) = (0, 0) \wedge \overline{\mathcal{H}}_{\text{bd}}) &= \mathbb{P}_0(\tilde{Z}_{ij}(t) = (0, 0) \wedge \tilde{\mathcal{H}}_{\text{bd}}) \\ &\leq \mathbb{P}_0(\tilde{Z}_{ij}(t) = (0, 0)). \end{aligned}$$

The return probability obeys $\mathbb{P}_0(\tilde{Z}_{ij}(t) = (0, 0)) \sim t^{-1}$ for a random walk over an infinite plane [36], and $\mathbb{P}_0(\mathcal{H}_{\text{bd}})$ will be bounded in Lemma 4. Summing up all $q_{ij}^0(t)$ yields (19). See Appendix E for detailed derivation. \blacksquare

In order to derive an estimate on the number of distinct nodes receiving M_i directly from source i , we need to calculate the number of slots where i fails to forward M_i to a new user. In addition to physical-layer outage events, some transmissions occur to users already possessing M_i , and hence are not successful. Recall that we are using one-sided push-only strategy, and hence, we cannot always send an innovative message. Denote by $F_i(t)$ the number of *wasted transmissions* from i to

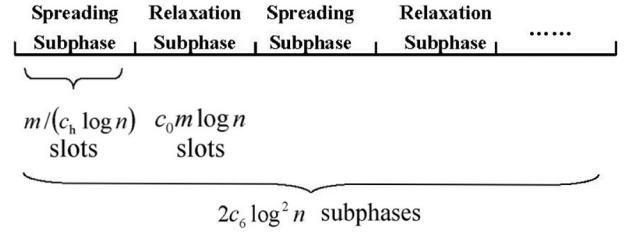


Fig. 3. Phase 1 is divided into $2c_6 \log^2 n$ subphases. Each odd subphase accounts for $m/(c_h \log n)$ slots, during which all nodes perform message spreading. Each even subphase contains $c_0 m \log n$ slots, during which no transmissions occur; it allows all nodes containing a typical message to be uniformly spread out.

some users already containing M_i by time t . This can be estimated as in the following lemma.

Lemma 6: For $t_0 = \frac{m}{c_h \log n}$, the number of wasted transmissions $F_i(t)$ defined above obeys

$$\mathbb{E}(F_i(t_0)) \leq c_5 \frac{m \log n}{n} t_0 \quad (21)$$

for some fixed constant c_5 with probability exceeding $1 - 3n^{-3}$.

Sketch of Proof of Lemma 6: Consider a particular pair of nodes i and j , where i is the source and j contains M_i . A wasted transmission occurs when (a) i and j meets in the same subsquare again, and (b) i is designated as a sender with j being the intended receiver. The probability of event (a) can be calculated using Lemma 5. Besides, the probability of (b) is $\Theta(m/n)$ due to sharp concentration on N_{A_i} . See Appendix F. \blacksquare

The above result is helpful in estimating the *expected number* of distinct users containing M_i . However, it is not obvious whether $F_i(t)$ exhibits desired sharp concentration. The difficulty is partly due to the dependence among $\{Z_{ij}(t)\}$ for different t arising from its Markov property. Due to their underlying relation with location of i , $Z_{ij_1}(t)$ and $Z_{ij_2}(t)$ are not independent either for $j_1 \neq j_2$. However, this difficulty can be circumvented by constructing different processes that exhibit approximate mutual independence as follows.

The time duration $[1, c_6(c_0 m \log n + m/(c_h \log n)) \log^2 n]$ of Phase 1 is divided into $c_6 \log^2 n$ nonoverlapping subphases $P_{1,j}$ ($1 \leq j \leq \log^2 n$) for some constant c_6 . Each odd subphase accounts for $m/(c_h \log n)$ time slots, whereas each even subphase contains $c_0 m \log n$ slots. See Fig. 3 for an illustration. Instead of studying the true evolution, we consider different evolutions for each subphase. In each odd subphase, source i attempts to transmit message M_i to its intended receiver as in the original process. But in every even subphase, all new transmissions will be immediately deleted. The purpose for constructing these *clearance* or *relaxation* processes in even subphases is to allow for approximately independent counting for odd subphases. The duration $c_0 m \log n$ of each even subphase, which is larger than the typical mixing time duration of the random walk, is sufficient to allow each user to move to everywhere almost uniformly likely.

Lemma 7: Set t to be $c_6 \left(c_0 m \log n + \frac{m}{c_h \log n} \right) \log^2 n$, which is the end time slot of Phase 1. The number of users containing each message M_i can be bounded as

$$\forall i, \quad N_i(t) > 32m \log n \quad (22)$$

with probability at least $1 - c_7 n^{-2}$.

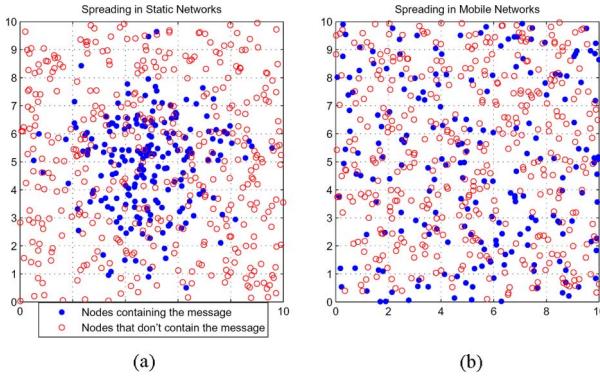


Fig. 4. Left plot illustrates the clustering phenomena of $\mathcal{N}_i(t)$ in the evolution in a static network. However, even restricted mobility may allow these nodes to spread out within the mixing time duration as illustrated in the right plot.

Proof: See Appendix G. ■

In fact, if $m \log^2 n \ll n$ holds, the above lemma can be further refined to $N_i(t) = \Theta(m \log^2 n)$. This implies that, by the end of Phase 1, each message has been flooded to $\Omega(m \log n)$ users. They are able to *cover* all subsquares (i.e., the messages' locations are roughly uniformly distributed over the unit square) after a further mixing time duration.

2) *Phase 2:* This phase starts from the end of Phase 1 and ends when $N_i(t) > n/8$ for all i . We use $t = 0$ to denote the starting slot of Phase 2 for convenience of presentation. Instead of directly looking at the original process, we generate a new process $\tilde{\mathcal{G}}$ which evolves slower than the original process \mathcal{G} . Define $\mathcal{S}_i(t)$ and $\tilde{\mathcal{S}}_i(t)$ as the set of messages that node i contains at time t in \mathcal{G} and $\tilde{\mathcal{G}}$, with $S_i(t)$ and $\tilde{S}_i(t)$ denoting their cardinality, respectively. For more clear exposition, we divide the entire phase into several time blocks each of length $k + c_0 \log n / v^2(n)$, and use t_B to label different time blocks. We define $\tilde{\mathcal{N}}_i^B(t_B)$ to denote $\tilde{\mathcal{N}}_i(t)$ with t being the starting time of time block t_B . $\tilde{\mathcal{G}}$ is generated from \mathcal{G} : everything in these two processes remains the same (including locations, movements, physical-layer outage events, etc.) except message selection strategies, detailed as follows:

Message Selection Strategy in the Coupled Process $\tilde{\mathcal{G}}$:

- 1) Initialize: At $t = 0$, for all i , copy the set $\mathcal{S}_i(t)$ of all messages that i contains to $\tilde{\mathcal{S}}_i(t)$. Set $t_B = 0$.
- 2) In the next $c_0 \log n / v^2(n)$ time slots, all new messages received in this subphase are immediately deleted, i.e., no successful forwarding occurs in this subphase regardless of the locations and physical-layer conditions.
- 3) In the next k slots, for every sender i , each message it contains is randomly selected with probability $1/k$ for transmission.
- 4) For all i , if the number of nodes containing M_i is larger than $2\tilde{N}_i^B(t_B)$, delete M_i from some of these nodes so that $\tilde{N}_i(t) = 2\tilde{N}_i^B(t_B)$ by the end of this time block.
- 5) Set $t_B \leftarrow t_B + 1$. Repeat from (2) until $N_i > n/8$ for all i .

