STAT 5005 Homework 2 Oct. 13 2022 ZHOU Xiaocheng 1155184323

1.4.1 N.T.S. if $f \geq 0$ and $\int f d\mu = 0 \Rightarrow f = 0$ a.e.

Take $A_n = \{\omega : f(\omega) \geq \frac{1}{n}\} \subset \Omega$, where Ω is the sample space

then $0 = \int_{\Omega} f d\mu \geq \int_{A_n} f d\mu \geq \int_{A_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(A_n)$

then $\mu(A_n) = 0$ for $\forall n \in \{1, 2, \cdots\}$

Denote $A = \{\omega : f(\omega) > 0\}$, easy to know $A_n \uparrow A$

$\mu(A) = \mu(\bigcup_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n) = 0$

i.e. $\mu(\{\omega : f(\omega) > 0\}) = 0$

$\Rightarrow f = 0$ a.e.

1.5.3 proof of Minkowski's inequality (using Hölder's ineq.: $\int |fg| d\mu \leq \|f\|_p \|g\|_q$, $1/p + 1/q = 1$ $p, q > 1$)
$\|f\|_p := (\int |f|^p d\mu)^{1/p}$

(i) Easy to know $|x|^p$ is convex for $p \in (1, \infty)$,

then by Jensen's ineq., $|f+g|^p \leq |f|^p + |g|^p \leq 2^p(|f|^p + |g|^p)$ holds

Take $q = \frac{p}{p-1}$, then $q \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$

then $\int |f||f+g|^{p-1} d\mu = \int ||f||f+g|^{p-1}| d\mu$

(by Hölder's ineq.) $\leq (\int |f|^p d\mu)^{1/p} (\int ||f+g|^{p-1}|^q d\mu)^{1/q}$ $\quad ||f+g|^{p-1}|^{\frac{p}{p-1}} = |f+g|^p$

$= \|f\|_p (\int |f+g|^p d\mu)^{1/q}$ $\quad 1/q = \frac{1}{p} \cdot (p-1)$

$= \|f\|_p (\|f+g\|_p)^{p-1}$

Similarly, we have $\int |g||f+g|^{p-1} d\mu \leq \|g\|_p (\|f+g\|_p)^{p-1}$

then $(\|f\|_p + \|g\|_p)(\|f+g\|_p)^{p-1}$

$= \|f\|_p (\|f+g\|_p)^{p-1} + \|g\|_p (\|f+g\|_p)^{p-1}$

$\geq \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu$

$= \int (|f| + |g|) |f+g|^{p-1} d\mu$

$\geq \int |f+g|^p d\mu$

$= (\|f+g\|_p)^p$

$\Rightarrow \|f\|_p + \|g\|_p \geq \|f+g\|_p$

(next page for question (ii))

(iii) For $p=1$,

then $\|f\|_1+\|g\|_1 = \int|f|d\mu + \int|g|d\mu$

$\geq \int|f+g|d\mu$

$= \|f+g\|_1$

For $p=\infty$, we know $\|f\|_\infty = \lim_{p\to\infty}\|f\|_p$ from Ex. 1.5.2.

then $\|f\|_\infty + \|g\|_\infty = \lim_{p\to\infty}\|f\|_p + \lim_{p\to\infty}\|g\|_p$

$= \lim_{p\to\infty}(\|f\|_p+\|g\|_p)$

$\geq \lim_{p\to\infty}\|f+g\|_p$

$= \|f+g\|_\infty$

1.5.4 Suppose E_m $m=1,2,\cdots$ are disjoint and $\bigcup_{m=1}^{\infty}E_m = E$

Denote $F_n = \bigcup_{m=0}^{n}E_m$, then $F_1 \subset \cdots \subset F_n \subset \cdots$, $\bigcup_{n=0}^{\infty}F_n = E$ i.e. $F_n \uparrow E$

then r.v.'s $0 \leq f1_{F_n} \uparrow f1_E$, for $f\geq 0$, by MCT

$\mathbb{E}f1_{F_n} \uparrow \mathbb{E}f1_E \Leftrightarrow \lim_{n\to\infty}\int_{F_n} f d\mu = \int_E f d\mu$ $\cdots$ (*)

thus, $\sum_{m=0}^{\infty}\int_{E_m} f d\mu = \sum_{m=0}^{\infty}\int f 1_{E_m} d\mu = \lim_{n\to\infty}\int f\sum_{m=0}^{n}1_{E_m}d\mu$

