STAT 5005 Homework 6 Dec. 1st 2022 Xiaocheng ZHOU 1155184323.

TODO List
4.1. (3, 5, 6, 13, 14)
5.1. (5, 7, 8, 9, 10, 11)
5.2. (4, 7, 9, 11, 13)
5.3. (4).

4.1.3 Suppose that S and T are stopping times

then for $\forall s,t \in \mathbb{N}$ and $0 < s,t < \infty$, we have $\{S=s\} \in \mathcal{F}_s$ and $\{T=t\} \in \mathcal{F}_t$

then $\{S \le n\} = \bigcup_{s=1}^n \{S=s\} \in \bigcup_{s=1}^n \mathcal{F}_s = \mathcal{F}_n$

similarly $\{T \le n\} \in \mathcal{F}_n$

(1) N.T.S $\{S \wedge T = n\} \in \mathcal{F}_n$

For $n=1$, $\{S\wedge T = n\} = \{S=1\} \cup \{T=1\} \in \mathcal{F}_1$

For $n \ge 2$, $\{S \wedge T = n\} = \{S \wedge T \le n\} \cap \{S \wedge T \le n-1\}^c$

Since $\{S \wedge T \le n\} = \{S \le n\} \cup \{T \le n\} \in \mathcal{F}_n$

and $\{S \wedge T \le n-1\} = \{S \le n-1\} \cup \{T \le n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$

then $\{S \wedge T = n\} \in \mathcal{F}_n$

Thus $\{S \wedge T = n\} \in \mathcal{F}_n$ for $\forall n \ge 1$

(2) N.T.S. $\{S \vee T = n\} \in \mathcal{F}_n$

For $n=1$, $\{S \vee T = n\} = \{S=1\} \cup \{T=1\} \in \mathcal{F}_1$

For $n \ge 2$, $\{S \vee T = n\} = \{S \vee T \le n\} \cap \{S \vee T \le n-1\}^c$

Since $\{S \vee T \le n\} = \{S \le n\} \cap \{T \le n\} \in \mathcal{F}_n$

$\{S \vee T \le n-1\} = \{S \le n-1\} \cap \{T \le n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$

then $\{S \vee T = n\} \in \mathcal{F}_n$

Thus $\{S \vee T = n\} \in \mathcal{F}_n$ for $\forall n \ge 1$

4.1.5 Suppose $Y_n \in \mathcal{F}_n$ and N is a stopping time.

By the definition, $\mathcal{F}_N := \{A : A \cap \{N=n\} \in \mathcal{F}_n \text{ for } \forall n < \infty\}$.

Since $Y_n : \mathcal{F}_n \to \mathcal{B}$, $\mathcal{B}$ is a Borel set.

then for $\forall B \in \mathcal{B}$ & $\forall n < \infty$, $Y_n^{-1}(B) \in \mathcal{F}_n$

then $Y_N^{-1}(B) \cap \{N=n\} = Y_n^{-1}(B) \in \mathcal{F}_n$

(Def $\mathcal{F}_N$) $\Rightarrow Y_N \in \mathcal{F}_N$

Corollary of the result above.

Suppose $f: S \to \mathbb{R}$ is measurable, $T_n = \sum_{m \le n} f(X_m)$ and $M_n = \max_{m \le n} T_m$

For $\forall m \le n$, $f(X_m) \in \mathcal{F}_m \subset \mathcal{F}_n$

then $T_n = \sum_{m \le n} f(X_m) \in \mathcal{F}_n$

then $M_n = \max_{m \le n} T_m \in \mathcal{F}_n$

by the result above,

$T_N, M_N \in \mathcal{F}_N$

4.1.6. Suppose $M \le N$ are stopping times.

N.T.S for $\forall A \in \mathcal{F}_M$, $A \in \mathcal{F}_N$

For $\forall A \in \mathcal{F}_M$, then $A \cap \{M = n\} \subset \mathcal{F}_n$ for any $n < \infty$

$A \cap \{M \le n\} = A \cap [\bigcup_{k=1}^{n} \{M = k\}]$

$= \bigcup_{k=1}^{n} [A \cap \{M = k\}] \in \mathcal{F}_n$

Since $M \le N$, then $N \le n \Rightarrow M \le n$ can imply $\{M \le n\} \supset \{N \le n\}$.

