STAT5005 Homework 3 Oct 20, 2022 ZHOU Xiaocheng 1155184323

To-Do List: 2.1.(15, 17), 2.2.(2,3,4,5,6,7,8), 2.3.(2,3,10,11,12,14,15)

2.1.15 (i) Suppose $X \sim$ Binomial $(n,p)$ and $Y \sim$ Binomial $(m,p)$ are independent, then we have

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} \quad \text{and} \quad P(Y=j) = \binom{m}{j} p^j (1-p)^{m-j}$$

Then

$$\begin{aligned} P(X+Y=k) &= \sum_{i=l}^{u} P(X=i, Y=k-i) \\ &\overset{\text{indep.}}{=} \sum_{i=l}^{u} P(X=i) P(Y=k-i) \\ &= \sum_{i=l}^{u} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \\ &= p^k (1-p)^{m+n-k} \sum_{i=l}^{u} \binom{n}{i}\binom{m}{k-i} \quad \text{where } l = \max\{0, k-m\},\ u = \min\{n, k\} \\ &= \binom{n+m}{k} p^k (1-p)^{m+n-k} \end{aligned}$$

$\begin{cases} i \geq 0, & k-i \geq 0 \\ i \leq n, & k-i \leq m \end{cases} \Rightarrow \begin{cases} i \geq \max\{0, k-m\} \\ i \leq \min\{n, k\} \end{cases}$

(the choices selecting k from m+n balls is equal to the choices selecting k from m balls mixed with n balls)

$\Rightarrow X+Y \sim$ Binomial $(m+n, p)$

(ii) Suppose $X_1, \cdots, X_n$ iid Bernoulli $(p)$, then $P(X_i=1) = p$, $P(X_i=0) = 1-p$.

we can know $X_1 \sim$ Binomial $(1, p)$

Assume $\sum_{i=1}^{n-1} X_i \sim$ Binomial $(n-1, p)$, then $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n-1} X_i + X_n$

by the conclusion of (i), we have $\sum_{i=1}^{n} X_i \sim$ Binomial $(n, p)$

Hence, by induction, $\sum_{i=1}^{n} X_i \sim$ Binomial $(n, p)$.

2.1.17 Suppose $X, Y \geq 0$ and $X \sim F$ and $Y \sim G$ are independent

Denote $f$ and $g$ are the pdf's of X and Y, respectively, which satisfying

$$F(x) = \int_0^x f(w)\,dw = P(X \leq x), \quad G(y) = \int_0^y g(w)\,dw = P(Y \leq y)$$

Take $M = XY$, $N = Y$, then $X = \frac{M}{N}$, $Y = N$,

whose Jacobian Matrix: $\begin{bmatrix} \frac{1}{N} & -\frac{M}{N^2} \\ 0 & 1 \end{bmatrix}$ with determinant $\frac{1}{N}$

$$f_{MN}(m,n) = f_{XY}(x,y) = f_{XY}\left(\frac{m}{n}, n\right)\frac{1}{n} = f\left(\frac{m}{n}\right) g(n) \frac{1}{n}$$

then

$$\begin{aligned} F_M(z) &= \int^z f_M(m)\,dm = \int^z \int f_{MN}(m,n)\,dn\,dm \\ &= \int^z \int f\left(\frac{m}{n}\right) g(n) \frac{1}{n}\,dn\,dm \\ &= \int g(n) \int^z f\left(\frac{m}{n}\right) \frac{1}{n}\,dm\,dn \\ &= \int g(n) F\left(\frac{z}{n}\right) dn \\ &= \int F\left(\frac{z}{n}\right) dG(n) \end{aligned}$$

i.e. $P(XY \leq z) = \int F\left(\frac{z}{y}\right) dG(y)$

Concern: $\mathbb{E}X_n = 0$ & $\mathbb{E}X_nX_m \le r(n-m)$ where is $n$ the fixed length of array or arbitrary value $1 \le m \le n \le$ the length of array? If it is the former case, this claim may not be true because of null statement of $\mathbb{E}X_i$ & $\mathbb{E}X_iX_j$ for $i,j<n$.