$= \lim_{n\to\infty}\int f 1_{\bigcup_{m=0}^{n}E_m} d\mu = \lim_{n\to\infty}\int f 1_{F_n}d\mu = \lim_{n\to\infty}\int_{F_n} f d\mu$

$\overset{(*)}{=} \int_E f d\mu$

Denote $\nu(E) = \int_E f d\mu$

(i) $\nu(E) = \int_E f d\mu = \int f 1_E d\mu \overset{f\geq 0}{\geq} \int f 1_\emptyset d\mu = \int f\cdot 0 d\mu = 0 = \nu(\emptyset)$

(ii) for $\bigcup_{i=1}^{\infty}E_m = E$,

$\nu(E) = \nu(\bigcup_{i=1}^{\infty}E_m) = \sum_{m=0}^{\infty}\int_{E_m} f d\mu = \sum_{m=0}^{\infty}\nu(E_m)$ (by the conclusion proofed above)

then $\nu(E) = \int_E f d\mu$ is a measure.

1.5.6 Suppose $g_m \geqslant 0$ and take $f_n = \sum_{m=0}^{n} g_m \geqslant 0$

we have $f := \lim_{n\to\infty} f_n = \lim_{n\to\infty} \sum_{m=0}^{n} g_m = \sum_{m=0}^{\infty} g_m$, i.e. $f_n \uparrow f$

By MCT, then $\int f_n d\mu \uparrow \int f d\mu$ i.e. $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$

then
$$\begin{aligned}\int \sum_{m=0}^{\infty} g_m d\mu &= \int \lim_{n\to\infty} \sum_{m=0}^{n} g_m d\mu \\ &= \int \lim_{n\to\infty} f_n d\mu \\ &= \int f d\mu \\ &= \lim_{n\to\infty} \int f_n d\mu \\ &= \lim_{n\to\infty} \int \sum_{m=0}^{n} g_m d\mu \\ &= \lim_{n\to\infty} \sum_{m=0}^{n} \int g_m d\mu \\ &= \sum_{m=0}^{\infty} \int g_m d\mu\end{aligned}$$

1.5.8 Suppose f is integrable on $[a,b]$, and $g(x) := \int_{[a,x]} f(y) dy$. WLOG, assume $f \geqslant 0$

(any f could be decomposed to $f^+ \geqslant 0$, $f^- \geqslant 0$,

$f = f^+ - f^-$, the continuity can be proof on both parts)

For $\forall$ fixed $x \in (a,b)$, take function $h_n(y) = f(y) 1_{[x-\frac{1}{n}, x+\frac{1}{n}]}$ $n \in \mathbb{N}$

$0 \leqslant h_n = f 1_{[x-\frac{1}{n}, x+\frac{1}{n}]} \leqslant f$, then $h_n \longrightarrow 0$ a.e.

By DCT, we have $\int h_n d\mu \longrightarrow \int 0 d\mu$ (i.e. $\lim_{n\to\infty} \int f 1_{[x-\frac{1}{n}, x+\frac{1}{n}]} d\mu = 0$).

$$\begin{aligned}\left| g(x+\tfrac{1}{n}) - g(x-\tfrac{1}{n}) \right| &= \left| \int_{[a, x+\frac{1}{n}]} f(y) dy - \int_{[a, x-\frac{1}{n}]} f(y) dy \right| \\ &= \left| \int f(y) 1_{[a, x+\frac{1}{n}]} dy - \int f(y) 1_{[a, x-\frac{1}{n}]} dy \right| \\ &= \left| \int f(y) 1_{[x-\frac{1}{n}, x+\frac{1}{n}]} dy \right| \\ &= \left| \int h_n dy \right| \\ &= \int h_n dy \longrightarrow 0 \quad \text{when } n \to 0\end{aligned}$$

Hence function $g(x) = \int_{[a,x]} f(y) dy$ is continuous on (a,b).