$A \cap \{N \le n\} = \underbrace{A \cap \{M \le n\}}_{\in \mathcal{F}_n} \cap \underbrace{\{N \le n\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n$ for $\forall n < \infty$

then $A \in \mathcal{F}_N$

Thus $\mathcal{F}_M \subset \mathcal{F}_N$

4.1.13. Asymmetric Simple Random Walk.

Let X_1, X_2 iid with $P(X_1 = 1) = p > \frac{1}{2}$ and $P(X_1 = -1) = 1-p$, and

$S_n = X_1 + \cdots + X_n$, $\alpha = \inf\{m : S_m > 0\}$ and $\beta = \inf\{n : S_n < 0\}$.

(i) $\mathbb{E}X_1 = p - (1-p) = 2p - 1 > 0$

Since $X_1, X_2, \cdots$ iid then by SLLN

$\frac{S_n}{n} \xrightarrow{a.s.} 2p-1$ as $n \to \infty$.

then $\lim_{n\to\infty} S_n \to \infty$ a.s.

$\Rightarrow P(\sup_n S_n = \infty) = 1$ & $P(\inf_n S_n = -\infty) = 0$.

then by Exercise 4.1.9

$P(\alpha < \infty) = 1$ & $P(\beta < \infty) < 1$.

Ex. 4.1.9.

Take $\alpha_0 = 0$, $\alpha_1 = \alpha$, $\alpha_k = \{n > \alpha_{k-1} : S_n - S_{\alpha_{k-1}} > 0\}$ $k \ge 2$

$\beta_0 = 0$, $\beta_1 = \beta$, $\beta_k = \{n > \beta_{k-1} : S_n - S_{\beta_{k-1}} < 0\}$

Since $S_{\alpha_k} - S_{\alpha_{k-1}} \overset{d}{=} S_{\alpha_{k-1}} - S_{\alpha_{k-2}} \overset{d}{=} \cdots \overset{d}{=} S_\alpha - S_0$

and $S_{\alpha_k} \perp\!\!\!\perp S_{\alpha_{k-1}}$

then $\alpha_1, \alpha_2 - \alpha_1, \cdots \overset{iid}{\sim} \alpha$

$\beta_1, \alpha_2 - \alpha_1 \cdots \overset{iid}{\sim} \beta$

Since $1 = P(\sup_n S_n = \infty) = P(\bigcap_{k=1}^{\infty} \{\alpha_k < \infty\})$

$= \lim_{k\to\infty} P(\alpha_k < \infty) = \lim_{k\to\infty} [P(\alpha < \infty)]^k$

$\Rightarrow P(\alpha < \infty) = 1$

and $0 = P(\inf_n S_n = -\infty) = P(\bigcap_{k=1}^{\infty} \{\beta_k < \infty\})$

$= \lim_{k\to\infty} P(\beta_k < \infty) = \lim_{k\to\infty} [P(\beta < \infty)]^k$

$\Rightarrow P(\beta < \infty) < 1$.

(ii) If $Y = \inf S_n$, then $P(Y \le -k) = P(\beta_k < \infty)$

$= P(\bigcap_{n=1}^{k} \{\beta_n - \beta_{n-1} < \infty\})$

$= \prod_{n=1}^{k} P(\beta_n - \beta_{n-1} < \infty)$

$= [P(\beta < \infty)]^k$

(iii) Since α is stopping time, $\alpha \wedge n$ is also stopping time with $\mathbb{E}(\alpha \wedge n) < \infty$.

By Wald's Equation, $\mathbb{E}S_{\alpha\wedge n} = \mathbb{E}(\alpha \wedge n)\,\mathbb{E}X_1$.

