1. $Y_{n\times 1} \sim N(\alpha 1, \sigma^2 I)$, $U = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/\sigma^2$ and $V = n(\bar{Y}-\alpha)^2/\sigma^2$.
Show the distribution of U and V, and $U \perp\!\!\!\perp V$.

U and V can be rewritten as matrix products.

Denote $Z_i = \frac{1}{\sigma}(Y_i - \alpha)$ then $Z = [Z_1, \dots, Z_n]^T$ follows $N(0, I_n)$

$$\Rightarrow \frac{1}{\sigma}(\bar{Y}-\alpha) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma}(Y_i - \alpha) = \frac{1}{n}1_n^T Z$$

$$U = \sum_{i=1}^{n}(Y_i - \alpha - (\bar{Y}-\alpha))^2/\sigma^2 = \sum_{i=1}^{n}\left(\frac{Y_i-\alpha}{\sigma}\right)^2 - n\left(\frac{\bar{Y}-\alpha}{\sigma}\right)^2$$

$$= Z^T Z - n \cdot \frac{1}{n^2} Z^T 1_n 1_n^T Z = Z^T(I_n - \frac{1}{n}J_n)Z$$

$$V = n\left(\frac{\bar{Y}-\alpha}{\sigma}\right)^2 = n \cdot \frac{1}{n^2} Z^T 1_n 1_n^T Z = Z^T \frac{1}{n} J_n Z$$

Denote $A = I_n - \frac{1}{n}J_n$ and $B = \frac{1}{n}J_n$, both symmetric.

Since $AI_n = A$ is idempotent and $\text{rank}(A) = n-1$

then $U = Z^T A Z \sim \chi^2_{n-1}$

$BI_n = B$ is idempotent and $\text{rank}(B) = 1$

then $V = Z^T B Z \sim \chi^2_1$

Notice $B\Sigma A = \frac{1}{n}J_n I_n(I_n - \frac{1}{n}J_n) = \frac{1}{n}J_n - \frac{1}{n}J_n = 0$

then $U \perp\!\!\!\perp V$.

2. $Y = (Y_1 \dots Y_n)^T \sim N(\mu 1, I)$, $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, $Q_1 = n\bar{Y}^2$, and $Q_2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$

(a) Prove $\bar{Y} \perp\!\!\!\perp Q_2$

Notice $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}1_n^T Y$ and

$$Q = \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2 = Y^T Y - \frac{1}{n} Y^T 1_n 1_n^T Y = Y^T(I_n - \frac{1}{n}J_n)Y$$

Since $\frac{1}{n}1_n^T I_n(I_n - \frac{1}{n}J_n) = \frac{1}{n}1_n^T - \frac{1}{n}1_n^T = 0$

then $\bar{Y} \perp\!\!\!\perp Q$

(b) Prove $Q_1 \perp\!\!\!\perp Q_2$

Notice $Q_1 = n\bar{Y}^2 = n\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2 = \frac{1}{n}(1_n^T Y)^2 = Y^T \frac{1}{n} J_n Y$

Since $\frac{1}{n} J_n I_n (I_n - \frac{1}{n} J_n) = \frac{1}{n} J_n - \frac{1}{n} J_n = 0$

then $Q_1 \perp\!\!\!\perp Q_2$

(c) Find the distribution of Q_1 & Q_2

Since $\frac{1}{n} J_n I_n = \frac{1}{n} J_n$ is idempotent, $\text{rank}(\frac{1}{n} J_n) = 1$ and

$$\lambda = \frac{1}{2}(\mu 1_n)^T \frac{1}{n} J_n (\mu 1_n) = \frac{1}{2n} \cdot \mu^2 n^2 = \frac{1}{2} n\mu^2$$

then $Q_1 = Y^T \frac{1}{n} J_n Y \sim \chi^2_{(1, \frac{1}{n} n\mu^2)}$

Since $(I_n - \frac{1}{n} J_n) I_n = I_n - \frac{1}{n} J_n$ is idempotent, $\text{rank}(I_n - \frac{1}{n} J_n) = n-1$ and