2.2.2. N.T.S. $\lim_{n\to\infty} P(|\frac{1}{n}\sum_{k=1}^n X_k| \ge \varepsilon) = 0$

By Chebyshev's Ineq. we have $P(|\frac{1}{n}\sum_{k=1}^n X_k| \ge \varepsilon) \le \dfrac{\mathbb{E}(\frac{1}{n}\sum_{k=1}^n X_k)^2}{\varepsilon^2}$

$$= \frac{1}{n^2\varepsilon^2}\sum_{i,j=1}^n \mathbb{E}(X_iX_j) = \frac{1}{n^2\varepsilon^2}\cdot 2\sum_{1\le i\le j\le n}\mathbb{E}(X_jX_i)$$

$$\le \frac{k(n)}{n^2\varepsilon^2}\cdot 2\cdot\frac{n(n-1)}{2} = \frac{k(n)}{\varepsilon^2}\left(1-\frac{1}{n}\right) \le \frac{k(n)}{\varepsilon^2}$$

Since $\lim_{k\to\infty} r(k) = 0$, then $\lim_{n\to\infty} P(|\frac{1}{n}\sum_{k=1}^n X_k| \ge \varepsilon) \le \lim_{n\to\infty}\frac{k(n)}{\varepsilon^2} = 0$ for $\forall$ fixed $\varepsilon > 0$.

i.e. $\frac{1}{n}\sum_{k=1}^n X_k \xrightarrow{P} 0$.

2.2.3. Monto Carlo integration.

(i) Suppose $f$ a measurable function on $[0,1]$ with $\int_0^1 |f(x)|dx < \infty$ and $\{U_i, i=1,\dots\}$ iid Uniform$([0,1])$

then $\{f(U_i), i=1,\dots\}$ iid

Take $I_n := \frac{1}{n}\sum_{i=1}^n f(U_i)$ and $\mathbb{E}f(U_i) = \int_0^1 f(x)dx =: I$

Since $\mathbb{E}|f(U_i)| = \int_0^1 |f(x)|dx < \infty$,

then by Thm. 2.2.9, $I_n \xrightarrow{P} I$

(ii) Suppose $\int_0^1 |f(x)|^2 dx < \infty$,

By Chebyshev's Ineq. $P(|I_n - I| \ge \frac{a}{\sqrt{n}}) = \frac{n}{a^2}\cdot\mathbb{E}(I_n - I)^2 = \frac{n}{a^2}\cdot\frac{1}{n^2}\cdot\sum_{i=1}^2[\mathbb{E}f^2(U_i) - \mathbb{E}f(U_i)]$

$$= \frac{1}{a^2}\left\{\int_0^1 f^2(x)dx - \left[\int_0^1 f(x)dx\right]^2\right\}$$

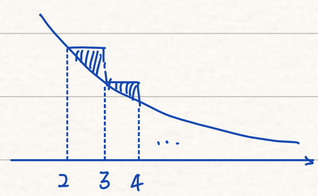
2.2.4 Suppose $\{X_i, i=1,\dots\}$ iid $P(X_i = (-1)^k k) = \frac{C}{k^2\log k}$ for $k \ge 2$, where $C$ is a probability normalizing constant

$$\mathbb{E}|X_i| = \sum_{k\ge2}\frac{C}{k^2\log k}|(-1)^k k| = \sum_{k\ge2}\frac{C}{k\log k}$$

$$\ge \int_2^\infty \frac{C}{k\log k}dk$$

$$= C\int_2^\infty \frac{1}{\log k}\,d\log k$$

$$= C\log x\Big|_{\log 2}^{\infty} = \infty$$




i.e. $\mathbb{E}|X_i| = \infty$

We notice that $NP(|X_i| > N) = N\sum_{k=N+1}^\infty \frac{C}{k^2\log k}$

$$\le N\sum_{k=N+1}^\infty \frac{C}{k(k-1)\log N}$$

$$= \frac{NC}{\log N}\sum_{k=N+1}^\infty\left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= \frac{C}{\log N} \to 0 \quad \text{as } N\to\infty$$

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let $\mu_n = \mathbb{E}(X_i 1_{|X_i|\le n}) = \sum_{k=2}^{n} (-1)^k k \cdot \frac{C}{k^2 \log k} = \sum_{k=2}^{n} (-1)^k \frac{C}{k\log k}$ , $S_n = \sum_{i=1}^{n} X_i$

convergence ?

By WLLN, $\frac{S_n}{n} \xrightarrow{P} \mu_n$

Take $\alpha_m = \mu_{2m+1} = C\sum_{k=1}^{m}\left(\frac{1}{2k\log 2k} - \frac{1}{(2k+1)(\log(2k+1))}\right)$, then $\alpha_m$ monotonely increasing.

