3.4.2. Let $X_1, X_2, \cdots$ be iid with $\mathbb{E}X_i = 0$, $\operatorname{Var} X_i = \sigma^2$, $0<\sigma^2<\infty$

$S_n = X_1 + \cdots + X_n$

(a) For $\forall$ fixed $k \geq 0$, denote $A = \{\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq k\}$ and $A_m = \{\limsup_{n\to\infty} \frac{S_n - S_m}{\sqrt{n}} \geq k\}$

Notice that $A_m = \{\limsup_{n\to\infty} \frac{X_{m+1} + \cdots + X_n}{\sqrt{n}} \geq k\}$ is independent with $\sigma(\{X_1, \cdots, X_m\})$

and for $\forall$ fixed m, $\frac{S_m}{\sqrt{n}} \xrightarrow{a.s.} 0$ as $n \to \infty$.

Then
$$
\begin{aligned}
A &= \{\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq k\} \\
&= \{\limsup_{n\to\infty} \frac{S_n - S_m}{\sqrt{n}} + \limsup_{n\to\infty} \frac{S_m}{\sqrt{n}} \geq k\} \\
&= \{\limsup_{n\to\infty} \frac{S_n - S_m}{\sqrt{n}} \geq k\} = A_m \quad \text{for } \forall \text{ fixed } m.
\end{aligned}
$$
$\Rightarrow A \in \mathcal{T}$, $\mathcal{T}$ is tail field.

By Kolmogorov's zero-one law, $P(A) = 0$ or 1, and

by CLT, $\frac{S_n}{\sqrt{n}\sigma} \Rightarrow \mathcal{X}$ as $n \to \infty$, where $\mathcal{X} \sim N(0,1)$.

Notice that
$$
\begin{aligned}
P(A) &= P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq k) \\
&= P(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\frac{S_n}{\sqrt{n}} \geq k\}) \\
&= \lim_{m\to\infty} P(\bigcup_{n=m}^{\infty} \{\frac{S_n}{\sqrt{n}} \geq k\}) \\
&\geq \limsup_{m\to\infty} P(\frac{S_n}{\sqrt{n}} \geq k) \\
&> 0
\end{aligned}
$$
then $P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq k) = P(A) = 1$ for $\forall\ k \geq 0$

By setting $k \to \infty$, $P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty) = 1$, i.e. $\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty$ a.s.

(b) Suppose that $\frac{S_n}{\sqrt{n}}$ converges in probability.

then for $\forall\ k_1, k_2 > 0$, $\left|\frac{S_{k_1}}{\sqrt{k_1}} - \frac{S_{k_2}}{\sqrt{k_2}}\right| \xrightarrow{P} 0$ as $k_1, k_2 \to \infty$.

Take $k_1 = m!$ $k_2 = (m+1)!$

then $\left|\frac{S_{m!}}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| \xrightarrow{P} 0$ as $m \to \infty$.

By CLT, $\frac{S_n}{\sqrt{n}} \Rightarrow \sigma\mathcal{X}$, $\mathcal{X} \sim N(0,1)$

Notice $S_{m!}$ and $[S_{(m+1)!} - S_{m!}]$ are independent.

then
$$
\begin{aligned}
& P(\left|\frac{S_{m!}}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| > 2) \\
\geq\ & P(\frac{S_{m!}}{\sqrt{m!}} > 1, \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} < -1) \\
\geq\ & P(1 < \frac{S_{m!}}{\sqrt{m!}} < 2, \frac{S_{(m+1)!} - S_{m!}}{\sqrt{(m+1)!}} < -1 - \frac{1}{\sqrt{m+1}} \cdot \frac{S_{m!}}{\sqrt{m!}}) \\
\geq\ & P(1 < \frac{S_{m!}}{\sqrt{m!}} < 2, \frac{\sqrt{m}}{\sqrt{m+1}} \cdot \frac{S_{(m+1)!} - S_{m!}}{\sqrt{m(m!)}} < -3) \\
\longrightarrow\ & P(1 < \sigma X < 2) P(\sigma X < -3) > 0
\end{aligned}
$$
Contradiction occurs.

3.4.4 Let $X_1, X_2, \cdots$ be iid with $X_i \geq 0$, $\mathbb{E}X_i = 1$ and $Var X_i = \sigma^2 \in (0,\infty)$

Denote $S_n = X_1 + \cdots + X_n$, then $\mathbb{E}S_n = n$ and $Var S_n = n\sigma^2 \in (0,\infty)$

By CLT, $\frac{S_n - n}{\sqrt{n}\sigma} \Rightarrow \mathcal{X}$, where $\mathcal{X} \sim N(0,1)$

then $\sqrt{n}(S_n/n - 1) \Rightarrow \sigma\mathcal{X}$

Take the function $f = \sqrt{x}$, which is differentiable within $x \in (0,\infty)$

by mean value Thm. $\exists\, \xi_n$ between $\frac{S_n}{n}$ and 1 s.t.

$$f'(\xi_n) = \frac{f(\frac{S_n}{n}) - f(1)}{\frac{S_n}{n} - 1} = \frac{\sqrt{\frac{S_n}{n}} - 1}{\frac{S_n}{n} - 1}$$

$$\Rightarrow \sqrt{\frac{S_n}{n}} - 1 = f'(\xi_n)\left(\frac{S_n}{n} - 1\right)$$

Since by SLLN, $\frac{S_n}{n} \xrightarrow{a.s.} 1$.

then $\xi_n \xrightarrow{a.s.} 1$, $f'(\xi_n) = \frac{1}{2}(\xi_n)^{-\frac{1}{2}} \xrightarrow{a.s.} \frac{1}{2}$ as $n \to \infty$.