$$\lambda = \frac{1}{2}(\mu 1_n)^T (I_n - \frac{1}{n} J_n)(\mu 1_n) = \frac{1}{2}\mu^2 (n - \frac{1}{n} \cdot n^2) = 0$$

then $Q_2 = Y^T (I_n - \frac{1}{n} J_n) Y \sim \chi^2_{n-1}$

3. $Y_{n\times 1} \sim N(\mu, \Sigma)$, $q_1 = Y^T A_1 Y$, $q_2 = Y^T A_2 Y$, and $T = BY$ ($B \in \mathbb{R}^{r\times n}$, A_1 and A_2 symmetric).

(a) $Eq_1 = tr(A_1 \Sigma) + \mu^T A_1 \mu$

$$\mathbb{E}q_1 = \mathbb{E}(Y^T A_1 Y) = \mathbb{E}tr(Y^T A_1 Y) = \mathbb{E}tr(A_1 Y Y^T)$$

$$= tr(\mathbb{E}(A_1 Y Y^T)) = tr(A_1 \mathbb{E}(YY^T)) = tr(A_1(\Sigma + \mu\mu^T)) = tr(A_1\Sigma) + tr(A_1 \mu\mu^T)$$

$$= tr(A_1\Sigma) + tr(\mu^T A_1 \mu) = tr(A_1 \Sigma) + \mu^T A_1 \mu.$$

(b) $Var\, q_1 = 2tr(A_1 \Sigma A_1 \Sigma) + 4\mu^T A_1 \Sigma A_1 \mu$

TO BE CONTINUE...

(c) $Cov(q_1, q_2) = 2tr(A_1\Sigma A_2\Sigma) + 4\mu^T A_1 \Sigma A_2 \mu$

Since $Var(q_1+q_2) = \mathbb{E}[q_1+q_2-\mathbb{E}(q_1+q_2)]^2 = \mathbb{E}[(q_1-\mathbb{E}q_1)+(q_2-\mathbb{E}q_2)]^2$

$$= Var q_1 + Var q_2 + 2Cov(q_1, q_2)$$

then $Cov(q_1, q_2) = \frac{1}{2}[Var(q_1+q_2) - Var q_1 - Var q_2]$

Since $q_1 = Y^T A_1 Y$, $q_2 = Y^T A_2 Y$, and $q_1+q_2 = Y^T(A_1+A_2)Y$

by (b), $Var q_1 = 2tr(A_1\Sigma A_1\Sigma) + 4\mu^T A_1 \Sigma A_1 \mu$

$Var q_2 = 2tr(A_2\Sigma A_2\Sigma) + 4\mu^T A_2 \Sigma A_2 \mu$

$$Var(q_1+q_2) = 2tr[(A_1+A_2)\Sigma(A_1+A_2)\Sigma] + 4\mu^T(A_1+A_2)\Sigma(A_1+A_2)\mu$$

$$= 2tr(A_1\Sigma A_1\Sigma) + 2tr(A_2\Sigma A_2\Sigma) + 4tr(A_1\Sigma A_2\Sigma)$$

$$+ 4\mu^T A_1\Sigma A_1\mu + 4\mu^T A_2\Sigma A_2\mu + 8\mu^T A_1\Sigma A_2\mu$$

then $Cov(q_1, q_2) = 2tr(A_1\Sigma A_2\Sigma) + 4\mu^T A_1\Sigma A_2\mu$.