$\beta_m = \mu_{2m+2} = \frac{C}{2\log 2} + C\sum_{k=1}^{m}\left[-\frac{1}{(2k+1)\log(2k+1)} + \frac{1}{(2k+2)(\log(2k+2))}\right]$, then $\beta_m$ monotonely decreasing

Since $\alpha_m \le \frac{C}{\log 3}\sum_{k=1}^{m}\frac{1}{2k(2k+1)} \le \frac{C}{4\log 3}\sum_{k=1}^{m}\frac{1}{k^2} \le \frac{C\pi^2}{24\log 3}$

then $\alpha_m$ converges to a constant value, we denote it $\mu_\alpha$

Similarly, since $\beta_m \ge \frac{C}{2\log 2} + \frac{C}{\log 3}\sum_{k=1}^{m}\frac{-1}{(2k+1)(2k+2)}$

$\ge \frac{C}{2\log 2} + \frac{C}{4\log 3}\sum_{k=1}^{m}\frac{-1}{k(k+1)}$

$\ge \frac{(2\log 3 - \log 2)}{4\log 2\log 3}C$

then $\beta_m$ converges to a constant value, we denote it $\mu_\beta$

Since $\mu_\beta - \mu_\alpha = \lim_{m\to\infty}\beta_m - \lim_{m\to\infty}\alpha_m = \lim_{m\to\infty}(\beta_m - \alpha_m)$

$= \lim_{m\to\infty}\frac{1}{(2m+2)\log(2m+2)} = 0$

then $\mu_\alpha = \mu_\beta =: \mu$, then $\lim_{n\to\infty}\mu_n = \mu$

Hence $\frac{S_n}{n} \xrightarrow{P} \mu$.

2.2.5 Suppose $\{X_i, i=1,\dots\}$ iid $P(X_i > x) = \frac{e}{x\log x} 1_{(x\ge e)}$

then $F_{X_i}(x) = P(X_i \le x) = \left(1 - \frac{e}{x\log x}\right) 1_{(X_i\ge e)}$

$\mathbb{E}|X_i| = \mathbb{E}X_i 1_{(X_i\ge e)} = \int_e^\infty x\, d\left(1-\frac{e}{x\log x}\right) = \left(1-\frac{e}{x\log x}\right)x\Big|_e^\infty - \int_e^\infty\left(1-\frac{e}{x\log x}\right)dx$

$= \left(\cancel{x} - \frac{e}{\log x} - \cancel{x} + e\log\log x\right)\Big|_e^\infty = e\log\log\infty - \frac{e}{\log\infty} - e\log\log e + \frac{e}{\log e}$

$= \infty - 0 - 0 + e = \infty$

However, $xP(|X_i| > x) = xP(X_i 1_{(X_i\ge e)} > x) = \frac{e}{\log x} \to 0$ as $x\to\infty$

take $S_n = \sum_{i=1}^{n} X_i$, $\mu_n = \mathbb{E}[X_1 1_{(|X_1|\le n)}]$

by WLLN, $S_n/n \xrightarrow{P} \mu_n$ as $n\to\infty$

where $\mu_n = \mathbb{E}[X_1 1_{(|X_1|\le n)}] \overset{\text{by Lemma 2.2.8 } p=1}{=\!=} \int_0^\infty 1_{(|X_1|\le n)} P(X_1 > x) 1_{(X_i\ge e)}dx$

$= \int_e^n \frac{e}{x\log x}dx$

$= \frac{e}{\log\log x}\Big|_e^n = \frac{e}{\log\log n} \to 0$, as $n\to 0$

2.2.6 (i) Suppose r.v. $X \in \mathbb{N}$,

then $\mathbb{E}X = \sum_{n=0}^{\infty} nP(X=n) = \lim_{N\to\infty} \sum_{n=1}^{N} nP(X=n)$

$= \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{m=1}^{N} 1(m\le n)\, P(X=n)$

$\overset{\text{transpose}}{=} \lim_{N\to\infty} \sum_{m=1}^{N} \sum_{n=1}^{N} P(X=n)\, 1(n\ge m)$

$= \sum_{m=1}^{\infty} P(X\ge m)$

$N\times N$

| | $m=1$ | $m=2$ | $m=3$ |
|---|---|---|---|
| $n=1$ | $P(X=1)$ | | |
| $n=2$ | $P(X=2)$ | $+\,P(X=2)$ | |
| $n=3$ | $P(X=3)$ | $+\,P(X=2)$ | $+\,P(X=3)$ |
| $\vdots$ | | | |
| $n=N$ | $P(X=N)$ | $+\,\dots$ | $+\,P(X=N)$ |

