1. Laplace's law of succession gives a distribution for Bernoulli variables $X_1, X_2, \cdots$ in which $\mathbb{P}(X_1 = 1) = 1/2$ and
$$\mathbb{P}(X_{j+1} = 1 | X_1 = x_1, \cdots, X_j = x_j) = \frac{1 + x_1 + \cdots + x_j}{j+2}, \qquad j \geq 1$$
Consider testing the hypothesis H_1 that $X_1, \cdots, X_n$ have this distribution against the null hypothesis H_0 that the variables are *i.i.d.* with $\mathbb{P}(X_i = 1) = 1/2$. If $n = 10$, find the best test with size $\alpha = 0.05$. What is the power of this test?

The likelihood ratio is
$$r(x) = \frac{P_1(x)}{O_0(x)} = \frac{T}{0.5^{10}},$$
with $T = \prod_{j=1}^{9} \left(\frac{1+S_j}{j+2}\right)^{X_{j+1}} \left(1 - \frac{1+S_j}{j+2}\right)^{1-X_{j+1}}$, $S_j = 1 + X_1 + \cdots + X_j$

Let
$$\phi(x) = \begin{cases} 1 & \text{if } T > c \\ \gamma & \text{if } T = c \\ 0 & \text{if } T < c \end{cases} \tag{0.1}$$
With $\alpha = 0.05$, we find $r = 0.3244$ and the power of test is 0.4266.

2. Suppose we have a family of tests φ_α, where $\alpha \in (0,1)$ indexed by level (so φ_α has level α), and that these tests are "nested" in the sense that $\varphi_\alpha(x)$ is nondecreasing as a function of α. We can then define the "p-value" or "attained significance" for observed data x as $\inf\{\alpha : \varphi_\alpha(x) = 1\}$, thought of as the smallest value for α where test φ_α rejects H_0. Supposewe are testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$ and that the densities for data X have monotone likelihood ratios in T. Further suppose T has continuous distribution.

   (a) Show that the family of uniformly most powerful tests are nested in the sense described.

   (b) Show that if $X = x$ is observed, the p-values $P(x)$ is
   $$\mathbb{P}_{\theta_0}\{T(X) > t\}$$
   where $t = T(x)$ is the observed value of T.

   (c) Determine the distribution of the p-value $P(X)$ when $\theta = \theta_0$.

(a) Define $F(t) = \mathbb{P}_{\theta_0}(T \le t)$. The uniformly most powerful level α test is
$$\psi_\alpha(x) = \begin{cases} 1, & T(x) > k(\alpha) \\ 0, & T(x) < k(\alpha) \end{cases} = \mathbb{1}\{T(x) > k(\alpha)\}$$
(No condition of $T(x) = k(\alpha)$ since $T(x)$ is continuous) with $k(\alpha)$ chosen so that $F\{k(\alpha)\} = 1-\alpha$. Suppose $\alpha_0 < \alpha_1$. Then, since F is non-decreasing, $k(\alpha_0) > k(\alpha_1)$. So if $T(x) = k(\alpha_0)$, $T(x)$ also exceeds $k(\alpha_1)$, and hence, $\varphi_{\alpha_1}(x) = 1$ whenever $\varphi_{\alpha_0}(x) = 1$. Thus, $\varphi_{\alpha_1}(x) \ge \varphi_{\alpha_0}(x)$ for all x, and since α_0 and α_1 are arbitrary, $\varphi_\alpha(x)$ is non-decreasing in α.

(b) Because F is non-decreasing and continuous, if $t > k(\alpha)$, then $F(t) \ge F\{k(\alpha)\} = 1-\alpha$, and so
$$P = \inf\{\alpha : t > k(\alpha)\} \ge \inf\{\alpha : F(t) \ge 1-\alpha\} = 1 - F(t)$$
However, if $F(t) > F\{k(\alpha)\} = 1-\alpha$, then $t > k(\alpha)$ and so
$$P = \inf\{\alpha : t > k(\alpha)\} \le \inf\{\alpha : F(t) > 1-\alpha\} = 1 - F(t)$$
Hence, the p-value must be $1 - F(t) = \mathbb{P}_{\theta_0}(T > t)$.

(c) Let $F^\uparrow$ denotes the largest inverse function of F:
$$F^\uparrow(c) = \sup\{t : F(t) = c\}, \qquad c \in (0,1)$$
Then $F(T) \le x$ if and only if $T \le F^\uparrow(x)$ and
$$\mathbb{P}_{\theta_0}\{F(T) \le x\} = \mathbb{P}_{\theta_0}\{T \le F^\uparrow(x)\} = F\{F^\uparrow(x)\} = x$$
Hence, $F(T)$ and the p-value $1 - F(T)$ are both uniformly distributed on (0, 1) under $\mathbb{P}_{\theta_0}$.