Thus $\sqrt{\frac{S_n}{n}} - 1 \xrightarrow{a.s.} \frac{1}{2}\left(\frac{S_n}{n} - 1\right)$

$\Rightarrow 2(\sqrt{S_n} - \sqrt{n}) \xrightarrow{a.s.} \sqrt{n}\left(\frac{S_n}{n} - 1\right) \Rightarrow \sigma\mathcal{X}$

Hence $2(\sqrt{S_n} - \sqrt{n}) \Rightarrow \sigma\mathcal{X}$

3.4.5 Let $X_1, \cdots$ be iid with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = \sigma^2 \in (0,\infty)$

By CLT, we have $\frac{\sum_{i=1}^n X_i}{\sqrt{n}\sigma} \Rightarrow \mathcal{X}$, $\mathcal{X} \sim N(0,1)$

By WLLN,

$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{P} \sigma^2$, as $n \to \infty$.

then $(\sum_{i=1}^n X_i^2)^{1/2}/\sqrt{n}\sigma \xrightarrow{P} 1$, as $n \to \infty$

By Slutsky's Thm.

$$\frac{\sum_{i=1}^n X_i}{(\sum_{i=1}^n X_i^2)^{1/2}} = \frac{\sum_{i=1}^n X_i/\sqrt{n}\sigma}{(\sum_{i=1}^n X_i^2)^{1/2}/\sqrt{n}\sigma} \Rightarrow \mathcal{X}$$

Hence. $\frac{\sum_{m=1}^n X_i}{(\sum_{i=1}^n X_i^2)^{1/2}} \Rightarrow \mathcal{X}$

3.4.6 Let $X_1, X_2, \cdots$ be iid with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = \sigma^2 \in (0, \infty)$ and let $S_n = X_1 + \cdots + X_n$

Let N_n be a sequence of nonnegative integer-valued r.v. and

a_n be a sequence of integers with $a_n \to \infty$ and $N_n / a_n \xrightarrow{P} 1$

Notice that for $\forall\ \varepsilon, \delta > 0$

$$P(|S_{N_n} - S_{a_n}| > \varepsilon) = P(|S_{N_n} - S_{a_n}| > \varepsilon, \left|\frac{N_n - a_n}{a_n}\right| \le \delta) + P(|S_{N_n} - S_{a_n}| > \varepsilon, \left|\frac{N_n - a_n}{a_n}\right| > \delta)$$

$$\le P(|S_{N_n} - S_{a_n}| > \varepsilon,\ (1-\delta)a_n \le N_n \le (1+\delta)a_n) + P(\left|\frac{N_n}{a_n} - 1\right| > \delta)$$

Since $\frac{N_n}{a_n} \xrightarrow{P} 1$,

then $P(\left|\frac{N_n}{a_n} - 1\right| > \delta) \to 0$ as $n \to \infty$.

Denote $G_n^\delta = \{g \in \mathbb{N} : (1-\delta)a_n \le g \le (1+\delta)a_n\}$ and $\underline{G}_n^\delta = \{g \in \mathbb{N} : (1-\delta)a_n \le g \le a_n\}$, $\overline{G}_n^\delta = \{g \in \mathbb{N} : a_n \le g \le (1+\delta)a_n\}$

$\Rightarrow G_n^\delta = \underline{G}_n^\delta \cup \overline{G}_n^\delta$

Since $P(|S_{N_n} - S_{a_n}| > \varepsilon,\ (1-\delta)a_n \le N_n \le (1+\delta)a_n)$

$$\le \sum_{m \in G_n^\delta} P(|S_{N_n} - S_{a_n}| > \varepsilon \mid N_n = m) P(N_n = m)$$

$$\le \sum_{m \in G_n^\delta} P(\max_{k \in G_n^\delta} |S_k - S_{a_n}| > \varepsilon \mid N_n = m) P(N_n = m)$$

$$= P(\max_{k \in G_n^\delta} |S_k - S_{a_n}| > \varepsilon,\ N_n \in G_n^\delta)$$

$$\le P(\max_{k \in G_n^\delta} |S_k - S_{a_n}| > \varepsilon)$$

$$= P(\{\max_{k \in \underline{G}_n^\delta} |S_k - S_{a_n}| > \varepsilon\} \cup \{\max_{k \in \overline{G}_n^\delta} |S_k - S_{a_n}| > \varepsilon\})$$

$$\le P(\{\max_{k \in \underline{G}_n^\delta} |S_k - S_{a_n}| > \varepsilon\}) + P(\{\max_{k \in \overline{G}_n^\delta} |S_k - S_{a_n}| > \varepsilon\})$$

$$\le \frac{1}{\varepsilon^2} Var(S_{a_n + \lfloor \delta a_n \rfloor} - S_{a_n}) + \frac{1}{\varepsilon^2} Var(S_{a_n} - S_{a_n - \lfloor \delta a_n \rfloor})$$

$$\le \frac{2}{\varepsilon^2} \lfloor \delta a_n \rfloor \sigma^2$$

$$\le \frac{2}{\varepsilon^2} \delta a_n \sigma^2$$

Take $Y_n = \frac{S_{N_n}}{\sigma\sqrt{a_n}}$, $Z_n = \frac{S_{a_n}}{\sigma\sqrt{a_n}}$

then $P(|Y_n - Z_n| > \varepsilon)$

$$= P(|S_{N_n} - S_{a_n}| > \sigma\sqrt{a_n}\,\varepsilon)$$

$$\le P(\left|\frac{N_n}{a_n} - 1\right| > \delta) + \frac{2}{\sigma^2 a_n \varepsilon^2} \delta a_n \sigma^2$$

$$= P(\left|\frac{N_n}{a_n} - 1\right| > \delta) + \frac{2}{\varepsilon^2}\delta$$

$$\to \frac{2}{\varepsilon^2}\delta \text{ as } n \to \infty \text{ for } \forall\ \varepsilon, \delta > 0$$