(ii) $\mathbb{E}X^2 = \sum_{n=0}^{\infty} n^2 P(X=n) = \lim_{N\to\infty} \sum_{n=1}^{N} n^2 P(X=n)$

$= \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{m=1}^{N^2} 1(m\le n^2)\, P(X=n)$

$\overset{\text{transpose}}{=} \lim_{N\to\infty} \sum_{m=1}^{N^2} \sum_{n=1}^{N} P(X=n)\, 1(n^2\ge m)$

$= \lim_{N\to\infty} \sum_{m=1}^{N^2} P(X\ge\sqrt{m})$

$= \sum_{m=1}^{\infty} P(X\ge\sqrt{m})$

$N\times N^2$

$m=1,1$ $\quad m=2,\ 2^2-1^2 \quad \dots$

$n=1$ $\quad P(X=1)$

$n=2$ $\quad P(X=2)+\dots+P(X=2) = 4P(X=2)$

$n=3$ $\quad P(X=3)+\dots+P(X=3)+\dots+P(X=3) = 9P(X=3)$

$\vdots$

$n=N$ $\quad P(X=N)+\dots+P(X=N) = N^2P(X=N)$

2.2.7 Take $H(x) = \int_{-\infty}^{x} h(y)\,dy$ with $h(y)\ge 0$

then $\mathbb{E}H(X) = \int_{\mathbb{R}} H(x)\, dP(x)$

$= \int_{\mathbb{R}} \int_{-\infty}^{x} h(y)\,dy\, dP(x)$

$= \int_{\mathbb{R}} \int_{\mathbb{R}} h(y)\, 1(y\le x)\, dy\, dP(x)$

$= \int_{\mathbb{R}} h(y) \int_{\mathbb{R}} 1(x\ge y)\, dP(x)\, d(y)$

$= \int_{\mathbb{R}} h(y)\, P(X\ge y)\, dy$

2.2.8 Suppose r.v.s $X_1, X_2, \dots \overset{iid}{\sim} P(X_i = x) = \begin{cases} p_0, & x=-1 \\ p_k, & x = 2^k-1,\ k\ge 1, \end{cases}$

where $p_k = \frac{1}{2^k k(k+1)}$, $p_0 = 1-\sum_{k\ge1} p_k$

$\mathbb{E}X_i = -1\cdot p_0 + \sum_{k\ge1}(2^k-1)p_k = -1 + \cancel{\sum_{k\ge1} p_k} + \sum_{k\ge1} 2^k p_k - \cancel{\sum_{k\ge1} p_k}$

$= -1 + \sum_{k\ge1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = -1+1 = 0$.

Let $S_n = \sum_{i=1}^{n} X_i$, $b_n = 2^{m_n}$, $m_n = \min\{m : 2^{-m} m^{-3/2} \le n^{-1}\} = \min\{m : 2^m m^{3/2} \ge n\}$, $b_n \to \infty$ as $n\to\infty$

and let $\bar{X}_i = X_i 1(|X_i|\le b_n)$.

Then $P(|X_i| > b_n) = \sum_{k>\log_2(2^{m_n}+1)} p_k = \sum_{k\ge m_n+1} \frac{1}{2^k k(k+1)} \le \frac{1}{(m_n+1)^2} \sum_{k\ge m_n+1} \frac{1}{2^k} = \frac{1}{(m_n+1)^2}\cdot\frac{1}{2^{m_n}}$

(Thm. 2.2.6 (ii)) $\sum_{i=1}^{n} P(|X_i| > b_n) \le \sum_{i=n}^{n} \frac{1}{(m_n+1)^2}\cdot\frac{1}{2^{m_n}} = \frac{n}{2^{m_n}(m_n+1)^2} \le \frac{2^{m_n} m_n^{3/2}}{2^{m_n}(m_n+1)^2} = \frac{1}{\sqrt{m_n}}\cdot\frac{1}{(1+\frac{1}{m_n})^2} \le \frac{1}{\sqrt{m_n}} \to 0$ as $n\to\infty$

