STAT5005
Homework 1 | 22/9/2022 | ZHOU Xiaocheng | 1155184323

2. proof for any events A_i, $i=1,\cdots,n$, where $n \geq 3$

(1) $P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1\leq i<j\leq n} P(A_iA_j)$

(2) $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \sum_{1\leq i<j\leq n} P(A_iA_j) + \sum_{1\leq i<j<k\leq n} P(A_iA_jA_k)$

3. Show that, the so called countable additivity or σ-additivity is equivalent to finite additivity plus continuity (if $A_n \downarrow \phi$, then $P(A_n) \to 0$).

4. If X_1, X_2 are r.v., so is X_1+X_2.

5. Textbook: 1.1.(5)
1.2.(3、5、6、7)
1.3.(1、7、8)

2. (1) We notice $A_i \setminus \bigcup_{1\le j<i} A_j$, $i=1,2,\cdots,n$ are disjoint (note $\bigcup_{j\in\emptyset} A_j = \emptyset$)

and $\bigcup_{i=1}^{n} A_i = \bigcup_{1\le i\le n} A_i \setminus \bigcup_{1\le j<i} A_j$

then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} A_i \setminus \bigcup_{1\le j<i} A_j$

$$\begin{aligned} P(A_i \setminus \bigcup_{1\le j<i} A_j) &= P(A_i \setminus \bigcup_{1\le j<i} A_iA_j) \\ &= P(A_i) - P(\bigcup_{1\le j<i} A_iA_j) \\ &\geq P(A_i) - \sum_{1\le j<i} P(A_iA_j) \end{aligned}$$

then $$\begin{aligned} P(\bigcup_{i=1}^{n} A_i) &\geq \sum_{i=1}^{n} (P(A_i) - \sum_{1\le j<i} P(A_iA_j)) \\ &= \sum_{i=1}^{n} P(A_i) - \sum_{1\le j<i\le n} P(A_iA_j) \end{aligned}$$

(2) $$\begin{aligned} P(\bigcup_{1\le j<i} A_iA_j) &= P(\bigcup_{1\le j<i} (A_iA_j \setminus \bigcup_{1\le k<j<i} A_iA_jA_k)) \\ &\overset{\text{same reason with (1)}}{=} \sum_{1\le j<i} P(A_iA_j \setminus \bigcup_{1\le k<j<i} A_iA_jA_k) \\ &\overset{(1)}{\geq} \sum_{1\le j<i} P(A_iA_j) - \sum_{1\le k<j<i} P(A_iA_jA_k) \end{aligned}$$

then $$\begin{aligned} P(\bigcup_{i=1}^{n} A_i) &= \sum_{i=1}^{n} [P(A_i) - P(\bigcup_{1\le j<i} A_iA_j)] \\ &\leq \sum_{i=1}^{n} [P(A_i) - \sum_{1\le j<i} P(A_iA_j) + \sum_{1\le k<j<i} P(A_iA_jA_k)] \\ &= \sum_{i=1}^{n} P(A_i) - \sum_{1\le j<i\le n} P(A_iA_j) + \sum_{1\le k<j<i\le n} P(A_iA_jA_k) \end{aligned}$$

3. proof: countable additivity $\Leftrightarrow$ finite additivity + continuity

$\Rightarrow$: Given a collection of set $\mathcal{A}$, derived from Ω, satisfies countable additivity

For $\forall$ disjoint $A_1, \cdots, A_n \in \mathcal{A}$, take $A_k = \emptyset$, $k \geq n+1$

then $P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{\infty} A_i) \xlongequal{A_i,\ i=1,\dots\ \text{disjoint}} \bigcup_{i=1}^{\infty} P(A_i) = \bigcup_{i=1}^{n} P(A_i)$

$\Rightarrow$ finite additivity

if $A_n \downarrow \emptyset$, that is $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} A_i = \emptyset$, $A_i \in \mathcal{A}$ $i=1,\cdots$

take $B_i = A_i^c$, $i=1,\cdots$,

then $B_n \uparrow \Omega$, that is $B_1 \subset B_2 \subset \cdots$ and $\bigcup_{i=1}^{\infty} B_i = \Omega$

construct $C_i = \begin{cases} B_1 & i=1 \\ B_i \setminus B_{i-1} & i \geq 2 \end{cases}$

then C_i, $i=1,\cdots$ are disjoint and $\bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} B_i = \Omega$

$$\lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} B_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} C_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(C_i)$$

$$= \sum_{i=1}^{\infty} P(C_i) = P(\bigcup_{i=1}^{\infty} C_i) = P(\Omega)$$

then $\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P(\Omega \setminus B_n) = \lim_{n\to\infty} [P(\Omega) - P(B_n)]$

$$= P(\Omega) - \lim_{n\to\infty} P(B_n) = 0$$

$\Rightarrow$ continuity

$\Leftarrow$: See the next page.

