

PTE 5TH EDITION SOLUTIONS MANUAL

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1. MEASURE THEORY

1.1. Probability Theory.

Exercise 1.1.1. (i) \mathcal{F} is a σ -algebra.

From the definition, \emptyset, Ω are in \mathcal{F} , and \mathcal{F} is closed under operation of complement. Now, suppose A_1, A_2, \dots are in \mathcal{F} . If all A_n 's are countable, then $\cup_{n=1}^{\infty} A_n$, which is countable union of countable sets, is also countable, and thus is in \mathcal{F} . Otherwise, $\cup_{n=1}^{\infty} A_n^c$ is countable, and thus $\cup_{n=1}^{\infty} A_n$ is in \mathcal{F} .

(ii) P is a probability measure.

By definition, $P(A) \geq P(\emptyset) = 0$, for any $A \in \mathcal{F}$. For disjoint $A_1, A_2, \dots \in \mathcal{F}$, if all A_n 's are countable, then

$$P(\cup_{n=1}^{\infty} A_n) = 0 = \sum_{n=1}^{\infty} P(A_n).$$

If one of A_n is uncountable (say A_k), then since A_k^c is countable and $A_j \subset A_k^c$ for $j \neq k$, $(\cup_{n=1}^{\infty} A_n)^c \subset A_k^c$ and A_j 's are countable for $j \neq k$. Thus,

$$P(\cup_{n=1}^{\infty} A_n) = 1 = P(A_k) + \sum_{n \neq k} P(A_n).$$

Therefore, (Ω, \mathcal{F}, P) is a probability space.

Exercise 1.1.2. Let $\mathcal{O}_d = \{(a_1, b_1) \times \dots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}\}$. It suffices to show that $\sigma(\mathcal{O}_d) = \sigma(\mathcal{S}_d)$. Since,

$$\begin{aligned} (a_1, b_1] \times \dots \times (a_d, b_d] &= \bigcap_{n=1}^{\infty} (a_1, b_1 + \frac{1}{n}) \times \dots \times (a_d, b_d + \frac{1}{n}). \\ (a_1, b_1) \times \dots \times (a_d, b_d) &= \bigcup_{n=1}^{\infty} (a_1, b_1 - \frac{1}{n}) \times \dots \times (a_d, b_d - \frac{1}{n}) \end{aligned}$$

Exercise 1.1.3. $\mathcal{C} = \{(a_1, b_1) \times \dots \times (a_d, b_d) : \text{rational numbers } a < b\}$.

Exercise 1.1.4. Let $\mathcal{F} = \cup_i \mathcal{F}_i$.

(i) Clearly, $\emptyset, \Omega \in \mathcal{F}$. If $A \in \mathcal{F}$, then there exists j such that $A \in \mathcal{F}_j$. Then, $A^c \in \mathcal{F}_j$, and thus $A^c \in \mathcal{F}$. Let A and B are in \mathcal{F} . Then, there exist j_1, j_2 such that $A \in \mathcal{F}_{j_1}$ and $B \in \mathcal{F}_{j_2}$. WLOG, let $j_1 \leq j_2$. Since \mathcal{F}_n are increasing sequence, A is also in \mathcal{F}_{j_2} . Thus, $A \cup B \in \mathcal{F}_{j_2} \subset \mathcal{F}$. Therefore, \mathcal{F} is an algebra.

(ii) Let $\Omega = (0, 1]$ and $I_n = \{(\frac{j-1}{2^n}, \frac{j}{2^n}] \text{ for } j = 1, \dots, 2^n\}$ for $\forall n$. Define

$$\mathcal{F}_n \equiv \sigma(I_n) \text{ for } n = 1, 2, \dots$$

Since $(\frac{j-1}{2^n}, \frac{j}{2^n}] = (\frac{2(j-1)}{2^{n+1}}, \frac{2j-1}{2^{n+1}}] \cup (\frac{2j-1}{2^{n+1}}, \frac{2j}{2^{n+1}}]$, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Now,

$$(0, 1) = \bigcup_{n=1}^{\infty} (\frac{2^n - 2}{2^n}, \frac{2^n - 1}{2^n}],$$

where $(\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}] \in \mathcal{F}_n$ for $\forall n$. However, since $(0,1) \notin \mathcal{F}_n$ for $\forall n$, $(0,1) \notin \mathcal{F}$.

Therefore, \mathcal{F} is not a σ -algebra.

Exercise 1.1.5. (Counterexample) Let A be the set of even numbers. Then, $A \in \mathcal{A}$ with $\theta = \frac{1}{2}$. Let $I_0 = \{1\}$, $I_k = \{n : 2^{k-1} < n \leq 2^k\}$, for $k = 1, \dots$. Construct B as follows.

$$B = \cup_{k=0}^{\infty} \{(I_{2k} \cap A^c) \cup (I_{2k+1} \cap A)\}$$

Then, $B \in \mathcal{A}$ with $\theta = \frac{1}{2}$. However, $A \cap B \notin \mathcal{A}$ for following reasons.

(1) $[n = 2^{2m}]$

$$|(A \cap B) \cap \{1, 2, \dots, 2^{2m}\}| = \sum_{k=1}^m \frac{1}{2} |I_{2k-1}|.$$

$$\begin{aligned} \frac{|(A \cap B) \cap \{1, 2, \dots, 2^{2m}\}|}{2^{2m}} &= \frac{\sum_{k=1}^m \frac{1}{2} |I_{2k-1}|}{2^{2m}} \\ &= \frac{1 + \sum_{k=2}^m 2^{2k-3}}{2^{2m}} \\ &= \frac{\frac{2^{2m+1}-1}{3}}{2^{2m}} \rightarrow \frac{1}{6}, \end{aligned}$$

(2) $[n = 2^{2m+1}]$

Similarly,

$$\begin{aligned} \frac{|(A \cap B) \cap \{1, 2, \dots, 2^{2m+1}\}|}{2^{2m+1}} &= \frac{\sum_{k=1}^{m+1} \frac{1}{2} |I_{2k-1}|}{2^{2m+1}} \\ &= \frac{1 + \sum_{k=2}^{m+1} 2^{2k-3}}{2^{2m+1}} \\ &= \frac{\frac{2^{2m+2}-1}{3}}{2^{2m+1}} \rightarrow \frac{1}{3} \end{aligned}$$

By (1), (2), $\limsup \frac{|(A \cap B) \cap \{1, 2, \dots, n\}|}{n} \geq \frac{1}{3}$, $\liminf \frac{|(A \cap B) \cap \{1, 2, \dots, n\}|}{n} \leq \frac{1}{6}$. Thus, $A \cap B \notin \mathcal{A}$. Therefore, \mathcal{A} is neither an algebra nor a σ -algebra.

1.2. Distributions.

Exercise 1.2.1. To show that Z is a r.v., it suffices to show that for any $B \in \mathcal{B}(\mathbb{R})$, $Z^{-1}(B) \in \mathcal{F}$.

$$\begin{aligned} Z^{-1}(B) &= [Z^{-1}(B) \cap A] \cup [Z^{-1}(B) \cap A^c] \\ &= [X^{-1}(B) \cap A] \cup [Y^{-1}(B) \cap A^c] \\ &\in \mathcal{B}(\mathbb{R}), \end{aligned}$$

since X, Y are r.v.s, and $A \in \mathcal{F}$.

Exercise 1.2.2. Omit.

Exercise 1.2.3. Let A be the discontinuity set of distribution function F . Since F is right-continuous, for $x \in A$, $F(x^-) < F(x) = F(x^+)$ holds. For $x < y$ who are in A , we have

$F(x^-) < F(x) \leq F(y^-) < F(y)$. Pick rationals q_x and q_y such that $q_x \in (F(x^-), F(x))$ and $q_y \in (F(y^-), F(y))$. Then, $q_x < q_y$, and thus $|A| \leq \aleph_0$. Therefore, there exist at most countably many discontinuities.

Exercise 1.2.4. Let $F^{-1}(y) = \inf\{x : y \leq F(x)\}$. Then followings hold.

- (i) $F(F^{-1}(y)) = y$ for $y \in (0, 1)$.
- (ii) F^{-1} is strictly increasing.
- (iii) $\{F^{-1}(F(X)) \leq F^{-1}(y)\} = \{X \leq F^{-1}(y)\}$

(i) comes from continuity of F , and (ii) comes from monotonicity of F . For (iii),

$$\begin{aligned} \{F^{-1}(F(X)) \leq F^{-1}(y)\} &= \{F(X) \leq y\} \quad (\text{by (ii)}) \\ &= \{X \leq F^{-1}(y)\}. \quad (\text{by definition}) \end{aligned}$$

Thus, for $0 < y < 1$,

$$\begin{aligned} P(F(X) \leq y) &= P(F^{-1}(F(X)) \leq F^{-1}(y)) && (\text{by (ii)}) \\ &= P(X \leq F^{-1}(y)) && (\text{by (iii)}) \\ &= F(F^{-1}(y)) \\ &= y && (\text{by (i)}) \end{aligned}$$

For $y = 0$, $P(F(X) \leq 0) = \lim_n P(F(X) \leq \frac{1}{n}) = \lim_n \frac{1}{n} = 0$.

For $y = 1$, $P(F(X) \leq 1) = \lim_n P(F(X) \leq 1 - \frac{1}{n}) = \lim_n 1 - \frac{1}{n} = 1$.

Thus, $F(X) = Y \sim \text{Unif}(0, 1)$.

Exercise 1.2.5. Omit.

Exercise 1.2.6. Omit.

Exercise 1.2.7. Omit.

1.3. Random Variables.

Exercise 1.3.1. We want to show that $\sigma(X) = \sigma(X^{-1}(\mathcal{A}))$.

- (\supset) By definition of $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$, it is clear.
- (\subset) It suffices to show that X is $\sigma(X^{-1}(\mathcal{A}))$ -measurable. Firstly, for any σ -field \mathcal{D} on Ω , $\mathcal{T} = \{B \in \mathcal{S} : \{X \in B\} \in \mathcal{D}\}$ is also a σ -field on S . Construct \mathcal{T} with $\mathcal{D} = \sigma(X^{-1}(\mathcal{A}))$. Then, \mathcal{T} is a σ -field containing \mathcal{A} . Since \mathcal{A} generates \mathcal{S} , $\mathcal{S} = \mathcal{T}$, which means that X is $\sigma(X^{-1}(\mathcal{A}))$ -measurable.

Exercise 1.3.2.

$$\begin{aligned} \{X_1 + X_2 < x\} &= \{X_1 < x - X_2\} \\ &= \bigcup_{q \in \mathbb{Q}} \{X_1 < q < x - X_2\} \\ &= \bigcup_{q \in \mathbb{Q}} [\{X_1 < q\} \cap \{X_2 < x - q\}] \in \mathcal{F}. \end{aligned}$$

Exercise 1.3.3. Since f is continuous, $f(X_n)$ and $f(X)$ are r.v. Moreover, on $\Omega_0 = \{\omega : X_n(\omega) \rightarrow X(\omega)\}$, $f(X_n)(\omega) \rightarrow f(X)(\omega)$, which means that $\Omega_0 \subset \Omega'_0 = \{\omega : f(X_n)(\omega) \rightarrow f(X)(\omega)\}$, since f is continuous. Thus, $P(\Omega'_0) \geq P(\Omega_0) = 1$, that is, $f(X_n) \rightarrow f(X)$ a.s.

Exercise 1.3.4. (i) Let f be a continuous function from $\mathbb{R}^d \rightarrow \mathbb{R}$ and let $\mathcal{O}, \mathcal{O}_d$ be collections of open sets in \mathbb{R}, \mathbb{R}^d , respectively. Since pre-image of open set in \mathbb{R} under f is also open (by continuity), $\sigma(f^{-1}(\mathcal{O})) \subset \sigma(\mathcal{O}_d) = \mathcal{B}(\mathbb{R}^d)$. Also, by Exercise 1.3.1, $\sigma(f^{-1}(\mathcal{O})) = f^{-1}(\mathcal{B}(\mathbb{R}))$. Thus, f is $(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ -measurable.

(ii) Let \mathcal{A} be the smallest σ -field that makes all the continuous functions measurable. By (i), we showed that $\mathcal{A} \subset \mathcal{B}(\mathbb{R}^d)$. To show the converse, define $f_i((x_1, \dots, x_d)) = x_i$, for $i = 1, \dots, d$, who are continuous. Then, $f_i^{-1}((a_i, b_i)) \in \mathcal{B}(\mathbb{R}^d)$, and thus,

$$f_1^{-1}((a_1, b_1)) \cap \dots \cap f_d^{-1}((a_d, b_d)) = (a_1, b_1) \times \dots \times (a_d, b_d),$$

for any $a_i, b_i \in \mathbb{R}$. This means that any open boxes in \mathbb{R}^d is in \mathcal{A} . Therefore, $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{A}$.

Exercise 1.3.5. Let f be a lower semicontinuous function. f is l.s.c if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that $f(y) > f(x) - \epsilon$ for $y \in B(x, \delta)$. We want to show that f is l.s.c if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$.

(\Rightarrow) Suppose not. Then, there exists $a_0 \in \mathbb{R}$ such that $\{x : f(x) \leq a\}$ is not closed, that is, there exists a limit point x_0 of $\{x : f(x) \leq a\}$ with $f(x_0) > a_0$. For $\epsilon = \frac{f(x_0) - a_0}{2}$, there exists $\delta > 0$ such that $f(y) > f(x_0) - \epsilon = \frac{f(x_0) + a_0}{2} > a_0$ for $y \in B(x_0, \delta)$. This contradicts that x_0 is a limit point of $\{x : f(x) \leq a\}$.

(\Leftarrow) Fix x_0 . Let's show that f is l.s.c. at x_0 . For any $\epsilon > 0$, let $a = f(x_0) - \epsilon$. Then, $x_0 \in \{x : f(x) > a\}$. Since $\{x : f(x) > a\} = \{x : f(x) \leq a\}^c$ is an open set, there exists $\delta > 0$ such that $B(x_0, \delta) \subset \{x : f(x) > a\}$. That is, $f(y) > f(x_0) - \epsilon$ for $y \in B(x_0, \delta)$. Thus, f is l.s.c. at x_0 . Since x_0 is arbitrary, f is l.s.c.

For measurability, by Exercise 1.3.1, it suffices to show that $f^{-1}((a, \infty))$ is in $\mathcal{B}(\mathbb{R})$. If f is l.s.c, then $f^{-1}((-\infty, a]) = \{x : f(x) \leq a\}$ is closed. Thus, $f \in \mathcal{B}(\mathbb{R})$.

Exercise 1.3.6. Let's show that $f^\delta = \sup\{f(y) : |y - x| < \delta\}$ is l.s.c. By Exercise 1.3.5, it suffices to show that $A_a \equiv \{x : f^\delta(x) \leq a\}$ is closed for any $a \in \mathbb{R}$. Let x_0 be a limit point of A_a . For any $y \in B(x_0, \delta)$, there exists $z_y \in A_a$ different from x_0 such that $|x_0 - z_y| < \delta - |x_0 - y|$. Since $|z_y - y| \leq |z_y - x_0| + |x_0 - y| < \delta$ and $z_y \in A_a$, $f(y) \leq a$. Since this holds for all $y \in B(x_0, \delta)$, $\sup\{f(y) : |y - x_0| < \delta\} \leq a$. That is, $x_0 \in A_a$, which implies that A_a is closed. Similar argument gives us that f_δ is u.s.c.

Since semicontinuous functions are measurable (Exercise 1.3.5), f^0 and f_0 , limits of semicontinuous functions, are also measurable. Thus,

$$\{f^0 \neq f_0\} = \bigcup_{n=0}^{\infty} \left\{x : f^0(x) - f_0(x) > \frac{1}{n}\right\}$$

is measurable. (Clearly, $f^0 \geq f_0$)

Exercise 1.3.7. (1) Obviously, simple functions are \mathcal{F} -measurable. And, for any $f_n \in \mathcal{F}$, $\limsup f_n$ is also \mathcal{F} -measurable.

(2) Any \mathcal{F} -measurable function is a pointwise limit of simple functions.

By (1) and (2), class of \mathcal{F} -measurable functions is the smallest class containing the simple functions and closed under pointwise limit.

Exercise 1.3.8.

$$(\Omega, \sigma(X)) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R})) \xrightarrow{f} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

(\Leftarrow) Since X is $\sigma(X)$ -measurable, $Y = f(X)$ is $\sigma(X)$ -measurable. (Composition of measurable functions is measurable.)

(\Rightarrow) If one can show that $\mathcal{T} = \{Y = f(X) : f \in \mathcal{B}(\mathbb{R})\}$ contains simple functions with respect to $(\Omega, \sigma(X))$ and is closed under pointwise limit, then by previous Exercise 1.3.7, $\{Y : Y \in \sigma(X)\} \subset \mathcal{T}$.

(a) Let φ be a simple function with respect to $(\Omega, \sigma(X))$. That is,

$$\varphi(\omega) = \sum_{m=1}^n c_m I_{A_m}(\omega) \quad \text{where disjoint } A_m \in \sigma(X).$$

Since $A_m \in \sigma(X)$, there exist disjoint $B_m \in \mathcal{B}(\mathbb{R})_X$ such that $A_m = X^{-1}(B_m)$. Thus,

$$\begin{aligned} \varphi(\omega) &= \sum_{m=1}^n c_m I_{A_m}(\omega) \\ &= \sum_{m=1}^n c_m I_{X^{-1}(B_m)}(\omega) \\ &= \sum_{m=1}^n c_m I_{B_m}(X(\omega)) \end{aligned}$$

Therefore, \mathcal{T} contains simple functions with respect to $(\Omega, \sigma(X))$.

(b) Now, let Y be a pointwise limit of $Y_n = f_n(X)$ where $f \in \mathcal{B}(\mathbb{R})$. That is, $Y(\omega) = \lim_n f_n(X(\omega))$ for all $\omega \in \Omega$. Define $f(x) \stackrel{(\star)}{=} \limsup_n f_n(x)$. Since $f_n \in \mathcal{B}(\mathbb{R})$, $f \in \mathcal{B}(\mathbb{R})$. Also, $Y(\omega) = f(X(\omega))$ for all $\omega \in \Omega$. Thus, \mathcal{T} is closed under pointwise limit.

(\star) : From the definition of f_n and f , we can ensure that f_n converges to f only on $x \in X(\Omega)$. Since we don't know whether convergence also holds on $\mathbb{R} \setminus X(\Omega)$ or not, we define $f(x)$ by limsup of $f_n(x)$.

Exercise 1.3.9. Omit.

1.4. Integration.

Exercise 1.4.1. Suppose not. Then, there exists n such that $\mu(A_n) > 0$, where $A_n = \{f(x) > \frac{1}{n}\}$. Since $f \geq 0$,

$$\int f d\mu \geq \int_{A_n} f d\mu \geq \frac{1}{n} \mu(A_n) > 0,$$

which is a contradiction.

Exercise 1.4.2. Define $f_n(x) = \sum_{m=1}^{\infty} \frac{m}{2^n} I_{E_{n,m}}(x)$. By MCT, $\int f_n(x) = \sum_{m=1}^{\infty} \frac{m}{2^n} \mu(E_{n,m})$. Note that if $x \in E_{n,m}$, then $x \in E_{n+1,2m} \cup E_{n+1,2m+1}$. Thus, $f_n(x) \leq f_{n+1}(x)$. And, since $f(x) - f_n(x) < \frac{1}{2^n}$, $f_n(x) \rightarrow f(x)$ for all x . Thus, by MCT, $\int f_n d\mu = \int f d\mu$. Combining two results, we get the desired result.

Exercise 1.4.3. (i) Let $g \geq 0$. Since $\int g dm = \sup\{\int s dm : 0 \leq s \leq f, s \text{ simple}\}$, there exists simple $0 \leq s \leq g$ such that $\int g dm - \epsilon < \int s dm$. Thus, $\int |g - s| dm = \int (g - s) dm < \epsilon$.

For integrable $g = g^+ - g^-$, find simple s^+ and s^- such that

$$\int |g^+ - s^+| dm < \frac{\epsilon}{2} \quad \int |g^- - s^-| dm < \frac{\epsilon}{2}.$$

Then,

$$\begin{aligned} \int |g - (s^+ - s^-)| dm &\leq \int |g - s^+| dm + \int |g - s^-| dm \\ &< \epsilon. \end{aligned}$$

$s^+ - s^-$ is the desired simple function.

(ii) Firstly, let $\varphi = I_A$, where A is measurable. By outer regularity of Lebesgue measure, there exists open $G \supset A$ such that $m(G - A) < \frac{\epsilon}{2}$. Since $G = \biguplus_{i=1}^{\infty} (a_i, b_i)$, by continuity of measure, there exists n such that $m(G \setminus \biguplus_{i=1}^n (a_i, b_i)) < \frac{\epsilon}{2}$. By relabeling (a_i, b_i) for $i = 1, \dots, n$, we can construct a step function q such that

$$\begin{aligned} \int |\varphi - q| dm &\leq \int |I_A - I_G| dm + \int |I_G - q| dm \\ &= m(G - A) + m(G \setminus \biguplus_{i=1}^n (a_i, b_i)) \\ &< \epsilon. \end{aligned}$$

Now, let $\varphi = \sum_{k=1}^n b_k I_{A_k}$. For each k , there exists step function q_k such that

$$\int |I_{A_k} - q_k| dm < \frac{\epsilon}{\sum_{k=1}^n |b_k|}.$$

Then,

$$\begin{aligned} \int |\varphi - \sum_{k=1}^n b_k q_k| dm &= \int \sum_{k=1}^n |b_k| |I_{A_k} - q_k| dm \\ &\leq \sum_{k=1}^n |b_k| \int |I_{A_k} - q_k| dm \end{aligned}$$

$$< \epsilon.$$

And we can construct a step function from $\sum_k b_k q_k$.

(iii) For given ϵ , pick sufficiently small positive $\delta < \frac{\epsilon}{\sum_{j=1}^k |c_j|}$. Define a continuous function r as follows.

On $x \in [a_{j-1} + \delta, a_j - \delta]$, $r(x) = c_j$.

On $x \in [a_j - \delta, a_j]$, connect $(a_j - \delta, c_j)$ and $(a_j, 0)$ continuously.

On $x \in [a_j, a_j + \delta]$, connect $(a_j, 0)$ and $(a_j + \delta, c_{j+1})$ continuously.

Then,

$$\int |q - r| d\mu < \delta \sum_{j=1}^n |c_j| < \epsilon.$$

Exercise 1.4.4. Suppose g is a step function. Then,

$$\begin{aligned} \int g(x) \cos nx dx &= \sum_{j=1}^k c_j \int_{a_{j-1}}^{a_j} \cos nx dx \\ &= \sum_{j=1}^k c_j \frac{1}{n} (\sin(na_j) - \sin(na_{j-1})) \\ &\leq \sum_{j=1}^k \frac{2|c_j|}{n} \rightarrow 0. \end{aligned}$$

For integrable g , previous exercise says there exists a step function q such that $\int |g - q| dx < \epsilon$, for given $\epsilon > 0$. Then,

$$\begin{aligned} \int |g(x) \cos nx| dx &= \int |g(x) \cos nx - q(x) \cos nx + q(x) \cos nx| dx \\ &\leq \int |g - q| |\cos nx| dx + \int |q(x) \cos nx| dx \\ &\leq \int |g - q| dx + \int |q(x) \cos nx| dx \\ &< \epsilon + \int |q(x) \cos nx| dx \rightarrow \epsilon. \end{aligned}$$

Since q is a step function, it goes to 0 as n increases. Letting $\epsilon \rightarrow 0$, we get the desired result.

1.5. Properties of Integral.

Exercise 1.5.1. Our claim is that if $\|g\|_\infty = M$, then $|g| \leq M$ a.e. Suppose not. Then,

$$\mu(\{x : |g(x)| > M\}) = \mu(\cup_{n=1}^\infty \{x : |g(x)| > M + \frac{1}{n}\}) > 0.$$

Thus, there exists $N \in \mathbb{N}$ such that $\mu(\{x : |g(x)| > M + \frac{1}{N}\}) > 0$, which contradicts on definition of $\|g\|_\infty$. Therefore, $|fg| \leq |f| \|g\|_\infty$ a.e. By integrating, we get the desired result.

Exercise 1.5.2. As shown in the previous exercise, we know that $|f| \leq \|f\|_\infty$ a.e. Then, $\int |f|^p d\mu \leq \|f\|_\infty^p$, and thus $\limsup_p \|f\|_p \leq \|f\|_\infty$. For the converse, let $B_\epsilon = \{x : |f(x)| >$

$\|f\|_\infty - \epsilon\}$. Then, for sufficiently small $\epsilon > 0$, $0 < \mu(B_\epsilon) \leq 1$. Note that

$$\begin{aligned} \int |f(x)|^p d\mu &\geq \int_{B_\epsilon} |f(x)|^p d\mu \\ &> \int_{B_\epsilon} (\|f\|_\infty - \epsilon)^p d\mu \\ &= \mu(B_\epsilon) (\|f\|_\infty - \epsilon)^p \end{aligned}$$

Letting $p \rightarrow \infty$ and letting $\epsilon \rightarrow 0^+$ give us $\liminf_p \|f\|_p \geq \|f\|_\infty$

Exercise 1.5.3. (i)

$$\|f + g\|_p^p = \int |f + g|^p d\mu \leq \int 2^p (|f|^p + |g|^p) d\mu = 2^p \|f\|_p^p + 2^p \|g\|_p^p < \infty.$$

Thus, $\|f + g\|_p < \infty$. If $\|f + g\|_p = 0$, then clearly $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. Suppose not. Applying Holder Inequality to $|f| |f + g|^{p-1}$ and $|g| |f + g|^{p-1}$ with $p, q = \frac{p}{p-1}$,

$$\begin{aligned} \int |f| |f + g|^{p-1} d\mu &\leq \|f\|_p \left(\int (|f + g|^{p-1})^{\frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} \\ &= \|f\|_p \|f + g\|_p^{p-1} \\ \int |g| |f + g|^{p-1} d\mu &\leq \|g\|_p \left(\int (|f + g|^{p-1})^{\frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} \\ &= \|g\|_p \|f + g\|_p^{p-1} \end{aligned}$$

Then,

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p d\mu \\ &\leq \int (|f| + |g|) |f + g|^{p-1} d\mu \\ &\leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}. \end{aligned}$$

Since $\|f + g\|_p < \infty$, we get $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

(ii) ($p = 1$) $|f + g| \leq |f| + |g|$

($p = \infty$) Let $\|f\|_\infty = M_1$ and $\|g\|_\infty = M_2$. Since

$$\begin{aligned} \{x : |f(x)| + |g(x)| > M_1 + M_2\} &\subset \{x : |f(x)| > M_1 + M_2\} \\ &\subset \{x : |f(x)| > M_1\} \cup \{x : |g(x)| > M_2\}, \end{aligned}$$

$\mu(\{x : |f(x)| + |g(x)| > M_1 + M_2\}) \leq 0$. Thus, $\|f + g\|_\infty \leq M_1 + M_2$.

Exercise 1.5.4. Let $f_n := \sum_{m=0}^n f \mathbf{1}_{E_m} = f \mathbf{1}_{\cup_{m=0}^n E_m}$. Then, $f_n \rightarrow f \mathbf{1}_E$ and $|f_n| \leq f$: integrable. By DCT, we get the desired result.

Exercise 1.5.5. Let $f_n := g_n + g_1^-$. Since $g_n \uparrow g$, $-g_n^- \geq -g_1^-$. Thus, $f_n = g_n^+ - g_n^- + g_1^- \geq 0$. By MCT,

$$\int f_n d\mu = \int g_n d\mu + \int g_1^- d\mu \rightarrow \int g d\mu + \int g_1^- d\mu.$$

Since $\int g_1^- d\mu < \infty$, by subtracting, we get the desired result.

Exercise 1.5.6. MCT

Exercise 1.5.7. (i) MCT

(ii) Step 1 $g \geq 0$.

By (i), for given ϵ , there exists N such that $\int g d\mu < \int (g \wedge N) d\mu + \frac{\epsilon}{2}$. Pick $\delta = \frac{\epsilon}{2N}$. Then, for A with $\mu(A) < \delta$,

$$\int_A g d\mu < \int_A (g \wedge N) d\mu + \frac{\epsilon}{2} \leq N\mu(A) + \frac{\epsilon}{2} < \epsilon.$$

Step 2 g : integrable.

$|g| = g^+ + g^-$. For given ϵ , there exists N such that

$$\begin{aligned} \int g^+ d\mu &< \int (g^+ \wedge N) d\mu + \frac{\epsilon}{4} \\ \int g^- d\mu &< \int (g^- \wedge N) d\mu + \frac{\epsilon}{4}. \end{aligned}$$

Pick $\delta = \frac{\epsilon}{4N}$. Then, for A with $\mu(A) < \delta$,

$$\begin{aligned} \int_A |g| d\mu &= \int_A g^+ d\mu + \int_A g^- d\mu \\ &< \int_A (g^+ \wedge N) d\mu + \int_A (g^- \wedge N) d\mu + \frac{\epsilon}{2} \\ &\leq 2N\mu(A) + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Exercise 1.5.8. Let $c \in (a, b)$. Since f is integrable, by previous exercise(1.5.7), for given $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\int_A |f| d\mu < \epsilon$. For $|x - c| < \delta$,

$$|g(x) - g(c)| = \left| \int_c^x f(y) dy \right| \leq \int_c^x |f(y)| dy < \epsilon.$$

Since c is arbitrary in (a, b) , g is continuous on (a, b) .

Exercise 1.5.9. (i) $(1 \leq p < \infty)$

Step 1 $f \geq 0$.

There exists $(\varphi_n)_{n \in \mathbb{N}}$ such that $0 \leq \varphi_n \leq f$ and $\varphi_n \uparrow f$. Let $g_n = |f - \varphi_n|^p$. $g_n \rightarrow 0$ and $|g_n| \leq (|f| + |\varphi_n|)^p \leq 2^p |f|^p$: integrable. By DCT, $\int g_n d\mu = \int |f - \varphi_n|^p \rightarrow 0$. Thus, $\|\varphi_n - f\|_p \rightarrow 0$.

Step 2 Integrable $f = f^+ - f^-$.

There exist $(\varphi_n^+)_{n \in \mathbb{N}}$, $(\varphi_n^-)_{n \in \mathbb{N}}$ such that

$$\begin{aligned} 0 &\leq \varphi_n^+ \leq f^+ \text{ and } \varphi_n^+ \uparrow f^+ \\ 0 &\leq \varphi_n^- \leq f^- \text{ and } \varphi_n^- \uparrow f^-. \end{aligned}$$

For $\varphi_n \equiv \varphi_n^+ - \varphi_n^-$,

$$\|\varphi_n - f\|_p \leq \|\varphi_n^+ - f^+\|_p + \|\varphi_n^- - f^-\|_p \rightarrow 0.$$

Exercise 1.5.10. By MCT, $\int \sum_{n=1}^{\infty} |f_n| d\mu = \sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$. For $g_m \equiv \sum_{n=1}^m f_n$, $g_m \rightarrow g \equiv \sum_{n=1}^{\infty} f_n$ and $|g_m| \leq \sum_{n=1}^m |f_n| \leq \sum_{n=1}^{\infty} |f_n|$: integrable. By DCT,

$$\int g_m d\mu = \int \sum_{n=1}^m f_n d\mu = \sum_{n=1}^m \int f_n d\mu \rightarrow \int g d\mu = \int \sum_{n=1}^{\infty} f_n d\mu.$$

1.6. Expected Value.

Exercise 1.6.1. For a support line $l(x) = a(x - x_0) + \varphi(x_0)$ at x_0 of $\varphi(x)$ ($l(x) \leq \varphi(x)$), strict convexity implies that (*) $l(x) < \varphi(x)$ if $x \neq x_0$. For $x_0 = EX$, $E\varphi(X) - El(X) = E\varphi(X) - l(EX) = E\varphi(X) - \varphi(EX) = 0$ implies $\varphi(X) = l(X)$ a.s. since $\varphi(X) \geq l(X)$. Thus, $X = EX$ a.s.

(*) : Suppose $l(y) = \varphi(y)$ for $y \neq x_0$. Then, with $\lambda = \frac{1}{2}$,

$$\begin{aligned} \varphi\left(\frac{x_0}{2} + \frac{y}{2}\right) &> \frac{1}{2}\varphi(x_0) + \frac{1}{2}\varphi(y) \\ &= \frac{1}{2}l(x_0) + \frac{1}{2}l(y) \\ &= l\left(\frac{x_0}{2} + \frac{y}{2}\right) \end{aligned}$$

which contradicts that $l(x)$ is a support line of $\varphi(x)$. The last equality comes from the linearity of $l(x)$.

Exercise 1.6.2. Let $l(x_1, \dots, x_n) = \varphi(EX_1, \dots, EX_n) + g_x^\top(x_1 - EX_1, \dots, x_n - EX_n)$, where $g_x \in \partial f(x)$. Then, $\varphi(x) \geq l(x)$ for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Taking expectation gives us the desired result.

$$E\varphi(X) \geq El(X) = \varphi(EX).$$

Exercise 1.6.3. (i) Let $\varphi(x) = x^2$ and $A = (-\infty, -a] \cup [a, \infty)$. Then, it is clear that $i_A = \inf\{\varphi(y) : y \in A\} = a^2$. By Chebyshev's inequality, we have

$$a^2 P(|X| \geq a) \leq EX^2 = b^2 \Rightarrow P(|X| \geq a) \leq \frac{b^2}{a^2}.$$

If some random variable X attains equality above, then X has to satisfy following identities.

$$\int_A dP = \frac{b^2}{a^2}, \quad \int_{A^c} dP = 1 - \frac{b^2}{a^2}, \quad \int x^2 dP = b^2.$$

It is clear that X with $P(X = a) = \frac{b^2}{a^2}$, $P(X = 0) = 1 - \frac{b^2}{a^2}$ satisfies identities. Therefore, inequality is sharp.

(ii) For $a > 0$, it's clear that if $x \in \{x : |x| \geq a\}$, then $x^2 \geq a^2$. Thus,

$$a^2 \mathbf{1}_{(|X| \geq a)} \leq X^2 \mathbf{1}_{(|X| \geq a)} = X^2 - X^2 \mathbf{1}_{(|X| < a)}.$$

Since EX^2 is finite, both functions on each sides are integrable. Thus, linearity of integration and order-preserving property of integration give us

$$\int a^2 \mathbf{1}_{(|X| \geq a)} dP \leq \int X^2 dP - \int X^2 \mathbf{1}_{(|X| < a)} dP.$$

We know that $\text{LHS} = a^2 P(|X| \geq a)$. Let's focus on RHS. The first term in RHS is clearly EX^2 . For the second term, we can apply MCT. (It is clear that conditions of MCT hold.) By MCT,

$$\begin{aligned} \lim_{a \rightarrow \infty} a^2 P(|X| \geq a) &\leq EX^2 - \lim_{a \rightarrow \infty} \int X^2 \mathbf{1}_{(|X| < a)} dP \\ &= EX^2 - \int \lim_{a \rightarrow \infty} X^2 \mathbf{1}_{(|X| < a)} dP \\ &= EX^2 - EX^2 = 0 \end{aligned}$$

The last equality holds since $EX^2 < \infty$. Therefore, we have

$$\lim_{a \rightarrow \infty} a^2 P(|X| \geq a) / EX^2 = 0.$$

However, Chebyshev's inequality with $\varphi(x) = x^2$ and $A = (-\infty, -a] \cup [a, \infty)$, so that $i_A = a^2$, gives us following inequality.

$$a^2 P(|X| \geq a) / EX^2 \leq 1,$$

which is not sharp.

Exercise 1.6.4. (i) Let $A = [a, \infty)$, $\varphi(y) = (y + b)^2$ where $0 < b < a$. It is clear that $i_A = \inf\{\varphi(y) : y \in A\} = (a + b)^2$. Then by Chebyshev's inequality,

$$(a + b)^2 P(Y \geq a) \leq E[\varphi(Y)] = E[Y^2 + 2bY + b^2] = E[X^2 + 2bX + b^2] = E[\varphi(X)].$$

The second equality holds since $EY = EX$, $\text{Var}(Y) = \text{Var}(X)$. Note that

$$E[(X + b)^2] = p(a + b)^2.$$

Thus,

$$P(Y \geq a) \leq p.$$

If $Y = X$, then equality holds.

(ii) Let $p = \frac{\sigma^2}{(a^2 + \sigma^2)}$, $b = \frac{p}{1-p}a$, $A = [a, \infty)$, and $\varphi(y) = (y + b)^2$. Note that $0 < p < 1$ and $b > 0$. Then, it is clear that $i_A = (a + b)^2$. By Chebyshev's inequality,

$$(a + b)^2 P(Y \geq a) \leq E[\varphi(Y)] = E[Y^2 + 2bY + b^2] = \sigma^2 + b^2,$$

since $E(Y) = 0$, $\text{Var}(Y) = E(Y^2) - E(Y)^2 = E(Y^2)\sigma^2$. Thus, we have

$$P(Y \geq a) \leq \frac{\sigma^2 + b^2}{(a + b)^2}.$$

Since $b = \frac{p}{1-p}a = \frac{\sigma^2}{a}$, we have

$$P(Y \geq a) \leq \frac{\sigma^2 + b^2}{(a + b)^2} = \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{(a + \frac{\sigma^2}{a})^2} = \frac{\frac{\sigma^2}{a^2}(a^2 + \sigma^2)}{\frac{1}{a^2}(a^2 + \sigma^2)^2} = \frac{\sigma^2}{(a^2 + \sigma^2)}.$$

And equality holds when Y has $P(Y = a) = p$ and $P(Y = -b) = 1 - p$, which satisfies

$$E(Y) = ap - b(1 - p) = 0$$

$$\text{Var}(Y) = E[(Y - 0)^2] = a^2p + b^2(1 - p) = \frac{a^2\sigma^2}{a^2 + \sigma^2} + \frac{\sigma^4}{a^2 + \sigma^2} = \sigma^2.$$

Exercise 1.6.5. (i) It suffices to show that $\forall n, \exists X$ such that $EX = 0$, $\text{Var}(X) = 1$ and $P(|X| > \epsilon) \leq \frac{1}{n}$. Construct X as follows.

$$P(X = x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x = 0 \\ \frac{1}{2n}, & \text{if } x = \pm\sqrt{n} \end{cases}$$

Then, $EX = 0$, $\text{Var}(X) = 1$ and $P(|X| > \epsilon) \leq P(X \neq 0) = \frac{1}{n}$.

(ii) Similarly,

$$P(X = x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x = 1 \\ \frac{1}{2n}, & \text{if } x = 1 \pm \sqrt{n}\sigma \end{cases}$$

Then, $EX = 1$, $\text{Var}(X) = \sigma^2$ and $P(|X| > y) \leq P(X \neq 1) = \frac{1}{n}$.