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$2^k - 2^{k+1} + 1$

Also $\mathbb{E}\bar{X}_i^2 = \mathbb{E}X_i^2 1_{(|X_i|\le b_n)} = p_0 + \sum_{k=1}^{m_n} (2^k-1)^2 \cdot \frac{1}{2^k k(k+1)}$

$$\le p_0 + \sum_{k=1}^{m_n} \frac{2^k}{k(k+1)}$$

(Thm. 2.2.6 (ii)) $b_n^{-2}\sum_{i=1}^n \mathbb{E}\bar{X}_i^2 \le \frac{n}{b_n^2}\left[p_0 + \sum_{k=1}^{m_n}\frac{2^k}{k(k+1)}\right] \le \frac{m_n^{3/2}}{2^{m_n}}\left[p_0 + \sum_{k=1}^{m_n}\frac{2^k}{k(k+1)}\right] \le m_n^{3/2}\sum_{k=1}^{m_n}\frac{2^{k-m_n}}{k(k+1)}$

$$= m_n^{3/2}\left[\sum_{k=1}^{\lfloor\frac12 m_n\rfloor}\frac{2^{k-m_n}}{k(k+1)} + \sum_{k\ge\lceil\frac12 m_n\rceil}^{m_n}\frac{2^{k-m_n}}{k(k+1)}\right]$$

$$\le m_n^{3/2}\left[\sum_{k=1}^{\lfloor\frac12 m_n\rfloor} 2^{\lfloor\frac12 m_n\rfloor - m_n}\left(\frac1k - \frac1{k+1}\right) + \sum_{k\ge\lceil\frac12 m_n\rceil}^{m_n}\frac{4}{m_n^2}\cdot 2^{k-m_n}\right]$$

$$\le m_n^{3/2}\left[2^{-m_n/2}\sum_{k=1}^{\infty}\left(\frac1k-\frac1{k+1}\right) + \frac{4}{m_n^2}\cdot\sum_{k=0}^{\infty}2^{-k}\right]$$

$$\le m_n^{3/2}\left(2^{-m_n/2} + \frac{4}{m_n^2}\cdot 2\right)$$

$$= \frac{m_n^{3/2}}{2^{m/2}} + \frac{8}{\sqrt{m_n}} \longrightarrow 0 \quad \text{as } n\to\infty$$

Put $a_n = \sum_{i=1}^n \mathbb{E}\bar{X}_i = \sum_{i=1}^n \mathbb{E}X_i 1_{(|X_i|\le b_n)} = \sum_{i=1}^n \mathbb{E}(X_i - X_i 1_{(|X_i|>b_n)})$

$$= 0 - n\sum_{k=m_n+1}^{\infty}(2^k-1)p_k = n\sum_{k=m_n+1}^{\infty}p_k - n\sum_{k=m_n+1}^{\infty}\left(\frac1k-\frac1{k+1}\right)$$

$$= nP(|X_1|>b_n) - n\cdot\frac{1}{m_n+1} = nP(X_1>b_n) - \frac{n}{m_n+1}$$

Then by Thm. Weak law for triangular arrays $\frac{S_n - a_n}{b_n} \xrightarrow{P} 0$.

2.3.2 Let $0 \le X_1 \le X_2 \le \cdots$ be r.v.s with $\mathbb{E}X_n \sim an^\alpha$, $a, \alpha > 0$, and $Var X_n \le Bn^\beta$ with $\beta < 2\alpha$

Notice $\mathbb{E}\frac{X_n}{n^\alpha} \sim a$,

$$Var\frac{X_n}{n^\alpha} = n^{-2\alpha} Var X_n \le Bn^{\beta-2\alpha} = Bn^{-\delta} \to 0 \text{ as } n \to \infty \text{ for } \delta > 0$$

By Chebyshev's Ineq. $P(|\frac{X_n}{n^\alpha} - a| > \varepsilon) \le \frac{\mathbb{E}(\frac{X_n}{n^\alpha} - a)^2}{\varepsilon^2} = \frac{Var\frac{X_n}{n^\alpha}}{\varepsilon^2}$

$$\le Bn^{\beta-2\alpha}/\varepsilon^2 \to 0 \quad \text{as } n \to \infty$$

so $\frac{X_n}{n^\alpha} \xrightarrow{P} a$

Take $p = \frac{1}{2\alpha - \beta} + 1$, then $p > 1$, $(2\alpha - \beta)p = 1 + 2\alpha - \beta > 1$

and take $n_k = \inf\{n : n \ge k^p\}$ $k = 1, 2, \cdots$,

then $k^p \le n_k < (k+1)^p$

we have $P(|\frac{X_{n_k}}{n_k^\alpha} - a| > \varepsilon) \le Bn_k^{\beta-2\alpha}/\varepsilon^2 \le \frac{B}{\varepsilon^2} \cdot \frac{1}{k^{p(\beta-2\alpha)}}$

then $\sum_{k=1}^{\infty} P(|\frac{X_{n_k}}{n_k^\alpha} - a| > \varepsilon) \le \frac{B}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^{p(\beta-2\alpha)}} < \infty$

then $P(|\frac{X_{n_k}}{n_k^\alpha} - a| > \varepsilon, \text{i.o.}) = 0 \Rightarrow \frac{X_{n_k}}{n_k^\alpha} \xrightarrow{a.s.} a$ as $k \to \infty$

