STAT 5010 Homework 1 10/3/2022 Xiaocheng Zhou 1155184323

1. Take $g(a) = \mathbb{E}|X-a|$, a function w.r.t. a, then we have

$$
\begin{aligned}
g(a) &= \int_{-\infty}^{+\infty} |x-a| \, dF(x) \\
&= -\int_{-\infty}^{a} (x-a) \, dF(x) + \int_{a}^{+\infty} (x-a) \, dF(x) \\
&= -\int_{-\infty}^{a} x \, dF(x) + \int_{a}^{+\infty} x \, dF(x) + \int_{-\infty}^{a} a \, dF(x) - \int_{a}^{+\infty} a \, dF(x) \\
&= -\int_{-\infty}^{a} x F'(x) \, dx + \int_{a}^{+\infty} x F'(x) \, dx + aF(a) - a + aF(a) \\
&= -\int_{-\infty}^{a} x F'(x) \, dx + \int_{a}^{+\infty} x F'(x) \, dx + 2aF(a) - a
\end{aligned}
$$

$$
g'(a) = -\cancel{aF'(a)} - \cancel{aF'(a)} + 2F(a) + \cancel{2aF'(a)} - 1 = 2F(a) - 1
$$

We have $g(a)$ reachs the minimum at m, then $g'(m) = 2F(m) - 1 = 0$
$\Rightarrow F(m) = \frac{1}{2}$, namely, m is the median of X.

2. Cauchy distribution : $f(x|\mu,\sigma) = \dfrac{1}{\sigma\pi\left(1+\frac{(x-\mu)^2}{\sigma}\right)}$, $-\infty < x < \infty$

(a) For $X, Y \overset{iid}{\sim} N(0,1)$, then $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$
a joint distribution $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$ can be constructed.

Take $G = \frac{X}{X+Y}$, $H = X+Y \in \mathbb{R}$. then $X = GH$, $Y = H - GH$
We note transformation function T : $[X, Y]^T = T(G,H) = [GH, H-GH]^T$
By Jacobian Matrix of the transformation : $\begin{bmatrix} H & G \\ -H & 1-G \end{bmatrix}$ with determinant H

then we have
$$
\begin{aligned}
f_{G,H}(g,h) &= f_{X,Y}(gh, h-gh)\, h \\
&= \frac{1}{2\pi} e^{-\frac{g^2h^2+(h-gh)^2}{2}}\, h \\
&= \frac{1}{2\pi} e^{-\frac{(2g^2-2g+1)h^2}{2}}\, h
\end{aligned}
$$

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$$f_G(g) = \int_{-\infty}^{+\infty} f_{G,H}(g,h)\,dh = \int_{-\infty}^{+\infty} \frac{1}{2\pi}\exp\{-\frac{(2g^2-2g+1)h^2}{2}\}h\,dh$$
$$= \int_{-\infty}^{+\infty} \frac{1}{4\pi}\exp\{-\frac{(2g^2-2g+1)}{2}h^2\}dh^2$$
$$= \frac{1}{4\pi}(-\frac{2}{2g^2-2g+1})\exp\{-\frac{(2g^2-2g+1)}{2}t\}\Big|_{t=0}^{+\infty}$$
$$= 0-(-\frac{1}{2\pi}\cdot\frac{1}{2g^2-2g+1})$$
$$= \frac{1}{\pi[4(g-\frac{1}{2})^2+1]}$$

By the definition of Cauchy distribution, $G \sim$ Cauchy distribution

namely $\frac{X}{X+Y} \sim$ Cauchy distribution.

(b) Let $Z = \frac{X}{|Y|} \in \mathbb{R}$ $\quad P(Z\leq z) = P(\frac{X}{|Y|}\leq z)$
$$= P(X\leq zY,\ Y\geq 0) + P(X\leq -zY,\ Y<0)$$
$$= P(X\leq zY,\ Y\geq 0) + P(-X\leq -zY,\ Y<0)$$
$$= P(X\leq zY,\ Y\geq 0) + P(X\geq zY,\ Y<0).$$
$$= P(\frac{X}{Y}\leq z,\ Y\geq 0) + P(\frac{X}{Y}\leq z,\ Y<0)$$
$$= P(\frac{X}{Y}\leq z)$$