Thus, each time block consists of a relaxation period and a spreading period. The key idea is to simulate an approxi-

mately spatially-uniform evolution, which is summarized as follows.

- 1) After each spreading subphase, we give the process a *relaxation* period to allow each node to move almost uniformly likely to all subsquares. This is similar to the relaxation period introduced in Phase 1.
- 2) Trimming the messages alone does *not* necessarily generate a slower process, because it potentially increases the selection probability for each message. Therefore, we force the message selection probability to be a lower bound $1/k$, which is *state-independent*. Surprisingly, this conservative bound suffices for our purpose because it is exactly one of the *bottlenecks* for the evolution.

The following lemma makes a formal comparison of \mathcal{G} and $\tilde{\mathcal{G}}$.

Lemma 8: $\tilde{\mathcal{G}}$ evolves stochastically slower than \mathcal{G} , i.e.,

$$\mathbb{P}(T_2 > x) < \mathbb{P}(\tilde{T}_2 > x), \quad \forall x > 0 \quad (23)$$

where $T_2 = \min\{t : N_i(t) > n/8, \forall i\}$ and $\tilde{T}_2 = \min\{t : \tilde{N}_i(t) > n/8, \forall i\}$ are the stopping time of Phase 2 for \mathcal{G} and $\tilde{\mathcal{G}}$, respectively.

Proof: Whenever a node i sends a message M_k to j in \mathcal{G} : (a) if $M_k \in \tilde{\mathcal{S}}_i$, then i selects M_k with probability S_i/k , and a random useless message otherwise; (b) if $M_k \notin \tilde{\mathcal{S}}_i$, i always sends a random noise message. The initial condition $\tilde{\mathcal{S}}_i = \mathcal{S}_i$ guarantees that $\tilde{\mathcal{S}}_i \subseteq \mathcal{S}_i$ always holds with this coupling method. Hence, the claimed stochastic order holds. ■

Lemma 9: Denote by $\tilde{T}_2^B := \min\{t_B : \tilde{N}_i^B(t_B) > n/8, \forall i\}$ the stopping time block of Phase 2 in $\tilde{\mathcal{G}}$. Then, there exists a constant c_{14} independent of n such that

$$\mathbb{P}(\tilde{T}_2^B \leq 4 \log_{c_{14}} n) \leq 1 - n^{-2}.$$

Sketch of Proof of Lemma 9: We first look at a particular message M_i and use union bound later after we derive the concentration results on the stopping time associated with this message. We observe the following facts: after a mixing time duration, the number of users $N_{i,A_k}(t)$ containing M_i at each subsquare A_k is approximately uniform. Since $N_i^B(t_B)$ is the lower bound on the number of copies of M_i across this time block, concentration results suggest that $N_{i,A_k}(t) = \Omega(N_i^B(t_B)/m)$. Observing from the mobility model that the position of any node inside a subsquare is i.i.d. chosen, we can derive

$$\mathbb{E}(\tilde{N}_i^B(t_B + 1) - \tilde{N}_i^B(t_B) | \tilde{N}_i^B(t_B)) \geq \frac{\tilde{c}_9}{2} \tilde{N}_i^B(t_B) \quad (24)$$

for some constant \tilde{c}_9 . A standard *martingale* argument then yields an upper bound on the stopping time. See Appendix H for detailed derivation. ■

This lemma implies that after at most $4 \log_{c_{14}} n$ time blocks, the number of nodes containing all messages will exceed $n/8$ with high probability. Therefore, the duration \tilde{T}_2 of Phase 2 of $\tilde{\mathcal{G}}$ satisfies $\tilde{T}_2 = O(k \log n)$ with high probability. This gives us an upper bound on T_2 of the original evolution \mathcal{G} .

3) *Phase 3:* This phase ends when $N_i(t) = n$ for all i with $t = 0$ denoting the end of Phase 2. Assume that $N_{i,A_j}(0) >$

$\frac{n}{16m}$ for all i and all j ; otherwise, we can let the process further evolve for another mixing time duration $\Theta(\log n/v^2(n))$.

Lemma 10: Denote by T_3 the duration of Phase 3, i.e., $T_3 = \min\{t : N_i(t) = n \mid N_i(0) \geq n/8, \forall i\}$. Then, there exists a constant c_{18} such that

$$\mathbb{P}\left(T_3 \leq \frac{64}{c_{18}} k \log n\right) \geq 1 - \frac{15}{16n^2}. \quad (25)$$

Sketch of Proof of Lemma 10: The random push strategies are efficient near the start (exponential growth), but the evolution will begin to slow down after Phase 2. The concentration effect allows us to obtain a different evolution bound as

$$\begin{aligned} & \mathbb{E}(N_i(t+1) - N_i(t) | N_i(t)) \\ &= \mathbb{E}(n - N_i(t) - (n - N_i(t+1)) | N_i(t)) \\ &\geq \frac{c_{18}}{16k} (n - N_i(t)). \end{aligned}$$

Constructing a different submartingale based on $n - N_i(t)$ yields the above results. See Appendix I. ■

4) *Discussion:* Combining the stopping time in all three phases, we can see that the spreading time $T_{\text{mp}}^d = \min\{t : \forall i, N_i(t) = n\}$ satisfies

$$T_{\text{mp}}^d \leq O\left(\frac{\log^3 n}{v^2(n)}\right) + O(k \log n) + O(k \log n) = O(k \log^2 n).$$

It can be observed that the mixing time bottleneck will not be critical in multimesage dissemination. The reason is that the mixing time in the regime $v(n) = \omega\left(\sqrt{\frac{\log n}{k}}\right)$ is much smaller than the optimal spreading time. Hence, the nodes have sufficient time to spread out to everywhere. The key step is how to seed the network with a sufficiently large number of copies at the initial stage of the spreading process, which is accomplished by the self-promotion phase of MOBILE PUSH.

Remark 5: It can be observed that the upper bounds on spreading time within Phases 2 and 3 are order-wise tight, since a gap of $\Omega(\log n)$ exists even for complete graphs [16]. The upper bound for Phase 1, however, might not be necessarily tight. We note that the $O(\log^2 n)$ factor arises in the analysis stated in Lemma 7, where we assume that each relaxation subphase is of duration $\Theta(m \log n)$ for ease of analysis. Since we consider $\Theta(\log^2 n)$ subphases in total, we do not necessarily need $\Theta(m \log n)$ slots for each relaxation subphase in order to allow spreading of all copies. We conjecture that with a finer tuning of the concentration of measures and coupling techniques, it may be possible to obtain a spreading time of $\Theta(k \log n)$.

IV. CONCLUDING REMARKS

In this paper, we design a simple distributed gossip-style protocol that achieves near-optimal spreading rate for multi-message dissemination, with the assistance of mobility. The key observation is that random gossiping over static geometric graphs is inherently constrained by the expansion property of the underlying graph—capacity loss occurs since the copies are spatially constrained instead of being spread out. Encouragingly, this bottleneck can indeed be overcome in mobile networks, even with fairly limited degree of velocity. In fact, the velocity-constrained mobility assists in achieving a large expansion property from a long-term perspective, which simulates a spatially uniform evolution.