For each $n$, take $t^{(n)} = \sup\{k : n_k \le n\}$ then $n_t \le n < n_{t+1}$

Since $1 \le (\frac{n_{t+1}}{n_t})^\alpha \le [\frac{(t+2)^p}{t^p}]^\alpha = [1 + \frac{2}{t}]^{\alpha p} \to 1$ as $n \to \infty$

$\Rightarrow (\frac{n_{t+1}}{n_t})^\alpha \to 1$ as $n \to \infty$

then $\frac{X_{n_t}}{n_t^\alpha}(\frac{n_t}{n_{t+1}})^\alpha \le \frac{X_{n_t}}{n_{t+1}^\alpha} \le \frac{X_n}{n^\alpha} \le \frac{X_{n_{t+1}}}{n_t^\alpha} = \frac{X_{n_{t+1}}}{n_{t+1}^\alpha}(\frac{n_{t+1}}{n_t})^\alpha$

Hence $\frac{X_n}{n^\alpha} \xrightarrow{a.s.} a$

2.3.3 Let $X_n$ be independent Poisson r.v.'s with $\mathbb{E}X_k = \lambda_k$ with $\sum_{k=1}^{\infty} \lambda_n = \infty$

Take $S_n = \sum_{k=1}^n X_k$, then $\mathbb{E}S_n = \sum_{k=1}^n \mathbb{E}X_k = \sum_{k=1}^n \lambda_k$ $Var S_n = \sum_{k=1}^n Var X_k = \sum_{k=1}^n \lambda_k$

For $\forall \varepsilon > 0$, by Chebyshev's Ineq. we have

$$P(|\frac{S_n}{\mathbb{E}S_n} - 1| > \varepsilon) \le \frac{\mathbb{E}(\frac{S_n}{\mathbb{E}S_n} - 1)^2}{\varepsilon^2} = \frac{\mathbb{E}(S_n - \mathbb{E}S_n)^2}{\mathbb{E}^2 S_n \cdot \varepsilon^2} = \frac{Var S_n}{\mathbb{E}^2 S_n \cdot \varepsilon^2} = \frac{1}{\varepsilon^2 \sum_{k=1}^n \lambda_k} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow \frac{S_n}{\mathbb{E}S_n} \xrightarrow{P} 1$

Take $n_k = \inf\{n : \sum_{i=1}^n \lambda_i \ge k^2\}$, then $k^2 \le \sum_{i=1}^{n_k} \lambda_k < (k+1)^2$

then $\sum_{k=1}^{\infty} P(|\frac{S_{n_k}}{\mathbb{E}S_{n_k}} - 1| > \varepsilon) \le \sum_{k=1}^{\infty} \frac{1}{\varepsilon^2 \sum_{i=1}^{n_k} \lambda_k} \le \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi}{6\varepsilon^2} < \infty$

then $P(|\frac{S_{n_k}}{\mathbb{E}S_{n_k}} - 1| > \varepsilon, \text{i.o.}) = 0 \Longrightarrow \frac{S_{n_k}}{\mathbb{E}S_{n_k}} \xrightarrow{a.s.} 1$ as $k \to \infty$

For each $n$, take $t(n) = \sup\{k : n_k \le n\}$ then $n_t \le n < n_{t+1}$

Since $1 \le \frac{\mathbb{E}S_{n_{t+1}}}{\mathbb{E}S_{n_t}} = \frac{\sum_{i=1}^{n_{t+1}} \lambda_i}{\sum_{i=1}^{n_t} \lambda_i} \le \frac{(t+2)^2}{t^2} = (1 + \frac{2}{t})^2$

$\Rightarrow \frac{\mathbb{E}S_{n_{t+1}}}{\mathbb{E}S_{n_t}} \to 1$ as $n \to \infty$

then $\frac{S_{n_t}}{\mathbb{E}S_{n_t}} \cdot \frac{\mathbb{E}S_{n_t}}{\mathbb{E}S_{n_{t+1}}} \le \frac{S_{n_t}}{\mathbb{E}S_{n_{t+1}}} \le \frac{S_n}{\mathbb{E}S_n} \le \frac{S_{n_{t+1}}}{\mathbb{E}S_{n_t}} = \frac{S_{n_{t+1}}}{\mathbb{E}S_{n_{t+1}}} \cdot \frac{\mathbb{E}S_{n_{t+1}}}{\mathbb{E}S_{n_t}}$

Hence $\frac{S_n}{\mathbb{E}S_n} \xrightarrow{a.s.} 1$ as $n \to \infty$.

