Measure Theory

Sec1. Probability Space

Def. σ-field $\mathcal{F}$: i) $\Omega \in \mathcal{F}$; ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
iii) if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

algebra: iii) if $A_1, \ldots A_n \in \mathcal{F}$, then $\cup_{i=1}^{n} A_i \in \mathcal{F}$.

Fact. if $\mathcal{F}_i, i \in I$ are all σ-fields, $\cap_{i\in I}\mathcal{F}_i$ is a σ-field.

Def. measure μ: i) $\mu(A) \geq 0, \forall A \in \mathcal{F}$; ii) $\mu(\emptyset) = 0$;
iii) if $A_1, A_2, \cdots \in \mathcal{F}$ disjoint, then $\mu(\cup_i A_i) = \sum_i \mu(A_i)$.

Thm 1.1.1. iii) continuity from below: if $A_i \uparrow A$, then $\mu(A_i) \uparrow \mu(A)$.
iv) continuity from above: if $A_i \downarrow A$ and $\mu(A_1) < \infty$, then $\mu(A_i) \downarrow \mu(A)$.

Def. Borel σ-field: $\mathcal{R} = \sigma(\{(a,b] : -\infty < a < b < \infty\})$.

Def. π-system $\mathcal{P}$: if $A, B \in \mathcal{P}$, then $A \cap B \in \mathcal{P}$.

λ-system $\mathcal{L}$: i) $\Omega \in \mathcal{L}$; ii) if $A, B \in \mathcal{L}$ and $A \subset B$, then $B \backslash A \in \mathcal{L}$;
iii) if $A_1, A_2, \cdots \in \mathcal{L}$, and $A_i \uparrow A$, then $A \in \mathcal{L}$.

Ex $\star$. if $\mathcal{F}$ is π-system and λ-system, then $\mathcal{F}$ is σ-field.

Thm 2.1.2. if $\mathcal{P}$ is π-system, $\mathcal{L}$ is λ-system, $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Sec2-1. Measurable Function

Def. f is meas- if $\{\omega_1 \in \Omega_1 : f(\omega_1) \in A\} = f^{-1}(A) \in \mathcal{F}_1, \forall A \in \mathcal{F}_2$.

Fact. gen- σ-field by f: $\sigma(f) = \{f^{-1}(A) : A \in \mathcal{F}_2\}$ is σ-field in Ω_1,
$\{A \subset \Omega_2 : f^{-1}(A) \in \mathcal{F}_1\}$ is a σ-field in Ω_2.

Thm 1.3.1. if $\mathcal{F}_2 = \sigma(\mathcal{A}_2)$, and $f^{-1}(\mathcal{A}_2) \subset \mathcal{F}_1$, then f is meas-.

Thm 1.3.2. if f_1 and f_2 are meas-, then $f_2 \circ f_1$ is meas-.

Def. induced measure: $\mu_2(A) = \mu_1(f^{-1}(A))$.

Sec2-2. Random Variable

Thm 1.3.5. $\inf_n X_n, \sup_n X_n, \limsup X_n, \liminf X_n$ are r-v-.

Ex 1.3.1. $\sigma(X^{-1}(\mathcal{A})) = X^{-1}(\sigma(\mathcal{A}))$
Hint: $\mathcal{C} = \{B \in \sigma(\mathcal{A}) : X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))\}$.

Sec2-3. Distribution

Thm 1.2.2. $\Omega = (0,1), \mathcal{F} = \mathcal{R}, P =$ Lebesgue measure, then $X(\omega) = F^{-1}(\omega) = \inf\{y \in \mathbb{R} : F(y) \geq \omega\} = \sup\{y \in \mathbb{R} : F(y) < \omega\}$ with dist- F.

Hint: $\{\omega : \omega \leq F(x)\} = \{\omega : X(\omega) \leq x\}$, right-continuous of F.

Sec3. Expectation

Def. indicator -> simple -> non-negative -> arbitrary.
case3: $\mathbb{E}[X] = \sup\{\mathbb{E}[Y] : 0 \leq Y \leq X, Y \text{ is simple}\}$.

Prop. a) monotonicity, b) linearity.
Hint: $Z_M^{(u)} = \frac{1}{2^M}\lceil 2^M Z\rceil, Z_M^{(l)} = \frac{1}{2^M}\lceil 2^M Z - 1\rceil$ for truc- case.