Exercise 1.6.6. Let $f = Y$ and $g = I_{(Y>0)}$. Then,

$$E|fg| = E|YI_{(Y>0)}| = EYI_{(Y>0)} = EY$$

$$E|f|^2 = EY^2$$

$$E|g|^2 = EI_{(Y>0)} = P(Y > 0).$$

By Cauchy-Schwarz Ineq., $(E|fg|)^2 \leq E|f|^2 E|g|^2$. Thus, $(EY)^2 \leq EY^2 P(Y > 0)$.

Exercise 1.6.7. Checking (i), (ii) in Theorem 1.6.8 is obvious. For (iii),

$$\begin{aligned} Eg(X_n) &= \int |X_n|^{\frac{2}{\alpha}} dm \\ &= \int n^2 I_{(\frac{1}{n+1}, \frac{1}{n})} dm \\ &= \frac{n}{n+1} < 1 \text{ for all } n. \end{aligned}$$

Thus, Theorem 1.6.8 can be applied.

Suppose that $Y \geq |X_n| = X_n$. Then,

$$\begin{aligned} \int Y dm &\geq \sum_{n=1}^{\infty} n^{\alpha} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{\infty} \frac{n^{\alpha-1}}{n+1} = \infty \end{aligned}$$

The last equality holds since $0 < \alpha - 1 < 1$. Therefore, X_n are not dominated by an integrable function.

Exercise 1.6.8. Step 1 g : Indicator function.

Let $g(x) = I_A(x)$ where $A \in \mathcal{B}(\mathbb{R})$. Then,

$$\int g(x)\mu(dx) = \int I_A(x)\mu(dx)$$

$$\begin{aligned}
&= \mu(A) \\
\int g(x)f(x)dx &= \int I_A(x)f(x)d(x) \\
&= \int_A f(x)dx \\
&= \mu(A).
\end{aligned}$$

Step 2 g : simple function.
Linearity of Integral.

Step 3 $g \geq 0$.

There exists non-negative simple function $s_n \uparrow g$. Then,

$$\begin{aligned}
\int g(x)\mu(dx) &= \int \lim_{n \rightarrow \infty} s_n(x)\mu(dx) \\
&\stackrel{(MCT)}{=} \lim_{n \rightarrow \infty} \int s_n(x)\mu(dx) \\
&\stackrel{Step 2}{=} \lim_{n \rightarrow \infty} \int s_n(x)f(x)dx \\
&\stackrel{(MCT)}{=} \int g(x)f(x)dx.
\end{aligned}$$

Step 4 g : Integrable.

$g = g^+ - g^-$. By Step 3, $\int g(x)\mu(dx) = \int g(x)f(x)dx$.

Exercise 1.6.9. Induction.

Exercise 1.6.10. Induction.

Exercise 1.6.11. (i) Let $A = (|X| > 1)$. Then, $|X|^j < |X|^k$ for $0 < j < k$ on A .

$$\begin{aligned}
\int |X|^j dP &= \int_A |X|^j dP + \int_{A^c} |X|^j dP \\
&\leq \int_A |X|^k dP + 1 \\
&\leq \int |X|^k dP + 1 < \infty.
\end{aligned}$$

(ii) Let $\varphi(x) = x^{\frac{k}{j}} I_{(x \geq 0)}$. It is easy to check that φ is convex. Let $Y = |X|^j$. Then, $|EY| = E|X|^j < \infty$ (by (i)) and $|E\varphi(Y)| = E|X|^k < \infty$. By Jensen's Ineq.,

$$(E|X|^j)^{\frac{k}{j}} = \varphi(EY) \leq E\varphi(Y) = E|X|^k.$$

Thus, we get $E|X|^j \leq (E|X|^k)^{\frac{j}{k}}$.

Exercise 1.6.12. $P(e^X = y_m) = P(X = \log y_m) = p(m)$.

$$Ee^X = \sum_{m=1}^n p(m)y_m$$

$$\begin{aligned}
e^{EX} &= \exp \left(\sum_{m=1}^n p(m) \log y_m \right) \\
&= \exp \left(\log \prod_{m=1}^n y_m^{p(m)} \right) \\
&= \prod_{m=1}^n y_m^{p(m)}.
\end{aligned}$$

By Jensen's Ineq., $Ee^X \geq \exp(EX)$. And, we get the desired result.

Exercise 1.6.13. Exercise 1.5.5

Exercise 1.6.14. (i)

$$E(1/X; X > y) = E(1/X; 1/X < 1/y) \leq \frac{1}{y} P(1/X < 1/y) = \frac{1}{y} P(X > y).$$

Thus, $\lim_{y \rightarrow \infty} yE(1/X; X > y) \leq \lim_{y \rightarrow \infty} P(X > y) = 0$.

(ii) Let $0 < y < \epsilon$.

$$\begin{aligned}
yE(1/X; X > y) &= E(y/X; X > y \text{ and } 0 \leq X < \epsilon) + E(y/X; X > y \text{ and } \epsilon \leq X). \\
&\leq P(X > y, 0 \leq X < \epsilon) + E(y/X; \epsilon \leq X). \\
&\leq P(0 < X < \epsilon) + E(y/X; \epsilon \leq X).
\end{aligned}$$

For the second term on right hand side, $\frac{y}{X} I_{\epsilon \leq X} \leq \frac{y}{\epsilon} I_{\epsilon \leq X} \leq 1$ and $\frac{y}{X} I_{\epsilon \leq X} \rightarrow 0$ as $y \rightarrow 0$.

Thus, by DCT,

$$\limsup_{y \rightarrow 0} yE(1/X; X > y) \leq P(0 < X < \epsilon).$$

Letting $\epsilon \rightarrow 0$, we get the desired result.

Exercise 1.6.15. Exercise 1.5.6

Exercise 1.6.16. Let $X_k = XI_{\biguplus_{n=0}^k A_n}$. Then, $X_k \rightarrow XI_A$ and $|X_k| \leq |X|$: integrable. By DCT, $EX_k \rightarrow E(X; A)$. Note that

$$EX_k = \int_{\biguplus_{n=0}^k A_n} X dP = \sum_{n=0}^k \int_{A_n} X dP = \sum_{n=0}^k E(X; A_n).$$

Thus, $E(X; A) = \lim_{k \rightarrow \infty} EX_k = \sum_{n=0}^{\infty} EX_k$.

1.7. Product Measures, Fubini's Theorem.

Exercise 1.7.1. Since $|f(x, y)| \geq 0$, by Fubini's Theorem, $\int_{X \times Y} |f| d\mu = \int_X \int_Y |f(x, y)| \mu_2(dy) \mu_1(dx) < \infty$. And by Fubini's Theorem again, we get the desired result.

Exercise 1.7.2. Construct a probability space $(Y, \mathcal{B}, \nu) = ([0, \infty), \mathcal{B}([0, \infty)), \lambda)$, where λ denotes a Lebesgue measure. Let $f(x, y) = I_{\{(x, y) : 0 \leq y < g(x)\}}$. Then,

$$\begin{aligned}
\int_X \int_Y f d\lambda d\mu &= \int_X g d\mu \\
\int_{X \times Y} f d(\mu \times \lambda) &= (\mu \times \lambda)(\{(x, y) : 0 \leq y < g(x)\})
\end{aligned}$$

$$\int_Y \int_X f d\mu d\lambda = \int_0^\infty \mu(\{x : g(x) > y\}) dy.$$

Since $f \geq 0$, Fubini's Theorem implies that

$$\int_X \int_Y f d\lambda d\mu = \int_{X \times Y} f d(\mu \times \lambda) = \int_Y \int_X f d\mu d\lambda.$$

And we get the desired result.

Exercise 1.7.3. (i) Fubini's Theorem with $f(x, y) = I_{\{(x, y) : a < x \leq y \leq b\}} \geq 0$.

$$\begin{aligned} \text{(ii)} \quad & \int_{(a, b]} F(y) dG(y) + \int_{(a, b]} G(x) dF(x) \\ &= \int_{(a, b]} \{F(y) - F(a)\} dG(y) + \int_{(a, b]} \{G(x) - G(a)\} dF(x) + \int_{(a, b]} F(a) dG(y) + \int_{(a, b]} G(a) dF(x) \\ &\stackrel{\text{(i)}}{=} \int I_{\{(x, y) : a < x \leq y \leq b\}} d(\mu \times \nu) + \int I_{\{(x, y) : a < y \leq x \leq b\}} d(\mu \times \nu) + F(a)(G(b) - G(a)) + G(a)(F(b) - F(a)) \\ &= (\mu \times \nu)((a, b] \times (a, b]) + (\mu \times \nu)(\{(x, y) : a < x = y \leq b\}) + F(a)G(b) + F(b)G(a) - 2F(a)G(a) \\ &= (F(b) - F(a))(G(b) - G(a)) + \sum_{x \in (a, b]} \mu(\{x\})\nu(\{x\}) + F(a)G(b) + F(b)G(a) - 2F(a)G(a) \\ &= F(b)(G(b) - F(a)G(a)) + \sum_{x \in (a, b]} \mu(\{x\})\nu(\{x\}). \end{aligned}$$

(iii) If $F=G$, then the third term in right hand side of (ii) equals zero. Thus,

$$\int_{(a, b]} 2F(y) dF(y) = F^2(b) - F^2(a).$$

Exercise 1.7.4. Let $f = I_{\{(x, y) : x < y \leq x+c\}}$. Then,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} f d\mu(y) dx &= \int_{\mathbb{R}} (F(x+c) - F(x)) dx \\ \int_{\mathbb{R}} \int_{\mathbb{R}} f dx d\mu(y) &= \int_{\mathbb{R}} c d\mu(y) \\ &= c\mu(\mathbb{R}). \end{aligned}$$

Since $f \geq 0$, Fubini's Theorem gives us that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f d\mu(y) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f dx d\mu$$

and we get the desired result.

Exercise 1.7.5.

$$\begin{aligned} \int_0^a \int_0^\infty |e^{-xy} \sin x| dy dx &= \int_0^a \int_0^\infty e^{-xy} |\sin x| dy dx \\ &= \int_0^a \frac{|\sin x|}{x} dx \\ &\leq \int_0^a 1 dx < \infty. \end{aligned}$$

By Exercise 1.7.1, $e^{-xy} \sin x I_{\{(x,y): 0 < x < a, 0 < y\}}$ is integrable and we can apply Fubini's Theorem.

$$\begin{aligned}
\int_0^a \frac{\sin x}{x} dx &= \int_0^a \int_0^\infty e^{-xy} \sin x dy dx = \int_0^\infty \int_0^a e^{-xy} \sin x dx dy \\
&= \int_0^\infty \left[\frac{e^{-xy}(-y \sin x - \cos x)}{1+y^2} \right]_0^a dy \\
&= \int_0^\infty \frac{1 - e^{-ay}(y \sin a + \cos a)}{1+y^2} dy \\
&= \frac{\pi}{2} - \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin a \int_0^\infty \frac{ye^{-ay}}{1+y^2} dy.
\end{aligned}$$

The last equality comes from the fact that $\int \frac{1}{1+y^2} dy = \tan^{-1}(y)$. Moreover,

$$\begin{aligned}
\left| \int_0^a \frac{\sin x}{x} dx - \frac{\pi}{2} \right| &= \left| \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2} dy + \sin a \int_0^\infty \frac{ye^{-ay}}{1+y^2} dy \right| \\
&\leq \int_0^\infty e^{-ay} \frac{|\cos a| + y|\sin a|}{1+y^2} dy \\
&\leq \int_0^\infty e^{-ay} \frac{1+y}{1+y^2} dy \\
&\leq \int_0^\infty e^{-ay} \cdot 2 dy = \frac{2}{a}.
\end{aligned}$$

Letting $a \rightarrow \infty$, we get

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Note that since $\frac{\sin x}{x}$ is not integrable on $\mathbb{R}_{\geq 0}$, we can't calculate its integration directly.

2. LAWS OF LARGE NUMBERS

2.1. Independence.

Exercise 2.1.1. By Thm 2.1.8, it suffices to show that $P(X_1 \leq z_1, \dots, X_n \leq z_n) = \prod_{i=1}^n P(X_i \leq z_i)$. Let $c_i = \int_{-\infty}^\infty g_i(x_i) dx_i$, for $i = 1, \dots, n$. Then, we have $c_1 \cdots c_n = 1$. Our claim is that

$$P(X_i \leq z_i) = \frac{1}{c_i} \int_{-\infty}^{z_i} g_i(x_i) dx_i.$$

WLOG, let $i = 1$.

$$\begin{aligned}
P(X_1 \leq z_1) &= P(-\infty < X_1 \leq z_1, -\infty < X_2 < \infty, \dots, -\infty < X_n < \infty) \\
&= (c_2 \cdots c_n) \int_{-\infty}^{z_1} g_1(x_1) dx_1 \\
&= \frac{1}{c_1} \int_{-\infty}^{z_1} g_1(x_1) dx_1.
\end{aligned}$$

Now,

$$P(X_1 \leq z_1, \dots, X_n \leq z_n) = \left(\int_{-\infty}^{z_1} g_1(x_1) dx_1 \right) \cdots \left(\int_{-\infty}^{z_n} g_n(x_n) dx_n \right)$$

$$\begin{aligned}
&= \left(\frac{1}{c_1} \int_{-\infty}^{z_1} g_1(x_1) dx_1 \right) \cdots \left(\frac{1}{c_n} \int_{-\infty}^{z_n} g_n(x_n) dx_n \right) \\
&= \prod_{i=1}^n P(X_i \leq z_i)
\end{aligned}$$

Exercise 2.1.2. Let $\mathcal{A}_i = \{X_i^{-1}(x_i) : x_i \in S_i\} \cup \{\emptyset, \Omega\}$. Let $I = \{n_1, \dots, n_k\} \subset \{1, \dots, n\}$ and $J = \{1, \dots, n\} \setminus I = \{m_1, \dots, m_{n-k}\}$. Then,

$$\begin{aligned}
P(X_{n_1} = x_{n_1}, \dots, X_{n_k} = x_{n_k}) &= \sum_{x_{m_1} \in S_{m_1}} \cdots \sum_{x_{m_{n-k}} \in S_{m_{n-k}}} P(X_1 = x_1, \dots, X_n = x_n) \\
&= \prod_{i \in I} P(X_i = x_i).
\end{aligned}$$

By Lemma 2.1.5, \mathcal{A}_i 's are independent. By Theorem 2.1.7, $\sigma(\mathcal{A}_i)$'s are independent. Since $X_i^{-1}(A) \in \sigma(\mathcal{A}_i)$ for all $A \in 2^{S_i}$, $\sigma(X_i) \subset \sigma(\mathcal{A}_i)$. Therefore, X_1, \dots, X_n are independent.

Exercise 2.1.3. (1) $h(\rho(x, y)) \rightarrow [0, \infty)$

(i) $h(\rho(x, y)) = 0 \iff \rho(x, y) = 0 \iff x = y$.

First \iff holds since $h(0) = 0$ and $h'(x) > 0$ for $x > 0$.

(ii) $h(\rho(x, y)) = h(\rho(y, x))$ by symmetry of metric ρ .

$$\begin{aligned}
\text{(iii) } h(\rho(x, z)) &= \int_0^{\rho(x, z)} h'(u) du \\
&\leq \int_0^{\rho(x, y) + \rho(y, z)} h'(u) du \\
&\leq \int_0^{\rho(x, y)} h'(u) du + \int_0^{\rho(y, z)} h'(u) du \\
&= h(\rho(x, y)) + h(\rho(y, z)).
\end{aligned}$$

The first inequality holds since ρ is metric. The second inequality holds since h' is decreasing.

(2) By simple calculus, one can show that $h(x) = \frac{x}{x+1}$ satisfies (1).

Exercise 2.1.4. It is easy to check the uncorrelatedness. To show that X_n 's are not independent, it suffices to show that there exist A and B such that $P(X_n \in A, X_m \in B) \neq P(X_n \in A)P(X_m \in B)$ for different m and n .

For $0 < \epsilon < 1$, let $\begin{cases} A = X_n^{-1}((0, \epsilon)) \\ B = X_m^{-1}(X_m(A)) \end{cases}$. Then, $A \subset B$. Note that we have $2n$ intervals in

A . Let l_ϵ be the length of one interval. Now, pick ϵ so that $2n l_\epsilon < \frac{1}{m}$. This is possible since $P(A) \searrow 0$ as $\epsilon \searrow 0$. Then, it is clear that $P(B) < 1$. Thus, $P(A \cap B) = P(A) > P(A)P(B)$. Therefore, X_n and X_m are not independent.

Exercise 2.1.5. (i) Let D_μ be a set of discontinuities of μ . By Exercise 1.2.3, D_μ is at most countable.

$$P(X + Y = 0) = \int \int \mathbf{1}_{(x+y=0)} d\mu d\nu$$

$$\begin{aligned}
&= \int \mu(\{-y\})d\nu \\
&= \int_{D_\mu} \mu(\{-y\})d\nu + \int_{D_\mu^c} \mu(\{-y\})d\nu.
\end{aligned}$$

Note that for $-y \in D_\mu^c$, $\mu(\{-y\}) = P(X \leq -y) - P(X < -y) = 0$. Thus,

$$\begin{aligned}
P(X + Y = 0) &= \int_{\mathbb{R}} \mu(\{-y\})\mathbf{1}_{D_\mu}(y)d\nu \\
&= \sum_{y \in D_\mu} \mu(\{-y\})\nu(\{y\}).
\end{aligned}$$

The second equality holds by Exercise 1.5.6.

(ii) With $Y = -Y$ in (i), $P(X = Y) = 0$ since $D_\mu = \emptyset$.

Exercise 2.1.6. Note that

$$\begin{aligned}
\sigma(f(X)) &= \{X^{-1}(f^{-1}(B)) : B \in \mathcal{B}(\mathbb{R})\} \\
\sigma(f(Y)) &= \{Y^{-1}(g^{-1}(B)) : B \in \mathcal{B}(\mathbb{R})\}
\end{aligned}$$

Since f and g are measurable, $f^{-1}(B)$ and $g^{-1}(B)$ are in $\mathcal{B}(\mathbb{R})$ for any $B \in \mathcal{B}(\mathbb{R})$. Thus, $\sigma(f(X)) \subset \sigma(X)$ and $\sigma(g(Y)) \subset \sigma(Y)$. Therefore, independence of X and Y implies the independence of $f(X)$ and $g(Y)$.

Exercise 2.1.7. By Exercise 2.1.2, it suffices to show that $P(Z_i = \alpha, Z_j = \beta) = P(Z_i = \alpha)P(Z_j = \beta)$ for all $\alpha, \beta \in \{0, 1, \dots, K-1\}$.

Firstly, for given $Y = y$, $P(Z_i = \alpha | Y = y) = \frac{1}{K}$ since K is a prime. Thus,

$$\begin{aligned}
P(Z_i = \alpha) &= \sum_{y=0}^{K-1} P(Z_i = \alpha | Y = y)P(Y = y) \\
&= K \cdot \frac{1}{K^2} = \frac{1}{K}.
\end{aligned}$$

Now, for given $Y = y$, $Z_i = \alpha$ and $Z_j = \beta$ means that $X + iy \equiv \alpha$ and $X + jy \equiv \beta$. That is,

$$(i - j)y \equiv \alpha - \beta \pmod{K}.$$

Since $\gcd(i - j, K) = 1$, there exists unique $y_0 \in \{0, 1, \dots, K-1\}$ satisfying $Z_i = \alpha$ and $Z_j = \beta$. And, with $y_0, x_0 \in \{0, 1, \dots, K-1\}$ uniquely determined. Thus,

$$P(Z_i = \alpha, Z_j = \beta) = \frac{1}{K^2}.$$

Therefore,

$$P(Z_i = \alpha, Z_j = \beta) = \frac{1}{K^2} = \frac{1}{K} \cdot \frac{1}{K} = P(Z_i = \alpha)P(Z_j = \beta) \text{ for all } \alpha, \beta \in \{0, 1, \dots, K-1\}.$$

Exercise 2.1.8. $X_1, X_2, X_3 \stackrel{i.i.d}{\sim} \frac{1}{2}\mathbf{1}_{\{1\}} + \frac{1}{2}\mathbf{1}_{\{-1\}}$ and $X_4 = X_1X_2X_3$.

Exercise 2.1.9. $\mathcal{A}_1 = \{\{1, 2\}, \Omega\}$ and $\mathcal{A}_2 = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \Omega\}$

Exercise 2.1.10.

$$P(X + Y = n) = \sum_m P(X = m, Y = n - m) = \sum_m P(X = m)P(Y = n - m)$$

The last equality holds by the independence.

Exercise 2.1.11. Omit.

Exercise 2.1.12. Omit.

Exercise 2.1.13. (a) $F(z) = \frac{1}{9}\mathbf{1}_{[0,1)}(z) + \frac{2}{9}\mathbf{1}_{[1,2)}(z) + \frac{1}{3}\mathbf{1}_{[2,3)}(z) + \frac{2}{9}\mathbf{1}_{[3,4)}(z) + \frac{1}{9}\mathbf{1}_{[4,\infty)}(z)$

(b) For some $c \in [0, \frac{2}{9}]$

	0	1	2
0	$\frac{1}{9}$	c	$\frac{2}{9} - c$
1	$\frac{2}{9} - c$	$\frac{1}{9}$	c
2	c	$\frac{2}{9} - c$	$\frac{1}{9}$

Exercise 2.1.14.

$$\begin{aligned} P(XY \leq z) &= \int \int \mathbf{1}_{(xy \leq z)} dF dG \\ &= \int \int (\mathbf{1}_{(x \leq z/y, y \neq 0)} + \mathbf{1}_{(0 \leq z, y=0)}) dF dG \\ &= \int_{\mathbb{R} \setminus \{0\}} F\left(\frac{z}{y}\right) dG + P(Y = 0)I(z \geq 0). \end{aligned}$$

Exercise 2.1.15. For simplicity, let $\Omega = [0, 1)$. We have

$$Y_n(\omega) = \begin{cases} 0, & \text{if } \omega \in [\frac{2m}{2^n}, \frac{2m+1}{2^n}) : m = 0, 1, \dots, 2^{n-1} - 1 \\ 1, & \text{if } \omega \in [\frac{2m+1}{2^n}, \frac{2m+2}{2^n}) : m = 0, 1, \dots, 2^{n-1} - 1 \end{cases}.$$

By construction, it is obvious that $P(Y_n = 0) = P(Y_n = 1) = \frac{1}{2}$. To show that Y_1, Y_2, \dots are independent, we have to show that Y_1, \dots, Y_n are independent for all $n \in \mathbb{N}$. By Exercise 2.1.2, it suffices to show that $P(Y_1 = y_1, \dots, Y_n = y_n) = \prod_{i=1}^n P(Y_i = y_i)$, where $y_i \in \{0, 1\}$. By induction, one can show that $\{\omega : Y_1(\omega) = y_1, \dots, Y_n(\omega) = y_n\} = [\frac{l}{2^n}, \frac{l+1}{2^n})$ for some $l \in \{0, 1, \dots, 2^n - 1\}$ for any y_i 's. Therefore, $P(Y_1 = y_1, \dots, Y_n = y_n) = \frac{1}{2^n} = \prod_{i=1}^n P(Y_i = y_i)$, where $y_i \in \{0, 1\}$.

2.2. Weak Laws of Large Numbers.

Exercise 2.2.1. We assume that $EX_i^2 < \infty$ so that $\text{Var}(X_i) < \infty$ for all i . Since L^2 convergence implies convergence in probability, it suffices to show that

$$E|S_n/n - \nu_n|^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Note that

$$E|S_n/n - \nu_n|^2 = \frac{1}{n^2} E|S_n - ES_n|^2$$

$$\begin{aligned}
&= \frac{1}{n^2} \text{Var}(S_n) \\
&= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \\
&\leq \frac{1}{n} \sum_{i=1}^n \frac{\text{Var}(X_i)}{i}
\end{aligned}$$

The last equality holds since X_i 's are uncorrelated. Now, fix $\epsilon > 0$. Since $\frac{\text{Var}(X_n)}{n} \rightarrow 0$, there exists $N \in \mathbb{N}$ such that $\frac{\text{Var}(X_n)}{n} < \epsilon$ for all $n > N$. Let $M = \sum_{i=1}^N \frac{\text{Var}(X_i)}{i} < \infty$. Then, for $n > N$,

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \frac{\text{Var}(X_i)}{i} &< \frac{1}{n} \left(M + \sum_{i=N+1}^n \epsilon \right) \\
&\leq \frac{1}{n} \cdot M + \epsilon
\end{aligned}$$

By taking limsup and letting $\epsilon \rightarrow 0$, we get

$$E|S_n/n - \nu_n|^2 \rightarrow 0,$$

which is the desired result.

Exercise 2.2.2. We assume that $r(0)$ is finite. Let $S_n = X_1 + \cdots + X_n$. Since $ES_n = 0$,

$$\begin{aligned}
E \left[\left(\frac{S_n}{n} \right)^2 \right] &= \text{Var} \left(\frac{S_n}{n} \right) \\
&= \frac{1}{n^2} \sum_{1 \leq i, j \leq n} EX_i X_j \\
&\leq \frac{1}{n^2} \left(\sum_{i=1}^n EX_i^2 + 2 \sum_{i=1}^{n-1} EX_i X_{i+1} + 2 \sum_{i=1}^{n-2} EX_i X_{i+2} + \cdots + 2 \sum_{i=1}^2 EX_i X_{i+n-2} + 2EX_1 X_n \right)
\end{aligned}$$

Now, since $r(k) \rightarrow 0$ as $k \rightarrow \infty$, for given $\epsilon > 0$, $\exists N$ such that for $n \geq N$, $|r(n)| < \epsilon$. Moreover, since $EX_i^2 \leq r(0) < \infty$, by Cauchy-Schwartz Inequality, $E|X_i X_j| \leq (EX_i^2)^{1/2} (EX_j^2)^{1/2} \leq r(0) < \infty$. Then, for $n \geq N + 1$, we have

$$\begin{aligned}
&E \left[\left(\frac{S_n}{n} \right)^2 \right] \\
&\leq \frac{1}{n^2} \left(\sum_{i=1}^n EX_i^2 + 2 \sum_{i=1}^{n-1} E|X_i X_{i+1}| + \cdots + 2 \sum_{i=1}^{n-N+1} E|X_i X_{i+N-1}| + 2 \sum_{i=1}^{n-N} EX_i X_{i+N} + \cdots + 2EX_1 X_n \right) \\
&\leq \frac{1}{n^2} \left(nr(0) + 2(n-1)r(0) + \cdots + 2(n-N+1)r(0) + 2 \sum_{i=1}^{n-N} r(N) + 2 \sum_{i=1}^{n-N-1} r(N+1) + \cdots + 2r(n-1) \right) \\
&\leq \frac{1}{n^2} \left(2Nnr(0) + 2 \sum_{i=1}^{n-N} |r(N)| + 2 \sum_{i=1}^{n-N-1} |r(N+1)| + \cdots + 2|r(n-1)| \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{n^2} \left(2Nnr(0) + 2 \sum_{i=1}^{n-N} i \cdot \epsilon \right) \\
&\leq \frac{1}{n^2} \left(2Nnr(0) + 2 \sum_{i=1}^n i \cdot \epsilon \right) \\
&\leq \frac{2Nr(0)}{n} + \frac{n(n+1)}{n^2} \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $E \left[\left(\frac{S_n}{n} \right)^2 \right] \rightarrow 0$, which means that $S_n/n \rightarrow 0$ in L^2 . Therefore, $S_n/n \rightarrow 0$ in probability, which is the desired result.

Exercise 2.2.3. (i) Since U_i 's are *i.i.d.* and f is a measurable function, $f(U_i)$'s are also *i.i.d.* Note that since U_i follows Uniform distribution on $[0,1]$, $Ef(U_1) = \int_0^1 f(x)dx$. By assumption, $E|f(U_1)| = \int_0^1 |f(x)|dx < \infty$. By WLLN,

$$I_n = \frac{f(U_1) + \cdots + f(U_n)}{n} \rightarrow Ef(U_1) = \int_0^1 f(x)dx = I \text{ in prob.}$$

(ii) Since $\int_0^1 |f(x)|^2 dx < \infty$,

$$\text{Var}(f(U_1)) = Ef(U_1)^2 - (Ef(U_1))^2 = \int_0^1 f(x)^2 dx - \left(\int_0^1 f(x)dx \right)^2 < \infty.$$

By Chebyshev's inequality with $\varphi(x) = x^2$ and $A = \{x \in \mathbb{R} : |x| > a/n^{1/2}\}$, we have $i_A = \inf\{\varphi(y) : y \in A\} = a^2/n$ and

$$\begin{aligned}
P(|I_n - I| > a/n^{1/2}) &\leq \frac{n}{a^2} E(I_n - I)^2 \\
&= \frac{n}{a^2} \text{Var}(I_n) \\
&= \frac{1}{a^2} \text{Var}(f(U_1)) \\
&= \frac{1}{a^2} \left[\int_0^1 f(x)^2 dx - \left(\int_0^1 f(x)dx \right)^2 \right]
\end{aligned}$$

Exercise 2.2.4. (1) $E|X_i| = \infty$.

Since $\sum_{k=2}^{\infty} \frac{1}{k \log k} \geq \int_2^{\infty} \frac{1}{x \log x} dx = [\log(\log x)]_2^{\infty} = \infty$, $E|X_i| = \sum_{k=2}^{\infty} \frac{C}{k \log k} = \infty$.

(2) There exists $\mu < \infty$ so that $S_n/n \rightarrow \mu$ in probability.

Since

$$nP(|X_1| > n) \leq n \sum_{k=n+1}^{\infty} \frac{C}{k^2 \log k} \leq \frac{n}{\log n} \sum_{k=n+1}^{\infty} \frac{C}{k^2} \leq \frac{n}{\log n} \int_n^{\infty} \frac{C}{x^2} dx = \frac{C}{\log n},$$

$nP(|X_1| > n) \rightarrow 0$ as $n \rightarrow \infty$. Also, there exists $\mu < \infty$ such that

$$\mu_n := E(X_1 \mathbf{1}_{(|X_1| \leq n)}) = \sum_{k=2}^{\infty} (-1)^k \cdot k \cdot \frac{C}{k^2 \log k} = \sum_{k=2}^{\infty} (-1)^k \frac{C}{k \log k} \rightarrow \mu$$

since it is alternating series. Therefore, by Theorem 2.2.12, $S_n/n \rightarrow \mu$ in probability.

Exercise 2.2.5. By definition, X_i takes only positive values.

(1) There exists a sequence $\mu_n \rightarrow \infty$ such that $S_n/n - \mu_n \rightarrow 0$ in probability.

Firstly, for $x \geq 2$, $xP(|X_i| > x) = xP(X_i > x) = \frac{2}{\log x} \rightarrow 0$ as $x \rightarrow \infty$. Let $\mu_n = E(X_1 \mathbf{1}_{(|X_1| \leq n)})$. Then, for $n \geq 3$,

$$\mu_n = \int_e^n \frac{e}{x \log x} dx = [e \log(\log x)]_e^n = e \log(\log n) \rightarrow \infty.$$

Therefore, by Theorem 2.2.12, $S_n/n - \mu_n \rightarrow 0$ in probability.

(2) $E|X_i| = \infty$.

Since $E|X_i| = EX_i \geq \mu_n$ for all n , $E|X_i| = \infty$.

Exercise 2.2.6. (i) $X = \sum_{n=1}^{\infty} \mathbf{1}_{(X \geq n)}$.

$$\begin{aligned} \text{(ii) } EX^2 &= \sum_{x=1}^{\infty} x^2 P(X = x) \\ &= P(X = 1) + (1 + 3)P(X = 2) + \cdots + (1 + 3 + \cdots + (2n - 1))P(X = n) + \cdots \\ &= P(X \geq 1) + 3P(X \geq 2) + \cdots + (2n - 1)P(X \geq n) + \cdots \\ &= \sum_{n=1}^{\infty} (2n - 1)P(X \geq n). \end{aligned}$$

Exercise 2.2.7.

$$\begin{aligned} \int_{-\infty}^{\infty} h(y)P(X \geq y)dy &= \int_{-\infty}^{\infty} \int_{\Omega} h(y)\mathbf{1}_{(X \geq y)}(\omega) dP dy \\ &= \int_{\Omega} \int_{-\infty}^{\infty} h(y)\mathbf{1}_{(X \geq y)}(\omega) dP dy \\ &= \int_{\Omega} H(X) dP = EH(X). \end{aligned}$$

The second equality holds by Fubini.

Exercise 2.2.8. Let's use Theorem 2.2.11 in Textbook with $X_{n,k} = X_k$ for $k = 1, \dots, n$ and $b_n = 2^{m(n)}$ where $m(n) = \min \{m : 2^{-m} m^{-3/2} \geq n^{-1}\}$. Let's check the conditions.

(0) $b_n > 0$ with $b_n \rightarrow \infty$.

By definition of $m(n)$.

(i) $\sum_{k=1}^n P(|X_{n,k}| > b_k) \rightarrow 0$.

$$\begin{aligned} \sum_{k=1}^n P(|X_{n,k}| > b_k) &= nP(|X_1| > 2^{m(n)}) = n \sum_{j=m(n)+1}^{\infty} p_j \\ &= \sum_{j=m(n)+1}^{\infty} \frac{n}{2^j \cdot j(j+1)} \leq \sum_{j=m(n)+1}^{\infty} \frac{n}{2^j \cdot m(n)^2} \\ &= \frac{n}{2^{m(n)} m(n)^2} \leq m(n)^{-\frac{1}{2}} \rightarrow 0 \end{aligned}$$

(ii) $b_n^{-2} \sum_{k=1}^n E \bar{X}_{n,k}^2 \rightarrow 0$.

$$\begin{aligned}
 b_n^{-2} \sum_{k=1}^n E \bar{X}_{n,k}^2 &= \frac{n}{2^{2m(n)}} E (X_1^2 \mathbf{1}_{\{|X_1| \leq 2^{m(n)}\}}) \\
 &\leq \frac{m(n)^{3/2}}{2^{m(n)}} E (X_1^2 \mathbf{1}_{\{|X_1| \leq 2^{m(n)}\}}) \\
 (1) \quad &= \frac{m(n)^{3/2}}{2^{m(n)}} \left(\sum_{k=1}^{m(n)} (2^k - 1)^2 p_k + p_0 \right).
 \end{aligned}$$

Note that

$$\begin{aligned}
 \left(\sum_{k=1}^{m(n)} (2^k - 1)^2 p_k + p_0 \right) &\leq 1 + \sum_{k=1}^{m(n)} 2^{2k} p_k \\
 &= 1 + \sum_{k=1}^{m(n)} 2^k \cdot \frac{1}{k(k+1)} \\
 &= 1 + \sum_{1 \leq k < \frac{m(n)}{2}} 2^k \cdot \frac{1}{k(k+1)} + \sum_{\frac{m(n)}{2} \leq k \leq m(n)} 2^k \cdot \frac{1}{k(k+1)} \\
 &\leq 1 + \sum_{1 \leq k < \frac{m(n)}{2}} 2^k + \frac{4}{m(n) \cdot (m(n) + 2)} \sum_{\frac{m(n)}{2} \leq k \leq m(n)} 2^k \cdot \frac{1}{k(k+1)} \\
 &= 1 + 2^{\frac{m(n)}{2} + 1} - 2 + \frac{4}{m(n) \cdot (m(n) + 2)} \left(2^{m(n)+1} - 2^{\frac{m(n)}{2} + 1} \right) \\
 &\leq \frac{2^{m(n)}}{m(n)^2} \cdot C \quad \text{for large } n.
 \end{aligned}$$

Thus,

$$(1) \leq \frac{m(n)^{\frac{3}{2}}}{2^{m(n)}} \cdot \frac{2^{m(n)}}{m(n)^2} \cdot C = C \cdot m(n)^{-\frac{1}{2}} \rightarrow 0$$

By (0)-(ii), Theorem 2.2.11 gives us that $\frac{S_n - a_n}{b_n} \xrightarrow{p} 0$, where $a_n = \sum_{k=1}^n E \bar{X}_{n,k}$.

Now,

$$\begin{aligned}
 a_n &= n E (X_1 \mathbf{1}_{\{|X_1| \leq 2^{m(n)}\}}) \\
 &= n \left[\sum_{k=1}^{m(n)} (2^k - 1) p_k - p_0 \right] \\
 &= n \left[-\frac{1}{m(n) + 1} + \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k \cdot k(k+1)} \right] \\
 &\leq n \left[-\frac{1}{m(n) + 1} + \frac{1}{m(n)^2} \sum_{k=m(n)+1}^{\infty} \frac{1}{2^k} \right]
 \end{aligned}$$

$$\sim -\frac{n}{m(n)}.$$

Since $n \leq 2^{m(n)} m(n)^{3/2} \leq Cn$, $\log n \sim m(n)$. Thus,

$$\begin{aligned} a_n &\sim \frac{n}{\log n} \\ b_n &= 2^{m(n)} \sim n(m(n))^{-3/2} \sim n(\log n)^{-3/2}. \end{aligned}$$

Thus,

$$\frac{S_n - a_n}{b_n} \sim \frac{S_n + \frac{n}{\log n}}{n(\log n)^{-\frac{3}{2}}} \xrightarrow{p} 0.$$

Since $(\log n)^{-\frac{1}{2}} \rightarrow 0$,

$$\frac{S_n + n(\log n)^{-1}}{n(\log n)^{-1}} \xrightarrow{p} 0.$$

Finally,

$$\frac{S_n}{n/\log n} \xrightarrow{p} 0.$$

Exercise 2.2.9.

2.3. Borel-Cantelli Lemmas.

Exercise 2.3.1. Since $\cup_{k \geq n} A_k \supset A_m$ for all $m \geq n$, $P(\cup_{k \geq n} A_k) \geq \sup_{k \geq n} P(A_k)$. Thus,

$$P(\limsup A_n) = P(\cap_{n=1}^{\infty} \cup_{k \geq n} A_k) = \lim_{n \rightarrow \infty} P(\cup_{k \geq n} A_k) \geq \lim_{n \rightarrow \infty} \sup_{k \geq n} P(A_k) = \limsup P(A_n).$$

By applying above result to $B_n = A_n^c$, we get

$$P(\liminf A_n) \leq \liminf P(A_n).$$

Exercise 2.3.2. $\epsilon > 0$ given. Since f is uniformly continuous on $[-M-1, M+1]$, there exists $\delta \in (0, 1)$ such that if $x, y \in [-M-1, M+1]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Then, $|f(x) - f(y)| \geq \epsilon$ implies $|x-y| \geq \delta$ or $|x| \geq M$. Thus,

$$P(|f(X_n) - f(X)| \geq \epsilon) \leq P(|X_n - X| \geq \delta) + P(|X| \geq M).$$

Since $X_n \rightarrow X$ in probability,

$$\limsup_n P(|f(X_n) - f(X)| \geq \epsilon) \leq P(|X| \geq M).$$

By letting $M \rightarrow \infty$, we get

$$\lim_n P(|f(X_n) - f(X)| \geq \epsilon) = 0.$$

Exercise 2.3.3. We know that

$$P(l_n = k) = \begin{cases} \frac{1}{2^{k+1}}, & k = 0, 1, \dots, n-1 \\ \frac{1}{2^n}, & k = n. \end{cases}$$

(1) $\limsup \frac{l_n}{\log_2 n} = 1$ a.s.