$\Leftarrow$: Given a collection of set $\mathcal{A}$, derived from Ω, satisfies finite additivity and continuity.

For $\forall$ disjoint $A_1, \cdots \in \mathcal{A}$, take $A = \bigcup_{i=1}^{\infty} A_i$, $B_n = A \setminus \bigcup_{i=1}^{n} A_i$

then $B_1 \supset B_2 \supset \cdots \supset B_n \to \emptyset$ when $n \to \infty$

then according to the definition of continuity, $\lim_{n\to\infty} P(B_n) \to 0$

by B_n and $\bigcup_{i=1}^{n} A_i$ disjoint

then $P(A) = P(B_n \cup (\bigcup_{i=1}^{n} A_i)) = P(B_n) + P(\bigcup_{i=1}^{n} A_i)$

$$= P(B_n) + \sum_{i=1}^{n} P(A_i) \to 0 + \sum_{i=1}^{\infty} P(A_i) \text{ when } n \to \infty.$$

$\Rightarrow$ countable additivity

4. X_1 and X_2 are random variables,

then we can construct a random vector (X_1, X_2)

Take functions $f_1 : \Omega \to \mathbb{R}^2 \quad (w \to (X_1(\omega), X_2(\omega)))$

$$f_2 : \mathbb{R}^2 \to \mathbb{R} \quad ((X_1, X_2) \to X_1 + X_2)$$

For $\forall\, a, b \in \mathbb{R}$: $f_1^{-1}((-\infty, a] \times (-\infty, b])$

$$= \{w : X_1(\omega) \leq a \text{ and } X_2(\omega) \leq b\}$$

$$= \underbrace{X_1^{-1}((-\infty, a])}_{\in \mathcal{F}} \cap \underbrace{X_2^{-1}((-\infty, b])}_{\in \mathcal{F}} \in \mathcal{F}$$

$\Rightarrow f_1$ is measurable ... ①

For $\forall\, c \in \mathbb{R}$: $f_2^{-1}((-\infty, c])$

$$= \{(X_1, X_2) : X_1 + X_2 \leq c\} \in \mathcal{B}$$

$\Rightarrow f_2$ is measurable ... ②

Combine conclusion ① & ②,

then the compostion of f_1 & f_2, $f_1 \circ f_2$ is measurable.

$\Rightarrow$ measurable funtion $X_1 + X_2 = f_1 \circ f_2 : \Omega \to \mathbb{R}$ is a r.v.

1.1.5 (i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ are σ-algebras, then $\bigcup_{i=1} \mathcal{F}_i$ is an algebra. (ii) Give an example to show that $\bigcup_{i=1} \mathcal{F}_i$ need not to be a σ-algebra.

(i) Given $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ are σ-algebras,

For $\forall A, B \in \bigcup_{i=1} \mathcal{F}_i$, $\exists a, b$ in $\{1, 2, \cdots\}$ s.t. $A \in \mathcal{F}_a$, $B \in \mathcal{F}_b$

then we have $A^c \in \mathcal{F}_a$

Take $c = \max\{a, b\}$, then $A \cup B \in \mathcal{F}_c \subset \bigcup_{i=1} \mathcal{F}_i$

$\Rightarrow \bigcup_{i=1} \mathcal{F}_i$ is an algebra.

(ii) Take $A_n = [0, \frac{1}{n}]$, $\mathcal{F}_n = \sigma(\{A_i\}_{i=1}^n)$, $n = 1, 2, \cdots$

then $\bigcup_{i=1} A_i = \bigcup_{i=1} [0, \frac{1}{i}] = \{0\}$ and $A_i \in \bigcup_{k=1} \mathcal{F}_k$ for $i = 1, 2, \cdots$

but $\{0\} \notin \mathcal{F}_n$ for $\forall n \in \{1, 2, \cdots\}$

then $\{0\} \notin \bigcup_{k=1} \mathcal{F}_k$

$\Rightarrow \bigcup_{k=1} \mathcal{F}_k$ is not a σ-algebra.

