STAT5005 Homework4 Nov 3, 2022 ZHOU Xiaocheng 1155184323

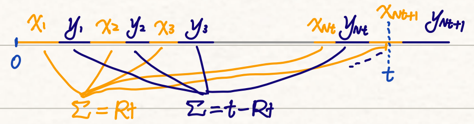
To-Do List: 2.4.(2,3,4), 2.5.(3,4,5,6,7,9,10), 3.1.(4), 3.2.(2,3,9,11,13,14), 3.3.(10,21)

2.4.2 Suppose $\{X_i\} \overset{iid}{\sim} F$, $\{Y_i\} \overset{iid}{\sim} G$, and $\{X_i\}$ and $\{Y_i\}$ are independent

Take $S_n = \sum_{i=1}^n X_i$, $T_n = \sum_{i=1}^n Y_i$, then by SLLN, we have

$S_n/n \xrightarrow{a.s.} \mathbb{E}X_1$ and $T_n/n \xrightarrow{a.s.} \mathbb{E}Y_1$

then $S_n/T_n \xrightarrow{a.s.} \mathbb{E}X_1/\mathbb{E}Y_1$




Take $N_t = \sup\{n: \sum_{i=1}^n (X_i + Y_i) \le t\}$, then we have

$$\min\left\{\frac{S_{N_t}}{S_{N_t}+T_{N_t}}, \frac{S_{N_t}+X_{N_t+1}}{S_{N_t}+X_{N_t+1}+T_{N_t}+Y_{N_t+1}}\right\} \le \frac{R_t}{t} \le \frac{S_{N_t}+X_{N_t+1}}{S_{N_t}+X_{N_t+1}+T_{N_t}}, \text{ i.e.}$$

$$\min\left\{\frac{S_{N_t}}{S_{N_t}+T_{N_t}}, \frac{S_{N_t+1}}{S_{N_t+1}+T_{N_t+1}}\right\} \le \frac{R_t}{t} \le \frac{S_{N_t+1}}{S_{N_t+1}+T_{N_t}} \quad \cdots (*)$$

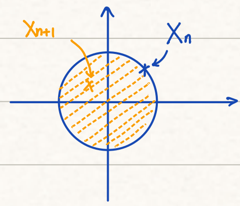
$$\frac{S_{N_t}}{S_{N_t}+T_{N_t}} = \frac{1}{1+T_{N_t}/S_{N_t}} \xrightarrow{a.s.} \frac{1}{1+\mathbb{E}Y_1/\mathbb{E}X_1} \text{ as } n\to\infty \ (N_t\to\infty) \quad (\text{limit of LHS}(*))$$

$$\frac{S_{N_t+1}}{S_{N_t+1}+T_{N_t}} = \frac{1}{1+T_{N_t}/S_{N_t+1}} = \frac{1}{1+\frac{T_{N_t}}{N_t}\cdot\frac{N_t+1}{S_{N_t+1}}\cdot\frac{N_t}{N_t+1}} \xrightarrow{a.s.} \frac{1}{1+\mathbb{E}Y_1\cdot\frac{1}{\mathbb{E}X_1}} = \frac{\mathbb{E}X_1}{\mathbb{E}X_1+\mathbb{E}Y_1}$$

as $n\to\infty$ ($N_t\to\infty$) (limit of RHS(*))

Thus $\frac{R_t}{t} \xrightarrow{a.s.} \frac{\mathbb{E}X_1}{\mathbb{E}X_1+\mathbb{E}Y_1}$ as $n\to\infty$ //

2.4.3 Let $X_0 = (1,0)$, and define $X_n \in \mathbb{R}^2$ inductively by declaring that $X_{n+1}$ is chosen at random from the ball of radius $|X_n|$ centered at the origin.




Notice that $\{X_n/|X_{n-1}|\}_{n=1,\dots}$ is uniformly distributed on the ball of radius 1 and mutually independent.

then we have r.v. $|X_n|/|X_{n-1}| \overset{iid}{\sim} |X_1|/|X_0|$,

We know $P(|X_1|/|X_0| \le r) = r^2$ for $r\in[0,1]$, then its pdf is $f(r) = 2r$.

Since $\mathbb{E}\log(|X_1|/|X_0|) = \int_0^1 2r\cdot\log r\,dr = \int_0^1 \log r\,dr^2 = r^2\log r\big|_0^1 - \int_0^1 r\,dr = 0-0-(\frac{1}{2}-0) = -\frac{1}{2}$

and $\log(|X_1|/|X_0|) < 0$ holds, then $\mathbb{E}|\log|X_1|/|X_0|| = |\mathbb{E}\log(|X_1|/|X_0|)| = |-\frac{1}{2}| = \frac{1}{2} < \infty$

Then $\frac{1}{n}\log|X_n| = \frac{1}{n}\log\left(\frac{|X_n|}{|X_{n-1}|}\cdot\frac{|X_{n-1}|}{|X_{n-2}|}\cdot\dots\cdot\frac{|X_1|}{|X_0|}\right)$