1.5.10 WLOG, take $f_n = f_n^+ - f_n^-$, $|f_n| = f_n^+ + f_n^-$ $(f_n^+ > 0, f_n^- > 0)$

Suppose $\sum_{n=1}^{\infty}\int |f_n|\, d\mu < \infty$,

then $\sum_{n=1}^{\infty}\int (f_n^+ + f_n^-)\, d\mu = \sum_{n=1}^{\infty}\int f_n^+ d\mu + \sum_{n=1}^{\infty}\int f_n^- d\mu < \infty$

$\Rightarrow \sum_{n=1}^{\infty}\int f_n^+ d\mu < \infty \quad \sum_{n=1}^{\infty}\int f_n^- d\mu < \infty$

By the conclusion of 1.5.6 , we have

$\sum_{n=1}^{\infty}\int f_n^+ d\mu = \int \sum_{n=1}^{\infty} f_n^+ d\mu$, $\sum_{n=1}^{\infty}\int f_n^- d\mu = \int \sum_{n=1}^{\infty} f_n^- d\mu$

then
$$
\begin{aligned}
\sum_{n=1}^{\infty}\int f_n\, d\mu &= \sum_{n=1}^{\infty}\int (f_n^+ - f_n^-)\, d\mu \\
&= \sum_{n=1}^{\infty}\int f_n^+ d\mu - \sum_{n=1}^{\infty}\int f_n^- d\mu \\
&= \int \sum_{n=1}^{\infty} f_n^+ d\mu - \int \sum_{n=1}^{\infty} f_n^- d\mu \\
&= \int \sum_{n=1}^{\infty} (f_n^+ - f_n^-)\, d\mu \\
&= \int \sum_{n=1}^{\infty} f_n\, d\mu
\end{aligned}
$$

1.6.2 Suppose $\phi: \mathbb{R}^n \to \mathbb{R}$ is convex, that is for $\forall\, x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$, we have
$$\lambda\phi(x) + (1-\lambda)\phi(y) \geq \phi(\lambda x + (1-\lambda)y)$$

Let $c = (\mathbb{E}X_1, \mathbb{E}X_2, \cdots, \mathbb{E}X_n)^T$ and $\ell(x) = a^T(x-c) + \phi(c)$, $a \in \mathbb{R}^n$

As convexity implies, we can find $a^T \in \mathbb{R}^n$ s.t. $\phi(x) \geq \ell(x)$ for $\forall\, x \in \mathbb{R}^n$

then
$$
\begin{aligned}
\mathbb{E}\phi(X) \geq \mathbb{E}\ell(X) &= \mathbb{E}\{a^T(X-c) + \phi(c)\} \\
&= a^T(\mathbb{E}X - c) + \phi(c) \\
&= 0 + \phi(\mathbb{E}X)
\end{aligned}
$$

i.e. $\mathbb{E}\phi(X_1, \cdots, X_n) \geq \phi(\mathbb{E}X_1, \cdots, \mathbb{E}X_n)$

1.6.3. (i) Construct a discrete r.v. X satisfying
$$P(X=x) = \begin{cases} \frac{b^2}{2a^2}, & x = \pm a \\ 1 - \frac{b^2}{a^2}, & x = 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{for fixed } 0 < b \leq a$$

such X has $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = b^2$

$$
\begin{aligned}
\Rightarrow a^2 P(|X| \geq a) &= a^2[P(X=1) + P(X=-1)] \\
&= a^2\left(\frac{b^2}{2a^2} + \frac{b^2}{2a^2}\right) \\
&= b^2 = \mathbb{E}X^2
\end{aligned}
$$

(question (ii) on the next page).