1.2.3 Suppose there are countable discontinuous points $D=\{x_\alpha\}_{\alpha\in P}$

where P is an uncountable set

Denote $F(x^-)=\lim_{a\uparrow x}F(a)$, $F(x)\overset{\text{right-continuous}}{=}\lim_{a\downarrow x}F(a)$

For $\forall\, x_\alpha\in D$, then $F(x_\alpha^-)<F(x_\alpha)$

Denote region $A_\alpha=(F(x_\alpha^-), F(x_\alpha))$, $\alpha\in P$

then $A_\alpha, \alpha\in P$ are disjoint.

For $\forall\, a<b$, $\exists\, q\in\mathbb{Q}$ s.t. $q\in(a,b)$

then we can find uncountable distinct rational numbers

$$\{q_\alpha: q_\alpha\in A_\alpha,\ q_\alpha\in\mathbb{Q}\}_{\alpha\in P}$$

which contradicts with the countability of $\mathbb{Q}$

$\Longrightarrow$ A distribution function has at most countably many discontinuity.

1.2.5 Take $Y = g(X)$, then we have $P(g(\alpha) \le Y \le g(\beta)) = 1$

$$F(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$

$$= \int_{\infty}^{g^{-1}(y)} f(x)dx = \int_{\alpha}^{g^{-1}(y)} f(x)dx$$

then $f(y) = \frac{dF(y)}{dy} = f(g^{-1}(y)) \cdot \frac{dg^{-1}(y)}{dy}$

we notice that $g(g^{-1}(y)) = y$

then $g'(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} = 1$

then $f_Y(y) = f(g^{-1}(y)) \cdot \frac{1}{g'(g^{-1}(y))}$

When $g(x) = ax + b$, then $g'(x) = a$, $g^{-1}(x) = \frac{x-b}{a}$

then $f_Y(y) = f(\frac{y-b}{a}) \cdot \frac{1}{a}$

1.2.6 Given $X \sim N(\mu, \sigma^2)$, then $f(X) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(X-\mu)^2}{2\sigma^2}\}$

Take $Y = g(X) = \exp\{X\}$, then $g^{-1}(X) = \log X$, $g'(X) = \exp\{X\}$

then, according to 1.2.5, the density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(\log y - \mu)^2}{2\sigma^2}\} \cdot \frac{1}{y}, \quad y > 0$$

1.2.7 Given that X has density function f, denote $Y = X^2$

then $F_{X^2}(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx$$

so $f_{X^2}(y) = \frac{d}{dy} F_{X^2}(y) = f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f(-\sqrt{y}) \cdot (-\frac{1}{2\sqrt{y}})$

$$= \frac{1}{2\sqrt{y}} \cdot [f(\sqrt{y}) + f(-\sqrt{y})]$$

1.3.1 proof if $S=\sigma(\mathscr{A}), X^{-1}(\mathscr{A}):=\{\{X\in A\}: A\in\mathscr{A}\}, \sigma(X):=\{\{X\in B\}: B\in S\}$

then $\sigma(X^{-1}(\mathscr{A}))=\sigma(X)$

Given $S=\sigma(\mathscr{A})$, then

$$X^{-1}(\mathscr{A}) := \{\{\omega: X(\omega)\in A\}: A\in\mathscr{A}\}$$
$$\subset \{\{\omega: X(\omega)\in A\}: A\in \underbrace{\sigma(\mathscr{A})}_{S}\}$$
$$= \{\{\omega: X(\omega)\in B\}: B\in S\}$$
$$=: \sigma(X)$$

$\sigma(X)$ is a σ-field

$\Rightarrow \quad \sigma(X^{-1}(\mathscr{A}))\subset\sigma(X)$

Denote $\mathcal{M}=\{B\in S: X^{-1}(B)\in\sigma(X^{-1}(\mathscr{A}))\}\subset S$

① We know $\phi\in S$ and $X^{-1}(\phi)=\phi\in\sigma(X^{-1}(\mathscr{A}))$

then $\phi\in\mathcal{M}$

② For $M\in\mathcal{M}$, then $M\in S$ and $X^{-1}(M)\in\sigma(X^{-1}(\mathscr{A}))$

by S and $\sigma(X^{-1}(\mathscr{A}))$ are σ-field.

then $M^c\in S$ and $X^{-1}(M^c)=(X^{-1}(M))^c\in\sigma(X^{-1}(\mathscr{A}))$

then $M^c\in\mathcal{M}$

③ For $M_1, M_2, \cdots \in\mathcal{M}$

$\bigcup_{i=1}^{\infty} M_i\in S$ and $X^{-1}(\bigcup_{i=1}^{\infty} M_i)=\bigcup_{i=1}^{\infty} X^{-1}(M_i)\in\sigma(X^{-1}(\mathscr{A}))$

then $\bigcup_{i=1}^{\infty} M_i\in\mathcal{M}$ (continue in the next page)

① + ② + ③ $\Rightarrow$ $\mathcal{M}$ is a σ-field.