(i) $\limsup \frac{l_n}{\log_2 n} \leq 1$ a.s.

Fix $\epsilon > 0$. Let $A_n = \{l_n > (1 + \epsilon) \log_2 n\}$ and $m = \lfloor (1 + \epsilon) \log_2 n \rfloor$. Then,

$$\begin{aligned} P(A_n) &= 1 - P(l_n \leq (1 + \epsilon) \log_2 n) \\ &= 1 - \sum_{k=1}^n \frac{1}{2^{k+1}} = \left(\frac{1}{2}\right)^{m+1} \\ &\leq \left(\frac{1}{2}\right)^{(1+\epsilon) \log_2 n} = n^{-(1+\epsilon)}. \end{aligned}$$

Since $1 + \epsilon > 1$, we get $\sum_{n=1}^{\infty} P(A_n) < \infty$. By Borel-Cantelli Lemma 1,

$$P(A_n \text{ i.o.}) = P\left(\frac{l_n}{\log_2 n} > 1 + \epsilon \text{ i.o.}\right) = 0,$$

that is,

$$P\left(\frac{l_n}{\log_2 n} \leq 1 + \epsilon \text{ eventually}\right) = 1.$$

Since $\epsilon > 0$ was arbitrarily chosen, we get $\limsup \frac{l_n}{\log_2 n} \leq 1$ a.s.

(ii) $\limsup \frac{l_n}{\log_2 n} \geq 1$ a.s.

Construct a sequence $(r_n)_{n \in \mathbb{N}}$ by $r_1 = 1, r_2 = 2, r_3 = r_{n-1} + \lfloor \log_2 n \rfloor$ for $r \geq 3$. Let $B_n = \{X_m = 1, \text{ for all } m \text{ such that } r_{n-1} \leq m \leq r_n\}$. Then,

$$P(B_n) = \left(\frac{1}{2}\right)^{r_n - r_{n-1}} = \left(\frac{1}{2}\right)^{\lfloor \log_2 n \rfloor} \geq \left(\frac{1}{2}\right)^{\log_2 n} = \frac{1}{n}.$$

Since $\sum_{n=1}^{\infty} P(B_n) = \infty$ and B_n 's are independent, by Borel-Cantelli Lemma 2, $P(B_n \text{ i.o.}) = 1$. Since $B_n \subset \{l_{r_n} \geq \lfloor \log_2 n \rfloor\}$, $P(l_{r_n} \geq \lfloor \log_2 n \rfloor \text{ i.o.}) = 1$. Note that $r_n \leq n \log_2 n$ for $n \geq 2$. (It can be shown easily by induction.) Then, since $\left\{\frac{l_{r_n}}{\log_2 r_n} \geq \frac{\lfloor \log_2 n \rfloor}{\log_2 r_n}\right\} \subset \left\{\frac{l_{r_n}}{\log_2 r_n} \geq \frac{\lfloor \log_2 n \rfloor}{\log_2(n \log_2 n)} = \frac{\lfloor \log_2 n \rfloor}{\log_2 n + \log_2 \log_2 n}\right\}$, we get

$$P\left(\frac{l_{r_n}}{\log_2 r_n} \geq \frac{\lfloor \log_2 n \rfloor}{\log_2 n + \log_2 \log_2 n} \text{ i.o.}\right) = 1.$$

Since $\lim_{n \rightarrow \infty} \frac{\lfloor \log_2 n \rfloor}{\log_2 n + \log_2 \log_2 n} = 1$,

$$\limsup \frac{l_n}{\log_2 n} \geq 1 \text{ a.s.}$$

(2) $\liminf \frac{l_n}{\log_2 n} = 0$ a.s.

It suffices to show that $P(l_n = 0 \text{ i.o.}) = 1$. Note that $\{l_n = 0\} = \{X_n = 0\}$ are independent and $P(X_n = 0) = \frac{1}{2}$. Thus, by Borel-Cantelli Lemma 2, $P(l_n = 0 \text{ i.o.}) = 1$.

Exercise 2.3.4. Firstly, find a subsequence $(X_{n(m)})_{m \in \mathbb{N}}$ such that $EX_{n(m)} \rightarrow \liminf EX_n$. Since $X_n \rightarrow X$ in probability, there exists a further subsequence $X_{n(m_k)}$ such that $X_{n(m_k)} \rightarrow X$ a.s. by Theorem 2.3.2 in Textbook. Then, by Fatou's Lemma,

$$EX = E(\liminf X_{n(m_k)}) \leq \liminf EX_{n(m_k)} = \liminf EX_n.$$

The inequality comes from Fatou's Lemma.

Exercise 2.3.5. (a) For all subsequence X_{n_k} , there exists a sub-subsequence $X_{m(n_k)}$ converging almost surely to X . Since $|X_{m(n_k)}| \leq Y$: integrable, by DCT, $EX_{m(n_k)} \rightarrow EX$. This means that every subsequence (EX_{n_k}) of (EX_n) has a sub-subsequence $EX_{m(n_k)}$ converging to EX . Therefore, $EX_n \rightarrow EX$.

(b) See Theorem 1.6.8.

Exercise 2.3.6. (a) Note that $f(x) = \frac{x}{1+x}$ is non-decreasing, $0 \leq f(x) \leq 1$ for all $x \geq 0$ and $f(x) = 0$ if and only if $x = 0$.

Firstly, $d(X, Y) = E(f(|X - Y|)) \geq [0, \infty)$.

(i)

$$\begin{aligned} d(X, Y) = E(f(|X - Y|)) = 0 &\iff f(|X - Y|) = 0 \text{ a.s.} \\ &\iff |X - Y| = 0 \text{ a.s.} \end{aligned}$$

(ii) Clear.

(iii) Since $|X - Z| \leq |X - Y| + |Y - Z|$,

$$\begin{aligned} \frac{|X - Z|}{1 + |X - Z|} &= f(|X - Z|) \\ &\leq f(|X - Y| + |Y - Z|) \\ &= \frac{|X - Y| + |Y - Z|}{1 + |X - Y| + |Y - Z|} \\ &= \frac{|X - Y|}{1 + |X - Y|} + \frac{|Y - Z|}{1 + |Y - Z|} \end{aligned}$$

Therefore,

$$d(X, Z) = E(f(|X - Z|)) \leq E(f(|X - Y|)) + E(f(|Y - Z|)) = d(X, Y) + d(Y, Z).$$

(b) (\Rightarrow) Fix $\epsilon > 0$. Let $\varphi(x) = \frac{x}{1+x} \geq 0$ and $A = (\epsilon, \infty)$. Then, we have $i_A = \inf\{\varphi(y) : y \in A\} = \frac{\epsilon}{1+\epsilon}$. By Chebyshev's inequality, we get

$$P(|X_n - X| > \epsilon) \leq \frac{1+\epsilon}{\epsilon} \cdot E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) \leq \frac{1+\epsilon}{\epsilon} \cdot d(X_n, X) \rightarrow 0.$$

Thus, $X_n \rightarrow X$ in probability.

(\Leftarrow) For any $\epsilon > 0$,

$$\begin{aligned} d(X_n, X) &= E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) \\ &= E\left(\frac{|X_n - X|}{1 + |X_n - X|}; |X_n - X| > \epsilon\right) + E\left(\frac{|X_n - X|}{1 + |X_n - X|}; |X_n - X| \leq \epsilon\right) \\ &\leq E(1; |X_n - X| > \epsilon) + E\left(\frac{\epsilon}{1 + \epsilon}; |X_n - X| \leq \epsilon\right) \\ &\leq P(|X_n - X| > \epsilon) + \frac{\epsilon}{1 + \epsilon}. \end{aligned}$$

Since $X_n \rightarrow X$ in probability, $\limsup_n d(X_n, X) = \frac{\epsilon}{1+\epsilon}$. Since $\epsilon > 0$ was arbitrarily chosen, by letting $\epsilon \rightarrow 0$, we get $\lim_n d(X_n, X) = 0$.

Exercise 2.3.7. By Theorem 2.3.2 in Textbook, it suffices to show that there exists a r.v. X_∞ such that for any subsequence $X_{n(m)}$, there exists a further subsequence $X_{n(m_k)}$ that converges almost surely to X_∞ . Since $d(X_m, X_n) \rightarrow 0$, there exists $N_k \in \mathbb{N}$ such that $d(X_m, X_n) < 2^{-k}$ if $m, n > N_k$. Now, fix a subsequence $X_{n(m)}$. Choose increasing subsequence (m_k) so that $n(m_k) \geq N_k$. Using Chebyshev's Inequality with $\varphi(y) = \frac{y}{1+y}$ and $A = (\frac{1}{k^2}, \infty)$, we get

$$P\left(|X_{n(m_k)} - X_{n(m_{k-1})}| > \frac{1}{k^2}\right) \leq \frac{k^2 + 1}{2^k}.$$

Since $\sum_{k=1}^{\infty} P(|X_{n(m_k)} - X_{n(m_{k-1})}| > \frac{1}{k^2}) \leq \sum_{k=1}^{\infty} \frac{k^2 + 1}{2^k} < \infty$, by Borel-Cantelli Lemma 1,

$$P\left(|X_{n(m_k)} - X_{n(m_{k-1})}| > \frac{1}{k^2} \text{ i.o.}\right) = 0,$$

that is, $P(|X_{n(m_k)} - X_{n(m_{k-1})}| \leq \frac{1}{k^2} \text{ eventually}) = 1$. For $\omega \in \Omega_0 := \{|X_{n(m_k)} - X_{n(m_{k-1})}| \leq \frac{1}{k^2} \text{ eventually}\}$, $(X_{n(m_k)}(\omega))_{k \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} . Thus, there exists $X_\infty(\omega) := \lim_k X_{n(m_k)}(\omega)$. Since $P(\Omega_0) = 1$, $X_{n(m_k)} \rightarrow X_\infty$ a.s.

It remains to check the well-definedness of X_∞ . Consider other subsequence $X_{n'(m')}$ such that $X_{n'(m'_k)} \rightarrow X'_\infty$. Since $d(X_{n(m_k)}, X_{n'(m'_k)}) \rightarrow 0$, by DCT, $d(X_\infty, X'_\infty) = 0$. By Exercise 2.3.6, it's equivalent to that $X_\infty = X'_\infty$.

Exercise 2.3.8. Since $P(\cup_n A_n) = 1$, $P(\cap_n A_n^c) = 0$. Moreover, since A_n 's are independent events, so are A_n^c . And this implies that

$$\begin{aligned} P\left(\bigcap_{n=1}^{\infty} A_n^c\right) &= \prod_{n=1}^{\infty} P(A_n^c) \\ &= \prod_{n=1}^{\infty} (1 - P(A_n)) = 0. \end{aligned}$$

Since $P(A_n) < 1$, $\prod_{n=1}^{m-1} (1 - P(A_n)) > 0$ for $m = 2, 3, \dots$. Thus, we have for all $m \in \mathbb{N}$,

$$P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = 0.$$

Since $\{A_n^c \text{ eventually}\} = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n^c$,

$$\begin{aligned} P(A_n^c \text{ eventually}) &= P\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n^c\right) \\ &\leq \sum_{m=1}^{\infty} P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = 0. \end{aligned}$$

Then, $P(A_n \text{ i.o.}) = 1$. Thus, by Borel-Cantelli Lemma 1(Theorem 2.3.1), $\sum_{n=1}^{\infty} P(A_n) = \infty$.

Exercise 2.3.9. (i) Let $B_n = A_n^c \cap A_{n+1}$. Then,

$$P(\cup_{m \geq n} A_m) \leq P(A_n) + \sum_{m=n}^{\infty} P(B_m) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $P(A_n \text{ i.o.}) = \lim_{n \rightarrow \infty} P(\cup_{m \geq n} A_m) = 0$.

(ii) $A_n = [0, \frac{1}{n})$ with Lebesgue measure.

Exercise 2.3.10. 10

Exercise 2.3.11. (i) For $0 < \epsilon < 1$, $P(|X_n| > \epsilon) = p_n \rightarrow 0 \iff p_n \rightarrow 0$.

(ii) Let $A_n = \{|X_n| > \epsilon\}$. Then, $P(A_n \text{ i.o.}) = 0 \iff \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} p_n < \infty$. Note that (\Rightarrow) holds since X_n 's are independent.

Exercise 2.3.12. Let's enumerate $\Omega = \{\omega_1, \omega_2, \dots\}$. Let $\Omega_0 = \{\omega_i : P(\omega_i) > 0\}$. Since $P(\Omega_0^c) = 0$, if one can show that $X_n(\omega) \rightarrow X(\omega)$ for $\omega \in \Omega_0$, then it's done. Fix $\epsilon > 0$. Since $X_n \rightarrow X$ in probability, for each $\omega_i \in \Omega_0$, there exists $N_i \in \mathbb{N}$ such that $P(|X_n - X| > \epsilon) < \frac{1}{2}P(\omega_i)$ if $n > N_i$. Since $\frac{1}{2}P(\omega_i) < P(\omega_i)$,

$$\omega_i \notin \{\omega : |X_n - X| > \epsilon \text{ for } n > N_i\}.$$

Thus, $X_n(\omega_i) \rightarrow X(\omega_i)$ as $n \rightarrow \infty$.

Exercise 2.3.13. Pick c_n such that $P(|X_n| > \frac{|c_n|}{n}) \leq \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} P(|\frac{X_n}{c_n}| > \frac{1}{n}) < \infty$, by Borel-Cantelli Lemma 1, $P(|\frac{X_n}{c_n}| > \frac{1}{n} \text{ i.o.}) = 0$. That is, $P(|\frac{X_n}{c_n}| \leq \frac{1}{n} \text{ eventually}) = 1$. Now, for given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, since $\{|X_n| \leq |c_n|\epsilon\} \supset \{|X_n| \leq |c_n|/N\}$,

$$P(|\frac{X_n}{c_n}| \leq \epsilon \text{ eventually}) \geq P(|\frac{X_n}{c_n}| \leq \frac{1}{N} \text{ eventually}) = 1.$$

Since ϵ is arbitrarily chosen, $X_n/c_n \rightarrow 0$ a.s.

Exercise 2.3.14. (\Rightarrow) Suppose $\sum_n P(X_n > A) = \infty$ for all A . Then, by Borel-Cantelli Lemma 2, $P(X_n > A \text{ i.o.}) = 1$ for all A . That is, $\sup X_n = \infty$ a.s. which is the contradiction.

(\Leftarrow) Since $\sum_n P(X_n > A) < \infty$ for some A , by Borel-Cantelli Lemma 1, $P(X_n > A \text{ i.o.}) = 0$. Thus, $\sup X_n \leq A < \infty$ a.s.

Exercise 2.3.15. (i) $\limsup_{n \rightarrow \infty} \frac{X_n}{\log n} = 1$ a.s.

(\leq) It suffices to show that $P(\frac{X_n}{\log n} > 1 + \epsilon \text{ i.o.}) = 0$ for any $\epsilon > 0$. By definition,

$$P(X_n > (1 + \epsilon) \log n) = n^{-(1+\epsilon)}. \text{ Since } 1 + \epsilon > 1, \text{ we have}$$

$$\sum_{n=1}^{\infty} P(X_n > (1 + \epsilon) \log n) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty.$$

By Borel-Cantelli Lemma 1, $P(X_n > (1 + \epsilon) \log n \text{ i.o.}) = 0$.

(\geq) It suffices to show that $P\left(\frac{X_n}{\log n} > 1 \text{ i.o.}\right) = 1$. Since X'_n 's are independent and $\sum_{n=1}^{\infty} P(X_n > \log n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, by Borel-Cantelli Lemma 2, we get

$$P(X_n > \log n \text{ i.o.}) = 1.$$

(ii) If one can show that

$$1 \stackrel{(a)}{\leq} \liminf_{n \rightarrow \infty} \frac{M_n}{\log n} \leq \limsup_{n \rightarrow \infty} \frac{M_n}{\log n} \stackrel{(b)}{\leq} 1 \quad \text{a.s.},$$

then it's done.

(a) It suffices to show that $P\left(\frac{M_n}{\log n} < 1 - \epsilon \text{ i.o.}\right) = 0$ for any $\epsilon > 0$. Note that

$$\begin{aligned} P\left(\frac{M_n}{\log n} < 1 - \epsilon\right) &= P\left(\max_{1 \leq m \leq n} X_m < (1 - \epsilon) \log n\right) \\ &= \prod_{m=1}^n P(X_m < (1 - \epsilon) \log n) \\ &= \left(1 - \frac{1}{n^{1-\epsilon}}\right)^n \\ &\leq e^{-n^\epsilon}. \end{aligned}$$

The second equality holds since X_i 's are *i.i.d.* By L'Hopital's rule, $e^{-n^\epsilon}/\frac{1}{n^2} \rightarrow 0$. Thus, by comparison test,

$$\sum_{n=1}^{\infty} P\left(\frac{M_n}{\log n} < 1 - \epsilon\right) < \infty.$$

By Borel-Cantelli Lemma 1, $P\left(\frac{M_n}{\log n} < 1 - \epsilon \text{ i.o.}\right) = 0$.

(b) It suffices to show that $P\left(\frac{M_n}{\log n} < 1 + \epsilon \text{ eventually}\right) = 1$ for any $\epsilon > 0$.

Fix $\epsilon > 0$. Let $\Omega_0 = \{\omega : \limsup_{n \rightarrow \infty} \frac{X_n(\omega)}{\log n} = 1\}$.

(\star) For $\omega \in \Omega_0$, there exists $N_1(\omega) \in \mathbb{N}$ such that $\frac{X_n(\omega)}{\log n} < 1 + \epsilon$ if $n > N_1(\omega)$.

Now, for $n > N_1(\omega)$,

$$\begin{aligned} \frac{M_n(\omega)}{\log n} &= \max_{1 \leq m \leq n} \frac{X_m(\omega)}{\log n} \\ &= \max \left\{ \max_{1 \leq m \leq N_1(\omega)} \frac{X_m(\omega)}{\log n}, \max_{N_1(\omega) < m \leq n} \frac{X_m(\omega)}{\log n} \right\}. \end{aligned}$$

- There exists $N_2(\omega) \in \mathbb{N}$ such that $\max_{1 \leq m \leq N_1(\omega)} \frac{X_m(\omega)}{\log n} < 1 + \epsilon$ if $n > N_2(\omega)$.

- By (\star), $\frac{X_m(\omega)}{\log n} \leq \frac{X_m(\omega)}{\log m} < 1 + \epsilon$ for $N_1(\omega) < m \leq n$.

Thus, for $n > N(\omega) = \max\{N_1(\omega), N_2(\omega)\}$, we have

$$\frac{M_n(\omega)}{\log n} < 1 + \epsilon.$$

That is, $\Omega_0 \subset \left\{ \frac{M_n}{\log n} < 1 + \epsilon \text{ eventually} \right\}$. Since $P(\Omega_0) = 1$ by (i),

$$P\left(\frac{M_n}{\log n} < 1 + \epsilon \text{ eventually}\right) = 1.$$

Exercise 2.3.16. (1) $\sum_n (1 - F(\lambda_n)) < \infty$.

Let $B_n = \{X_n > \lambda_n\}$. Since $1 - F(\lambda_n) = P(X_n > \lambda_n)$, by Borel-Cantelli Lemma 1, $P(X_n > \lambda_n \text{ i.o.}) = 0$. That is, $P(X_n \leq \lambda_n \text{ eventually}) = 1$. Note that (\star) for $\omega \in \Omega_0 := \{\omega : X_n(\omega) \leq \lambda_n \text{ eventually}\}$, there exists $N_1(\omega) \in \mathbb{N}$ such that $X_n(\omega) \leq \lambda_n$ if $n > N_1(\omega)$.

Now, for $n > N_1(\omega)$,

$$\max_{1 \leq m \leq n} X_m(\omega) = \max \left\{ \max_{1 \leq m \leq N_1(\omega)} X_m(\omega), \max_{N_1(\omega) < m \leq n} X_m(\omega) \right\}.$$

- Since $\lambda_n \uparrow \infty$, there exists $N_2(\omega) \in \mathbb{N}$ such that $\max_{1 \leq m \leq N_1(\omega)} X_m(\omega) \leq \lambda_n$ if $n > N_2(\omega)$.

- By (\star) , $X_m(\omega) \leq \lambda_m \leq \lambda_n$ for $N_1(\omega) < m \leq n$.

Thus, for $n > N(\omega) = \max\{N_1(\omega), N_2(\omega)\}$, we have

$$\max_{1 \leq m \leq n} X_m(\omega) \leq \lambda_n.$$

That is, $\Omega_0 \subset \{\max_{1 \leq m \leq n} X_m \leq \lambda_n \text{ eventually}\}$.

Since $P(\Omega_0) = 1$, $P(\max_{1 \leq m \leq n} X_m \leq \lambda_n \text{ eventually}) = 1$, which means that

$$P\left(\max_{1 \leq m \leq n} X_m > \lambda_n \text{ i.o.}\right) = 0.$$

(2) $\sum_n (1 - F(\lambda_n)) = \infty$.

By Borel-Cantelli Lemma 2, $P(X_n > \lambda_n \text{ i.o.}) = 1$. Then, clearly, $P(\max_{1 \leq m \leq n} X_m > \lambda_n \text{ i.o.}) = 1$.

Exercise 2.3.17. The answers are

- (i) $\mathbb{E}|Y_1| < \infty$,
 - (ii) $\mathbb{E}Y_1^+ < \infty$,
 - (iii) $n\mathbb{P}(|Y_1| > n) \rightarrow 0$, and
 - (iv) $\mathbb{P}(|Y_1| < \infty) = 1$.
- (i) If $\mathbb{E}|Y_1| < \infty$, then

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{Y_1}{n}\right| > \epsilon\right) = \sum_{n=1}^{\infty} \mathbb{P}(|Y_1| > \epsilon n) \leq \int_0^{\infty} \mathbb{P}(|Y_1| > t) dt = \mathbb{E}|Y_1| < \infty$$

since $\mathbb{P}(|Y_1| > \cdot)$ decreases. By Borel-Cantelli, we have the convergence. Conversely, if $\mathbb{E}|Y_1| = \infty$, then

$$\infty = \mathbb{E}|Y_1| = \int_0^{\infty} \mathbb{P}(|Y_1| > t) dt \leq \sum_{n=0}^{\infty} \mathbb{P}(|Y_1| > n)$$

since $\mathbb{P}(|Y_1| > \cdot)$ decreases. By the second Borel-Cantelli, the convergence fails.

- (ii) If $\mathbb{E}Y_1^+ < \infty$, then $Y_n^+/n \rightarrow 0$ a.s. from (i). Thus for a given $\epsilon > 0$, there exists N such that $Y_n/n^+ < \epsilon$ for all $n > N$. Then,

$$\frac{\max_{m \leq n} Y_m^+}{n} \leq \max \left\{ \frac{Y_1^+}{n}, \dots, \frac{Y_N^+}{n}, \epsilon \right\} \rightarrow 0$$

almost surely as $n \rightarrow \infty$. That is,

$$\lim_{n \rightarrow \infty} \frac{\max_{m \leq n} Y_m^+}{n} = 0$$

almost surely. Now we have

$$\limsup_{n \rightarrow \infty} \frac{\max_{m \leq n} Y_m}{n} \leq \limsup_{n \rightarrow \infty} \frac{\max_{m \leq n} Y_m^+}{n} = 0.$$

We claim that

$$\limsup_{n \rightarrow \infty} \frac{\max_{m \leq n} Y_m}{n} \geq 0.$$

Note that

$$\mathbb{P} \left(\frac{\max_{m \leq n} Y_m}{n} < -\epsilon \right) = F(-n\epsilon)^n$$

where F is the distribution of Y_1 . Let M be large so that $F(-n\epsilon) < \delta < 1$ for all $n > M$.

Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P} \left(\frac{\max_{m \leq n} Y_m}{n} < -\epsilon \right) &\leq \sum_{n=1}^M \mathbb{P} \left(\frac{\max_{m \leq n} Y_m}{n} < -\epsilon \right) + \sum_{n=M+1}^{\infty} \delta^n \\ &\leq M + \frac{\delta^{M+1}}{1-\delta} \\ &< \infty. \end{aligned}$$

Therefore, by Borel-Cantelli,

$$\limsup_{n \rightarrow \infty} \frac{\max_{m \leq n} Y_m}{n} \geq -\epsilon$$

almost surely. This proves our claim.

Conversely, if $\mathbb{E}Y_1^+ = \infty$, then

$$\infty = \mathbb{E}Y_1^+ = \int_0^{\infty} \mathbb{P}(Y_1 > t) dt \leq \sum_{n=0}^{\infty} \mathbb{P}(Y_1 > n),$$

and thus $Y_n/n > 1$ infinitely often.

- (iii) If

- (iv) By definition, $Y_n/n \rightarrow 0$ in probability if and only if

$$\mathbb{P}(|Y_n| > n\epsilon) \rightarrow 0$$

for any $\epsilon > 0$. This is equivalent to $\mathbb{P}(|Y_1| = \infty) = 0$.

Exercise 2.3.18 and 2.3.19 will be solved by the way we did in Theorem 2.3.9 in Textbook.

Exercise 2.3.18. Step 1. $\frac{X_n}{an^\alpha} \rightarrow 1$ in probability.

Fix $\epsilon > 0$. Since $\beta < 2\alpha$,

$$P\left(\left|\frac{X_n - EX_n}{EX_n}\right| > \epsilon\right) \leq \frac{\text{Var}(X_n)}{\epsilon^2(EX_n)^2} \leq \frac{Bn^\beta}{\epsilon^2(EX_n)^2} \sim \frac{B}{a^2\epsilon^2} \cdot n^{\beta-2\alpha} \rightarrow 0.$$

Thus, $\frac{X_n - EX_n}{EX_n} \rightarrow 0$ in probability. Therefore, $\frac{X_n}{EX_n} \rightarrow 1$ in probability.

Step 2. Find subsequence (n_k) .

Let $n_k = k^{\lceil 2/(2\alpha-\beta) \rceil + 1}$ (+1 to avoid the case where $\lceil \frac{2}{2\alpha-\beta} \rceil = 0$.) and $T_k = X_{n_k}$.

Then,

$$\begin{aligned} P\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \epsilon\right) &\leq \frac{\text{Var}(T_k)}{\epsilon^2(ET_k)^2} \leq \frac{Bn_k^\beta}{\epsilon^2(ET_k)^2} \sim \frac{B}{a^2\epsilon^2} \cdot n_k^{\beta-2\alpha} \\ &\leq \frac{B}{a^2\epsilon^2} \cdot k^{\left(\frac{2}{2\alpha-\beta}\right) \cdot (\beta-2\alpha)} \\ &= Ck^{-2} \end{aligned}$$

The last inequality holds by construction of n_k . Since $\sum_{k=1}^{\infty} P\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \epsilon\right) < \infty$, by Borel-Cantelli Lemma 1,

$$P\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \epsilon \text{ i.o.}\right) = 0.$$

That is, $\frac{T_k}{ET_k} \rightarrow 1$ a.s.

Step 3. $\frac{X_n}{an^\alpha} \rightarrow 1$ a.s.

For sufficiently large n such that $n_k \leq n < n_{k+1}$ and $\omega \in \Omega_0 := \{\omega : \frac{T_k}{ET_k} \rightarrow 1\}$, we have

$$\frac{ET_k}{ET_{k+1}} \cdot \frac{T_k(\omega)}{ET_k} = \frac{T_k(\omega)}{ET_{k+1}} \leq \frac{X_n(\omega)}{EX_n} \leq \frac{T_{k+1}(\omega)}{ET_k} = \frac{T_{k+1}(\omega)}{ET_{k+1}} \cdot \frac{ET_{k+1}}{ET_k}.$$

Note that

$$\frac{ET_{k+1}}{ET_k} \sim \frac{an_{k+1}^\alpha}{an_k^\alpha} = \left[\left(1 + \frac{1}{k}\right)^{\lceil \frac{2}{2\alpha-\beta} \rceil + 1}\right]^\alpha \rightarrow 1.$$

Therefore, we have $\frac{X_n(\omega)}{EX_n} \rightarrow 1$. Since $P(\Omega_0) = 1$ by Step 2, we conclude that $\frac{X_n}{EX_n} \rightarrow 1$ a.s.

Step 4. $\frac{X_n}{n^\alpha} \rightarrow a$ a.s.

$$\frac{X_n}{n^\alpha} = a \cdot \left(\frac{EX_n}{an^\alpha}\right) \cdot \left(\frac{X_n}{EX_n}\right) \rightarrow a \quad \text{a.s.}$$

Exercise 2.3.19. Suppose $Y \sim \text{Poisson}(\lambda)$, where $\lambda > 1$. Then, we can decompose Y into $Y = Y_1 + \dots + Y_n$, where Y_i 's are independent and $Y_i \sim \text{Poisson}(\lambda_i)$. Thus, without loss of generality, we may assume that X_n are independent Poisson r.v.'s with $EX_n = \lambda_n$, where $\lambda_n \leq 1$.

Step 1. $\frac{S_n}{ES_n} \rightarrow 1$ in probability.

Fix $\epsilon > 0$. Since $\sum_n \lambda_n = \infty$,

$$P\left(\left|\frac{S_n - ES_n}{ES_n}\right| > \epsilon\right) \leq \frac{\text{Var}(S_n)}{\epsilon^2(ES_n)^2} = \frac{\sum_n \lambda_n}{\epsilon^2(\sum_n \lambda_n)^2} = \frac{1}{\epsilon^2 \sum_n \lambda_n} \rightarrow 0.$$

Thus, $\frac{S_n - ES_n}{ES_n} \rightarrow 0$ in probability. Therefore, $\frac{S_n}{ES_n} \rightarrow 1$ in probability.

Step 2. Find subsequence (n_k) .

Since $\sum_n \lambda_n = \infty$, one can find subsequence (n_k) such that $n_k = \inf\{n : \sum_{i=1}^n \lambda_i \geq k^2\}$. Let $T_k = S_{n_k}$. Then,

$$P\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \epsilon\right) \leq \frac{\text{Var}(T_k)}{\epsilon^2(ET_k)^2} = \frac{1}{\epsilon^2 \sum_{i=1}^{n_k} \lambda_i} \leq \frac{1}{\epsilon^2 k^2}.$$

Since $\sum_{k=1}^{\infty} P\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \epsilon\right) < \infty$, by Borel-Cantelli Lemma 1,

$$P\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \epsilon \text{ i.o.}\right) = 0.$$

That is, $\frac{T_k}{ET_k} \rightarrow 1$ a.s.

Step 3. $\frac{S_n}{ES_n} \rightarrow 1$ a.s.

For $n \in \mathbb{N}$ such that $n_k \leq n < n_{k+1}$, we have

$$k^2 \leq \sum_{i=1}^{n_k} \lambda_i = ET_k \leq k^2 + 1 \leq (k+1)^2 \leq ET_{k+1} \leq (k+1)^2 + 1.$$

Note that the assumption that $\lambda_n \leq 1$ is used here.

With such n and $\omega \in \Omega_0 := \{\omega : \frac{T_k(\omega)}{ET_k} \rightarrow 1\}$,

$$\frac{ET_k}{ET_{k+1}} \cdot \frac{T_k(\omega)}{ET_k} = \frac{T_k(\omega)}{ET_{k+1}} \leq \frac{S_n(\omega)}{ES_n} \leq \frac{T_{k+1}(\omega)}{ET_k} = \frac{T_{k+1}(\omega)}{ET_{k+1}} \cdot \frac{ET_{k+1}}{ET_k}.$$

Note that

$$\frac{ET_{k+1}}{ET_k} \leq \frac{(k+1)^2 + 1}{k^2} = 1 + \frac{2(k+1)}{k^2} \rightarrow 1.$$

Therefore, we have $\frac{S_n(\omega)}{ES_n} \rightarrow 1$. Since $P(\Omega_0) = 1$ by Step 2, we conclude that $\frac{S_n}{ES_n} \rightarrow 1$ a.s.

Exercise 2.3.20. It suffices to show that for any $M > 0$, $P\left(\frac{X_n}{n \log_2 n} \geq M \text{ i.o.}\right) = 1$. Fix $M > 0$. Let $k_n = \lfloor \log_2(M \cdot n \log_2 n) \rfloor + 1$. Then, for $n \geq 2$,

$$\begin{aligned} P(X_1 \geq M \cdot n \log_2 n) &= \sum_{j=k_n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k_n-1}} \\ &= \frac{1}{2^{\lfloor \log_2(M \cdot n \log_2 n) \rfloor}} \geq \frac{1}{M \cdot n \log_2 n}. \end{aligned}$$

Then, by Exercise 2.2.4,

$$\sum_{n=2}^{\infty} P\left(\frac{X_1}{n \log_2 n} \geq M\right) \geq \sum_{n=2}^{\infty} \frac{1}{M \cdot n \log_2 n} = \infty.$$

Since X_n 's are independent, by Borel-Cantelli Lemma 2, we get $P\left(\frac{X_n}{n \log_2 n} \geq M \text{ i.o.}\right) = 1$.

2.4. Strong Law of Large Numbers.

Exercise 2.4.1. $X_i, Y_i \geq 0$. $X_i \stackrel{i.i.d}{\sim} F$, $Y_i \stackrel{i.i.d}{\sim} G$. $EX_1, EY_1 < \infty$.

Let $T_n := (X_1 + Y_1) + \dots + (X_n + Y_n)$ and $N_t := \min\{n : T_n \leq t\}$. Note that $N_t \rightarrow \infty$ as $t \rightarrow \infty$. Then,

$$\frac{\sum_{i=1}^{N_t} X_i}{t} \leq \frac{R_t}{t} \leq 1 - \frac{\sum_{i=1}^{N_t} Y_i}{t}.$$

By SLLN and Theorem 2.4.7, we have

$$\frac{\sum_{i=1}^{N_t} X_i}{N_t} \xrightarrow{\text{a.s.}} EX_1, \quad \frac{\sum_{i=1}^{N_t} Y_i}{N_t} \xrightarrow{\text{a.s.}} EY_1, \quad \frac{N_t}{t} \xrightarrow{\text{a.s.}} \frac{1}{EX_1 + EY_1}.$$

Thus, we get

$$\frac{R_t}{t} \xrightarrow{\text{a.s.}} \frac{EX_1}{EX_1 + EY_1}.$$

Exercise 2.4.2. Let $Y_{n+1} = X_{n+1}/|X_n|$. Since Y_n is independent of X_1, \dots, X_n , $(Y_n)_n$ is an iid sequence. Note that

$$\frac{\log |X_n|}{n} = \frac{\log |Y_1| + \dots + \log |Y_n|}{n}$$

converges to $\mathbb{E} \log |Y_1|$ whenever it exists by the Strong Law of Large Numbers. Now to find the expectation,

$$\mathbb{E} \log |Y_1| = \int_{B_1} \frac{1}{\pi} \log |x| dx = \int_0^1 \int_{\partial B_r} \frac{1}{\pi} \log r dS dr = \int_0^1 2r \log r dr = -\frac{1}{2}.$$

Remark. For d -dimensional vectors, one can derive that it converges to $-1/d$ by a similar argument. \square

Exercise 2.4.3. (i)

$$\begin{aligned} W_n &= (ap + (1-p)V_{n-1})W_{n-1} \\ &= (ap + (1-p)V_{n-1})(ap + (1-p)V_{n-2})W_{n-2} \\ &\quad \vdots \\ &= \prod_{i=1}^{n-1} (ap + 1 - p)V_i. \end{aligned}$$

Thus, $\log W_n = \sum_{i=1}^{n-1} \log(ap + (1-p)V_i)$. By SLLN,

$$\frac{\log W_n}{n} = \frac{n-1}{n} \cdot \frac{\sum_{i=1}^{n-1} \log(ap + (1-p)V_i)}{n-1} \xrightarrow{\text{a.s.}} E\left(\log(ap + (1-p)V_1)\right) =: c(p).$$

(ii) $c(p) = \int \log(ap + (1-p)\nu)\mu(d\nu)$, where μ : distribution of V . By Theorem A.5.1,

$$c''(p) = - \int \left(\frac{a - \nu}{ap + (1-p)\nu} \right)^2 \mu(d\nu) < 0.$$

(iii)

$$c'(0) = \int \frac{a - \nu}{\nu} \mu(d\nu) = -1 + aE(V_1^{-1})$$

$$c'(1) = \int \frac{a - \nu}{a} \mu(d\nu) = 1 - \frac{1}{a}EV_1.$$

To guarantee that the optimal choice of p lies in $(0,1)$, $E(V_1^{-1}) > \frac{1}{a}$ and $E(V_1) > a$ should hold.

(iv) $p = \frac{8-5a}{2a^2-10a+8}.$

3. CENTRAL LIMIT THEOREMS

3.1. The De Moivre-Laplace Theorem.

3.2. Weak Convergence.

Exercise 3.2.1. Omit.