$= \frac{1}{n}\sum_{i=1}^n \log\frac{|X_i|}{|X_{i-1}|} \xrightarrow{a.s.} \mathbb{E}\log(|X_1|/|X_0|) = -\frac{1}{2}$ //

2.4.4 Suppose $\{V_n\}_{n=1,\dots}$ are iid with $\mathbb{E}V_n^2 < \infty$ and $\mathbb{E}(V_n^2) < \infty$, and $W_{n+1} = [ap + (1-p)V_n]W_n$

(i) Notice that $\frac{W_{n+1}}{W_n} = ap + (1-p)V_n$ are iid and $\mathbb{E}\log\frac{W_{n+1}}{W_n} \le \mathbb{E}\left(\frac{W_{n+1}}{W_n} - 1\right) = \mathbb{E}[ap+(1-p)V_n - 1] < \infty$

By SLLN, we have $\frac{1}{n}\log W_n = \frac{1}{n}\log\left(\frac{W_n}{W_{n-1}}\cdot\dots\cdot\frac{W_1}{W_0}\right) = \frac{1}{n}\cdot\sum_{k=1}^n \log\frac{W_k}{W_{k-1}} \xrightarrow{a.s.} c(p)$ where $c(p) = \mathbb{E}\log[ap+(1-p)V]$

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(ii) Since $\mathbb{E} V_n^2 < \infty$ and $\mathbb{E}(V_n^{-2}) < \infty$

then the first- and second-order derivative of $c(p)$ exist, i.e.

$$\frac{d}{dp} c(p) = \mathbb{E} \frac{a - V_n}{ap + (1-p)V_n} \quad \text{and} \quad \frac{d^2}{dp^2} c(p) = -\mathbb{E} \left( \frac{a - V_n}{ap + (1-p)V_n} \right)^2 \leq 0$$

thus $c(p)$ is concave

(iii) Since $c(p)$ is concave, the sufficient conditions on $V$ that guarantee that the optional choice of $p$ is in $(0,1)$

$$c'(0) = \mathbb{E} \frac{a-V}{V} = a \mathbb{E} \frac{1}{V} - 1 > 0 \Rightarrow \mathbb{E} \frac{1}{V} > \frac{1}{a}$$

$$c'(1) = \mathbb{E} \frac{a-V}{a} = 1 - \frac{1}{a} \mathbb{E} V < 0 \Rightarrow \mathbb{E} V > a$$

(iv) Suppose $P(V=1) = P(V=4) = \frac{1}{2}$

Since $c'(p) = \mathbb{E} \frac{a-V}{ap + (1-p)V} = \frac{1}{2} \left[ \frac{a-1}{ap + (1-p)} + \frac{a-4}{ap + (1-p)4} \right]$,

then the optimal $p$ is

$$p(a) = \begin{cases} 0 & , a \leq \frac{8}{5} \\ -\frac{5a-8}{(a-1)(a-4)} & , \frac{8}{5} < a < \frac{5}{2} \\ 1 & , a \geq \frac{5}{2} \end{cases}$$ //

2.5.3 Suppose $\{X_n\}_{n=1,\dots} \stackrel{iid}{\sim} N(0,1)$

Since $\mathbb{E}\left[X_n \cdot \frac{\sin(n\pi t)}{n}\right] = \frac{\sin(n\pi t)}{n} \mathbb{E} X_n = 0$ and

$$\sum_{n=1}^{\infty} Var\left[X_n \cdot \frac{\sin(n\pi t)}{n}\right] = \sum_{n=1}^{\infty} \frac{\sin^2 n\pi t}{n^2} \cdot Var X_n \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \text{hold for } \forall t$$

then by Thm 2.5.3, we have

$$\sum_{n=1}^{\infty} X_n \cdot \frac{\sin(n\pi t)}{n} \text{ converges a.s.} //$$

2.5.4 Suppose $\{X_n\}_{n=1,\dots}$ are independent with $\mathbb{E} X_n = 0$, $Var X_n = \sigma_n^2$

(i) Assume that $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty$.

Since $\mathbb{E} \frac{X_n}{n} = 0$ and $\sum_{n=1}^{\infty} Var \frac{X_n}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} Var X_n = \sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty$

then by Thm 2.5.3, we have $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s. and

by Thm 2.5.5 (Kronecker's Lemma), we have $\frac{1}{n} \sum_{m=1}^{n} X_m \to 0$ a.s.

