STAT 5010 Homework 2 Oct. 24, 2022 ZHOU Xiaocheng. 1155184323

1. (a) Suppose samples $\{X_i\}_{i=1}^n \overset{iid}{\sim} N(\mu, \sigma^2)$, where μ and σ^2 are unknown.

then $p(x_1, x_2, \cdots, x_n; \mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\{-(x_i-\mu)^2/2\sigma^2\}$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\{-[\sum_{i=1}^n (x_i-\mu)^2]/2\sigma^2\}$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^n x_i - \frac{n}{2\sigma^2}\mu^2\}$$

Take $T_1(X) = \sum_{i=1}^n X_i^2$, $T_2(X) = \sum_{i=1}^n X_i$,

$g_\theta(T_1(x), T_2(x)) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^n x_i - \frac{n}{2\sigma^2}\mu^2\}$, and $h(x) = 1$.

by NFFC. $(\sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i)$ is a sufficient statistic for (μ, σ^2)

(b) Suppose samples $\{X_i\}_{i=1}^n \overset{iid}{\sim}$ Uniform $[\theta-\frac{1}{2}, \theta+\frac{1}{2}]$

then $p(x_1, \cdots, x_n; \theta) = \prod_{i=1}^n 1\{\theta-\frac{1}{2} \le x_i \le \theta+\frac{1}{2}\}$

$$= 1\{\theta-\tfrac{1}{2} \le \min\{x_{1:n}\} \le \theta+\tfrac{1}{2}\} \cdot 1\{\theta-\tfrac{1}{2} \le \max\{x_{1:n}\} \le \theta+\tfrac{1}{2}\}$$

Take $T_1(X) = \min\{X_{1:n}\}$, $T_2(X) = \max\{X_{1:n}\}$

$g_\theta(T_1(x), T_2(x)) = 1\{\theta-\frac{1}{2} \le \min\{x_{1:n}\} \le \theta+\frac{1}{2}\} \cdot 1\{\theta-\frac{1}{2} \le \max\{x_{1:n}\} \le \theta+\frac{1}{2}\}$ and $h(x)=1$

by NFFC, $(\min\{X_{1:n}\}, \max\{X_{1:n}\})$ is a sufficient statistic for θ.

2. Suppose samples $\{X_i\}_{i=1}^n \overset{iid}{\sim} F$, $\{X_{(k)}\}_{k=1}^n$ are the kth order statistic of samples, and $\{\xi_\tau : F(\xi_\tau) = \tau, \tau \in (0,1)\}$ are the τ-th quantiles of F, where F' is continuous.

then, by Thm. 8.18 in the textbook: Theoretical Statistics: Topics for a Core Course,

we have $\sqrt{n}(X_{(k)} - \xi_\tau) \xrightarrow{d} N(0, \frac{\tau(1-\tau)}{[F'(\xi_\tau)]^2})$

Notice that $\frac{1}{\sqrt{n}} \to 0$, then, by Slutsky's Thm.

we have $\sqrt{n}(X_{(k)} - \xi_\tau) \cdot \frac{1}{\sqrt{n}} = X_{(k)} - \xi_\tau \xrightarrow{d} 0$,

Since 0 is a constant, then $X_{(k)} - \xi_\tau \xrightarrow{P} 0$

then $X_{(k)} \xrightarrow{P} \xi_\tau$

3. Suppose that X is one observation from a $N(0, \sigma^2)$ population.

then $p(x; \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-x^2/2\sigma^2\} = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(\frac{1}{\sigma}|x|)^2\}$

Take $T(X) = |X|$, $g_\theta(T(x)) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(\frac{1}{\sigma}|x|)^2\}$, and $h(x) = 1$,

by NFFC, $|X|$ is a sufficient statistic for σ

4. Suppose $\{X_i\}_{i=1}^n$ are random samples from pdf $f(x|\mu,\sigma) = \frac{1}{\sigma}\exp\{-(x-\mu)/\sigma\}$, $\mu<x<\infty$, $0<\sigma<\infty$

then $p(x_1, x_2, \cdots, x_n; \mu, \sigma) = \prod_{i=1}^n \frac{1}{\sigma}\exp\{-(x_i-\mu)/\sigma\} 1(x_i>\mu)$

$= \frac{1}{\sigma^n}\exp\{-\frac{1}{\sigma}\sum_{i=1}^n x_i + \frac{n\mu}{\sigma}\} 1(\min\{x_{1:n}\}>\mu)$

Take $T_1(X) = \sum_{i=1}^n X_i$, $T_2(X) = \min\{X_{1:n}\}$.