(d) $Cov(Y, q_1) = 2\Sigma A_1 \mu$

$$Cov(Y, q_1) = \mathbb{E}(Y-\mathbb{E}Y)(q_1-\mathbb{E}q_1)$$

$$= \mathbb{E}(Y-\mu)(Y^T A_1 Y - tr(A_1\Sigma) - \mu^T A_1 \mu)$$

$$= \mathbb{E}(Y-\mu)[(Y-\mu)^T A_1 (Y-\mu) + \mu^T A_1 Y + Y^T A_1 \mu - tr(A_1\Sigma) - 2\mu^T A \mu]$$

$$= \mathbb{E}(Y-\mu)(Y-\mu)^T A_1 (Y-\mu) + 2\mathbb{E}(Y-\mu)(Y-\mu)^T A_1 \mu - tr(A_1\Sigma)\mathbb{E}(Y-\mu) \quad \cdots (*)$$

Easy to know $tr(A_1\Sigma)\mathbb{E}(Y-\mu) = 0$ and

any third central moment of multi-variate normal distribution is 0.

then $(*) = 0 + 2\Sigma A_1 \mu - 0 = 2\Sigma A_1 \mu$.

(e) $Cov(T, q_1) = 2B\Sigma A_1 \mu$.

$Cov(T, q_1) = \mathbb{E}(BY - \mathbb{E}BY)(q_1 - \mathbb{E}q_1) = B\mathbb{E}(Y-\mathbb{E}Y)(q_1-\mathbb{E}q_1)$

$$= B\,Cov(Y, q_1) = 2B\Sigma A_1 \mu.$$

4. $y \sim N_3(\mu, \sigma^2 I)$, $\mu = [3\ -2\ 1]^T$, $A = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, and $B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

(a) Find the distribution of y^TAy/σ^2

Since $y \sim N_3(\mu, \sigma^2 I)$, then $y/\sigma \sim N_3(\mu/\sigma, I_3)$

Notice $AA = \frac{1}{9}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = A$.

then A is idempotent. (AI_3 is idempotent) with $\text{rank}(A) = 2$

$$\lambda = \frac{1}{2\sigma^2}\mu^TA\mu = \frac{1}{6\sigma^2}[7\ -8\ 1][3\ -2\ 1] = \frac{19}{3\sigma^2}$$

then $y^TAy/\sigma^2 \sim \chi^2_{(2, \frac{19}{3\sigma^2})}$

(b) Are y^TAy and By independent?

$$B\sigma^2 I_3 A = \sigma^2 BA = \frac{\sigma^2}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \frac{\sigma^2}{9}\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & -3 \end{bmatrix} \neq 0$$

then $y^TAy \not\perp\!\!\!\perp By$

(c) Are y^TAy and $y_1+y_2+y_3$ independent?

$y_1+y_2+y_3$ can be rewritten as 1^Ty

Since $1^T\sigma^2 I_3 A = \sigma^2[1\ 1\ 1]\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 0$

then $y^TAy \perp\!\!\!\perp 1^Ty$ i.e. y^TAy and $y_1+y_2+y_3$ independent.

5 $X_1, X_2, X_3, X_4 \overset{iid}{\sim} N(0, \sigma^2)$, $Q = X_1X_2 - X_3X_4$, Does Q/σ^2 has a χ^2 distribution?

The range of Q covers $\mathbb{R}^-$,

while r.v. following χ^2 distribution should be non-negative

then Q cannot has a χ^2 distribution.

6. $y \sim N_n(\mu 1, \Sigma)$ with $\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$ derive the distribution of $\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2(1-\rho)}$

$$\sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n y_i^2 - n\bar{y}^2 = y^T(I_n - \tfrac{1}{n}J_n)y$$

Denote $A = \frac{1}{\sigma^2(1-\rho)}(I_n - \frac{1}{n}J_n)$, then $\frac{\sum_{i=1}^n (y_i-\bar{y})^2}{\sigma^2(1-\rho)} = y^T A y$