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(ii) Suppose $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n} = \infty$ and WLOG $\sigma_n^2 \le n^2$

Construct independent $\{X_n\}_{n=1,\cdots}$ satisfying $X_n \sim N(0, n^2)$

Take $T_n = \frac{1}{n}\sum_{m=1}^{n} X_m$ , $c(n) = Var(T_n) = \sum_{m=1}^{n} (\frac{m}{n})^2$

Since $c(n) \ge \int_0^n (\frac{x}{n})^2 dx = \frac{1}{3n^2} x^3 \Big|_0^n = \frac{n}{3}$ and

$c(n) \le \int_1^n (\frac{x}{n})^2 dx = \frac{1}{3n^2} x^3 \Big|_1^{n+1} = \frac{n}{3} + 1 + \frac{1}{n}$

For $\forall \varepsilon > 0$, we have $P(|T_n| \ge \varepsilon) = P(\frac{|T_n|}{\sqrt{c(n)}} \ge \frac{\varepsilon}{\sqrt{c(n)}}) = 2[1 - \Phi(\frac{\varepsilon}{\sqrt{c(n)}})] \to 1$ as $n \to \infty$

then $|T_n| \ge \varepsilon$ a.s. as $n \to \infty$

then $\frac{1}{n}\sum_{m=1}^{n} X_m$ does not converge to 0 in probability. and then does not converge to 0 almost surely. //

2.5.5 Suppose $X_n \ge 0$ are independent for $n \ge 1$. N.T.S. The following are equivalent.

(i) $\sum_{n=1}^{\infty} X_n < \infty$ a.s.

(ii) $\sum_{n=1}^{\infty} [P(X_n > 1) + \mathbb{E}(X_n \mathbb{1}(X_n \le 1))] < \infty$

(iii) $\sum_{n=1}^{\infty} \mathbb{E}(X_n/(1+X_n)) < \infty$

(i) $\Leftrightarrow$ (ii) Take $Y_n = X_n \mathbb{1}(X_n \le 1)$ then condition (ii) is equivalent to

$$\begin{cases} \sum_{n=1}^{\infty} P(X_n \ne Y_n) = \sum_{n=1}^{\infty} P(X_n > 1) < \infty \\ \sum_{n=1}^{\infty} \mathbb{E} Y_n = \sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{1}(X_n \le 1)] < \infty \end{cases}$$

which implies $\sum_{n=1}^{\infty} Var\, Y_n \le \sum_{n=1}^{\infty} \mathbb{E} Y_n^2 = \sum_{n=1}^{\infty} \mathbb{E}[X_n^2 \cdot \mathbb{1}(X_n \le 1)] \le \sum_{n=1}^{\infty} \mathbb{E}[X_n \cdot \mathbb{1}(X_n \le 1)] < \infty$

By Thm 2.5.4 (Kolmogorov's Three-Series Theorem),

condition (ii) is sufficient and necessary for (i).

(ii) $\Leftrightarrow$ (iii) Since we have

$$\mathbb{E}\frac{X_n}{1+X_n} = \mathbb{E}\frac{X_n}{1+X_n}\mathbb{1}(X_n > 1) + \mathbb{E}[\frac{X_n}{1+X_n}\mathbb{1}(X_n \le 1)]$$
$$\le \mathbb{E}[1 \cdot \mathbb{1}(X_n > 1)] + \mathbb{E}[X_n \mathbb{1}(X_n \le 1)]$$
$$= P(X_n > 1) + \mathbb{E}[X_n \mathbb{1}(X_n \le 1)]$$

$$\mathbb{E}\frac{X_n}{1+X_n} = \mathbb{E}[\frac{X_n}{1+X_n} \cdot \mathbb{1}(X_n > 1)] + \mathbb{E}[\frac{X_n}{1+X_n} \cdot \mathbb{1}(X_n \le 1)]$$
$$\ge \mathbb{E}[\frac{1}{2} \cdot \mathbb{1}(X_n > 1)] + \mathbb{E}[\frac{X_n}{2} \cdot \mathbb{1}(X_n \le 1)]$$
$$= \frac{1}{2} P(X_n > 1) + \mathbb{E}[X_n \cdot \mathbb{1}(X_n \le 1)]$$

then $\frac{1}{2}\sum_{n=1}^{\infty}\{P(X_n > 1) + \mathbb{E}[X_n \cdot \mathbb{1}(X_n \le 1)]\} \le \sum_{n=1}^{\infty} \mathbb{E}\frac{X_n}{1+X_n} \le \sum_{n=1}^{\infty}\{P(X_n > 1) + \mathbb{E}[X_n \cdot \mathbb{1}(X_n \le 1)]\}$

which implies condition (ii) and (iii) are equivalent. //

2.5.6. Take $\varphi(x) = \begin{cases} x^2 & |x|\le 1 \\ |x| & |x|>1 \end{cases}$

Suppose that $\{X_n\}_{n=1,\cdots}$ are independent with $\mathbb{E}X_n=0$ and $\sum_{n=1}^{\infty} \mathbb{E}\varphi(X_n)<\infty$.