Take $\delta \to 0$, then $P(|Y_n - Z_n| > \varepsilon) \to 0$ as $n \to \infty$ for $\forall \varepsilon > 0$.

then $Y_n \xrightarrow{P} Z_n$

By CLT, $\frac{S_n}{\sigma\sqrt{n}} \Rightarrow \chi$, $\chi \sim N(0, 1)$ and $a_n \to \infty$ as $n \to \infty$

then $Z_n = \frac{S_{a_n}}{\sigma\sqrt{a_n}} \Rightarrow \chi$

Hence $Y_n = \frac{S_{N_n}}{\sigma\sqrt{a_n}} \Rightarrow \chi$

3.4.7. Let $Y_1, \cdots$ be iid positive r.v.s with $\mathbb{E}Y_i = \mu$ and $Var\, Y_i = \sigma^2 \in (0,\infty)$

Let $S_n = Y_1 + \cdots + Y_n$ and $N_t = \sup\{m : S_m \le t\}$

then we have $S_{N_t} \le t < S_{N_t+1}$

For $\forall$ integer M, $P(N_t \ge M) = P(S_M \le t) = 1 - P(S_M > t)$

$$\ge 1 - \frac{1}{t^2}\mathbb{E}S_M^2 = 1 - \frac{1}{t^2}[Var S_M + (\mathbb{E}S_M)^2]$$

$$= 1 - \frac{1}{t^2}(M\sigma^2 + M^2\mu^2)$$

Since $\{N_t \ge M\}_{t=1}^{\infty}$ is increasing, then $P(N_t \ge M \text{ i.e.}) = \lim_{m\to\infty} P(\bigcup_{t=m}^{\infty}\{N_t \ge M\})$

$$\ge \lim_{m\to\infty} P(N_m \ge M) = 1$$

then $N_t \ge M$ a.s. as $\to \infty$ for $\forall$ integer M.

then $N_t \to \infty$ a.s. as $t \to \infty$.

By SLLN, $\frac{S_n}{n} \to \mu$ a.s. as $n \to \infty$

then $\frac{S_{N_t}}{N_t} \to \mu$ a.s. as $t \to \infty$ ($N_t \to \infty$)

then $\frac{S_{N_t}}{N_t} \le \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t} = \frac{S_{N_t+1}}{N_t+1} \cdot (1 + \frac{1}{N_t})$

$\Rightarrow \frac{t}{N_t} \to \mu$ a.s. as $t \to \infty$, i.e.

$\frac{N_t}{t/\mu} \to 1$ a.s. as $t \to \infty$

Take $X_i = Y_i - \mu$, then $\mathbb{E}X_i = 0$, $Var X_i = \sigma^2$

Take $T_n = X_1 + \cdots + X_n = S_n - n\mu$, then $T_{N_t} \le t - N_t\mu < T_{N_t+1}$

Take $a_t = t/\mu$, then $\frac{N_t}{a_t} \to 1$ a.s. as $t \to \infty$

Apply the result of Exercise 3.4.6 to X_i,

then $\frac{T_{N_t}}{(\sigma^2 \cdot a_t)^{1/2}} \Rightarrow \chi$, as $t \to \infty$, where $\chi \sim N(0,1)$

Since $a_t = t/\mu \to \infty$ as $t \to \infty$

then $\frac{X_{N_t+1}}{(\sigma^2 \cdot a_t)^{1/2}} \overset{d}{=} \frac{X_1}{(\sigma^2 \cdot a_t)^{1/2}} \xrightarrow{P} 0$ as $t \to \infty$

then $0 \le \frac{T_{N_t}}{(\sigma^2 \cdot a_t)^{1/2}} - \frac{t - N_t \cdot \mu}{(\sigma^2 \cdot a_t)^{1/2}} \le \frac{X_{N_t+1}}{(\sigma^2 \cdot a_t)^{1/2}} \xrightarrow{P} 0$

$\Rightarrow \frac{t - N_t \cdot \mu}{(\sigma^2 \cdot t/\mu)^{1/2}} \xrightarrow{P} \frac{T_{N_t}}{(\sigma^2 \cdot a_t)^{1/2}}$

Hence $\frac{t - N_t \cdot \mu}{(\sigma^2 \cdot t/\mu)^{1/2}} \Rightarrow \chi$.

3.4.9. Suppose that $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots + X_n$
and $P(X_m = m) = P(X_m = -m) = m^{-2}/2$
for $m \geq 2$, $P(X_m = 1) = P(X_m = -1) = (1 - m^{-2})/2$

Then we have $\mathbb{E}X_m = 0$, $Var X_1 = 1$, and
for $m \geq 2$, $Var X_m = m^2 \cdot \frac{1}{m^2} + 1 \cdot (1 - \frac{1}{m^2}) = 2 - \frac{1}{m^2}$
then, $\frac{1}{n} Var S_n = \frac{1}{n} \sum_{m=1}^{n} Var X_m = 2 - \sum_{m=1}^{n} \frac{1}{m^2} \longrightarrow 2$ as $n \to \infty$

Take $Y_m = X_m \cdot \mathbb{1}(|X_m| \leq 2)$, $T_n = \sum_{m=1}^{n} Y_m$ and
$Z_m = X_m - Y_m$, $K_n = \sum_{m=1}^{n} Z_m$
then $\frac{S_n}{\sqrt{n}} = \frac{T_n}{\sqrt{n}} + \frac{K_n}{\sqrt{n}}$

