symmetric

If G is a generalized inverse of X^TX, then

(i) G^T is a generalized inverse of X^TX.

(ii) $XGX^TX = X$, i.e., GX^T is a generalized inverse of X.

$X(X^TX)^-X^T : K$ if $KX = X$, then GX^T is G-Inverse of X.

(iii) XGX^T is invariant with respect to the choice of G. Recall G is G-Inverse of X^TX.

(iv) XGX^T is symmetric.

Proof:

(i) Since G is a generalized inverse of (X^TX), $(X^TX)G(X^TX) = X^TX$.
Taking the transpose of both sides

NTS G^T is G-inverse of X^TX
i.e. $(X^TX)G^T(X^TX) = (X^TX)$

$$\begin{aligned} \left[X^TX\right]^T &= \left[(X^TX)G(X^TX)\right]^T \\ &= (X^TX)^TG^T(X^TX)^T \end{aligned}$$

But $(X^TX)^T = X^T(X^T)^T = X^TX$,
hence $(X^TX)G^T(X^TX) = (X^TX)$.

NTS $XGX^TX = X$

(ii) From (i) $(X^TX)G^T(X^TX) = (X^TX)$.
Denote $(X^TX)G^T$ by B. $B^T = G(X^TX)$

WTS $A=0$
NTS $AA^T=0$.

Then

$(X^TX)G^T(X^TX) = X^TX$
$\Rightarrow (X^TX)G^TX^T = X^T$
$\overset{T}{\Rightarrow} XGX^TX = X$
GX^T is G-inverse of X.

any non-zero matrix

$$\begin{aligned} 0 &= BX^TX - X^TX \\ &= (BX^TX - X^TX)(B^T - I) \\ &= BX^TXB^T - X^TXB^T - BX^TX - X^TX \\ &= (BX^T - X^T)(BX^T - X^T)^T \end{aligned}$$

Hence, $0 = BX^T - X^T$
$\Rightarrow \quad BX^T = X^T$
$\Rightarrow \quad X^TXG^TX^T = X^T$
Taking the transpose

$$\begin{aligned} X &= (X^TXG^TX^T)^T \\ &= XGX^TX \end{aligned}$$

Hence, GX^T is a generalized inverse for X.

NTS $XGX^T \equiv$? G is any G-inverse of X^TX

(iii) Suppose F and G are generalized inverses for X^TX. Then, from (ii)

$$XGX^TX = X$$

and

$$XFX^TX = X$$

It follows that

$$\begin{aligned}
0 &= X - X \\
&= (XGX^TX - XFX^TX) \quad \text{multiply some non-zero term.} \\
&= (XGX^TX - XFX^TX)\underline{(G^TX^T - F^TX^T)} \\
&= (XGX^T - XFX^T)X(G^TX^T - F^TX^T) \\
&= (XGX^T - XFX^T)(XG^TX^T - XF^TX^T) \\
&= (XGX^T - XFX^T)(XGX^T - XFX^T)^T
\end{aligned}$$

Since the (i,i) diagonal element of the result of multiplying a matrix by its transpose is the sum of the squared entries in the *i-th* row of the matrix, the diagonal elements of the product are all zero only if all entries are zero in every row of the matrix. Consequently,

$$(XGX^T - XFX^T) = 0$$

(iv) For any generalized inverse G, NTS X^TGX is symmetric.

$$T = GX^TXG^T$$ build a special symmetric matrix.

is a symmetric generalized inverse. Then $XTX^T = XGX^TXG^TX^T$

How

$$XTX^T$$

is symmetric and from (iii),

$$XGX^T = XTX^T.$$