STAT 5010 Homework 3 Nov, 14th, 2022 Zhou Xiaocheng 1155184323.

1. (a) Gamma (a,b) has density function $f(x|b) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-\frac{x}{b}}$

Let $\eta = -\frac{1}{b}$, then

$$f(x|\eta) = \frac{(-\eta)^a}{\Gamma(a)} x^{a-1} e^{\eta x} = \frac{1}{\Gamma(a)} \cdot x^{a-1} \cdot \exp\{x \cdot \eta + a \ln(-\eta)\}$$

Then the likelihood function of $\{X_i\}_{i=1}^n$ is

$$f(X_1, \dots, X_n | \eta) = \left[\frac{1}{\Gamma(a)}\right]^n \cdot \left[\prod_{i=1}^n X_i\right]^{a-1} \cdot \exp\{\eta \sum_{i=1}^n X_i + na \ln(-\eta)\}$$

$$\Rightarrow f(\eta | x_1, \dots, x_n) = \left[\frac{1}{\Gamma(a)}\right]^n \cdot \left[\prod_{i=1}^n x_i\right]^{a-1} \cdot \exp\{\eta \sum_{i=1}^n x_i + na \ln(-\eta)\}$$

thus its conjugate family can be expressed as

$$\pi(\eta | k, \mu) = c(k,\mu) \exp\{k\mu\eta + ka \ln(-\eta)\}.$$

where $c(k,\mu)$ plays as a normalization term to make sure total probability is 1. //

(b) $$\pi(\eta | \vec{x}, k, \mu) = f(\vec{x}|\eta)\pi(\eta|k,\mu) / \int f(\vec{x}|\eta)\pi(\eta|k,\mu)\, d\eta$$

$$\propto f(\vec{x}|\eta)\pi(\eta|k,\mu)$$

$$\propto \exp\{\eta \sum_{i=1}^n x_i + na \ln(-\eta)\} \cdot \exp\{k\mu\eta + ka\ln(-\eta)\}$$

$$= \exp\{\eta(n\bar{x}_n + k\mu) + (n+k)a \ln(-\eta)\}, \text{ where } \bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i$$

by TPE Thm. 4.1.2

$$\delta_B(\vec{X}) = \min_{\delta} \mathbb{E}\{L(b,\delta) | \vec{X} = \vec{x}\}$$

(1) $L(b,\delta) = (b-\delta)^2$, by TPE Corollary 4.1.2 (a)

$$\delta_B(\vec{x}) = \mathbb{E}(b | \vec{X} = \vec{x}) = \mathbb{E}(\tfrac{1}{\eta} | \vec{X} = \vec{x})$$

Since $\int \pi(\eta|\vec{x},k,\mu)\, d\eta = 1 \Rightarrow \frac{\partial}{\partial \eta}\int \pi(\eta|\vec{x},k,\mu)\, d\eta = 0$

then $\int \frac{\partial}{\partial\eta} \pi(\eta|\vec{x},k,\mu)\, d\eta = 0$

$$\Rightarrow \int \left[(n\bar{x}_n + k\mu) - \frac{(n+k)a}{\eta}\right] \pi(\eta|\vec{x},k,\mu)\, d\eta = 0$$

then, we have $\mathbb{E}(\frac{1}{\eta} | \vec{X} = \vec{x}) = \frac{n\bar{x}_n + k\mu}{(n+k)a}$ is the Bayes estimator of b.

(2) $L(b,\delta) = (1-\delta/b)^2 = \frac{1}{b^2}(b-\delta)^2$, by TPE Corollary 4.1.2 (b)

$$\delta_B(\vec{x}) = \mathbb{E}(b^{-1}|\vec{X}=\vec{x}) / \mathbb{E}(b^{-2}|\vec{X}=\vec{x}) = \mathbb{E}(-\eta|\vec{X}=\vec{x}) / \mathbb{E}(\eta^2|\vec{X}=\vec{x})$$

Since $\pi(\eta|\vec{x},k,\mu) \propto \exp\{\eta(n\bar{x}_n + k\mu) + (n+k)a\ln(-\eta)\}$ has the same kernel with Gamma(α,β) distribution.