2.3.10 Suppose $\{X_n, n=1,\cdots\}$ is any sequence of r.v.'s,

take constants $\{C_n\} \geq n \to \infty$, as $n \to \infty$

$$\text{s.t. } P(|\frac{X_n}{C_n}| \geq \frac{1}{n}) \leq \frac{1}{2^n} \to 0, \text{ as } n \to \infty$$

thus $\frac{X_n}{C_n} \xrightarrow{P} 0$ as $n \to \infty$ holds.

For $\forall \varepsilon > 0$, denote $N := \lceil \frac{1}{\varepsilon} \rceil$, then $N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$

$$\begin{aligned}
\text{then } \sum_{n=1}^{\infty} P(|\frac{X_n}{C_n}| \geq \varepsilon) &= \sum_{n=1}^{N-1} P(|\frac{X_n}{C_n}| \geq \varepsilon) + \sum_{n=N}^{\infty} P(|\frac{X_n}{C_n}| \geq \varepsilon) \\
&\leq N-1 + \sum_{n=N}^{\infty} P(|\frac{X_n}{C_n}| \geq \frac{1}{n}) \\
&\leq N-1 + \sum_{n=N}^{\infty} \frac{1}{2^n} \\
&= N-1 + \frac{1}{2^{N-1}} < \infty
\end{aligned}$$

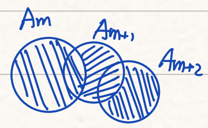
By Borel-Cantelli Lemma,

$$P(|\frac{X_n}{C_n}| \geq \varepsilon \text{ i.o.}) = 0 \quad \text{i.e. } \frac{X_n}{C_n} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

2.3.11 (i) Suppose $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$

By Borel-Cantelli Lemma

$$P(A_n^c \cap A_{n+1}, \text{ i.o.}) = 0$$




For $\forall m > 0$, $A_m \cup (\bigcup_{n=m}^{\infty} (A_n^c \cap A_{n+1})) = \bigcup_{n=m}^{\infty} A_n$

then $P(A_m) + P(\bigcup_{n=m}^{\infty} (A_n^c \cap A_{n+1})) \geq P(\bigcup_{n=m}^{\infty} A_n)$

$$P(\bigcup_{n=m}^{\infty} (A_n^c \cap A_{n+1})) \geq P(\bigcup_{n=m}^{\infty} A_n) - P(A_m)$$

$$\begin{aligned}
0 = P(A_n^c \cap A_{n+1} \text{ i.o.}) &= \lim_{m\to\infty} P(\bigcup_{n=m}^{\infty} (A_n^c \cap A_{n+1})) \\
&\geq \lim_{m\to\infty} (P(\bigcup_{n=m}^{\infty} A_n) - P(A_m)) = \lim_{m\to\infty} P(\bigcup_{n=m}^{\infty} A_n) - \lim_{m\to\infty} P(A_m) \\
&= P(A_n \text{ i.o.}) \quad \text{i.e. } P(A_n \text{ i.o.}) = 0.
\end{aligned}$$

(ii) To construct a sequence $A_n$ satisfying $\sum_{n=1}^{\infty} P(A_n) < \infty$, $P(A_n) \to 0$, $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$.

We take a probability space $(\Omega, \mathcal{B}, P)$, where $\Omega = [0,1]$, $P$ is Lebesgue measure

and take $A_n = [1 - \frac{2}{n+1}, 1 - \frac{1}{n+1}]$ $n = 1, \cdots$

then $P(A_n) = \frac{1}{n+1}$ and $P(A_n^c \cap A_{n+1}) = \frac{1}{n+1} - \frac{1}{n+2}$

The result in (i) can be applied to sequence $\{A_n\}$, but Borel-Cantelli Lemma cannot.