Thm MC. X_n n-n seq- r-v-. if $X_n \uparrow X$, then $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$.

Hint: $Y_\epsilon = \sum_i (b_i - \epsilon/2)1_{B_i}$.

Thm Fatou's L. if $X_n \geq 0$, then $\liminf_n \mathbb{E}[X_n] \geq \mathbb{E}[\liminf_n X_n]$.

Thm DC. if $X_n \to X$, $|X_n| \leq Y$ with $\mathbb{E}[Y] < \infty$, then $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$.

Thm Jensen. if $\mathbb{E}[X] < \infty$, φ is convex, then $\mathbb{E}[|\varphi(X)|] \leq \infty$.

Thm Hölder. if $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathbb{E}[|XY|] \leq \|X\|_p + \|Y\|_q$.
Hint: $x \cdot y \leq x^p/p + x^q/q$ via concav- of log.

Thm Minkowski. for $p \geq 1$, $\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$.

Thm Markov. if r-v- $X \geq 0$ and $a > 0$, then $P(X \geq a) \leq \frac{1}{a}\mathbb{E}[X]$.

Thm Chebyshev. if $\exists$ var, then $P(|X - \mathbb{E}[X]| \geq a) \leq \frac{1}{a^2}\operatorname{Var}(X)$.

Law of Large Number

Sec1. Independence

Def. inde- events -> collections (σ-fields) -> random variables.

Thm 2.1.3. if π-sys- $\mathcal{A}_i|_{i=1}^n$ are inde-, then $\sigma(\mathcal{A}_i)|_{i=1}^n$ are indep-.

Thm 2.1.4. r-v- $X_i|_{i=1}^n$ are inde- if-f- $P(\cap_i\{X_i \leq x_i\}) = \prod_i P(X_i \leq x_i)$.

Thm -. if $X_i|_{i=1}^n$ are inde-, then $\sigma(X_i : i \in I) \perp\!\!\!\perp \sigma(X_j : j \in I^c)$.

Thm 2.1.5. if $X_i|_{i=1}^n$ inde-, then $g(X_i, i \in I) \perp\!\!\!\perp h(X_j, j \in I^c)$, g, h meas-.

Thm 2.1.8. if X, Y inde-, $\mathbb{E}[\cdot] < \infty$ or ≥ 0, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Thm Kolmogorov's 0-1 Law. if X_i's inde-,
tail σ-field $\mathcal{T} = \cap_n \sigma(X_k, k \geq n)$, then $P(A) = 0, 1$ for $A \in \mathcal{T}$.

Sec2-1. Weak Law of Large Number

Def. converges in prob: if $\forall\ \epsilon > 0$, $P(|Y_n - Y| > \epsilon) \to 0$, as $n \to \infty$.

Thm 2.2.6. (WLLN for triangular arrays)
suppose that $\{X_{n,k} : 1 \leq k \leq n\}$ are inde-, $\bar{X}_{n,k} = X_{n,k}1_{\{|X_{n,k}| \leq b_n\}}$,
if i) $\sum_{k=1}^n P(|X_{n,k}| \geq b_n) \to 0$, ii) $b_n^{-2}\sum_{k=1}^n \mathbb{E}[(\bar{X}_{n,k})^2] \to 0$,
$a_n = \sum_{k=1}^n \mathbb{E}[]$, then $(S_n - a_n)/b_n \to 0$ in prob-.

Thm 2.2.7. (WLLN without moment assumption) $\mu_n = \mathbb{E}[X1_{\{X \leq n\}}]$.
i.i.d. if $xP(|X1| > x) \to 0$ as $x \to \infty$, then $\frac{1}{n}S_n - \mu_n \to 0$ in prob-.

Remark: for $0 < \epsilon < 1$, $\mathbb{E}[|X|^{1-\epsilon}] < \infty$.

Lem 2.2.8. if $Y \geq 0$ and $p > 0$, then $\mathbb{E}[Y^p] = \int_0^\infty py^{p-1}P(Y > y)dy$.

Thm 2.2.9. (WLLN with finite 1st moment)
i.i.d. $\mu = \mathbb{E}[X_1]$. if $\mathbb{E}[|X_1|] < \infty$, then $\frac{1}{n}S_n - \mu \to 0$ in prob.

