STAT5030 Assignment 4 Solution

1. (a) $\sum_{i=1}^{n} \hat{y}_i(y_i - \hat{y}_i) = HY(I-H)Y = 0$.

   (b) $\sum_{i=1}^{n} Var(\hat{y}_i) = tr(Var(HY)) = p\sigma^2$.

2. Note that

$$
\begin{aligned}
& M(\hat{\boldsymbol{\beta}}, \boldsymbol{\beta}) - M(\hat{\boldsymbol{\beta}}_1, \boldsymbol{\beta}) \\
=& E(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top - E(\hat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta})^\top \\
=& (X^\top X)^{-1}\sigma^2 - \left(\frac{1}{\rho+1}\right)^2 (X^\top X)^{-1}\sigma^2 - \left(\frac{\rho}{\rho+1}\right)^2 \beta\beta^\top \\
=& \frac{\rho^2+2\rho}{(\rho+1)^2}(X^\top X)^{-1}\sigma^2 - \frac{\rho^2}{(\rho+1)^2}\beta\beta^\top.
\end{aligned}
$$

Since $\boldsymbol{X}$ has full column rank, for any vector $\boldsymbol{v}$, there exist a vector $\boldsymbol{u}$ such that $\boldsymbol{v} = \rho^{-1}(1+\rho)(\boldsymbol{X}^\top\boldsymbol{X})^{1/2}\boldsymbol{u}$, and

$$
\begin{aligned}
\boldsymbol{v}^\top(M(\hat{\boldsymbol{\beta}}, \boldsymbol{\beta}) - M(\hat{\boldsymbol{\beta}}_1, \boldsymbol{\beta}))\boldsymbol{v} &= \frac{(1+\rho)^2}{\rho^2}\boldsymbol{u}^\top(\boldsymbol{X}^\top\boldsymbol{X})^{1/2}(M(\hat{\boldsymbol{\beta}}, \boldsymbol{\beta}) - M(\hat{\boldsymbol{\beta}}_1, \boldsymbol{\beta}))(\boldsymbol{X}^\top\boldsymbol{X})^{1/2}\boldsymbol{u} \\
&\geq \sigma^2\boldsymbol{I} - (\boldsymbol{X}^\top\boldsymbol{X})^{1/2}\boldsymbol{\beta}\boldsymbol{\beta}^\top(\boldsymbol{X}^\top\boldsymbol{X})^{1/2}.
\end{aligned}
$$

By the given theorem, $\sigma^2\boldsymbol{I} - (\boldsymbol{X}^\top\boldsymbol{X})^{1/2}\boldsymbol{\beta}\boldsymbol{\beta}^\top(\boldsymbol{X}^\top\boldsymbol{X})^{1/2} \geq 0$ if and only if $\boldsymbol{\beta}^\top\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{\beta} \leq \sigma^2$.

3. (a) $E(\boldsymbol{\lambda}^\top\hat{\boldsymbol{\beta}}) = E(r^\top X^\top X\hat{\beta}) = E(r^\top X^\top X(X^\top X)^- X^\top Y) = E(\boldsymbol{r}^\top\boldsymbol{X}^\top\boldsymbol{Y}) = \boldsymbol{\lambda}^\top\boldsymbol{\beta}$.

   (b) $\boldsymbol{r}^\top\boldsymbol{X}^\top\boldsymbol{Y} = \lambda^\top(X^\top X)^- X^\top Y$. Since $\lambda^\top\beta$ is estimable, there exists a vector t such that $\lambda^\top\beta = t^\top X$. Then $r^\top X^\top Y = t^\top X(X^\top X)^- X^\top Y$ is invariant to the choice of $\boldsymbol{r}$.

4. $Y = X\beta + \varepsilon$, where $X = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$, and $\beta = (\mu, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)^\top$.

   (a) $\lambda_0\mu + \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 + \lambda_4\beta_1 + \lambda_5\beta_2 + \lambda_6\beta_3$ is estimable only when $\lambda^\top H = \lambda^\top$, where $H = (X^\top X)^- X^\top X$.

   (b) Not estimable.

   (c) Not estimable.

(d) $\mu + \alpha_2$ is not estimable.

(e) $6\mu + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\beta_1 + 3\beta_3$ is estimable.

(f) $\alpha - 2\alpha_2 + \alpha_3$ is estimable.

5. $Y = X\beta + \varepsilon$, where $X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. $q^\top \beta$ is estimable if and only if $q^\top H = q^\top$, where $H = GX^\top X$.

   $\lambda_1\beta_1 + \lambda_2\beta_2 + \lambda_3\beta_3 = q^\top \beta$ is estimable if and only if $q^\top H = q^\top$. Therefore $\lambda_1 = \lambda_2 + \lambda_3$.

6. $\beta = 2\pi\sqrt{1/g}$. $\hat{\beta} = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} \sqrt{l_i} t_{ij}}{\sum_{i=1}^k n_i l_i}$, and $Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^k n_i l_i}$.

7. (a) The null hypothesis

$$H_0 : \frac{\mu + \alpha_1}{a_1} = \frac{\mu + \alpha_2}{a_2} = \cdots = \frac{\mu + \alpha_k}{a_k}$$

   can also be presented as $K^\top \beta = m$, where $\beta = 0$ and $K = \begin{pmatrix} a_2 - a_1 & a_2 - a_1 & 0 & \dots & 0 \\ a_3 - a_2 & 0 & a_3 - a_2 & \dots & 0 \\ \vdots & & & & \\ a_k - a_{k-1} & 0 & 0 & \dots & a_k - a_{k-1} \end{pmatrix}$.

   Since $H = GX^\top X = \begin{pmatrix} 0 & 0 \\ I_{1\times k} & I_{k\times k} \end{pmatrix}$, then $K^\top H = K^\top$. Therefore, H_0 is testable.

   (b) $F(H) \sim F_{k-1,n-k}$.

8. (a) $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2)^\top$ is not identifiable.

   (b) $\boldsymbol{\beta}$ is identifiable.

   (c) $\boldsymbol{\beta}$ is identifiable.

9. $A = E(\dot{g}(x)) = E(XX^\top Dirac(\varepsilon_\tau))$. $B = E\{XX^\top(\tau^2 1(\varepsilon_\tau > 0) + (\tau - 1)^2 1(\varepsilon_\tau \leq 0))\}$.