$g_\theta(T_1(x), T_2(x)) = \frac{1}{\sigma^n}\exp\{-\frac{1}{\sigma}\sum_{i=1}^n x_i + \frac{n\mu}{\sigma}\} 1(\min\{x_{1:n}\}>\mu)$, and $h(x)=1$

by NFFC, $(\sum_{i=1}^n X_i, \min\{X_{1:n}\})$ is a sufficient statistic for (μ, σ).

5. Suppose $\{X_j\}_{j=1}^n \overset{iid}{\sim} f(x|\theta)$, pdf or pmf $f(x|\theta)$ belongs to an exponential family

$f(x|\theta) = h(x)c(\theta)\exp\{\sum_{i=1}^k w_i(\theta)t_i(x)\}$

then $p(x_1, \cdots, x_n; \theta) = \prod_{j=1}^n h(x_j)c(\theta)\exp\{\sum_{i=1}^k w_i(\theta)t_i(x_j)\}$

$= [\prod_{j=1}^n h(x_j)] \cdot [c(\theta)]^n \cdot \exp\{\sum_{i=1}^k w_i(\theta)[\sum_{j=1}^n t_i(x_j)]\}$

Take $\{T_i(X) = \sum_{j=1}^n t_i(X_j)\}_{i=1}^k$, $g_\theta(T_1(x), \cdots, T_k(x)) = [c(\theta)]^n \exp\{\sum_{i=1}^k w_i(\theta)[\sum_{j=1}^n t_i(x_j)]\}$, $h(x) = \prod_{j=1}^n h(x_j)$

by NFFC, $[\sum_{j=1}^n t_1(X_j), \cdots, \sum_{j=1}^n t_k(X_j)]$ is a sufficient statistic for θ.

6. Suppose $\{X_i\}_{i=1}^n$ are independent with pdf's $f(x_i|\theta)$, where

$$f(x_i|\theta) = \begin{cases} \frac{1}{2i\theta}, & \text{if } -i(\theta-1) < x_i < i(\theta+1) \text{ and } \theta>0. \\ 0, & \text{otherwise.} \end{cases}$$

then $p(x_1, \cdots, x_n; \theta) = \prod_{i=1}^n \frac{1}{2i\theta} 1\{-i(\theta-1) < x_i < i(\theta+1)\}$

$= (2\theta)^{-1}(n!)^{-1}\prod_{i=1}^n 1\{-\theta+1 < x_i/i < \theta+1\}$

$= (2\theta)^{-1}(n!)^{-1} \cdot 1\{-\theta+1 < \min\{\{x_i/i\}_{i=1}^n\} < \theta+1\} \cdot 1\{-\theta+1 < \max\{\{x_i/i\}_{i=1}^n\} < \theta+1\}$

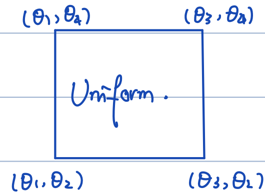
Take $T_1(X) = \min\{\{X_i\}_{i=1}^n\}$, $T_2(X) = \max\{\{X_i\}_{i=1}^n\}$.

$g_\theta(T_1(x), T_2(x)) = (2\theta)^{-1}(n!)^{-1} \cdot 1\{-\theta+1 < \min\{\{x_i/i\}_{i=1}^n\} < \theta+1\} \cdot 1\{-\theta+1 < \max\{\{x_i/i\}_{i=1}^n\} < \theta+1\}$

and $h(x) = 1$.

by NFFC, $(\min\{\{X_i\}_{i=1}^n\}, \max\{\{X_i\}_{i=1}^n\})$ is a two-dimensional sufficient statistic for θ.

7. Suppose the given condition, we have



$$f(x,y|\theta_1,\theta_2,\theta_3,\theta_4) = \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)}, \quad (\theta_1<\theta_3,\ \theta_2<\theta_4)$$

then $p((x_1,y_1),\dots,(x_n,y_n);\theta_1,\theta_2,\theta_3,\theta_4)$

$$= \prod_{i=1}^{n} \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} \mathbb{1}\{\theta_1\le x_i\le\theta_3,\ \theta_2\le y_i\le\theta_4\}$$

$$= (\theta_3-\theta_1)^{-n}(\theta_4-\theta_2)^{-n}\prod_{i=1}^{n}\mathbb{1}\{\theta_1\le x_i\le\theta_3\}\prod_{i=1}^{n}\mathbb{1}\{\theta_2\le y_i\le\theta_4\}$$

$$= (\theta_3-\theta_1)^{-n}(\theta_4-\theta_2)^{-n}\,\mathbb{1}\{\theta_1\le\min\{x_{1:n}\}\le\theta_3\}\cdot\mathbb{1}\{\theta_1\le\max\{x_{1:n}\}\le\theta_3\}\cdot$$

$$\mathbb{1}\{\theta_2\le\min\{y_{1:n}\}\le\theta_4\}\cdot\mathbb{1}\{\theta_2\le\max\{y_{1:n}\}\le\theta_4\}$$