then Z has the same distribution with $\frac{X}{Y}$

Take $G = \frac{X}{Y}$, $H = Y$, then $X = GH$, $Y = H$

We note transformation function T: $[X,Y]^T = T(G,H) = [GH, H]^T$

By Jacobian Matrix of T: $\begin{bmatrix} H & G \\ 0 & 1 \end{bmatrix}$ with determinant H

then we have $f_{G,H}(g,h) = f_{X,Y}(gh,h) = \frac{1}{2\pi}\exp\{-\frac{1}{2}(g^2h^2+h^2)\}h$
$$= \frac{1}{2\pi}\exp\{-\frac{1}{2}(g^2+1)h^2\}h$$

$$f_G(g) = \int_{-\infty}^{+\infty} f_{G,H}(g,h)\,dh = \int_{-\infty}^{+\infty}\frac{1}{2\pi}\exp\{-\frac{1}{2}(g^2+1)h^2\}h\,dh$$
$$= \int_{-\infty}^{+\infty}\frac{1}{4\pi}\exp\{-\frac{1}{2}(g^2+1)h^2\}dh^2$$
$$= \frac{1}{4\pi}\cdot\frac{-2}{g^2+1}\exp\{-\frac{1}{2}(g^2+1)t\}\Big|_{t=0}^{+\infty}$$
$$= \frac{1}{2\pi(g^2+1)}$$

then $\frac{X}{|Y|}$ follows the distribution $f_Z(z) = \frac{1}{2\pi(z^2+1)}$

3. (a) Take a sequence of countable sets $A_1, A_2, \cdots, A_n, \cdots$ for $\forall i \neq j \in \{1, 2, \cdots, n\}$, $A_i \cap A_j = \emptyset$, and for $i > n$, $A_i = \emptyset$.

$$
\begin{aligned}
P\{\bigcup_{i=1}^{n} A_i\} &= P\{(\bigcup_{i=1}^{n} A_i) \cup \emptyset\} \\
&= P\{(\bigcup_{i=1}^{n} A_i) \cup (\bigcup_{i=n+1}^{\infty} A_i)\} \\
&= P\{\bigcup_{i=1}^{\infty} A_i\} \\
&= \sum_{i=1}^{\infty} P(A_i) \qquad \text{(by Axiom of Countable Additivity)} \\
&= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} \underbrace{P(A_i)}_{=0} \\
&= \sum_{i=1}^{n} P(A_i)
\end{aligned}
$$

Namely, Countable Additivity $\Rightarrow$ Finite Additivity.

(b) N.T.S. finite additivity: $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.
continuity: $A_1 \supset A_2 \supset \cdots$ $\lim_{n\to\infty} A_n = \emptyset$, then $\lim_{n\to\infty} P(A_n) = 0$.
$\Big\} \Rightarrow P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Take a sequence of disjoint sets, $A_1, A_2, \cdots \subset A$ satisfying $\bigcup_{i=1}^{\infty} A_i = A$

Let $B_n = A \setminus \bigcup_{i=1}^{n} A_i$, then $B_1 \supset B_2 \supset \cdots$

with $\lim_{n\to\infty} B_n = \emptyset$, $A = B_n \cup (\bigcup_{i=1}^{n} A_i)$ $B_n \cap (\bigcup_{i=1}^{n} A_n) = \emptyset$

$$
\begin{aligned}
P(A) = P(B_n \cup (\bigcup_{i=1}^{n} A_i)) &= P(B_n) + P(\bigcup_{i=1}^{n} A_i) \\
\text{(by finite additivity)} &= P(B_n) + \sum_{i=1}^{n} P(A_i) \\
\text{(by continuity)} &\to 0 + \sum_{i=1}^{\infty} P(A_i) \quad \text{when } n \to \infty
\end{aligned}
$$

Namely $P(A) = P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$\Longrightarrow$ countabe additivity.