TABLE I
SUMMARY OF NOTATION

$v(n)$	velocity
m	the number of subsquares; $m = 1/v^2(n)$
n	the number of users/nodes
k	the number of distinct messages
M_i	the message of source i
A_k	subsquare k
$N_i(t), \mathcal{N}_i(t)$	the number, and the set of nodes containing M_i at time t
$N_{i,A_k}(t), \mathcal{N}_{i,A_k}(t)$	the number, and the set of nodes containing M_i at subsquare A_k at time t
$S_i(t), \mathcal{S}_i(t)$	the number, and the set of messages node i has at time t
α	path loss exponent
β	SINR requirement for single-hop success

APPENDIX

A. Proof of Lemma 1

Let us look at a typical time slot t at subsquare A_i . We know that $2b/3m \leq \mathbb{E}(N_{A(i)}(t)) \leq 4b/3m$. For each node j , define the indicator variable $X_{j,A_i}(t) := \mathbb{1}\{j \text{ lies in } A_i \text{ at } t\}$. Then, $\{X_{j,A_i}(t) : 1 \leq j \leq b\}$ forms a set of i.i.d. random variables each satisfying $\frac{2}{3m} \leq \mathbb{P}(X_{j,A_i}(t) = 1) \leq \frac{4}{3m}$.

Define another two sets of i.i.d. Bernoulli random variables $\{X_{j,A_i}^{\text{ub}}(t) : 1 \leq j \leq b\}$ and $\{X_{j,A_i}^{\text{lb}}(t) : 1 \leq j \leq b\}$ such that

$$\begin{cases} \mathbb{P}(X_{j,A_i}^{\text{ub}}(t) = 1) = \frac{4}{3m} \\ \mathbb{P}(X_{j,A_i}^{\text{lb}}(t) = 1) = \frac{2}{3m}. \end{cases}$$

We also define $N_{A(i)}^{\text{ub}}(t) := \sum_{1 \leq j \leq b} X_{j,A_i}^{\text{ub}}(t)$ and $N_{A(i)}^{\text{lb}}(t) := \sum_{1 \leq j \leq b} X_{j,A_i}^{\text{lb}}(t)$. The following stochastic orders can be immediately observed through simple coupling arguments:

$$N_{A(i)}^{\text{lb}}(t) \preceq N_{A(i)}(t) \preceq N_{A(i)}^{\text{ub}}(t) \quad (26)$$

i.e., for any positive d , we have

$$\mathbb{P}(N_{A(i)}^{\text{lb}}(t) > d) \leq \mathbb{P}(N_{A(i)}(t) > d) \leq \mathbb{P}(N_{A(i)}^{\text{ub}}(t) > d).$$

Applying Chernoff bound yields

$$\begin{aligned} & \mathbb{P}\left(N_{A(i)}(t) \geq \frac{(1+\epsilon)4b}{3m}\right) \\ & \leq \mathbb{P}\left(\sum_{j=1}^b X_{j,A_i}^{\text{ub}}(t) \geq \frac{(1+\epsilon)4b}{3m}\right) \\ & \leq \exp\left(-\frac{2b\epsilon^2}{3m}\right). \end{aligned}$$

We can thus observe through union bound that

$$\begin{aligned} & \mathbb{P}\left(\exists(i, t) \text{ such that } N_{A(i)}(t) \geq \frac{(1+\epsilon)4b}{3m}\right) \\ & \leq mn^2 \mathbb{P}\left(N_{A(i)}(t) \geq \frac{(1+\epsilon)4b}{3m}\right) \\ & \leq n^3 \exp\left(-\frac{2b\epsilon^2}{3m}\right). \end{aligned}$$

Similarly, the stochastic order implies that

$$\begin{aligned} \mathbb{P}\left(N_{A(i)}(t) \leq \frac{(1-\epsilon)2b}{3m}\right) & \leq \mathbb{P}\left(N_{A(i)}^{\text{lb}}(t) \leq \frac{(1-\epsilon)2b}{3m}\right) \\ & \leq \exp\left(-\frac{b\epsilon^2}{3m}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \mathbb{P}\left(\exists(i, t) \text{ such that } N_{A(i)}(t) \leq \frac{(1-\epsilon)2b}{3m}\right) \\ & \leq n^3 \exp\left(-\frac{b\epsilon^2}{3m}\right). \end{aligned}$$

When $b = \omega(m \log n)$, $\exp\left(-\frac{\epsilon^2 b}{3m}\right) \ll n^{-c}$ holds for any positive constant c ; when $b = \Theta(m \log n)$ and $b > 32m \log n$, taking $\epsilon = \frac{3}{4}$ completes our proof.

B. Proof of Lemma 2

Consider first the vertical strip V_l . Obviously, the spreading within V_l will be influenced by nodes residing in adjacent strips V_{l-1} , V_{l+1} , and V_l itself. Define a set of i.i.d. Bernoulli random variables $\{X_{v,t}\}$ such that

$$\mathbb{P}(X_{v,t} = 1) = \frac{1}{w}. \quad (27)$$

For any $v \in \mathcal{N}_{V_{l-1}}(t_X) \cup \mathcal{N}_{V_l}(t_X) \cup \mathcal{N}_{V_{l+1}}(t_X)$, the probability of node v selecting M^* for transmission at time t can be bounded above by $\mathbb{P}(X_{v,t} = 1)$. Simple coupling argument yields the following stochastic order:

$$\begin{aligned} & N_{V_l}(t_X) \\ & \leq \sum_{t=1}^{t_X} \sum_{v \in \bigcup_{i=l-1}^{l+1} \mathcal{N}_{V_i}(t_X)} \mathbb{1}(v \text{ selects } M^* \text{ for transmission at } t) \\ & \preceq^{\text{st}} \sum_{t=1}^{t_X} \sum_{v=1}^{N_{V_{l-1}}(t_X) + N_{V_l}(t_X) + N_{V_{l+1}}(t_X)} X_{v,t} \end{aligned} \quad (28)$$

where \preceq^{st} denotes stochastic order. If $N_{V_{l-1}}(t_X) + N_{V_{l+1}}(t_X) > \omega(w^\epsilon \log n)$, then we have

$$\begin{aligned} & \mathbb{P}\left(N_{V_l}(t_X) > \frac{2}{w^\epsilon} (N_{V_{l-1}}(t_X) + N_{V_{l+1}}(t_X))\right) \\ & \leq \mathbb{P}\left(\left(1 - \frac{3}{2} \frac{1}{w^\epsilon}\right) N_{V_l}^x(t_X) > \frac{3}{2} \frac{1}{w^\epsilon} (N_{V_{l-1}}(t_X) + N_{V_{l+1}}(t_X))\right) \\ & = \mathbb{P}\left(N_{V_l}(t_X) > \frac{3}{2} \frac{1}{w^\epsilon} \left(\sum_{j=l-1}^{l+1} N_{V_j}(t_X)\right)\right) \\ & \leq \mathbb{P}\left(\sum_{t=1}^{t_X} \sum_{v=1}^{N_{V_{l-1}}(t_X) + N_{V_l}(t_X) + N_{V_{l+1}}(t_X)} X_{v,t} > \frac{3}{2} \mathbb{E}\left(\sum_{t=1}^{t_X} \sum_{v=1}^{N_{V_{l-1}}(t_X) + N_{V_l}(t_X) + N_{V_{l+1}}(t_X)} X_{v,t}\right)\right) \\ & \leq \frac{1}{n^5} \end{aligned} \quad (30)$$

where (30) follows from the stochastic order (28), and the last inequality follows from large deviation bounds and the observation that

$$\mathbb{E}\left(\sum_{t=1}^{t_X} \sum_{v=1}^{N_{V_{l-1}}(t_X) + N_{V_l}(t_X) + N_{V_{l+1}}(t_X)} X_{v,t}\right) = \omega(\log n).$$

