STAT5005 Final Exam 2022/23
[Totally 100 marks] (1:30-4:30pm, 8 December 2022)

Instructions:

1. Turn off all the communication devices during the examination.
2. This is a closed book examination. Only one A4-sized help sheet is allowed.
3. Cheating is a serious offence. Students who commit the offence may score no mark in the examination. Furthermore, more serious penalty may be imposed.

Question 1: [15 marks] We say a random variable X has a sub-exponential tail if there exists a positive constant c_0 such that $P(|X| \geqslant t) \leqslant 2\exp(-t/c_0)$ for all $t \geqslant 0$. Prove that for such X, its L^p norm grows at most linearly in p, that is, there exists a positive constant C such that $[\mathbb{E}(|X|^p)]^{1/p} \leqslant Cp$ for all $p \geqslant 1$. [You may use Stirling's approximation: $n! \sim \sqrt{2\pi n}(n/e)^n$.]

Question 2: [10 marks] Let $X_1, X_2, \ldots$ be a sequence of random variables and $p > 0$ a constant. Suppose $\sup_{n\geqslant 1} \mathbb{E}|X_n|^p < \infty$ and X_n converges a.s. to a limiting random variable X. Does $\mathbb{E}(|X_n - X|^p) \to 0$? If so, prove it. If not, what additional condition you would need?

Question 3: [15 marks] Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$. Let $c_1, c_2, \ldots$ be a sequence of real numbers such that $\sup_{i\geqslant 1} |c_i| \leqslant 1$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} c_i X_i \to 0 \text{ in probability as } n \to \infty.$$

Question 4: [10 marks] Let φ be the characteristic function of a random variable X. Prove that for any constant $x > 0$, we have

$$P(|X| > x) \leqslant \frac{x}{2}\int_{-\frac{2}{x}}^{\frac{2}{x}} (1 - \varphi(t))dt.$$

Question 5: [20 marks] (a) Let X, Y be two random variables defined on the sample probability space such that $\mathbb{E}|Y| < \infty$. Prove that if $g : \mathbb{R} \to \mathbb{R}$ satisfies $\mathbb{E}[f(X)g(X)] = \mathbb{E}[f(X)Y]$ for every bounded function $f : \mathbb{R} \to \mathbb{R}$, then $\mathbb{E}[Y|X] = g(X)$ a.s.

(b) Let X, Y be two independent random variables and let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\mathbb{E}|f(X,Y)| < \infty$. We set, for $x \in \mathbb{R}$, $g(x) = \mathbb{E}[f(x,Y)]$. Prove that $\mathbb{E}[f(X,Y)|X] = g(X)$. [You may first consider the case $f(x,y) = 1_{\{x\in A, y\in B\}}$ for two Borel sets A, B.]

(c) Prove that if Z is independent of $\{X, Y\}$, then $\mathbb{E}[f(X,Z)|X,Y] = \mathbb{E}[f(X,Z)|X]$.

Question 6: [10 marks] Let M be a large, fixed integer. Let

$$A = \mathbb{Z}^3 \cap [-M, M]^3 = \{i, j, k \in \mathbb{Z} : -M \leqslant i, j, k \leqslant M\},$$

that is, A is the 3-dimensional integer lattice restricted to be inside of a large box. Consider a simple random walk $\{S_n : n \geqslant 1\}$ in A which starts from the origin and moves in each of the possible directions (restricted to be inside of A) uniformly at random in each step. Prove that this random walk is recurrent.

Question 7: [20 marks] (a) Recall that in coupon collector's problem, we have $n \geqslant 1$ cards. $X_1, X_2, \ldots$ are i.i.d. uniformly distributed on $\{1, 2, \ldots, n\}$. For $m \geqslant 1$, denote by $|\{X_1, \ldots, X_m\}|$ the number of distinct cards in the first m draws. Let $Y_m = n - |\{X_1, \ldots, X_m\}|$ for $m \geqslant 1$. Show that $S_m := (\frac{n}{n-1})^m Y_m$, $m \geqslant 1$, is a martingale with respect to $\mathcal{F}_m = \sigma(X_1, \ldots, X_m)$.

(b) Let $S_n, n \geqslant 1$, be a one-dimensional simple random walk. Let $\tau := \inf\{n \geqslant 1 : S_n = 1\}$ be the first time the simple random walk hits 1. Let $T_n := S_{\tau\wedge n}$. Prove that (1) T_n is a martingale, (2) $\liminf_{n\to\infty} \mathbb{E}T_n^+ < \infty$, (3) $T_n \to 1$ a.s., and (4) $\mathbb{E}T_n \to 0$. Therefore, this gives the desired counter-example in the L^1 martingale convergence theorem.