Exercise 3.2.2. (i) $P\left(M_n/n^{1/\alpha} \leq y\right) = P\left(M_n \leq yn^{1/\alpha}\right)$

$$= \left[P\left(X_i \leq yn^{1/\alpha}\right)\right]^n$$

$$= \left(1 - \left(yn^{1/\alpha}\right)^{-\alpha}\right)^n$$

$$= \left[1 - \frac{y^{-\alpha}}{n}\right]^n \rightarrow \exp(-y^{-\alpha}).$$

(ii) $P\left(n^{1/\beta}M_n \leq y\right) = P\left(M_n \leq yn^{-1/\beta}\right)$

$$= \left[P\left(X_i \leq yn^{-1/\beta}\right)\right]^n$$

$$= \left(1 - \frac{|y|^\beta}{n}\right)^n \rightarrow \exp(-|y|^\beta).$$

(iii) $P(M_n - \log n \leq y) = P(M_n \leq y + \log n)$

$$= [P(X_i \leq y + \log n)]^n$$

$$= (1 - e^{-y - \log n})^n$$

$$= \left(1 - \frac{e^{-y}}{n}\right)^n \rightarrow \exp(-e^{-y}).$$

Exercise 3.2.3. (i)
$$\int_x^\infty e^{-\frac{y^2}{2}} dy = \left[-\frac{1}{y} e^{-\frac{y^2}{2}} \right]_x^\infty - \int_x^\infty \frac{1}{y^2} e^{-\frac{y^2}{2}} dy$$

$$= \left[-\frac{1}{y} e^{-\frac{y^2}{2}} \right]_x^\infty - \left[\left[-\frac{1}{y^3} e^{-\frac{y^2}{2}} \right]_x^\infty - \int_x^\infty \frac{3}{y^4} e^{-\frac{y^2}{2}} dy \right].$$

The first equality gives us

$$\int_x^\infty e^{-\frac{y^2}{2}} dy \leq \frac{1}{x} e^{-\frac{x^2}{2}}.$$

The second equality gives us

$$\left(\frac{1}{x} - \frac{1}{x^3} \right) e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{y^2}{2}} dy.$$

Thus, $P(X_i > x) \sim \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Then,

$$\frac{P(X_i > x + \theta/x)}{P(X_i > x)} \sim \frac{\frac{1}{\sqrt{2\pi}(x + \frac{\theta}{x})} \exp\left(-\frac{1}{2}\left(x + \frac{\theta}{x}\right)^2\right)}{\frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{1}{2}x^2\right)} \rightarrow e^{-\theta}$$

(ii) It's clear that $b_n \rightarrow \infty$ as $n \rightarrow \infty$. By (i), $P\left(X_i > b_n + \frac{x}{b_n}\right) \cdot n \rightarrow e^{-x}$.

$$\begin{aligned} P(b_n(M_n - b_n) \leq x) &= P\left(M_n \leq b_n + \frac{x}{b_n}\right) \\ &= \left[P\left(X_i \leq b_n + \frac{x}{b_n}\right) \right]^n \\ &= \left[1 - P\left(X_i > b_n + \frac{x}{b_n}\right) \right]^n \\ &= \left[1 - P(X_i > b_n) \cdot \frac{P\left(X_i > b_n + \frac{x}{b_n}\right)}{P(X_i > b_n)} \right]^n \\ &= \left[1 - \frac{1}{n} \cdot \frac{P\left(X_i > b_n + \frac{x}{b_n}\right)}{P(X_i > b_n)} \right]^n \\ &\rightarrow \exp(-e^{-x}). \end{aligned}$$

(iii) **Claim.** $b_n \sim \sqrt{2 \log n}$.

Proof. (\leq) By (i), $P(X_i > \sqrt{2 \log n}) \sim \frac{1}{n\sqrt{4\pi \log n}}$. Thus, for large n , $b_n \leq \sqrt{2 \log n}$.

(\geq) Similar argument with $\sqrt{2 \log n} - 2 \log \log n$

□

Now, by (iii), $P(b_n(M_n - b_n) \leq x) \rightarrow \exp(-e^{-x})$. Let x_n and y_n be $x_n \rightarrow \infty$ and $y_n \rightarrow -\infty$. Then,

$$P\left(M_n - b_n \leq \frac{x_n}{b_n}\right) \rightarrow 1$$

$$\begin{aligned}
& P\left(M_n - b_n \leq \frac{y_n}{b_n}\right) \rightarrow 0 \\
& \Rightarrow P\left(\frac{y_n}{b_n} \leq M_n - b_n \leq \frac{x_n}{b_n}\right) \rightarrow 1 \\
& \Rightarrow P\left(\frac{y_n}{b_n \sqrt{2 \log n}} \leq \frac{M_n}{\sqrt{2 \log n}} - \frac{b_n}{\sqrt{2 \log n}} \leq \frac{x_n}{b_n \sqrt{2 \log n}}\right) \rightarrow 1.
\end{aligned}$$

By choosing x_n, y_n with $o(b_n^2)$, we can show that

$$\frac{M_n}{\sqrt{2 \log n}} \xrightarrow{P} 1.$$

Exercise 3.2.4. By Skorohod's Representation Theorem (Thm 3.2.8 in Textbook), there exists Y_n and Y such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ with $Y_n \xrightarrow{a.s.} Y$. By Continuous Mapping Theorem (Exercise 1.3.3), $Eg(Y_n) \xrightarrow{a.s.} Eg(Y)$. By Fatou's Lemma, we get

$$\liminf_n Eg(X_n) = \liminf_n Eg(Y_n) \geq Eg(Y) = Eg(X).$$

Exercise 3.2.5. By Skorohod's Representation Theorem, there exists Y_n and Y with distribution F_n and F , respectively, such that $Y_n \xrightarrow{a.s.} Y$. By Theorem 1.6.8 in Textbook,

$$\int h(x) dF_n(x) \rightarrow \int h(x) dF(x).$$

Exercise 3.2.6. Define $L(F, G) = \{c : F(x - c) - c \leq G(x) \leq F(x + c) + c \text{ for all } x\}$.

Since any distribution is non-decreasing,

- (a) if $a \in L(F, G)$, then for any $b \geq a$ are also in $L(F, G)$
- (b) $L(F, G)$ contains only non-negative constants.

(1) ρ is a metric.

- (i) $\rho(F, G) = 0 \iff F = G$.

(\Leftarrow) Trivial.

(\Rightarrow) For any $\epsilon > 0$, $F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon$ for all x . Letting $\epsilon \searrow 0$, we get $F(x^-) \leq G(x) \leq F(x)$ for all x . Thus, for $x \in C(F)$, $F(x) = G(x)$.

For $x \in D(F)$, since $D(F)$ is at most countable (Exercise 1.2.3), we can find $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in C(F)$ with $x_n \downarrow x$. Since $F(x_n) = G(x_n)$, $F(x) = \lim_n F(x_n) = \lim_n G(x_n) = G(x)$. Therefore, $F = G$.

- (ii) $\rho(F, G) = \rho(G, F)$.

Suppose $a \in L(F, G)$. Then, for all x ,

$$F(x) - a \leq G(x + a) \leq F(x + 2a) + a$$

$$F(x - 2a) - a \leq G(x - a) \leq F(x) + a.$$

Thus, $G(x - a) - a \leq F(x) \leq G(x + a) + a$ for all x . That is, $a \in L(G, F)$. Therefore, $\rho(G, F) \leq \rho(F, G)$. By symmetry, it also holds that $\rho(F, G) \leq \rho(G, F)$.

- (iii) $\rho(F, H) \leq \rho(F, G) + \rho(G, H)$.

Suppose $a \in L(F, G)$ and $b \in L(G, H)$. Then, for all x ,

$$F(x - a - b) - a - b \leq G(x - b) - b \leq H(x) \leq G(x + b) + b \leq F(x + a + b) + a + b.$$

That is, $a + b \in L(F, H)$. Thus, $\rho(F, H) \leq \rho(F, G) + \rho(G, H)$.

By (i)-(iii), ρ is a metric.

(2) $\rho(F_n, F) \rightarrow 0$ if and only if $F_n \Rightarrow F$.

(\Rightarrow) Since $\rho(F_n, F) \rightarrow 0$, for given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\rho(F_n, F) < \epsilon$ for $n > N_\epsilon$. By (a), $\epsilon \in L(F_n, F)$ for $n > N_\epsilon$. That is, if $n > N_\epsilon$,

$$F(x - \epsilon) - \epsilon \leq F_n(x) \leq F(x + \epsilon) + \epsilon, \quad \forall x.$$

Letting $n \rightarrow \infty$,

$$F(x - \epsilon) - \epsilon \leq \liminf F_n(x) \leq \limsup F_n(x) \leq F(x + \epsilon) + \epsilon, \quad \forall x.$$

For $x \in C(F)$, letting $\epsilon \searrow 0$,

$$F(x) \leq \liminf F_n(x) \leq \limsup F_n(x) \leq F(x).$$

Thus, $F_n(x) \rightarrow F(x)$.

(\Leftarrow) Since $F_n \Rightarrow F$, $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ for $x \in C(F)$. Fix $\epsilon > 0$. Since $D(F)$ is at most countable, we can find $\{x_1, \dots, x_{k_\epsilon}\} \in C(F)$ such that

$$F(x_1) < \epsilon, \quad F(x_{k_\epsilon}) > 1 - \epsilon \quad \text{and} \quad |x_i - x_{i-1}| < \epsilon \quad \text{for } i = 2, \dots, k_\epsilon.$$

For each x_i , there exists $N_i(\epsilon) \in \mathbb{N}$ such that $F(x_i) - \epsilon < F_n(x_i) < F(x_i) + \epsilon$ if $n > N_i(\epsilon)$. Let $N(\epsilon) = \max\{N_1(\epsilon), \dots, N_{k_\epsilon}(\epsilon)\}$. Then, for $n > N(\epsilon)$,

(i) if $x < x_1$,

$$F(x - 2\epsilon) - 2\epsilon \leq F(x_1) - 2\epsilon < 0 \leq F_n(x) \leq F_n(x_1) < F(x_1) + \epsilon < 2\epsilon \leq F(x + 2\epsilon) + 2\epsilon.$$

(ii) if $x > x_{k_\epsilon}$,

$$F(x - 2\epsilon) - 2\epsilon \leq 1 - 2\epsilon < F(x_{k_\epsilon}) - \epsilon < F_n(x_{k_\epsilon}) \leq F_n(x) \leq 1 < F(x_{k_\epsilon}) + 2\epsilon \leq F(x + 2\epsilon) + 2\epsilon.$$

(iii) if $x_{i-1} \leq x \leq x_{i+1}$ for $i = 2, \dots, k_\epsilon$,

$$F(x - \epsilon) - \epsilon \leq F(x_{i-1}) - \epsilon < F_n(x_{i-1}) \leq F_n(x) \leq F_n(x_i) < F(x_i) + \epsilon \leq F(x + \epsilon) + \epsilon.$$

By (i)-(iii), $2\epsilon \in L(F, F_n)$ for $n > N(\epsilon)$. Since $\epsilon > 0$ is arbitrarily chosen, we can conclude that $\rho(F_n, F) \rightarrow 0$.

Exercise 3.2.7. $X \sim F$ and $Y \sim G$. Recall $L(F, G)$ defined in Exercise 3.2.6.

Define $K(X, Y) = \{\epsilon \geq 0 : P(|X - Y| > \epsilon) \leq \epsilon\}$. If one can show that $K(X, Y) \subset L(F, G)$, then it's done. Suppose $a \in K(X, Y)$. Then, $P(|X - Y| > a) \leq a$. Since

$$P(X \leq x - a) - P(|X - Y| > a) \leq P(Y \leq x) \leq P(X \leq x + a) + P(|X - Y| > a) \quad \forall x,$$

we get

$$P(X \leq x - a) - a \leq P(Y \leq x) \leq P(X \leq x + a) + a, \quad \forall x.$$

Thus, $a \in L(F, G)$

Exercise 3.2.8. Recall $K(X, Y)$ defined in Exercise 3.2.7. It is easy to check that if $a \in K(X, Y)$, then any $b > a$ is also in $K(X, Y)$.

If $a = 0$, then $X = Y$, and thus $\beta(X, Y) = 0$. Now, suppose $a > 0$.

(1) $\frac{a^2}{1+a} \leq \beta(X, Y)$.

Fix $0 < \epsilon < a$. Since $\epsilon \notin K(X, Y)$, $\epsilon < P(|X - Y| > \epsilon)$. By Exercise 2.3.6 (b) (\Rightarrow), we get

$$\epsilon < P(|X - Y| > \epsilon) \leq \frac{1 + \epsilon}{\epsilon} E \left(\frac{|X - Y|}{1 + |X - Y|} \right).$$

Thus, $\frac{\epsilon^2}{1 + \epsilon} < \beta(X, Y)$. Letting $\epsilon \nearrow a$, we get

$$\frac{a^2}{1 + a^2} \leq \beta(X, Y).$$

(2) $\beta(X, Y) \leq \frac{2a}{1+a}$.

For all $b > a$,

$$\begin{aligned} \beta(X, Y) &= \int \frac{|X - Y|}{1 + |X - Y|} dP \\ &= \int_{|X - Y| > b} \frac{|X - Y|}{1 + |X - Y|} dP + \int_{|X - Y| \leq b} \frac{|X - Y|}{1 + |X - Y|} dP \\ &\leq \int_{|X - Y| > b} 1 dP + \int_{|X - Y| \leq b} \frac{b}{1 + b} dP \\ &= P(|X - Y| > b) + \frac{b}{1 + b} \cdot P(|X - Y| \leq b) \\ &= P(|X - Y| > b) \cdot \left(1 - \frac{b}{1 + b} \right) + \frac{b}{1 + b} \\ &\leq \frac{b}{1 + b} + \frac{b}{1 + b} = \frac{2b}{1 + b}. \end{aligned}$$

The last inequality holds since $b \in K(X, Y)$. Letting $b \searrow a$, we get $\beta(X, Y) \leq \frac{2a}{1+a}$.

Exercise 3.2.9. Since $F_n \Rightarrow F$ and F is continuous, $F_n(x) \rightarrow F(x)$, $\forall x$. Fix $k \in \mathbb{N}$. Pick $x_{j,k} = F^{-1} \left(\frac{j}{k} \right)$ for $j = 1, \dots, k-1$ with $x_{0,k} = -\infty$ and $x_{k,k} = \infty$. For each j , there exists $N_k(j) \in \mathbb{N}$ such that $|F_n(x_{j,k}) - F(x_{j,k})| < \frac{1}{k}$ if $n > N_k(j)$. Let $N_k = \max \{N_k(1), \dots, N_k(k-1)\}$. Then, for $n > N_k$, if $x_{j-1,k} < x < x_{j,k}$, then

$$F(x) - \frac{2}{k} \leq F(x_{j-1,k}) - \frac{1}{k} < F_n(x_{j-1,k}) \leq F_n(x) \leq F_n(x_{j,k}) < F(x_{j,k}) + \frac{1}{k} \leq F(x) + \frac{2}{k}.$$

That is, $|F_n(x) - F(x)| < \frac{2}{k}$. Thus, $\sup_x |F_n(x) - F(x)| \rightarrow 0$ as $n \rightarrow \infty$.

Exercise 3.2.10. Let $X_i \stackrel{iid}{\sim} F$ and $F_n(\omega, x) = \frac{1}{n} \sum_{i=1}^n I(X_i(\omega) \leq x)$. By Glivenko-Cantelli Theorem (Thm 2.4.9), for each $x \in \mathbb{R}$, $F_n(\omega, x) \rightarrow F(x)$ a.s. Let Ω_x be a set where $F_n(\omega, x) \rightarrow F(x)$ and let $\Omega_0 = \cap_{q \in \mathbb{Q}} \Omega_q$. Then, $P(\Omega_0) = 1$ and for each $\omega_0 \in \Omega_0$, we have $F_n(\omega, q) \rightarrow F(q)$, $\forall q \in \mathbb{Q}$. Now, for $x \in C(F)$, choose $r, s \in \mathbb{Q}$ s.t. $r < x < s$ and

$$|F(r) - F(x)| < \epsilon$$

$$|F(s) - F(x)| < \epsilon.$$

Then,

$$\begin{aligned} F_n(x) - F(x) &\leq F_n(s) - F(r) \rightarrow F(s) - F(r) < 2\epsilon \\ F_n(x) - F(x) &\geq F_n(r) - F(s) \rightarrow F(r) - F(s) > -2\epsilon. \end{aligned}$$

Therefore, $-2\epsilon \leq \liminf(F_n(x) - F(x)) \leq \limsup(F_n(x) - F(x)) \leq 2\epsilon$.

Exercise 3.2.11. Since X_n , $1 \leq n \leq \infty$, are integer valued, $\mathbb{R} \setminus \mathbb{Z} \subset C(F_n)$, where F_n is df of X_n .

(\Rightarrow) For $1 \leq n < \infty$,

(i) $m \in \mathbb{R} \setminus \mathbb{Z}$

$$P(X_n = m) = 0 = P(X_\infty = m).$$

(ii) $m \in \mathbb{Z}$

$$\begin{aligned} P(X_n = m) &= F_n\left(m + \frac{1}{2}\right) - F_n\left(m - \frac{1}{2}\right) \\ &\rightarrow F_\infty\left(m + \frac{1}{2}\right) - F_\infty\left(m - \frac{1}{2}\right) \\ &= P(X_\infty = m). \end{aligned}$$

By (i), (ii), $P(X_n = m) \rightarrow P(X_\infty = m)$, $\forall m$.

(\Leftarrow) Want to show that $F_n(x) \rightarrow F_\infty(x)$, $\forall x \in C(F_\infty)$.

Fix $\epsilon > 0$. Since F_∞ can have positive probability only on \mathbb{Z} , which is countable, we can pick $K(\epsilon) := \{k_1, \dots, k_{l_\epsilon}\} \subset \mathbb{Z}$ such that

$$(11a) \quad \sum_{i=1}^{l_\epsilon} P(X_\infty = k_i) > 1 - \epsilon.$$

By assumption, for each k_i , there exists $N_\epsilon(i) \in \mathbb{N}$ such that for $n > N_\epsilon(i)$,

$$|P(X_n = k_i) - P(X_\infty = k_i)| < \frac{\epsilon}{l_\epsilon}.$$

Let $N_\epsilon = \max\{N_\epsilon(1), \dots, N_\epsilon(l_\epsilon)\}$. Then, for $n > N_\epsilon$,

$$\left| \sum_{i=1}^{l_\epsilon} [P(X_n = k_i) - P(X_\infty = k_i)] \right| \leq \sum_{i=1}^{l_\epsilon} |P(X_n = k_i) - P(X_\infty = k_i)| =: P_0 < \epsilon$$

Also, by (11a), we have

$$P_1 := \sum_{k \in \mathbb{Z} \setminus K(\epsilon)} P(X_\infty = k) < \epsilon.$$

Thus,

$$P_2 := \left| \sum_{k \in \mathbb{Z} \setminus K(\epsilon)} P(X_n = k) \right|$$

$$\begin{aligned}
&= \left| 1 - \sum_{i=1}^{l_\epsilon} P(X_n = k_i) \right| \\
&= \left| 1 - \sum_{i=1}^{l_\epsilon} P(X_\infty = k_i) + \sum_{i=1}^{l_\epsilon} P(X_\infty = k_i) - \sum_{i=1}^{l_\epsilon} P(X_n = k_i) \right| \\
&\leq \left| \sum_{k \in \mathbb{Z} \setminus K(\epsilon)} P(X_\infty = k) \right| + \left| \sum_{i=1}^{l_\epsilon} [P(X_n = k_i) - P(X_\infty = k_i)] \right| \\
&< 2\epsilon.
\end{aligned}$$

Now, for $x \in C(F_\infty)$, let $Z_x = \mathbb{Z} \cap (-\infty, x]$.

$$\begin{aligned}
F_n(x) &= \sum_{k \in Z_x} P(X_n = k) \\
&= \sum_{k \in Z_x \cap K(\epsilon)} P(X_n = k) + \sum_{k \in Z_x \setminus K(\epsilon)} P(X_n = k).
\end{aligned}$$

Similarly,

$$F_\infty(x) = \sum_{k \in Z_x \cap K(\epsilon)} P(X_\infty = k) + \sum_{k \in Z_x \setminus K(\epsilon)} P(X_\infty = k).$$

For $n > N_\epsilon$,

$$\begin{aligned}
|F_n(x) - F_\infty(x)| &= \left| \sum_{k \in Z_x \cap K(\epsilon)} [P(X_n = k) - P(X_\infty = k)] + \sum_{k \in Z_x \setminus K(\epsilon)} [P(X_n = k) - P(X_\infty = k)] \right| \\
&\leq \left| \sum_{k \in Z_x \cap K(\epsilon)} [P(X_n = k) - P(X_\infty = k)] \right| + \left| \sum_{k \in Z_x \setminus K(\epsilon)} P(X_n = k) \right| + \left| \sum_{k \in Z_x \setminus K(\epsilon)} P(X_\infty = k) \right| \\
&\leq \sum_{k \in Z_x \cap K(\epsilon)} |P(X_n = k) - P(X_\infty = k)| + \left| \sum_{k \in Z_x \setminus K(\epsilon)} P(X_n = k) \right| + \left| \sum_{k \in Z_x \setminus K(\epsilon)} P(X_\infty = k) \right|.
\end{aligned}$$

Note that $\begin{cases} Z_x \cap K(\epsilon) \subset K(\epsilon) = \{k_1, \dots, k_{l_\epsilon}\} \\ Z_x \setminus K(\epsilon) \subset \mathbb{Z} \setminus K(\epsilon) \end{cases}$. Thus,

$$|F_n(x) - F_\infty(x)| \leq P_0 + P_1 + P_2 < 4\epsilon.$$

Therefore, $F_n(x) \rightarrow F_\infty(x)$, and thus $X_n \Rightarrow X_\infty$.

Exercise 3.2.12. (1) If $X_n \rightarrow X$ in probability, then $X_n \Rightarrow X$.

Since $X_n \xrightarrow{P} X$, for given $\epsilon > 0$, $P(|X_n - X| > \epsilon) \rightarrow 0$.

$$\begin{aligned}
\text{(i) } P(X_n \leq x) &= P(X_n \leq x, |X_n - X| \leq \epsilon) + P(X_n \leq x, |X_n - X| > \epsilon) \\
&\leq P(X \leq x + \epsilon) + P(|X_n - X| > \epsilon).
\end{aligned}$$

Then, $\limsup_n P(X_n \leq x) \leq P(X \leq x + \epsilon)$. By letting $\epsilon \searrow 0$, we get

$$\limsup_n P(X_n \leq x) \leq P(X \leq x).$$

$$\begin{aligned}
\text{(ii) } P(X_n > x) &= P(X_n > x, |X_n - X| \leq \epsilon) + P(X_n > x, |X_n - X| > \epsilon) \\
&\leq P(X > x - \epsilon) + P(|X_n - X| > \epsilon).
\end{aligned}$$

Then,

$$\begin{aligned}
\limsup_n P(X_n > x) &\leq P(X > x - \epsilon) \\
&\Rightarrow \limsup_n [1 - P(X_n \leq x)] \leq 1 - P(X \leq x - \epsilon) \\
&\Rightarrow \liminf_n P(X_n \leq x) \geq P(X \leq x - \epsilon).
\end{aligned}$$

For $x \in C(F_X)$, by letting $\epsilon \searrow 0$, we get

$$\liminf_n P(X_n \leq x) \geq P(X \leq x).$$

By (i), (ii), for $x \in C(F_X)$,

$$F_n(x) = P(X_n \leq x) \rightarrow P(X \leq x) = F(x).$$

Thus, $X_n \Rightarrow X$.

$$(2) \text{ Since } X_n \Rightarrow c, P(X_n \leq y) \rightarrow \begin{cases} 0, & y < c \\ 1, & y > c \end{cases}. \text{ Thus, for any } \epsilon > 0,$$

$$\begin{aligned}
P(|X_n - c| > \epsilon) &= P(X_n > c + \epsilon) + P(X_n < c - \epsilon) \\
&= 1 - P(X_n \leq c + \epsilon) + P(X_n < c - \epsilon) \\
&\rightarrow 0.
\end{aligned}$$

Exercise 3.2.13. Since $\begin{cases} X_n \Rightarrow X, & P(X_n \leq x) \rightarrow P(X \leq x), \quad \forall x \in C(F_X). \\ Y_n \xrightarrow{P} c, & P(|Y_n - c| > \epsilon) \rightarrow 0, \quad \forall \epsilon > 0. \end{cases}$

$$\begin{aligned}
\text{(i) } P(X_n + Y_n \leq z) &= P(X_n + Y_n \leq z, |Y_n - c| \leq \epsilon) + P(X_n + Y_n \leq z, |Y_n - c| > \epsilon) \\
&\geq P(X_n \leq z - c + \epsilon) + P(|Y_n - c| > \epsilon).
\end{aligned}$$

Since $D(F_X)$ is at most countable, we may assume that $z - c + \epsilon \in C(F_X)$. Then, $\limsup_n P(X_n + Y_n \leq z) \leq P(X \leq z - c + \epsilon)$. Letting $\epsilon \searrow 0$,

$$\limsup_n P(X_n + Y_n \leq z) \leq P(X \leq z - c) = P(X + c \leq z).$$

$$\begin{aligned}
\text{(ii) } P(X_n + Y_n > z) &= P(X_n + Y_n > z, |Y_n - c| \leq \epsilon) + P(X_n + Y_n > z, |Y_n - c| > \epsilon) \\
&\leq P(X_n > z - c - \epsilon) + P(|Y_n - c| > \epsilon).
\end{aligned}$$

By similar argument, we get

$$\liminf_n P(X_n + Y_n \leq z) \geq P(X \leq z - c - \epsilon).$$

For $z - c \in C(F_X)$, by letting $\epsilon \rightarrow 0$,

$$\liminf_n P(X_n + Y_n \leq z) \geq P(X \leq z - c) = P(X + c \leq z).$$

Note that $z \in C(F_{X+c})$ if and only if $z - c \in C(F_X)$. By (i), (ii), we get

$$P(X_n + Y_n \leq z) \rightarrow P(X + c \leq z).$$

Exercise 3.2.14. Since $\begin{cases} X_n \Rightarrow X, & P(X_n \leq x) \rightarrow P(X \leq x), \quad \forall x \in C(F_X). \\ Y_n \xrightarrow{P} c, & P(|Y_n - c| > \epsilon) \rightarrow 0, \quad \forall \epsilon > 0. \end{cases}$

For simple proof, let's assume $0 < \epsilon < c$.

$$\begin{aligned} \text{(i)} \quad P(X_n Y_n \leq z) &= P(X_n Y_n \leq z, |Y_n - c| \leq \epsilon) + P(X_n Y_n \leq z, |Y_n - c| > \epsilon) \\ &\leq P\left(X_n \leq \frac{z}{c - \epsilon}\right) + P(|Y_n - c| > \epsilon). \end{aligned}$$

Let $\frac{z}{c - \epsilon} \in C(F_X)$. Then,

$$\limsup_n P(X_n Y_n \leq z) \leq P\left(X \leq \frac{z}{c - \epsilon}\right).$$

Letting $\epsilon \rightarrow 0$,

$$\limsup_n P(X_n Y_n \leq z) \leq P\left(X \leq \frac{z}{c}\right) = P(cX \leq z).$$

(ii) $P(X_n Y_n > z) = P(X_n Y_n > z, |Y_n - c| \leq \epsilon) + P(X_n Y_n > z, |Y_n - c| > \epsilon)$ Similarly,

$$\leq P\left(X_n > \frac{z}{c + \epsilon}\right) + P(|Y_n - c| > \epsilon).$$

$$\liminf_n P(X_n Y_n \leq z) \geq P\left(X \leq \frac{z}{c + \epsilon}\right).$$

For $\frac{z}{c} \in C(F_X)$, by letting $\epsilon \rightarrow 0$,

$$\liminf_n P(X_n Y_n \leq z) \geq P\left(X \leq \frac{z}{c}\right) = P(cX \leq z).$$

Note that $z \in C(F_{cX})$ if and only if $\frac{z}{c} \in C(F_X)$. By (i),(ii), we get

$$P(X_n Y_n \leq z) \rightarrow P(cX \leq z).$$

Exercise 3.2.15. Step 1. $X_n^i \stackrel{d}{=} Y_i / (\sum_{m=1}^n Y_m^2 / n)^{1/2}$.

Note that uniformly distributed over the surface of the unit sphere is the only distribution which is invariant to rotation and belongs to unit sphere.

Since $Y_i \stackrel{iid}{\sim} N(0, 1)$, $Y = (Y_1, \dots, Y_n) \sim N(\mathbf{0}, I_n)$. For any orthogonal matrix A , $AY \sim N(\mathbf{0}, I_n) \stackrel{d}{=} Y$. Then, $\frac{Y}{\|Y\|_2} \stackrel{d}{=} \frac{AY}{\|AY\|_2}$. Thus, $\frac{Y}{\|Y\|_2} = (Y_1 / \sum_{m=1}^n Y_m^2, \dots, Y_n / \sum_{m=1}^n Y_m^2)$ is uniformly distributed over the surface of the unit sphere. Then, $X_n \stackrel{d}{=} \sqrt{n} \frac{Y}{\|Y\|_2}$, and thus

$$X_n^i \stackrel{d}{=} Y_i / \left(\sum_{m=1}^n Y_m^2 / n \right)^{1/2}.$$

Step 2. $Y_i/(\sum_{m=1}^n Y_m^2/n)^{1/2} \Rightarrow N(0, 1)$.

For $Y_i \sim N(0, 1)$, $EY_i^2 = 1 < \infty$. Thus, by WLLN,

$$\sum_{m=1}^n Y_m^2 / n \xrightarrow{P} 1.$$

By Exercise 3.2.12(1), $\sum_{m=1}^n Y_m^2/n \Rightarrow 1$. Note that $f(x) = \frac{1}{\sqrt{x}}I_{(x>0)}$ is continuous on $\mathbb{R} \setminus \{0\}$ and $P(\sum_{m=1}^n Y_m^2/n = 0) = P(Y_1 = \dots = Y_n = 0) = 0$. By Continuous Mapping Theorem (Theorem 3.2.10), we get

$$\left(\sum_{m=1}^n Y_m^2 / n \right)^{-1/2} \Rightarrow 1.$$

By Exercise 3.2.14,

$$Y_i \cdot \left(\sum_{m=1}^n Y_m^2 / n \right)^{-1/2} \Rightarrow N(0, 1).$$

Thus, $X_n^1 \Rightarrow$ standard normal.

Exercise 3.2.16. For $1 < M < \infty$, since $EY_n^\beta \rightarrow 1$, there exists $N \in \mathbb{N}$ such that for $n > N$, $EY_n^\beta \leq M$. Therefore, we can drop finitely many Y_n 's and say $\{F_n\}$ is tight. (with $\varphi(y) = |y|^\beta$)

For every subsequence (F_{n_k}) , by Helly and Tightness Theorem, there exists $(F_{n(m_k)})$ such that $F_{n(m_k)} \Rightarrow F$, where F is a df. Let Y be a r.v. with F .

For $g(y) = |y|^\beta$ and $h(y) = |y|^\alpha$, we have

(0) g, h are continuous.

(i) $g(y) > 0$

(ii) $|h(y)|/g(y) = |y|^{\alpha-\beta} \rightarrow 0$ as $|y| \rightarrow \infty$

(iii) $F_{n(m_k)} \Rightarrow F$

(iv) $\int |y|^\beta dF_{n(m_k)}(y) = EY_{n(m_k)}^\beta \leq M < \infty$.

By Exercise 3.2.5,

$$\int |y|^\alpha dF_{n(m_k)}(y) = EY_{n(m_k)}^\alpha \rightarrow EY^\alpha = \int |y|^\alpha dF(y).$$

Since convergent real sequence has the unique limit, $EY^\alpha = 1$. Also, similar argument for $h(y) = |y|^\gamma$ with $\gamma \in (\alpha, \beta)$ gives us that

$$EY_{n(m_k)}^\gamma \rightarrow EY^\gamma.$$

From Jensen's Inequality with following functions, we get $EY_n^\gamma \rightarrow 1$.

(1) $\varphi_1(y) = |y|^{\frac{\gamma}{\alpha}}$.

$$(EY_n^\alpha)^{\frac{\gamma}{\alpha}} \leq E(Y_n^\alpha)^{\frac{\gamma}{\alpha}} = EY_n^\gamma \Rightarrow 1 \leq \liminf_n EY_n^\gamma.$$

$$(2) \quad \varphi_2(y) = |y|^{\frac{\beta}{\gamma}}.$$

$$(EY_n^\gamma)^{\frac{\beta}{\gamma}} \leq E(Y_n^\gamma)^{\frac{\beta}{\gamma}} = EY_n^\gamma \quad \Rightarrow \quad \limsup_n EY_n^\gamma \leq 1.$$

Therefore, $EY^\gamma = 1$.

For strictly convex $\varphi(y) = |y|^{\frac{\gamma}{\alpha}}$, we have $(EY^\alpha)^{\frac{\gamma}{\alpha}} = 1 = E(Y^\alpha)^{\frac{\gamma}{\alpha}} = EY^\gamma$. By Exercise 1.6.1, $Y^\alpha = 1$ a.s. Since $\alpha > 0$, $Y = 1$ a.s. Since $Y_{n(m_k)} \xrightarrow{P} 1$, by Theorem 2.3.2,

$$Y_n \xrightarrow{P} 1.$$

Exercise 3.2.17. Suppose not. Then, for some $K < \infty$ and $y < 1$, there exists a r.v. X with $EX^2 = 1$ and $EX^4 \leq K$ such that $P(|X| > y) < c$ for all $c > 0$. With $c = \frac{1}{n}$, $n \in \mathbb{N}$, we can construct a sequence (X_n) such that $P(|X_n| > y) \rightarrow 0$, $EX_n^2 = 1$ and $EX_n^4 \leq K$. Theorem 3.2.14 with $\varphi(x) = x^2$ gives us that $\{X_n\}$ is tight. Then, by Helly and Tightness Theorem, there exists (X_{n_m}) such that $X_{n_m} \Rightarrow X_\infty$. By Generalized DCT with $h(x) = x^2$ and $g(x) = x^4$, we get $EX_\infty^2 = \lim_{m \rightarrow \infty} EX_{n_m}^2 = 1$. However, since $P(|X_n| > y) \rightarrow 0$, $P(|X_n| \leq y) \rightarrow 1$. Then, $EX_\infty^2 \leq y^2 < 1$, which is a contradiction.

3.3. Characteristic Functions.

Exercise 3.3.1. Let X be a r.v. with ch.f φ and X' be an independent copy of $-X$. Then, its ch.f $\varphi'(t) = Ee^{it(-X)} = \overline{\varphi(t)}$. By Theorem 3.3.2 and Lemma 3.3.9 in Textbook, $\text{Re } \varphi$ is ch.f of $\frac{1}{2}(X + X')$ and $|\varphi|^2$ is ch.f of $X + X'$.

Exercise 3.3.2. (i) Let

$$\begin{aligned} I_T &= \frac{1}{2T} \int_{-T}^T e^{-ita} \varphi(t) dt \\ &= \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} e^{-ita} e^{itx} \mu(dx) dt \\ &= \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} e^{it(x-a)} \mu(dx) dt \end{aligned}$$

Since $|e^{it(x-a)}| \leq 1$ and $[-T, T]$ is a finite interval, by Fubini,

$$I_T = \frac{1}{2T} \int_{-\infty}^{\infty} \int_{-T}^T \cos(t(x-a)) + i \sin(t(x-a)) dt \mu(dx).$$

Since $\sin t$ is symmetric about the origin,

$$I_T = \frac{1}{2T} \int_{-\infty}^{\infty} \int_{-T}^T \cos(t(x-a)) dt \mu(dx)$$

Since $|\int_0^x \cos(y) dy| = |\sin(y)| \leq |x|$, $\left| \frac{1}{2T} \int_{-T}^T \cos(t(x-a)) dt \right| \leq 1$: integrable. By DCT,

$$\lim_{T \rightarrow \infty} I_T = \int_{-\infty}^{\infty} \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos(t(x-a)) dt \right] \mu(dx).$$

(1) $x \neq a$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos(t(x-a)) dt = \lim_{T \rightarrow \infty} \frac{\sin((x-a)T)}{(x-a)T} = 0.$$

(2) $x = a$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos(t(x-a)) dt = 1.$$

By (1) and (2), we get

$$\lim_{T \rightarrow \infty} I_T = \mu(\{a\}).$$

(ii) Let μ be the distribution of X .

$$\begin{aligned} \varphi\left(\frac{2\pi}{h} + t\right) &= \int e^{i(\frac{2\pi}{h} + t)x} \mu(dx) \\ &= \int_{h\mathbb{Z}} e^{i(\frac{2\pi}{h} + t)x} \mu(dx) \\ &= \int_{h\mathbb{Z}} e^{itx} \mu(dx) \\ (2a) \quad &= \int e^{itx} \mu(dx) = \varphi(t). \end{aligned}$$

By (i),

$$P(X = x) = \mu(\{x\}) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-itx} \varphi(t) dt.$$

Let $T_n = \frac{\pi}{h} + \frac{2\pi}{h} \cdot n$, with $n \geq 0$. Then, since limit of RHS above exists,

$$P(X = x) = \lim_{n \rightarrow \infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} e^{-itx} \varphi(t) dt.$$

Now, for $1 \leq k \in \mathbb{N}$,

$$\begin{aligned} \int_{T_{k-1}}^{T_k} e^{-itx} \varphi(t) dt &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i(t + \frac{2k\pi}{h})x} \cdot \varphi\left(t + \frac{2k\pi}{h}\right) dt \\ &= e^{-i(\frac{2k\pi x}{h})} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-it} \varphi(t) dt \end{aligned}$$

by (2a). Since $x \in h\mathbb{Z}$, $\frac{2k\pi x}{h} \in \mathbb{N}$. Thus, $e^{-i(\frac{2k\pi x}{h})} = 1$. Therefore,

$$\begin{aligned} \int_{T_{k-1}}^{T_k} e^{-itx} \varphi(t) dt &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-it} \varphi(t) dt \\ &= \int_{-T_0}^{T_0} e^{-itx} \varphi(t) dt. \end{aligned}$$

This also holds for $\int_{-T_k}^{-T_{k-1}} e^{-itx} \varphi(t) dt = \int_{-T_0}^{T_0} e^{-itx} \varphi(t) dt$. Then,

$$\begin{aligned} \frac{1}{2T_n} \int_{-T_n}^{T_n} e^{-itx} \varphi(t) dt &= \frac{1}{2 \cdot \frac{(2n+1)}{h} \pi} \cdot (2n+1) \int_{-T_0}^{T_0} e^{-itx} \varphi(t) dt \\ &= \frac{h}{2\pi} \int_{-T_0}^{T_0} e^{-itx} \varphi(t) dt \\ &= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} \varphi(t) dt. \end{aligned}$$

Therefore,

$$P(X = x) = \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} \varphi(t) dt.$$

(iii) $X = Y + b$. If $P(X \in b + h\mathbb{Z}) = 1$, then $P(Y \in h\mathbb{Z}) = 1$. For $x \in b + h\mathbb{Z}$, $P(X = x) = P(Y = x - b)$. Then,

$$\begin{aligned} P(Y = x - b) &= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-it(x-b)} \varphi_Y(t) dt \\ &= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} \cdot e^{itb} \cdot e^{-itb} \varphi_X(t) dt \\ &= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-itx} \varphi_X(t) dt. \end{aligned}$$

Exercise 3.3.3. Since X and Y are independent, X and $-Y$ are also independent. Then, ch.f of $X - Y$ is given by $\varphi(t) \cdot \overline{\varphi(t)} = |\varphi(t)|^2$. By Exercise 3.3.2,

$$P(X - Y = 0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\varphi(t)|^2 dt.$$

By Exercise 2.1.5,

$$P(X - Y = 0) = \sum_x \mu(\{x\})^2.$$

Exercise 3.3.4. For exponential distribution,

$$\begin{aligned} f(y) &= e^{-y} I_{(y>0)} \\ \varphi(t) &= \frac{1}{1 - it} \end{aligned}$$

And,

$$\int |\varphi(t)| dt = \int \left| \frac{1}{1 - it} \right| dt = \int \frac{1}{\sqrt{1 + t^2}} dt = \infty.$$

Exercise 3.3.5. ch.f of X_1 is given by

$$\begin{aligned} \varphi_1(t) &= E e^{itX} \\ &= \int_{-1}^1 e^{itx} \cdot \frac{1}{2} dx = \left[\frac{1}{2it} e^{itx} \right]_{-1}^1 \\ &= \frac{1}{2it} (e^{it} - e^{-it}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2it}(\cos t + i \sin t - \cos t + i \sin t) \\
&= \frac{\sin t}{t}.
\end{aligned}$$

Thus, ch.f of $X_1 + \cdots + X_n$ is given by $\varphi(t) = \left(\frac{\sin t}{t}\right)^n$.