2.3.12 Suppose the event sequences $\{A_n\}$ is independent and with $P(A_n)<1$ for $\forall n$ and $P(\bigcup_{n=1}^{\infty} A_n)=1$

For $m<n<\infty$

$$P(\bigcup_{k=m}^{n} A_k) = 1-P(\bigcap_{k=m}^{n} A_k^c) = 1-\prod_{k=m}^{n} P(A_k^c)$$

$\{A_n\}$ indep. $\Rightarrow$ $\{A_n^c\}$ indep.

$$= 1-\prod_{k=1}^{n} P(A_k^c)/\prod_{k=1}^{m-1} P(A_k^c)$$

$$= 1-P(\bigcap_{k=1}^{n} A_k^c)/\prod_{k=1}^{m-1} P(A_k^c)$$

$$= 1-[1-P(\bigcup_{k=1}^{n} A_k)]/\prod_{k=1}^{m-1} P(A_k^c)$$

then $\lim_{n\to\infty} P(\bigcup_{k=m}^{n} A_k) = 1-[1-\underbrace{\lim_{n\to\infty} P(\bigcup_{k=1}^{n} A_k)]}_{=0}/\prod_{k=1}^{m-1} P(A_k^c) = 1$

$P(A_n \text{ i.o.}) = P(\limsup_{k\to\infty} A_k) = P(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k) = \lim_{m\to\infty} P(\bigcup_{k=m}^{\infty} A_k) = 1.$

2.3.14 Suppose $\{X_n, n=1,\dots\}$ independent with $P(X_n=1)=p_n$ and $P(X_n=0)=1-p_n$

(i) N.T.S $X_n \xrightarrow{P} 0 \Leftrightarrow p_n \to 0$

"$\Rightarrow$": Assume $X_n \xrightarrow{P} 0$

$p_n = P(X_n=1) = P(|X_n|>\varepsilon) \longrightarrow 0$ holds for $\forall \varepsilon>0$ as $n\to\infty$

"$\Leftarrow$": Assume $p_n \to 0$

$P(|X_n|>\varepsilon) = P(X_n=1) = p_n \to 0$ holds for $\forall \varepsilon>0$ as $n\to\infty$

thus $X_n \xrightarrow{P} 0$.

(ii) N.T.S $X_n \xrightarrow{a.s.} 0 \Leftrightarrow \sum p_n<\infty$

"$\Leftarrow$": Assume $\sum_{n=1}^{\infty} p_n<\infty$

we have $\sum_{n=1}^{\infty} P(|X_n|>\varepsilon) = \sum_{n=1}^{\infty} P(X_n=1) = \sum_{n}^{\infty} p_n <\infty$ for $\forall\ 0<\varepsilon<1$

by Borel-Cantelli Lemma.

then $P(|X_n|>\varepsilon \text{ i.o.})=0$ for $\forall\ 0<\varepsilon<1$

thus $X_n \xrightarrow{a.s.} 0$ as $n\to\infty$

"$\Rightarrow$" Assume that $X_n \xrightarrow{a.s.} 0$

Take $\varepsilon_0=\frac{1}{2}$ then $0=P(|X_n|>\varepsilon \text{ i.o.}) = P(X_n=1, \text{ i.o.})$

Denote $A_n=\{w: X_n(w)=1\}$ $P(A_n)=p_n$.

$$0=P(X_n=1, \text{ i.o.}) = P(A_n \text{ i.o.}) = P(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k) = \lim_{m\to\infty} P(\bigcup_{k=m}^{\infty} A_k)$$

then $\exists M>0$. s.t. $\forall m>M$ $0<P(\bigcup_{k=m}^{\infty} A_k)<\frac{1}{2}$.

(continue on the next page)

Let $M < m < n < \infty$

then $P(\bigcup_{k=m}^{n} A_k) = 1 - P(\bigcap_{k=m}^{n} A_k^c) = 1 - \prod_{k=m}^{n} P(A_k^c)$

$= 1 - \prod_{k=m}^{n} (1 - p_k)$

$\geq 1 - \exp\{-\sum_{k=m}^{n} p_k\}$

then $\sum_{k=m}^{n} p_k \leq -\ln(1 - P(\bigcup_{k=m}^{n} A_k))$

$\leq -\ln(1 - P(\bigcup_{k=m}^{\infty} A_k))$

$\leq -\ln(1 - \frac{1}{2}) = \ln 2$

$\Rightarrow \sum_{k=m+1}^{\infty} p_k \leq \ln 2$

thus $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{M} p_k + \sum_{k=M+1}^{\infty} p_k < M + \ln 2 < \infty$

2.3.15 (i)