# Progressive Choices in School Choice Problems

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**Abstract**

A common assumption of different mechanisms in a school choice problem or a college admission problem is to have an equitable preference for students, while in reality students' preferences are treated asymmetrically. To capture this feature, extended from two similar class of Constrained Choices mechanisms introduced by Haeringer and Klijn (2009) and Chen and Kesten(2017) respectively, this paper introduce a class of mechanism in a college admission problem with strict and homogeneous priorities, which is called the class of Progressive Choices mechanisms (PC). The class of Progressive Choices mechanisms could be regarded as transition mechanisms between the Immediate Acceptance (IA) mechanism and the Serial Dictatorship mechanism (SD). PCM falls into the class of mechanisms that satisfies Pareto efficient. I also find this class of mechanisms is less manipulable than IA but more manipulable than SD and has less justified envy than IA but has more justified envy than SD. In all, this paper makes a first attempt to examine a serial mechanisms with asymmetric treatment of students.

*Keywords:* homogeneous priorities; college admission; stability

## 1 Introduction

School Choice problem has been among the most widely debated education policies in various countries, which is expected to allocate students to their ideal schools as well as create healthy competitive environment between schools. A widely used school choice mechanism is Boston mechanism, which was once used to match medical graduates to positions in regions of the British National Health Service (Roth 1990, 1991) and in Boston school choice. One of the most obvious shortcomings of the Boston mechanism is that if a student is not get enrolled into his first choice school, his second option may already taken by other students who listed it as their first choice. Meanwhile, presented with theoretical analysis (Ergin and Sönmez 2006) and experimental evidence (Chen and Sönmez 2006) that the Boston mechanism is vulnerable to strategic manipulation. Boston School Committee in later years decided to replace it with an alternative mechanism that removed the incentives to manipulations that handicapped the Boston mechanism. To minimize opportunities to manipulate, New York City has turned largely to rules based on the deferred acceptance mechanism (Gale and Shapley 1962), which has a profound merit of strategy-proof. In 2003, a paper produced by Abdulkadiroğlu and Sönmez (2003) formulated school choice as a mechanism design problem and proposed two alternative mechanisms, which is motivated by K-12 public school admissions in the United States. Since that paper was published, more cities in the United States considered to reform their school choice systems including New Orleans, Denver, and Washington DC. New systems have also been developed in Brazil, Britain, Netherlands, a number of Asian countries, and elsewhere.

Similar with school choice in the United States, the university and college admissions are among the most intensively debated public policies in the past four decades in China. Unlike university admission in North America, university and college admissions in China is a centralized school choice problem which first appeared in late 1980s and began to take shape in the early 1990s. The *GaoKao* or the *College Entrance Exam*, is an examination that is taken by Chinese students in their third and final year of high school typically from June 7 to June 8 or 9. It is also the lone criterion for admission into Chinese universities. Based on the outcome of the college entrance exam, The sequential mechanism, which until recently used to be the only mechanism used in Chinese student assignments both at the high school and university level, is what is often referred as the Boston mechanism in the school choice literature (Nie 2007).

Just like common drawbacks of the Boston mechanism, the previous college entrance examination enrolment system, existed several prominent shortcomings, for example: insufficient information for candidates, high-score candidates have risk to be admitted into a relatively low rank university or fail to be admitted, since only when the first choices are not satisfied with the admission, can the second-choice students be enrolled. As there's a general dissatisfaction with the Sequential mechanism, the Parallel mechanism was proposed. This mechanism improves satisfactory of candidates. Based on survey and interview data from Shanghai in 2008, the first year when Shanghai adopted the Parallel mechanism for college admissions, Hou et al. (2009) find there is a 40.6 % decrease in the number of candidates who refused to go to the universities they were matched with, compared to the year before when the Sequential mechanism was in place.

Theoretically, the college and university admission in China can be considered as a special case of matching game in the one-sided market with identical sequential priorities for every universities and colleges, since the priorities for each universities and colleges depend and only depend on the scores of students in the College Entrance Examination. On the other side of the market, students tend to evaluate different universities with quite similar standards such as academic reputation, employment rate after graduation, and locations of the universities in order to maximize their achievements from the university-level studies, which causes them to have similar ordinal preferences.

Application strategies and admission mechanism are the key and complex subsystems in the enrolment of universities. It is also the focus and difficulty of the college entrance examination system reform. In this paper, I call such a centralized college admission problem as *a school choice model with homogeneous priorities*. What's more, instead of following a common assumption of equitable students' preference, a class of Progressive Choices mechanisms with asymmetrical students' preference is introduced in this paper. In this class of progressive choices mechanisms, comparing with class of Constrained

Choices mechanisms, by grouping students based on their scores, high scoring students have fewer choices that could be considered than low scoring students, while students in the same group share the same number of choices. The intuition behind that is for high-scoring students they are less risky than those low-scoring students since they know they are more likely to get admitted. By examining the properties, my main finding is that: the class of Progressive Choices mechanisms could be regarded as transition mechanisms between the Immediate Acceptance (IA) mechanism and the Serial Dictatorship mechanism (SD). PCM falls into the class of mechanisms that satisfies Pareto efficient. I also find this class of mechanisms is less manipulable than IA but more manipulable than SD, while has less justified envy than IA but has more justified envy than SD. There are several merits of these mechanisms such as it encourages low-scoring students to have courageous options and relieve application difficulties to some extent.