Note that

$$\int |\varphi(t)| dt = 2 \int_0^\infty \left| \frac{\sin t}{t} \right|^n dt.$$

For $n \geq 2$,

$$\begin{aligned}
\int_0^\infty \left| \frac{\sin t}{t} \right|^n dt &= \int_0^1 \left| \frac{\sin t}{t} \right|^n dt + \int_1^\infty \left| \frac{\sin t}{t} \right|^n dt \\
&\leq \int_0^1 1 dt + \int_1^\infty \frac{1}{t^n} dt < \infty.
\end{aligned}$$

Therefore, by Thm 3.3.14,

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itx} \cdot \left(\frac{\sin t}{t}\right)^n dt \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \left[\left(\frac{\sin t}{t}\right)^n \cdot \cos tx - i \left(\frac{\sin t}{t}\right)^n \cdot \sin tx \right] dt \\
&= \frac{1}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^n \cos tx dt.
\end{aligned}$$

Exercise 3.3.6. In Example 3.3.16, we showed that ch.f of Cauchy distribution is given by $\exp(-|t|)$.

Since X_1, \dots, X_n are independent, by Thm 3.3.2,

$$\varphi_{X_1+\dots+X_n}(t) = \prod_{i=1}^n \varphi_{X_i}(t) = \prod_{i=1}^n \exp(-|t|) = \exp(-n|t|)$$

By Thm 3.3.1(e),

$$\varphi_{\frac{X_1+\dots+X_n}{n}}(t) = \varphi_{X_1+\dots+X_n}\left(\frac{t}{n}\right) = \exp(-|t|),$$

which is ch.f of Cauchy. Thus, by Inversion Formula, $\frac{X_1+\dots+X_n}{n} \stackrel{d}{=} X_1$.

Exercise 3.3.7. $\varphi_{X_n}(t) = \exp\left(-\frac{\sigma_n^2 t^2}{2}\right)$.

Since $X_n \Rightarrow X$, $\varphi_{X_n}(t) \rightarrow \varphi_\infty(t)$, $\forall t$. For $t = 1$, $\varphi_{X_n}(1) = \exp\left(-\frac{\sigma_n^2}{2}\right)$. Thus,

$$\begin{aligned}
\varphi_{X_n}(1) \text{ converges} &\iff \log \varphi_{X_n}(1) \text{ converges} \\
&\iff -\frac{\sigma_n^2}{2} \text{ converges.}
\end{aligned}$$

Therefore, there exists σ^2 such that $\sigma_n^2 \rightarrow \sigma^2$. Since $\sigma_n^2 \in (0, \infty)$, we can say that $\sigma^2 \in [0, \infty]$. Now, suppose $\sigma^2 = \infty$. Then $\varphi_{X_n}(t) = \exp\left(-\frac{\sigma_n^2 t^2}{2}\right) \rightarrow 0$ if $t \neq 0$. However, $\varphi_{X_n}(0) = 1$, $\forall n$. Thus, continuity of φ_X at 0 is violated, which means that $X_n \not\Rightarrow X$. Thus, $\sigma_n^2 \rightarrow \sigma^2 \in [0, \infty)$.

Exercise 3.3.8. Let $\varphi_n(t)$ be the ch.f of X_n and $\psi_n(t)$ be the ch.f of Y_n , for $1 \leq n \leq \infty$. By Theorem 3.3.17(Continuity Theorem),

$$\begin{aligned}\varphi_n(t) &\rightarrow \varphi_\infty(t), \forall t \\ \psi_n(t) &\rightarrow \psi_\infty(t), \forall t.\end{aligned}$$

Since X_n and Y_n are independent, ch.f of $X_n + Y_n = \varphi_n(t)\psi_n(t)$, for $1 \leq n \leq \infty$. Then, $\varphi_n(t)\psi_n(t) \rightarrow \varphi_\infty(t)\psi_\infty(t)$, $\forall t$. Since $\varphi_\infty, \psi_\infty$ are ch.f, they are continuous at 0. Thus, $\varphi_\infty\psi_\infty$ is also continuous at 0. By Thm 3.3.17, $X_n + Y_n \Rightarrow X_\infty + Y_\infty$.

Exercise 3.3.9. By Thm 3.3.2, ch.f of S_n is given by

$$\varphi_{S_n}(t) = \prod_{j=1}^n \varphi_j(t).$$

Since a.s. convergence implies weak convergence, $S_n \Rightarrow S_\infty$. By Thm 3.3.17(i),

$$\varphi_{S_\infty}(t) = \lim_{n \rightarrow \infty} \varphi_{S_n}(t) = \prod_{j=1}^{\infty} \varphi_j(t), \forall t.$$

Exercise 3.3.10. By Exercise 3.3.5, $\frac{\sin t}{t}$ is ch.f of $Unif(-1, 1)$. Let Y_1, \dots, Y_n be *i.i.d* symmetric Bernoulli random variables. $\left(P(Y_1 = 1) = P(Y_1 = -1) = \frac{1}{2}\right)$. Define $X_i = \frac{Y_i}{2^i}$. Then, ch.f of $X_i = \cos \frac{t}{2^i}$, and thus ch.f of $\sum_{i=1}^{\infty} X_i = \prod_{i=1}^{\infty} \cos \frac{t}{2^i}$. **Claim.** $\sum_{i=1}^n X_i \Rightarrow Unif(-1, 1)$.

$$P\left(\sum_{i=1}^n X_i = \frac{2k+1}{2^n}\right) = \frac{1}{2^n} \text{ for } k = -2^{n-1}, \dots, -1, 0, 1, \dots, 2^{n-1} - 1.$$

Then,

$$\begin{aligned}P\left(\sum_{i=1}^n X_i \leq x\right) &= \max\left[\frac{1}{2^n} \left\{\left\lfloor \frac{2^n x}{2} \right\rfloor + 2^{n-1}\right\}, 0\right] \\ &= \max\left[\frac{1}{2} + \frac{1}{2^n} \left\lfloor \frac{2^n x}{2} \right\rfloor, 0\right]\end{aligned}$$

Now, for $x \in (-1, 1)$, letting $n \rightarrow \infty$ gives us

$$P\left(\sum_{i=1}^{\infty} X_i \leq x\right) = \frac{1}{2} + \frac{1}{2}x.$$

Note that for $x \in (-1, 1)$, $P(Unif(-1, 1) \leq x) = \frac{1}{2} + \frac{1}{2}x$. Therefore, $\sum_{i=1}^n X_i \Rightarrow Unif(-1, 1)$. By Theorem 3.3.17, we get the desired result.

Exercise 3.3.11. ch.f of $\frac{2X_j}{3^j} = Ee^{it\frac{2X_j}{3^j}} = \frac{1}{2} + \frac{1}{2} \cdot e^{\frac{2it}{3^j}}$. By Exercise 3.3.9, ch.f of X is given by

$$\varphi_X(t) = \prod_{j=1}^{\infty} \left[\frac{1}{2} + \frac{1}{2}e^{\frac{2it}{3^j}}\right].$$

Then,

$$\begin{aligned}\varphi_X(3^k\pi) &= \prod_{j=1}^{\infty} \left[\frac{1}{2} + \frac{1}{2} \exp\left(\frac{2i3^k\pi}{3^j}\right) \right] \\ &= \prod_{l=1}^{\infty} \left[\frac{1}{2} + \frac{1}{2} \exp(2\pi \cdot 3^{-l}i) \right] \cdot \prod_{j=1}^k \left[\frac{1}{2} + \frac{1}{2} \exp(2\pi \cdot 3^{k-j}i) \right]\end{aligned}$$

Since $3^{k-j} \in \mathbb{N} \cup \{0\}$, we get

$$\varphi_X(3^k\pi) = \prod_{j=1}^{\infty} \left[\frac{1}{2} + \frac{1}{2} \exp(2\pi 3^{-l}i) \right].$$

Exercise 3.3.12. We have $\int_{-\infty}^{\infty} |x|e^{-\frac{1}{2}x^2} dx = 2 \int_0^{\infty} xe^{-\frac{1}{2}x^2} dx < \infty$. By induction, for $n \geq 2$,

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^n e^{-\frac{1}{2}x^2} dx &= 2 \int_0^{\infty} x^n e^{-\frac{1}{2}x^2} dx \\ &= 2 \left[-x^{n-1} e^{-\frac{1}{2}x^2} \right]_0^{\infty} + 2 \int_0^{\infty} x^{n-1} e^{-\frac{1}{2}x^2} dx \\ &= 0 + 2 \int_0^{\infty} x^{n-1} e^{-\frac{1}{2}x^2} dx < \infty.\end{aligned}$$

Note that

$$\begin{aligned}\varphi^{(n)}(t) &= \exp^{-\frac{1}{2}t^2} \\ &= 1 - \frac{1}{2}t^2 + \frac{1}{2!} \cdot \left(-\frac{1}{2}t^2\right)^2 + \frac{1}{3!} \cdot \left(-\frac{1}{2}t^2\right)^3 + \dots \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k! \cdot (-2)^k} t^{2k}.\end{aligned}$$

By Thm 3.3.18, we have $i^n EX^n = \varphi^{(n)}(t)|_{t=0}$. Therefore,

$$EX^{2n} = \frac{(2n)!}{2^n n!}.$$

Exercise 3.3.13. (i) Fix $\epsilon > 0$. For $0 < h < \frac{\epsilon^2}{4}$,

$$\begin{aligned}|\varphi_i(t+h) - \varphi_i(t)| &\leq E|e^{ihX_i} - 1| \\ &\leq E \min(|hX_i|, 2) \\ &\leq E \left(|hX_i|; |X_i| \leq \frac{2}{\sqrt{h}} \right) + E \left(2; |X_i| > \frac{2}{\sqrt{h}} \right) \\ &\leq 2\sqrt{h} + 2P \left(|X_i| > \frac{2}{\sqrt{h}} \right) \\ &\leq 2\sqrt{h} + \epsilon \\ &\leq 2\epsilon.\end{aligned}$$

The first inequality comes from Theorem 3.3.1(d). The second inequality comes from (3.3.3). The fifth inequality comes from tightness.

(ii) Since $\mu_n \Rightarrow \mu_\infty$, $\{\mu_n\}_{n \in \mathbb{N}}$ is tight, and thus $\{\mu_n\}_{1 \leq n \leq \infty}$ is tight.

Then, by (i), for $\epsilon > 0$, choose $\delta > 0$ so that

$$\text{if } |h| < \delta, \text{ then } |\varphi_n(t+h) - \varphi_n(t)| < \epsilon, \text{ for } 1 \leq n \leq \infty.$$

Without loss of generality, assume the compact set is $[-K, K]$.

Choose integer $m > \frac{1}{\delta}$ and let $x_j = \frac{j}{m} \cdot K$ for $-m \leq j \leq m$. By Continuity Theorem, we have $\varphi_n(t) \rightarrow \varphi_\infty(t)$ for all t . Thus, for x_j , there exists $n_j \in \mathbb{N}$ such that if $n > n_j$, then $|\varphi_n(x_j) - \varphi_\infty(x_j)| < \epsilon$. Let $n_0 = \max\{n_{-m}, \dots, n_m\}$. Then, for all $t \in [-K, K]$, there exists j such that $\left|t - \frac{jK}{m}\right| < \delta$. Now, for $n > n_0$,

$$\begin{aligned} |\varphi_n(t) - \varphi_\infty(t)| &\leq \left| \varphi_n(t) - \varphi_n\left(\frac{jK}{m}\right) \right| + \left| \varphi_n\left(\frac{jK}{m}\right) - \varphi_\infty\left(\frac{jK}{m}\right) \right| + \left| \varphi_\infty\left(\frac{jK}{m}\right) - \varphi_\infty(t) \right| \\ &< 3\epsilon. \end{aligned}$$

Therefore, $\varphi_n \rightarrow \varphi_\infty$ uniformly on a compact set.

(iii) Let $X_n = \frac{1}{n}$. Then, $\varphi_n(t) = e^{\frac{it}{n}}$

Exercise 3.3.14. (i) ch.f of $\begin{cases} \frac{S_n}{n} = \varphi\left(\frac{t}{n}\right)^n \\ a = e^{iat} \end{cases}$.

Claim. $\varphi\left(\frac{t}{n}\right) \rightarrow e^{iat}$, for all t .

Proof. Since $\varphi'(0) = ia$,

$$\lim_{n \rightarrow \infty} \frac{\varphi\left(\frac{t}{n}\right) - \varphi(0)}{t/n} = ia \Rightarrow \lim_{n \rightarrow \infty} n \left(\varphi\left(\frac{t}{n}\right) - 1 \right) = iat.$$

Thus, for all t ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi\left(\frac{t}{n}\right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{n(\varphi\left(\frac{t}{n}\right) - 1)}{n} \right)^n \\ &= e^{iat}. \end{aligned}$$

The second equality holds by Theorem 3.4.2. □

By *Claim* and Theorem 3.3.17, $\frac{S_n}{a} \Rightarrow a$, and thus $\frac{S_n}{a} \xrightarrow{P} a$.

(ii) In probability convergence implies weak convergence. By Theorem 3.3.17, we get the desired result.

(iii) (I think $(\varphi(h) - 1)/h \rightarrow +ia$.)

(a) Since $\log z$ is differentiable at $z = 1$,

$$\lim_{n \rightarrow \infty} \frac{\log \varphi\left(\frac{t}{n}\right) - \log 1}{\varphi\left(\frac{t}{n}\right) - 1} = \lim_{n \rightarrow \infty} \frac{\log \varphi\left(\frac{t}{n}\right)}{\varphi\left(\frac{t}{n}\right) - 1} = (\log z)'|_{z=1} = 1.$$

(b) By (ii), $\varphi\left(\frac{t}{n}\right)^n \rightarrow e^{iat}$. By taking logarithm, we get

$$n \log \varphi\left(\frac{t}{n}\right) \rightarrow iat.$$

By (a), (b), we get

$$\begin{aligned} n \left(\varphi \left(\frac{t}{n} \right) - 1 \right) &= \frac{\varphi(\frac{t}{n}) - 1}{\log \varphi(\frac{t}{n})} \cdot n \log \varphi \left(\frac{t}{n} \right) \\ &\rightarrow iat. \end{aligned}$$

By Exercise 3.3.13, $\frac{S_n}{n} \Rightarrow a$ implies that $\varphi(\frac{t}{n})^n \rightarrow e^{iat}$ uniformly on a compact set. Fix small $0 < \delta < 1$. For any $h > 0$, we can express $h = \frac{t}{n}$ with large n and $t \in [1 - \delta, 1 + \delta]$. Then,

$$\begin{aligned} \left| \frac{\varphi(h) - 1}{h} - ia \right| &= \left| \frac{\varphi(\frac{t}{n}) - 1}{t/n} - ia \right| \\ &= \left| \frac{1}{t} (n(\varphi(t/n) - 1) - iat) \right| \\ &\leq \sup_{t' \in [1 - \delta, 1 + \delta]} \left| \frac{1}{t'} \right| \cdot |n(\varphi(t'/n) - 1) - iat'| \\ &\leq \frac{1}{1 - \delta} \sup_{t' \in [1 - \delta, 1 + \delta]} |n(\varphi(t'/n) - 1) - iat'|. \end{aligned}$$

Since $n(\varphi(t/n) - 1) \rightarrow iat$ uniformly on compact set,

$$\limsup_{h \rightarrow 0} \left| \frac{\varphi(h) - 1}{h} - ia \right| \leq \frac{1}{1 - \delta} \limsup_{n \rightarrow \infty} \left[\sup_{t \in [1 - \delta, 1 + \delta]} |n(\varphi(t/n) - 1) - iat| \right] = 0.$$

Therefore, $\frac{\varphi(h) - 1}{h} \rightarrow ia$ as $h \downarrow 0$.

Exercise 3.3.15. From Example 3.3.15, $\int_{-\infty}^{\infty} \frac{1 - \cos x}{\pi x^2} dx = 1$.

Let $x = |y|t$. Then,

$$\int_{-\infty}^{\infty} \frac{1 - \cos |y|t}{\pi |y|^2 t^2} |y| dt = \frac{1}{|y|} \int_{-\infty}^{\infty} \frac{1 - \cos yt}{\pi t^2} dt = 1.$$

Thus, $2 \int_0^{\infty} \frac{1 - \cos yt}{\pi t^2} dt = |y|$.

Now, integrate with respect to $F(y)$. Then, we get

$$2 \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1 - \cos yt}{\pi t^2} dt dF(y) = \int_{-\infty}^{\infty} |y| dF(y).$$

Since $\frac{1 - \cos yt}{\pi t^2} \geq 0$, by Fubini,

$$2 \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1 - \cos yt}{\pi t^2} dF(y) dt = 2 \int_0^{\infty} \frac{1 - \operatorname{Re}(\varphi(t))}{\pi t^2} dt = \int_{-\infty}^{\infty} |y| dF(y).$$

Exercise 3.3.16. Since $\lim_{t \downarrow 0} \frac{\varphi(t) - 1}{t^2} = c > -\infty$,

$$\lim_{t \downarrow 0} \overline{\left(\frac{\varphi(t) - 1}{t^2} \right)} = \lim_{t \downarrow 0} \frac{\overline{\varphi(t)} - 1}{t^2} = \lim_{t \downarrow 0} \frac{\varphi(-t) - 1}{t^2} = c.$$

Thus,

$$\lim_{t \downarrow 0} \frac{\varphi(t) + \varphi(-t) - 2\varphi(0)}{t^2} = 2c > -\infty.$$

By Theorem 3.3.21, $EX^2 < \infty$.

By Theorem 3.3.20, $\varphi(t) = 1 + itEX - \frac{t^2}{2}EX^2 + o(t^2)$.

Then,

$$\begin{aligned}\lim_{t \downarrow 0} \frac{\varphi(t) - 1}{t^2} &= \lim_{t \downarrow 0} \frac{itEX - \frac{t^2}{2}EX^2 + o(t^2)}{t^2} \\ &= \lim_{t \downarrow 0} \frac{iEX}{t} - \frac{1}{2}EX^2 \\ &= c > -\infty.\end{aligned}$$

Thus, $EX = 0$ and $EX^2 = -2c < \infty$.

Especially, if $\varphi(t) = 1 + o(t^2)$, we get $\lim_{t \downarrow 0} \frac{\varphi(t) - 1}{t^2} = \lim_{t \downarrow 0} \frac{o(t^2)}{t^2} = 0$. Then, $EX = 0$ and $EX^2 = 0$. Thus, $\text{Var}(X) = 0$, which means that X is constant a.s. Therefore, $X = 0$ a.s.

Thus, $\varphi(t) = Ee^{itX} = 1$.

Exercise 3.3.17.

(\Rightarrow) By Continuity Theorem, $\varphi_n(t) \rightarrow \varphi_0(t)$, for all t , where φ_0 is ch.f of 0. Since $\varphi_0(t) = Ee^{it0} = 1$, $\varphi_n(t) \rightarrow 1$, for all t .

(\Leftarrow) **Step 1.** $\{Y_n\}$ is tight.

Let $f_n(t) = \frac{1}{\alpha}(1 - \varphi_n(t))I_{[-\alpha, \alpha]}(t)$ for $\alpha < \delta$. Then, $\begin{cases} |f_n(t)| \leq \frac{2}{\alpha}I_{[-\alpha, \alpha]}(t) : \text{integrable} \\ f_n(t) \rightarrow 0 \text{ by assumption} \end{cases}$.

By DCT, we have

$$\int f_n(t) dt = \frac{1}{\alpha} \int_{-\alpha}^{\alpha} (1 - \varphi_n(t)) dt \rightarrow 0.$$

Note that

$$\begin{aligned}\frac{1}{u} \int_{-u}^u (1 - \varphi_n(t)) dt &= \frac{1}{u} \int_{-u}^u \left[1 - \int_{-\infty}^{\infty} e^{ity} d\mu_n(y) \right] dt \\ &= \frac{1}{u} \int_{-\infty}^{\infty} \int_{-u}^u 1 - e^{ity} dt d\mu_n(y) \\ &= \frac{1}{u} \int_{-\infty}^{\infty} \left[t - \frac{1}{iy} e^{ity} \right]_{-u}^u d\mu_n(y) \\ &= \frac{1}{u} \int_{-\infty}^{\infty} \left[\left(u - \frac{1}{iy} (\cos uy + i \sin uy) \right) \right. \\ &\quad \left. - \left(-u - \frac{1}{iy} (\cos uy - i \sin uy) \right) \right] d\mu_n(y) \\ &= \frac{1}{u} \int_{-\infty}^{\infty} \left[2u - \frac{2}{y} \sin uy \right] d\mu_n(y) \\ &= 2 \int_{-\infty}^{\infty} \left[1 - \frac{\sin uy}{uy} \right] d\mu_n(y) \\ &\geq 2 \int_{|y| \geq \frac{2}{u}} \left[1 - \frac{1}{u|y|} \right] d\mu_n(y)\end{aligned}$$

$$\begin{aligned}
&\geq \int_{|y| \geq \frac{2}{u}} 1 d\mu_n(y) \\
&= \mu_n \left\{ y : |y| \geq \frac{2}{u} \right\} \\
&\geq \mu_n \left\{ y : |y| > \frac{2}{u} \right\}.
\end{aligned}$$

The second equality holds by Fubini. The first inequality holds since $\frac{\sin uy}{uy} \leq 1$ and $\frac{\sin uy}{uy} \leq \frac{1}{u|y|}$.

Step 2. $Y_n \Rightarrow 0$.

Fix a subsequence $\{Y_{n_k}\}$. By Helly and Tightness Theorem, there exists sub-subsequence $\{Y_{n_{m(k)}}\}$ such that $Y_{n_{m(k)}} \Rightarrow X$. Since $\varphi_{n_{m(k)}} \rightarrow 1$ for $|t| < \delta$, $\varphi_X(t) = 1$ for $|t| < \delta$. Then, for $|t| < \delta$, $\frac{\varphi_X(t)-1}{t^2} = 0$, and thus

$$\lim_{t \downarrow 0} \frac{\varphi(t) - 1}{t^2} = 0.$$

By Exercise 3.3.16, $\varphi_X(t) \equiv 1$. This means that $X \equiv 0$. Since this holds for all subsequence, by Theorem 3.2.15, $Y_n \Rightarrow 0$.

Exercise 3.3.18. Step 1. $S_m - S_n \xrightarrow{P} 0$.

Let $\varphi(t)$ be ch.f of X_1 . Then, ch.f of S_n is given by $\varphi(t)^n$ and it converges to for some ch.f $\psi(t)$. Since ψ is ch.f, there exists $\delta > 0$ such that $\psi(t) > 0$ for $|t| \leq \delta$. Now, suppose $m > n$. ch.f of $S_m - S_n = \sum_{l=n+1}^m X_l$ is given by

$$\varphi(t)^{m-n} = \frac{\psi(t)^m}{\psi(t)^n}.$$

For $|t| \leq \delta$, as $m, n \rightarrow \infty$, $\varphi(t)^{m-n} \rightarrow 1$.

By Exercise 3.3.17, $S_m - S_n \Rightarrow 0$.

By Exercise 3.2.12, $S_m - S_n \xrightarrow{P} 0$.

Step 2. S_n converges in prob.

By Exercise 2.3.6,

$$d(S_n - S_m, 0) \rightarrow 0 \iff d(S_n, S_m) \rightarrow 0.$$

By Exercise 2.3.7, there exists a r.v. S_∞ such that $S_n \xrightarrow{P} S_\infty$.

Exercise 19-25 (★)

Exercise 3.3.19.

Exercise 3.3.20.

Exercise 3.3.21.

Exercise 3.3.22.

Exercise 3.3.23.

Exercise 3.3.24.

Exercise 3.3.25.

3.4. Central Limit Theorems.

Exercise 3.4.1. $\Phi(-1)$.

Exercise 3.4.2. (a) Let $\sigma^2 = \text{Var}(X_i) \in (0, \infty)$. By CLT, we have $\frac{S_n}{\sqrt{n}\sigma} \Rightarrow Z \sim N(0, 1)$.

Then, for any M ,

$$\limsup_n P\left(\frac{S_n}{\sqrt{n}} > M\right) = \limsup_n P\left(\frac{S_n}{\sqrt{n}\sigma} > \frac{M}{\sigma}\right) = P(Z > M/\sigma) > 0.$$

Therefore,

$$P\left(\frac{S_n}{\sqrt{n}} > M \text{ i.o.}\right) \geq \limsup_n P\left(\frac{S_n}{\sqrt{n}} > M\right) > 0.$$

Note that $\left\{\limsup_n \frac{S_n}{\sqrt{n}} > M\right\} \in \mathcal{T}$: tail σ -algebra.

By Kolmogorov's 0-1 law, $P\left(\limsup_n \frac{S_n}{\sqrt{n}} > M\right) = 1$, for all M .

Thus, $\limsup_n \frac{S_n}{\sqrt{n}} = \infty$ a.s.

(b) Assume that $\frac{S_n}{\sqrt{n}} \xrightarrow{P} U$ for some U .

Then, as $m \rightarrow \infty$,

$$\left|\frac{S_m!}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| \xrightarrow{P} 0.$$

Since $S_m!$ and $S_{(m+1)!} - S_m!$ are independent, we get

$$\begin{aligned} P\left(1 < \frac{S_m!}{\sqrt{m!}} < 2, \frac{S_{(m+1)!} - S_m!}{\sqrt{(m+1)!}} < -3\right) &= P\left(1 < \frac{S_m!}{\sqrt{m!}} < 2\right) P\left(\frac{S_{(m+1)!} - S_m!}{\sqrt{(m+1)!}} < -3\right) \\ (2a) \quad &\rightarrow P(1 < T < 2)P(T < -3) > 0, \end{aligned}$$

where $T \sim N(0, \sigma^2)$. For the second term, $\frac{1}{\sqrt{(m+1)!}}S_m! = \frac{1}{\sqrt{(m+1)!}} \cdot \frac{S_m!}{\sqrt{m!}} \xrightarrow{P} 0$.

Also,

$$\begin{aligned} P\left(1 < \frac{S_m!}{\sqrt{m!}} < 2, \frac{S_{(m+1)!} - S_m!}{\sqrt{(m+1)!}} < -3\right) &= P\left(1 < \frac{S_m!}{\sqrt{m!}} < 2, \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} < -3 + \frac{S_m!}{\sqrt{(m+1)!}}\right) \\ &\leq P\left(1 < \frac{S_m!}{\sqrt{m!}} < 2, \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} < -3 + \frac{S_m!}{\sqrt{m!}}\right) \\ &\leq P\left(1 < \frac{S_m!}{\sqrt{m!}}, \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} < -1\right) \\ (2b) \quad &\leq P\left(\left|\frac{S_m!}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| > 2\right). \end{aligned}$$

By combining (2a) and (2b), we get

$$\liminf_m P\left(\left|\frac{S_m!}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| > 2\right) > 0,$$

which contradicts to the assumption that $\frac{S_m!}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} \xrightarrow{P} 0$.

Exercise 3.4.3.

$$\begin{cases} X'_i : \text{independent copy of } X_i \\ Y_i = X_i - X'_i \\ U_i = Y_i \mathbf{1}_{(|Y_i| \leq A)} \\ V_i = Y_i \mathbf{1}_{(|Y_i| > A)} \end{cases}$$

We know that $(X_i), (X'_i), (Y_i), (U_i), (V_i)$ are *i.i.d* sequences. Suppose $EX_i^2 = \infty$.

Since there exists some r.v. W such that $\frac{S_n}{\sqrt{n}} \Rightarrow W$, $S'_n = \sum_{i=1}^n X'_i$ also weakly converges to W . That is, for all t , $\varphi_W(t) = \lim_{n \rightarrow \infty} \varphi_{S_n/\sqrt{n}}(t) = \lim_{n \rightarrow \infty} \varphi_{S'_n/\sqrt{n}}(t)$. Then, ch.f of $\frac{\sum_{m=1}^n Y_m}{\sqrt{n}}$ is given by

$$\begin{aligned} \varphi_{\frac{\sum_{m=1}^n Y_m}{\sqrt{n}}}(t) &= E \exp \left(it \left(\frac{S_n}{\sqrt{n}} - \frac{S'_n}{\sqrt{n}} \right) \right) \\ &= \varphi_{\frac{S_n}{\sqrt{n}}}(t) \cdot \varphi_{\frac{S'_n}{\sqrt{n}}}(-t) \\ &\rightarrow \varphi_W(t) \cdot \varphi_W(-t), \quad \forall t. \end{aligned}$$

Since $\varphi_W(t) \cdot \varphi_W(-t)$ is continuous at 0, by Continuity Theorem, there exists a r.v. T such that $\frac{\sum_{m=1}^n Y_m}{\sqrt{n}} \Rightarrow T$. By hint, for any K , which is a continuity point of d.f of T ,

$$P(Y \geq K) = \lim_{n \rightarrow \infty} P \left(\frac{\sum_{m=1}^n Y_m}{\sqrt{n}} \geq K \right) \geq \frac{1}{5}.$$

This means that $\lim_{K \rightarrow \infty} P(T \geq K) \geq \frac{1}{5}$ since discontinuity points of d.f of T is at most countable, which is a contradiction. One can derive hint from CLT and MCT.

Exercise 3.4.4. Let $S_n = X_1 + \dots + X_n$. CLT for *i.i.d* sequence implies that

$$(4a) \quad \frac{S_n - ES_n}{\sqrt{n}\sigma} = \frac{S_n - n}{\sqrt{n}\sigma} \Rightarrow \chi.$$

Note that $\frac{S_n - n}{\sqrt{n}\sigma} = \frac{\sqrt{S_n} + \sqrt{n}}{\sqrt{n}\sigma} \cdot (\sqrt{S_n} - \sqrt{n})$.

By WLLN, we have $\frac{S_n}{n} \xrightarrow{P} 1$. ($EX_1 = 1, \text{Var}(X_1) < \infty$).

By Continuous Mapping Theorem,

$$(4b) \quad \frac{1}{\sigma} \left(\frac{\sqrt{S_n}}{\sqrt{n}} + 1 \right) \xrightarrow{P} \frac{2}{\sigma}.$$

By Converging Together Lemma with (4a) and (4b),

$$\sqrt{S_n} - \sqrt{n} = \left(\frac{S_n - n}{\sqrt{n}\sigma} \right) \cdot \left(\frac{1}{\sigma} \left(\frac{\sqrt{S_n}}{\sqrt{n}} + 1 \right) \right)^{-1} \Rightarrow \frac{\sigma}{2} \chi$$

Exercise 3.4.5. By CLT,

$$(5a) \quad \frac{\sum_{m=1}^n X_m}{\sqrt{n}\sigma} \Rightarrow \chi.$$

By WLLN,

$$\frac{\sum_{m=1}^n X_m^2}{n} \xrightarrow{P} \sigma^2.$$

By Continuous Mapping Theorem,

$$(5b) \quad \sqrt{\frac{\sum_{m=1}^n X_m^2}{n}} \xrightarrow{P} \sigma^2.$$

By Converging Together Lemma with (5a) and (5b),

$$\frac{\sum_{m=1}^n X_m}{(\sum_{m=1}^n X_m^2)^{1/2}} = \frac{\sum_{m=1}^n X_m}{\sqrt{n}\sigma} \cdot \sqrt{\frac{n\sigma^2}{\sum_{m=1}^n X_m^2}} \Rightarrow \chi.$$

Exercise 3.4.6. By CLT for *i.i.d* sequence, we know that $\frac{S_n}{\sigma\sqrt{n}} \Rightarrow N(0, 1)$. Thus, its subsequence $\frac{S_{a_n}}{\sigma\sqrt{a_n}} \Rightarrow N(0, 1)$. For Y_n and Z_n defined above, if one can show that $Y_n - Z_n \xrightarrow{P} 0$,

$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} = \frac{S_{a_n}}{\sigma\sqrt{a_n}} + \frac{S_{a_n} - S_{N_n}}{\sigma\sqrt{a_n}} = \frac{S_{a_n}}{\sigma\sqrt{a_n}} + (Y_n - Z_n) \Rightarrow N(0, 1)$$

by Converging Together Lemma. **Claim.** $Y_n - Z_n \xrightarrow{P} 0$.

Fix $\epsilon > 0$. Since $\frac{N_n}{a_n} \xrightarrow{P} 1$, $P\left(\left|\frac{N_n}{a_n} - 1\right| > \epsilon\right) = P(|N_n - a_n| > a_n\epsilon) \rightarrow 0$. Then,

$$\begin{aligned} \limsup_n P(|Y_n - Z_n| > \delta) &= \limsup_n \left[P(|Y_n - Z_n| > \delta, |N_n - a_n| > a_n\epsilon) + P(|Y_n - Z_n| > \delta, |N_n - a_n| \leq a_n\epsilon) \right] \\ &\leq \limsup_n P(|Y_n - Z_n| > \delta, |N_n - a_n| > a_n\epsilon) + \limsup_n P(|Y_n - Z_n| > \delta, |N_n - a_n| \leq a_n\epsilon) \\ &= \limsup_n P(|Y_n - Z_n| > \delta, |N_n - a_n| \leq a_n\epsilon). \end{aligned}$$

Note that

$$P(|Y_n - Z_n| > \delta, |N_n - a_n| \leq a_n\epsilon) = P(|S_{N_n} - S_{a_n}| > \delta \cdot \sigma\sqrt{a_n}, |N_n - a_n| \leq a_n\epsilon).$$

Since $|N_n - a_n| \leq a_n\epsilon$,

$$\begin{aligned} P\left[|S_{N_n} - S_{a_n}| > \delta\sigma\sqrt{a_n}, |N_n - a_n| \leq a_n\epsilon\right] &\leq P\left[\max_{(1-\epsilon)a_n \leq m \leq (1+\epsilon)a_n} |S_m - S_{[(1-\epsilon)a_n]}| > \frac{\delta\sigma\sqrt{a_n}}{2}\right] \\ &\leq \frac{4}{\delta^2\sigma^2a_n} \text{Var}(S_{[(1+\epsilon)a_n]} - S_{[(1-\epsilon)a_n]}) \\ &= \frac{4}{\delta^2\sigma^2a_n} \sigma^2 \left([(1+\epsilon)a_n] - [(1-\epsilon)a_n] \right) \\ &\leq \frac{8\epsilon}{\delta^2} \end{aligned}$$

The second inequality holds by Kolmogorov's Maximal Inequality.

Thus, $\limsup_n P(|Y_n - Z_n| > \delta) \leq \frac{8\epsilon}{\delta^2}$. Since ϵ was arbitrarily chosen, $P(|Y_n - Z_n| > \delta) \rightarrow 0$, for any $\delta > 0$. Therefore, $Y_n \xrightarrow{P} Z_n$.

Exercise 3.4.7. 7

Exercise 3.4.8. 8

Exercise 3.4.9. (1) $\text{Var}(S_n)/n \rightarrow 2$.

$\text{Var}(X_n) = EX_n^2 = 2 - \frac{1}{m^2}$. Then, $\text{Var}(S_n) = 2n - \sum_{m=1}^n \frac{1}{m^2}$. Since $\sum_{m=1}^n \frac{1}{m^2}$ converges, we get the desired result.

(2) $S_n/\sqrt{n} \Rightarrow \chi$.

Define $Y_m = \begin{cases} 1 & , X_m > 0 \\ -1 & , X_m < 0 \end{cases}$. Then, Y_m 's are *i.i.d* sequence with $EY_m = 0$ and $\text{Var}(Y_m) = 1$. By CLT for *i.i.d*, we have

$$\sum_{m=1}^n Y_m / \sqrt{n} \Rightarrow \chi.$$

Note that $P(X_m \neq Y_m) = P(X_m = \pm m) = \frac{1}{m^2}$. Then, $\sum_{m=1}^{\infty} P(X_m \neq Y_m) = \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty$. By Borel-Cantelli Lemma 1, $P(X_m \neq Y_m \text{ i.o.}) = 0$. Thus, $P(X_m = Y_m \text{ eventually}) = 1$. This means that

$$\frac{S_n}{\sqrt{n}} = \frac{\sum_{m=1}^n Y_m}{\sqrt{n}} + \frac{S_n - \sum_{m=1}^n Y_m}{\sqrt{n}} \Rightarrow \chi$$

by converging together Lemma since the second term converges to 0.

Exercise 3.4.10. Let $s_n^2 = \sum_{m=1}^n \text{Var}(X_m)$ and $Y_{nm} = \frac{X_m - EX_m}{s_n}$. Then, $EY_{nm} = 0$ and $t_n^2 := \sum_{m=1}^n \text{Var}(Y_{nm}) = 1$. For Y_{nm} ,

- (i) $\sum_{m=1}^n EY_{nm}^2 = t_n^2 = 1$
- (ii) For $\epsilon > 0$, $|Y_{nm}| > \epsilon \iff |X_m - EX_m| > \epsilon s_n$.

Note that since $|X_i| \leq M$, $|X_m - EX_m| \leq 2M$, $\forall m$. However, since $\sum_{n=1}^{\infty} \text{Var}(X_n) = \infty$, $\epsilon s_n > 2M$ for sufficiently large n . This means that $\{|Y_{nm}| > \epsilon\}$ does not occur eventually. Thus,

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \int_{|Y_{nm}| > \epsilon} |Y_{nm}|^2 dP = 0.$$

By (i) and (ii), Lindeberg Theorem gives us that $\sum_{m=1}^n Y_{nm} \Rightarrow \chi$. Therefore,

$$\frac{\sum_{m=1}^n X_m - E(\sum_{m=1}^n X_m)}{\sqrt{\sum_{m=1}^n \text{Var}(X_m)}} = \frac{S_n - ES_n}{\sqrt{\text{Var}(S_n)}} \Rightarrow \chi.$$

Exercise 3.4.11. By Exercise 3.4.12, it suffices to show that $\{X_n\}$ satisfies Lyapounov's condition.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^n E(|X_m - EX_m|^{2+\delta}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^{1+\delta/2}} \sum_{m=1}^n E|X_m|^{2+\delta} \\ &= \lim_{n \rightarrow \infty} \frac{C}{n^{\delta/2}} \\ &= 0. \end{aligned}$$

Exercise 3.4.12. Let $X_{n,m} = \frac{X_m - EX_m}{\alpha_n}$. It suffices to show that $\{X_{n,m}\}$ satisfies the Lindeberg condition.

- (i) $\sum_{m=1}^n EX_{n,m}^2 = 1$.