Notice that $\mathbb{E}\varphi(X) = \mathbb{E}[X^2 \cdot 1_{(X\le 1)}] + \mathbb{E}[|X|\cdot 1_{(X>1)}]$

$\ge \mathbb{E}[X^2\cdot 1^2_{(X\le 1)}] + \mathbb{E}[1\cdot 1_{(X>1)}]$

$= Var[X\cdot 1_{(X\le 1)}] + P(X>1)$

thus $P(X>1)\le \mathbb{E}\varphi(X)$ and $Var[X\cdot 1_{(X\le 1)}]\le \mathbb{E}\varphi(X)$

Since $X = X\cdot 1_{(X\le 1)} + X\cdot 1_{(X>1)}$ and $\mathbb{E}X=0$

then $|\mathbb{E}[X\cdot 1_{(X\le 1)}]| = |\mathbb{E}[X\cdot 1_{(X>1)}]| \le \mathbb{E}[|X|\cdot 1_{(X>1)}] \le \mathbb{E}\varphi(X)$

Take $Y_n = X_n\cdot 1_{(X_n\le 1)}$, then

(i) $\sum_{n=1}^{\infty} P(X_n>1) \le \sum_{n=1}^{\infty}\mathbb{E}\varphi(X_n)<\infty$

(ii) $|\sum_{n=1}^{\infty}\mathbb{E}[X_n\cdot 1_{(X_n\le 1)}]| \le \sum_{n=1}^{\infty}|\mathbb{E}[X_n\cdot 1_{(X_n\le 1)}]| \le \sum_{n=1}^{\infty}\mathbb{E}\varphi(X_n)<\infty$

then $\sum_{n=1}^{\infty}\mathbb{E}Y_n$ converges.

(iii) $\sum_{n=1}^{\infty} Var\,Y_n = \sum_{n=1}^{\infty} Var[X_n\cdot 1_{(X_n\le 1)}] \le \sum_{n=1}^{\infty}\mathbb{E}\varphi(X_n)<\infty$

by Thm. 2.5.4 (Kolmogorov's Three-Series Theorem)

$\sum_{n=1}^{\infty} X_n$ converges a.s. //

2.5.7 Let $X_n$ be independent.

Suppose $\sum_{n=1}^{\infty}\mathbb{E}|X_n|^{p(n)}<\infty$ where $0<p(n)\le 2$ for $\forall n$ and $\mathbb{E}X_n=0$ when $p(n)>1$.

Take $A_1=\{n: 1<p(n)\le 2\}$ and $A_2=\{n: 0<p(n)\le 1\}$, and take $Y_n = X_n\cdot 1_{(X_n\le 1)}$ for $\forall n$

Since $\sum_{n\in A_1}\mathbb{E}|X_n|^{p(n)} + \sum_{n\in A_2}\mathbb{E}|X_n|^{p(n)} = \sum_{n=1}^{\infty}\mathbb{E}|X_n|^{p(n)}<\infty$

then $\sum_{n\in A_1}\mathbb{E}|X_n|^{p(n)}<\infty$, $\sum_{n\in A_2}\mathbb{E}|X_n|^{p(n)}<\infty$

For $n\in A_1$, we have $1<p(n)\le 2$ and $\mathbb{E}X_n=0$

then $P(X_n>1) = \mathbb{E}[1_{(X_n>1)}] \le \mathbb{E}[|X_n|^{p(n)}\cdot 1_{(X_n>1)}] \le \mathbb{E}|X_n|^{p(n)}$

$|\mathbb{E}[X_n\cdot 1_{(X_n\le 1)}]| = |\mathbb{E}[X_n\cdot 1_{(X_n\le 1)}]| \le \mathbb{E}[|X_n|\cdot 1_{(X_n>1)}] \le \mathbb{E}[|X_n|^{p(n)}\cdot 1_{(X_n>1)}] \le \mathbb{E}|X_n|^{p(n)}$

$Var[X_n\cdot 1_{(X_n\le 1)}] \le \mathbb{E}[X_n^2\cdot 1_{(X_n\le 1)}] \le \mathbb{E}[|X_n|^{p(n)}\cdot 1_{(X_n\le 1)}] \le \mathbb{E}|X_n|^{p(n)}$

For $n\in A_2$, we have $0<p(n)\le 1$

then $P(X_n>1) = \mathbb{E}[1_{(X_n>1)}] \le \mathbb{E}[|X_n|^{p(n)}\cdot 1_{(X_n>1)}] \le \mathbb{E}|X_n|^{p(n)}$

$|\mathbb{E}[X_n\cdot 1_{(X_n\le 1)}]| \le \mathbb{E}[|X_n|\cdot 1_{(X_n\le 1)}] \le \mathbb{E}[|X_n|^{p(n)}\cdot 1_{(X_n>1)}] \le \mathbb{E}|X_n|^{p(n)}$

$Var[X_n\cdot 1_{(X_n\le 1)}] \le \mathbb{E}[X_n^2\cdot 1_{(X_n\le 1)}] \le \mathbb{E}[|X_n|^{p(n)}\cdot 1_{(X_n\le 1)}] \le \mathbb{E}|X_n|^{p(n)}$

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Thus, we have

$$\sum_{n=1}^{\infty} P(X_n > 1) \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$$

$$\left|\sum_{n=1}^{\infty} \mathbb{E}[X_n \cdot 1(X_n \le 1)]\right| \le \sum_{n=1}^{\infty} |\mathbb{E}[X_n \cdot 1(X_n \le 1)]| \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$$

$$\sum_{n=1}^{\infty} \mathrm{Var}[X_n 1(X_n \le 1)] \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{p(n)} < \infty$$

by Thm 2.5.4 (Kolmogorov's Three-Series Theorem)