4. N.T.S. $\lim_{n\to\infty} \frac{n!}{n^{(n+1/2)} e^{-n}} = C$, $C \in \mathbb{R}^+$ $\Leftrightarrow$ $\lim_{n\to\infty} \{\sum_{i=1}^{n} \log i - (n+\frac{1}{2}) \log n + n\} = C'$, $C' \in \mathbb{R}$

Note that $a_n = \sum_{i=1}^{n} \log i - (n+\frac{1}{2}) \log n + n$

$$
\begin{aligned}
\text{then } a_{n+1} - a_n &= \log(n+1) - (n+\tfrac{3}{2}) \log(n+1) + (n+\tfrac{1}{2}) \log n + 1 \\
&= (n+\tfrac{1}{2}) \log \frac{n}{n+1} + 1
\end{aligned}
$$

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Consider the Taylor expansion of $\log\frac{1-x}{1+x}$ for $|x|<1$:

$$\begin{aligned}\log\frac{1-x}{1+x} &= \log(1-x)-\log(1+x)\\ &= \sum_{i=1}^{\infty}\frac{(-1)^i x^i}{i}-\sum_{i=1}^{\infty}\frac{x^i}{i}\\ &= -2\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1}\end{aligned}$$

then replace x with $\frac{1}{2n+1}\in(0,1)$

$$\log\frac{1-\frac{1}{2n+1}}{1+\frac{1}{2n+1}}=\log\frac{n}{n+1}=-2\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2n+1)^{2k-1}}$$

then
$$\begin{aligned}a_{n+1}-a_n &= (n+\tfrac{1}{2})\left(-2\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2n+1)^{2k-1}}\right)+1\\ &= -\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2n+1)^{2k-2}}+1\\ &= -1-\sum_{k=2}^{\infty}\frac{1}{(2k-1)(2n+1)^{2k-2}}+1\\ &= -\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2n+1)^{2k}}<0\end{aligned}$$

$\Rightarrow a_{n+1}<a_n$, $\{a_n\}$ is a decreasing sequence.

$$\begin{aligned}a_{n+1}-a_n &= -\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2n+1)^{2k}}\\ &> -\sum_{k=1}^{\infty}\frac{1}{3(2n+1)^{2k}} \qquad \text{(sum of geometric progression)}\\ &= -\frac{1}{3}\frac{\frac{1}{(2n+1)^2}}{1-\frac{1}{(2n+1)^2}} \qquad (2n+1)^2-1\\ &= -\frac{1}{12n(n+1)} \qquad 4n^2+4n\\ &= -\frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)\end{aligned}$$

$\Rightarrow a_{n+1}-\frac{1}{12(n+1)}>a_n-\frac{1}{12n}$, $\{a_n-\frac{1}{12n}\}$ is a increasing sequence.

$\Rightarrow a_n-\frac{1}{12n}>a_1-\frac{1}{12}=\frac{11}{12}$

$\Rightarrow a_n>\frac{11}{12}$

combining with the monotonely decreasity of $\{a_n\}$

then $\{a_n\}$ converges to a constant $C'\in\mathbb{R}^+$. $\Leftrightarrow \lim_{n\to\infty}\frac{n!}{n^{(n+1/2)}e^{-n}}=C$, $C\in\mathbb{R}^+$

5. Take a sequence of m-dependent r.v. $X_1, X_2, \cdots$ with $\mathbb{E}X_i = \mu$, $Var X_i = \sigma^2 < \infty$

then $Cov(X_i, X_j) = 0$ for $|i-j| \geq m$ and $|Cov(X_i, X_j)| \leq \sigma^2$ for $|i-j| < m$

we have
$$
\begin{aligned}
Var \bar{X}_n &= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n} Var X_i + \sum_{|i-j|<m} Cov(X_i, X_j)\right] \\
&\leq \frac{1}{n^2}\left\{n\sigma^2 + [(2m+1)n - 2m]\sigma^2\right\} \\
&= \frac{2(mn+n-m)}{n^2}\sigma^2
\end{aligned}
$$

By Chebyshev's Inequality, for $\forall \delta > 0$,
$$
\begin{aligned}
P(|\bar{X}_n - \mu| \geq \delta) &\leq \frac{Var \bar{X}_n}{\delta^2} \\
&\leq \frac{2(mn+n-m)}{n^2\delta^2}\sigma^2 \\
&= 2\left(\frac{m}{n} + \frac{1}{n} - \frac{m}{n^2}\right)\frac{\sigma^2}{\delta^2} \\
&\to 0 \quad \text{when } n \to \infty. \quad m \ll n
\end{aligned}
$$

Hence, $\bar{X}_n \xrightarrow{P} \mu$.