(ii) For given $\epsilon > 0$,

$$\begin{aligned}
\sum_{m=1}^n E(X_{n,m}^2; |X_{n,m}| > \epsilon) &= \sum_{m=1}^n \int_{|X_{n,m}| > \epsilon} X_{n,m}^2 dP \\
&= \frac{1}{\alpha_n^2} \sum_{m=1}^n \int_{|X_m - EX_m| > \epsilon \alpha_n} |X_m - EX_m|^2 dP \\
&\leq \frac{1}{\alpha_n^2} \sum_{m=1}^n \int_{|X_m - EX_m| > \epsilon \alpha_n} \left(\frac{|X_m - EX_m|}{\epsilon \alpha_n} \right)^\delta |X_m - EX_m|^2 dP \\
&\leq \frac{1}{\epsilon^\delta} \cdot \left[\frac{1}{\alpha_n^{2+\delta}} \sum_{m=1}^n \int |X_m - EX_m|^{2+\delta} dP \right] \rightarrow 0
\end{aligned}$$

by assumption.

By (i) and (ii), we get the desired result.

Exercise 3.4.13. (i) $\beta > 1$.

$P(X_j \neq 0) = P(X_j = \pm j) = \frac{1}{j^\beta}$. Thus,

$$\sum_{j=1}^{\infty} P(X_j \neq 0) = \sum_{j=1}^{\infty} \frac{1}{j^\beta} < \infty.$$

By Borel-Cantelli Lemma 1, $P(X_j = 0 \text{ eventually}) = 1$. For $\omega \in \Omega_0 := \{\omega : X_j(\omega) = 0 \text{ eventually}\}$, $\sum_{j=1}^{\infty} X_j(\omega)$ converges. Thus, $S_n \rightarrow S_\infty$ a.s., where S_∞ is a r.v. such that $S_\infty(\omega) = \sum_{j=1}^{\infty} X_j(\omega)$ for $\omega \in \Omega_0$

(ii) $\beta < 1$.

Let

$$\begin{cases} X_{nm} &= X_m \text{ for } 1 \leq m \leq n, \forall n \\ Y_{nm} &= \frac{X_{nm}}{n^{(3-\beta)/2}} \\ T_n &= \sum_{m=1}^n Y_{nm} = \frac{S_n}{n^{(3-\beta)/2}} \end{cases}$$

Then, $EY_{nm} = 0$ and $EY_{nm}^2 = \frac{1}{n^{(3-\beta)}} m^{2-\beta}$.

(a) $\sum_{m=1}^n Y_{nm}^2 = \frac{1}{n^{(3-\beta)}} \sum_{m=1}^n m^{2-\beta} \leq \frac{1}{n^{(3-\beta)}} \cdot n \cdot n^{2-\beta} = 1$.

Since $\sum_{m=1}^n Y_{nm}^2$ is bounded and monotonically increasing, $\sum_{m=1}^n Y_{nm}^2 \rightarrow c^2 \leq 1$ for some $c > 0$.

(b) $\lim_{n \rightarrow \infty} \sum_{m=1}^n E(|Y_{nm}|^2; |Y_{nm}| > \epsilon) = \lim_{n \rightarrow \infty} \sum_{m=1}^n E(|Y_{nm}|^2; |X_{nm}| > \epsilon \cdot n^{(3-\beta)/2})$.

Since $\beta < 1$, $\frac{3-\beta}{2} > 1$. Then, for large n , $\epsilon \cdot n^{(3-\beta)/2} > n$. Then, $\{|X_{nm}| > \epsilon \cdot n^{(3-\beta)/2}\} = \emptyset$. Thus, $\lim_{n \rightarrow \infty} \sum_{m=1}^n E(|Y_{nm}|^2; |X_{nm}| > \epsilon \cdot n^{(3-\beta)/2}) = 0$.

By (a) and (b), Lindeberg Theorem implies that $T_n = \frac{S_n}{n^{(3-\beta)/2}} \Rightarrow c\chi$.

(iii) $\beta = 1$.

$$\varphi_{S_n/n}(t) = \prod_{j=1}^n \varphi_{X_j}(t/n)$$

$$\begin{aligned}
&= \prod_{j=1}^n \left(1 - \frac{1}{j} \left(1 - \cos \frac{jt}{n} \right) \right) \\
&= \prod_{j=1}^n \left(1 - \frac{1}{n} \cdot \left(\frac{j}{n} \right)^{-1} \cdot \left(1 - \cos \frac{jt}{n} \right) \right).
\end{aligned}$$

Note that

$$\sum_{j=1}^n \frac{1}{n} \left(\frac{j}{n} \right)^{-1} \left(1 - \cos \frac{jt}{n} \right) \rightarrow \int_0^1 \frac{1}{x} (1 - \cos xt) dx.$$

Also, $\frac{1}{j} (\cos \frac{jt}{n} - 1) \leq 0$ for all j and $\lim_{n \rightarrow \infty} \frac{1}{j} (\cos \frac{jt}{n} - 1) \rightarrow 0$.

Thus, $\max_{1 \leq j \leq n} \frac{1}{j} (\cos \frac{jt}{n} - 1) \rightarrow 0$.

Exercise ?? with $c_{j,n} = -\frac{1}{j} (1 - \cos \frac{jt}{n})$ implies

$$\begin{aligned}
\varphi_{\frac{S_n}{\sqrt{n}}}(t) &= \prod_{j=1}^n \left(1 + \left(-\frac{1}{j} \left(1 - \cos \frac{jt}{n} \right) \right) \right) \\
&\rightarrow \exp \left[- \int_0^1 \frac{1}{x} (1 - \cos tx) dx \right].
\end{aligned}$$

Since the last one is the ch.f of χ , it is continuous at $t = 0$. By Continuity theorem, we get $S_n/n \Rightarrow \chi$.

3.5. **Local Limit Theorems.** No exercise

3.6. **Poisson Convergence.** No exercise

3.7. **Poisson Processes.**

Exercise 3.7.1. 1

Exercise 3.7.2. 2

Exercise 3.7.3. 3

Exercise 3.7.4. 4

Exercise 3.7.5. 5

Exercise 3.7.6. 6

Exercise 3.7.7. 7

Exercise 3.7.8. 8

Exercise 3.7.9. 9

3.8. **Stable Laws*.**

Exercise 3.8.1. 1

Exercise 3.8.2. 2

Exercise 3.8.3. 3

Exercise 3.8.4. 4

Exercise 3.8.5. 5

Exercise 3.8.6. 6

Exercise 3.8.7. 7

3.9. **Infinitely Divisible Distributions*.**

Exercise 3.9.1. 1

Exercise 3.9.2. 2

Exercise 3.9.3. 3

Exercise 3.9.4. 4

3.10. Limit Theorems in \mathbb{R}^d .

Exercise 3.10.1.

$$\begin{aligned} P(X_i \leq x) &= P(X_i \leq x, X_j \in (-\infty, \infty) \text{ for } j \neq i) \\ &= \lim_{y \rightarrow \infty} P(X_1 \leq y, \dots, X_{i-1} \leq y, X_i \leq x, X_{i+1} \leq y, \dots, X_d \leq y) \\ &= \lim_{y \rightarrow \infty} F(y, \dots, y, x, y, \dots, y), \end{aligned}$$

where x lies at i -th coordinate.

Exercise 3.10.2. Omit.

Exercise 3.10.3. Omit.

Exercise 3.10.4. Portmanteu

Exercise 3.10.5. Let X be a r.v. whose ch.f is φ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be a random vector whose ch.f is ψ .

Then,

$$\begin{aligned} \varphi(t_1 + \dots + t_d) &= E \exp[i(t_1 + \dots + t_d)X] \\ \psi(t_1, \dots, t_d) &= E \exp[i\mathbf{t}^T \mathbf{Y}] \\ &= E \exp[i(t_1 Y_1 + \dots + t_d Y_d)]. \end{aligned}$$

Thus, $\mathbf{Y} = (X, \dots, X)$

Exercise 3.10.6. Let $\mathbf{X} = (X_1, \dots, X_k)^T$.

$$\begin{aligned} (\Rightarrow) \varphi_{X_1, \dots, X_k}(t) &= E e^{i\mathbf{t}^T \mathbf{X}} \\ &= E e^{it_1 X_1 + \dots + t_k X_k} \\ &= E e^{it_1 X_1} \dots E e^{it_k X_k} \\ &= \prod_{j=1}^k \varphi_{X_j}(t_j) \end{aligned}$$

The third equality holds since X_i 's are independent.

(\Leftarrow) For $A = [a_1, b_1] \times \dots \times [a_k, b_k]$ with $\mu_X(\partial A) = 0$,

$$\begin{aligned} \mu_{\mathbf{X}}(A) &= \lim_{T \rightarrow \infty} \left[(2\pi)^{-k} \int_{[-T, T]^k} \left(\prod_{j=1}^k \psi_j(t_j) \right) \varphi_{\mathbf{X}}(\mathbf{t}) d\mathbf{t} \right] \\ &= \lim_{T \rightarrow \infty} \left[(2\pi)^{-k} \int_{[-T, T]^k} \prod_{j=1}^k \left(\psi_j(t_j) \varphi_{X_j}(t_j) \right) dt_1 \dots dt_j \right] \\ &= \lim_{T \rightarrow \infty} \left[\prod_{j=1}^k \left((2\pi)^{-1} \int_{[-T, T]} \psi_j(t_j) \varphi_{X_j}(t_j) dt_j \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^k \left[\lim_{T \rightarrow \infty} (2\pi)^{-1} \int_{[-T, T]} \psi_j(t_j) \varphi_{X_j}(t_j) dt_j \right] \\
&= \prod_{j=1}^k \mu_{X_j}([a_j, b_j]).
\end{aligned}$$

Since it holds for any A , X_1, \dots, X_k are independent.

Exercise 3.10.7. Let $\mathbf{X} = (X_1, \dots, X_d)^T$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)^T$ and $\boldsymbol{\Gamma} = (\Gamma_{ij})$

$$\begin{aligned}
(\Rightarrow) \quad \varphi_{\mathbf{X}}(\mathbf{t}) &= \exp \left[i\mathbf{t}^T \boldsymbol{\theta} - \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Gamma_{ij} t_i t_j \right] \\
&= E \exp(i\mathbf{t}^T \mathbf{X}) \\
&= E \exp(it_1 X_1 + \dots + it_d X_d) \\
&= E \exp(it_1 X_1) \cdots E \exp(it_d X_d) \\
&= \exp \left[i\mathbf{t}^T \boldsymbol{\theta} - \frac{1}{2} \sum_{l=1}^d \Gamma_{ll} t_l^2 \right]
\end{aligned}$$

The fourth equality holds since X_i 's are independent. The fifth equality comes from Exercise 3.10.8 with $\mathbf{c} = \mathbf{e}_i$. Thus, $\Gamma_{ij} = 0$ for $i \neq j$.

(\Leftarrow) Similarly,

$$\begin{aligned}
\varphi_{\mathbf{X}}(\mathbf{t}) &= \exp \left[i\mathbf{t}^T \boldsymbol{\theta} - \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Gamma_{ij} t_i t_j \right] \\
&= \exp \left[i\mathbf{t}^T \boldsymbol{\theta} - \frac{1}{2} \sum_{l=1}^d \Gamma_{ll} t_l^2 \right] \\
&= \prod_{i=1}^d \exp \left(it_i \theta_i - \frac{1}{2} \Gamma_{ii} t_i^2 \right) \\
&= \prod_{i=1}^d \varphi_{X_i}(t_i)
\end{aligned}$$

The last equality comes from Exercise 3.10.8 with $\mathbf{c} = \mathbf{e}_i$.

Exercise 3.10.8. Let $\mathbf{X} = (X_1, \dots, X_d)^T$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)^T$, $\mathbf{c} = (c_1, \dots, c_d)^T$ and $\boldsymbol{\Gamma} = (\Gamma_{ij})$: $d \times d$ matrix.

We want to show that

$$X \sim N_d(\boldsymbol{\theta}, \boldsymbol{\Gamma}) \iff \forall \mathbf{c} \in \mathbb{R}^d, \mathbf{c}^T \mathbf{X} \sim N(\mathbf{c}^T \boldsymbol{\theta}, \mathbf{c}^T \boldsymbol{\Gamma} \mathbf{c}).$$

Note that ch.f of $\mathbf{X} = N_d(\boldsymbol{\theta}, \boldsymbol{\Gamma})$ is given by

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp \left[i\mathbf{t}^T \boldsymbol{\theta} - \frac{1}{2} \mathbf{t}^T \boldsymbol{\Gamma} \mathbf{t} \right].$$

$$\begin{aligned}
(\Rightarrow) \quad \varphi_{\mathbf{c}^T \mathbf{X}}(t) &= E e^{i t \mathbf{c}^T \mathbf{X}} \\
&= E e^{i(t\mathbf{c})^T \mathbf{X}} \\
&= \exp \left[i(t\mathbf{c})^T \boldsymbol{\theta} - \frac{1}{2} (t\mathbf{c})^T \boldsymbol{\Gamma} (t\mathbf{c}) \right] \\
&= \exp \left[i t (\mathbf{c}^T \boldsymbol{\theta}) - \frac{1}{2} (\mathbf{c}^T \boldsymbol{\Gamma} \mathbf{c}) t^2 \right],
\end{aligned}$$

which is ch.f of $N(\mathbf{c}^T \boldsymbol{\theta}, \mathbf{c}^T \boldsymbol{\Gamma} \mathbf{c})$. Thus, $\mathbf{c}^T \mathbf{X} = c_1 X_1 + \cdots + c_d X_d \sim N(\mathbf{c}^T \boldsymbol{\theta}, \mathbf{c}^T \boldsymbol{\Gamma} \mathbf{c})$.

$$\begin{aligned}
(\Leftarrow) \quad \varphi_{\mathbf{X}}(\mathbf{t}) &= E e^{i \mathbf{t}^T \mathbf{X}} \\
&= E e^{i \cdot 1 \cdot (\mathbf{t}^T \mathbf{X})} \\
&= \varphi_{\mathbf{t}^T \mathbf{X}}(1) \\
&= \exp \left[i \cdot 1 \cdot (\mathbf{t}^T \boldsymbol{\theta}) - \frac{1}{2} \mathbf{t}^T \boldsymbol{\Gamma} \mathbf{t} \cdot 1^2 \right] \\
&= \exp \left[i \mathbf{t}^T \boldsymbol{\theta} - \frac{1}{2} \mathbf{t}^T \boldsymbol{\Gamma} \mathbf{t} \right],
\end{aligned}$$

which is ch.f of $N_d(\boldsymbol{\theta}, \boldsymbol{\Gamma})$. Thus, $\mathbf{X} \sim N_d(\boldsymbol{\theta}, \boldsymbol{\Gamma})$.

4. MARTINGALES

4.1. Conditional Expectation.

Exercise 4.1.1. Since $G, \Omega \in \mathcal{G}$,

$$\begin{aligned}
\int_G \mathbb{P}(A|\mathcal{G}) d\mathbb{P} &= \int_G \mathbf{1}_A d\mathbb{P} = \mathbb{P}(A \cap G) \\
\int_\Omega \mathbb{P}(A|\mathcal{G}) d\mathbb{P} &= \int_\Omega \mathbf{1}_A d\mathbb{P} = \mathbb{P}(A)
\end{aligned}$$

and the result follows.

Exercise 4.1.2. Let $F \in \mathcal{F}$.

$$\int_F \mathbb{P}(|X| \geq a | \mathcal{F}) d\mathbb{P} = \int_F \mathbf{1}_{|X| \geq a} d\mathbb{P} \leq \int_F \frac{|X|^2}{a^2} d\mathbb{P} = \int_F a^{-2} \mathbb{E}[|X|^2 | \mathcal{F}] d\mathbb{P}.$$

Since $F \in \mathcal{F}$ was arbitrary, the result follows.

Exercise 4.1.3. Almost surely, we have

$$0 \leq \mathbb{E}^\mathcal{G}[(|X| + \theta|Y|)^2] = \mathbb{E}^\mathcal{G}[X^2] + 2\theta \mathbb{E}^\mathcal{G}[XY] + \theta^2 \mathbb{E}^\mathcal{G}[Y^2].$$

Thus, almost surely,

$$D/4 = (\mathbb{E}^\mathcal{G}[XY])^2 - \mathbb{E}^\mathcal{G}[X^2] \mathbb{E}^\mathcal{G}[Y^2] \leq 0.$$

Exercise 4.1.4. Let $\mu : \Omega \times \mathcal{F} \rightarrow [0, 1]$ be an r.c.p. given \mathcal{G} .

$$\mathbb{E}^\mathcal{G}[XY] = \int |X(\omega)Y(\omega)| \mu(\tilde{\omega}, d\omega)$$

$$\begin{aligned}
&\leq \left(\int |X(\omega)|^p \mu(\tilde{\omega}, d\omega) \right)^{1/p} \left(\int |Y(\omega)|^q \mu(\tilde{\omega}, d\omega) \right)^{1/q} \\
&= (\mathbb{E}^{\mathcal{G}} |X|^p)^{1/p} (\mathbb{E}^{\mathcal{G}} |Y|^q)^{1/q}.
\end{aligned}$$

Exercise 4.1.5. Let $\mathbb{P}(a) = \mathbb{P}(b) = \mathbb{P}(c) = 1/3$ and let $\mathcal{F}_1 = \sigma(\{a\})$, $\mathcal{F}_2 = \sigma(\{b\})$. Define X as

$$X = \begin{cases} a \mapsto 1 \\ b \mapsto 2 \\ c \mapsto 3 \end{cases}.$$

Then,

$$\mathbb{E}^{\mathcal{F}_1} X = \begin{cases} a \mapsto 1 \\ b \mapsto 2.5 \\ c \mapsto 2.5 \end{cases}, \quad \mathbb{E}^{\mathcal{F}_2} X = \begin{cases} a \mapsto 2 \\ b \mapsto 2 \\ c \mapsto 2 \end{cases}$$

and

$$\mathbb{E}^{\mathcal{F}_2} \mathbb{E}^{\mathcal{F}_1} X = \begin{cases} a \mapsto 1.75 \\ b \mapsto 2.5 \\ c \mapsto 1.75 \end{cases}, \quad \mathbb{E}^{\mathcal{F}_1} \mathbb{E}^{\mathcal{F}_2} X = \begin{cases} a \mapsto 2 \\ b \mapsto 2 \\ c \mapsto 2 \end{cases}.$$

Exercise 4.1.6.

$$\begin{aligned}
&\mathbb{E}[(X - \mathbb{E}^{\mathcal{F}} X)^2] + \mathbb{E}[(\mathbb{E}^{\mathcal{F}} X - \mathbb{E}^{\mathcal{G}} X)^2] \\
&= \mathbb{E}X^2 + \mathbb{E}[(\mathbb{E}^{\mathcal{F}} X)^2] - 2\mathbb{E}[X \mathbb{E}^{\mathcal{F}} X] + \mathbb{E}[(\mathbb{E}^{\mathcal{F}} X)^2] + \mathbb{E}[(\mathbb{E}^{\mathcal{G}} X)^2] - 2\mathbb{E}[(\mathbb{E}^{\mathcal{F}} X)(\mathbb{E}^{\mathcal{G}} X)].
\end{aligned}$$

Note that $\mathbb{E}[(\mathbb{E}^{\mathcal{F}} X)^2] = \mathbb{E}[(\mathbb{E}^{\mathcal{F}} X)(\mathbb{E}^{\mathcal{F}} X)] = \mathbb{E}[\mathbb{E}^{\mathcal{F}}[X \mathbb{E}^{\mathcal{F}} X]] = \mathbb{E}[X \mathbb{E}^{\mathcal{F}} X]$, and $\mathbb{E}[(\mathbb{E}^{\mathcal{F}} X)(\mathbb{E}^{\mathcal{G}} X)] = \mathbb{E}[\mathbb{E}^{\mathcal{F}}[X \mathbb{E}^{\mathcal{G}} X]] = \mathbb{E}[X \mathbb{E}^{\mathcal{G}} X] = \mathbb{E}[(\mathbb{E}^{\mathcal{G}} X)^2]$. Thus the above becomes

$$= \mathbb{E}X^2 - \mathbb{E}[(\mathbb{E}^{\mathcal{G}} X)^2] = \mathbb{E}[(X - \mathbb{E}^{\mathcal{G}} X)^2].$$

Exercise 4.1.7. Note that

$$\text{Var}[X|\mathcal{F}] = \mathbb{E}^{\mathcal{F}} X^2 - (\mathbb{E}^{\mathcal{F}} X)^2 = \mathbb{E}^{\mathcal{F}}[(X - \mathbb{E}^{\mathcal{F}} X)^2].$$

Thus,

$$\mathbb{E}[\text{Var}[X|\mathcal{F}]] = \mathbb{E}[(X - \mathbb{E}^{\mathcal{F}} X)^2].$$

Also,

$$\text{Var}[\mathbb{E}^{\mathcal{F}} X] = \mathbb{E}[(\mathbb{E}^{\mathcal{F}} X - \mathbb{E}X)^2].$$

Thus the result follows from Exercise 4.1.6.

Exercise 4.1.8. Let $\mathcal{F} = \sigma(N)$. First, we need to check that $\mathbb{E}|X|^2 < \infty$. Indeed,

$$\begin{aligned}
\mathbb{E}|X|^2 &= \sum_n \mathbb{E}[|X|^2 \mathbf{1}_{\{N=n\}}] \\
&= \sum_n \mathbb{E}[(Y_1 + \cdots + Y_n)^2 \mathbf{1}_{\{N=n\}}] \\
&= \sum_n \mathbb{E}[(Y_1 + \cdots + Y_n)^2] \mathbb{E}[\mathbf{1}_{\{N=n\}}] \\
&= \sum_n (n\sigma^2 + n^2\mu^2) \mathbb{P}[N = n] \\
&\leq \sum_n Cn^2 \mathbb{P}[N = n] \\
&= C\mathbb{E}N^2 < \infty
\end{aligned}$$

where for the third equality we used the independence of Y_i and N . Now we claim that $\mathbb{E}^{\mathcal{F}}X = N\mu$ and $\text{Var}[X|\mathcal{F}] = N\sigma^2$. Note that these are \mathcal{F} -measurable. For $A_n = \{N = n\}$, we have

$$\begin{aligned}
\int_{A_n} X \, d\mathbb{P} &= \int_{A_n} (Y_1 + \cdots + Y_n) \, d\mathbb{P} \\
&= \mathbb{E}[(Y_1 + \cdots + Y_n) \mathbf{1}_{A_n}] \\
&= \mathbb{E}[Y_1 + \cdots + Y_n] \mathbb{E}[\mathbf{1}_{A_n}] \\
&= n\mu \mathbb{E}[\mathbf{1}_{A_n}] \\
&= \int_{A_n} N\mu \, d\mathbb{P}
\end{aligned}$$

and

$$\begin{aligned}
\int_{A_n} X^2 \, d\mathbb{P} &= \int_{A_n} (Y_1 + \cdots + Y_n)^2 \, d\mathbb{P} \\
&= \mathbb{E}[(Y_1 + \cdots + Y_n)^2 \mathbf{1}_{A_n}] \\
&= \mathbb{E}[(Y_1 + \cdots + Y_n)^2] \mathbb{E}[\mathbf{1}_{A_n}] \\
&= (n\sigma^2 + n^2\mu^2) \mathbb{E}[\mathbf{1}_{A_n}] \\
&= \int_{A_n} (N\sigma^2 + N^2\mu^2) \, d\mathbb{P}.
\end{aligned}$$

Since \mathcal{F} is generated by $A_n, n \geq 1$ by complement and countable disjoint union (cf. Dynkin's π - λ theorem), we have the above equalities for all $A \in \mathcal{F}$. Thus $\mathbb{E}^{\mathcal{F}}X = N\mu$ and $\text{Var}[X|\mathcal{F}] = \mathbb{E}^{\mathcal{F}}X^2 - (\mathbb{E}^{\mathcal{F}}X)^2 = N\sigma^2$. Finally, from Exercise 4.1.7,

$$\text{Var}[X] = \text{Var}[\mathbb{E}^{\mathcal{F}}X] + \mathbb{E}[\text{Var}[X|\mathcal{F}]] = \mu^2 \text{Var}[N] + \sigma^2 \mathbb{E}N.$$

Exercise 4.1.9. From Exercise 4.1.7,

$$\mathbb{E}[(Y - X)^2] + \mathbb{E}[(X - \mathbb{E}X)^2] = \mathbb{E}[(Y - \mathbb{E}Y)^2].$$

Note that $\mathbb{E}[(X - \mathbb{E}X)^2] = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \mathbb{E}[(Y - \mathbb{E}Y)^2]$, where the second equality holds since $\mathbb{E}X^2 = \mathbb{E}Y^2$ and $\mathbb{E}X = \mathbb{E}Y$. Thus $\mathbb{E}[(Y - X)^2] = 0$ and $X = Y$ a.s.

Exercise 4.1.10. Fix $c \in \mathbb{Q}$. Since $\mathbb{E}[Y|\mathcal{G}] - c \stackrel{d}{=} Y - c$, we have $\mathbb{E}|\mathbb{E}[Y|\mathcal{G}] - c| = \mathbb{E}|Y - c|$. Meanwhile, we also have

$$\mathbb{E}|\mathbb{E}[Y|\mathcal{G}] - c| = \mathbb{E}|\mathbb{E}[Y - c|\mathcal{G}]| \leq \mathbb{E}\mathbb{E}[|Y - c||\mathcal{G}] = \mathbb{E}|Y - c|$$

and thus

$$\mathbb{E}|\mathbb{E}[Y - c|\mathcal{G}]| = \mathbb{E}\mathbb{E}[|Y - c||\mathcal{G}] \quad \text{a.s.}$$

Since $|\mathbb{E}[Y - c|\mathcal{G}]| \leq \mathbb{E}[|Y - c||\mathcal{G}]$, it follows that

$$|\mathbb{E}[Y - c|\mathcal{G}]| = \mathbb{E}[|Y - c||\mathcal{G}] \quad \text{a.s.}$$

Now let $A = \{\mathbb{E}[Y - c|\mathcal{G}] > 0\} \in \mathcal{G}$. Multiplying $\mathbf{1}_A$ both sides, the last display becomes

$$\begin{aligned} \mathbb{E}[(Y - c)\mathbf{1}_A|\mathcal{G}] &= \mathbb{E}[Y - c|\mathcal{G}]\mathbf{1}_A \\ &= |\mathbb{E}[Y - c|\mathcal{G}||\mathbf{1}_A \\ &= \mathbb{E}[|Y - c||\mathcal{G}]\mathbf{1}_A \\ &= \mathbb{E}[|Y - c|\mathbf{1}_A|\mathcal{G}]. \end{aligned}$$

Since $(Y - c)\mathbf{1}_A \leq |Y - c|\mathbf{1}_A$, it again implies that $(Y - c)\mathbf{1}_A = |Y - c|\mathbf{1}_A$ a.s. A similar argument with $B = \{\mathbb{E}[Y - c|\mathcal{G}] \leq 0\}$ in place of A will show that $-(Y - c)\mathbf{1}_B = |Y - c|\mathbf{1}_B$ a.s. Thus $\text{sgn}(Y - c) = \text{sgn}(\mathbb{E}[Y - c|\mathcal{G}]) = \text{sgn}(\mathbb{E}[Y|\mathcal{G}] - c)$. Now suppose that $\{Y \neq \mathbb{E}[Y|\mathcal{G}]\}$ has positive probability. Without loss of generality we may assume that $\{Y < \mathbb{E}[Y|\mathcal{G}]\}$ has a positive probability. Since $\{Y < \mathbb{E}[Y|\mathcal{G}]\} = \bigcup_{c \in \mathbb{Q}} \{Y < c < \mathbb{E}[Y|\mathcal{G}]\}$, there must exist some $c \in \mathbb{Q}$ such that $\{Y < c < \mathbb{E}[Y|\mathcal{G}]\}$ has a positive probability. This is a contradiction because then with a positive probability, $\text{sgn}(Y - c) \neq \text{sgn}(\mathbb{E}[Y|\mathcal{G}] - c)$. Therefore $Y = \mathbb{E}[Y|\mathcal{G}]$ a.s.

4.2. Martingales, Almost Sure Convergence.

Exercise 4.2.1. Since X_1, \dots, X_n are measurable with respect to \mathcal{G}_n , $\mathcal{F}_n \subset \mathcal{G}_n$. Note that $\mathbb{E}^{\mathcal{F}_n}[X_{n+1}] = \mathbb{E}^{\mathcal{F}_n}[\mathbb{E}^{\mathcal{G}_n}X_{n+1}] = \mathbb{E}^{\mathcal{F}_n}X_n = X_n$. Thus X_n is a martingale with respect to \mathcal{F}_n .

Exercise 4.2.2. Let $X_n = -1/n$. Then $\mathbb{E}^{\mathcal{F}_n}X_{n+1} = -1/(n+1) > -1/n = X_n$ and $\mathbb{E}^{\mathcal{F}_n}X_{n+1}^2 = 1/(n+1)^2 < 1/n^2 = X_n^2$.

Exercise 4.2.3. Note that, almost surely,

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_n}[X_{n+1} \vee Y_{n+1}] &= \mathbb{E}^{\mathcal{F}_n}[X_{n+1}\mathbf{1}_{\{X_{n+1} \geq Y_{n+1}\}}] + \mathbb{E}^{\mathcal{F}_n}[Y_{n+1}\mathbf{1}_{\{X_{n+1} < Y_{n+1}\}}] \\ &\geq \mathbb{E}^{\mathcal{F}_n}[Y_{n+1}\mathbf{1}_{\{X_{n+1} \geq Y_{n+1}\}}] + \mathbb{E}^{\mathcal{F}_n}[Y_{n+1}\mathbf{1}_{\{X_{n+1} < Y_{n+1}\}}] \\ &= \mathbb{E}^{\mathcal{F}_n}[Y_{n+1}] = Y_n, \end{aligned}$$

and similarly $\mathbb{E}^{\mathcal{F}_n}[X_{n+1} \vee Y_{n+1}] \geq X_n$ a.s. Thus $\mathbb{E}^{\mathcal{F}_n}[X_{n+1} \vee Y_{n+1}] \geq X_n \vee Y_n$.

Exercise 4.2.4. Let $T_M = \inf\{n \leq 0: X_n^+ > M\}$. Consider the stopped process $X_{T_M \wedge n}$. Note that $X_{T_M \wedge n}^+ \leq M + \sup_j \xi_j^+$, which leads to $\mathbb{E}X_{T_M \wedge n}^+ \leq M + \mathbb{E} \sup_j \xi_j^+ < \infty$ for all n . Thus $X_{T_M \wedge n}$ converges a.s. That is, X_n converges on the event $\{T_M = \infty\}$. Since we have $\sup_n X_n < \infty$, if we let $M \rightarrow \infty$, $\{T_M = \infty\} \uparrow \Omega$ up to a measure zero set. Therefore, X_n converges a.s.

Exercise 4.2.5. Let ξ_j be defined as

$$\xi_j = \begin{cases} -1 & \text{with probability } \frac{1}{\epsilon_j + 1} \\ \frac{1}{\epsilon_j} & \text{with probability } \frac{\epsilon_j}{\epsilon_j + 1} \end{cases}$$

where $\{\epsilon_j\}$ satisfies $\sum_j \epsilon_j < \infty$. By the Borel-Cantelli lemma, $\mathbb{P}(\xi_j \neq -1 \text{ i.o.}) = 0$ since $\sum \mathbb{P}(\xi_j \neq -1) < \infty$. That is, $\xi_j = -1$ eventually a.s. and thus $X_n \rightarrow -\infty$ a.s.

Exercise 4.2.6. (i) Note that $X_n \geq 0$. Since X is a martingale, it is a supermartingale, and thus X_n converges a.s. to an L^1 random variable X . To see $X = 0$ a.s., note that

$$\begin{aligned} \mathbb{P}(|X_{n+1} - X_n| > \delta\epsilon) &= \mathbb{P}(X_n |Y_{n+1} - 1| > \delta\epsilon) \\ &\geq \mathbb{P}(X_n > \delta, |Y_{n+1} - 1| > \epsilon) \\ &= \mathbb{P}(X_n > \delta) \mathbb{P}(|Y_{n+1} - 1| > \epsilon). \end{aligned}$$

Here, $\mathbb{P}(|X_{n+1} - X_n| > \delta\epsilon)$ converges to 0 since X_n converges a.s. Since $\mathbb{P}(|Y_{n+1} - 1| > \epsilon) > 0$ for small ϵ , this implies that $\mathbb{P}(X_n > \delta) \rightarrow 0$. That is, $X_n \xrightarrow{P} 0$. Therefore $X = 0$ a.s.

(ii) By applying Jensen's inequality to $\log(Y \vee \delta)$, we have $\mathbb{E} \log(Y \vee \delta) \leq \log \mathbb{E}[Y \vee \delta]$. Letting $\delta \rightarrow 0$, we have $\mathbb{E} \log Y \leq \log \mathbb{E} Y = 0$. By the strong law of large numbers, it follows that

$$\frac{1}{n} \log X_n = \frac{1}{n} \sum_{m \leq n} \log Y_m \rightarrow \mathbb{E} \log Y \leq 0.$$

(Note that this holds even when $\mathbb{E} \log Y = -\infty$.)

Exercise 4.2.7. Confer Proposition 3.1 in Chapter 5 of Complex Analysis by Stein and Shakarchi.

Exercise 4.2.8. Observe that, from the given inequality,

$$\mathbb{E} \left[\frac{X_{n+1}}{\prod_{m=1}^n (1 + Y_m)} \middle| \mathcal{F}_n \right] \leq \frac{X_n}{\prod_{m=1}^{n-1} (1 + Y_m)}.$$

Thus the random variable

$$W_n = \frac{X_n}{\prod_{m=1}^{n-1} (1 + Y_m)},$$

which is also adapted and L^1 , is a nonnegative supermartingale, and thus converges a.s. Since $\prod_{m=1}^{n-1} (1 + Y_m)$ converges a.s. by Exercise 4.2.7, it follows that X_n converges a.s.

Exercise 4.2.9. (i) On $\{N > n\}$,

$$\begin{aligned}
\mathbb{E}[Y_{n+1} \mathbf{1}_{N>n} | \mathcal{F}_n] &= \mathbb{E}[X_{n+1}^1 \mathbf{1}_{N>n+1} + X_{n+1}^2 \mathbf{1}_{n<N \leq n+1} | \mathcal{F}_n] \\
&\leq \mathbb{E}[X_{n+1}^1 \mathbf{1}_{N>n+1} | \mathcal{F}_n] + \mathbb{E}[X_{n+1}^1 \mathbf{1}_{n<N \leq n+1} | \mathcal{F}_n] \\
&= \mathbb{E}[X_{n+1}^1 \mathbf{1}_{N>n} | \mathcal{F}_n] \\
&= \mathbb{E}[X_{n+1}^1 | \mathcal{F}_n] \mathbf{1}_{N>n} \\
&= X_n^1 \mathbf{1}_{N>n}.
\end{aligned}$$

Meanwhile, on $\{N \leq n\}$,

$$\begin{aligned}
\mathbb{E}[Y_{n+1} \mathbf{1}_{N \leq n} | \mathcal{F}_n] &= \mathbb{E}[X_{n+1}^2 \mathbf{1}_{N \leq n} | \mathcal{F}_n] \\
&= \mathbb{E}[X_{n+1}^2 | \mathcal{F}_n] \mathbf{1}_{N \leq n} \\
&\leq X_n^2 \mathbf{1}_{N \leq n}.
\end{aligned}$$

Thus $\mathbb{E}[Y_{n+1} | \mathcal{F}_n] \leq X_n^1 \mathbf{1}_{N>n} + X_n^2 \mathbf{1}_{N \leq n} = Y_n$, and Y_n is a supermartingale.

(ii) Use the similar argument as the above.

Exercise 4.2.10. content

4.3. Examples.

Exercise 4.3.1. Let $X_0 = 0$. Define X_{n+1} as follows: If $X_n = 0$, then $X_{n+1} = 1$ or -1 with probability $1/2$ each. If $X_n \neq 0$, then $X_{n+1} = 0$ or $n^2 X_n$ with probability $1 - 1/n^2$ and $1/n^2$ respectively. Note that $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$. i.e. X_n is indeed a martingale. Note also that, since $\sum_n 1/n^2 < \infty$, by Borel-Cantelli, $\mathbb{P}(X_{n+1} = n^2 X_n \text{ i.o.} | X_n \neq 0) = 0$. That is, given $X_n \neq 0$, $X_{n+1} = 0$ except for finitely many times. Thus $\mathbb{P}(X_n = a \text{ i.o.}) = 1$ for $a = -1, 0, 1$.

Exercise 4.3.2.

Exercise 4.3.3. Observe that from the given inequality,

$$\mathbb{E}[X_{n+1} - \sum_{k=1}^n Y_k | \mathcal{F}_n] \leq X_n - \sum_{k=1}^{n-1} Y_k.$$

Thus the random variable

$$Z_n = X_n - \sum_{k=1}^{n-1} Y_k,$$

which is also adapted and L^1 , is a supermartingale. Let $N_M = \inf\{\sum_{k=1}^n Y_k > M\}$. Then $Z_{N_M \wedge n}$ is also a supermartingale. Since $Z_{N_M \wedge n} + M = X_{N_M \wedge n} + M - \sum_{k=1}^{N_M \wedge n} Y_k \geq 0$, it is a nonnegative supermartingale, and thus converges a.s. to an L^1 limit. That is, Z_n converges to an L^1 limit on the event $\{N_M = \infty\}$. Since $\sum Y_k < \infty$ a.s., if we let $M \rightarrow \infty$, $\{N_M = \infty\} \uparrow \Omega$ up to a measure zero set. Therefore, Z_n converges a.s. to an L^1 limit Z_∞ and thus $X_n = Z_n + \sum_{k=1}^{n-1} Y_k \xrightarrow{\text{a.s.}} Z_\infty + \sum Y_k < \infty$.

Exercise 4.3.4. Let $(A_m)_m$ be a sequence of independent events with $\mathbb{P}(A_m) = p_m$. By the Borel-Cantelli lemma, we know that $\sum p_m = \infty$ if and only if $\mathbb{P}(A_m \text{ i.o.}) = 1$. Thus we show that $\mathbb{P}(A_m \text{ i.o.}) = 1$ if and only if $\prod(1 - p_m) = 0$. Suppose first that $\mathbb{P}(A_m \text{ i.o.}) = 1$. Then

$$\begin{aligned} 0 &= \mathbb{P}(A_m^c \text{ eventually}) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{m \geq n} A_m^c\right) \\ &= \lim_{n \rightarrow \infty} \prod_{m \geq n} (1 - p_m). \end{aligned}$$

Note that

$$\prod_{m=1}^{\infty} (1 - p_m) \leq \prod_{m \geq n} (1 - p_m) \rightarrow 0$$

and thus $\prod(1 - p_m) = 0$. Now for the other direction, suppose that $\prod(1 - p_m) = 0$. Since $p_m < 1$, $\prod_{1 \leq m \leq n} (1 - p_m) \neq 0$, and thus it follows that

$$0 = \prod_{m \geq n+1} (1 - p_m) = \mathbb{P}\left(\bigcap_{m \geq n+1} A_m^c\right)$$

for all n . Letting $n \rightarrow \infty$, we get $\mathbb{P}(A_m^c \text{ eventually}) = 0$ and thus $\mathbb{P}(A_m \text{ i.o.}) = 1$.