where $\frac{1}{\beta} = n\bar{x}_n + k\mu$, $\alpha - 1 = (n+k)a$,

then $\mathbb{E}(-\eta|\vec{X}=\vec{x}) = \alpha\beta = \frac{(n+k)a+1}{n\bar{x}_n + k\mu}$ $\quad Var(-\eta|\vec{X}=\vec{x}) = \mathbb{E}(\eta^2|\vec{X}=\vec{x}) - [\mathbb{E}(-\eta|\vec{X}=\vec{x})]^2 = \alpha\beta^2 = \frac{(n+k)a+1}{(n\bar{x}_n + k\mu)^2}$

Then, we have

$$\mathbb{E}(\eta|\vec{X}=\vec{x}) / \mathbb{E}(\eta^2|\vec{X}=\vec{x}) = \alpha\beta / (\alpha\beta^2 + \alpha^2\beta^2) = \frac{1}{\beta}/(\alpha+1) = \frac{n\bar{x}_n + k\mu}{(n+k)a+2}$$

(c) $L(b,\delta) = (1-\delta/b)^2$. where $\delta = \nabla \ln m(\vec{x}) - \nabla \ln h(\vec{x})$, $h(\vec{x}) = \left[\frac{1}{\Gamma(a)}\right]^n \left[\prod_{i=1}^n x_i\right]^{a-1}$

and $m(\vec{x})$ is the marginal distribution of $\vec{x}$, i.e.

$$m(\vec{x}) = \int f(\vec{x}|\eta)\pi(\eta|k,\mu)\, d\eta = h(\vec{x}) \cdot c(k,\mu) \cdot \int \exp\{\eta\sum_{i=1}^n x_i + na\ln(-\eta)\}\exp\{k\mu\eta + ka\ln(-\eta)\}\, d\eta$$

then, in the situation of this problem

we have $$\mathbb{E}(\eta|\vec{X}=x) = \frac{\partial}{\partial x_j}\ln m(\vec{x}) - \frac{\partial}{\partial x_j}\ln h(\vec{x})$$

$$= \frac{\partial}{\partial x_j} \ln \int \exp\{\eta\sum_{i=1}^n x_i + na\ln(-\eta)\}\exp\{k\mu\eta + ka\ln(-\eta)\}\, d\eta$$

$$= \int \eta\, g(\vec{x},\eta)\, d\eta \Big/ \int g(\vec{x},\eta)\, d\eta$$

(where $g(\vec{x},\eta) = \exp\{\eta(n\bar{x}_n + k\mu) + (n+k)a\ln(-\eta)\}$, then $\pi(\eta|\vec{X}=x,\mu,k) = \frac{g(\vec{x},\eta)}{\int g(\vec{x},\eta)d\eta}$)

which has the same formation with (b)(ii),

then the answer is YES. //

2. Proof of TPE Corollary 4.3.3

First, by TPE Thm. 4.3.2, If $p_\eta(\vec{x}) = \exp\{\sum_{i=1}^{s}\eta_i T_i(\vec{x}) - A(\vec{\eta})\} h(\vec{x})$, with prior $\pi(\vec{\eta})$ of $\vec{\eta}$,

then $\mathbb{E}\left(\sum_{i=1}^{s}\eta_i \frac{\partial}{\partial x_j} T_i(\vec{x}) \mid \vec{x}\right) = \frac{\partial}{\partial x_j}\log m(\vec{x}) - \frac{\partial}{\partial x_j}\log h(\vec{x})$

where $m(\vec{x}) = \int p_\eta(\vec{x})\pi(\vec{\eta})d\vec{\eta}$ is the marginal distribution of $\vec{X}$.

by TPE Corollary 4.1.2 (b), the Bayes estimator of η under MSE is given by $\mathbb{E}(\eta_i \mid \vec{X}=\vec{x})$

In the case of this question, $T_i(\vec{x}) = x_i$ for $i=1,2,\cdots,p$

then $\mathbb{E}(\eta_j \mid \vec{X}=\vec{x}) = \mathbb{E}\left(\sum_{i=1}^{p}\eta_i \frac{\partial}{\partial x_j} x_i \mid \vec{X}=\vec{x}\right)$

$= \frac{\partial}{\partial x_j}\log m(\vec{x}) - \frac{\partial}{\partial x_j}\log h(\vec{x})$. //

3. TPE Example 4.3.6 is continuation of Example 4.3.4:

$X_i \mid \theta_i \sim N(\theta_i, \sigma^2)$, $i=1,\cdots,p$, independent.