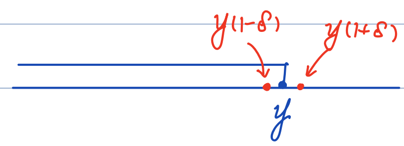
6. N.T.S. $\left.\begin{array}{l} A_n \xrightarrow{P} 1 \\ Y_n \xrightarrow{d} Y \end{array}\right\} \Rightarrow A_n Y_n \xrightarrow{d} Y$

$\Leftrightarrow \left.\begin{array}{l} \lim_{n\to\infty} P(|A_n - 1| > \delta) = 0 \text{ for } \forall \delta > 0 \\ \lim_{n\to\infty} P(Y_n \leq y) = P(Y \leq y) \text{ for } \forall y \text{ continuity point} \end{array}\right\} \Rightarrow \lim_{n\to\infty} P(A_n Y_n \leq y) = P(Y \leq y)$

$$
\begin{aligned}
\lim_{n\to\infty} P(|A_n - 1| > \delta) = 0 &\Leftrightarrow \lim_{n\to\infty} P(|A_n - 1| \leq \delta) = 1 \\
&\Leftrightarrow \lim_{n\to\infty} (1-\delta \leq A_n \leq \delta + 1) = 1 \quad \text{for } \forall \delta > 0
\end{aligned}
$$

$\lim_{n\to\infty} P(Y_n \leq y) P(1-\delta \leq A_n \leq \delta+1) = P(Y \leq y) \cdot 1 = P(Y \leq y)$

where
$$
\begin{aligned}
&P(Y_n \leq y) P(1-\delta \leq A_n \leq \delta+1) \\
&= P(Y_n \leq y, 1-\delta \leq A_n \leq \delta+1) \\
&= P(A_n Y_n \leq y(\delta+1)) \quad \text{for } \delta > 0
\end{aligned}
$$



Let $\delta \to 0$, then $P(Y_n \leq y) P(1-\delta \leq A_n \leq \delta+1) = P(A_n Y_n \leq y)$

then $\lim_{n\to\infty} P(A_n Y_n \leq y) = P(Y \leq y)$

Namely, $A_n Y_n \xrightarrow{d} Y$.

7 Let $Y_n = X_n^{-1}$, and $G_n = H_n^{-1} = \frac{1}{n}\sum_{i=1}^{n} X_i^{-1} = \frac{1}{n}\sum_{i=1}^{n} Y_n$

$F_Y(y) = P(Y \le y) = P(X \ge y^{-1}) = 1 - P(X \le y^{-1}) = 1-(y^{-1}-1) = 2-y^{-1}$

so $\mathbb{E} Y_n = \int_{\frac{1}{2}}^{1} y \, dF_Y(y) = \int_{\frac{1}{2}}^{1} y \cdot \frac{1}{y^2} dy = \int_{\frac{1}{2}}^{1} \frac{1}{y} dy = \ln 1 - \ln\frac{1}{2} = \ln 2.$

$\mathbb{E} Y_n^2 = \int_{\frac{1}{2}}^{1} y^2 \, dF_Y(y) = \int_{\frac{1}{2}}^{1} y^2 \cdot \frac{1}{y^2} dy = \frac{1}{2}$

$\mathrm{Var}\, Y_n = \mathbb{E} Y_n^2 - (\mathbb{E} Y_n)^2 = \frac{1}{2} - \ln 2$

By SLLN $P(\lim_{n\to\infty} |G_n - \ln 2| < \varepsilon) = 1$, namely $G_n \xrightarrow{a.s.} \ln 2$

$\Rightarrow G_n \xrightarrow{P} \ln 2 \Rightarrow H_n^{-1} \xrightarrow{P} \ln 2$

$\lim_{n\to\infty} P(|H_n^{-1} - \ln 2| \le \varepsilon) = 1$

$\Leftrightarrow \lim_{n\to\infty} P(-\varepsilon + \ln 2 \le H_n^{-1} \le \varepsilon + \ln 2) = 1$ $\qquad Y_n \in [\frac{1}{2}, 1] \Rightarrow H_n^{-1} = \frac{1}{n}\sum_{i=1}^{n} Y_n \in [\frac{1}{2}, 1]$