Since $\{S_{\alpha\wedge n}\}$ & $\{\alpha \wedge n\}$ are increasing,

then by MCT.

$1 = \mathbb{E}S_\alpha = \mathbb{E}\lim_{n\to\infty} S_{\alpha\wedge n}$

$= \lim_{n\to\infty} \mathbb{E}S_{\alpha\wedge n}$

$= \lim_{n\to\infty} \mathbb{E}(\alpha \wedge n)\,\mathbb{E}X_1$

$= \mathbb{E}X_1 \cdot \mathbb{E}(\lim_{n\to\infty} \alpha \wedge n)$

$= \mathbb{E}X_1 \cdot \mathbb{E}\alpha$

then $\mathbb{E}\alpha = \frac{1}{\mathbb{E}X_1} = \frac{1}{2p-1}$.

4.1.14. An Optimal Stopping Problem.

Let X_n, $n \geq 1$ be iid with $\mathbb{E} X_1^+ < \infty$, and

$$Y_n = \max_{1 \leq m \leq n} X_m - cn$$

(i) Let $T = \inf\{n : X_n > a\}$, $p = P(X_n > a)$

Since X_T follows truncated distribution of X_1

then $\mathbb{E} X_T = a + \dfrac{\mathbb{E}(X_1 - a)^+}{P(X_1 > a)} = a + \frac{1}{p} \mathbb{E}(X_1 - a)^+$

$P(T=k) = \prod_{i=1}^{k-1} P(X_i < a) \cdot P(X_k > a) = (1-p)^{k-1} p \quad k = 1, 2, \cdots$

then $T \sim \text{Geometric}(p)$

Since $Y_T = \max_{1 \leq m \leq T} X_m - cT = X_T - cT$

then $\mathbb{E} Y_T = \mathbb{E} X_T - c \cdot \mathbb{E} T = a + \frac{1}{p} \mathbb{E}(X_1 - a)^+ - \frac{c}{p}$.

(ii) Let α (possibly < 0) be the unique solution of $\mathbb{E}(X_1 - \alpha)^+ = c$

If $a = \alpha$, then $\mathbb{E} Y_T = \alpha + \frac{1}{p}(\mathbb{E}(X_1 - \alpha)^+ - c) = \alpha$

Consider ineq.:

$$\begin{aligned} Y_n = \max_{1 \leq m \leq n} X_m - cn &= \alpha + \max_{1 \leq m \leq n} (X_m - \alpha) - cn \\ &\leq \alpha + \max_{1 \leq m \leq n} (X_m - \alpha)^+ - cn \\ &\leq \alpha + \sum_{m=1}^{n} [(X_m - \alpha)^+ - c] \end{aligned}$$

For $\forall\ \tau \geq 1$ with $\mathbb{E}\tau < \infty$

$$\begin{aligned} \mathbb{E} Y_\tau &\leq \alpha + \mathbb{E}\left[\sum_{m=1}^{\tau} (X_m - \alpha)^+ - c\right] \\ &= \alpha + \mathbb{E}\tau \cdot \mathbb{E}[(X_m - \alpha)^+ - c] \\ &= \alpha. \end{aligned}$$

Thus $\mathbb{E} Y_\tau = \alpha$.

5.1.5 For $\forall\ A \in \mathbb{R}$.

$$\begin{aligned} 0 &\leq \mathbb{E}[(X + AY)^2 | \mathcal{G}] \\ &= \mathbb{E}[X^2 + 2AXY + A^2 Y^2 | \mathcal{G}] \\ &= \mathbb{E}[Y^2 | \mathcal{G}] A^2 + 2\mathbb{E}[XY | \mathcal{G}] A + \mathbb{E}[X^2 | \mathcal{G}] \end{aligned}$$

(rewrite) $= aA^2 + bA + c$

$aA^2 + \ A + c = 0$ has at most one real value root.

then $b^2 - 4ac = 4\,\mathbb{E}^2[XY|\mathcal{G}] - 4\,\mathbb{E}[Y^2|\mathcal{G}] \cdot \mathbb{E}[X^2|\mathcal{G}] \leq 0$

Thus $\mathbb{E}^2[XY|\mathcal{G}] \leq \mathbb{E}[Y^2|\mathcal{G}] \cdot \mathbb{E}[X^2|\mathcal{G}]$.

