Problem 3.16 Based on X with distribution indexed by $\theta \in \Omega$, the problem is to test $\theta \in \omega$ versus $\theta \in \omega'$. Suppose there exists a test ϕ such that $E_\theta[\phi(X)] \le \beta$ for all θ in ω, where $\beta < \alpha$. Show there exists a level α test $\phi^*(X)$ such that

$$E_\theta[\phi(X)] \le E_\theta[\phi^*(X)],$$

for all θ in ω' and this inequality is strict if $E_\theta[\phi(X)] < 1$.

if $E_\theta[\phi(X)] < 1$

$\phi^*(X) = \min\{\phi(X) + \alpha - \beta, 1\}$

$\forall \theta \in \omega$, $E_\theta \phi^*(X) \le E_\theta[\phi(X) + \alpha - \beta] \le \beta + \alpha - \beta = \alpha$

$\forall \theta \in \omega'$ $\because E_\theta \phi(X) < 1 \Rightarrow P_\theta(\phi(X) < 1) > 0$

$\exists \varepsilon > 0$ s.t. $P_\theta(\phi(X) < 1 - \varepsilon) > 0$ and $\varepsilon < \alpha - \beta$

$$E_\theta \phi^*(X) = E_\theta \phi^*(X) 1_{\{\phi(X) < 1-\varepsilon\}} + E_\theta \phi^*(X) 1_{\{1 \ge \phi(X) \ge 1-\varepsilon\}}$$

$$\ge E_\theta \phi(X) 1_{\{\phi(X) < 1-\varepsilon\}} + \varepsilon E_\theta 1_{\{\phi(X) < 1-\varepsilon\}} + E_\theta \phi(X) 1_{\{1 \ge \phi(X) \ge 1-\varepsilon\}}$$

$\phi(X) + \alpha - \beta > \phi(X) + \varepsilon$ $\qquad$ $\phi(X) + \alpha - \beta > \phi(X)$

$1 > \phi(X)$ $\qquad$ $1 > \phi(X)$

$$= E_\theta(\phi(X)) + \varepsilon P(\phi(X) < 1-\varepsilon) > E_\theta \phi(X)$$

Exercise 13 (#6.18). Let $(X_1, ..., X_n)$ be a random sample from the uniform distribution on $(\theta, \theta+1)$, $\theta \in \mathcal{R}$. Suppose that $n \ge 2$.
(i) Show that a UMP test of size $\alpha \in (0,1)$ for testing $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$ is of the form

$$T_*(X_{(1)}, X_{(n)}) = \begin{cases} 0 & X_{(1)} < 1 - \alpha^{1/n},\ X_{(n)} < 1 \\ 1 & \text{otherwise,} \end{cases}$$

where $X_{(j)}$ is the jth order statistic.
(ii) Does the family of all densities of $(X_{(1)}, X_{(n)})$ have monotone likelihood

Solution A. (i) The Lebesgue density of $(X_{(1)}, X_{(n)})$ is

$$f_\theta(x,y) = n(n-1)(y-x)^{n-2} I_{(\theta,y)}(x) I_{(x,\theta+1)}(y).$$

A direct calculation of $\beta_{T_*}(\theta) = \int T_*(x,y) f_\theta(x,y)\,dx\,dy$, the power function of T_*, leads to

$$\beta_{T_*}(\theta) = \begin{cases} 0 & \theta < -\alpha^{1/n} \\ (\theta + \alpha^{1/n})^n & -\alpha^{1/n} \le \theta \le 0 \\ 1 + \alpha - (1-\theta)^n & 0 < \theta \le 1 - \alpha^{1/n} \\ 1 & \theta > 1 - \alpha^{1/n}. \end{cases}$$

For any $\theta_1 \in (0, 1 - \alpha^{1/n}]$, by the Neyman-Pearson Lemma, the UMP test T of size α for testing $H_0 : \theta = 0$ versus $H_1 : \theta = \theta_1$ is

$$T = \begin{cases} 1 & X_{(n)} > 1 \\ \alpha/(1-\theta_1)^n & \theta_1 < X_{(1)} < X_{(n)} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The power of T at θ_1 is computed as

$$\beta_T(\theta_1) = 1 - (1-\theta_1)^n + \alpha,$$

which agrees with the power of T_* at θ_1. When $\theta > 1 - \alpha^{1/n}$, T_* has power 1. Therefore T_* is a UMP test of size α for testing $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$.
(ii) The answer is no. Suppose that the family of densities of $(X_{(1)}, X_{(n)})$ has monotone likelihood ratio. By the theory of UMP test (e.g., Theorem 6.2 in Shao, 2003), there exists a UMP test T_0 of size $\alpha \in (0, \frac{1}{2})$ for testing $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$ and T_0 has the property that, for $\theta_1 \in (0, 1-\alpha^{1/n})$, T_0 is UMP of size $\alpha_0 = 1 + \alpha - (1-\theta_1)^n$ for testing $H_0 : \theta \le \theta_1$ versus $H_1 : \theta > \theta_1$. Using the transformation $X_i - \theta_1$ and the result in (i), the test

$$T_{\theta_1}(X_{(1)}, X_{(n)}) = \begin{cases} 0 & X_{(1)} < 1 + \theta_1 - \alpha_0^{1/n},\ X_{(n)} < 1 + \theta_1 \\ 1 & \text{otherwise} \end{cases}$$

is a UMP test of size α_0 for testing $H_0 : \theta \le \theta_1$ versus $H_1 : \theta > \theta_1$. At $\theta = \theta_2 \in (\theta_1, 1-\alpha^{1/n}]$, it follows from part (i) of the solution that the power of T_0 is $1+\alpha-(1-\theta_2)^n$ and the power of T_{θ_1} is $1+\alpha_0-[1-(\theta_2-\theta_1)]^n$. Since both T_0 and T_{θ_1} are UMP tests, $1+\alpha-(1-\theta_2)^n = 1+\alpha_0-[1-(\theta_2-\theta_1)]^n$. Because $\alpha_0 = 1+\alpha-(1-\theta_1)^n$, this means that

$$1 = (1-\theta_1)^n - (1-\theta_2)^n + [1-(\theta_2-\theta_1)]^n$$

holds for all $0 < \theta_1 < \theta_2 \le 1-\alpha^{1/n}$, which is impossible. This contradiction proves that the family of all densities of $(X_{(1)}, X_{(n)})$ does not have monotone likelihood ratio.