For $\forall \varepsilon > 0$, $P(|Y_m - X_m| > \varepsilon) \leq P(X_m \neq Y_m) = \frac{1}{m^2}$
then $\sum_{m=1}^{n} P(|Y_m - X_m| > \varepsilon) \leq \sum_{m=1}^{n} \frac{1}{m^2} < \infty$
then by Borel-Cantelli Lemma,
$P(|Y_m - X_m| > \varepsilon \text{ i.o.}) = 0$ for $\forall \varepsilon > 0$.
$\Rightarrow Y_m \xrightarrow{a.s.} X_m$

Notice that $\mathbb{E}Y_m^2 = 1 - \frac{1}{m^2} \to 1$.
then take $H_{n,m} = \frac{1}{\sqrt{n}} Y_m$, $1 \leq m \leq n$ $\Rightarrow |H_{n,m}| \leq \frac{1}{\sqrt{n}}$
we have: (1) $\sum_{m=1}^{n} \mathbb{E}H_{n,m} = \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}Y_m^2 = 1 - \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m^2} \to 1$.
(2) for $\forall \varepsilon > 0$, if $n > \frac{1}{\varepsilon^2}$ then $P(|H_{n,m}| < \varepsilon) = 1$,
$\sum_{m=1}^{n} \mathbb{E}[|H_{n,m}|^2 \cdot \mathbb{1}(|H_{n,m}| > \varepsilon)] = 0$ and $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}[|H_{n,m}|^2 \cdot \mathbb{1}(|H_{n,m}| > \varepsilon)] = 0$

Thus, by Lindeberg-Feller Thm. we have
$\frac{1}{\sqrt{n}} \cdot T_n = \frac{1}{\sqrt{n}} \cdot \sum_{m=1}^{n} Y_m = \sum_{m=1}^{n} H_{n,m} \Rightarrow \mathcal{X}$ as $n \to \infty$, where $\mathcal{X} \sim N(0,1)$
Since $\mathbb{E}Z_m = 0$, $Var Z_m = m^2 \cdot \frac{1}{m^2} = 1$ for $m \geq 2$
then $Var \frac{K_n}{\sqrt{n}} = \frac{1}{n} \sum_{m=1}^{n} Var Z_m = \frac{n-1}{n} \to 1$?

3.4.10 Suppose that $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots + X_n$ and $|X_i| \le M$ and $\sum_n Var X_n = \infty$.

Denote $\lambda_n = Var S_n = \sum_{i=1}^n Var X_i$, then $\lim_{n\to\infty} \lambda_n = \infty$

Take $Y_{n,m} = \frac{X_m - \mathbb{E}X_m}{\sqrt{\lambda_n}}$ for $1 \le m \le n$ and $T_n = \sum_{m=1}^n Y_{n,m}$

then $\mathbb{E} Y_{n,m} = 0$

Notice that $\sum_{m=1}^n \mathbb{E} Y_{n,m}^2 = \sum_{m=1}^n Var Y_{n,m} = \frac{1}{\lambda_n} \sum_{m=1}^n Var X_m = 1$.

and $|Y_{n,m}| = \frac{|X_m - \mathbb{E}X_m|}{\sqrt{\lambda_n}} \le \frac{|X_m| + |\mathbb{E}X_m|}{\sqrt{\lambda_n}} \le \frac{|X_m| + \mathbb{E}|X_m|}{\sqrt{\lambda_n}} \le \frac{2M}{\sqrt{\lambda_n}}$

For $\forall \varepsilon > 0$, take $N = \inf\{n : \lambda_n = \sum_{m=1}^n Var X_m > \frac{4M^2}{\varepsilon^2}\}$

and when $n > N$, we have $|Y_{n,m}| \le \frac{2M}{\sqrt{\lambda_n}} < \varepsilon$ and $\sum_{m=1}^n \mathbb{E}[Y_{n,m}^2 1(|Y_{n,m}| > \varepsilon)] = 0$

By Lindeberg-Feller Thm. $T_n \Rightarrow \mathcal{X}$, $\mathcal{X} \sim N(0,1)$

Since $T_n = \sum_{m=1}^n Y_{n,m} = \frac{\sum_{m=1}^n X_m - \sum_{m=1}^n \mathbb{E}X_m}{\sqrt{Var S_n}} = \frac{S_n - \mathbb{E}S_n}{\sqrt{Var S_n}}$, then $\frac{S_n - \mathbb{E}S_n}{\sqrt{Var S_n}} \Rightarrow \mathcal{X}$

3.4.11 Suppose that $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots + X_n$ and $\mathbb{E}X_i = 0$ $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}|X_i|^{2+\delta} \le C$ for some $0 < \delta, C < \infty$

Take $Y_{n,m} = \frac{X_m}{\sqrt{n}}$ for $1 \le m \le n$ and $T_n = \sum_{m=1}^n Y_{n,m} = \frac{1}{\sqrt{n}} S_n$.

Notice that

$$\sum_{m=1}^n \mathbb{E} Y_{nm}^2 = \sum_{m=1}^n \mathbb{E} \frac{X_m^2}{n} = \frac{1}{n} \sum_{m=1}^n \mathbb{E} X_m^2 = 1$$

and for $\forall \varepsilon > 0$

$$\begin{aligned}
\sum_{m=1}^n \mathbb{E} Y_{nm}^2 \cdot 1(|Y_{nm}| > \varepsilon) &= \frac{1}{n} \sum_{m=1}^n \mathbb{E} X_m^2 \cdot 1(|X_m| > \sqrt{n}\cdot\varepsilon) \\
&\le \frac{1}{n} \sum_{m=1}^n \mathbb{E} X_m^2 \left(\frac{|X_m|}{\sqrt{n}\cdot\varepsilon}\right)^\delta 1(|X_m| > \sqrt{n}\cdot\varepsilon) \\
&= \frac{1}{n} \sum_{m=1}^n \frac{1}{\varepsilon^\delta n^{\delta/2}} \mathbb{E}|X_m|^{2+\delta} \cdot 1(|X_m| > \sqrt{n}\cdot\varepsilon) \\
&\le \frac{1}{\varepsilon^\delta n^{\delta/2}} \cdot \frac{1}{n} \sum_{m=1}^n \mathbb{E}|X_m|^{2+\delta} \\
&= \frac{C}{\varepsilon^\delta n^{\delta/2}} \to 0 \text{ as } n \to \infty
\end{aligned}$$