$\sum_{n=1}^{\infty} X_n$ converges a.s. //

2.5.9. Let $\{X_n\}_{n=1,\cdots}$ be independent and $S_{m,n} = X_{m+1} + \cdots + X_n$

Take $A_{m,k} = \{|S_{m,k}| : |S_{m,k}| \geq 2a,\ |S_{m,j}| < 2a \text{ for } m<j<k\}$

Notice that for $k = m+1, \cdots, n$. we have

$$A_k \cap \{|S_{m,k}| > 2a\} \in \sigma(X_{m+1}, \cdots, X_k)$$

$$S_{k,n} \in \sigma(X_{k+1}, \cdots, X_n)$$

$$\sigma(X_{m+1}, \cdots, X_k) \perp\!\!\!\perp \sigma(X_{k+1}, \cdots X_n)$$

then $A_k \cap \{|S_{m,k}| > 2a\} \perp\!\!\!\perp S_{k,n}$

Then
$$\begin{aligned}
P(|S_{m,n}| > a) &\geq \sum_{k=m+1}^{n} P(A_k \cap \{|S_{m,n}| > a\}) \\
&\geq \sum_{k=m+1}^{n} P(A_k \cap \{|S_{m,k}| > 2a\} \cap \{|S_{k,n}| > a\}) \\
&\geq \sum_{k=m+1}^{n} [P(A_k \cap \{|S_{m,k}| > 2a\}) \cdot P(|S_{k,n}| > a)] \\
&\geq \left[\sum_{k=m+1}^{n} P(A_k \cap \{|S_{m,k}| > 2a\})\right] \cdot \min_{m<j\le n} P(|S_{j,n}| > a) \\
&= P(\max_{m<k\le n} |S_{m,k}| > 2a) \min_{m<j\le n} P(|S_{j,n}| > a) \qquad //
\end{aligned}$$

2.5.10 Let $\{X_n\}_{n=1,2,\cdots}$ be independent and $S_n = X_1 + \cdots + X_n$

Suppose that $\lim_{n\to\infty} S_n$ exists in probability.

then $\lim_{m\to\infty} \lim_{n\to\infty} P(|S_n - S_m| > \varepsilon) = 0$ for $\forall \varepsilon > 0$.

By the result of 2.5.9,

for $\forall \varepsilon > 0$, $P(\max_{m<j\le n} |S_j - S_m| > \varepsilon) \le P(|S_n - S_m| > \frac{\varepsilon}{2}) / \min_{m<k\le n} P(|S_n - S_k| \le \frac{\varepsilon}{2})$

$\longrightarrow 0$ as $n, m \to \infty$.

Thus $S_m - S_n \xrightarrow{a.s.} 0 \Rightarrow \lim_{n\to\infty} S_n$ exists a.s. //

3.1.4. Suppose $P(X_i \leq k) = \frac{e^{-1}}{k!}$ for $\forall k=0,1,\cdots$, i.e. $\{X_i\}_{i=1,2,\cdots} \overset{iid}{\sim}$ Poisson (1)

then $S_n = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n)$

Since $P(S_n = k) = n^k \cdot \frac{e^{-n}}{k!}$ and $P(S_n = k+t) = n^{k+t}\frac{e^{-n}}{(k+t)!} = n^t \cdot \frac{k!}{(k+t)!} \cdot P(S_n = k)$

then $P(S_n \geq k) = \sum_{t=0}^{\infty} P(S_n = k+t) = \left(\sum_{t=0}^{\infty} n^t \frac{k!}{(k+t)!}\right) P(S_n = k) =: c(n,k) \cdot P(S_n = k)$

where $c(n.k) = \sum_{t=0}^{\infty} n^t \cdot \frac{k!}{(k+t)!}$

and then $n \cdot \log P(S_n \geq n \cdot a) = n\log c(n.na) + n\log P(S_n = n \cdot a)$

For $a > 1$, $1 \leq c(n.na) \leq \sum_{t=0}^{\infty} (\frac{1}{a})^t = \frac{a}{a-1}$ holds.

$\Rightarrow 0 \leq \frac{1}{n}\log c(n,na) \leq \frac{1}{n} \cdot \log \frac{a}{a-1} \to 0$ as $n \to \infty$

and $\frac{1}{n}\log P(S_n = na) = \frac{n \cdot a}{n}\log n - 1 - \frac{1}{n}\log (na)!$

$\sim a\log n - 1 - \frac{1}{n}(na\log na - na + \frac{1}{2}\log 2\pi n)$

$= -1 - a\log a + a - \frac{1}{2n}\log 2\pi n$

$\longrightarrow a - 1 - a\log a$ as $n \to \infty$

Hence $n \cdot \log P(S_n \geq na) \to a - 1 - a\log a$ //

3.2.2 Let $X_1, X_2 \cdots$ be independent with distribution $F$ and $M_n = \max_{m \leq n}\{X_m\}$

(i) If $F(x) = 1 - x^{-\alpha}$, for $x \geq 1$, where $\alpha > 0$, then for $y > 0$.