$H_n > 0$, take $\varepsilon < \ln 2$ $\Rightarrow \lim_{n\to\infty} P\left(\frac{-\varepsilon}{(\varepsilon + \ln 2)\ln 2} \le H_n - \frac{1}{\ln 2} \le \frac{\varepsilon}{(-\varepsilon + \ln 2)\ln 2}\right) = 1$

take $\delta = \max\left\{-\frac{-\varepsilon}{(\varepsilon+\ln 2)\ln 2}, \frac{\varepsilon}{(-\varepsilon+\ln 2)\ln 2}\right\} = \frac{\varepsilon}{(-\varepsilon+\ln 2)\ln 2} > 0$

then $\lim_{n\to\infty} P(|H_n - \frac{1}{\ln 2}| \le \delta) = 1$

$\lim_{n\to\infty} P(|H_n - \frac{1}{\ln 2}| > \delta) = 0$ namely $H_n \xrightarrow{P} \frac{1}{\ln 2}$

8. (a) $f_{Y_1,Y_2,\cdots,Y_n}(y_1,y_2,\cdots,y_n) := n!\, f(y_1)f(y_2)\cdots f(y_n)\, I(0\le y_1\le y_2\le\cdots\le y_n)$

$$= n!\,\frac{1}{(2\theta)^n}\exp\left\{-\frac{\sum_{i=1}^n y_i}{2\theta}\right\} I(0\le y_1\le y_2\le\cdots\le y_n)$$

Integrate the joint density w.r.t. $y_{r+1},\cdots,y_n$

then marginal distribution of $Y_1\le Y_2\le\cdots\le Y_r$ has density:

$$f_{Y_1,Y_2,\cdots,Y_n}(y_1,y_2,\cdots,y_r) = n!\,\frac{1}{(2\theta)^n}\int_{y_r}^{\infty}\cdots\int_{y_{n-2}}^{\infty}\int_{y_{n-1}}^{\infty}\exp\left\{-\frac{\sum_{i=1}^n y_i}{2\theta}\right\}dy_n\,dy_{n-1}\cdots dy_{r+1}$$

$$= n!\,\frac{1}{(2\theta)^{n-1}}\int_{y_r}^{\infty}\cdots\int_{y_{n-2}}^{\infty}\exp\left\{-\frac{\sum_{i=1}^{n-2} y_i+2y_{n-1}}{2\theta}\right\}dy_{n-1}\cdots dy_{r+1}$$

$$= n!\,\frac{1}{(2\theta)^{n-2}}\int_{y_r}^{\infty}\cdots\int_{y_{n-3}}^{\infty}\frac{1}{2!}\exp\left\{-\frac{\sum_{i=1}^{n-3} y_i+3y_{n-2}}{2\theta}\right\}dy_{n-2}\cdots dy_{r+1}$$

$$=\cdots$$

$$= n!\,\frac{1}{(2\theta)^{r+1}}\int_{y_r}^{\infty}\frac{1}{(n-r-1)!}\exp\left\{-\frac{\sum_{i=1}^{r} y_i+(n-r)y_{r+1}}{2\theta}\right\}dy_{r+1}$$

$$= \frac{1}{(2\theta)^r}\cdot\frac{n!}{(n-r)!}\exp\left\{-\frac{\sum_{i=1}^{r} y_i+(n-r)y_r}{2\theta}\right\}\quad \text{for } 0\le y_1\le\cdots\le y_r.$$

(b) Take $M_1 = Y_1\in\mathbb{R}^+$, $M_2 = Y_2 - Y_1\in\mathbb{R}^+$, $\cdots$, $M_r = Y_r - Y_{r-1}\in\mathbb{R}^+$

then we have $Y_1 = M_1$, $Y_2 = M_1+M_2$, $\cdots$, $Y_r = M_1+M_2+\cdots+M_r$

By the Jacobian Matrix of transformation: $\begin{bmatrix}1&0&\cdots&0\\1&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\end{bmatrix}$ with determinant 1

then the joint density of $M_1,\cdots,M_r$ is

$$f_{M_1,\cdots,M_r}(m_1,\cdots,m_r) = f_{Y_1,\cdots,Y_r}(m_1,\cdots,m_r)\cdot 1$$

$$= \frac{1}{(2\theta)^r}\frac{n!}{(n-r)!}\exp\left\{-\frac{\sum_{k=1}^r (n-k+1)m_k}{2\theta}\right\}$$

$$= \prod_{k=1}^r\left[\frac{n-k+1}{2\theta}\exp\left(-\frac{n-k+1}{2\theta}m_k\right)\right]$$

so $\{M_i\}$ is independent and each M_i has an exponential distribution with mean $\frac{2\theta}{n-k+1}$, $k=1,\cdots,r$

then each in $\left\{\frac{n-k-1}{\theta}M_k\right\}_{k=1}^r \overset{iid}{\sim} Exp(2)$, which is also a chi-square distribution with degree of freedom 2

namely each in $\left\{\frac{n-k-1}{\theta}M_k\right\}_{k=1}^r \overset{iid}{\sim} \chi^2_2$