By the Lindeberg-Feller Thm., $T_n \Rightarrow \mathcal{X}$, $\mathcal{X} \sim N(0,1)$

i.e. $S_n/\sqrt{n} \Rightarrow \mathcal{X}$

3.4.12 Suppose that $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots + X_n$

Let $\alpha_n = \{Var S_n\}^{1/2}$, $\exists \delta > 0$ s.t. $\lim_{n\to\infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^n \mathbb{E}(|X_m - \mathbb{E}X_m|^{2+\delta}) = 0$.

Take $Y_{n,m} = \frac{X_m - \mathbb{E}X_m}{\alpha_n}$ for $\forall 1 \le m \le n$

then $\mathbb{E} Y_{nm} = 0$, $\lim_{n\to\infty} \sum_{m=1}^n \mathbb{E}|Y_{n,m}|^{2+\delta} = 0$

Take $T_n = \sum_{m=1}^n Y_{n,m}$

then $T_n = \frac{S_n - \mathbb{E}S_n}{\alpha_n}$

Notice that $\sum_{m=1}^n \mathbb{E} Y_{n,m}^2 = \sum_{m=1}^n Var Y_{n,m}^2 = \sum_{m=1}^n \frac{Var X_m}{\alpha_n^2} = 1$

and for $\varepsilon > 0$,

$$\begin{aligned}
\sum_{m=1}^n \mathbb{E} Y_{n,m}^2 \cdot 1(|Y_{n,m}| > \varepsilon) &\le \sum_{m=1}^n \mathbb{E}\left[\left|\frac{Y_{n,m}}{\varepsilon}\right|^\delta \cdot Y_{n,m}^2 \cdot 1(|Y_{n,m}| > \varepsilon)\right] \\
&= \frac{1}{\varepsilon^\delta} \sum_{m=1}^n \mathbb{E}\left[|Y_{n,m}|^{2+\delta} \cdot 1(|Y_{n,m}| > \varepsilon)\right] \\
&\le \frac{1}{\varepsilon^\delta} \sum_{m=1}^n \mathbb{E}\left[|Y_{n,m}|^{2+\delta}\right] \to 0 \text{ as } n \to \infty
\end{aligned}$$

By Lindeberg-Feller Thm. $T_n \Rightarrow \mathcal{X}$, $\mathcal{X} \sim N(0,1)$

i.e. $(S_n - \mathbb{E}S_n)/\alpha_n \Rightarrow \mathcal{X}$.

3.4.13 Suppose that $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots + X_n$ and X_j satisfies $P(X_j = x) = \begin{cases} 1-\frac{1}{j^\beta}, & x=0 \\ \frac{1}{2j^\beta}, & x = \pm j \end{cases}$ $\quad \beta > 0$

(i) If $\beta > 1$ then $\sum_{k=1}^{\infty} P(X_k \neq 0) = \sum_{k=1}^{\infty} \frac{1}{k^\beta} < \infty$

By Borel-Cantelli Lemma. $P(X_k \neq 0 \text{ i.o.}) = 0$

then $P(X_k \neq 0 \text{ finitely occurs}) = 1$

$\Rightarrow S_n$ converges a.s.

(ii) Assume that $\beta < 1$

then $Var\, S_n = \sum_{m=1}^{n} Var\, X_m = \sum_{m=1}^{n} m^{2-\beta}$

Since $\sum_{m=1}^{n} m^{2-\beta} \geq \int_0^n m^{2-\beta}\, dm = \frac{n^{3-\beta}}{3-\beta}$

and $\sum_{m=1}^{n} m^{2-\beta} \leq \int_1^{n+1} m^{2-\beta}\, dm = \frac{(n+1)^{3-\beta}-1}{3-\beta}$

then $Var\, S_n = \frac{n^{3-\beta}}{3-\beta} + o(n^{3-\beta})$

Take $Y_{n,m} = \frac{X_m}{n^{(3-\beta)/2}}$, we have

(1) $\sum_{m=1}^{n} \mathbb{E}\, Y_{n,m}^2 = \sum_{m=1}^{n} Var\, Y_{n,m} = \frac{1}{n^{3-\beta}} \cdot Var\, S_n \to \frac{1}{3-\beta}$

(2) for $\forall \varepsilon > 0$ and $\forall n > \varepsilon^{-\frac{2}{1-\beta}}$, $|Y_{n,m}| < \frac{n}{n^{(3-\beta)/2}} < \varepsilon$

then $\sum_{m=1}^{n} \mathbb{E}[|Y_{n,m}|^2 \cdot 1(|Y_{n,m}| > \varepsilon)] = 0$

i.e. for $\forall \varepsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}[|Y_{n,m}|^2 \cdot 1(|Y_{n,m}| > \varepsilon)] = 0$.

