A Note on High Order Finite Volume Methods

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Abstract

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1 Introduction

This is a note of high order finite volume methods of textbook[**?**]. There are three steps of a finite volume scheme: the reconstruction, the evolution and the projection step.

2 Preliminaries

2.1 1D nonuniform grids

Grven a grid

$$a = x_{\frac{1}{2}} < \ldots < x_{N+\frac{1}{2}} = b, \tag{1}$$

We define cells, cell centers, and cell sizes by,

$$I_i \equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], x_i \equiv \frac{1}{2}\left(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}\right), \Delta x_i \equiv x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, i = 1, 2, \ldots, N \tag{2}$$

We denote the maximum cell size by

$$\Delta x \equiv \max_i \Delta x_i \tag{3}$$

2.2 2D Cartesian grids

The computational domain is a rectangle

$$[a, b] \times [c, d] \tag{4}$$

covered by cells

$$I_{ij} \equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}], 1 \le i \le N_x, 1 \le j \le N_y, \tag{5}$$

where

$$a = x_{\frac{1}{2}} < \ldots < x_{N_x+\frac{1}{2}} = b,$$

and

$$c = y_{\frac{1}{2}} < \ldots < y_{N_y+\frac{1}{2}} = b.$$

The centers of the cells are

$$x_{ij}^c \equiv (x_i, y_j), x_i \equiv \frac{1}{2}\left(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}\right), y_j \equiv \frac{1}{2}\left(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}\right) \tag{6}$$

and we use

$$\Delta x_i \equiv x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, i = 1, 2, \ldots, N_x \tag{7}$$

and

$$\Delta y_j \equiv y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}, j = 1, 2, \ldots, N_y \tag{8}$$

to denote the grid sizes.

We denote the maximum gird sizes by

$$\Delta x \equiv \max_i \Delta x_i, \Delta y \equiv \max_j \Delta y_j, \tag{9}$$

and assume that Δx and Δy are of the same magnitude.Finally,

$$\Delta \equiv \max(\Delta x, \Delta y)$$

2.3 Unstructured mesh

A computational mesh $\mathcal{T}$ for Ω is a set of compact polygons or polyhedrons, if $d = 2$ or $d = 3$, respectively, such that

$$\overline{\Omega} = \cup_{K \in \mathcal{T}} K \tag{10}$$

Here K is called an element of a computational mesh $\mathcal{T}$. The boundary ∂K of an element K is the union of its faces or edges.

We require the following properties:

- For any $i \neq j, K_i \cap K_j$ is of empty interior,
- K_i is connected.
- The boundary of K_i is a polygonal line with at most N_0 vertices.

2.4 Structured mesh

3 Equations

Here are the Euler equations:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{0} \tag{11}$$

As usual, $\boldsymbol{U}$ stands for the vector of conservable variable and $\boldsymbol{F}$ is the flux:

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}, \boldsymbol{F} = [F, G, H] = \begin{bmatrix} \rho u & \rho v & \rho w \\ \rho u^2 + p & \rho uv & \rho uw \\ \rho uv & \rho v^2 + p & \rho vw \\ \rho uw & \rho vw & \rho w^2 + p \\ (E+p)u & (E+p)v & (E+p)w \end{bmatrix} \tag{12}$$

It is well known that the system defined by equations (11) is hyperbolic: for any non-zero vector $\boldsymbol{n} = (n_1, n_2, n_3)$, the matrix,

$$\boldsymbol{A_n} = \boldsymbol{n} \cdot (\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}) = n_1 \frac{\partial F}{\partial \boldsymbol{U}} + n_2 \frac{\partial G}{\partial \boldsymbol{U}} + n_3 \frac{\partial H}{\partial \boldsymbol{U}} \tag{13}$$

is diagonalizable with distinctive real eigenvalues and eigenvectors.

4 Finite volume formulation

A mesh $\mathcal{T}$ consisting of control volume $\{T_i\}$ is considered here. The semi-discrete finite volume formulation of (11) is,

4.1 Approximation of the evolution operator

The set of equations to be solved are supposed to be written in the form,

$$\frac{\mathrm{d}\boldsymbol{U}_h}{\mathrm{d}t} = \mathcal{L}(\boldsymbol{U}_h). \tag{14}$$

where the operator $\mathcal{L}$ contains spatial derivatives.