$\Theta_i \sim N(\mu, \tau^2)$, $i=1,\cdots,p$, independent.

where σ^2, τ^2, and μ are known, $\eta_i = \theta_i/\sigma^2$ and the Bayes estimator of θ_i is

$$\mathbb{E}(\Theta_i \mid \vec{x}) = \sigma^2 \mathbb{E}(\eta_i \mid \vec{x}) = \sigma^2\left[\frac{\partial}{\partial x_i}\log m(\vec{x}) - \frac{\partial}{\partial x_i}\log h(\vec{x})\right]$$
$$= \frac{\tau^2}{\sigma^2+\tau^2}x_i + \frac{\sigma^2}{\sigma^2+\tau^2}\mu,$$

Since $\frac{\partial}{\partial x_i}\log m(\vec{x}) = \frac{\partial}{\partial x_i}\log\left[\exp\left\{\frac{1}{2(\sigma^2-\tau^2)}\sum_{i=1}^{p}(x_i-\mu)^2\right\}\right] = -\frac{x_i-\mu}{\sigma^2+\tau^2}$

$\frac{\partial}{\partial x_i}\log h(\vec{x}) = \frac{\partial}{\partial x_i}\log\left[\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{p}x_i^2\right\}\right] = -\frac{x_i}{\sigma^2}$

then $\frac{\partial^2}{\partial x_i^2}\log m(\vec{x}) = -\frac{1}{\sigma^2+\tau^2}$

$R(\eta, \mathbb{E}(\eta \mid \vec{X})) = R(\eta, -\nabla\log h(\vec{X})) - \frac{2p}{\sigma^2+\tau^2} + \sum_{i=1}^{p}E_\eta\left(\frac{X_i-\mu}{\sigma^2+\tau^2}\right)^2$

the best unbiased estimator of $\eta_i = \frac{\theta_i}{\sigma^2}$ is $-\frac{\partial}{\partial x_i}\log h(\vec{X}) = \frac{X_i}{\sigma^2}$ with risk $R(\eta, -\nabla\log h(\vec{X})) = \frac{p}{\sigma^2}$

(a) For SE loss, by Corollary 4.1.2,

$\delta = \mathbb{E}(\Theta \mid \vec{x})$ and Bayes estimator of η is $\mathbb{E}(\eta \mid \vec{x}) = \mathbb{E}\left(\frac{\Theta}{\sigma^2} \mid \vec{x}\right) = \frac{1}{\sigma^2}\mathbb{E}(\Theta \mid \vec{x}) = \frac{\delta}{\sigma^2}$

then $R(\theta, \delta) = \mathbb{E}[L(\theta,\delta) \mid \theta] = \mathbb{E}\left[\sum_{i=1}^{p}(\theta_i - \delta_i)^2 \mid \theta\right] = \mathbb{E}\left[\sum_{i=1}^{p}\sigma^4\left(\frac{\theta_i}{\sigma^2} - \frac{\delta_i}{\sigma^2}\right)^2 \mid \theta\right]$

$= \sigma^4\,\mathbb{E}\left[\sum_{i=1}^{p}(\eta_i - \delta_i')^2 \mid \theta\right] = \sigma^4 R(\eta, \delta')$

(b) $\mathbb{E}_\eta\left(\frac{X_i-\mu}{\sigma^2+\tau^2}\right)^2 = \frac{1}{(\sigma^2+\tau^2)^2}E_\eta(X_i-\mu)^2 = \frac{1}{(\sigma^2+\tau^2)^2}\left[\underbrace{\mathbb{E}_\eta(X_i-\theta_i)^2}_{\sigma^2} + 2(\theta_i-\mu)\underbrace{\mathbb{E}(X_i-\theta_i)}_{0} + (\underbrace{\theta_i}_{\sigma^2\eta_i}-\mu)^2\right]$

$= \frac{1}{(\sigma^2+\tau^2)^2}\left[\sigma^2 + \sigma^4(\eta_i - \mu/\sigma^2)^2\right]$

then $R(\eta, \delta') = \frac{p}{\sigma^2} - \frac{2p}{\sigma^2+\tau^2} + \sum_{i=1}^{p}\frac{1}{(\sigma^2+\tau^2)^2}\left[\sigma^2 + \sigma^4(\eta_i - \mu/\sigma^2)^2\right]$