By Lindeberg-Feller Thm. $\frac{S_n}{n^{(3-\beta)/2}} = \sum_{m=1}^{n} Y_{n,m} \Rightarrow \frac{1}{3-\beta} \mathcal{X}$, $\mathcal{X} \sim N(0,1)$

(iii) Assume that $\beta = 1$, then the ch.f. of X_m is

$$\mathbb{E} \exp\{itX_m\} = \frac{1}{2m} \cdot \exp\{itm\} + \frac{1}{2m} \exp\{-itm\} + 1 - \frac{1}{m}$$

$$= \frac{1}{2m}[\cos(tm) + i\sin(tm)] + \frac{1}{2m}[\cos(tm) - i\sin(tm)] + 1 - \frac{1}{m}$$

$$= 1 - \frac{1}{m}[1-\cos(tm)]$$

Since $X_1, X_2, \cdots$ are independent, then the ch.f. of $\frac{S_n}{n}$ is

$$\mathbb{E} \exp\{it \tfrac{S_n}{n}\} = \mathbb{E} \exp\{i \tfrac{t}{n} \cdot S_n\}$$

$$= \prod_{m=1}^{n} \left\{1 - \frac{1}{m}[1 - \cos(\tfrac{t}{n} m)]\right\}$$

$$= \prod_{m=1}^{n} \left\{1 - \frac{\frac{n}{m}[1-\cos(t \cdot \frac{m}{n})]}{n}\right\}$$

$$\to \exp\left\{1 - \int_0^1 \frac{1}{x}(1-\cos(tx))\, dx\right\}$$

Hence, $\frac{S_n}{n} \to \mathcal{X}$, where the ch.f. of $\mathcal{X}$ is $\mathbb{E} \exp\{it\mathcal{X}\} = \exp\{1 - \int_0^1 \frac{1}{x}(1-\cos(tx))\, dx\}$

3.6.8 Let T_n be the time of n-th arrival in a rate 1 Poisson process.
Taking V_k^n as the k-th order statistic in $\{u_1, \cdots, u_n\}$ where $u_1, \cdots, u_n \overset{iid}{\sim}$ Uniform(0,1)
and $V_0^n = 0$, $V_{n+1}^n = 1$.

Based on Exercise 3.6.7, we have $V_k^n \overset{d}{=} \frac{T_k}{T_{n+1}}$
Notice that $T_1, (T_2 - T_1), (T_3 - T_2), \cdots \overset{iid}{\sim} Exp(1)$
then by LNN, $\frac{1}{n} T_n = \frac{1}{n}(T_1 + (T_2 - T_1) + \cdots + (T_n - T_{n-1})) \to 1$ a.s.
and then $\frac{1}{n} T_{n+1} = \frac{n+1}{n} \cdot \frac{1}{n+1} T_{n+1} \to 1$ a.s.

Thus $n \cdot V_k^n \overset{d}{=} \frac{n}{n+1} T_k \Rightarrow T_k$ as $n \to \infty$.

3.6.9. Let T_n be the time of n-th arrival in a rate 1 Poisson process.
Taking V_k^n as the k-th order statistic in $\{u_1, \cdots, u_n\}$, where $u_1, \cdots, u_n \overset{iid}{\sim}$ Uniform(0,1)
and $V_0^n = 0$ $V_{n+1}^n = 1$

Take $Z_{n,m} = 1\{n \cdot (V_m^n - V_{m-1}^n) > x\}$, then
$$\mathbb{E} Z_{n,m} = P(nV_m^n - nV_{m-1}^n > x) = P\left(\frac{n}{T_{n+1}}(T_m - T_{m-1}) > x\right)$$

Since $\frac{T_{n+1}}{n} \to 1$ in probability by LLN, then
for $\forall \varepsilon > 0$, we have
$$\begin{aligned}
& P\left(\frac{n}{T_{n+1}} \cdot (T_m - T_{m-1}) > x\right) \\
= \ & P\left(T_m - T_{m-1} > x \cdot \frac{T_{n+1}}{n}, \left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right) + P\left(T_m - T_{m-1} > x \cdot \frac{T_{n+1}}{n}, \left|\frac{T_{n+1}}{n} - 1\right| \le \varepsilon\right) \\
\le \ & P\left(\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right) + P(T_m - T_{m-1} > x(1-\varepsilon)) \\
= \ & P\left(\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right) + e^{-x(1-\varepsilon)} \longrightarrow e^{-x(1-\varepsilon)} \text{ as } n \to \infty
\end{aligned}$$
and
$$\begin{aligned}
& P\left(\frac{n}{T_{n+1}}(T_m - T_{m-1}) > x\right) \\
\ge \ & P\left(T_m - T_{m-1} > x \frac{T_{n+1}}{n}, \left|\frac{T_{n+1}}{n} - 1\right| \le \varepsilon\right) \\
\ge \ & 1 - P\left(\left\{T_m - T_{m-1} \le x \cdot \frac{T_{n+1}}{n}\right\} \cup \left\{\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right\}\right) \\
\ge \ & 1 - \left[P\left(T_m - T_{m-1} \le x \cdot \frac{T_{n+1}}{n}\right) + P\left(\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right)\right] \\
= \ & P\left(T_m - T_{m-1} > x \frac{T_{n+1}}{n}\right) - P\left(\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right) \\
\ge \ & P(T_m - T_{m-1} > x(1+\varepsilon)) - P\left(\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right) \\
= \ & e^{-x(1+\varepsilon)} - P\left(\left|\frac{T_{n+1}}{n} - 1\right| > \varepsilon\right) \longrightarrow e^{-x(1+\varepsilon)} \text{ as } n \to \infty
\end{aligned}$$

Thus $\limsup_{n \to \infty} \mathbb{E} Z_{n,m} \le e^{-x(1-\varepsilon)}$ $\liminf_{n \to \infty} \mathbb{E} Z_{n,m} \ge e^{-x(1+\varepsilon)}$ for $\forall \varepsilon > 0$
then $\lim_{n \to \infty} \mathbb{E} Z_{n,m} = e^{-x}$

By WLLN, $\frac{1}{n} \sum_{m=1}^{n} 1\{n(V_m^n - V_{m-1}^n) > x\} \to e^{-x}$ in probability as $n \to \infty$.