The rest of this paper is organized as follows. Section 2 formally set up the model of school choice problem. Section 3 introduces related mechanisms illustratively with detailed examples and procedures. Section 4 summarizes properties of the class of Progressive Choices mechanisms. Section 5 presents conclusions.

# 2 Model

A School Choice problem (Abdulkadiroğlu and Sönmez 2003) studies an allocation strategy to assign a number of students to schools. Furthermore, a student could not accept more than one school at the same time, while a school could not accept students more than its maximum capacity.

Denote the number of students as $\mathbf{n}$ and the number of schools as $\mathbf{k}$. Each student $i$ shall be assigned to a school $s$ or remain unmatched. I refer an unmatched student $i$ as being matched with himself. Each school $s$ has a maximum number of seats $q_s$.

Formally, the finite set of students is denoted as $\boldsymbol{I} = \{i_1,i_2,...,i_n\}$, the finite set of schools is denoted as $\boldsymbol{S} = \{s_1,s_2,...,s_k\}$ and the *maximum seats* of each school: $\boldsymbol{Q} = \{q_{s_1}, q_{s_2}, ..., q_{s_k}\}$. For simplicity, I denote $i$ as an arbitrary student that $i \in I$, denote $s$ as an arbitrary school that $s \in S$ and denote $q_s$ as the maximum seats of school $s$ that $q_s \in Q$.

Furthermore, each school $s$ has a strict priority order $\succ_s$ over the set of students and each student $i$ has a strict preference ordering profile $P_i$ over the set of schools. Particularly, a weak preference ordering profile for each student is denoted as $R_i$. That is, for any $s$ and $s$' $\in S$, $sR_is'$ implies either $sP_is'$ or $s = s'$. Formally, the *strict school priorities* over an individual student $\succ_s$ is denoted as $\succ = \{\succ_{s_1}, \succ_{s_2}, ..., \succ_{s_k}\}$, the *strict student preference profiles* $\boldsymbol{P}$ is denoted as $\boldsymbol{P} = \{P_{i_1}, P_{i_2}, ..., P_{i_n}\}$, and the *weak student preference profiles* $\boldsymbol{R}$ is denoted as $\boldsymbol{R} = \{R_{i_1}, R_{i_2}, ..., R_{i_n}\}$. For simplicity, I denote $\succ_s$ as the strict priorities the school $s$ and denote $P_i$ as strict preferences of student $i$ and denote $R_i$ as weak preferences of student $i$.

In addition to above, students $\forall i \in I$ are partitioned and formed into *groups*, and the maximum number of groups should not exceed the number of students. Denote the $\boldsymbol{G}$ as the finite set of groups and the maximum number of groups as $\beta$ , $\boldsymbol{G} = \{g_1, ..., g_\beta\}$, where $\beta \leq n$. The *size of the groups* restricts the maximum number of students within the same group. Formally, the finite set of the group size for each group $g$ is denoted as $\boldsymbol{M} = \{m^1, ..., m^g\}$, where $\Sigma_\alpha^g m^\alpha = n$.

Denote $i^g$ as a specific student in group $g$, where $i \in I$, $g \in G$. The alternative students set under grouping could be denoted as $\mathbf{I^g} = \{i_1^1, ..., i_n^g\}$. By partitioning students, students' preferences are treated asymmetrically. Students in separate groups are restricted with different maximum number of proposals in each step of a matching rule. Once getting rejected, if available proposals have not exhausted, a student could make another proposal in the same step instead of dropping into the next step. A student can at most propose to all schools in the same step. Formally, given the set of student preference profiles $\boldsymbol{P}$, the auxiliary preference profiles set $\hat{\mathbf{P}}$ is jointly introduced to describe the *maximum number of proposals* allowed for each student in the same step: $\hat{\mathbf{P}} = \{P^1, ..., P^g\}$, where $P^g \leq k$.

A *standard school choice problem*, or simply a *standard problem*, could be denoted as a priority - preference pair $(\succ, \boldsymbol{P})$ given a three-tuple $(\boldsymbol{I}, \boldsymbol{S}, \boldsymbol{Q})$.