For $\forall A \in \mathcal{A}$, we have $A \in \mathcal{A} \subset \mathcal{S}$ and $X^{-1}(A) \in \sigma(X^{-1}(\mathcal{A}))$

then $A \in \mathcal{M}$ then $\mathcal{A} \subset \mathcal{M}$

Since $\mathcal{A} \subset \mathcal{M} \subset \mathcal{S} = \sigma(\mathcal{A})$ and $\mathcal{M}$ is a σ-field

then $\mathcal{M} = \mathcal{S}$

For $\forall B \in \mathcal{S} = \mathcal{M}$, $X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))$

then $\underline{\underline{X^{-1}(\mathcal{S})}} \subset \sigma(X^{-1}(\mathcal{A}))$

$\equiv \{\{\omega : X(\omega) \in B\} : B \in \mathcal{S}\}$

$\Rightarrow \sigma(X) \subset \sigma(X^{-1}(\mathcal{A}))$

Hence $\sigma(X) = \sigma(X^{-1}(\mathcal{A}))$

1.37 Denote class C contains the simple function and closed under pointwise limits and $f: \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable function

For $\forall f$ (f is non-negative), we have $f = f^+ - f^-$

then we construct convergencing functions respectively for f^+ and f^-

For $\forall N \in \mathbb{N}$, we define

$$E_{l,N} = \{\omega \in \Omega : \frac{l}{N} < f(\omega) \leq \frac{l+1}{N}\} = f^{-1}(\frac{l}{N}, \frac{l+1}{N}], \quad 0 \leq l < N^2$$

Since f is measurable with respect to $\mathcal{F}$

then $E_{l,N} \in \mathcal{F}$

Construct simply function

$$g_N(\omega) = \sum_{l=0}^{N^2-1} \frac{l}{N} \cdot \mathbb{1}_{E_{l,N}}(\omega) \quad \omega \in \Omega$$

The value range of g_N is $[0, N]$ and

$$|g_N(\omega) - f(\omega)| \leq \frac{1}{N}, \quad \forall \omega \in f^{-1}[0, N]$$

For $\forall \omega \in \Omega$, $\exists N_1$, s.t. $N_1 < f(\omega) \leq N_1 + 1$

then for $\forall n > N_1$, then we have $f(\omega) \leq n$ and $|f(\omega) - g_n(\omega)| \leq \frac{1}{n}$

Hence, for $\forall f$, the pointwise limits of $\{g_n\}$ is f

Therefore, for $\forall C$, $\mathcal{F} \subset C$

$\Rightarrow$ class $\mathcal{F}$ is contained by the intersection of all Cs.

1.3.8 proof: Y is measurable w.r.t. $\sigma(X) \Leftrightarrow Y = f(X)$

where $f: \mathbb{R} \to \mathbb{R}$ is measurable.

$\Rightarrow$: Suppose Y is measurable w.r.t. $\sigma(X)$

Then for $\forall B_1 \in \mathcal{B}$, we have $Y^{-1}(B_1) \in \sigma(X)$

By the definition of $\sigma(X)$,

then $\exists B_2 \in \mathcal{B}$, s.t. $Y^{-1}(B_1) = X^{-1}(B_2)$.

Suppose there is a function $f: \mathbb{R} \to \mathbb{R}$. s.t. $Y = f(X)$

then $f^{-1}(B_1) = B_2 \in \mathcal{B}$ for $\forall B_1 \in \mathcal{B}$.

$\Rightarrow$ f is measurable.

$\Leftarrow$: Suppose X is a r.v. and f is measurable.

Since
$$\left.\begin{array}{l} X: (\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{B}) \\ f: (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B}) \end{array}\right\} \text{both measurable.}$$

then the composition $Y = f \circ X$ is also measurable w.r.t. $\sigma(X)$