$P(\frac{M_n}{n^{1/\alpha}} \leq y) = P(M_n \leq y \cdot n^{1/\alpha}) = [1 - (y \cdot n^{1/\alpha})^{-\alpha}]^n = [1 - \frac{y^{-\alpha}}{n}]^n$

$\longrightarrow \exp\{-y^{-\alpha}\}$.

(ii) If $F(x) = 1 - |x|^\beta$, for $-1 \leq x \leq 0$ where $\beta > 0$, then for $y < 0$

$P(n^{1/\beta} M_n \leq y) = P(M_n \leq y/n^{1/\beta}) = [1 - |y|^\beta/n]^n$

$\longrightarrow \exp\{-|y|^\beta\}$.

(iii) If $F(x) = 1 - e^{-x}$, for $x \geq 0$, then for all $y \in (-\infty, +\infty)$

$P(M_n - \log n \leq y) = P(M_n \leq y + \log n) = [1 - \exp\{-y - \log n\}]^n = [1 - \frac{1}{n}e^{-y}]^n$

$\longrightarrow \exp\{-e^{-y}\}$. //

3.2.3 Let $X_1, X_2, \cdots$ be iid and have the standard normal distribution.

(i) From Thm 1.2.3, we have $P(X_i > x) \sim \frac{1}{\sqrt{2\pi}\cdot x}\cdot e^{-\frac{x^2}{2}}$, as $n \to \infty$

For any real number $\theta$

$$P(X_i > x + \theta/x)/P(X_i > x) \sim \frac{x}{x+\frac{\theta}{x}}\cdot \exp\{-\tfrac{1}{2}\cdot(x+\tfrac{\theta}{x})^2 + \tfrac{1}{2}x^2\} \sim \frac{x}{1+\frac{\theta}{x^2}}\cdot \exp\{-\theta - \tfrac{1}{2}\tfrac{\theta^2}{x^2}\}$$

$$\longrightarrow e^{-\theta} \text{ as } x \to \infty$$

(ii) Take $b_n$ satisfying $P(X_i > b_n) = \frac{1}{n}$

$$P(b_n(M_n - b_n) \le x) = P(M_n \le \tfrac{x}{b_n} + b_n) = [P(X_i \le \tfrac{x}{b_n} + b_n)]^n = [1 - P(X_i > \tfrac{x}{b_n} + b_n)]^n$$

$$\longrightarrow [1 - e^{-x}\cdot P(X_i > b_n)]^n = [1 - \tfrac{1}{n}e^{-x}]^n$$

$$\longrightarrow \exp\{-e^{-x}\} \text{ as } n \to \infty$$

(iii) Since we have

$$P(X_i > (2\log n)^{1/2}) \sim \frac{1}{\sqrt{2\pi\cdot 2\log n}}\exp\{-\tfrac{1}{2}\cdot 2\log n\} = \frac{1}{\sqrt{4\pi\log n}}\cdot\frac{1}{n} \le \frac{1}{n}$$

and
$$P(X_i > (2\log n - 2\log\log n)^{1/2}) \sim \frac{1}{\sqrt{2\pi(2\log n - 2\log\log n)}}\cdot\exp\{-\log n + \log\log n\}$$

$$= \frac{\log n}{\sqrt{4\pi(\log n - \log\log n)}}\cdot\frac{1}{n} \ge \frac{1}{n}$$

Hence $(2\log n - 2\log\log n)^{1/2} \le b_n \le (2\log n)^{1/2}$ and $\frac{(2\log n)^{1/2}}{(2\log n - 2\log\log n)^{1/2}} \longrightarrow 1$

$\Rightarrow b_n \sim (2\log n)^{1/2}$

3.2.9 Suppose that $F_n \Rightarrow F$ and $F$ is continuous

For $\forall k > 0$, take $x_{k,i} = \inf\{x : F(x) > \frac{i}{k}\}$, $i = 1, 2, \cdots, k-1$

Since $F$ is continuous then $F(x_{k,i}) = \frac{i}{k}$

Since $F_n \Rightarrow F$ converges pointwisely,

then for each $i$, $\exists N_{k,i} > 0$, s.t. $\forall n > N_{k,i}$ $|F(x_{k,i}) - F_n(x_{k,i})| < \frac{1}{k}$

take $N_k = \max_{1\le i\le k} N_{k,i}$ then $|F(x_{k,i}) - F_n(x_{k,i})| < \frac{1}{k}$, for $\forall n > N_k$ and $\forall i$

(1) For $x \in [x_{k,1}, x_{k,k-1}]$, $\exists i$, s.t. $x_{k,i} \le x \le x_{k,i+1}$

when $n > N_k$,
$$F_n(x) \le F_n(x_{k,i+1}) \le F(x_{k,i+1}) + \tfrac{1}{k} \le F(x_{k,i}) + \tfrac{2}{k} \le F(x) + \tfrac{2}{k}$$
$$F_n(x) \ge F_n(x_{k,i}) \ge F(x_{k,i}) - \tfrac{1}{k} \ge F(x_{k,i+1}) - \tfrac{2}{k} \ge F(x) - \tfrac{2}{k}$$

then $|F_n(x) - F(x)| \le \frac{2}{k}$ for $\forall x \in [x_{k,1}, x_{k,k-1}]$

(2) For $x < x_{k,1}$

when $n > N_k$,
$$F_n(x) \le F_n(x_{k,1}) \le F(x_{k,1}) + \tfrac{1}{k} \le \tfrac{2}{k} \le F(x) + \tfrac{2}{k}$$
$$F_n(x) \ge 0 = F(x_{,1}) - \tfrac{1}{k} \ge F(x) - \tfrac{1}{k}$$

then $|F_n(x) - F(x)| \le \frac{2}{k}$ for $\forall x < x_{k,1}$

(continue on the next page)