Besides, if $N_{V_{l-1}}(t_X) + N_{V_{l+1}}(t_X) = O(w^\epsilon \log n)$, then we have

$$\begin{aligned} & \mathbb{P}\{N_{V_l}(t_X) > 3 \log^2 n\} \\ & \leq \mathbb{P}\left\{N_{V_l}(t_X) > \frac{2}{w^\epsilon} (N_{V_{l-1}}(t_X) + N_{V_{l+1}}(t_X)) + 2 \log^2 n\right\} \\ & \leq \mathbb{P}\left\{N_{V_l}(t_X) > \frac{1}{w^\epsilon} \left(\sum_{j=l-1}^{l+1} N_{V_j}(t_X)\right) + \log^2 n\right\} \\ & \leq \mathbb{P}\left\{\sum_{t=1}^{t_X} \sum_{v=1}^{N_{V_{l-1}}(t_X) + N_{V_l}(t_X) + N_{V_{l+1}}(t_X)} Y_{v,t} > \log^2 n\right\} \end{aligned} \quad (31)$$

where $Y_{v,t} \stackrel{\Delta}{=} X_{v,t} - 1/w$. Hence, $Y_{v,t} = 1 - 1/w$ with probability $1/w$. Let $\bar{\beta} = 1 - 1/w := w^\epsilon \log^2 n / (N_{V_1}(t_X) + N_{V_2}(t_X) + N_{V_3}(t_X)) = \Omega(\log n)$. By applying [35, Th. A.1.12], we can show that there exist constants c_{21} and \tilde{c}_{21} such that

$$(31) \leq \left(\frac{c_{21}}{\log n}\right)^{\log^2 n} \leq (\tilde{c}_{21} e)^{-\log^2 n \cdot \log \log n} \leq \frac{1}{n^5}. \quad (32)$$

Define $l^* := \min\{l : N_{V_l}(t_X) = O(w^\epsilon \log n)\}$ and $L^* := \min\{l^*, 0.5 \log n\}$. Since $N_{V_{L^*+1}}(t_X) + N_{V_{L^*-1}}(t_X) \geq N_{V_{L^*-1}}(t_X) = \omega(w^\epsilon \log n)$, we can derive with similar spirit that

$$N_{V_{L^*}}(t_X) \leq \frac{2}{w^\epsilon} (N_{V_{L^*+1}}(t_X) + N_{V_{L^*-1}}(t_X))$$

with probability at least $1 - n^{-5}$. Similarly, if $N_{V_{L^*}}(t_X) + N_{V_{L^*+2}}(t_X) = \omega(w^\epsilon \log n)$, we can derive

$$N_{V_{L^*+1}}(t_X) \leq \frac{2}{w^\epsilon} (N_{V_{L^*}}(t_X) + N_{V_{L^*+2}}(t_X))$$

with high probability. But if $N_{V_{L^*}}(t_X) + N_{V_{L^*+2}}(t_X) = O(w^\epsilon \log n)$, it can still be shown that

$$N_{V_{L^*+1}}(t_X) \leq \log^2 n \leq \frac{\log n}{w^\epsilon} N_{V_{L^*-2}}(t_X) \quad (33)$$

by observing that $N_{V_{L^*-2}}(t_X) = \omega(w^\epsilon \log n)$. Combining all these facts yields

$$\begin{aligned} & N_{V_{L^*-1}}(t_X) + N_{V_{L^*+1}}(t_X) \\ & \leq \frac{2}{w^\epsilon} (N_{V_{L^*-2}}(t_X) + N_{V_{L^*}}(t_X)) + \frac{\log n}{w^\epsilon} (N_{V_{L^*+2}}(t_X) + N_{V_{L^*}}(t_X) + N_{V_{L^*-2}}(t_X)) \\ & \leq \frac{3 \log n}{w^\epsilon} (N_{V_{L^*-2}}(t_X) + N_{V_{L^*}}(t_X) + N_{V_{L^*+2}}(t_X)) \\ & \leq \frac{3 \log n}{w^\epsilon} (N_{V_{L^*-2}}(t_X) + N_{V_{L^*+2}}(t_X)) + \frac{6 \log n}{w^{2\epsilon}} (N_{V_{L^*+1}}(t_X) + N_{V_{L^*-1}}(t_X)). \end{aligned}$$

Simple manipulation gives us

$$\begin{aligned} & N_{V_{L^*-1}}(t_X) + N_{V_{L^*+1}}(t_X) \\ & \leq \frac{4 \log n}{w^\epsilon} (N_{V_{L^*-2}}(t_X) + N_{V_{L^*+2}}(t_X)) \end{aligned}$$

with high probability. Proceeding with similar spirit gives us: for all s ($0 \leq s < L^* \leq \log n/2$)

$$\begin{aligned} & N_{V_{L^*-s}}(t_X) + N_{V_{L^*+s}}(t_X) \\ & \leq \frac{4 \log n}{w^\epsilon} (N_{V_{L^*-s-1}}(t_X) + N_{V_{L^*+s+1}}(t_X)) \end{aligned} \quad (34)$$

holds with probability at least $1 - n^{-4}$. By iteratively applying (34) we can derive that for any s ($1 \leq s < L^*$),

$$\begin{aligned} & N_{V_s}(t_X) \\ & \leq N_{V_s}(t_X) + N_{V_{2L^*-s}}(t_X) \\ & \leq \left(\frac{4 \log n}{w^\epsilon}\right)^{s-1} (N_{V_1}(t_X) + N_{V_{2L^*-1}}(t_X)) \\ & \leq \left(\frac{4 \log n}{w^\epsilon}\right)^{s-1} (c_{31} \sqrt{n} \log n) \end{aligned}$$

where the last inequality arises from the fact that there are at most $O(\sqrt{n} \log n)$ nodes residing in each strip with high probability.

This shows the geometric decaying rate of $N_{V_s}(t_X)$ in s . Suppose that $L^* > \log n/2$; then, we have

$$\begin{aligned} N_{V_{\frac{1}{2} \log n}}(t_X) &= \left(\frac{4 \log n}{w^\epsilon}\right)^{\frac{1}{2} \log n - 1} (c_{31} \sqrt{n} \log n) \\ &\ll 1 = o(w^{-\epsilon} \log n) \end{aligned}$$

where contradiction arises. Hence, $L^* < \log n/2$ with high probability.

Additionally, we can derive an upper bound on $N_{V_{\log n/2}}$ using the same fixed-point arguments as follows:

$$\begin{aligned} & N_{V_{\log n/2}}(t_X) \\ & \leq \max \left\{ \log^2 n, \left(\frac{4 \log n}{w^\epsilon}\right)^{\frac{1}{2} \log n - 1} (N_{V_1}(t_X) + N_{V_{\log n-1}}(t_X)) \right\} \\ & = \log^2 n \end{aligned}$$

with probability at least $1 - n^{-3}$.