5.1.7. Suppose that $\mathbb{E}|X|, \mathbb{E}|Y|, \mathbb{E}|X\cdot Y| < \infty$.

consider the following statements.

(i) X and Y indep.

(ii) $\mathbb{E}(Y|X) = \mathbb{E}Y$

(iii) $\mathbb{E}(X\cdot Y) = \mathbb{E}X\mathbb{E}Y$.

"(i) ⇒ (ii)" If $X \perp\!\!\!\perp Y$, then

constant $\mathbb{E}Y \in \sigma(X)$ and

for $\forall A \in \sigma(X)$, we have $A \perp\!\!\!\perp Y$, then

$$\int_A Y\,dP = \int Y\cdot 1_A\,dP = \mathbb{E}(Y1_A)$$

$$= \mathbb{E}Y\,\mathbb{E}1_A = \mathbb{E}Y\int_A dP = \int_A \mathbb{E}Y\,dP$$

Thus $\mathbb{E}Y = \mathbb{E}(Y|X)$

"(ii) ⇒ (iii)" If $\mathbb{E}(Y|X) = \mathbb{E}Y$, then

$$\mathbb{E}XY = \mathbb{E}[\mathbb{E}(XY|X)] = \mathbb{E}[X\,\mathbb{E}(Y|X)]$$

$$= \mathbb{E}(X\,\mathbb{E}Y) = \mathbb{E}X\,\mathbb{E}Y.$$

5.1.8 Suppose that $\mathcal{G} \subset \mathcal{F}$ and $\mathbb{E}X^2 < \infty$

We know $\mathbb{E}[X - \mathbb{E}(X|\mathcal{G})]^2$

$$= \mathbb{E}[X - \mathbb{E}(X|\mathcal{F}) + \mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G})]^2$$

$$= \mathbb{E}[X - \mathbb{E}(X|\mathcal{F})]^2 + \mathbb{E}[\mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G})]^2 + 2\mathbb{E}[(X - \mathbb{E}(X|\mathcal{F}))(\mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G}))] \cdots (1)$$

where $\mathbb{E}[(X - \mathbb{E}(X|\mathcal{F}))(\mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G}))]$

$$= \mathbb{E}[\mathbb{E}\{(X - \mathbb{E}(X|\mathcal{F}))(\mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G}))|\mathcal{F}\}] \qquad \mathcal{F}, \mathcal{G} \subset \mathcal{F}$$

$$= \mathbb{E}[\{\mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G})\}\mathbb{E}\{X - \mathbb{E}(X|\mathcal{F})|\mathcal{F}\}] \cdots (2)$$

where $\mathbb{E}\{X - \mathbb{E}(X|\mathcal{F})|\mathcal{F}\}$

$$= \mathbb{E}\{X|\mathcal{F}\} - \mathbb{E}(X|\mathcal{F}) = 0$$

then $(2) = 0$

then $(1) = \mathbb{E}[X - \mathbb{E}(X|\mathcal{F})]^2 + \mathbb{E}[\mathbb{E}(X|\mathcal{F}) - \mathbb{E}(X|\mathcal{G})]^2$