Alternatively, a *school choice problem with grouping*, or simply a *generalized problem*, could be denoted as a priority - preference pair ($\succ$, $\boldsymbol{P}$) given a six-tuple ($\mathbf{I^g}$, $\boldsymbol{S}$, $\boldsymbol{Q}$, $\boldsymbol{G}$, $\boldsymbol{M}$, $\mathbf{\hat{P}^g}$ ). Let $\mathcal{Y}$ be the set of these particular problems.

Considering a *generalized problem* ($\succ$, $\boldsymbol{P}$) and two matchings $\mu$ and $\nu$:

A *matching pair* or simply a *matching* $\mu$ is set-valued mapping of students with schools such that each student is assigned to one school and the maximum students that a school could admit should not exceed its maximum seats. Formally, a matching is any mapping $\mu : \mathbf{I^g} \rightarrow \boldsymbol{S}$ such that $\forall\ s \in \boldsymbol{S}$, $|\mu^{-1}(s)| \leq q_s$. Note that, in a matching $\mu$, denote $\mu(i^g)$ to be the matched school of student $i^g$, particularly $\mu(i^g) = i^g$ indicates an unmatched student $i^g$. Denote $\mu^{-1}(s) = i^g$ to be the matched student of school $s$. Let $\mathcal{X}$ be the set of all matchings. For simplicity, we denote $\mu(i^g)$ and $\mu^{-1}(s)$ as $\mu_i$ and $\mu_s$ respectively.

A matching $\mu$ violates *individually rational* if it is blocked by an individual. Formally, a matching violates individually rational if and only if $\exists$ $i \in \mathbf{I^g}$, $i^g P_i \mu_i$. A matching $\mu$ is *non-wasteful* if no student prefers a school with available seats to his current matching. Formally, $\forall\ i^g \in \mathbf{I^g}$, $sP_i\mu_i$ implies $|\mu_s| = q_s$; A matching $\mu$ violates *justified envy-free* if there exists a student-school pair ($i$, $s$) such that $sP_i\mu_i$ and $i \succ_s j$ for some $j$, $\mu_j = s$. A matching is *stable* if it is individually rational, non-wasteful and satisfies justified envy-free.

A matching $\mu$ *weakly Pareto dominates* $\nu$ if $\forall i \in \mathbf{I^g}$, $\mu_i R_i \nu_i$ and matching $\mu$ *Pareto dominates* $\nu$ if $\forall i \in \mathbf{I^g}$, $\mu_i R_i \nu_i$ and for at least one student $\exists\ i' \in \mathbf{I^g}$, $\mu_{i'} P_{i'} \nu_{i'}$. The matching $\mu$ is *Pareto efficient* if it is not Pareto dominated by any other matchings.

A *matching rule* $\phi$, or simply a *mechanism* or an *algorithm*, is a mechanism that solves each school choice problem by assigning matchings: $\phi : \mathcal{X} \rightarrow \mathcal{Y}$. The set of selected matching is *the matching outcomes* which is denoted as $\phi(\succ, P)$. For a student $i \in \mathbf{I^g}$ whose matching outcomes selected by $\phi$ is denoted as $\phi_i(\succ, P)$ . A matching rule is *Pareto efficient* if the finalized matching outcomes are Pareto efficient. Formally, there is no $\bar{\mu} \in \mathcal{X}$, such that $\bar{\mu}_i \bar{R}_i \mu_i$ for all $i \in \mathbf{I^g}$, and $\bar{\mu}_j P_j \mu_j$ for some $j \in \mathbf{I^g}$. A matching rule $\phi$ is *strategy-proof* if reporting true preferences is each student's weakly dominant strategy. Formally, a mechanism $\phi$ is strategy-proof if and only if $\nexists\ i \in \mathbf{I^g}$ and $P'_i \in \boldsymbol{P}$ such that $\phi_i(\succ, P'_i, P_{-i})P_i\phi_i(\succ, P)$.

In a mechanism $\phi$, for a problem $(\succ, P)$, a school $s$ is *manipulable* to student $i$ if there exists at least a preference profile $P_i$ and an alternative preference $P'_i$ such that: $s = \phi_i(\succ, P'_i, P_{-i})P_i\phi_i(\succ, P)$. Thus, such a problem is said to be a *manipulatable problem.* A mechanism $\phi$ is *a manipulable mechanism* if there exists at least one manipulatable problem $(\succ, P)$. If (1) all manipulable problems in a mechanism $\phi$ is also manipulable in a mechanism $\nu$ and (2) there exists at least one manipulable problem in the mechanism $\nu$ is unmanipulable in the mechanism $\phi$, then I say a mechanism $\phi$ is *less manipulable* then a mechanism $\nu$. By the definition of stability above, it is straightforward to conclude: If all matching outcomes in a mechanism $\phi(\succ, P)$ are stable, then I say a mechanism $\phi$ is a *stable mechanism.* Further, If (1) all stable matching outcomes in a mechanism $\phi$ are also stable in a mechanism $\nu$ and (2) there exists at least one stable outcome in the mechanism $\nu$ that is unstable in the mechanism $\phi$, then I say a mechanism $\phi$ is *less stable* then a mechanism $\nu$.