Notice that $\sum_{k=1}^r\frac{n-k-1}{\theta}M_k = \frac{\sum_{k=1}^r Y_i+(n-r)Y_r}{\theta}$ and by the property of iid chi-square r.v.'s

then we have $\frac{1}{\theta}\left[\sum_{k=1}^r Y_i+(n-r)Y_r\right]\sim\chi^2_{2r}$

(c) By the property of Poisson process $N(t)$ with parameter $\lambda = \frac{1}{2\theta'}$

we have
$$\begin{aligned} P(Z_1 \geq z) &= P(Y_1 \geq z\theta') \\ &= P(N(z\theta')=0) \\ &= \frac{(\lambda z\theta')^0}{0!} \exp(-\lambda z\theta') \\ &= \exp(-\tfrac{z}{2}) \end{aligned}$$

then $Z_1 \sim Exp(\frac{1}{2})$, namely $Z_1 \sim \chi_2^2$

$$\begin{aligned} P(Z_2 \geq z_2, Z_1 \geq z_1) &= \int_{z_1}^{\infty} P(Z_2 \geq z_2 \mid Z_1 = z_0) P(Z_1 = z_0) dz_0 \\ &= \int_{z_1}^{\infty} P(Y_2 - Y_1 \geq z_2\theta' \mid Y_1 = z_0\theta') f_{Z_1}(z_0) dz_0 \\ &= P(N(z_2\theta')=0) \int_{z_1}^{\infty} f_{Z_1}(z_0) dz_0 \\ &= \exp(-\tfrac{z_2}{2}) \exp(-\tfrac{z_1}{2}) \end{aligned}$$

then $f_{Z_1 Z_2}(z_1, z_2) = \frac{1}{2}\exp(-\frac{z_2}{2}) \cdot \frac{1}{2}\exp(-\frac{z_1}{2})$,

so $Z_1 \perp\!\!\!\perp Z_2$ and $Z_2 \sim \exp\{\frac{1}{2}\}$, namely $Z_2 \sim \chi_2^2$

Similarly, we can show that $Z_3, \cdots, Z_r \sim \chi_2^2$,

then their joint distribution is
$$f_{Z_1, \cdots, Z_r}(z_1, \cdots, z_r) = \frac{1}{2^r} \exp\left\{-\frac{\sum_{i=1}^{r} z_i}{2}\right\}$$

We know $Z_1 = \frac{1}{\theta'} Y_1 \quad Z_k = \frac{1}{\theta'}(Y_k - Y_{k-1}),\ k = 2, \cdots, r$

by the Jacobian Matrix of transformation: $\begin{bmatrix} \frac{1}{\theta'} & \cdots & & 0 \\ -\frac{1}{\theta'} & \frac{1}{\theta'} & \cdots & 0 \\ 0 & -\frac{1}{\theta'} & \frac{1}{\theta'} \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdot \ -\frac{1}{\theta'} & \frac{1}{\theta'} \end{bmatrix}$ with determinant $(\frac{1}{\theta'})^r$

then the joint density of $Y_1, \cdots, Y_r$ is
$$\begin{aligned} f_{Y_1, \cdots, Y_r}(y_1, \cdots, y_r) &= f_{Z_1, \cdots, Z_r}(\tfrac{1}{\theta'} y_1, \cdots, \tfrac{1}{\theta'}(y_k - y_{k-1}))(\tfrac{1}{\theta'})^r \\ &= \frac{1}{(2\theta')^r} \exp(-\tfrac{y_r}{2\theta'}), \quad 0 \leq y_1 \leq \cdots \leq y_r \end{aligned}$$

Considering $\sum_{i=1}^{r} Z_i = \frac{1}{\theta'}(Y_r - Y_{r-1}) + \frac{1}{\theta'}(Y_{r-1} - Y_{r-2}) + \cdots + \frac{1}{\theta'} Y_1 = \frac{1}{\theta'} Y_r$,

we have $\frac{1}{\theta'} Y_r$ is the sum of r iid χ_2^2 distribution.

then $\frac{1}{\theta'} Y_r \sim \chi_{2r}^2$

9. (a) Suppose the population with values is $y_1, y_2, \cdots, y_N$, and the samples drawn without replacement is noted by $X_1, X_2, \cdots, X_n$.