Since $A\Sigma = \frac{1}{\sigma^2(1-\rho)}(I_n - \frac{1}{n}J_n)\sigma^2[(1-\rho)I_n + \rho J_n]$

$$= I_n + \frac{\rho}{1-\rho}J_n - \frac{1}{n}J_n - \frac{\rho}{1-\rho}J_n = I_n - \frac{1}{n}J_n$$

and $(A\Sigma)(A\Sigma) = I_n - \frac{1}{n}J_n$ then $A\Sigma$ is idempotent

We know that $\text{rank}(A\Sigma) = n-1$ and $\lambda = \frac{1}{2}(\mu 1)^T(I_n - \frac{1}{n}J_n)(\mu 1) = \frac{\mu^2}{2}(n-n) = 0$

Thus $y^TAy \sim \chi^2_{n-1}$

7. $Y = [Y_1 \; Y_2 \; Y_3]^T$ $\mathbb{E}Y = [2 \; 3 \; 4]^T$ $\Sigma = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

$U = \sum_{i=1}^3 (Y_i - \bar{Y})^2$, Find $\mathbb{E}U$

We know $U = Y^T(I_3 - \frac{1}{3}J_n)Y$

$$\mathbb{E}U = \mathbb{E}Y^T(I_3 - \tfrac{1}{3}J_n)Y = tr((I_3 - \tfrac{1}{3}J_n)\Sigma) + (\mathbb{E}Y)(I_3 - \tfrac{1}{3}J_n)(\mathbb{E}Y).$$

$$= tr\left(\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}\right) + [2\;3\;4]\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= (\tfrac{5}{3} + \tfrac{1}{3} + 2) + 2 = 6$$

8. $Y = [Y_1 \cdots Y_n]^T$ $\mathbb{E}Y = \mu 1$ $\text{Var}Y = \sigma^2 I$, $U = \sum_{i<j}(Y_i - Y_j)^2$

(a) Find $\mathbb{E}U$

$$\mathbb{E}U = \mathbb{E}\sum_{i<j}(Y_i - Y_j)^2 = \sum_{i<j}(\mathbb{E}Y_i^2 + \mathbb{E}Y_j^2 - 2\mathbb{E}Y_iY_j)$$

$$= \sum_{i<j}[\text{Var}Y_i + (\mathbb{E}Y_i)^2 + \text{Var}Y_j + (\mathbb{E}Y_j)^2 - 2\mathbb{E}Y_i\mathbb{E}Y_j]$$

$$= \sum_{i<j} 2\sigma^2 = \frac{(n-1)n}{2}\cdot 2\sigma^2 = (n-1)n\sigma^2$$

(b) Find k such that $\mathbb{E}kU = \sigma^2$

$$\sigma^2 = \mathbb{E}kU = k\mathbb{E}U = k(n-1)n\sigma^2$$

$$\Rightarrow \quad k = \frac{1}{n-1} - \frac{1}{n}$$

9. $Y = [Y_1 \cdots Y_n]^T$, $\mathbb{E}Y = \mu\mathbb{1}$, $\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$ ρ known. $U = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$

Find k such that $\mathbb{E}kU = \sigma^2$

$U = Y^T(I_n - \frac{1}{n}J_n)Y$, and Σ can be rewritten as $\Sigma = \sigma^2[(1-\rho)I_n + \rho J_n]$

then $(I_n - \frac{1}{n}J_n)\Sigma = \sigma^2[(1-\rho)I_n + \rho J_n - \frac{1}{n}(1-\rho)J_n - \rho J_n] = \sigma^2(1-\rho)(I_n - \frac{1}{n}J_n)$

then $\mathbb{E}U = \mathbb{E}Y^T(I_n - \frac{1}{n}J_n)Y$

$$= tr((I_n - \tfrac{1}{n}J_n)\Sigma) + (\mu\mathbb{1})^T(I_n - \tfrac{1}{n}J_n)(\mu\mathbb{1})$$

$$= \sigma^2(1-\rho)tr(I_n - \tfrac{1}{n}J_n) + \mu^2(n-n)$$

$$= (n-1)\sigma^2(1-\rho)$$

$$\sigma^2 = \mathbb{E}kU = k(n-1)\sigma^2(1-\rho) \quad \Rightarrow \quad k = \frac{1}{(n-1)(1-\rho)}$$

3.(b) $Var(q_1) = 2tr(A_1\Sigma A_1\Sigma) + 4\mu^T A_1\Sigma A_1\mu$

(From Linear Model in Statistics by Alvin C. Rencher and G. Bruce Schaalje, 2008).