$= \frac{p}{\sigma^2} - \frac{2p}{\sigma^2+\tau^2} + \frac{p\sigma^2}{(\sigma^2+\tau^2)^2} + \frac{\sigma^4}{(\sigma^2+\tau^2)^2}\sum_{i=1}^{p}(\underbrace{\eta_i - \mu/\sigma^2}_{a_i})^2$

which can be rewritten as

$$R(\eta, \delta') = \frac{p\tau^2}{\sigma^2(\sigma^2+\tau^2)^2} + \frac{\sigma^4}{(\sigma^2+\tau^2)^2}\sum_{i=1}^{p}a_i^2$$

(c) If $\sum_{i=1}^{p}a_i^2 = k$ is a fixed constant, then $R(\eta, \delta') = \frac{p\tau^2}{\sigma^2(\sigma^2+\tau^2)^2} + \frac{\sigma^4}{(\sigma^2+\tau^2)^2}\sum_{i=1}^{p}a_i^2$ is fixed

$\sum_{i=1}^{p}a_i^2 \geq \frac{1}{n}(\Sigma a_i)^2 = \frac{1}{n}\left(\sum_{i=1}^{p}\eta_i - \frac{p\mu}{\sigma^2}\right)^2$ equality holds when $a_i = a_j$ for $i,j \in \{1,\cdots,p\}$

then $pa_i^2 = k \Rightarrow p\left(\eta_i - \frac{\mu}{\sigma^2}\right)^2 = k \Rightarrow \eta_i = \frac{\mu}{\sigma^2} + \sqrt{\frac{k}{p}}$ this case $R(\eta, \delta')$ reaches minimium. //

4. Suppose $\{X_i\}_{i=1}^n \overset{iid}{\sim} f$, f belongs to exponential family.

(a) Assume $\mathbb{E}X_i = \mu$, $Var\, X_i = \sigma^2$, we have $\mathbb{E}\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i = \mu$

then $\mathbb{E}\{(a\bar{X}+b)-\mu\}^2 = \mathbb{E}\{a(\bar{X}-\mu) + (a-1)\mu+b\}^2$

$$= a^2\,\mathbb{E}(\bar{X}-\mu)^2 + 2a[(a-1)\mu+b]\,\mathbb{E}(\bar{X}-\mu) + [(a-1)\mu+b]^2$$

$$= a^2\, Var\,\bar{X} + [(a-1)\mu+b]^2$$

(b) If $a \neq 1$ and μ is unbounded, then $\mathbb{E}\{(a\bar{X}+b)-\mu\}^2$ is dominated by μ, which reachs finite as $\mu \to \infty$.

(c) Consider $\{X_i\}_{i=1}^n \overset{iid}{\sim} N(\mu, 1)$, $\mu \sim N(\alpha, \tau^2)$,

under SE loss, the Bayes estimator is δ_B

$\delta_B = \frac{\tau^2}{1+\tau^2}\bar{X}_n + \frac{1}{1+\tau^2}\alpha$ has the form of $a\bar{X}+b$, the same with (a) and (b)

then a conjugate-prior Bayes estimator in an exponential family can have finite squared error. //

5. Suppose $\log \Theta \sim N(\mu_0, \sigma_0^2)$ with $\mu_0, \sigma_0^2 > 0$ known, $X_i|\Theta=\theta \overset{iid}{\sim} Uniform(0,\theta)$

then $\pi(\theta) = \frac{1}{\sqrt{2\pi}\sigma_0}\exp\{-(\ln\theta-\mu_0)^2/2\sigma_0^2\}\cdot 1\{\theta>0\}$ $\quad f(X|\theta) = \frac{1}{\theta}\cdot 1\{0<X<\theta\} \Rightarrow f(\vec{X}|\theta) = \frac{1}{\theta^n}\cdot 1\{X_{(1)}>0,\ X_{(n)}<\theta\}$

(a) we have $f(\vec{X}|\theta)\pi(\theta) = \frac{1}{\theta^n}1\{X_{(1)}\geq 0,\ X_{(n)}\leq\theta\}\cdot\frac{1}{\sqrt{2\pi}\sigma_0}\exp\{-(\ln\theta-\mu_0)^2/2\sigma_0^2\}\cdot 1\{\theta>0\}$

$$= \frac{1}{\sqrt{2\pi}\sigma_0}\exp\{-n\ln\theta - (\ln\theta-\mu_0)^2/2\sigma_0^2\}\cdot 1\{X_{(1)}\geq 0,\ X_{(n)}\leq\theta\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_0}\exp\{-[\ln\theta-(\mu_0-n\sigma_0^2)]^2/2\sigma_0^2\}\exp\{-n\mu_0+\tfrac{1}{2}n^2\sigma_0^2\}\ 1\{X_{(1)}\geq 0,\ X_{(n)}\leq\theta\}$$

then $\int f(\vec{x}|\theta)\pi(\theta)\,d\theta = \exp\{-n\mu_0+\frac{1}{2}n^2\sigma_0^2\}\ 1\{\theta\geq X_{(n)}\}$