(ii) $a^2 P(|X|\geq a) = \int_{|X|\geq a} a^2 f(x)\,d\mu \leq \int_{|X|\geq a} x^2 f(x)\,d\mu$

then $\mathbb{E}X^2 = \int x^2 f(x)\,d\mu = \int_{|x|<a} x^2 f(x)\,d\mu + \int_{|x|\geq a} x^2 f(x)\,d\mu$

$\geq \int_{|x|<a} x^2 f(x)\,d\mu + a^2 P(|X|\geq a)$

Suppose $\mathbb{E}X^2 < \infty$, then $\exists\, M>0$, s.t. $\mathbb{E}X^2 = M < \infty$

then $\dfrac{a^2 P(|X|\geq a)}{\mathbb{E}X^2} = 1 - \dfrac{\int_{|x|<a} x^2 f(x)\,d\mu}{M}$

Since $\lim_{a\to\infty} \int_{|x|<a} x^2 f(x)\,d\mu = M$

then $\lim_{a\to\infty} \dfrac{a^2 P(|X|\geq a)}{\mathbb{E}X^2} = 1 - \dfrac{\lim_{a\to\infty} \int_{|x|<a} x^2 f(x)\,d\mu}{M} = 0$

1.6.4 (i) Suppose $a>b>0$, $0<p<1$ $\quad P(X=a)=p$, $P(X=-b)=1-p$

the $\mathbb{E}(X^2) = a^2 p + b^2(1-p)$ $\quad \mathbb{E}X = ap - b(1-p)$

By Thm. 1.6.4, we have $i_A P(Y\in A) \leq \mathbb{E}\phi(Y)$, $\quad A=[a,+\infty)$

where $i_A P(Y\in A) = \inf\{(y+b)^2 : y\geq a\}\, P(Y\geq a)$

$= (a+b^2)\, P(Y\geq a)$

$\mathbb{E}\phi(Y) = \mathbb{E}(Y+b)^2 = \mathbb{E}(Y^2) + 2b\,\mathbb{E}Y + b^2$

$= \mathrm{Var}\,Y + (\mathbb{E}Y)^2 + 2b\,\mathbb{E}Y + b^2$

$= \mathrm{Var}\,X + (\mathbb{E}X)^2 + 2b\,\mathbb{E}X + b^2$

$= \mathbb{E}(X)^2 + 2b\,\mathbb{E}X + b^2$

$= a^2 p + b^2(1-p) + 2b[ap - b(1-p)] + b^2$

$= (a+b)^2 p$

$\Rightarrow (a+b)^2 P(Y\geq a) \leq (a+b)^2 p$

$\Rightarrow P(Y\geq a) \leq p$

Since $P(X\geq a) = P(X=a) = p$, then the equality holds when $Y=X$

(ii) To construct a $\phi(y) = my^2 + ly + n$ satisfying $\dfrac{\mathbb{E}\varphi(Y)}{\varphi(a)} = \dfrac{\sigma^2}{a^2+\sigma^2}$

(m, l, n to be determined)

we prefer $\dfrac{m\mathbb{E}Y^2 + l\mathbb{E}Y + n}{ma^2 + la + n} = \dfrac{m\sigma^2 + l \cdot 0 + n}{ma^2 + la + n} = \dfrac{\sigma^2}{a^2+\sigma^2}$ $\qquad \mathbb{E}Y^2 = \sigma^2 \quad \mathbb{E}Y = 0.$

and $\begin{cases} m\sigma^2 + n = \sigma^2 \\ ma^2 + la + n = a^2 + \sigma^2 \end{cases}$ let $m = 1$, then $\begin{cases} n = 0 \\ l = \dfrac{\sigma^2}{a} \end{cases}$

then take $\phi(y) = y^2 + \frac{\sigma^2}{a} y$ as constructed

Bring $\phi(y)$ above into Thm 1.6.4, we have

$$\inf\{y^2 + \tfrac{\sigma^2}{a} y : y \geq a > 0\} P(Y \geq a) \leq \mathbb{E}(Y^2 + \tfrac{\sigma^2}{a} Y)$$

$$(a^2+\sigma^2) P(Y \geq a) \leq \mathbb{E}Y^2 + \tfrac{\sigma^2}{a}\mathbb{E}Y = \sigma^2$$