Exercise 4.3.5. Suppose first that $\mathbb{P}(A_n | \cap_{m=1}^{n-1} A_m^c) = 1$ for some n . This means that $A_n \supset \cap_{m=1}^{n-1} A_m^c$ up to a measure zero set. That is, $\cap_{m=1}^n A_m^c = \cap_{m=1}^{n-1} A_m^c \setminus A_n$ is measure zero. Then we immediately have $\mathbb{P}(\cap_{m=1}^{\infty} A_m^c) = 0$. Thus we may assume that $\mathbb{P}(A_n | \cap_{m=1}^{n-1} A_m^c) < 1$ for all n . By Exercise 4.3.4, it follows that

$$\begin{aligned} 0 &= \prod_{n=2}^{\infty} \mathbb{P}(A_n^c | \cap_{m=1}^{n-1} A_m^c) \\ &= \prod_{n=2}^{\infty} \frac{\mathbb{P}(\cap_{m=1}^n A_m^c)}{\mathbb{P}(\cap_{m=1}^{n-1} A_m^c)} \\ &= \frac{\mathbb{P}(\cap_{m=1}^{\infty} A_m^c)}{\mathbb{P}(A_1^c)} \end{aligned}$$

and thus $\mathbb{P}(\cap_{m=1}^{\infty} A_m^c) = 0$. (Note that $\mathbb{P}(A_1^c) \neq 0$ from the above assumption.)

Exercise 4.3.6. (i) Note first that X_n is \mathcal{F}_n -measurable and in L^1 . To see it is a martingale, we show that

$$\int_{I_{k,n}} X_{n+1} d\nu = \int_{I_{k,n}} X_n d\nu$$

for all k . If $\nu(I_{k,n}) = 0$, then $X_n \mathbf{1}_{I_{k,n}} = 0$ and $X_{n+1} \mathbf{1}_{I_{k,n}} = 0$, and thus the above holds. Now assume $\nu(I_{k,n}) \neq 0$ and suppose that $I_{k,n} = I_{p+1,n+1} \cup \cdots \cup I_{q,n+1} \cup I_{q+1,n+1} \cup \cdots \cup I_{r,n+1}$ where $\nu(I_{p+1,n+1}), \dots, \nu(I_{q,n+1})$ are nonzero and $\nu(I_{q+1,n+1}) = \cdots = \nu(I_{r,n+1}) =$

0. Then we have

$$\begin{aligned}
\int_{I_{k,n}} X_{n+1} d\nu &= \mu(I_{p+1,n+1}) + \cdots + \mu(I_{q,n+1}) \\
&= \mu(I_{p+1,n+1}) + \cdots + \mu(I_{q,n+1}) + \mu(I_{q+1,n+1}) + \cdots + \mu(I_{r,n+1}) \\
&= \mu(I_{k,n}) \\
&= \int_{I_{k,n}} X_n d\nu.
\end{aligned}$$

Thus X_n is a martingale.

(ii) Again, note that X_n is \mathcal{F}_n -measurable and in L^1 . Here, we show that

$$\int_{I_{k,n}} X_{n+1} d\nu \leq \int_{I_{k,n}} X_n d\nu$$

for all k . If $\nu(I_{k,n}) = 0$, then the inequality trivially holds as the above. Assume $\nu(I_{k,n}) \neq 0$. With the same notation as the above, we have

$$\begin{aligned}
\int_{I_{k,n}} X_{n+1} d\nu &= \mu(I_{p+1,n+1}) + \cdots + \mu(I_{q,n+1}) \\
&\leq \mu(I_{p+1,n+1}) + \cdots + \mu(I_{q,n+1}) + \mu(I_{q+1,n+1}) + \cdots + \mu(I_{r,n+1}) \\
&= \mu(I_{k,n}) \\
&= \int_{I_{k,n}} X_n d\nu.
\end{aligned}$$

Exercise 4.3.7. We first assume that μ and ν are finite measures. Let $\mathcal{F}_n = \sigma(A_m : 1 \leq m \leq n)$ and let μ_n and ν_n be restrictions of μ and ν to \mathcal{F}_n respectively. Obviously $\mu_n \ll \nu_n$. Since \mathcal{F}_n is finitely generated for each n , there exists a family of partitions $(I_{k,n})_{k,n}$ such that $\mathcal{F}_n = \sigma(I_{k,n} : 1 \leq k \leq K_n)$ and $\{I_{k,n+1} : 1 \leq k \leq K_{n+1}\}$ is a refinement of $\{I_{k,n} : 1 \leq k \leq K_n\}$. (Check this.) Define X_n as in Example 4.3.7. Then by definition, X_n satisfies

$$(*) \quad \mu_n(A) = \int_A X_n d\nu_n$$

for $A \in \mathcal{F}_n$. (We don't define X_n as the Radon-Nikodym derivative because we are in the way of proving the Radon-Nikodym theorem.) Moreover, as we saw in Exercise 4.3.6, X_n is a martingale, and in particular it is a nonnegative supermartingale. By Theorem 4.2.12, $X_n \xrightarrow{\text{a.s.}} X$ where $X \in L^1(\nu)$. Now by the proof of Theorem 4.3.5,

$$\mu(A) = \int_A X d\nu + \mu(A \cap \{X = \infty\}).$$

(Even though X_n is not a Radon-Nikodym derivative, it plays the same role as in the proof of Theorem 4.3.5 because of $(*)$ above. Thus the proof works.) Since $X \in L^1(\nu)$, $\nu(\{X = \infty\}) = 0$ and by the absolute continuity $\mu(\{X = \infty\}) = 0$. Therefore,

$$\mu(A) = \int_A X d\nu.$$

Finally, note that X is measurable with respect to $\mathcal{F} = \sigma(A_n : n \in \mathbb{N})$. This proves the Radon-Nikodym theorem for finite measures.

Now, let μ and ν be σ -finite. Let $\Omega_k \uparrow \Omega$ be satisfying $\mu(\Omega_k) < \infty$ and $\nu(\Omega_k) < \infty$ for all k . (Check that we can always find such Ω_k 's.) Define $\mu^{(k)}(\cdot) = \mu(\cdot \cap \Omega_k)$ and $\nu^{(k)}(\cdot) = \nu(\cdot \cap \Omega_k)$. Then $\mu^{(k)}$ and $\nu^{(k)}$ are finite measures. By the above, we have

$$\mu^{(k)}(A) = \int_A X^{(k)} d\nu^{(k)}$$

Where $X^{(k)}$ is \mathcal{F} -measurable and supported on Ω_k . Define X to be equal a.s. to X_k on Ω_k . Note that for $j < k$,

$$\mu^{(j)}(A) = \mu^{(k)}(A \cap \Omega_j) = \int_{A \cap \Omega_j} X^{(k)} d\nu^{(k)} = \int_A X^{(k)} d\nu^{(j)}.$$

Because of the uniqueness of (almost sure) limit, $X^{(k)} = X^{(j)}$ on Ω_j . Thus X is well-defined. Letting $k \rightarrow \infty$ and using the continuity of measures and the definition of Lebesgue integral,

$$\mu(A) = \lim_{k \rightarrow \infty} \mu^{(k)}(A) = \lim_{k \rightarrow \infty} \int_A X d\nu^{(k)} = \int_A X d\nu.$$

Finally it is obvious that X is \mathcal{F} -measurable.

Exercise 4.3.8. (i) Note that

$$\begin{aligned} 1 - \alpha_n &= F_n(0) = \int_{(-\infty, 0]} q_n(x) dG_n(x) = q_n(0)(1 - \beta_n), \\ 1 &= F_n(1) = \int_{(-\infty, 1]} q_n(x) dG_n(x) = q_n(0)(1 - \beta_n) + q_n(1)\beta_n. \end{aligned}$$

Thus $q_n(0) = (1 - \alpha_n)/(1 - \beta_n)$ and $q_n(1) = \alpha_n/\beta_n$. By Theorem 4.3.8, $\mu \ll \nu$ if and only if

$$\prod_{m=1}^{\infty} \int \sqrt{q_m} dG_m > 0.$$

Note that

$$\begin{aligned} \prod_{m=1}^{\infty} \int \sqrt{q_m} dG_m &= \prod_{m=1}^{\infty} \left(\sqrt{\frac{1 - \alpha_m}{1 - \beta_m}} (1 - \beta_m) + \sqrt{\frac{\alpha_m}{\beta_m}} \beta_m \right) \\ &= \prod_{m=1}^{\infty} \left(\sqrt{(1 - \alpha_m)(1 - \beta_m)} + \sqrt{\alpha_m \beta_m} \right). \end{aligned}$$

Thus the condition becomes

$$\prod_{m=1}^{\infty} \left(\sqrt{(1 - \alpha_m)(1 - \beta_m)} + \sqrt{\alpha_m \beta_m} \right) > 0.$$

(ii)

Exercise 4.3.9. By the Borel-Cantelli lemma, $\sum \alpha_n < \infty$ implies that $\xi_n = 0$ eventually μ -a.s. and $\sum \beta_n = \infty$ implies that $\xi_n = 1$ infinitely often ν -a.s. Since $\{\xi_n = 0 \text{ eventually}\} \cap \{\xi_n = 1 \text{ i.o.}\} = \emptyset$, we have $\mu \perp \nu$.

Exercise 4.3.10. content

Exercise 4.3.11. content

Exercise 4.3.12. content

Exercise 4.3.13. content

4.4. Doob's Inequality, Convergence in L^p , $p > 1$.

Exercise 4.4.1. Since X is a submartingale, $\mathbb{E}[X_k \mathbf{1}_{N=j} | \mathcal{F}_j] = \mathbb{E}[X_k | \mathcal{F}_j] \mathbf{1}_{N=j} \geq X_j \mathbf{1}_{N=j}$. Now take the expectation of both sides.

Exercise 4.4.2. On $\{M = N\}$, the result is trivial. On $\{M < N\}$, let $K_n = \mathbf{1}_{M < n \leq N}$. Then K is predictable. Note that $(K \cdot X)_n = X_{N \wedge n} - X_{M \wedge n}$. Since $(K \cdot X)$ is a submartingale, from $\mathbb{E}(K \cdot X)_0 \leq \mathbb{E}(K \cdot X)_k$, we have $\mathbb{E}X_M \leq \mathbb{E}X_N$.

(Another proof) Let $Y_n = X_{N \wedge n}$. Then Y is a submartingale. By Theorem 4.4.1, $\mathbb{E}X_M = \mathbb{E}Y_M \leq \mathbb{E}Y_k = \mathbb{E}X_N$.

Exercise 4.4.3. We first observe that $A \in \mathcal{F}_N$. To see this, note that $\{N \leq n\} \cap A = \{N \leq n\} \cap \{M \leq n\} \cap A$, and that $\{M \leq n\} \cap A \in \mathcal{F}_n$ and $\{N \leq n\} \in \mathcal{F}_n$. Thus $\{N \leq n\} \cap A \in \mathcal{F}_n$, meaning that $A \in \mathcal{F}_N$. Now consider $\{L \leq n\}$. Observe that

$$\begin{aligned} \{L \leq n\} &= (\{L \leq n\} \cap A) \cup (\{L \leq n\} \cap A^c) \\ &= (\{M \leq n\} \cap A) \cup (\{N \leq n\} \cap A^c) \in \mathcal{F}_n \end{aligned}$$

since $A \in \mathcal{F}_M$ and $A \in \mathcal{F}_N$. Thus L is a stopping time.

Exercise 4.4.4. Let $A \in \mathcal{F}_M$ and L be the stopping time defined as above. Since $L \leq N$, by Exercise 4.4.2, $\mathbb{E}X_L \leq \mathbb{E}X_N$. Note that $\mathbb{E}X_L = \mathbb{E}X_L \mathbf{1}_A + \mathbb{E}X_L \mathbf{1}_{A^c} = \mathbb{E}X_M \mathbf{1}_A + \mathbb{E}X_N \mathbf{1}_{A^c}$. Plugging this into the inequality, we have $\mathbb{E}X_N \geq \mathbb{E}X_M \mathbf{1}_A + \mathbb{E}X_N \mathbf{1}_{A^c}$, which leads to $\mathbb{E}X_N \mathbf{1}_A \geq \mathbb{E}X_M \mathbf{1}_A$. Since $A \in \mathcal{F}_M$ was arbitrary, it follows that $\mathbb{E}[X_N | \mathcal{F}_M] \geq X_M$.

Exercise 4.4.5. We show that $\mathbb{E}[\mathbb{E}[Y | \mathcal{G}] \mathbb{E}[Y | \mathcal{F}]] = \mathbb{E}[\mathbb{E}[Y | \mathcal{F}]^2]$. Since $\mathcal{F} \subset \mathcal{G}$,

$$\begin{aligned} \mathbb{E}[\mathbb{E}[Y | \mathcal{G}] \mathbb{E}[Y | \mathcal{F}]] &= \mathbb{E}[\mathbb{E}[Y \mathbb{E}[Y | \mathcal{F}] | \mathcal{G}]] \\ &= \mathbb{E}[Y \mathbb{E}[Y | \mathcal{F}]] \\ &= \mathbb{E}[\mathbb{E}[Y \mathbb{E}[Y | \mathcal{F}] | \mathcal{F}]] \\ &= \mathbb{E}[\mathbb{E}[Y | \mathcal{F}]^2]. \end{aligned}$$

This proves the result.

Exercise 4.4.6. Let $A = \{\max_{1 \leq m \leq n} |S_m| > x\}$, and let $N = \inf\{m \geq 0: |S_m| > x\}$ and $T = N \wedge n$. Then N and T are stopping times. Since $S_n^2 - s_n^2$ is a martingale, by Exercise 4.4.1,

$$\begin{aligned} 0 &= \mathbb{E}[S_0^2 - s_0^2] = \mathbb{E}[S_T^2 - s_T^2] \\ &= \mathbb{E}[(S_N^2 - s_N^2)\mathbf{1}_A] + \mathbb{E}[(S_n^2 - s_n^2)\mathbf{1}_{A^c}] \\ &\leq (x + K)^2 \mathbb{P}(A) + (x^2 - s_n^2) \mathbb{P}(A^c). \end{aligned}$$

Thus we have

$$0 \leq (x + K)^2 - (x + K)^2 \mathbb{P}(A^c) + (x^2 - s_n^2) \mathbb{P}(A^c)$$

and it follows that

$$\mathbb{P}(A^c) \leq \frac{(x + K)^2}{s_n^2 + (x + K)^2 - x^2} \leq \frac{(x + K)^2}{s_n^2}.$$

Exercise 4.4.7. Let $c \geq -\lambda$. Since $(X_n + c)^2$ is a nonnegative submartingale, by Doob's inequality,

$$\begin{aligned} \mathbb{P}(\max_{1 \leq m \leq n} X_m \geq \lambda) &= \mathbb{P}(\max_{1 \leq m \leq n} (X_m + c)^2 \geq (\lambda + c)^2) \\ &\leq \frac{\mathbb{E}(X_n + c)^2}{(\lambda + c)^2} \\ &= \frac{\mathbb{E}X_n^2 + c^2}{(\lambda + c)^2} \end{aligned}$$

Optimizing over $c \geq -\lambda$, the right-hand-side is minimized when $c = \mathbb{E}X_n^2/\lambda$. Plugging this c into the inequality, the result follows.

Exercise 4.4.8. (i)

$$\begin{aligned} \mathbb{E}(\bar{X}_n \wedge M) &= \int_0^\infty \mathbb{P}(\bar{X}_n \wedge M > x) dx \\ &\leq \int_0^1 dx + \int_1^\infty \mathbb{P}(\bar{X}_n \wedge M > x) dx \\ &\leq 1 + \int_1^\infty \frac{1}{x} \mathbb{E}(\bar{X}_n^+ \mathbf{1}_{\bar{X}_n \wedge M > x}) dx \\ &= 1 + \int_1^\infty \int \frac{1}{x} \bar{X}_n^+ \mathbf{1}_{\bar{X}_n \wedge M > x} d\mathbb{P} dx \\ &= 1 + \int \int_1^{\bar{X}_n \wedge M} \frac{1}{x} \bar{X}_n^+ dx d\mathbb{P} \\ &= 1 + \int \bar{X}_n^+ \log(\bar{X}_n \wedge M) d\mathbb{P} \end{aligned}$$

(ii) Let $f(a, b) = (a \log b - a \log a)/b$. From $\nabla f = 0$ we get $\log b = \log a + 1$, or simply $b = ea$. Thus the maximum of f is

$$f(a, ea) = 1/e$$

which proves the result.

Exercise 4.4.9. From $\mathbb{E}X_n < \infty$ and $\mathbb{E}Y_n < \infty$, all the expectations are finite. Note that $\mathbb{E}X_{m-1}Y_m = \mathbb{E}[\mathbb{E}[X_{m-1}Y_m|\mathcal{F}_{m-1}]] = \mathbb{E}[X_{m-1}\mathbb{E}[Y_m|\mathcal{F}_{m-1}]] = \mathbb{E}X_{m-1}Y_{m-1}$. Thus $\sum_{m=1}^n \mathbb{E}(X_m - X_{m-1})(Y_m - Y_{m-1}) = \sum_{m=1}^n (\mathbb{E}X_m Y_m - \mathbb{E}X_{m-1} Y_{m-1}) = \mathbb{E}X_n Y_n - \mathbb{E}X_0 Y_0$.

Exercise 4.4.10. Note that for all n ,

$$\begin{aligned} \mathbb{E}X_n^2 &= \mathbb{E}(X_0 + \xi_1 + \cdots + \xi_n)^2 \\ &= \mathbb{E}X_0^2 + \sum_{m=1}^n \mathbb{E}\xi_m^2 + 2 \sum_{m=1}^n \mathbb{E}X_0 \xi_m + 2 \sum_{m,k=1}^n \mathbb{E}\xi_m \xi_k \\ &= \mathbb{E}X_0^2 + \sum_{m=1}^n \mathbb{E}\xi_m^2 \\ &\leq \mathbb{E}X_0^2 + \sum_{m=1}^{\infty} \mathbb{E}\xi_m^2 < \infty \end{aligned}$$

where the third equality is due to Theorem 4.4.7. By Theorem 4.4.6, the result follows.

Exercise 4.4.11. For all n ,

$$\mathbb{E} \frac{X_n^2}{b_n^2} = \frac{1}{b_n^2} \mathbb{E}X_0^2 + \frac{1}{b_n^2} \sum_{m=1}^n \mathbb{E}\xi_m^2.$$

By Kronecker's lemma (Theorem 2.5.9), $\sum_{m=1}^n \mathbb{E}\xi_m^2/b_m^2$ converges to 0. Also, since $\mathbb{E}X_0^2 < \infty$ and $b_n \uparrow \infty$, $\mathbb{E}X_0^2/b_n^2$ converges to 0. Therefore $\mathbb{E}X_n^2/b_n^2$ converges to 0, which implies that $\sup \mathbb{E}X_n^2/b_n^2 < \infty$. Since X_n/b_n is a submartingale, by Theorem 4.4.6 X_n/b_n a.s. and in L^2 . As $X_n/b_n \rightarrow 0$ in L^2 , $X_n/b_n \rightarrow 0$ a.s.

4.6. Uniform Integrability, Convergence in L^1 .

Exercise 4.6.1. Let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ and define $X_n = \mathbb{E}[\theta|\mathcal{F}_n]$. Then X is a martingale with filtration (\mathcal{F}_n) . By Theorem 4.6.8, X_n converges a.s. to $\mathbb{E}[\theta|\mathcal{F}_\infty]$. Now it remains to show that $\theta = \mathbb{E}[\theta|\mathcal{F}_\infty]$. i.e. θ is \mathcal{F}_∞ -measurable. It follows by the strong law of large numbers that almost surely

$$\frac{Y_1 + \cdots + Y_n}{n} \rightarrow \theta$$

and thus θ is \mathcal{F}_∞ -measurable.

Exercise 4.6.2. Obviously X_n is \mathcal{F}_n -measurable and $X_n \in L^1$ for all n since $|X_n| \leq K$ as f is Lipschitz. Note that

$$\mathbb{E}X_{n+1} \mathbf{1}_{I_{k,n}} = \mathbb{E}X_{n+1} \mathbf{1}_{I_{2k+1,n+1}} + \mathbb{E}X_{n+1} \mathbf{1}_{I_{2k,n+1}}$$

$$\begin{aligned}
&= \frac{f\left(\frac{2k+2}{2^{n+1}}\right) - f\left(\frac{2k+1}{2^{n+1}}\right)}{\frac{1}{2^{n+1}}} \frac{1}{2^{n+1}} + \frac{f\left(\frac{2k+1}{2^{n+1}}\right) - f\left(\frac{2k}{2^{n+1}}\right)}{\frac{1}{2^{n+1}}} \frac{1}{2^{n+1}} \\
&= f\left(\frac{2k+2}{2^{n+1}}\right) - f\left(\frac{2k}{2^{n+1}}\right)
\end{aligned}$$

and that

$$\begin{aligned}
\mathbb{E}X_n \mathbf{1}_{I_{k,n}} &= \frac{f\left(\frac{k+1}{2^n}\right) - f\left(\frac{k}{2^n}\right)}{\frac{1}{2^n}} \frac{1}{2^n} \\
&= f\left(\frac{2k+2}{2^{n+1}}\right) - f\left(\frac{2k}{2^{n+1}}\right).
\end{aligned}$$

Thus $\mathbb{E}X_{n+1} \mathbf{1}_{I_{k,n}} = \mathbb{E}X_n \mathbf{1}_{I_{k,n}}$ for all k , and by the definition of conditional expectation we have $X_n = \mathbb{E}[X_{n+1} | \mathcal{F}_n]$. That is, (X_n) is a martingale. Since $|X_n| \leq K$ for all n , the family (X_n) is uniformly integrable. Therefore, by Theorem 4.6.7, X_n converges a.s. and in L^1 to an L^1 random variable X_∞ . Let a_k and b_k be endpoints of some $I_{k,n}$. Observe that

$$f(b_k) - f(a_k) = \int_{a_k}^{b_k} X_n(\omega) d\omega.$$

By the dominated convergence, it follows that

$$f(b_k) - f(a_k) = \int_{a_k}^{b_k} X_\infty(\omega) d\omega$$

for all such points a_k and b_k . Since f is continuous and $|X_\infty| \leq K$, letting $a_k \rightarrow a$ and $b_k \rightarrow b$ we have the last assertion.

Exercise 4.6.3. By Theorem 4.6.8, $\mathbb{E}[f | \mathcal{F}_n] \rightarrow \mathbb{E}[f | \mathcal{F}_\infty]$ a.s. and in L^1 . Now it remains to show that f is \mathcal{F}_∞ -measurable, which is obvious.

Exercise 4.6.4. Let $\omega \in \{\limsup X_n < \infty\}$. We will show that $\omega \in D$. Let $M(\omega) = \limsup X_n(\omega) < \infty$. Then there exists $N(\omega)$ such that $X_n(\omega) < 2M(\omega)$ for all $n \geq N(\omega)$. From the given assumption, it follows that

$$\mathbb{P}(D | X_1, \dots, X_n)(\omega) \geq \delta(x)$$

for all $n \geq N(\omega)$. Since $D \in \sigma(X_1, X_2, \dots)$, by Theorem 4.6.9, the left-hand-side of the above display converges to $\mathbf{1}_D(\omega)$ for all $\omega \in \{\limsup X_n < \infty\}$ except on a measure zero set. That is, we have $\mathbf{1}_D \geq \delta(x) > 0$ for all $\omega \in \{\limsup X_n < \infty\} \setminus E$ where $\mathbb{P}(E) = 0$. It follows that $\mathbf{1}_D(\omega) = 1$ and thus $\omega \in D$. i.e. $\{\limsup X_n < \infty\} \setminus E \subset D$. The result now immediately follows.

Exercise 4.6.5. content

Exercise 4.6.6. Note that X_n is a martingale as $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n(\alpha + \beta X_n) + (1 - X_n)\beta X_n = X_n$. Since $|X_n| \leq 1$ for all n , it is a uniformly integrable martingale. Thus X_n converges to an L^1 random variable X almost surely (and in L^1).

To see $X \in \{0, 1\}$, we first let $B_n = \{X_{n+1} = \alpha + \beta X_n\}$ and $B = \limsup B_n$. For $\omega \in B$, infinitely often,

$$\begin{aligned} X_{n+1}(\omega) - X_n(\omega) &= \alpha + \beta X_n(\omega) - X_n(\omega) \\ &= \alpha(1 - X_n(\omega)). \end{aligned}$$

Since X_n converges almost surely, $|X_{n+1}(\omega) - X_n(\omega)| < \epsilon$ for large enough n . That is, infinitely often we have

$$\alpha|1 - X_n(\omega)| < \epsilon.$$

Thus $X(\omega) = 1$ for almost every $\omega \in B$. Now consider $C = B^c = \liminf B_n^c$. For $\omega \in C$, eventually,

$$X_{n+1} = \beta X_n$$

which then immediately implies that $X(\omega) = 0$. Thus $X(\omega) = 0$ for almost every $\omega \in C$. This proves that $X \in \{0, 1\}$

Finally, since X_n is a martingale, we have $\mathbb{E}X_0 = \mathbb{E}X_n$ and by the dominated convergence $\mathbb{E}X_n \rightarrow \mathbb{E}X$. Thus

$$\theta = \mathbb{E}X_0 = \mathbb{E}X = 1 \cdot \mathbb{P}(X = 1) + 0 \cdot \mathbb{P}(X = 0) = \mathbb{P}(X = 1).$$

Exercise 4.6.7. Note that

$$\begin{aligned} \mathbb{E}|\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y|\mathcal{F}_\infty]| &\leq \mathbb{E}|\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y|\mathcal{F}_n]| + \mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n] - \mathbb{E}[Y|\mathcal{F}_\infty]| \\ &\leq \mathbb{E}|Y_n - Y| + \mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n] - \mathbb{E}[Y|\mathcal{F}_\infty]| \end{aligned}$$

By Theorem 4.6.8, $\mathbb{E}[Y|\mathcal{F}_n] \rightarrow \mathbb{E}[Y|\mathcal{F}_\infty]$ in L^1 and $Y_n \rightarrow Y$ in L^1 by the assumption.

4.7. Backwards Martingales.

Exercise 4.7.1. By the L^p maximum inequality, for all $n \leq 0$,

$$\mathbb{E} \left[\sup_{n \leq m \leq 0} |X_m|^p \right] \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}|X_0|^p.$$

Letting $n \rightarrow -\infty$, it follows that $\sup_{m \leq 0} |X_m| \in L^p$. Let $X_{-\infty}$ be the limit of the backward martingale X_n (which exists by Theorem 4.7.1). Noting that $|X_n - X_{-\infty}|^p \leq (2 \sup_{m \leq 0} |X_m|)^p$, by the dominated convergence we have

$$\mathbb{E}|X_n - X_{-\infty}|^p \rightarrow 0.$$

Exercise 4.7.2. Note that

$$\begin{aligned} |\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_{-\infty}]| &\leq |\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_n]| + |\mathbb{E}[Y_{-\infty}|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_{-\infty}]| \\ &\leq \mathbb{E}|Y_n - Y_{-\infty}||\mathcal{F}_n| + |\mathbb{E}[Y_{-\infty}|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_{-\infty}]|. \end{aligned}$$

By Theorem 4.7.3, $|\mathbb{E}[Y_{-\infty}|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_{-\infty}]| \rightarrow 0$. To see the convergence of the first term, let $W_N = \sup_{m,n \leq N} |Y_m - Y_n|$. Then $W_N \leq 2Z \in L^1$ and $W_N \rightarrow 0$. It follows that

$$\begin{aligned} \limsup_{n \rightarrow -\infty} \mathbb{E}[|Y_n - Y_{-\infty}||\mathcal{F}_n] &\leq \limsup_{n \rightarrow -\infty} \mathbb{E}[W_N|\mathcal{F}_n] \\ &= \mathbb{E}[W_N|\mathcal{F}_{-\infty}] \\ &\rightarrow 0 \end{aligned}$$

where the last convergence is due to the dominated convergence theorem.

Exercise 4.7.3. Let $p = \mathbb{P}(X_1 = 1, \dots, X_m = 1)$. Because (X_n) is exchangeable, $\mathbb{P}(X_{n_1} = 1, \dots, X_{n_m} = 1) = p$ for any subsequence (n_k) . Thus $\mathbb{P}(S_n = m) = \binom{n}{m}p$. Similarly, $\mathbb{P}(X_1 = 1, \dots, X_k = 1, X_{n_1} = 1, \dots, X_{n_{m-k}} = 1) = p$ for any (n_k) , and thus $\mathbb{P}(X_1 = 1, \dots, X_k = 1, S_n = m) = \binom{n-k}{m-k}p$. Therefore,

$$\mathbb{P}(X_1 = 1, \dots, X_k = 1 | S_n = m) = \binom{n-k}{m-k} / \binom{n}{m} = \binom{n-k}{n-m} / \binom{n}{m}$$

Exercise 4.7.4. Note that from the exchangeability,

$$0 \leq \mathbb{E}(X_1 + \dots + X_n)^2 = n\mathbb{E}X_1^2 + n(n-1)\mathbb{E}X_1X_2$$

and thus

$$\mathbb{E}X_1X_2 \geq -\frac{\mathbb{E}X_1^2}{(n-1)} \rightarrow 0.$$

Exercise 4.7.5. Let $\varphi(x, y) = (x - y)^2$ and let

$$A_n(\varphi) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \varphi(X_i, X_j).$$

Since $A_n(\varphi) \in \mathcal{E}_n$, we have

$$\begin{aligned} A_n(\varphi) &= \mathbb{E}[A_n(\varphi)|\mathcal{E}_n] \\ &= \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \mathbb{E}[\varphi(X_i, X_j)|\mathcal{E}_n] \\ &= \mathbb{E}[\varphi(X_1, X_2)|\mathcal{E}_n]. \end{aligned}$$

By Theorem 4.7.3, $\mathbb{E}[\varphi(X_1, X_2)|\mathcal{E}_n] \rightarrow \mathbb{E}[\varphi(X_1, X_2)|\mathcal{E}]$. (Note that this is possible because $\varphi(X_1, X_2)$ is integrable.) Now since \mathcal{E} is trivial due to Example 4.7.6, it follows that

$$\begin{aligned} \mathbb{E}[\varphi(X_1, X_2)|\mathcal{E}] &= \mathbb{E}\varphi(X_1, X_2) \\ &= \mathbb{E}X_1^2 + \mathbb{E}X_2^2 - 2\mathbb{E}X_1\mathbb{E}X_2 \\ &= 2\sigma^2. \end{aligned}$$

4.8. Optional Stopping Theorems.

Exercise 4.8.1. Let $A \in \mathcal{F}_L$ and let $N = L\mathbf{1}_A + M\mathbf{1}_{A^c}$. Note that N is a stopping time according to Exercise 4.4.3 and that $N \leq M$. Since $Y_{M \wedge n}$ is a uniformly integrable submartingale, applying Theorem 4.8.3 to this leads to

$$\mathbb{E}Y_0 \leq \mathbb{E}Y_N \leq \mathbb{E}Y_M.$$

Observe that $\mathbb{E}Y_N = \mathbb{E}Y_L\mathbf{1}_A + \mathbb{E}Y_M\mathbf{1}_{A^c}$. It follows that

$$\mathbb{E}Y_L\mathbf{1}_A \leq \mathbb{E}Y_M\mathbf{1}_A.$$

Since $A \in \mathcal{F}_L$ was arbitrary, by the definition of conditional expectation,

$$Y_L \leq \mathbb{E}[Y_M | \mathcal{F}_L]$$

and also $\mathbb{E}Y_L \leq \mathbb{E}Y_M$.

Exercise 4.8.2. Let $N = \inf\{m: X_m > \lambda\}$. Since $X_N\mathbf{1}_{N < \infty} \geq \lambda$,

$$\lambda\mathbb{P}(N < \infty) \leq \mathbb{E}X_N\mathbf{1}_{N < \infty} \leq \mathbb{E}X_0$$

by Theorem 4.8.4. Observe that $\{N < \infty\} = \{\sup_m X_m > \lambda\}$.

Exercise 4.8.3. Let $X_n = S_n^2 - n\sigma^2$. Note that

$$\begin{aligned} \mathbb{E}|X_T| &\leq \mathbb{E}S_T^2 = \mathbb{E}S_T^2\mathbf{1}_{T < \infty} + \mathbb{E}S_T^2\mathbf{1}_{T = \infty} \\ &\leq \mathbb{E}(a + \xi)^2\mathbf{1}_{T < \infty} + a^2 \\ &\leq 2a^2 + \sigma^2 < \infty. \end{aligned}$$

To see

$$\mathbb{E}[|X_n\mathbf{1}_{T > n}|; |X_n\mathbf{1}_{T > n}| > a^2 + M] = 0$$

for all n , we observe that $|X_n\mathbf{1}_{T > n}| \leq a^2\mathbf{1}_{T > n} \leq a^2$ for all n and thus $\{|X_n\mathbf{1}_{T > n}| > a^2 + M\} = \emptyset$. By Theorem 4.8.2, $X_{T \wedge n}$ is a uniformly integrable martingale. It follows that

$$0 = \mathbb{E}X_{T \wedge n} \rightarrow \mathbb{E}X_T = \mathbb{E}[S_T^2 - T\sigma^2].$$

If $\mathbb{E}T = \infty$, then the result is trivial. For $\mathbb{E}T < \infty$, we have $T < \infty$ a.s. and thus

$$\sigma^2\mathbb{E}T = \mathbb{E}S_T^2 \geq a^2.$$

Exercise 4.8.4. Let $X_n = S_n^2 - n\sigma^2$. Then X_n is a martingale. Since T is a stopping time with $\mathbb{E}T < \infty$, by the monotone convergence,

$$\mathbb{E}S_{T \wedge n}^2 = \sigma^2\mathbb{E}[T \wedge n] \rightarrow \sigma^2\mathbb{E}T.$$

In particular, $\mathbb{E}S_{T \wedge n}^2 \leq \sigma^2\mathbb{E}T$ for all n . i.e. $\sup_n \mathbb{E}S_{T \wedge n}^2 < \infty$. Since $S_{T \wedge n}$ is a martingale, this implies that $S_{T \wedge n}$ converges a.s. and in L^2 to S_T . Therefore, $\mathbb{E}S_{T \wedge n}^2 \rightarrow \mathbb{E}S_T^2$ and the result follows.

Exercise 4.8.5. (a) Let $A = p - q$ and $B = 1 - (p - q)^2$. Using the same argument as in Exercise 4.8.3, it follows that $(S_{V_0 \wedge n} - (V_0 \wedge n)A)^2 - (V_0 \wedge n)B$ is a uniformly integrable martingale. Thus we have

$$\begin{aligned} B\mathbb{E}V_0 &= \mathbb{E}(S_{V_0} - V_0A)^2 \\ &= \mathbb{E}S_{V_0}^2 - 2A\mathbb{E}S_{V_0}V_0 + A^2\mathbb{E}V_0^2. \end{aligned}$$

Now since $p < 1/2$, $V_0 < \infty$ a.s. and $S_{V_0} = 0$. Thus the above display becomes

$$B\mathbb{E}V_0 = A^2\mathbb{E}V_0^2.$$

Therefore, the second moment of V_0 is (not the variance; probably a typo)

$$\mathbb{E}V_0^2 = \frac{B}{A^2}\mathbb{E}V_0 = \frac{1 - (p - q)^2}{(q - p)^3}x.$$

Exercise 4.8.6. (a) Let $\phi(\theta) = \mathbb{E}\exp(\theta\xi_i) = pe^\theta + qe^{-\theta}$ and $X_n = \exp(\theta S_n)/\phi(\theta)^n$. Then X_n is a martingale and thus

$$\begin{aligned} e^{\theta x} &= \mathbb{E}X_0 = \mathbb{E}X_{V_0 \wedge n} \\ &= \mathbb{E}\frac{\exp(\theta S_{V_0 \wedge n})}{\phi(\theta)^{V_0 \wedge n}}. \end{aligned}$$

Now note that, since $\theta \leq 0$ and $S_{V_0 \wedge n} \geq 0$, $\exp(\theta S_{V_0 \wedge n}) \leq 1$ and $\phi(\theta) \geq 1$. By the bounded convergence theorem, it follows that

$$\begin{aligned} e^{\theta x} &= \mathbb{E}\frac{\exp(\theta S_{V_0 \wedge n})}{\phi(\theta)^{V_0 \wedge n}} \\ &\rightarrow \mathbb{E}\frac{\exp(\theta S_{V_0})}{\phi(\theta)^{V_0}} \\ &= \mathbb{E}[\phi(\theta)^{-V_0}]. \end{aligned}$$

(b) From $\phi(\theta) = 1/s$, we get $pe^{2\theta} - e^\theta/s + q = 0$. Solving this with respect to e^θ and noting that $e^\theta \leq 1$, we have

$$e^\theta = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}.$$

Thus

$$\mathbb{E}[s^{V_0}] = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2ps} \right)^x.$$

Exercise 4.8.7. Note that $\mathbb{E}[S_{n+1}^4|\mathcal{F}_n] = S_n^4 + 6S_n^2 + 1$ and $\mathbb{E}[S_{n+1}^2|\mathcal{F}_n] = S_n^2 + 1$. From $\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n$, we get $(2b - 6)n + (b + c - 5) = 0$ and thus $b = 3$ and $c = 2$. Now from $\mathbb{E}Y_0 = \mathbb{E}Y_{T \wedge n}$,

$$0 = \mathbb{E}S_{T \wedge n}^4 - 6\mathbb{E}(T \wedge n)S_{T \wedge n}^2 + 3\mathbb{E}(T \wedge n)^2 + 2\mathbb{E}(T \wedge n).$$

Noting that $|S_{T \wedge n}| \leq a$ and that $\mathbb{E}T < \infty$ (from the proof of Theorem 4.8.7), apply the DCT and MCT to get

$$0 = a^4 - 6a^2\mathbb{E}T + 3\mathbb{E}T^2 + 2\mathbb{E}T.$$

Now $\mathbb{E}T = a^2$ leads to

$$\mathbb{E}T^2 = \frac{5a^4 - 2a^2}{3}.$$

Exercise 4.8.8. Note that $S_{\tau \wedge n} \geq a$, and hence $X_n \leq e^{a\theta_0}$ for all n . Thus $X_{\tau \wedge n}$ is a uniformly integrable martingale, and by Theorem 4.8.2, $1 = \mathbb{E}X_0 = \mathbb{E}X_\tau$. Now note that

$$\begin{aligned} 1 = \mathbb{E}X_\tau &\geq \mathbb{E}[X_\tau; S_\tau \leq a] \\ &\geq e^{a\theta_0}\mathbb{P}(S_\tau \leq a) \end{aligned}$$

from which the result follows.