3.6.10 Let T_n be the time of n-th arrival in a rate 1 Poisson Process

Taking V_k^n as the k-th order statistic in $\{U_1, \cdots, U_n\}$ where $U_1, \cdots, U_n \overset{iid}{\sim}$ Uniform(0,1)

and $V_0^n = 0$ $V_{n+1}^n = 1$ $T_0 = 0$

Since $\frac{n}{\log n} \cdot \max\limits_{1\le m\le n+1} (V_m^n - V_{m-1}^n) \overset{d}{=} \frac{n}{\log n} \max\limits_{1\le m\le n+1} \left(\frac{T_m}{T_{n+1}} - \frac{T_{m-1}}{T_{n+1}}\right) = \frac{n}{T_{n+1}} \max\limits_{1\le m\le n+1} (T_m - T_{m-1}) / \log n$

denote $Y_n = T_n - T_{n-1}$ $(n=1,2,\cdots)$ and

$M_n = \max\limits_{1\le m\le n} Y_m$

For $\forall\ 0<\varepsilon<1$, $P(\frac{M_n}{\log n} < 1-\varepsilon) = P(M_n < (1-\varepsilon)(\log n))$

$= [P(X_1 < (1-\varepsilon)(\log n))]^n$

$= [1 - e^{-(1-\varepsilon)(\log n)}]^n$

$= [1 - \frac{1}{n^{1-\varepsilon}}]^n$

$\le \exp\{-n^\varepsilon\}$

Notice that $\frac{\log \frac{1}{n^2}}{\log(\exp\{-n^\varepsilon\})} = \frac{-2\log n}{-n^\varepsilon} = 2 \cdot \frac{\frac{1}{n}}{\varepsilon \cdot n^{\varepsilon-1}} = \frac{2}{\varepsilon} \cdot \frac{1}{n^\varepsilon} \to 0$

thus when n is large enough, $\exp\{-n^\varepsilon\} \le \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, then $\sum_{n=1}^{\infty} P(\frac{M_n}{\log n} < 1-\varepsilon) \le \sum_{n=1}^{\infty} \exp\{-n^\varepsilon\} < \infty$

By Borel-Cantelli Lemma, for $\forall\ \varepsilon > 0$

$P(\frac{M_n}{\log n} < 1-\varepsilon \text{ i.o.}) = 0$

$\Rightarrow P(\frac{M_n}{\log n} < 1-\varepsilon \text{ finitely occurs}) = 1$

then $\liminf\limits_{n\to\infty} \frac{M_n}{\log n} \ge 1-\varepsilon$ a.s. for $\forall\ \varepsilon > 0$

then $\liminf\limits_{n\to\infty} \frac{M_n}{\log n} \ge 1$ a.s.

3.6.11 Let T_n be the time of n-th arrival in a rate 1 Poisson process

Taking V_k^n as the k-th order statistic in $\{U_1, \cdots, U_n\}$ where $U_1, \cdots, U_n$ iid Uniform $(0,1)$

and $V_0^n = 0$, $V_{n+1}^n = 1$.

Notice that $(T_m - T_{m-1})$ iid $Exp(1)$ and $nV_k^n \Rightarrow T_k$ by Excerice 3.6.8

then $P(n^2 \min_{1 \le m \le n} (V_m^n - V_{m-1}^n) > x)$

$= P(n \min_{1 \le m \le n} (nV_m^n - nV_{m-1}^n) > x)$

$\xrightarrow{a.s.} P(n \cdot \min_{1 \le m \le n} (T_m - T_{m-1}) > x)$

$= P(\min_{1 \le m \le n} (T_m - T_{m-1}) > \frac{x}{n})$

$= \prod_{m=1}^{n} P(T_m - T_{m-1} > \frac{x}{n})$

$= (e^{-\frac{x}{n}})^n$

$= e^{-x}$

3.6.12 Let N have a Poisson distribution with mean λ

and $X_1, X_2, \cdots$ be iid sequence with $P(X_i = j) = p_j$ for $j = 0, 1, \cdots, k$

$N_j = |\{m \le N : X_m = j\}|$

Notice that $P(N_0 = m_0, \cdots, N_k = m_k)$

$= P(N_0 = m_0, \cdots, N_k = m_k, N = m_0 + \cdots + m_k)$

$= P(N_0 = m_0, \cdots, N_k = m_k \mid N = m_0 + \cdots + m_k) P(N = m_0 + \cdots + m_k)$

$\overset{\text{multinormial}}{=} \frac{m!}{m_0! \cdots m_k!} p_0^{m_0} \cdots p_k^{m_k} e^{-\lambda} \cdot \frac{\lambda^m}{m!}$

$= \prod_{j=0}^{k} \left[\frac{(p_j \cdot \lambda)^{m_j}}{m_j!} e^{-\lambda p_j} \right]$ where $m := m_0 + \cdots + m_k$

Thus $N_0, \cdots, N_k$ are independent and $N_i \sim \text{Poisson}(\lambda p_i)$

3.9.2 Let $F_1, \cdots, F_d$ be distributions on $\mathbb{R}$

then WTS : for any $\alpha \in [-1, 1]$

$F(x_1, \cdots, x_d) = \left[1 + \alpha \prod_{i=1}^{d} (1 - F_i(x_i))\right] \cdot \prod_{j=1}^{d} F_j(x_j)$

is a distribution function with given marginals

(i) $\log F(x_1, \cdots, x_d) = \log\left[1 + \alpha \prod_{i=1}^{d} (1 - F_i(x_i))\right] + \sum_{j=1}^{d} \log F_j(x_j)$

$\Rightarrow \frac{\partial}{\partial x_k} \log F(x_1, \cdots, x_d) = \frac{-\alpha f_k(x_k) \prod_{i \ne k} (1 - F_i(x_i))}{1 + \alpha \prod_{i=1}^{d} (1 - F_i(x_i))} + \frac{f_k(x_k)}{F_k(x_k)}$

$= \frac{f_k(x_k) \left[1 + \alpha (1 - 2F_k(x_k)) \cdot \prod_{i \ne k} (1 - F_i(x_i))\right]}{\left[1 + \alpha \prod_{i=1}^{d} (1 - F_i(x_k))\right] \cdot F_k(x_k)} \ge 0$ for each k

thus $F(x_1, \cdots, x_d)$ is non-decreasing.