Sec2-2. Strong Law of Large Number

Def. events A_n occurs infinitely often $\{A_n \text{ i.o.}\} = \cap_n \cup_k A_k$.

Fact. $Y_n \to Y$ a.s. if-f- $\forall \epsilon > 0, P(|Y_n - Y| > \epsilon \text{ i.o.}) = 0$

Thm 2.3.1/6. (Borel-Cantelli Lemma)
i) if $\sum_n P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.
ii) if $\sum_n P(A_n) = \infty$ and A_n's indep-, then $P(A_n \text{ i.o.}) = 1$.

Thm 2.3.5. (SLLN with 4M) i.i.d. $\mathbb{E}[X_i^4] < \infty$, then $S_n/n \to \mu$ a.s.

Thm 2.3.3. if $Y_n \to Y$ in prob. then $\exists\ n(k)$ s.t. $Y_{n(k)} \to Y$ a.s.

Thm 2.3.8. (gen B-C(ii)) $\sum_n P(A_n) = \infty$, then $\frac{\sum_i 1\{A_i\}}{\sum_i P(A_i)} \to 1$ a.s.

Thm SLLN. i.i.d. $\mathbb{E}[|X_i|] < \infty$, then $S_n/n \to \mu$ a.s.

Hint: n-trunc -> subseq $k(n) = \lfloor \alpha^n \rfloor, \forall \alpha > 1$. $\sum_i \operatorname{Var}(Y_i)/i^2 < \infty$.

Remark: if $S_n/n \to \mu$ a.s. then $\mathbb{E}[X_i] = \mu < \infty$. via B-C(ii) + 2.2.8.

Thm 2.4.5. i.i.d. $\mathbb{E}[X_i^+] = \infty, \mathbb{E}[X_i^-] < \infty$, then $S_n/n \to \infty$. (M-trunc.)

Thm 2.4.7. r-v- $F_n(x) = \frac{1}{n}\sum_i 1_{\{X_i \leq x\}}$, then $\sup_x |F_n(x) - F(x)| \to 0$.

Sec3. Convergence of Random Series

Thm 2.5.2. (Kolmogorov's Maximal Inequality)
indep-, $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] < \infty$, then $P(\max_{k \leq n}|S_k| \geq x) \leq \mathbb{E}[S_n^2]/x^2$.

Hint: $A_k = \{|S_i| \text{ for } i < k, |S_k| \geq x\}$.

Thm 2.5.3. indep-, $\mathbb{E}[X_i] = 0$. $\sum_i \mathbb{E}[X_i^2] \leq \infty$ => $\sum_i X_i$ converges a.s.
Hint: $\omega_M = \sup_{m,n \geq M}|S_m - S_n| \downarrow 0$ a.s. as $M \to \infty$.

Thm 2.5.4. (Kolmo-'s three-series thm) X_i indep-, $Y_i = X_i 1_{\{|X_i| < A\}}$,
if i) $\sum P(|X_n| > A) < \infty$, ii) $\sum \mathbb{E}[Y_n]$ converges, iii) $\sum \operatorname{Var}(Y_n) < \infty$,
then $\sum X_n$ converges a.s.

Thm 2.2.5. (Kronecker's Lemma) if $a_n \uparrow \infty$ and $\sum_n (x_n/a_n)$ converges,
then $(\sum_{m=1}^n x_m)/a_n \to 0$. Remark: second proof of SLLN.

Thm 2.2.8. (M-Z SLLN) i.i.d. $\mathbb{E}[X_i] = 0$, $\mathbb{E}[|X_i|^p] < \infty$ for $1 < p < 2$,
then $S_n/n^{1/p} \to 0$ a.s. Remark: also true for $0 < p < 1$.

Central Limit Theorem

Sec1. Convergence in Distribution

Thm . (Stirling's Formula) $n! \sim n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$.

Def. d.f.s $F_n \Rightarrow F$ weakly conv, if $F_n(y) \to F(y)$ at $\forall$ cont-point y of F.

Fact. i) if $X_n \to X$ in p., then $X_n \Rightarrow X$. ii) if $X_n \Rightarrow c$, then $X_n \to c$ in p.