(3) For $x > x_{k,k-1}$

when $n > N_k$, $F_n(x) \le 1 = F(x_{k,k-1}) + \frac{1}{k} \le F(x) + \frac{1}{k}$

$F_n(x) \ge F_n(x_{k,k-1}) \ge F(x_{k,k-1}) - \frac{1}{k} = 1 - \frac{2}{k} \ge F(x) - \frac{2}{k}$

then $|F_n(x) - F(x)| \le \frac{2}{k}$ for $\forall\, x > x_{k,k-1}$

Hence $\sup_x |F_n(x) - F(x)| \le \frac{2}{k}$ as $n \to \infty$ for $\forall\, k > 0$

$\Rightarrow \sup_x |F_n(x) - F(x)| = 0$ as $n \to \infty$.

3.2.11 Let $X_n$, $1 \le n \le \infty$ be integer valued with distribution function $F_n(x)$

NTS. $X_n \Rightarrow X_\infty$ iff $P(X_n = m) \to P(X_\infty = m)$ for all $m$.

($\Rightarrow$) Since $F_n(x) \to F_\infty(x)$ for $\forall\, x$, then for $\forall$ integer $m$

$$P(X_n = m) = F_n(m + \tfrac{1}{2}) - F_n(m - \tfrac{1}{2})$$
$$\to F_\infty(m + \tfrac{1}{2}) - F_n(m - \tfrac{1}{2})$$
$$= P(X_\infty = m)$$

($\Leftarrow$) For $\forall\, x, y \in \mathbb{R}$. with $x > y$, take $m(x) = \lfloor x \rfloor$ $n(y) = \lfloor y \rfloor + 1$,

$$F_n(x) - F_n(y) = P(y < X_n \le x) = \sum_{k=n}^{m} P(X_n = k)$$
$$\to \sum_{k=n}^{m} P(X_\infty = k) = P(n \le X_\infty \le m)$$
$$= P(y < X_\infty \le x) = F_\infty(x) - F_\infty(y) \quad \text{for } \forall\, x, y \in \mathbb{R}.$$

Since $\lim_{y \to -\infty} F_n(y) = \lim_{y \to -\infty} F_\infty(y) = 0$, then $F_n(x) \to F_\infty(x)$ for $\forall\, x \in \mathbb{R}$.

Hence $X_n \Rightarrow X$. //

3.2.13 Suppose that $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ ($c$ is constant)

For $\forall\, \varepsilon > 0$, $P(|Y_n - c| > \varepsilon) = P(Y_n > c + \varepsilon) + P(Y_n < c - \varepsilon)$

$$\to P(c > c + \varepsilon) + P(c < c - \varepsilon) = 0.$$

$\Rightarrow Y_n \xrightarrow{P} c$

For $\forall\, x$ and $\forall\, \varepsilon > 0$,

$$P(X_n + Y_n \le x + c) = P(X_n + Y_n \le x + c, |Y_n - c| \le \varepsilon) + P(X_n + Y_n \le x + c, |Y_n - c| > \varepsilon)$$
$$\le P(X_n \le x + c - (c - \varepsilon)) + P(|Y_n - c| > \varepsilon)$$
$$= P(X_n \le x + \varepsilon) + P(|Y_n - c| > \varepsilon)$$
$$\to P(X + c \le x + c + \varepsilon) + 0 \quad \text{as } n \to \infty \text{ for } \forall\, \varepsilon > 0.$$

then $\lim_{n \to \infty} P(X_n + Y_n \le x + c) \le P(X + c \le x + c)$ (continue on the next page).

$$P(X_n+Y_n \le x+c) \ge P(\{X_n+Y_n \le x+c\} \cap \{|Y_n-c| \le \varepsilon\})$$
$$\ge P(\{X_n \le x+c-(c+\varepsilon)\} \cap \{|Y_n-c| \le \varepsilon\})$$
$$= 1-P(\{X_n > x-\varepsilon\} \cup \{|Y_n-c| > \varepsilon\})$$
$$\ge 1-P(X_n > x-\varepsilon) - P(|Y_n-c| > \varepsilon)$$
$$= P(X_n+c \le x+c-\varepsilon) - P(|Y_n-c| > \varepsilon)$$
$$\to P(X+c \le x+c-\varepsilon) - 0 \quad \text{as } n\to\infty \quad \text{for } \forall \varepsilon>0$$

then $\lim_{n\to\infty} P(X_n+Y_n \le x+c) \ge P(X+c \le x+c)$

Hence $\lim_{n\to\infty} P(X_n+Y_n \le x+c) = P(X+c \le x+c)$ i.e. $X_n+Y_n \Rightarrow X+c$