Take $T_1(X,Y)=\min\{X_{1:n}\}$, $T_2(X,Y)=\min\{Y_{1:n}\}$. $T_3(X,Y)=\max\{X_{1:n}\}$, $T_4(X,Y)=\max\{Y_{1:n}\}$

$$g_\theta(T_1,T_2,T_3,T_4) = (\theta_3-\theta_1)^{-n}(\theta_4-\theta_2)^{-n}\times$$

$$\mathbb{1}\{\theta_1\le\min\{x_{1:n}\}\le\theta_3\}\cdot\mathbb{1}\{\theta_1\le\max\{x_{1:n}\}\le\theta_3\}\times$$

$$\mathbb{1}\{\theta_2\le\min\{y_{1:n}\}\le\theta_4\}\cdot\mathbb{1}\{\theta_2\le\max\{y_{1:n}\}\le\theta_4\}, \text{ and } h(x)=1$$

by NFFC, $(\min\{X_{1:n}\}, \min\{Y_{1:n}\}, \max\{X_{1:n}\}, \max\{Y_{1:n}\})$ is a four dimensional sufficient statistic for $(\theta_1,\theta_2,\theta_3,\theta_4)$.

8. Suppose the given conditions, the density can be reorganized

$$p(x;\eta) = \exp\Big\{\sum_{i=1}^{s}\eta_i T_i(x) - A(\eta)\Big\}h(x)$$

$$= h(x)\exp\{-A(\eta)\}\exp\Big\{\sum_{i=1}^{s}\eta_i T_i(x)\Big\}$$

$$= h(x)c(\eta)\exp\Big\{\sum_{i=1}^{s}\eta_i T_i(x)\Big\}, \quad \text{where } c(\eta):=\exp\{-A(\eta)\}$$

Take two param. points $\eta,\eta'\in\Theta\subset\mathbb{R}^s$, $\eta\ne\eta'$, and $\forall\, 0<\alpha<1$, and μ a measure

$$\int \exp\Big\{\sum_{i=1}^{s}[\alpha\eta_i+(1-\alpha)\eta_i']T_i(x)\Big\}h(x)\,d\mu(x)$$

$$= \int \Big(\exp\Big\{\sum_{i=1}^{s}\eta_iT_i(x)\Big\}\Big)^{\alpha}\Big(\exp\Big\{\sum_{i=1}^{s}\eta_i'T_i(x)\Big\}^{1-\alpha}\Big)h(x)\,d\mu(x)$$

$$\le\Big[\int\exp\Big\{\sum_{i=1}^{s}\eta_iT_i(x)\Big\}h(x)\,d\mu(x)\Big]^{\alpha}\Big[\int\exp\Big\{\sum_{i=1}^{s}\eta_i'T_i(x)\Big\}h(x)\,d\mu(x)\Big]^{1-\alpha}<\infty \quad \text{(by Hölder's Ineq.)}.$$

Since $\Theta := \{\theta=(\theta_1,\dots,\theta_s): \int\exp\{\sum_{i=1}^{s}\theta_iT_i(x)\}h(x)\,d\mu(x)<\infty\}$.

then $\alpha\eta_i+(1-\alpha)\eta_i'\in\Theta$ for $\forall\,\alpha\in(0,1)$.

thus Θ is convex.

9. Proof of Rao-Blackwell Thm. (Cited from TPE, Thm. 1.7.8.)

Let $\phi(d) = L(\theta, d)$ and δ depend on X, i.e. $\delta = \delta(X)$

Let $X \sim P_{X|t}$, i.e. conditional dist. of X given $T=t$

Then, by Jensen's Ineq., we have

$$L[\theta, \eta(t)] = L\{\theta, \mathbb{E}[\delta(X) \mid t]\}$$

$$\leqslant \mathbb{E}\{L[\theta, \delta(X)] \mid t\} \qquad (L(\theta, d) \text{ is a \underline{strictly} convex function of } d)$$

where the equal sign holds only when $\eta(T) = \delta(X)$ with probability of 1.

Take expectation of both sides of the inequality above, we have

$R(\theta, \eta) < R(\theta, \delta)$ unless $\eta(T) = \delta(X)$ with probability of 1.