# 3 Mechanism Procedures

A national wide college admission problem creates a homogeneous priority for each school, since the only judging criteria for each school is the outcome of some typical exams such as the College Entrance Exam in China. On the other hand, families tend to value similar qualities about schools (e.g. academic reputation, employment rate after graduation and etc.), which causes them to have similar ordinal preferences. The common evaluations on different mechanisms such as Pareto efficiency or stability then loses its relevance; For instance, if all students have the common ordinal preferences and schools have the same priorities on every student, then any arbitrary assignment will meet these standard, and mechanisms become indistinguishable on these criteria. (Abdulkadiroğlu et al. 2011).

## 3.1 Boston Mechanism (The Sequential Choice mechanism)

The Boston mechanism, which was an equivalent allocation mechanism to the Sequential Choice mechanism in China's college admission problem in the 1980s and 1990s, is a popular and widely debated mechanism in the school choice problem. The outcome of Boston mechanism can be formalized as follows:

*Step 1*: In Step 1, for each school $s$, only the students who have listed it as their first choice are considered; A student $i$ with the highest scores $T_i$ among all proposer for school $s$ is admitted until all capacity $Q_s$ of school $s$ is filled and reject the rest students.

In general,
*Step k* $(k \geq 2)$: In Step k $(k \geq 2)$, the rejected students from previous steps are sent to their $k$-th choice colleges; If a college still has available seats, it accepts the remaining highest-scored students until all seats are filled, and the students left behind are rejected.

This mechanism terminates when all students have been matched or when colleges leave no capacity. The following example demonstrates how the Sequential Choice mechanism works.

**Example 1**
*(Refer to Appendix.A)*

The main idea of using this mechanism is to give students their first choices to the greatest extent possible, which could also be regarded as a preference-first mechanism. Its most important feature is every assignment during every step is ultimate. Therefore, Boston mechanism is also known as Immediate Acceptance mechanism (IA)(Abdulkadiroğlu and Sönmez 2003).

Although the criticisms of the Boston mechanism are multi-angled, essentially they are mostly triggered by its poor resistance of manipulation.

## 3.2 The Class of Constrained Choices mechanism

Abdulkadiroğlu et al. (2006) provide a potential evidence that some players may have behaved naively and suffered as a consequence under the Boston mechanism. They find that as much as 20% of the applicants ranked two over-demanded schools as their first and second choices. In order to alleviate the incentives of manipulation, in practice, the number of schools that each student allowed to list is constrained. This practice is first studied by Haeringer and Klijn (2009). Meanwhile, Chen and Kesten (2017) described the parametric mechanisms that many Chinese provinces have been using, which could be regarded as a Constrained Gale-Shapley mechanism, which could be formalized as follows:

*Step 1*: In Step 1, for each school $s$, only the students who have listed it as their first choice are considered; Then those students with the highest scores for each school are tentatively enrolled until all its capacity $Q_s$ is filled and reject the rest students.

*Sub-Step 1*: Each rejected student, who is yet to apply to his $\alpha^{th}$ choice school, his next choice is automatically considered. The allocation is finalized every $\alpha^{th}$ choices. That is, if a student is rejected by all her $\alpha$ choices after the Sub-Step 1, then he moves onto step 2 together with other students who have also been rejected from their first $\alpha^{th}$ choices. The matching is final for matched students, while unmatched students proceed to Step 2.

In general,

*Step k* ($k \geq 2$): Each unassigned student from the previous round applies to his $[\alpha(k-1)+1]^{th}$-choice school. Each school $s$ considers its applicants. Those students with the highest scores for each school are tentatively enrolled until all its capacity $Q_s$ is filled and reject the rest students.If all preferences of a student have been considered but the student still cannot get matched, he will be matched with *the null school* $\varnothing$.

*Sub-Step k*: Procedures are similar with Sub-Step 1.

The process terminates when all students have been matched or when colleges leave no capacity. All tentative matches turn to be the final ones. The following example demonstrates how Gale-Shaply-Type Constrained Choices mechanism works.

**Example 2**

*(Refer to Appendix.A)*

Pathak and Sönmez(2013) described the assignment problem for Chicago's selective high schools under a Constrained Serial-Dictatorship mechanism, which is similar with the Constrained Gale-Shapley mechanism. These two mechanisms are versions of widely studied assignment mechanisms for assigning students to schools. Any version of the Boston mechanism, including the version abandoned in Chicago, is manipulable (Pathak and Sönmez 2013), however, all these newly-proposed mechanism is less vulnerable to manipulation than Boston mechanism.

## 3.3 The Class of Progressive Choices mechanism

Inspired by these existed mechanisms, I now introduce a class of Progressive Choices mechanism (PC), which is a general Constrained Choices of mechanism.