Suppose $Y_n = Z_n - X_n$, $c=0$, then $X_n+Y_n = Z_n \Rightarrow X$ //

3.2.14 Suppose $X_n \Rightarrow X$, $Y_n \ge 0$ and $Y_n \Rightarrow c$, where $c>0$ is constant.

For $\forall \varepsilon>0$, $P(|Y_n-c|>\varepsilon) = P(Y_n > c+\varepsilon) + P(Y_n < c-\varepsilon)$

$$\to P(c > c+\varepsilon) + P(c < c-\varepsilon) = 0$$

$\Rightarrow Y_n \xrightarrow{P} c$.

For $\forall x>0$ and $\forall \varepsilon>0$

$$P(X_n \cdot Y_n \le cx) = P(X_nY_n \le c\cdot x, |Y_n-c| \le \varepsilon) + P(X_nY_n \le cx, |Y_n-c|>\varepsilon)$$
$$\le P(X_n \le \frac{c}{c-\varepsilon}x, |Y_n-c| \le \varepsilon) + P(|Y_n-c|>\varepsilon)$$
$$\le P(X_n \le \frac{c}{c-\varepsilon}x) + P(|Y_n-c|>\varepsilon)$$
$$\to P(cX \le \frac{c}{c-\varepsilon}\cdot cx) + 0 \quad \text{as } n\to\infty \quad \text{for } \forall\varepsilon>0$$

then $\lim_{n\to\infty} P(X_n\cdot Y_n \le cx) \le P(cX \le cx)$

$$P(X_n\cdot Y_n \le cx) \ge P(\{X_nY_n \le c\cdot x\} \cap \{|Y_n-c| \le \varepsilon\})$$
$$\ge P(\{X_n \le \frac{c}{c+\varepsilon}x\} \cap \{|Y_n-c| \le \varepsilon\})$$
$$= 1-P(\{X_n > \frac{c}{c+\varepsilon}x\} \cup \{|Y_n-c|>\varepsilon\})$$
$$\ge 1-P(X_n > \frac{c}{c+\varepsilon}x) - P(|Y_n-c|>\varepsilon)$$
$$= P(X_n \le \frac{c}{c+\varepsilon}x) - P(|Y_n-c|>\varepsilon)$$
$$\to P(X \le \frac{c}{c+\varepsilon}x) - 0 \quad \text{as } n\to\infty, \text{ for } \forall\varepsilon>0$$

Hence $\lim_{n\to\infty} P(X_nY_n \le c\cdot x) = P(c\cdot X \le c\cdot x)$ for $x>0$

the conclusion is still valid for $x \le 0$

then $X_nY_n \Rightarrow cX$ //

3.3.10 Suppose that $X_n$ and $Y_n$ are independent for $1 \le n \le \infty$ and $X_n \Rightarrow X_\infty$, $Y_n \Rightarrow Y_\infty$

Denote the ch.f. of $X_n, Y_n$ as $\varphi_n(t)$ and $\phi_n(t)$

By Thm 3.3.6 continuity thm.

$\varphi_n(t) \to \varphi_\infty(t)$ and $\phi_n(t) \to \phi_\infty(t)$ pointwisely

then we have
$$\begin{aligned} \mathbb{E}\exp\{it(X_n+Y_n)\} &= \mathbb{E}\exp\{itX_n\} \cdot \mathbb{E}\exp\{itY_n\} \quad (\text{by indep.}) \\ &= \varphi_n(t) \cdot \phi_n(t) \\ &\to \varphi_\infty(t)\,\phi_\infty(t) = \mathbb{E}\exp\{it(X+Y)\} \end{aligned}$$

By Thm 3.3.6 again, $X_n + Y_n \Rightarrow X+Y$ as $n \to \infty$.

3.3.21 Let $X_1, X_2, \cdots$ be independent.

Suppose $S_n = \sum_{m \le n} X_m$ Convergens in distribution.

Denote $\varphi_X(t)$ as the ch.f. of r.v. $X$

For $m > n$, since $(S_m - S_n) = (X_{n+1} + \cdots + X_m) \perp\!\!\!\perp (X_1 + \cdots + X_n) = S_n$

then $\varphi_{S_n}(t) \cdot \varphi_{S_m - S_n}(t) = \varphi_{S_m}(t)$
$$\varphi_{S_m - S_n}(t) = \frac{\varphi_{S_m}(t)}{\varphi_{S_n}(t)} \to \frac{\varphi_S(t)}{\varphi_S(t)} = 1 \quad \text{as } m,n \to \infty$$
where $\varphi_S(t)$ is the limit of $\varphi_{S_n}(t)$ as $n \to \infty$

By the result of 3.3.20, $S_m - S_n \Rightarrow 0$ as $m, n \to \infty$

then $S_m - S_n \to 0$ in probability as $m, n \to \infty$

Thus $S_n$ converges in probability.