STAT5030 Assigment 2

Due: Feb 19, 2024

1. Let the $n \times 1$ vector $\boldsymbol{Y} = (Y_1, \cdots, Y_n)^\top \sim N(\alpha\mathbf{1}, \sigma^2\boldsymbol{I})$. Define $U = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/\sigma^2$ and $V = n(\bar{Y} - \alpha)^2/\sigma^2$. Find the distributions of U and V, and show that these two random variables are independent.

2. Let the $n \times 1$ vector $\boldsymbol{Y} = (Y_1, \cdots, Y_n)^\top \sim N(\mu\mathbf{1}, \boldsymbol{I})$. Let

$$\begin{aligned} \bar{Y} &= \frac{\sum_{i=1}^{n} Y_i}{n}, \\ Q_1 &= n\bar{Y}^2, \\ Q_2 &= \sum_{i=1}^{n}(Y_i - \bar{Y})^2. \end{aligned}$$

   (a) Prove that $\bar{Y}$ and Q_2 are independent.

   (b) Prove that Q_1 and Q_2 are independent.

   (c) Find the distributions of Q_1 and Q_2.

3. Let the $n \times 1$ vector $\boldsymbol{Y} = (Y_1, \cdots, Y_n)^\top \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $q_1 = \boldsymbol{Y}^\top \boldsymbol{A_1} \boldsymbol{Y}$, $q_2 = \boldsymbol{Y}^\top \boldsymbol{A_2} \boldsymbol{Y}$ and $\boldsymbol{T} = \boldsymbol{B}\boldsymbol{Y}$, where $\boldsymbol{B}$ is $r \times n$ and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are symmetric. Prove the followings:

   (a) $E(q_1) = tr(\boldsymbol{A_1\Sigma}) + \boldsymbol{\mu}^\top \boldsymbol{A_1}\boldsymbol{\mu}$.

   (b) $Var(q_1) = 2tr(\boldsymbol{A_1\Sigma A_1\Sigma}) + 4\boldsymbol{\mu}^\top \boldsymbol{A_1\Sigma A_1}\boldsymbol{\mu}$.

   (c) $Cov(q_1, q_2) = 2tr(\boldsymbol{A_1\Sigma A_2\Sigma}) + 4\boldsymbol{\mu}^\top \boldsymbol{A_1\Sigma A_2}\boldsymbol{\mu}$.

   (d) $Cov(\boldsymbol{Y}, q_1) = 2\boldsymbol{\Sigma A_1}\boldsymbol{\mu}$.

   (e) $Cov(\boldsymbol{T}, q_1) = 2\boldsymbol{B\Sigma A_1}\boldsymbol{\mu}$.

4. Suppose $\boldsymbol{y}$ is $N_3(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$ and let $\boldsymbol{\mu}^\top = [3, -2, 1]$ and

$$\boldsymbol{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$
$$\boldsymbol{B} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

   (a) Find the distribution of $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}/\sigma^2$.

   (b) Are $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{B}\boldsymbol{y}$ independent?

   (c) Are $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}$ and $y_1 + y_2 + y_3$ independent?

5. X_1, X_2, X_3, X_4 be a random sample of size 4 from a distribution which is $N(0, \sigma^2)$. Let $Q = X_1X_2 - X_3X_4$. Does Q/σ^2 has a χ^2 distribution? Justify your answer.

6. Suppose $\boldsymbol{y}$ is $N_n(\mu\mathbf{1}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\Sigma} = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}.$$

Derive the distribution of

$$\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2(1-\rho)}.$$

7. Let $\boldsymbol{Y} = (Y_1, Y_2, Y_3)^\top$. $E(\boldsymbol{Y}) = (2, 3, 4)^\top$ and the covariance matrix of $\boldsymbol{Y}$ is

$$\boldsymbol{\Sigma} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Let $U = \sum\limits_{i=1}^{3} (Y_i - \bar{Y})^2$. Find the expected value of U.

8. Let $\boldsymbol{Y} = (Y_1, \cdots, Y_n)^\top$ with $E(\boldsymbol{Y}) = \mu\mathbf{1}$ and the covariance matrix of $\boldsymbol{Y}$ be $\sigma^2\boldsymbol{I}$. Let

$$U = \sum_{i<j} (Y_i - Y_j)^2.$$

   (a) Find the expected value of U.

   (b) Find a constant k such that kU is an unbiased estimator of σ^2.

9. Let $Y_1, \cdots, Y_n$ be random variables with equal means μ, variances σ^2 and covariances $\rho\sigma^2$. Assume ρ known. Let

$$U = \sum_{i=1}^{n} (Y_i - \bar{Y})^2.$$

Find a constant k such that kU is an unbiased estimator of σ^2.