5.1.9. Let $Var(X|\mathcal{F}) = \mathbb{E}(X^2|\mathcal{F}) - (\mathbb{E}(X|\mathcal{F}))^2$

Since $\mathbb{E}\,Var(X|\mathcal{F}) = \mathbb{E}(\mathbb{E}(X^2|\mathcal{F})) - \mathbb{E}(\mathbb{E}(X|\mathcal{F}))^2$

$$= \mathbb{E}X^2 - \mathbb{E}(\mathbb{E}(X|\mathcal{F}))^2$$

$$Var\,\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}))^2 - [\mathbb{E}(\mathbb{E}(X|\mathcal{F}))]^2$$

$$= \mathbb{E}(\mathbb{E}(X|\mathcal{F}))^2 - (\mathbb{E}X)^2$$

then $\mathbb{E}\,Var(X|\mathcal{F}) + Var\,\mathbb{E}(X|\mathcal{F})$

$$= \mathbb{E}X^2 - \mathbb{E}(\mathbb{E}(X|\mathcal{F}))^2 + \mathbb{E}(\mathbb{E}(X|\mathcal{F}))^2 - (\mathbb{E}X)^2$$

$$= \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$= Var\,X$$

Thus $Var\,X = \mathbb{E}\,Var(X|\mathcal{F}) + Var\,\mathbb{E}(X|\mathcal{F})$.

5.1.10 Let $Y_1, Y_2, \cdots$ iid with mean μ & variance σ^2

N be an independent positive integer valued r.v. with $\mathbb{E}N^2 < \infty$, and

$X = Y_1 + \cdots + Y_N$

Since $X = \sum_{k=1}^{N} Y_k = \sum_{k=1}^{\infty} Y_k \cdot 1_{(N \geq k)}$

then $\mathbb{E}(X|N) = \mathbb{E}(\sum_{k=1}^{\infty} Y_k \cdot 1_{(N \geq k)}|N) = \sum_{k=1}^{\infty} \mathbb{E}(Y_k \cdot 1_{(N \geq k)}|N)$

$$= \sum_{k=1}^{\infty} 1_{(N \geq k)} \mathbb{E}(Y_k|N) = \sum_{k=1}^{\infty} 1_{(N \geq k)} \mathbb{E}Y_k$$

$$= \sum_{k=1}^{\infty} 1_{(N \geq k)} \mu$$

$$= \mu \cdot N.$$

$$Var(X|N) = Var(\sum_{k=1}^{\infty} Y_k \cdot 1_{(N \geq k)}|N)$$

$$= \sum_{k=1}^{\infty} Var(Y_k \cdot 1_{(N \geq k)}|N)$$

$$= \sum_{k=1}^{\infty} \{\mathbb{E}(Y_k^2 \cdot 1_{(N \geq k)}|N) - (\mathbb{E}Y_k \cdot 1_{(N \geq k)}|N)^2\}$$

$$= \sum_{k=1}^{\infty} \{\mathbb{E}(Y_k^2|N) \cdot 1_{(N \geq k)} - (\mathbb{E}(Y_k|N))^2 1_{(N \geq k)}\}$$

$$= \sum_{k=1}^{\infty} Var\,Y_k^2 \cdot 1_{(N \geq k)} = \sigma^2 N.$$

then $Var\,X = Var\,\mathbb{E}(X|N) + \mathbb{E}\,Var(X|N)$

$$= Var(\mu N) + \mathbb{E}(\sigma^2 N)$$

$$= \mu^2 \cdot Var N + \sigma^2 \mathbb{E}N$$

5.1.11 Suppose that X, Y are r.v.s with $\mathbb{E}(Y|\mathcal{G}) = X$ and $\mathbb{E}Y^2 = \mathbb{E}X^2 < \infty$

Since $\mathbb{E}XY = \mathbb{E}(\mathbb{E}(XY|\mathcal{G})) = \mathbb{E}(X\,\mathbb{E}(Y|\mathcal{G})) = \mathbb{E}X^2$

then $\mathbb{E}(X-Y)^2 = \mathbb{E}X^2 + \mathbb{E}Y^2 - 2\,\mathbb{E}XY$

$= \mathbb{E}X^2 + \mathbb{E}X^2 - 2\,\mathbb{E}X^2 = 0$

Hence $X = Y$ a.s.