$$\Rightarrow \quad P(Y \geq a) \leq \frac{\sigma^2}{a^2+\sigma^2}$$

Let $P(Y=a) = \frac{\sigma^2}{a^2+\sigma^2}$ $(\in (0,1))$, to construct Y satisfing $\mathbb{E}Y = 0$ $\mathbb{E}Y^2 = \sigma^2$

we let $b < a$ s.t. $\begin{cases} \frac{\sigma^2}{a^2+\sigma^2} a + (1 - \frac{\sigma^2}{a^2+\sigma^2}) b = 0 \\ \frac{\sigma^2}{a^2+\sigma^2} a^2 + (1 - \frac{\sigma^2}{a^2+\sigma^2}) b^2 = \sigma^2 \end{cases}$

$$\Rightarrow \quad b = -\frac{\sigma^2}{a}$$

so there is a Y for which equality holds. such Y satisfing

$$P(Y=y) = \begin{cases} \dfrac{\sigma^2}{a^2+\sigma^2}, & y = a \\ \dfrac{a^2}{a^2+\sigma^2}, & y = -\dfrac{\sigma^2}{a} \\ 0, & \text{otherwise} \end{cases}$$

1.6.6 Suppose $Y \geq 0$ and $\mathbb{E}Y^2 < \infty$,

then by Cauchy-Schwarz Inequality

$$(\mathbb{E}Y)^2 \overset{\text{omit}\{0\}}{=} (\mathbb{E}Y \cdot 1_{(Y>0)})^2 \leq (\mathbb{E}Y^2) \cdot (\mathbb{E}1^2_{(Y>0)})$$

$$= (\mathbb{E}Y^2) \cdot \mathbb{E}1_{(Y>0)}$$

$$= (\mathbb{E}Y^2) \cdot P(Y>0)$$

$$\Rightarrow \quad P(Y>0) \geq \frac{(\mathbb{E}Y)^2}{\mathbb{E}Y^2}$$

1.6.10 $B_1 = A_1$, $B_m = A_m \setminus [A_m \cap (\bigcup_{i=1}^{m-1} A_i)] = A_m \setminus [\bigcup_{i=1}^{m-1} (A_m \cap A_i)]$ for $m \geq 2$

we know B_i disjoint, $A = \bigcup_{i=1}^{n} B_i$, and $B_i \subset A_i$ for $i = 1, 2, \cdots, n$

Then $1_A = \sum_{i=1}^{n} 1_{B_i} \leq \sum_{i=1}^{n} 1_{A_i}$

$\Rightarrow \mathbb{E} 1_A \leq \mathbb{E} \sum_{i=1}^{n} 1_{A_i} = \sum_{i=1}^{n} \mathbb{E} 1_{A_i}$

$\Leftrightarrow P(A) \leq \sum_{i=1}^{n} P(A_i)$

Denote $C_j^{(m)} = A_m \cap A_j$ $\quad C^{(m)} = \bigcup_{j=1}^{m-1} C_j^{(m)}$, then we directly have

$1_{C^{(m)}} \leq \sum_{j=1}^{m-1} 1_{C_j^{(m)}}$ i.e. $1_{\bigcup_{j=1}^{m-1} A_m \cap A_j} \leq \sum_{j=1}^{m-1} 1_{A_m \cap A_j}$

then $1_{B_m} = 1_{A_m \setminus [\bigcup_{j=1}^{m-1} (A_m \cap A_j)]}$

(Since $\bigcup_{j=1}^{m-1} (A_m \cap A_j)$ and $A_m \setminus [\bigcup_{j=1}^{m-1} (A_m \cap A_j)]$ disjoint and their union is A_m

then $1_{A_m} = 1_{\bigcup_{j=1}^{m-1} (A_m \cap A_j)} + 1_{A_m \setminus [\bigcup_{j=1}^{m-1} (A_m \cap A_j)]}$)