The main difference is that all students are divided into $\theta$ groups ($\theta \in \{1, ..., \boldsymbol{N}\}$) and are ranked by their outcome of the exam $\boldsymbol{T}$. The highest-score student(s) are categorized in the $1^{st}$ group, while the lowest-score student(s) are in the $\theta^{st}$ group. The group size[1] $m$ ranges from $0$[2] to $\boldsymbol{N}$. Denote the group size $m$ of group $g$ by $m^g$. The group number is denoted as $g$ ($g \in [1, \theta]$). Denote student $i^g$ ($i^g \in \boldsymbol{I}$, $g \in [1, \theta]$ ) for a student from group $g$. The matches $\mu$ with a student $i^g$ from group $g$ is denoted as $\mu^g$.

In this class of mechanisms, contrary to the Boston mechanism and the

[1] The group size of different groups does not have to be equal.

[2] This implies that empty group is allowed.

class of Constrained Choices mechanisms which students are equitable in preference, students' preferences are treated asymmetrically under Progressive Choice mechanism. I denote $p^g$ as the number of preference which could be considered in one step.

**Remark.** A general students' preferences could be considered in one step under the Class of Progressive Choices mechanisms $\phi^\theta$ is equivalent to:

For students $i^g$ and $i^{g'}$, if $g < g'$, it implies that $p^g < p^{g'}$, while if $g = g'$, it implies that $p^g = p^{g'}$, ($g \in [1, \theta]$, $p^g \in [1, \boldsymbol{S}]$ )

Intuitively, high scoring students have fewer choices that could be considered than low scoring students, while students in the same group share the same number of choices. We now could also realize another feature of this class of mechanisms is that within the same groups, the game between the students are exactly the class of Constrained Choices mechanisms.

**Remark.** The class of Constrained Choices mechanisms could be regraded as the sub-mechanisms or the inner-group mechanisms if the group size is larger than 1(m >1), since students within the same groups have identical number of choices which could be considered in the same step.

The process could be formalized as follows:

*Step 0*: Partition students based on exam scores $T$ from high to low into groups.

*Step 1*: Consider the $1^{st}$ to $(p^g)^{th}$ choice(s) of each student. Each school $s$ tentatively enrolls students until its capacity is filled. Redundant students are rejected.

*Sub-Step 1*: Consider the next choice of each rejected student up to his $(p^g)^{th}$ choice. Each school $s$ tentatively enrolls higher-priority students and rejects lower-priority ones if its capacity is used up.

All matches become final;unmatched students proceed to Step 2.

From *Step 2*: Repeat the algorithm until either all students have been matched or when schools use up their capacity.

In general,

*Step k* ($k \geq 2$): Each unassigned student from the previous round applies to his $[p^g(k-1)+1]^{th}$-choice school. Each school $s$ considers its applicants. Those students with the highest scores for each school are tentatively enrolled until all its capacity $Q_s$ is filled and reject the rest students. If all preferences of a student have been considered but the student still cannot get matched, he will be matched with *the null school* $\varnothing$.

*Sub-Step k*: Procedures are similar with Sub-Step 1.

The process terminates when all students have been matched or when colleges leave no capacity. All tentative matches turn to be the final ones. The following example demonstrates how Gale-Shaply-Type Constrained Choices mechanism works.

**Example 3**

*(Refer to Appendix.A)*

The following section will theoretically analyze the properties of the class of Progressive Choices mechanism.

# 4 Properties

The Class of Constrained Choices mechanisms nests the Immediate Acceptance mechanism (IA) and the Deferred Acceptance mechanism (DA) as extreme cases (Chen et al. 2017). The Class of Progressive Choices mechanisms (PCM) introduced in this paper could be regraded as a transition between Immediate Acceptance mechanism (IA) and Serial Dictatorship mechanism (SD). I denote such a mechanism under strict and homogeneous priorities as $\phi^\theta$.

**Remark.** The Class of Progressive Choices mechanisms $\phi^\theta$ is equivalent to:

- the Immediate Acceptance mechanism (IA), when g = 1.
- the Serial Dictatorship mechanism (SD) , when g = $\mathbf{N}$.

A brief proposition is illustrated below with a first glance at this class of mechanisms.

**Proposition 4.1.** Within all the class of Progressive Choices mechanisms $\phi^{\theta}$, $\theta \in (1, N)$ :

- All mechanisms within the class are non-wasteful.
- All mechanisms within the class are Pareto efficient.
- There exists one and only one mechanism that is strategy-proof, which is the SD.
- There exists one and only one mechanism that satisfies justified envy-free, which is the SD[3].
- There exists one and only one mechanism that is stable, which is the SD.

*Proof. (Refer to Appendix.B)* □

The major feature of this class of Progressive Choices mechanisms is that it allows lower-scoring students have more chances to propose if get rejected than high-scoring students, which progressively decreases potential loss of priority advantage.