The moment generating function of $q_1 = y^T A_1 y$ is

$$M_{q_1}(t) = |I - 2tA_1\Sigma|^{-\frac{1}{2}} \exp\{-\mu^T[I - (I - 2tA_1\Sigma)^{-1}]\Sigma^{-1}\mu/2\}$$

Denote $C = I - 2tA\Sigma$, then take

$$k(t) := \ln M_{q_1}(t) = -\tfrac{1}{2}\ln|C| - \tfrac{1}{2}\mu^T(I - C^{-1})\Sigma^{-1}\mu.$$

then $k''(t) = \frac{1}{2}\cdot\frac{1}{|C|^2}\cdot\left(\frac{d|C|}{dt}\right)^2 - \frac{1}{2}\cdot\frac{1}{|C|}\cdot\frac{d^2|C|}{dt^2} - \frac{1}{2}\mu^T C^{-1}\frac{d^2C}{dt^2}C^{-1}\Sigma^{-1}\mu$

$$+\mu\left(C^{-1}\frac{dC}{dt}\right)^2 C^{-1}\Sigma^{-1}\mu \qquad \cdots (*)$$

If the eigenvalue of $A_1\Sigma$ are λ_i, $i=1,\cdots,n$, we have

$$|C| = \prod_{i=1}^{p}(1-2t\lambda_i) = 1-2t\sum_{i=1}^{n}\lambda_i + 4t^2\sum_{i\neq j}\lambda_i\lambda_j - \cdots + (-1)^n\cdot 2^n\cdot t^n\lambda_1\lambda_2\cdots\lambda_n$$

Then $$\frac{d|C|}{dt} = -2\sum_{i=1}^{n}\lambda_i + 8t\sum_{i\neq j}\lambda_i\lambda_j + o(t)$$

$$\frac{d^2|C|}{dt^2} = 8\sum_{i\neq j}\lambda_i\lambda_j + o(1)$$

$$\Rightarrow \begin{cases} |C|\big|_{t=0} = 1, \\ \frac{d|C|}{dt}\Big|_{t=0} = -2\sum_{i=1}^{n}\lambda_i = -2tr(A\Sigma) \\ \frac{d^2|C|}{dt^2}\Big|_{t=0} = 8\sum_{i\neq j}\lambda_i\lambda_j \end{cases}$$

and $$\begin{cases} C\big|_{t=0} = I \\ C^{-1}\big|_{t=0} = I \\ \frac{dC}{dt}\Big|_{t=0} = 2A_1\Sigma \\ \frac{d^2C}{dt^2}\Big|_{t=0} = 0 \end{cases}$$

Plug the above result in $k''(t)$, we have

$$k''(0) = 2[tr(A_1\Sigma)]^2 - 4\sum_{i\neq j}\lambda_i\lambda_j + 0 + 4\mu^T A_1\Sigma A_1\mu$$

$$= 2\left([tr(A_1\Sigma)]^2 - 2\sum_{i\neq j}\lambda_i\lambda_j\right) + 4\mu^T A_1\Sigma A_1\mu$$

By $[tr(A)]^2 = tr(A^2) + 2\sum_{i\neq j}\lambda_i\lambda_j$ for λ_i, $i=1,\cdots,n$ are the eigenvalues of A

$$[tr(A\Sigma)]^2 - 2\sum_{i\neq j}\lambda_i\lambda_j = tr[(A\Sigma)^2] = tr(A_1\Sigma A_1\Sigma)$$

Hence $$Var(q_1) = k''(0) = 2tr(A_1\Sigma A_1\Sigma) + 4\mu^T A_1\Sigma A_1\mu \quad //$$