C. Proof of Lemma 3

Define a set of Bernoulli random variables $\{X_t\}$ such that $X_t = 1$ if there is at least one node inside these substrips selecting M^* for transmission at time t and $X_t = 0$ otherwise. By observing that the size of each “substrip” will not exceed $\Theta(\log^3 n)$ tiles before T_2^* and that each tile contains $O(\log n)$ nodes, the probability $\mathbb{P}(X_t = 1)$ can be bounded above by $1 - (1 - \frac{1}{w})^{\Theta(\log^3 n)} \leq \frac{c_{40} \log^3 n}{w}$ for some constant c_{40} . This inspires us to construct the following set of i.i.d. Bernoulli random variables $\{\bar{X}_t\}$ through coupling as follows:

$$\begin{cases} \text{if } X_t = 1, \text{ then } \bar{X}_t = 1 \\ \text{if } X_t = 0, \text{ then } \bar{X}_t = \begin{cases} 1, & \text{w.p. } \frac{c_{40} \log^3 n}{w} - \mathbb{P}(X_t = 1) \\ 0, & \text{otherwise.} \end{cases} \end{cases} \quad (35)$$

That said

$$\bar{X}_t = \begin{cases} 1, & \text{with probability } \frac{c_{40} \log^3 n}{w} \\ 0, & \text{otherwise.} \end{cases} \quad (36)$$

Our way for constructing $\{\bar{X}_t\}$ implies that

$$\sum_{t=1}^{T_2^*} \bar{X}_t \geq \sum_{t=1}^{T_2^*} X_t = \frac{\log n}{2}. \quad (37)$$

Additionally, large deviation bounds yields

$$\begin{aligned} & \mathbb{P} \left(\sum_{t=1}^{T_X} \bar{X}_t \geq \frac{\log n}{2} \right) \\ & \leq \mathbb{P} \left(\sum_{t=1}^{T_X} (\bar{X}_t - \mathbb{E}(\bar{X}_t)) \geq \frac{\log n}{2} - \frac{\log^3 n}{w^\epsilon} \right) \\ & \leq \mathbb{P} \left(\sum_{t=1}^{T_X} (\bar{X}_t - \mathbb{E}(\bar{X}_t)) \geq \frac{\log n}{3} \right). \end{aligned} \quad (38)$$

Defining $\tilde{\beta} - 1 := \frac{w^\epsilon \log n}{3c_{40} \log^3 n} = \frac{w^\epsilon}{3c_{40} \log^2 n}$, [35, Th. A.1.12] gives

$$(38) \leq \left(\frac{e}{\tilde{\beta}}\right)^{\frac{\log n}{3}} \leq \frac{1}{n^5} \quad (39)$$

which further results in

$$\mathbb{P}(T_2^* > t_X) \geq 1 - \frac{1}{n^5}. \quad (40)$$

To conclude, the message M^* is unable to cross V_1^B by time $t_X = w^{1-\epsilon}$ with high probability.

D. Proof of Lemma 4

Define two sets of random variables $\{X_{ij}(t)\}$ and $\{Y_{ij}(t)\}$ to represent the coordinates of $Z_{ij}(t)$ in two dimensions, respectively, i.e., $Z_{ij}(t) = (X_{ij}(t), Y_{ij}(t))$. Therefore, for any $A \in \mathcal{A}_{bd}$, we can observe

$$\begin{aligned} & \mathbb{P}_0(Z_{ij}(t) = A) \\ & \leq \mathbb{P}_0 \left(X_{ij}(t) = \pm \frac{\sqrt{m}}{2} \cup Y_{ij}(t) = \pm \frac{\sqrt{m}}{2} \right) \\ & \leq 4 \mathbb{P}_0 \left(X_{ij}(t) \geq \frac{\sqrt{m}}{2} \right). \end{aligned}$$

Besides, we notice that $|X_{ij}(t+1) - X_{ij}(t)| \leq 2$ and $\mathbb{E}(X_{ij}(t+1) - X_{ij}(t)) = 0$; then, large deviation results implies

$$\mathbb{P}_0 \left(X_{ij}(t) \geq \frac{\sqrt{m}}{2} \right) \leq \exp \left(-\frac{6m}{c_h} \right) \quad (41)$$

for some constant c_h . We thus derive $\mathbb{P}_0 \left(X_{ij}(t) \geq \frac{\sqrt{m}}{2} \right) \leq n^{-6}$ for any $t < \frac{m}{c_h \log n}$, which leads to

$$\begin{aligned} & \mathbb{P}_0 \left(T_{hit} < \frac{m}{c_h \log n} \right) \\ & \leq \mathbb{P}_0 \left(\exists t < \frac{m}{c_h \log n} \text{ and } \exists A \in \mathcal{A}_{bd} \text{ s.t. } Z_{ij}(t) = A \right) \\ & \leq \frac{m}{c_h \log n} \cdot 4\sqrt{m} \cdot 4n^{-6} \leq \frac{1}{n^4}. \end{aligned}$$

E. Proof of Lemma 5

It can be observed that $Z_{ij}(t)$ is a discrete-time random walk which at each step randomly moves to one of 25 sites each with some constant probability. Formally, we can express it as follows: for $|a| \leq 2, |b| \leq 2$:

$$\mathbb{P}(Z_{ij}(t+1) - Z_{ij}(t) = (a, b) \mid Z_{ij}(t)) = p_{|a|, |b|} \quad (42)$$

holds before $Z_{ij}(t)$ hits the boundary, where $p_{|a|, |b|}$ are fixed constants independent of n , and $\sum_{|a| \leq 2, |b| \leq 2} p_{|a|, |b|} = 1$.

We construct a new process $\tilde{Z}_{ij}(t)$ such that $\tilde{Z}_{ij}(t)$ is a random walk over an “infinite” plane with $\mathbb{P}(\tilde{Z}_{ij}(t+1) - \tilde{Z}_{ij}(t) = (a, b) \mid \tilde{Z}_{ij}(t)) = p_{|a|, |b|}$.

Define the event $\mathcal{H}_{\text{bd}} = \{T_{\text{hit}} < m/(c_h \log n)\}$ where T_{hit} is the hitting time to the boundary as defined in Lemma 4. When restricted to the sample paths where $Z_{ij}(t)$ does not hit the boundary by t , we can couple the sample paths of $Z_{ij}(t)$ to those of $\tilde{Z}_{ij}(t)$ before the corresponding hitting time to the boundary.

It is well known that for a random walk $\tilde{Z}_{ij}(t)$ over an infinite 2-D plane, the return probability obeys $\tilde{q}_{ij}^0 \sim t^{-1}$ [36]. Specifically, there exists a constant \tilde{c}_3 such that

$$\tilde{q}_{ij}^0(t) \stackrel{\Delta}{=} \mathbb{P}_0(\tilde{Z}_{ij}(t) = (0, 0)) \leq \tilde{c}_3/t. \quad (43)$$