**Theorem 4.2.** (Manipulation between-mechanisms)

Let $\phi$ be a mechanism within the class of Progressive mechanisms. For any $\theta$ ($\theta \in [N]$); $\phi^{\theta}$ is more manipulable than $\phi^{\theta'}$ where $\theta$ <$\theta$'.

**Corollary 4.2.1.** Among IA, PCM and SD, IA is the most manipulable mechanism and SD is a unmanipulable mechanism.

**Theorem 4.3.** (Manipulation between-groups)

For a certain $\phi^{\theta}$; If $m^g$>0, $i^g$ has more incentives to manipulate than $i^{g'}$ where g <g'.

[3]Serial Dictatorship is not stable under heterogeneous priority since it cannot eliminate justified envy-free.

*Proof.* *(Refer to Appendix.B)* □

Beside the degree of manipulation, we could make a conjecture on the properties of *justified envy-free*. My problem-wise comparison is related to Abdulkadiroğlu et al.(2020), who define minimal justified envy-free mechanisms and compare the class of mechanisms that satisfy pareto efficient and strategy-proof.

**Conjecture 4.4.** (Justified envy-free between mechanisms)

Let $\phi$ be a mechanism within the class of Progressive mechanisms. For any $\theta$; $\phi^{\theta}$ has more justified envy-free than $\phi^{\theta'}$ where $\theta$ <$\theta$'.

**Corollary 4.4.1.** Among IA, PCM and SD, IA has the most justified envy mechanism and SD eliminates justified envy.

**Conjecture 4.5.** (Justified envy-free between groups)

For a certain $\phi^{\theta}$; student-school matches $\mu^{g}$ has more justified envy than $\mu^{g'}$ where g <g'.

*Proof.* (to be done in the future) □

If I could prove the previous theorem of manipulation and justified envy, the following theorem is quite straightforward.

**Theorem 4.6.** (Stability between-mechanisms)

Let $\phi$ be a mechanism within the class of Progressive mechanisms. If any $\theta$ ($\theta \in [N]$); $\phi^{\theta}$ is more manipulable than $\phi^{\theta'}$, while $\phi^{\theta}$ has more justified envy-free than $\phi^{\theta'}$, $\phi^{\theta}$ is less stable than $\phi^{\theta'}$, where $\theta$ <$\theta$'.

**Corollary 4.6.1.** (Stability between mechanisms)

Among IA, PCM and SD, IA is the least stable mechanism and SD is a stable mechanism.

**Theorem 4.7.** (Stability between-groups) Group $g$ is less stable than Group *g'* under the class of PCM if:

If $m^{g}$>0, $i^{g}$ has more incentives to manipulate than $i^{g'}$, while student-school matches $\mu^{g}$ has more justified envy than $\mu^{g'}$, where g <g'.

# 5 Conclusion

In this paper, I compare and introduce a class of mechanism which is called Progressive Choices mechanism. The largest features of Progressive Choices mechanism is grouping. By grouping students based on their scores, high scoring students have fewer choices that could be considered in one step than low scoring students. I showed that the class of Progressive Choices mechanisms which nests the IA and the SD mechanisms as extreme cases is less manipulable than IA but more manipulable than SD, while students with higher-scoring have more incentives to manipulate than lower-scoring students. In the meantime, PCM has less justified envy than IA but has more justified envy than SD. We should also notice that the inner-group game between students are exactly operated as the Constrained Gale-Shaply Mechanisms introduced by Chen and Kesten (2017). So far, I now extend the class of Constrained Gale-Shaply into a more general one with asymmetric mechanisms.

There's some merits of this class of mechanisms:

- **Securing students who under-performed accidentally**: If the application process begins before the exam, students have to predict how much they will score. However, everything might happen in the exam, if a good student submit a risky preference but didn't do well in the exam, he might be get unmatched under Boston or Constrained Choices mechanism. By grouping, under constrained Choices mechanism, those students will have more chances to get admitted.

- **Encourage courageous choices for lower-score students**: Contrary to Boston mechanism or the Class of Constrained Choices mechanism which identical options are considered for every student in one single step, lower-ranked students under the class of Progressive Choices mechanisms have less incentives to manipulate.

- **Protect naive applicants and relieve application struggling**: Grouping reduces everyone's difficulties of making applications. For high-scoring students, they could narrow down their competitors on over-demanding schools. For lower-scoring students, their choices are more guaranteed since they obtain extra options.

- **Convey signals to everyone**: Besides the test scores, students have extra information of their positions and may more aware about others which contributes to make better applications
- **Diverse grouping methods**: A personalized number of groups or group size could be set according to real needs.

In all, this paper is the first attempt to take a close look into an mechanism with asymmetric students' preference and this paper restricts the strict priority of different schools within the homogeneity. More studies could extend into heterogeneous priorities as well as in other aspects.