See the solutions in book written by Keener for more details.

3. Suppose X has a Poisson distribution with parameter λ. Determine the uniformly most powerful test of $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$ with level $\alpha = 0.05$.

For $\lambda_1 > \lambda_0$, the likelihood ratio is

$$\frac{p_{\lambda_1}(x)}{p_{\lambda_0}(x)} = \frac{e^{-\lambda_1}\lambda_1^x/x!}{e^{-\lambda_0}\lambda_0^x/x!} = \left(\frac{\lambda_1}{\lambda_0}\right)^x e^{\lambda_0-\lambda_1}$$

Since $\lambda_1 > \lambda_0$, $\lambda_1/\lambda_0 > 1$. Thus, it is non-decreasing in x. Hence, it has MLR for $T(x) = x$. The UMP test exists and is defined by

$$\psi(X) = \begin{cases} 1, & X > k \\ \gamma, & X = k \\ 0, & X < k \end{cases} = \mathbb{1}\{X > k\} + \gamma \mathbb{1}\{X = k\}$$

Test size $\alpha = 0.05$ is found around $k = 3$, thus,

$$0.05 = \alpha = \mathbb{E}_0\{\psi(X)\} = \mathbb{P}_0(X > 3) + \gamma\mathbb{P}_0(X = 3) = 1 - \sum_{x=0}^{3} \frac{e^{-1}1^x}{x!} + \gamma\frac{e^{-1}1^3}{3!}$$

Solving it gives $\gamma = 0.506$. Therefore, the UMP test with $\alpha = 0.05$ is

$$\psi(X) = \begin{cases} 1, & X > 3 \\ 0.506, & X = 3 \\ 0, & X < 3 \end{cases}$$

4. Suppose we observe a single observation X from $N(\theta, \theta^2)$.
   (a) Do the densities for X have monotone likelihood ratios?
   (b) Let ϕ^* be the best level alpha test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$. Is ϕ^* also the best level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 4$?

(a) For $\theta_2 > \theta_1$, the likelihood ratio is

$$\begin{aligned}
\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} &= \frac{1/(2\pi\theta_2^2)^{1/2}\exp\{-(x-\theta_2)^2/(2\theta_2^2)\}}{1/(2\pi\theta_1^2)^{1/2}\exp\{-(x-\theta_1)^2/(2\theta_1^2)\}} \\
&= \left(\frac{\theta_2}{\theta_1}\right)\exp\left\{-\frac{1}{2\theta_1^2\theta_2^2}\left(\theta_1^2(x-\theta_2)^2-\theta_2^2(x-\theta_1)^2\right)\right\} \\
&= \left(\frac{\theta_2}{\theta_1}\right)\exp\left\{-\frac{1}{2\theta_1^2\theta_2^2}\left((\theta_1^2-\theta_2^2)x^2-2(\theta_1^2\theta_2-\theta_2^2\theta_1)x\right)\right\} \\
&\propto_x \exp\left\{-\frac{\theta_1^2-\theta_2^2}{2\theta_1^2\theta_2^2}\left(x-\frac{\theta_1^2\theta_2-\theta_2^2\theta_1}{\theta_1^2-\theta_2^2}\right)^2\right\} \\
&= \exp\left\{-\frac{\theta_1^2-\theta_2^2}{2\theta_1^2\theta_2^2}\left(x-\frac{\theta_1\theta_2}{\theta_1+\theta_2}\right)^2\right\}
\end{aligned}$$

Since $\theta_2 > \theta_1$ and the square term, the exponent is positive. If $x < \theta_1\theta_2/(\theta_1+\theta_2)$, it is decreasing in x; If $x > \theta_1\theta_2/(\theta_1+\theta_2)$, it is increasing in x. Therefore, it does not have MLR in $T(X) = X$, but has MLR in $T(X) = |X|$.

(b) If both are MP test, then $P_{\theta_1'} > c' P_{\theta_0}(x)$ iff $P_{\theta_1} > cP_{\theta_0}(x)$ and we have $x^2 - \frac{8}{5}x > k'$ iff $x^2 - \frac{4}{3}x > k$. The iff is valid only when $k = k' < -\frac{16}{25}$. So it not MP for the two cases.

5. Suppose Y_1 and Y_2 are independent variables, both uniformly distributed on $(0, \theta)$, but our observation is $X = Y_1 + Y_2$.