Hence, the return probability $q_{ij}^0(t) \stackrel{\Delta}{=} \mathbb{P}_0(Z_{ij}(t) = (0, 0))$ of the original walk $Z_{ij}(t)$ satisfies

$$\begin{aligned} q_{ij}^0(t) &= \mathbb{P}_0(Z_{ij}(t) = (0, 0) \mid \mathcal{H}_{\text{bd}}) \mathbb{P}(\mathcal{H}_{\text{bd}}) \\ &\quad + \mathbb{P}_0(Z_{ij}(t) = (0, 0) \wedge \overline{\mathcal{H}}_{\text{bd}}) \\ &\leq \mathbb{P}_0(\mathcal{H}_{\text{bd}}) + \mathbb{P}_0(\tilde{Z}_{ij}(t) = (0, 0) \wedge \overline{\mathcal{H}}_{\text{bd}}) \\ &\leq \mathbb{P}_0(\mathcal{H}_{\text{bd}}) + \mathbb{P}_0(\tilde{Z}_{ij}(t) = (0, 0)) \\ &\leq \frac{1}{n^4} + \frac{\tilde{c}_3}{t} \leq \frac{2\tilde{c}_3}{t} \end{aligned} \quad (45)$$

where (45) arises from the coupling of $Z_{ij}(t)$ and $\tilde{Z}_{ij}(t)$, and the upper bound on $\mathbb{P}(\mathcal{H}_{\text{bd}})$ is derived in Lemma 4. Hence, the expected number of time slots in which i and j move to the same subsquare by time t ($1 < t < \frac{m}{c_h \log n}$) can be bounded above as

$$\begin{aligned} &\mathbb{E}\left(\sum_{l=1}^t \mathbb{1}(Z_{ij}(l) = (0, 0)) \mid Z_{ij}(0) = (0, 0)\right) \\ &\leq 2\tilde{c}_3 \sum_{l=1}^t \frac{1}{l} \leq c_3 \log t \end{aligned}$$

for some constant c_3 .

F. Proof of Lemma 6

Define $t_i(j)$ as the first time slot that j receives M_i . A “conflict” event related to M_i is said to occur if at any time slot the

source i moves to a subsquare that coincides with any user j already possessing M_i . Denote by $C_i(t)$ the total amount of conflict events related to M_i that happen before t ($t \leq \frac{m}{c_h \log n}$), which can be characterized as follows:

$$\begin{aligned} &\mathbb{E}(C_i(t)) \\ &= \mathbb{E}\left(\sum_{j \in \mathcal{N}_i(t)} \sum_{k=t_i(j)+1}^t \mathbb{1}(Z_{ij}(k) = 0 \mid Z_{ij}(t_i(j)) = 0)\right) \end{aligned} \quad (46)$$

$$\leq t \mathbb{E}\left(\sum_{k=1}^t \mathbb{1}(Z_{ij}(k) = 0 \mid Z_{ij}(0) = 0)\right) \quad (47)$$

$$\leq c_3 t \log t \quad (48)$$

where (47) arises from the facts that $N_i(t) \leq t$ and Z_{ij} is stationary, and (48) follows from Lemma 5. Lemma 1 implies that there will be more than $n/6m$ users residing in each subsquare with high probability. If this occurs for every subsquare in each of t slots, whenever i and j happens to stay in the same subsquare, the probability that i can successfully transmit M_i to j can be bounded above as $\theta(1-\theta) \cdot 6m/n$. Therefore, the amount of successful “retransmissions” can be bounded as

$$\mathbb{E}(F_i(t)) \leq \theta(1-\theta) \cdot \frac{6m}{n} \mathbb{E}(C_i(t)) \leq c_5 \frac{mt \log t}{n} \quad (49)$$

for some fixed constant c_5 . Setting $t = t_0$ yields

$$\mathbb{E}(F_i(t_0)) \leq c_5 \frac{m \log n}{n} t_0 = o(t_0). \quad (50)$$

This inequality is based on the assumption that $T_{\text{hit}} > t_0$ and that concentration effect occurs, which happens with probability at least $1 - 3n^{-3}$.

G. Proof of Lemma 7

Recall that $P_{1,j}$ denotes the j th subphase of Phase 1. Therefore, in the first odd subphase, the total number of successful transmissions from i (including “retransmissions”), denoted by G_i , can be bounded as

$$G_i \geq \tilde{c}_7 m / \log n \quad (51)$$

with probability exceeding $1 - c_7/n^4$ for some constants and c_7 and \tilde{c}_7 . This follows from simple concentration inequality and the fact that $\mathbb{E}(G_i) \geq c(1-\theta)\theta t$. It can be noted that G_i exhibits sharp concentration based on simple large deviation argument. Also, Markov bound yields

$$\mathbb{P}\left(F_i(t_0) > 65c_5 \frac{m \log n}{n} t_0\right) \leq \frac{\mathbb{E}(F_i(t_0))}{65c_5 m t_0 \log n / n} = \frac{1}{65}. \quad (52)$$

Since the duration of any odd subphase $t_0 = m/(c_h \log n)$ obeys $mt_0 \log n / n \ll t_0$, this implies that the number of distinct new users that can receive M_i within this odd subphase, denoted by $\overline{N}_i(P_{1,1})$, can be stochastically bounded as

$$\mathbb{P}\left(\overline{N}_i(P_{1,1}) < \tilde{c}_8 \frac{m}{\log n}\right) < \frac{1}{64} \quad (53)$$

for some constant \tilde{c}_8 . Note that we use the bound $\frac{1}{64}$ here instead of $\frac{1}{65}$ in order to account for the deviation of G_i .

We observe that there are in total $c_6 m \log n / c_h$ slots for spreading in Phase 1; therefore, the total number of distinct nodes receiving M_i in Phase 1 is bounded above by $c_6 m \log n / c_h \ll n/2$. Mixing behavior combined with concentration effect thus indicates that during any odd subphase, there are at least $n/12m$ nodes not containing M_i in each subsquare, and each transmitter is able to contact a new receiver (who does not have M_i) in each slot with probability at least $1/12$. This motivates us to construct the following stochastically slower process for odd subphase j : 1) there are $n/2$ nodes residing in the entire square; 2) each transmission event is declared “failure” regardless any other state with probability $11/12$; 3) the evolution in different odd subphases are independent; 4) other models and strategies remain the same as the original process.

Obviously, this constructed process allows us to derive a lower bound on $N_i(t)$ by the end of Phase 1. Proceeding with similar spirit in the analysis for first odd subphase, we can see that the number of distinct new users receiving M_i in odd subphase j of our constructed process, denoted by $\tilde{N}_i(P_{1,j})$, can be lower bounded as

$$\mathbb{P}\left(\tilde{N}_i(P_{1,j}) < \hat{c}_8 \frac{m}{\log n}\right) < \frac{1}{64} \quad (54)$$

for some constant \hat{c}_8 . By noticing that $\{\tilde{N}_i(P_{1,j})\}$ are mutually independent, simple large deviation inequality yields

$$\begin{aligned} & \mathbb{P}\left(\exists \frac{c_6}{2} \log^2 n \text{ odd subphases s.t. } \tilde{N}_i(P_{1,j}) > \hat{c}_8 \frac{m}{\log n}\right) \\ & > 1 - \frac{1}{n^5}. \end{aligned}$$

Taking c_6 such that $c_6 \hat{c}_8 / 2 > 32$, we can see that

$$\sum_{\text{odd subphase } j} \tilde{N}_i(P_{1,j}) > 32m \log n \quad (55)$$

holds with probability at least $1 - o(n^{-3})$. Since $N_i(t)$ in the original process is stochastically larger than the total number of distinct users receiving M_i in any subphase $\sum_{\text{odd subphase } j} \tilde{N}_i(P_{1,j})$, we can immediately observe: $N_i(t) > 32m \log n$ by the end of Phase 1 with high probability. Furthermore, applying union bound over all distinct messages completes the proof of this lemma.