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# A Appendix: Illustrative Examples

Consider the following example with a given homogeneous priority structure and the profile of preferences. The set of students is $\boldsymbol{I} = \{i_1,i_2,i_3,i_4,i_5\}$, and the set of schools is $\boldsymbol{S} = \{s_1, s_2, s_3, s_4, s_5\}$. Each school, $s_1$, $s_2$, $s_3$, $s_4$ and $s_5$, has a quota of one. The homogeneous priority $\boldsymbol{T}$ for each school is $T_s : i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5$ and the students' profile of preferences $\boldsymbol{P}$ is listed below:

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_5$ |
| $s_3$ | $s_3$ | $s_2$ | $s_4$ | $s_3$ |
| ... | $s_5$ | $s_5$ | $\varnothing$ | $\varnothing$ |
| | ... | $\varnothing$ | ... | ... |

I now illustrate the steps and outcomes under different mechanisms. The grey shades represent for all potential preferences that could be considered in the same step.

**Example 1**:*Boston mechanism*

*Step 1:* Consider the first option of all students.

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_5$ |
| ... | ... | ... | ... | ... |

In Step 1, Student $i_1$ and $i_2$ propose to School $s_2$; Student $i_3$ proposes to School $s_1$; Student $i_4$ propose to School $s_3$; $i_5$ propose to School $s_4$.

Student $i_1$ is matched with School $s_2$; Student $i_3$ is matched with School $s_1$; Student $i_4$ is matched with School $s_3$; Student $i_5$ is matched with School $s_4$; All matches are **final** matches; Student $i_2$ is rejected and moves to the next step.

*Step 2:* Consider the second option of remained unmatched student $i_2$.

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_5$ |
| ... | ... | ... | ... | ... |

In Step 2, Student $i_2$ proposes to School $s_1$ but gets rejected and moves to the next step.

*Step 3:* Consider the third option of the remained unmatched student $i_2$.

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_5$ |
| ... | $s_5$ | ... | ... | ... |
| ... | ... | ... | ... | ... |

In Step 3, Student $i_2$ proposes to School $s_5$ and get matched. The mechanism terminates. The final outcome of Boston mechanism is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_2 & s_5 & s_1 & s_3 & s_4 \end{pmatrix}$$

**Example 2**: *The Class of Constrained Choices mechanism*

Consider a case when $\alpha = 2$

*Step 1* Consider the first option of all students.

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_5$ |
| ... | ... | ... | ... | ... |

In Step 1, Student $i_1$ and $i_2$ propose to School $s_2$; Student $i_3$ proposes to

School $s_1$; Student $i_4$ proposes to School $s_3$; Student $i_5$ proposes to School $s_4$;

Student $i_1$ is matched with School $s_2$; Student $i_3$ is matched with School $s_1$; Student $i_4$ is matched with School $s_3$; Student $i_5$ is matched with School $s_4$; All matches are **tentative** matches; Student $i_2$ is rejected by School $s_2$ and moves to the sub-step 1.

*Sub-Step 1* Consider the first two options of rejected students.

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_5$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

In Sub-Step 1, Student $i_2$ proposes to School $s_1$ and takes the place of Student $i_3$ who is previously matched. Student $i_3$ will then propose to School $s_3$ and take place of Student $i_4$. Then, Student $i_4$ will then propose to School $s_2$ but gets rejected since School $s_2$ has a preferred student $i_1$. All matches are now **final** matches; Student $i_4$ is rejected by both his top two choices and moves to the step 2.

*Step 2:* Consider the third option of remained unmatched student $i_3$.

*Sub-Step 2:* Consider the second two options of remained unmatched student $i_3$.

| $P_{i_1}$ | $P_{i_2}$ | $P_{i_3}$ | $P_{i_4}$ | $P_{i_5}$ |
|---|---|---|---|---|
| $s_2$ | $s_2$ | $s_1$ | $s_3$ | $s_4$ |
| $s_1$ | $s_1$ | $s_3$ | $s_2$ | $s_3$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $s_4$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\varnothing$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

In Step 2, Student $i_4$ proposes to School $s_4$. Although School $s_4$ prefers

Student $i_4$ to its current pair Student $i_5$, it will reject Student $i_4$ since its match with Student $i_5$ is finalized, which implies that it forms a potential **blocking pair** ($i_4$,$s_4$).

In Sub-Step 2, Student $i_4$ proposes to null School $\varnothing$ and gets matched.

In Step 3, Student $i_2$ proposes to School $s_5$ and get matched. This match is the **final** match. The mechanism terminates. The final outcome of this Constrained Choices mechanism is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_2 & s_1 & s_3 & \varnothing & s_4 \end{pmatrix}$$

**Example 3**:*The Class of Progressive Choices mechanism*

Consider a case when $\theta = 3$ with a grouping formulated as: Group 1= $\{i_1^1\}$, Group 2= $\{i_2^2$,$i_3^2\}$, and Group 3= $\{i_4^3$, $i_5^3\}$,while $p^{g^1} = 1$, $p^{g^2} = 2$, $p^{g^3} = 3$.