10. $X_{(k)} \sim$

11 Suppose $\{X_i\}_{i=1}^n \overset{iid}{\sim} L(\theta, 1)$, i.e. $f_X(x) = \dfrac{\exp\{-(x-\theta)\}}{(1+\exp\{-(x-\theta)\})^2}$, $\theta \in \mathbb{R}$.

then $p(x_{1:n};\theta) = \prod_{i=1}^n \dfrac{\exp\{-(x_i-\theta)\}}{(1+\exp\{-(x_i-\theta)\})^2}$

Consider the subfamily P_0 provided, the by Thm. 6.12 in TPE,

the minimal sufficient statistic for P_0 is $T(X) = [T_1(X), \cdots, T_{n+1}(X)]$, where

$$T_j(X) = \frac{P(X_{1:n};\theta_j)}{P(X_{1:n};\underset{0}{\underset{\shortparallel}{\theta_0}})} = \exp\{n\theta_j\} \prod_{i=1}^n \left(\frac{1+\exp\{-x_i\}}{1+\exp\{-(x_i-\theta_j)\}}\right)^2$$

$T(x) = T(y)$ iff $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ have the same order statistic, i.e.

$$T_j(x) = T_j(y) \Leftrightarrow \prod_{i=1}^n \left(\frac{1+\exp\{-x_i\}}{1+\exp\{-(x_i-\theta_j)\}}\right)^2 = \prod_{i=1}^n \left(\frac{1+\exp\{-y_i\}}{1+\exp\{-(y_i-\theta_j)\}}\right)^2$$

the equation is established iff x and y are the permutation of each other.

Hence, $T_j(x) = T_j(y) \overset{\sqrt{\cdot}}{\Rightarrow} \prod_{i=1}^n \dfrac{1+\xi_j u_i}{1+u_i} = \prod_{i=1}^n \dfrac{1+\xi_j v_i}{1+v_i}$ $\cdots$ (*)

where $\xi_j = e^{\theta_j}$, $u_i = e^{-x_i}$, $v_i = e^{-y_i}$

Since LHS(*) and RHS(*) are both n-degree polynomials in ξ_j with $n+1$ terms.

LHS(*) = RHS(*) iff every term of ξ^r in both sides have the same coefficient.

For $r=0$, it implies $\prod_{i=1}^n (1+u_i) = \prod_{i=1}^n (1+v_i)$, then (*) can be rewritten as

$\prod_{i=1}^n (1+\xi_j u_i) = \prod_{i=1}^n (1+\xi_j v_i) \Leftrightarrow \prod_{i=1}^n (\eta_j + u_i) = \prod_{i=1}^n (\eta_j + v_i)$, $\eta_j = \frac{1}{\xi_j}$ for $j = 1, \cdots, n+1$

then both polynomials in η_j have the same roots, i.e.

x's and y's having the same order statistics.

Hence, proof is complete.

12. Suppose $\{X_i\}_{i=1}^n \overset{iid}{\sim}$ Uniform $(-\theta, \theta)$

(a) then $\frac{p(x_{1:n};\theta)}{p(y_{1:n};\theta)} = \prod_{i=1}^n \frac{1}{2\theta} \mathbb{1}\{|x_i|<\theta\} \Big/ \prod_{i=1}^n \frac{1}{2\theta} \mathbb{1}\{|y_i|<\theta\}$

$= \mathbb{1}\{\max\{|x_i|\}_{i=1}^n < \theta\} / \mathbb{1}\{\max\{|y_i|\}_{i=1}^n < \theta\}$

$= \mathbb{1}\{|x|_{(n)} < \theta\} / \mathbb{1}\{|y|_n < \theta\}$

where $|X|_{(k)}$ is the k-th order statistic for $\{|X_i|\}_{i=1}^n$,

then ratio shall be constant, i.e., 1, iff $|x|_{(n)} = |y|_{(n)}$

Take $T(X) = |X|_{(n)}$, by Thm. 6.2.13 of Statistical Inference.

$|X|_{(n)}$ is a minimal sufficient statistic for θ.

(b) Suppose $V = \frac{\bar{X}_n}{\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i} = \frac{n^{-1}\sum_{i=1}^n X_i}{X_{(n)} - X_{(1)}}$

then we can know V is ancillary

$g_\theta(T(X)) = (2\theta)^{-n} \mathbb{1}\{|X|_{(n)} < \theta\}$ from (a)

then $\mathbb{E}_\theta g(T) = \int (2\theta)^{-n} \mathbb{1}\{|x|_{(n)}<\theta\} \frac{1}{2\theta} dx$

$= (2\theta)^{-n-1} \int_{-\theta}^{\theta} dx = 0$

thus $|X|_{(n)}$ is a complete statistic

then by Basu's Thm., $V \perp\!\!\!\perp T$.