1. Second order scheme: It is TVD or SSP under CFL=1.

$$\begin{cases} U_h^{(1)} = U_h^n + \Delta t \mathcal{L}(U_h^n), \\ U_h^{(2)} = U_h^{(1)} + \Delta t \mathcal{L}(U_h^{(1)}), \\ U_h^{n+1} = \frac{1}{2}\left(U_h^n + U_h^{(2)}\right) \end{cases} \tag{15}$$

2. Third order scheme:

$$\begin{cases} U_h^{(1)} = U_h^n + \Delta t \mathcal{L}(U_h^n), \\ U_h^{(2)} = \frac{3}{4}U_h^n + \frac{1}{4}U_h^{(1)} + \frac{1}{4}\Delta t \mathcal{L}(U_h^{(1)}), \\ U_h^{n+1} = \frac{1}{3}U_h^n + \frac{2}{3}U_h^{(2)} + \frac{2}{3}\Delta t \mathcal{L}(U_h^{(2)}), \end{cases} \tag{16}$$

4.2 MUSCL

4.3 ENO

ENO scheme uses an adaptive stencil. The basic idea is to avoid including the discontinuous cell in the stencil, if possible.

Given the divided differences:

$$U[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] = \frac{U(x_{i+\frac{1}{2}}) - U(x_{i-\frac{1}{2}})}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \tag{17}$$

, in general the j-th degree divided differences, for j ≥ 1, are defined By

$$U[x_{i-\frac{1}{2}}, \ldots, x_{i+j-\frac{1}{2}}] \equiv \frac{U[x_{i+\frac{1}{2}}, \ldots, x_{i+j-\frac{1}{2}}] - U[x_{i-\frac{1}{2}}, \ldots, x_{i+j-\frac{3}{2}}]}{x_{i+j-\frac{1}{2}} - x_{i-\frac{1}{2}}} \tag{18}$$

4.4 WENO

It achieves this effect by adaptively choosing the stencil based on the absolute values of divided differences. However, one could make the following remarks about ENO reconstruction, indicating rooms for improvements,

References

Algorithm 1 1D ENO reconstruction

Require: the cell averages $\{\overline{u}_i\}$.

Ensure: A piecewise polynomial reconstruction of degree at most $k-1$ by ENO.

1: Compute the divided differences of the primitive function $U(x)$, for degree 1 to k, using $\overline{u}_i$

2: From the cell I_i with a two points stencil

$$S_1(i) = \{I_i\}$$

to get the reconstruction polynomial $P_1(x)$.

3: **for** l=2,...,k **do**, assuming

$$\overline{S}_l(i) = \{x_{j+\frac{1}{2}}, \dots, x_{j+l-\frac{1}{2}}\}$$

is known, add one of the two neighboring points $x_{j-\frac{1}{2}}, x_{j+l+\frac{1}{2}}$, to the stencil, following the ENO produce:

4: **if**

$$\left| U[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}, \dots, x_{j+l-\frac{1}{2}}] \right| < \left| U[x_{j-\frac{1}{2}}, \dots, x_{j+l-\frac{1}{2}}, x_{j+l+\frac{1}{2}}] \right|$$

then

5: add $x_{j-\frac{1}{2}}$ to the stencil $\overline{S}_l(i)$ to obtain

$$\overline{S}_{l+1}(i) = \{x_{j-\frac{1}{2}}, \dots, x_{j+l-\frac{1}{2}}\}$$

6: **else**

7: add $x_{j+l+\frac{1}{2}}$ to the stencil $\overline{S}_l(i)$ to obtain

$$\overline{S}_{l+1}(i) = \{x_{j+\frac{1}{2}}, \dots, x_{j+l+\frac{1}{2}}\}$$

8: **end if**

9: **end for**

10: Use the Lagrange form or the Newton form to obtain $p_i(x)$ which is a polynomial of degree at most $k-1$ in I_i. Get get approximations at the cell boundaries:

$$u^-_{i+\frac{1}{2}} = p_i(x_{i+\frac{1}{2}}), \quad u^+_{i-\frac{1}{2}} = p_i(x_{i-\frac{1}{2}})$$