*Step 1* Consider the first option of all students.

| $P_{i_1^1}$ | $P_{i_2^2}$ | $P_{i_3^2}$ | $P_S$ |
|---|---|---|---|
| $s_2$ | $s_2$ | $s_2$ | $i_1$ |
| $\varnothing$ | $s_3$ | $s_3$ | $i_2$ |
| ... | $s_1$ | $s_1$ | $i_3$ |

The outcome of Step 1 in this case is the same as the one in previous example. Student $i_1^1$ is matched with School $s_2$; Student $i_3^2$ is matched with School $s_1$; Student $i_4^3$ is matched with School $s_3$; Student $i_5^3$ is matched with School $s_4$; All matches are **tentative** matches; Student $i_2^1$ is rejected by School $s_2$ and move to the sub-step 1.

*Sub-Step 1* Consider the first m options of rejected students in Group m.

| $P_{i_1^1}$ | $P_{i_2^2}$ | $P_{i_3^2}$ | $P_S$ |
|---|---|---|---|
| $s_2$ | $s_2$ | $s_2$ | $i_1$ |
| $\varnothing$ | $s_3$ | $s_3$ | $i_2$ |
| ... | $s_1$ | $s_1$ | $i_3$ |

In Sub-Step 1, Student $i_2^2$ proposes to School $s_1$ and takes the place of Student $i_3^2$ who is previously matched. Student $i_3^2$ will then propose to School $s_3$ and replace $i_4^3$ , Student $i_4^2$ will then propose to School $s_2$ but gets rejected since School $s_2$ has a preferred student $i_1^1$. Then, he will continue to propose to $s_4$ and replace Student $i_5^3$. After rejected by School $s_3$ and $s_4$, Student $i_5^3$ get matched with School $s_5$ in his final proposal in this step. All matches are now **final** matches and the mechanism terminates. The final outcome of this Constrained Choices mechanism is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_2 & s_1 & s_3 & s_4 & s_5 \end{pmatrix}$$

# B Appendix: Omitted Proofs

*Proof.* (Proof of Proposition 1)

*Non-wasteful*: It is straightforward to verify intuitively that all members of the class of Constrained Progressive mechanisms are non-wasteful. The only possibility that a school rejects a student is $i$ when there's no seats left of this school and students it tentatively matched with have higher priorities than $i$. Thus, if a student when proposes to his more favourite school than his current match, it's impossible for that school has seats but turns down his offer or running out seats but matching with a lower-priority student. The Boston and SD are known as non-wasteful mechanisms.

*Pareto efficient*: With a strict and homogeneous preference, no matter under which mechanisms, the highest priority student always get his first choice, the second-highest priority student will make his best choice excluding the choice made by highest priority student, and so on. Therefore, the

Boston, SD and the class of Constrained Progressive mechanisms are Pareto efficient.

*Justified envy-free*: Under strict and homogeneous priority where student proposals are ranked with their scores, the serial dictatorship eliminates justified envy. We can prove that by contradiction. If serial dictatorship cannot eliminate justified envy, it implies that there exists at least one blocking pair (i,s). For school $s$, if it forms a block pair with student $i$, it implies its current match student *i'* has a low-priority than student $i$. Under the procedure of SD, a higher-priority student will always get matched earlier than lower-priority students, then it is not possible that a lower-priority student *i'* is preferred and get matched with school $s$ before the proposal of a higher-priority student $i$. Thus, by contradiction, SD eliminates justified envy. In my example 2, I have already showed that Constrained Choices mechanisms can form a blocking pair. As a sub-game of Constrained Progressive mechanisms, students in the same group where group size is larger than one could also form a blocking pair. Thus, the Constrained Progressive mechanisms like Boston mechanism cannot eliminate justified envy.

*Stable*: Since SD is the only one mechanism that is strategy-proof, non-wasteful and justified-envy, SD is the only mechanism that is stable. □

*Proof.* (Proof of Theorem 4.1) (to be done in future) □

*Proof.* (Proof of Theorem 4.2)

Bonkoungou et. al (2020) prove that the constrained Gale-Shapley mechanism $GS^k$ is less strategically accessible than the constrained Gale-Shapley mechanism $GS^l$, when $k > l$. In this paper, We could simply consider every separate group to be a sub-mechanism of PCM, which operates exactly the same as Constrained Gale-Shapley mechanism if group size is larger than zero. Denote such sub-mechanism as $phi^g$. For any $g < g'$, $p^g < p^{g'}$. Thus, $phi^g = GS^l$ $phi^{g'} = GS^k$. Therefore, for a certain $\phi^\theta$; If $m^g$>0, $i^g$ has more incentives to manipulate than $i^{g'}$ where g <g'. □