H. Proof of Lemma 9

We consider the evolution for a typical time block $[t_B, t_B + 1]$. During any time slot in this time block, each node $l \notin \tilde{N}_i^B(t_B)$ lying in subsquare A_k will receive M_i from a node in $\tilde{N}_i^B(t_B)$ with probability at least

$$\begin{aligned} & \mathbb{P}(l \text{ receives } M_i \text{ at } t \text{ from a node in } \tilde{N}_i^B(t_B)) \\ & \geq c_8 \frac{\tilde{N}_{i,A_k}(t)}{2n/m} \frac{1}{k} \geq \frac{c_9 \tilde{N}_i^B(t_B)}{nk} \end{aligned}$$

for constants c_8 and c_9 , because concentration results imply that: 1) there will be at least $\tilde{N}_i^B(t_B)/3m$ nodes belonging to

$\tilde{N}_i^B(t_B)$ residing in each subsquare A_j ; 2) there are at most $2n/m$ nodes in each subsquare; 3) each successful transmission allows l to receive a specific message M_i with state-independent probability $1/k$. Therefore, each node $j \notin \tilde{N}_i^B(t_B)$ will receive M_i from a node in $\tilde{N}_i^B(t_B)$ by $t_B + 1$ with probability exceeding $1 - \left(1 - \frac{c_9 \tilde{N}_i^B(t_B)}{nk}\right)^k \geq \tilde{c}_9 \tilde{N}_i^B(t_B)/n$ for some constant \tilde{c}_9 . Since there are in total $n - \tilde{N}_i^B(t_B)$ nodes not containing M_i at the beginning of this time block, we have

$$\begin{aligned} & \mathbb{P}(l \text{ receives } M_i \text{ from a node in } \tilde{N}_i^B(t_B) \text{ by } t_B + 1) \\ & \geq \left(n - \tilde{N}_i^B(t_B)\right) \tilde{c}_9 \tilde{N}_i^B(t_B)/n \\ & \geq \frac{\tilde{c}_9}{2} \tilde{N}_i^B(t_B). \end{aligned}$$

Moreover, after the relaxation period of $c_{13} \log n / v^2(n)$ time slots, all these nodes containing M_i will be spread out to all subsquares; then, similar arguments will hold for a new time block. The mixing time period plays an important role in maintaining an approximately uniform distribution of the locations of each node. Thus, we can derive the following evolution equation:

$$\mathbb{E}\left(\frac{1}{\tilde{N}_i^B(t_B + 1)} - \frac{1}{\tilde{N}_i^B(t_B)} \middle| \tilde{N}_i^B(t_B)\right) \quad (56)$$

$$= -\mathbb{E}\left(\frac{\tilde{N}_i^B(t_B + 1) - \tilde{N}_i^B(t_B)}{\tilde{N}_i^B(t_B + 1) \tilde{N}_i^B(t_B)} \middle| \tilde{N}_i^B(t_B)\right) \quad (57)$$

$$\leq -\mathbb{E}\left(\frac{\tilde{N}_i^B(t_B + 1) - \tilde{N}_i^B(t_B)}{2 [\tilde{N}_i^B(t_B)]^2} \middle| \tilde{N}_i^B(t_B)\right) \quad (58)$$

$$\leq -c_{13} \frac{1}{\tilde{N}_i^B(t_B)} \quad (59)$$

for some constant $c_{13} < 1$, where the inequality (58) follows from the fact $\tilde{N}_i^B(t_B + 1) \leq 2\tilde{N}_i^B(t_B)$ guaranteed by Step (4). Let $\tilde{T}_{i,2}^B = \min\{t_B : \tilde{N}_i^B(t_B) > n/8\}$, and define $\tilde{Z}_i(t_B) = (1 - c_{13})^{-t_B} / \tilde{N}_i^B(t_B)$. Simple manipulation yields

$$\mathbb{E}(\tilde{Z}_i((t_B + 1) \wedge \tilde{T}_{i,2}^B) | \tilde{Z}_i(t_B \wedge \tilde{T}_{i,2}^B)) \leq \tilde{Z}_i(t_B \wedge \tilde{T}_{i,2}^B)$$

which indicates that $\{\tilde{Z}_i(t_B \wedge \tilde{T}_{i,2}^B), t_B \geq 0\}$ forms a non-negative supermartingale. We further define a stopping time $\hat{T}_{i,2}^B = \tilde{T}_{i,2}^B \wedge n$, which satisfies $\hat{T}_{i,2}^B \leq n < \infty$. The Stopping Time Theorem yields [37, Th. 5.7.6]

$$\begin{aligned} \mathbb{E}(\tilde{Z}_i(\hat{T}_{i,2}^B)) &= \mathbb{E}(\tilde{Z}_i(\hat{T}_{i,2}^B \wedge \tilde{T}_{i,2}^B)) \\ &\leq \mathbb{E}(\tilde{Z}_i(0)) = \frac{1}{\tilde{N}_i^B(0)}, \\ \Rightarrow \mathbb{E}\left((1 - c_{13})^{-\hat{T}_{i,2}^B}\right) &\leq \frac{\max \tilde{N}_i^B(\hat{T}_{i,2}^B)}{\tilde{N}_i^B(0)} < n. \end{aligned} \quad (60)$$

Set $c_{14} = (1 - c_{13})^{-1}$; then, we have

$$\mathbb{P}\{\hat{T}_{i,2}^B > 4 \log_{c_{14}} n\} = \mathbb{P}\{\hat{T}_{i,2}^B > n^4\} \leq \frac{\mathbb{E}(\hat{T}_{i,2}^B)}{n^4} < \frac{1}{n^3}.$$

The fact that $\hat{T}_{i,2}^B = \tilde{T}_{i,2}^B \wedge n$ gives

$$\mathbb{P}\left\{\hat{T}_{i,2}^B \leq 4 \log_{c_{14}} n\right\} = \mathbb{P}\left\{\tilde{T}_{i,2}^B \leq 4 \log_{c_{14}} n\right\} \geq 1 - \frac{1}{n^3}.$$

Finally, applying union bound yields

$$\mathbb{P}\left\{\exists i : \tilde{T}_{i,2}^B > 4 \log_{c_{14}} n\right\} \leq \frac{1}{n^2}.$$

I. Proof of Lemma 10

Define $T_{i,3} = \min\{t : N_i(t) = n\}$ as the stopping time for all i . In each time slot, any node $l \notin \mathcal{N}_i(t)$ will receive M_i with probability at least $c_{18}/16n$ for a constant c_{18} simply because each transmitter in $\mathcal{N}_i(t)$ will select M_i for transmission to l with probability at least $1/k$. This yields the following inequality:

$$\mathbb{E}(N_i(t+1) - N_i(t) | N_i(t)) \geq \frac{c_{18}}{16k} (n - N_i(t)). \quad (61)$$

Define $Z_{i,3}(t) = (n - N_i(t)) \left(1 - \frac{c_{18}}{16k}\right)^{-t}$; then by manipulation, we get $\mathbb{E}(Z_{i,3}(t+1) | Z_{i,3}(t)) \geq Z_{i,3}(t)$, which indicates that $\{Z_{i,3}(t)\}$ forms a submartingale. Taking $t_4 = 64k \log n / c_{18}$, we have

$$\begin{aligned} \mathbb{E}[n - N_i(t_4)] &\leq \left(1 - \frac{c_{18}}{16k}\right)^{t_4} \frac{15n}{16} \\ \therefore \mathbb{P}(N_i(t_4) \leq n-1) &\leq \mathbb{E}[(n - N_i(t_4))] \leq \frac{15}{16n^3}. \end{aligned}$$

Moreover, applying union bound yields

$$\mathbb{P}(\exists i : N_i(t_4) \leq n-1) \leq n \mathbb{P}(N_i(t_4) \leq n-1) \leq \frac{15}{16n^2}.$$

Let $T_3 = \max_i T_{i,3}$. Combining the above results, we can see that the duration T_3 of Phase 3 satisfies $T_3 = O(k \log n)$ with probability at least $1 - \frac{15}{16n^2}$.