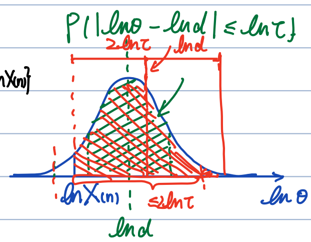
thus $\pi(\theta|\vec{x}) = \dfrac{f(\vec{x}|\theta)\pi(\theta)}{\int f(\vec{x}|\theta)\pi(\theta)\,d\theta} = \frac{1}{\sqrt{2\pi}\sigma_0}\exp\{-[\ln\theta-(\mu_0-n\sigma_0^2)]^2/2\sigma_0^2\}\ 1\{\theta\geq X_{(n)}\}$

(b) Consider loss $L(\theta,d) = \begin{cases} 0 & \text{if } \frac{1}{\tau}\leq\frac{\theta}{d}\leq\tau \\ 1 & \text{otherwise}\end{cases} \Rightarrow L(\theta,d) = \begin{cases} 0 & \text{if } |\ln\theta-\ln d|\leq\ln\tau. \\ 1 & \text{if } |\ln\theta-\ln d|>\ln\tau.\end{cases}$

by TPE Corollary 4.1.2, the Bayes loss is

$$\delta_\tau = \arg\min_d \mathbb{E}(L(\theta,d)|\vec{X}) = \arg\min_d [1-P(|\ln\theta-\ln d|\leq\ln\tau)]1\{\ln\theta\geq\ln X_{(n)}\}$$

$$= \arg\max_d P(|\ln\theta-\ln d|\leq\ln\tau)1\{\ln\theta\geq\ln X_{(n)}\}$$



If $\ln X_{(n)} \leq \mathbb{E}\ln\theta - \ln\tau = \mu_0 - n\sigma_0^2 - \ln\tau$, then $\ln\delta_\tau = \mu_0 - n\sigma_0^2$

If $\ln X_{(n)} > \mathbb{E}\ln\theta - \ln\tau = \mu_0 - n\sigma_0^2 - \ln\tau$, then $\ln\delta_\tau = \ln X_{(n)} + \ln\tau$

as $\tau\to 1$ then $\ln\tau\to 0$, $\ln\delta_\tau \to \max\{X_{(n)}, \mu_0-n\sigma_0^2\}$

that is $\delta_\tau \to \delta = \exp\{\max(X_{(n)}, \mu_0-n\sigma_0^2)\}$ as $\tau\to 1$. //

6. Suppose $\Theta \sim \Pi(\theta)$ continuous, $\Theta|X=x \sim F(\theta|X=x)$

$R(\theta,a) = \mathbb{E}(L(\theta,a)|\vec{X}) = \int_a^{+\infty} k_1|\theta-a|\,dF(\theta|X=x) + \int_{-\infty}^a k_2|\theta-a|\,dF(\theta|X=x)$

Denote δ_B is a Bayes estimator of θ. then $\delta_B = \arg\min_a R(\theta,a)$

$$\frac{\partial}{\partial a}R(\theta,a) = \frac{\partial}{\partial a}\int_a^{+\infty} k_1(\theta-a)\,dF(\theta|X) + \frac{\partial}{\partial a}\int_{-\infty}^a k_2(a-\theta)\,dF(\theta|X)$$

$$= \int_a^{+\infty} k_1(-1)\,dF(\theta|X) + \int_{-\infty}^a k_2\,dF(\theta|X)$$

$$= -k_1P(\theta>a|X) + k_2P(\theta\leq a|X) = (k_1+k_2)P(\theta\leq a|X) - k_1(P(\theta>a|X)+P(\theta\leq a|X))$$

$= (k_1+k_2)P(\theta\leq a|X) - k_1$ let it equal 0

then δ_B s.t. $P(\theta\leq\delta_B|X) = \frac{k_1}{k_1+k_2}$, let $p = \frac{k_1}{k_1+k_2}$

then δ_B is the p-th quantile of posterior distribution $P(\theta|X)$ //