(ii) Since $\lim_{x_k \to \infty} F_k(x_k) = 1$ and $\lim_{x_k \to -\infty} F_k(x_k) = 0$ for each k

then $\lim_{x \to \infty} F(x_1, \cdots, x_d) = 1$ and $\lim_{x \to -\infty} F(x_1, \cdots, x_d) = 1$

(iii) Since $F_k(x_k)$ is right-continuous for each k

then $F(x_1, \cdots, x_d)$ is right-continuous.

(iv) Continue on the next page.

(iv) Let $A = (a_1, b_1] \times \cdots \times (a_d, b_d]$

then
$$\begin{aligned}
\Delta_A F &= \prod_{i=1}^{d} [F_i(b_i) - F_i(a_i)] + \alpha \cdot \prod_{i=1}^{d} [F_i(b_i)(1-F_i(b_i)) - F_i(a_i)\cdot(1-F_i(a_i))] \\
&= \prod_{i=1}^{d} [F_i(b_i) - F_i(a_i)] - \alpha \cdot \prod_{i=1}^{d} [(F_i(b_i) - F_i(a_i))(1 - F_i(b_i) - F_i(a_i))] \\
&\geq \prod_{i=1}^{d} [F_i(b_i) - F_i(a_i)] - \min_k \{\alpha(1 - F_k(b_k) - F_k(a_k))\} \prod_{i=1}^{d} [F_i(b_i) - F_i(a_i)] \\
&= [1 + \min_k \{\alpha[1 - F_k(b_k) - F_k(a_k)]\}] \cdot \prod_{i=1}^{d} [F_i(b_i) - F_i(a_i)] \geq 0 \quad \text{for } \forall \text{ rectangular } A.
\end{aligned}$$

Hence $F(x_1, \cdots, x_d)$ is a distribution function and case $\alpha = 0$ corresponds to independent r.v.s

3.9.7 Assume that multivariable random vector $X = (X_1, \cdots, X_d)$ follows multi-normal distribution with mean θ and covariance Γ

Notice that the ch.f. of X is $\mathbb{E}\exp\{it^TX\} = \exp\{it^T\theta - \frac{1}{2}t^T\Gamma t\}$

(i) Assume that $X_1, \cdots, X_d$ are independent, then
$$\begin{aligned}
\mathbb{E}\exp\{it^TX\} &= \prod_{k=1}^{d} \mathbb{E}\exp\{it_kX_k\} \\
&= \exp\{it^TX - \tfrac{1}{2}\sum_{k=1}^{d} t_k^2\Gamma_{kk}\}
\end{aligned}$$
Since $\Gamma > 0$ thus we have $\Gamma = \mathrm{diag}(\Gamma_{11}, \cdots, \Gamma_{dd})$

(ii) Assume that $\Gamma = \mathrm{diag}(\Gamma_{11}, \cdots, \Gamma_{dd})$, then
$$\begin{aligned}
\mathbb{E}\exp\{i\sum_{k=1}^{d} t_kX_k\} &= \mathbb{E}\exp\{it^TX\} \\
&= \exp\{it^T\theta - \tfrac{1}{2}t^T\Gamma t\} \\
&= \exp\{i\sum_{k=1}^{d} t_k\theta_k - \tfrac{1}{2}\sum_{k=1}^{d} t_k^2\Gamma_{kk}\} \\
&= \prod_{k=1}^{d} \exp\{it_k\theta_k - \tfrac{1}{2}t_k^2\Gamma_{kk}\} \\
&= \prod_{k=1}^{d} \mathbb{E}\exp\{it_kX_k\}
\end{aligned}$$
Thus $X_1, \cdots, X_d$ are independent.

3.9.8 Assume that the multivariate random vector $X = (X_1, \cdots, X_d)$ follows multi-normal distribution with mean θ and covariance Γ

The ch.f. of X is
$$\mathbb{E}\exp\{it^TX\} = \exp\{it^T\theta - \tfrac{1}{2}t^T\Gamma t\}$$
The ch.f. of any linear combination of X : $c^TX = \sum_{k=1}^{d} c_kX_k$ is
$$\begin{aligned}
\mathbb{E}\exp\{it(c^TX)\} &= \mathbb{E}\exp\{i(tc)^TX\} \\
&= \exp\{i(tc)^T\theta - \tfrac{1}{2}(tc)^T \cdot \Gamma(tc)\} \\
&= \exp\{it(c^T\theta) - \tfrac{1}{2}t^2(c^T\Gamma c)\}
\end{aligned}$$
then $c^TX \sim \text{Normal}(c^T\theta, c'\Gamma c)$

Assume that any linear combination of X follows normal distribution

i.e. $c^TX \sim \text{Normal}(c^T\theta, c^T\Gamma c)$,

then the ch.f of X is
$$\mathbb{E}\exp\{it^TX\} = \exp\{it^T\theta - \tfrac{1}{2}t'\Gamma t\}$$

Hence the random vector X follows multinormal distribution with mean θ & covariance matrix Γ.