Given that m is the population mean, then $\frac{1}{N}\sum_{i=1}^{N} y_i = m$.

Since each individual in the population is sampled in the sample with the same probability,

then $P(X_i = y_j) = \frac{1}{N}$, so $\mathbb{E}(X_i) = \sum_{k=1}^{N}(y_k \cdot \frac{1}{N}) = \frac{1}{N}\sum_{k=1}^{N} y_k = m$

We have $\mathbb{E}(\bar{X}_n) = \mathbb{E}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{1}{n}\cdot\sum_{i=1}^{n} m = m$.

$\Rightarrow \bar{X}_n$ is an unbiased estimator of the population mean m.

(b) Suppose the samples drawn with replacement is noted by $R_1, R_2, \cdots, R_n$, and the population variance is noted by $\sigma^2 := \frac{1}{N}\sum_{i=1}^{N}(y_i - m)^2$

Consider X_i is sampled without replacement, then

$$P(X_i = y_s, X_j = y_t) = \frac{1}{N(N-1)} \quad s \neq t \quad i \neq j \in \{1, 2, \cdots, n\},\ s, t \in \{1, 2, \cdots, N\}$$

$$P(X_i = y_s, X_j = y_t) = 0 \quad s = t.$$

$$\frac{1}{N(N-1)}\sum_{s\neq t}^{N} y_s y_t$$

$$\begin{aligned}
Cov(X_i, X_j) &= \mathbb{E}(X_i - m)(X_j - m) = \mathbb{E}(X_i X_j) - m^2 \\
&= -m^2 + \sum_{s=1}^{N}\sum_{t=1}^{N} y_s y_t P(X_i = y_s, X_j = y_t) \\
&= -m^2 + \frac{1}{N(N-1)}\sum_{s\neq t}^{N} y_s y_t \\
&= -m^2 + \frac{1}{N(N-1)}\left[(\sum_{i=1}^{N} y_i)^2 - \sum_{i=1}^{N} y_i^2\right] \\
&= -m^2 + \frac{1}{N(N-1)}\left[N^2 m^2 - \sum_{i=1}^{N}(y_i - m + m)^2\right] \\
&= -m^2 + \frac{1}{N(N-1)}\left[N^2 m^2 - \underbrace{\sum_{i=1}^{N}(y_i - m)^2}_{N\sigma^2} \underbrace{-2m\sum_{i=1}^{N}(y_i - m)}_{0} - Nm^2\right] \\
&= -m^2 + \frac{1}{N(N-1)}\left[N^2 m^2 - N\sigma^2 - Nm^2\right] \\
&= -m^2 + \frac{Nm^2 - \sigma^2 - m^2}{N-1} \\
&= -\frac{\sigma^2}{N-1}
\end{aligned}$$

$$\begin{aligned}
\text{so } Var(\bar{X}_n) &= \frac{1}{n^2}\left(\sum_{i=1}^{n} Var(X_i) + \sum_{i\neq j} Cov(X_i, X_j)\right) \\
&= \frac{1}{n^2}\left(n\sigma^2 - n(n-1)\frac{\sigma^2}{N-1}\right) \\
&= \left(1 - \frac{n-1}{N-1}\right)\frac{\sigma^2}{n}
\end{aligned}$$

(continue on the next page).

Then consider R_i is sampled with replacement. then

$$P(R_i = y_j) = \frac{1}{N} \quad \text{and} \quad R_i \text{ iid for } i \in \{1, \dots, n\}$$

then $\mathbb{E}R_i = \sum_{j=1}^{N} y_j \frac{1}{N} = m$ and $Var(R_i) = \mathbb{E}(R_i - m)^2 = \frac{1}{N}\sum_{j=1}^{N}(y_j - m)^2 = \sigma^2$

so $Var(\bar{R}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(R_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n}\sigma^2$

$$\Rightarrow \left(1 - \frac{n-1}{N-1}\right)\frac{\sigma^2}{n} =: MSE(\bar{X}_n) < MSE(\bar{R}_n) := \frac{\sigma^2}{n}$$

(c) $MSE(\bar{R}_n) - MSE(\bar{X}_n) = \frac{n-1}{N-1} \cdot \frac{\sigma^2}{n} = O\left(\frac{1}{N}\right).$