$1_{B_m} = 1_{A_m} - 1_{\bigcup_{j=1}^{m-1} (A_m \cap A_j)}$

$\geq 1_{A_m} - \sum_{j=1}^{m-1} 1_{A_m \cap A_j}$

then $1_A = \sum_{i=1}^{n} 1_{B_m} \geq \sum_{i=1}^{n} 1_{A_i} - \sum_{i<j} 1_{A_m \cap A_j}$

Similarly, $P(A) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$

Then N.T.S. $P(A) = P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$

$\Leftarrow 1_A \leq \sum_{i=1}^{n} 1_{A_i} - \sum_{i<j} 1_{A_i \cap A_j} + \sum_{i<j<k} 1_{A_i \cap A_j \cap A_k} \quad \cdots \quad (*)$

Define $I(\omega) = \{i : \omega \in A_i\}$, for $\omega \in A$

for $\omega \in \{\omega : |I(\omega)| = 1\}$, $*RHS(\omega) = 1 - 0 + 0 = *LHS(\omega)$

for $\omega \in \{\omega : |I(\omega)| = 2\}$, $*RHS(\omega) = 2 - 1 + 0 = *LHS(\omega)$

for $\omega \in \{\omega : |I(\omega)| = k \geq 3\}$, $*RHS(\omega) = C_k^1 - C_k^2 + C_k^3$

$$= k - \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{3!}$$

$$= \frac{k[(k-3)^2+2]}{6}$$

$$\geq \frac{3[0^2+2]}{6} = 1 = *LHS(\omega)$$

so $(*)$ satisfied, and we take its expectation

$\Rightarrow P(A) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) + \sum_{i<j<k}^{n} P(A_i \cap A_j \cap A_k)$

Similarly, we can check every step of inclusion-exclusion formula after an even (odd) number of sums and confirm that an lower (upper) bound of $P(A)$ is obtained.

1.6.14 Suppose $X \geq 0$, $y > 0$

then $\lim_{y\to\infty} y\,\mathbb{E}(\frac{1}{X}; X > y) = \lim_{y\to\infty} \mathbb{E}(\frac{y}{X}; X>y) = \lim_{y\to\infty} \int_{\frac{y}{X}<1} \frac{y}{x}\, d\mu$

$\leq \lim_{y\to\infty} \int 1(X>y)\, d\mu$

$= \lim_{y\to\infty} P(X>y) = 0$

and considering $X\geq 0$ $y>0$, then $\lim_{y\to\infty} y\,\mathbb{E}(\frac{1}{X}; X>y) \geq 0$

$\Rightarrow \lim_{y\to\infty} y\,\mathbb{E}(\frac{1}{X}; X>y) = 0$

Construct r.v. $G(y) = \frac{y}{X} 1(X>y) = \frac{y}{X} 1(\frac{y}{X}<1)$ for $y>0$

$0 < G(y) = \frac{y}{X} 1(X>y) < 1$ for $y>0$

$\mathbb{E}G(y) < \int \frac{y}{x} 1(X>y>0)\, d\mu < P(X>y) < \infty.$

$\lim_{y\to 0} G(y) = G(0) = 0$

then by DCT, we have

$\lim_{y\to 0} \mathbb{E}G(y) = \mathbb{E} \lim_{y\to 0} G(y) = 0$

i.e. $\lim_{y\to 0} y\,\mathbb{E}(\frac{1}{X}; X>y) = 0$

2.1.4 Suppose $(X_1, \cdots, X_n)$ has density $f(x_1, \cdots, x_n)$, i.e.

$$P((X_1, \cdots, X_n) \in A) = \int_A f(x)\, dx \quad \text{for } A \in \mathbb{R}^n$$

where $f(x)$ can be writen as $g_1(x_1)\cdots g_n(x_n)$, $g_m(x_m) \geq 0$ measuroable for $m = 1, \cdots, n$