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REFERENCES

- [1] Y. Chen, S. Shakkottai, and J. Andrews, "Sharing multiple messages over mobile networks," in *Proc. 30th IEEE Int. Conf. Comput. Commun.*, Apr. 2011, pp. 658–666.
- [2] D. Qiu and R. Srikant, "Modeling and performance analysis of bittorrent-like peer-to-peer networks," in *Proc. ACM SIGCOMM*, 2004, pp. 367–378.
- [3] X. Yang and G. de Veciana, "Service capacity of peer to peer networks," in *Proc. IEEE Int. Conf. Comput. Commun.*, Mar. 2004, vol. 4, pp. 2242–2252, vol. 4.
- [4] B. Pittel, "On spreading a rumor," *SIAM J. Appl. Math.*, vol. 47, no. 1, pp. 213–223, 1987.
- [5] R. Karp, C. Schindelhauer, S. Shenker, and B. Vocking, "Randomized rumor spreading," in *Proc. 41st Annu. Symp. Found. Comput. Sci.*, 2000, pp. 565–574.
- [6] J. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Lab. Inf. Decision Syst., Massachusetts Inst. Technology, Cambridge, USA, 1984.
- [7] A. Kashyap and R. Srikant, "Quantized consensus," *Automatica*, vol. 43, pp. 1192–1203, 2007.
- [8] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2508–2530, Jun. 2006.
- [9] A. Nedic and A. Ozdaglar, "Distributed subgradient methods for multi-agent optimization," *IEEE Trans. Automat. Control*, vol. 54, no. 1, pp. 48–61, Jan. 2009.
- [10] K. Jung, D. Shah, and J. Shin, "Distributed averaging via lifted Markov chains," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 634–647, Jan. 2010.
- [11] E. Modiano, D. Shah, and G. Zussman, "Maximizing throughput in wireless networks via gossiping," in *Proc. ACM SIGMETRICS*, 2006, pp. 27–38.
- [12] A. Eryilmaz, A. Ozdaglar, D. Shah, and E. Modiano, "Distributed cross-layer algorithms for the optimal control of multihop wireless networks," *IEEE/ACM Trans. Netw.*, vol. 18, no. 2, pp. 638–651, Apr. 2010.
- [13] A. Demers, D. Greene, C. Houser, W. Irish, J. Larson, S. Shenker, H. Sturgis, D. Swinehart, and D. Terry, "Epidemic algorithms for replicated database maintenance," in *Proc. 6th Annu. ACM Symp. Principles Distrib. Comput.*, 1987, pp. 1–12.
- [14] B. Pittel, "On spreading a rumor," *SIAM J. Appl. Math.*, vol. 47, no. 1, pp. 213–223, 1987.
- [15] D. Shah, "Gossip algorithms," *Found. Trends Netw.*, vol. 3, no. 1, pp. 1–125, Apr. 2009.
- [16] S. Sanghavi, B. Hajek, and L. Massoulie, "Gossiping with multiple messages," *IEEE Trans. Inf. Theory*, vol. 53, no. 12, pp. 4640–4654, Dec. 2007.
- [17] D. Mosk-Aoyama and D. Shah, "Fast distributed algorithms for computing separable functions," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 2997–3007, Jul. 2008.
- [18] A. Sarwate and A. Dimakis, "The impact of mobility on gossip algorithms," in *Proc. IEEE INFOCOM*, Apr. 2009, pp. 2088–2096.
- [19] A. Clementi, A. Monti, F. Pasquale, and R. Silvestri, "Information spreading in stationary Markovian evolving graphs," *IEEE Trans. Parallel Distrib. Syst.*, vol. 22, no. 9, pp. 1425–1432, Sep. 2011.
- [20] H. Baumann, P. Crescenzi, and P. Fraignaud, "Parsimonious flooding in dynamic graphs," in *Proc. 28th ACM Symp. Principles Distrib. Comput.*, 2009, pp. 260–269.
- [21] A. Clementi, R. Silvestri, and L. Trevisan, "Information Spreading in Dynamic Graphs," in *Proc. ACM Symp. Principles of Distributed Computing*, 2012, pp. 37–46.
- [22] A. Pettarin, A. Pietracaprina, G. Pucci, and E. Upfal, "Tight bounds on information dissemination in sparse mobile networks," in *Proc. 30th Annu. ACM SIGACT-SIGOPS Symp. Princip. Distrib. Comput.*, 2011, pp. 355–362.
- [23] S. Deb, M. Medard, and C. Choute, "Algebraic gossip: A network coding approach to optimal multiple rumor mongering," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2486–2507, Jun. 2006.
- [24] B. Haupler, "Analyzing network coding gossip made easy," in *Proc. 43rd ACM Symp. Theory Comput.*, Jun. 2011, pp. 2088–2096.
- [25] R. Zheng, "Information dissemination in power-constrained wireless networks," in *Proc. IEEE INFOCOM*, Apr. 2006, pp. 1–10.
- [26] S. Subramanian, S. Shakkottai, and A. Arapostathis, "Broadcasting in sensor networks: The role of local information," *IEEE/ACM Trans. Netw.*, vol. 16, no. 5, pp. 1133–1146, Oct. 2008.
- [27] G. Resta and P. Santi, "On the fundamental limits of broadcasting in wireless mobile networks," in *Proc. IEEE INFOCOM*, 14–19, 2010, pp. 1–5.
- [28] M. Neely and E. Modiano, "Capacity and delay tradeoffs for ad hoc mobile networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 6, pp. 1917–1937, Jun. 2005.
- [29] M. Grossglauser and D. Tse, "Mobility increases the capacity of ad hoc wireless networks," *IEEE/ACM Trans. Netw.*, vol. 10, no. 4, pp. 477–486, Aug. 2002.
- [30] A. E. Gamal, J. Mammen, B. Prabhakar, and D. Shah, "Optimal throughput-delay scaling in wireless networks—Part I: The fluid model," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2568–2592, Jun. 2006.
- [31] L. Ying, S. Yang, and R. Srikant, "Optimal delay-throughput tradeoffs in mobile ad hoc networks," *IEEE Trans. Inf. Theory*, vol. 54, no. 9, pp. 4119–4143, Sep. 2008.
- [32] X. Lin, G. Sharma, R. Mazumdar, and N. Shroff, "Degenerate delay-capacity tradeoffs in ad-hoc networks with Brownian mobility," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2777–2784, Jun. 2006.
- [33] C. Avin and G. Ercal, "On the cover time and mixing time of random geometric graphs," *Theor. Comput. Sci.*, vol. 380, no. 1–2, pp. 2–22, 2007.
- [34] R. Durrett, *Random Graph Dynamics*, ser. Cambridge Series in Statistical and Probabilistic Mathematics. New York, NY, USA: Cambridge Univ. Press.

- [35] N. Alon and J. H. Spencer, *The Probabilistic Method*, ser. Wiley-Interscience Series in Discrete Mathematics and Optimization. New York, NY, USA: Wiley, 2008.
- [36] F. G. Foster and I. J. Good, "On a generalization of Pólya's random-walk theorem," *Quart. J. Math.*, vol. 4, pp. 120–126, Jun. 1953.
- [37] R. Durrett, *Probability: Theory and Examples*, 4th ed. Cambridge, U.K.: Cambridge Univ. Press, 2010.

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