Exercise 4.8.9. By the same argument as in Exercise 4.8.8, $X_{T \wedge n}$ is a uniformly integrable martingale and we have $1 = \mathbb{E}X_0 = \mathbb{E}X_T$. Now observe that

$$\begin{aligned} 1 = \mathbb{E}X_T &= \mathbb{E}[X_T; T < \infty] + \mathbb{E}[X_T; T = \infty] \\ &= e^{a\theta_0}\mathbb{P}(T < \infty) + \mathbb{E}[X_\infty; T = \infty]. \end{aligned}$$

Here, $X_\infty = \exp(\theta_0 S_\infty) = 0$ since $S_\infty = \lim S_n = \infty$ a.s. by the strong law of large numbers. Therefore,

$$\mathbb{P}(T < \infty) = e^{-a\theta_0}.$$

Exercise 4.8.10. Note that $\phi(\theta) = \mathbb{E}\exp(\theta\xi) = (e^{-\theta} + e^\theta + e^{2\theta})/3$. Solving $\phi(\theta) = 1$ with $\theta \leq 0$, we get $e^\theta = \sqrt{2} - 1$. By Exercise 4.8.9, $\mathbb{P}(T < \infty) = e^{\theta i} = (\sqrt{2} - 1)^i$.

Exercise 4.8.11. Let $\phi(\theta) = \mathbb{E}\exp(\theta\xi) = \exp((c - \mu)\theta + \sigma^2\theta^2/2)$. Note that $\theta_0 = -2(c - \mu)/\sigma^2 \leq 0$ satisfies $\phi(\theta_0) = 1$. For this θ_0 , $X_n = \exp(\theta_0 S_n)$ is a martingale. Now let $T = \inf\{S_n \leq 0\}$. By the same argument as in Exercise 4.8.8, $X_{T \wedge n}$ is a uniformly integrable martingale. Thus

$$\begin{aligned} \mathbb{E}\exp(\theta_0 S_0) &= \mathbb{E}\exp(\theta_0 S_T) \\ &\geq \mathbb{E}[\exp(\theta_0 S_T); T < \infty] \\ &\geq \mathbb{P}(T < \infty). \end{aligned}$$

Now plugging $\theta_0 = -2(c - \mu)/\sigma^2$ into the last display, the result follows.

4.9. Combinatorics of Simple Random Walk.

Exercise 4.9.1. content

5. MARKOV CHAINS

5.1. Examples.

Exercise 5.1.1.

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \begin{cases} 1 - \frac{i}{N} & \text{for } j = i + 1, \\ \frac{i}{N} & \text{for } j = i. \end{cases}$$

$$=: p(i, j)$$

Exercise 5.1.2. Note that

$$\begin{aligned} \mathbb{P}(X_4 = 2 | X_3 = 1, X_2 = 0, X_1 = 0) &= \frac{\mathbb{P}(X_4 = 2, X_3 = 1, X_2 = 0, X_1 = 0)}{\mathbb{P}(X_3 = 1, X_2 = 0, X_1 = 0)} \\ &= \frac{\mathbb{P}(S_4 = 2, S_3 = 1, S_2 = 0, S_1 = -1)}{\mathbb{P}(S_3 = 1, S_2 = 0, S_1 = -1)} = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} \mathbb{P}(X_4 = 2 | X_3 = 1, X_2 = 1, X_1 = 1) &= \frac{\mathbb{P}(X_4 = 2, X_3 = 1, X_2 = 1, X_1 = 1)}{\mathbb{P}(X_3 = 1, X_2 = 1, X_1 = 1)} \\ &= \frac{\mathbb{P}(S_4 = 2, S_3 = 1, S_2 = 0, S_1 = 1)}{\mathbb{P}(S_2 = 0, S_1 = 1)} = \frac{1}{4}. \end{aligned}$$

Thus X_n is not a Markov chain.

Exercise 5.1.3. Note that $X_i \in \sigma(\xi_n, \dots, \xi_0)$ for $i \leq n - 1$. Since $X_{n+1} = (\xi_{n+1}, \xi_{n+2})$, X_{n+1} is independent of $\{X_{n-1}, \dots, X_0\}$. Thus,

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j | X_n = i).$$

That is, X_n is a Markov chain. The transition probabilities are

p	HH	HT	TH	TT
HH	1/2	1/2	0	0
HT	0	0	1/2	1/2
TH	1/2	1/2	0	0
TT	0	0	1/2	1/2

and

p^2	HH	HT	TH	TT
HH	1/4	1/4	1/4	1/4
HT	1/4	1/4	1/4	1/4
TH	1/4	1/4	1/4	1/4
TT	1/4	1/4	1/4	1/4

Indeed, the distribution of $X_{n+2} = (\xi_{n+2}, \xi_{n+3})$ is independent of $X_n = (\xi_n, \xi_{n+1})$.

Exercise 5.1.4. The transition probabilities are

	AA,AA	AA,Aa	AA,aa	Aa,Aa	Aa,aa	aa,aa
AA,AA	1	0	0	0	0	0
AA,Aa	1/4	1/2	0	1/4	0	0
AA,aa	0	0	0	1	0	0
Aa,Aa	1/16	1/4	1/8	1/4	1/4	1/16
Aa,aa	0	0	0	1/4	1/2	1/4
aa,aa	0	0	0	0	0	1

Exercise 5.1.5.

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \begin{cases} \frac{m-i}{m} \times \frac{b-i}{m} & \text{for } j = i + 1 \\ \frac{m-i}{m} \times \frac{m-b+i}{m} + \frac{i}{m} \times \frac{b-i}{m} & \text{for } j = i \\ \frac{i}{m} \times \frac{m-b+i}{m} & \text{for } j = i - 1 \end{cases}$$

Exercise 5.1.6. Let $K_n = |\{1 \leq i \leq n : X_i = 1\}|$. Note that $S_n = K_n - (n - K_n) = 2K_n - n$, or $K_n = (S_n + n)/2$. From the elementary definition of conditional probability,

$$\mathbb{P}(X_{n+1} = 1 | X_n, \dots, X_1) = \frac{\int_0^1 \theta^{K_n+1} (1-\theta)^{n-K_n}}{\int_0^1 \theta^{K_n} (1-\theta)^{n-K_n}}.$$

Here, integrating by parts,

$$\int_0^1 \theta^m (1-\theta)^k = (-1)^{2m} \int_0^1 \frac{m!}{(k+1) \cdots (k+m)} (1-\theta)^{k+m} = \frac{m!k!}{(k+m+1)!}.$$

Thus,

$$\mathbb{P}(X_{n+1} = 1 | X_n, \dots, X_1) = \frac{K_n + 1}{n + 2} = \frac{S_n + n + 2}{2(n + 2)}.$$

Now, note that

$$\begin{aligned} \mathbb{P}(S_{n+1} = s_{n+1} | S_n, \dots, S_1) &= \begin{cases} \mathbb{P}(X_{n+1} = 1 | X_n, \dots, X_1) & \text{for } s_{n+1} = S_n + 1 \\ 1 - \mathbb{P}(X_{n+1} = 1 | X_n, \dots, X_1) & \text{for } s_{n+1} = S_n - 1 \end{cases} \\ &= \begin{cases} \frac{S_n + n + 2}{2(n + 2)} & \text{for } s_{n+1} = S_n + 1 \\ 1 - \frac{S_n + n + 2}{2(n + 2)} & \text{for } s_{n+1} = S_n - 1. \end{cases} \end{aligned}$$

That is, the transition probability depends only on S_n . Therefore, S_n is a Markov chain.

5.2. Construction, Markov Properties.

Exercise 5.2.1.

$$\begin{aligned} \mathbb{P}_\mu(A \cap B | X_n) &= \mathbb{E}_\mu [\mathbb{E}_\mu [\mathbf{1}_A \mathbf{1}_B | \mathcal{F}_n] | X_n] \\ &= \mathbb{E}_\mu [\mathbf{1}_A \mathbb{E}_\mu [\mathbf{1}_B | \mathcal{F}_n] | X_n] \\ &\stackrel{(*)}{=} \mathbb{E}_\mu [\mathbf{1}_A \mathbb{E}_\mu [\mathbf{1}_B | X_n] | X_n] \\ &= \mathbb{E}_\mu [\mathbf{1}_A | X_n] \mathbb{E}_\mu [\mathbf{1}_B | X_n] \end{aligned}$$

Here, $(*)$ is from the Markov property.

Exercise 5.2.2. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $\mathcal{F}_\infty = \sigma(X_1, X_2, \dots)$. Note that $\bigcup_{m \geq n+1} \{X_m \in B_m\} \in \mathcal{F}_\infty$. By Lévy's 0-1 law, on $\{X_n \in A_n\}$,

$$\begin{aligned} \delta &\leq \mathbb{P}\left(\bigcup_{m \geq n+1} \{X_m \in B_m\} | X_n\right) \stackrel{(*)}{=} \mathbb{P}\left(\bigcup_{m \geq n+1} \{X_m \in B_m\} | \mathcal{F}_n\right) \\ &\rightarrow \mathbf{1}_{\bigcup_{m \geq n+1} \{X_m \in B_m\}} \end{aligned}$$

where $(*)$ is from the Markov property. That is, for $\omega \in \{X_n \in A_n\}$, $\mathbf{1}_{\bigcup_{m \geq n+1} \{X_m \in B_m\}}(\omega) = 1$, or in other words,

$$\{X_n \in A_n\} \subset \bigcup_{m \geq n+1} \{X_m \in B_m\}.$$

Thus,

$$\bigcup_{n \geq k} \{X_n \in A_n\} \subset \bigcup_{m \geq k+1} \{X_m \in B_m\}$$

and letting $k \rightarrow \infty$, $\{X_n \in A_n \text{ i.o.}\} \subset \{X_n \in B_n \text{ i.o.}\}$.

Exercise 5.2.3. Let $A_\delta = \{x: \mathbb{P}_x(D) \geq \delta\}$. Then obviously $\mathbb{P}_{X_n}(D) \geq \delta$ on $\{X_n \in A_\delta\}$. Note that

$$\mathbb{P}_{X_n}(D) = \mathbb{P}\left(\bigcup_{m \geq n+1} \{X_m = a\} | X_n\right) \geq \delta$$

on $\{X_n \in A_\delta\}$. By the previous exercise,

$$\mathbb{P}(\{X_n \in A_\delta \text{ i.o.}\} - \{X_n = a \text{ i.o.}\}) = 0.$$

From $\{X_n = a \text{ i.o.}\} \subset D$, we have

$$\{X_n \in A_\delta \text{ i.o.}\} \subset \{X_n = a \text{ i.o.}\} \subset D.$$

This implies that $\mathbb{P}(X_n \in A_\delta^c \text{ eventually}) = 1$ on D^c . That is, on D^c , $\mathbb{P}_{X_n}(D) \leq \delta$ eventually almost surely. Since δ is arbitrary, we have the desired result.

Exercise 5.2.4.

$$\begin{aligned} p^n(x, y) &= \mathbb{P}_x(X_n = y) = \sum_{m=1}^n \mathbb{P}_x(T_y = m) \mathbb{P}_x(X_n = y | T_y = m) \\ &= \sum_{m=1}^n \mathbb{P}_x(T_y = m) \mathbb{P}_x(X_n = y | X_1 \neq y, \dots, X_{m-1} \neq y, X_m = y) \\ &\stackrel{(*)}{=} \sum_{m=1}^n \mathbb{P}_x(T_y = m) \mathbb{P}_x(X_n = y | X_m = y) \\ &= \sum_{m=1}^n \mathbb{P}_x(T_y = m) p^{n-m}(y, y) \end{aligned}$$

where $(*)$ is by the Markov property.

Exercise 5.2.5. Let $T^{(k)} = \inf\{i \geq k : X_i = x\} \geq k$. Then, similarly to the previous exercise,

$$\mathbb{P}_x(X_m = x) = \sum_{j=k}^m \mathbb{P}_x(T^{(k)} = j) \mathbb{P}_x(X_{m-j} = x).$$

Summing over m and applying Fubini,

$$\begin{aligned} \sum_{m=k}^{n+k} \mathbb{P}_x(X_m = x) &= \sum_{m=k}^{n+k} \sum_{j=k}^m \mathbb{P}_x(T^{(k)} = j) \mathbb{P}_x(X_{m-j} = x) \\ &= \sum_{j=k}^{n+k} \sum_{m=j}^{n+k} \mathbb{P}_x(T^{(k)} = j) \mathbb{P}_x(X_{m-j} = x) \\ &= \sum_{j=k}^{n+k} \mathbb{P}_x(T^{(k)} = j) \sum_{m=0}^{n-(j-k)} \mathbb{P}_x(X_m = x) \\ &\leq \sum_{j=k}^{n+k} \mathbb{P}_x(T^{(k)} = j) \sum_{m=0}^n \mathbb{P}_x(X_m = x) \\ &= \mathbb{P}_x(T^{(k)} \leq n+k) \sum_{m=0}^n \mathbb{P}_x(X_m = x) \\ &\leq \sum_{m=0}^n \mathbb{P}_x(X_m = x). \end{aligned}$$

Exercise 5.2.6. Since $\mathbb{P}_x(T_C < \infty) > 0$, there exist n_x and ϵ_x such that $\mathbb{P}_x(T_C \leq n_x) = \epsilon_x > 0$. Let $N = \max_{s \in S-C} n_x$ and $\epsilon = \min_{s \in S-C} \epsilon_x$. Then for all $y \in S-C$, $\mathbb{P}_y(T_C > N) \leq 1 - \epsilon$, and thus the desired result holds for $k = 1$. Now we proceed by induction. Note that

$$\begin{aligned} \mathbb{P}_y(T_C > kN) &= \mathbb{P}_y(X_1 \notin C, \dots, X_{kN} \notin C) \\ &= \mathbb{P}_y(X_{N+1} \notin C, \dots, X_{kN} \notin C | X_1 \notin C, \dots, X_N \notin C) \mathbb{P}_y(X_1 \notin C, \dots, X_N \notin C) \\ &\leq (1 - \epsilon) \mathbb{P}_y(X_{N+1} \notin C, \dots, X_{kN} \notin C | X_N \notin C) \end{aligned}$$

Here,

$$\begin{aligned} \mathbb{P}_y(X_{N+1} \notin C, \dots, X_{kN} \notin C | X_N \notin C) &= \frac{\mathbb{P}_y(X_N \notin C, \dots, X_{kN} \notin C)}{\mathbb{P}_y(X_N \notin C)} \\ &= \frac{1}{\mathbb{P}_y(X_N \notin C)} \int_{\{X_N \notin C\}} \mathbf{1}_{\{X_{N+1} \notin C, \dots, X_{kN} \notin C\}} d\mathbb{P}_y \\ &= \frac{1}{\mathbb{P}_y(X_N \notin C)} \int_{\{X_N \notin C\}} \mathbb{E}_y[\mathbf{1}_{\{X_{N+1} \notin C, \dots, X_{kN} \notin C\}} | X_N] d\mathbb{P}_y \\ &= \frac{1}{\mathbb{P}_y(X_N \notin C)} \int_{\{X_N \notin C\}} \mathbb{E}_{X_N}[\mathbf{1}_{\{X_1 \notin C, \dots, X_{(k-1)N} \notin C\}}] d\mathbb{P}_y. \end{aligned}$$

By the induction hypothesis, $\mathbb{E}_{X_N}[\mathbf{1}_{\{X_1 \notin C, \dots, X_{(k-1)N} \notin C\}}] \leq (1 - \epsilon)^{k-1}$ on $\{X_N \notin C\}$ \mathbb{P}_y -almost surely. Thus the desired result follows for general k .

Exercise 5.2.7.

5.3. Recurrence and Transience.

5.4. Recurrence of Random Walks.

5.5. Stationary Measures.

5.6. Asymptotic Behavior.

5.7. Periodicity, Tail σ -Field.

5.8. General State Space.

6. ERGODIC THEOREMS

6.1. Definitions and Examples.

Exercise 6.1.1. (i) Since $\varphi^{-1}\Omega \subset \Omega$ and $\varphi^{-1}\Omega$ has probability 1, $\varphi^{-1}\Omega = \Omega$ up to a null set. Thus $\Omega \in \mathcal{I}$.

(ii) Let $A \in \mathcal{I}$. Then up to a null set,

$$\varphi^{-1}(\Omega \setminus A) = \varphi^{-1}\Omega \setminus \varphi^{-1}A = \Omega \setminus A,$$

and thus $\Omega \setminus A \in \mathcal{I}$.

(iii) Let $A_j \in \mathcal{I}$ for $j \geq 1$. Then up to a null set,

$$\varphi^{-1}\left(\bigcup_j A_j\right) = \bigcup_j \varphi^{-1}A_j = \bigcup_j A_j,$$

and thus $\bigcup_j A_j \in \mathcal{I}$.

Now to see that $X \in \mathcal{I}$ if and only if $X \circ \varphi = X$ a.s., suppose first that $X \in \mathcal{I}$. Then up to a null set,

$$\{X \circ \varphi > a\} = \varphi^{-1}\{X > a\} = \{X > a\}$$

since $\{X > a\} \in \mathcal{I}$. Since this is true for arbitrary a , $X \circ \varphi = X$ a.s. (Confer the argument in Exercise 4.1.10.) For the other direction, suppose $X \circ \varphi = X$ a.s. Then up to a null set,

$$\{X > a\} = \{X \circ \varphi > a\} = \varphi^{-1}\{X > a\},$$

and thus $\{X > a\} \in \mathcal{I}$.

Exercise 6.1.2. (i)

$$\varphi^{-1}(B) = \varphi^{-1}\left(\bigcup_{n=0}^{\infty} \varphi^{-n}(A)\right) = \bigcup_{n=1}^{\infty} \varphi^{-n}(A) \subset B$$

(ii)

$$\varphi^{-1}(C) = \varphi^{-1}\left(\bigcap_{n=0}^{\infty} \varphi^{-n}(B)\right) = \bigcap_{n=1}^{\infty} \varphi^{-n}(B) = \bigcap_{n=1}^{\infty} (\varphi^{-n}(B) \cap B) = C$$

since $\varphi^{-n}(B) \cap B = \varphi^{-n}(B)$.

- (iii) Suppose that A is almost invariant. Define $B = \bigcup_{n=0}^{\infty} \varphi^{-n}(A)$ and $C = \bigcap_{n=0}^{\infty} \varphi^{-n}(B)$. Then by the above, C is invariant in the strict sense. Note that

$$\begin{aligned} C &= \bigcap_{n=0}^{\infty} \varphi^{-n} \left(\bigcup_{k=0}^{\infty} \varphi^{-k}(A) \right) \\ &= \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{\infty} \varphi^{-(n+k)}(A) \\ &= \bigcap_{n=0}^{\infty} \bigcup_{k \geq n} \varphi^{-k}(A) \\ &= \limsup_{n \rightarrow \infty} \varphi^{-n}(A). \end{aligned}$$

Thus both $C \setminus A = \limsup(\varphi^{-n}(A) \setminus A)$ and $A \setminus C = \liminf(A \setminus \varphi^{-n}(A))$ are measure-zero. Conversely, suppose that there exists C invariant in the strict sense with $\mathbb{P}(A \Delta C) = 0$. Then both $\varphi^{-1}(A \setminus C) = \varphi^{-1}(A) \setminus C$ and $\varphi^{-1}(C \setminus A) = C \setminus \varphi^{-1}(A)$ are measure-zero. It follows that both $\varphi^{-1}(A) \setminus A \subset (\varphi^{-1}(A) \setminus C) \cup (C \setminus A)$ and $A \setminus \varphi^{-1}(A) \subset (A \setminus C) \cup (C \setminus \varphi^{-1}(A))$ are measure-zero.

Exercise 6.1.3. (i) Suppose for the sake of contradiction that $x_n = x_m$ for some $n \neq m$. Then $(n - m)\theta = k \in \mathbb{N}$ and $\theta \in \mathbb{Q}$, which is a contradiction. Thus x_n are distinct in $[0, 1)$, and hence for any N , $|x_m - x_n| < 1/N$ for some $m < n \leq N$. Now let $\epsilon > 0$ and $x \in [0, 1)$. Fix $N > 1/\epsilon$. Let k be satisfying $x \in [k/N, (k+1)/N)$.

(ii)

(iii)

Exercise 6.1.4. Let $(Y_k)_{k \in \mathbb{Z}}$ be a sequence of random variables with joint distribution defined by

$$(Y_{-k}, \dots, Y_k) \stackrel{d}{=} (X_0, \dots, X_{2k}).$$

To see (Y_k) is well-defined, note that, by the stationarity of X ,

$$(Y_{-k}, \dots, Y_k) \stackrel{d}{=} (X_1, \dots, X_{2k+1}) \stackrel{d}{=} (X_0, \dots, X_{2k}).$$

Thus, distributions of $(Y_{-k}, \dots, Y_k)_k$ are consistent, and by the Kolmogorov extension theorem, the sequence (Y_k) is well-defined. To check (Y_k) is stationary, observe that

$$(Y_0, \dots, Y_k) \stackrel{d}{=} (X_k, \dots, X_{2k}) \stackrel{d}{=} (X_{2n+k}, \dots, X_{2n+2k}) \stackrel{d}{=} (Y_n, \dots, Y_{n+k}).$$

Finally, note that $(X_0, \dots, X_k) \stackrel{d}{=} (X_k, \dots, X_{2k}) \stackrel{d}{=} (Y_0, \dots, Y_k)$, and therefore (X_k) is indeed embedded in (Y_k) .

Exercise 6.1.5. To see that Y_k is stationary, note that

$$\begin{aligned} (Y_0, \dots, Y_k) &= (g((X_j)_{j \geq 0}), \dots, g((X_{k+j})_{j \geq 0})) \\ &\stackrel{d}{=} (g((X_{n+j})_{j \geq 0}), \dots, g((X_{n+k+j})_{j \geq 0})) \end{aligned}$$

$$= (Y_n, \dots, Y_{n+k}).$$

To see that Y_k is ergodic, note that

$$Y_{k+1} = g(X_{k+1}, X_{k+2}, \dots) = g(X_k \circ \varphi, X_{k+1} \circ \varphi, \dots) = Y_k \circ \varphi.$$

Thus, for $B \in \mathcal{B}(\mathbb{R}^{\{0,1,\dots\}})$,

$$\begin{aligned} \{(Y_1, Y_2, \dots) \in B\} &= \{(Y_0, Y_1, \dots) \circ \varphi \in B\} \\ &= \varphi^{-1}\{(Y_0, Y_1, \dots) \in B\} \end{aligned}$$

Thus, if we let $A = \{(Y_0, Y_1, \dots) \in B\}$, then since X_n is ergodic (or in other words, since φ is ergodic), $A = \varphi^{-1}A$ implies that A is trivial. Therefore Y_n is also ergodic.

Exercise 6.1.6. Fix $N > n$. Let x_ν and y_ν be the smallest integer ≥ 0 that satisfy $\nu + x_\nu \equiv 0 \pmod n$ and $\nu + N - y_\nu \equiv 0 \pmod n$. Then x_ν and y_ν are uniformly distributed on $\{1, \dots, n\}$. Note that

$$\begin{aligned} (Z_1, \dots, Z_N) &= (Y_{\nu+1}, \dots, Y_{\nu+N}) \\ &= (Y_{\nu+1}, \dots, Y_{\nu+x_\nu}, \text{iid blocks of length } n, Y_{\nu+N-y_\nu+1}, \dots, Y_{\nu+N}) \end{aligned}$$

and all the blocks that appear in the above (including the first and the last incomplete blocks) are independent. Now fix k and let \tilde{x}_ν and \tilde{y}_ν be the smallest integer ≥ 0 that satisfy $k + \nu + \tilde{x}_\nu \equiv 0 \pmod n$ and $k + \nu + N - \tilde{y}_\nu \equiv 0 \pmod n$. Then again \tilde{x}_ν and \tilde{y}_ν are uniformly distributed on $\{1, \dots, n\}$. Note again that

$$\begin{aligned} (Z_{k+1}, \dots, Z_{k+N}) &= (Y_{k+\nu+1}, \dots, Y_{k+\nu+N}) \\ &= (Y_{k+\nu+1}, \dots, Y_{k+\nu+\tilde{x}_\nu}, \text{iid blocks of length } n, Y_{k+\nu+N-\tilde{y}_\nu+1}, \dots, Y_{k+\nu+N}). \end{aligned}$$

Since ν is independent of Y , so is $x_\nu, y_\nu, \tilde{x}_\nu$, and \tilde{y}_ν . Thus, the distribution of (Z_1, \dots, Z_N) and $(Z_{k+1}, \dots, Z_{k+N})$ are equal. To see that Z is ergodic, note that the tail σ -field \mathcal{T} of Z is contained in the tail σ -field of blocks $(Y_{kn+1}, \dots, Y_{kn+n})$. Since the blocks $(Y_{kn+1}, \dots, Y_{kn+n})$ are iid, \mathcal{T} is trivial. As the invariant σ -field is contained in \mathcal{T} , it is also trivial.

Exercise 6.1.7. Note first that

$$\mu([a, b]) = \frac{1}{\log 2} \int_a^b \frac{dx}{1+x} = \frac{1}{\log 2} \log \frac{1+b}{1+a}.$$

From the definition of φ , note that $x = 1/(\varphi(x) + n)$ for $n \in \mathbb{N}$. It follows that

$$\begin{aligned} \mu(\varphi^{-1}[a, b]) &= \frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{\frac{1}{n+b}}^{\frac{1}{n+a}} \frac{dx}{1+x} \\ &= \frac{1}{\log 2} \sum_{n=1}^{\infty} \log \frac{1+n+a}{n+a} \frac{n+b}{1+n+b} \\ &= \frac{1}{\log 2} \lim_{n \rightarrow \infty} \log \frac{1+n+a}{1+a} \frac{1+b}{1+n+b} \end{aligned}$$

$$= \frac{1}{\log 2} \log \frac{1+b}{1+a}.$$

Thus $\mu(\varphi^{-1}[a, b]) = \mu([a, b])$. Now let $\mathcal{L} = \{A \subset (0, 1) : \mu(\varphi^{-1}A) = \mu(A)\}$. Then \mathcal{L} is a λ -system (check) and contains all closed intervals. By Dynkin's theorem, $\mu(\varphi^{-1}A) = \mu(A)$ for any Borel set A .

6.2. Birkhoff's Ergodic Theorem.

Exercise 6.2.1. As in the proof of Theorem 6.2.1, let $X'_M = X \mathbf{1}_{|X| \leq M}$ and $X''_M = X - X'_M$. Following the proof, we know that

$$\frac{1}{n} \sum_{m=0}^{n-1} X'_M(\varphi^m \omega) \rightarrow \mathbb{E}[X'_M | \mathcal{I}] \quad \text{a.s.}$$

Since

$$\left| \frac{1}{n} \sum_{m=0}^{n-1} X'_M(\varphi^m \omega) - \mathbb{E}[X'_M | \mathcal{I}] \right|^p \leq (2M)^p,$$

by the bounded convergence theorem, the convergence occurs in L^p as well. To handle X''_M part, observe that,

$$\left\| \frac{1}{n} \sum_{m=0}^{n-1} X''_M(\varphi^m \omega) - \mathbb{E}[X''_M | \mathcal{I}] \right\|_p \leq \frac{1}{n} \sum_{m=0}^{n-1} \|X''_M(\varphi^m \omega)\|_p + \|\mathbb{E}[X''_M | \mathcal{I}]\|_p \leq 2 \|X''_M\|_p.$$

Let

$$A = \frac{1}{n} \sum_{m=0}^{n-1} X'_M(\varphi^m \omega) - \mathbb{E}[X'_M | \mathcal{I}] \quad \text{and}$$

$$B = \frac{1}{n} \sum_{m=0}^{n-1} X''_M(\varphi^m \omega) - \mathbb{E}[X''_M | \mathcal{I}].$$

It follows that

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{m=0}^{n-1} X(\varphi^m \omega) - \mathbb{E}[X | \mathcal{I}] \right\|_p \leq \limsup_{n \rightarrow \infty} (\|A\|_p + \|B\|_p) \leq 2 \|X''_M\|_p.$$

Letting $M \rightarrow \infty$ and using the dominated convergence theorem, $\mathbb{E}[X''_M]^p \rightarrow 0$ and the result follows.

Exercise 6.2.2. (i) Let $h_N = \sup_{m, n \geq N} |g_m - g_n|$. $h_N \leq 2 \sup_k |g_k|$, so $\mathbb{E}h_N < \infty$. Note that

$$\begin{aligned} & \left| \frac{1}{n} \sum_{m=0}^{n-1} g_m(\varphi^m \omega) - \mathbb{E}[g | \mathcal{I}] \right| \\ & \leq \left| \frac{1}{n} \sum_{m=0}^{N-1} g_m(\varphi^m \omega) \right| + \frac{1}{n} \sum_{m=N}^{n-1} |g_m(\varphi^m \omega) - g(\varphi^m \omega)| + \left| \frac{1}{n} \sum_{m=N}^{n-1} g(\varphi^m \omega) - \mathbb{E}[g | \mathcal{I}] \right| \end{aligned}$$

$$\leq \left| \frac{1}{n} \sum_{m=0}^{N-1} g_m(\varphi^m \omega) \right| + \frac{1}{n} \sum_{m=N}^{n-1} h_N(\varphi^m \omega) + \left| \frac{1}{n} \sum_{m=N}^{n-1} g(\varphi^m \omega) - \mathbb{E}[g|\mathcal{I}] \right|.$$

Since $\mathbb{E} \sup_k |g_k| < \infty$, $g_m < \infty$ a.s. and thus the first term $\rightarrow 0$ a.s. as $n \rightarrow \infty$. Also, since h_N and g are integrable, the second term $\rightarrow \mathbb{E}[h_N|\mathcal{I}]$ and the third term $\rightarrow 0$ a.s. Thus

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{m=0}^{n-1} g_m(\varphi^m \omega) - \mathbb{E}[g|\mathcal{I}] \right| \leq \mathbb{E}[h_N|\mathcal{I}].$$

Now letting $N \rightarrow \infty$ and using the dominated convergence, the result follows.

(ii) Note that

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} g_m(\varphi^m \omega) - \mathbb{E}[g|\mathcal{I}] \right| \\ & \leq \mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} g_m(\varphi^m \omega) - \frac{1}{n} \sum_{m=0}^{n-1} g(\varphi^m \omega) \right| + \mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} g(\varphi^m \omega) - \mathbb{E}[g|\mathcal{I}] \right| \\ & \leq \frac{1}{n} \sum_{m=0}^{n-1} \mathbb{E} |g_m(\varphi^m \omega) - g(\varphi^m \omega)| + \mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} g(\varphi^m \omega) - \mathbb{E}[g|\mathcal{I}] \right| \end{aligned}$$

Since $g_m \rightarrow g$ in L^1 , the first term $\rightarrow 0$ as $n \rightarrow \infty$. The second term $\rightarrow 0$ from Theorem 6.2.1.

Exercise 6.2.3. Let $X' = X - \alpha$, $X'_j(\omega) = X'(\varphi^j \omega)$, $S'_k = X'_0 + \cdots + X'_{k-1}$, and $M'_k = \max(0, S'_1, \dots, S'_k)$. By Lemma 6.2.2, $\mathbb{E}[X'; M'_k > 0] \geq 0$. That is, $\mathbb{E}[X; M'_k > 0] \geq \alpha \mathbb{P}(M'_k > 0)$. Note that $M'_k > 0$ if and only if $D_k > \alpha$. It follows that

$$\mathbb{E}|X| \geq \mathbb{E}[X; M'_k > 0] \geq \alpha \mathbb{P}(M'_k > 0) = \alpha \mathbb{P}(D_k > \alpha).$$

6.3. Recurrence.

Exercise 6.3.1. Note that

$$\begin{aligned} R_n(\omega) &= \mathbf{1}_{\{S_2 - S_1 \neq 0, \dots, S_n - S_1 \neq 0\}}(\omega) + \cdots + \mathbf{1}_{\{S_n - S_{n-1} \neq 0\}}(\omega) + 1 \\ &= \mathbf{1}_{\{S_1 \neq 0, \dots, S_{n-1} \neq 0\}}(\varphi \omega) + \cdots + \mathbf{1}_{\{S_1 \neq 0\}}(\varphi^{n-1} \omega) + 1. \end{aligned}$$

Taking the expectation, we get

$$\begin{aligned} \mathbb{E}R_n &= \mathbb{P}(S_1 \neq 0, \dots, S_{n-1} \neq 0) + \cdots + \mathbb{P}(S_1 \neq 0) + 1 \\ &= \sum_{m=1}^n g_{m-1}. \end{aligned}$$

Exercise 6.3.2. Note that, since $\mathbb{P}(X_i > 1) = 0$,

$$\left\{ 1, \dots, \max_{m \leq n} S_m \right\} \leq \{S_1, \dots, S_n\} \leq \left\{ \min_{m \leq n} S_m, \dots, \max_{m \leq n} S_m \right\},$$

and thus

$$\max_{m \leq n} S_m \leq R_n \leq \max_{m \leq n} S_m - \min_{m \leq n} S_m$$

Since our sequence is ergodic, $S_n/n \rightarrow \mathbb{E}X_i > 0$ by the ergodic theorem. This implies that $S_n \rightarrow \infty$ and that $\min_m S_m > -\infty$ a.s. Thus,

$$\limsup_{n \rightarrow \infty} \max_{m \leq n} \frac{S_m}{n} \leq \lim_{n \rightarrow \infty} \frac{R_n}{n} \leq \liminf_{n \rightarrow \infty} \max_{m \leq n} \frac{S_m}{n},$$

and by Theorem 6.3.1,

$$\mathbb{P}(A) = \lim_{n \rightarrow \infty} \max_{m \leq n} \frac{S_m}{n}.$$

To evaluate the right-hand-side, note that, for some K ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{S_n}{n} &\leq \liminf_{n \rightarrow \infty} \max_{m \leq n} \frac{S_m}{n} \\ &\leq \limsup_{n \rightarrow \infty} \max_{m \leq n} \frac{S_m}{n} \\ &\leq \limsup_{n \rightarrow \infty} \max_{K \leq m \leq n} \frac{S_m}{n} \\ &\leq \max_{K \leq m} \frac{S_m}{m}. \end{aligned}$$

Since K is arbitrary, by letting $K \rightarrow \infty$, $\lim_{n \rightarrow \infty} \max_{m \leq n} S_m/n = \mathbb{E}X_i$. This completes the proof.

Exercise 6.3.3. Note that

$$\begin{aligned} \mathbb{E} \left[\sum_{1 \leq m \leq T_1} \mathbf{1}_{\{X_m \in B\}} \middle| X_0 \in A \right] &= \sum_{1 \leq m} \mathbb{E} [\mathbf{1}_{\{X_m \in B\}} \mathbf{1}_{\{T_1 \geq m\}} | X_0 \in A] \\ &= \sum_{1 \leq m} \mathbb{P}(X_m \in B, T_1 \geq m | X_0 \in A) \\ &= \mathbb{P}(X_0 \in A)^{-1} \sum_{1 \leq m} \mathbb{P}(X_m \in B, T_1 \geq m, X_0 \in A). \end{aligned}$$

Let $C_m = \{X_{-m} \in A, X_{-m+1} \notin A, \dots, X_{-1} \notin A, X_0 \in B\}$. Observe that

$$\begin{aligned} \left(\bigcup_{m=1}^K C_m \right)^c &= \{X_j \in A \text{ for some } -K \leq j \leq -1, X_0 \in B\}^c \\ &= \{X_j \notin A \text{ for all } -K \leq j \leq -1\} \cup \{X_0 \notin B\}. \end{aligned}$$

As $K \rightarrow \infty$, from $\mathbb{P}(X_n \in A \text{ at least once}) = 1$, it follows that $\mathbb{P}(X_j \notin A \text{ for all } -K \leq j \leq -1) = \mathbb{P}(X_j \notin A \text{ for all } 1 \leq j \leq K) \rightarrow 0$. Thus $\mathbb{P}(\bigcup C_m) = \mathbb{P}(X_0 \in B)$. Now the first display becomes

$$= \mathbb{P}(X_0 \in A)^{-1} \sum_{1 \leq m} \mathbb{P}(C_m) = \frac{\mathbb{P}(X_0 \in B)}{\mathbb{P}(X_0 \in A)}.$$

Exercise 6.3.4. From Theorem 6.3.3,

$$\begin{aligned}\frac{\bar{\mathbb{P}}(T_1 \geq n)}{\mathbb{E}T_1} &= \mathbb{P}(T_1 \geq n | X_0 = 1) \mathbb{P}(X_0 = 1) \\ &= \mathbb{P}(T_1 \geq n, X_0 = 1).\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{P}(T_1 = n) &= \mathbb{P}(X_1 = \cdots = X_{n-1} = 0, X_n = 1) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(X_{-m} = 1, X_{-m+1} = \cdots = X_{n-1} = 0, X_n = 1) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(X_0 = 1, X_1 = \cdots = X_{m+n-1} = 0, X_{m+n} = 1) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(X_0 = 1, T_1 = m + n) \\ &= \mathbb{P}(X_0 = 1, T_1 \geq n)\end{aligned}$$

which completes the proof.

6.5. Applications.

Exercise 6.5.1. content

Exercise 6.5.2. content

Exercise 6.5.3. content

Exercise 6.5.4. content

Exercise 6.5.5. content

7. BROWNIAN MOTION

7.1. Definition and Construction.

Exercise 7.1.1. content

Exercise 7.1.2. Let $X = B_1$, $Y = B_2 - B_1$, and $Z = B_3 - B_2$. Then X, Y, Z are iid normal random variables with mean 0 and variance 1. Note that

$$\begin{aligned}B_1^2 B_2 B_3 &= X^2(X + Y)(X + Y + Z) \\ &= X^4 + 2X^3Y + X^2Y^2 + X^3Z + X^2YZ.\end{aligned}$$

Taking the expectation, it follows that

$$\mathbb{E}B_1^2 B_2 B_3 = \mathbb{E}X^4 + \mathbb{E}X^2 Y^2 = 4.$$

7.2. Markov Property, Blumenthal's 0-1 Law.

7.3. Stopping Times, Strong Markov Property.

7.4. Path Properties.

7.5. Martingales.

7.6. Itô's Formula.

8. APPLICATIONS TO RANDOM WALK

8.1. Donsker's Theorem.

8.4. Empirical Distributions, Brownian Bridge.

8.5. Laws of Iterated Logarithm.

9. MULTIDIMENSIONAL BROWNIAN MOTION

9.1. Martingales.

9.5. Dirichlet Problem. end of document

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