Take $A = (-\infty, a_1] \times \cdots \times (-\infty, a_n]$ $a_i \in \mathbb{R}^n$ $i = 1, \cdots, n$

the $P((X_1, \cdots, X_n) \in A) = P(X_1 \leq a_1, \cdots, X_n \leq a_n)$

$$= \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} f(x)\, dx_n \cdots dx_1$$

$$= \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} g_1(x_1)\cdots g_n(x_n)\, dx_n \cdots dx_1$$

$$= \int_{-\infty}^{a_1} g_1(x_1)\, dx_1 \cdots \int_{-\infty}^{a_n} g_n(x_n)\, dx_n$$

$$\left(\begin{aligned} \int_{\mathbb{R}^n} f(x)\, dx &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_1(x_1)\cdots g_n(x_n)\, dx_n \cdots dx_1 \\ &= \int_{-\infty}^{+\infty} g_1(x_1)\, dx_1 \cdots \int_{-\infty}^{+\infty} g_n(x_n)\, dx_n \\ &= 1 \end{aligned} \right)$$

$$= \frac{\int_{-\infty}^{a_1} g_1(x_1)\, dx_1}{\int_{-\infty}^{+\infty} g_1(x)\, dx_1} \cdots \frac{\int_{-\infty}^{a_n} g_n(x_n)\, dx_n}{\int_{-\infty}^{+\infty} g_n(x_n)\, dx_n}$$

(contine on the next page)

Since $P(X_i \le a_i) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{a_i} \cdots \int_{-\infty}^{+\infty} g_1(x_1) \cdots g_i(x_i) \cdots g_n(x_n)\, dx_n \cdots dx_i \cdots dx_1$

$$= \left[\prod_{j \ne i}^{n} \int_{-\infty}^{+\infty} g_j(x_j)\,dx_j\right] \int_{-\infty}^{a_i} g_i(x_i)\,dx_i$$

$$= \int_{-\infty}^{a_i} g_i(x_i)\,dx_i \Big/ \int_{-\infty}^{+\infty} g_i(x_i)\,dx_i$$

then $P(X_1 \le a_1, \ldots, X_n \le a_n) = \prod_{i=1}^{n} P(X_i \le a_i)$

$\Rightarrow$ $X_1, \ldots, X_n$ are independent by Thm. 2.1.4.

2.1.9 Suppose X and Y are independent and f and g are measurable functions.

For $\forall A_1, A_2 \in \mathcal{R}$, then $B_1 = f^{-1}(A_1)$, $B_2 = g^{-1}(A_2) \in \mathcal{B}$

$$P(f(X) \in A_1, g(Y) \in A_2) = P(X \in B_1, Y \in B_2)$$

$$\overset{(X, Y \text{ indep.})}{=} P(X \in B_1) \cdot P(Y \in B_2)$$

$$= P(f(X) \in A_1) \cdot P(g(Y) \in A_2)$$

Hence $f(X)$ and $g(Y)$ are independent

2.1.12 Suppose $\Omega = \{1,2,3,4\}$ then $\mathcal{F} = 2^{\Omega}$

Take $\mathcal{A}_1 = \{\{1,2\}, \{2,4\}\}$, $\mathcal{A}_2 = \{\{1,4\}\}$

then $P(\{1,2\} \cap \{1,4\}) = P(\{1\}) = \frac{1}{4}$, $P(\{1,2\})P(\{1,4\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(\{2,4\} \cap \{1,4\}) = P(\{4\}) = \frac{1}{4}$, $P(\{2,4\})P(\{1,4\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

i.e. for $\forall A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, $P(A_1 \cap A_2) = P(A_1)P(A_2)$

$\Rightarrow \mathcal{A}_1$ and $\mathcal{A}_2$ are independent

$\sigma(\mathcal{A}_1) = 2^{\Omega} = \mathcal{F}$, $\sigma(\mathcal{A}_2) = \{\emptyset, \{1,4\}, \{2,3\}, \Omega\}$

we have $\{1,4\} \in \sigma(\mathcal{A}_1)$ and $\{1,4\} \in \sigma(\mathcal{A}_2)$

then $P(\{1,4\} \cap \{1,4\}) = P(\{1,4\}) = \frac{1}{2}$

but $P(\{1,4\})P(\{1,4\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \ne \frac{1}{2}$

$\Rightarrow \sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are not independent.