   (a) Show that the densities for X have monotone likelihood ratios.

   (b) Find the UMP level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ based on X.

(a) By convolution, the denisty for X is

$$f_X(x) = \int f_{Y_2}(x - y_1) f_{Y_1}(y_1)\, dy_1 = \frac{1}{\theta} \int_0^\theta f_{Y_2}(x - y_1)\, dy_1$$

i. If $0 \le x \le \theta$,

$$\frac{1}{\theta} \int_0^\theta f_{Y_2}(x - y_1)\, dy_1 = \frac{1}{\theta} \int_0^x \frac{1}{\theta} dy_1 = \frac{x}{\theta^2}$$

ii. If $\theta < x < 2\theta$,

$$\frac{1}{\theta} \int_0^\theta f_{Y_2}(x - y_1)\, dy_1 = \frac{1}{\theta} \int_{x-\theta}^\theta \frac{1}{\theta}\, dy_1 = \frac{1}{\theta^2}(2\theta - x)$$

The likelihood ratio for x is

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \frac{(x/\theta_2^2)^{\mathbb{1}\{0 \le x \le \theta_2\}} (1/\theta_2^2 (2\theta_2 - x))^{\mathbb{1}\{\theta_2 \le x \le 2\theta_2\}}}{(x/\theta_1^2)^{\mathbb{1}\{0 \le x \le \theta_1\}} (1/\theta_1^2 (2\theta_1 - x))^{\mathbb{1}\{\theta_1 \le x \le 2\theta_1\}}} = \left(\frac{\theta_1^2}{\theta_2^2}\right) \frac{x^{\mathbb{1}\{0 \le x \le \theta_2\}} (2\theta_2 - x)^{\mathbb{1}\{\theta_2 \le x \le 2\theta_2\}}}{x^{\mathbb{1}\{0 \le x \le \theta_1\}} (2\theta_1 - x)^{\mathbb{1}\{\theta_1 \le x \le 2\theta_1\}}}$$

With $\theta_2 > \theta_1$,

i. If $0 \le x \le \theta_1 < \theta_2$, the likelihood ratio becomes

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \left(\frac{\theta_1^2}{\theta_2^2}\right)$$

i.e. non-decreasing in x.

ii. If $\theta_1 < x \le \min(\theta_2, 2\theta_1)$, the likelihood ratio becomes

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \left(\frac{\theta_1^2}{\theta_2^2}\right) \left(\frac{x}{2\theta_1 - x}\right) \qquad \Rightarrow \qquad \frac{x}{2\theta_1 - x} = 1 + \frac{2\theta_1}{2\theta_1 - x}$$

i.e. non-decreasing in x.

iii. If $\theta_1 < \min(\theta_2, 2\theta_1) < x \le 2\theta_2$, the likelihood ratio becomes

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \left(\frac{\theta_1^2}{\theta_2^2}\right) \frac{2\theta_2 - x}{2\theta_1 - x} \qquad \Rightarrow \qquad \frac{2\theta_2 - x}{2\theta_1 - x} = 1 + \frac{2(\theta_2 - \theta_1)}{2\theta_1 - x}$$

i.e. non-decreasing in x.

Hence, it has MLR in $T(X) = X$.

(b) With $T(X) = X$, the UMP test is

$$\psi(X) = \begin{cases} 1, & X > k \\ 0, & X < k \end{cases}$$

With test size α,

$$\alpha = \mathbb{E}_0\{\psi(X)\} = \mathbb{P}_0(X > k) = 1 - \mathbb{P}_0(X \le k)\,.$$

So

$$k = F_0^{-1}(1 - \alpha) = \begin{cases} \theta_0 \sqrt{2\,(1 - \alpha)}, & \alpha \ge \frac{1}{2} \\ \left(2 - \sqrt{2\alpha}\right), & \alpha \le \frac{1}{2}. \end{cases}$$

6. Let the variables X_i, $1 \leq i \leq n$ be independently distributed with distribution $Poisson(\lambda_i)$, $1 \leq i \leq n$ respectively. For testing the hypothesis
$$H_0 \ : \sum_{i=1}^{n} \lambda_i \leq a \quad v.s. \quad H_1 \ : \sum_{i=1}^{n} \lambda_i > a \ .$$
(for example, that the combined radioactivity of a number of pieces of radioactive material does not exceed a), show that there exists a UMP test, which rejects when $\sum_{i=1}^{n} X_i > C$.

The key is Poisson distribution is additive. $\sum_{i=1}^{n} X_i \sim P(\sum_{i=1}^{n} \lambda_i)$, and we can show the UMP test using the argument in Question 4.