Thm 3.2.2. (Skorokhod's Theorem) if $F_n \Rightarrow F_\infty$,
then $\exists\ Y_n$ on the same prob-space, Y_n has d.f. F_n and $Y_n \to Y_\infty$ a.s.

Hint: $\Omega_0 = \{\text{preimage of } F \text{ is either empty or unique real number}\}$.

Thm 3.2.3. $X_n \Rightarrow X$ if-f- $\forall\ g$ bounded and conti-, $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$.

Thm -. (CLT with finite 3-rd moment) i.i.d. $\mathbb{E}[X_1] = \mu$, $\operatorname{Var}(X_1) = \sigma^2$,
if $\mathbb{E}[|X_1|^2] < \infty$, then $W_n = \sum_i (X_i - \mu)/\sigma\sqrt{n} \Rightarrow Z \sim N(0,1)$.

Hint: Lindeberg's Swap- Argument, T-expan for bd- cont- deriv- 3 order.

Sec2. Characteristic Functions

Def. ch.f. $\varphi(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i \cdot \mathbb{E}[\sin(tX)]$.

Prop. iii) (uniformly cont-) $\sup_t |\varphi(t+h) - \varphi(t)| \to 0$ as $h \to 0$.

Lem 3.3.7. $|e^{ix} - \sum_{m=0}^{n}\frac{(ix)^m}{m!}| \leq \min(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!})$.

Thm 3.3.8. if $\mathbb{E}[X^2] < \infty$, then $\varphi(t) = 1 + it\mathbb{E}[X] - \frac{t^2}{2}\mathbb{E}[X^2] + o(t^2)$.

Thm 3.3.4. (Inversion Formula)

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita} - e^{-itb}}{it}\varphi(t)\,dt = P(a < X < b) + \frac{1}{2}P(X = a) + \frac{1}{2}P(X = b).$$

Hint: i) $\lim_T \int_0^T \frac{\sin(tc)}{t}\,dt = \frac{\pi}{2}\cdot\operatorname{sgn}(c)$. ii) $|\int_0^T \frac{\sin(tc)}{t}\,dt| \leq 4$.

Thm 3.3.5. if $\int|\varphi(t)|dt < \infty$, then X has bd-ct- den- $f = \frac{1}{2\pi}e^{-itx}\varphi(t)dt$.

Thm 3.3.6. i) if $X_n \Rightarrow X$, then $\varphi_{X_n}(t) \to \varphi_X(t), \forall\, t \in \mathbb{R}$.
ii) if $\varphi_{X_n}(t) \to \varphi_X(t)$ for $\forall\, t$ and φ is cont- at 0, then $X_n \Rightarrow X$.

Thm 3.4.1. (CLT) i.i.d. $\sim (\mu, \sigma^2)$, then $(S_n - n\mu)/\sigma\sqrt{n} \Rightarrow \chi$.

Thm 3.4.2. if $c_n \to c \in \mathbb{C}$, then $(1 + c_n/n)^n \to e^c$.

Lem 3.4.3. $|z_1|, |w_n| \leq \theta$, then $|\prod_m z_m - \prod_m w_m| \leq \theta^{n-1}\sum_m |z_m - w_m|$.

Lem 3.4.4. if $b \in \mathbb{C}$ and $|b| \leq 1$, then $|e^b - (1+b)| \leq |b|^2$.

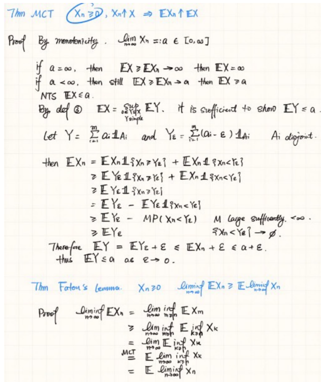
Lem 3.4.5. (Lindeberg-Feller Theorem)
for each n, $\{\xi_{n,i}\}_{k=1}^n$ are indep- with $\mathbb{E}[\xi_{n,i}] = 0, \mathbb{E}[\sum_i \xi_{n,i}^2] = 1$.
if (L's cond-) $\forall\ \epsilon > 0, \sum_i \mathbb{E}[\xi_{n,i}^2 1_{\{\xi_{n,i} > \epsilon\}}] \to 0$, then $\sum_i \xi_{n,i} \Rightarrow \chi$.

Hint: $\phi_{n,i} = 1 - \frac{1}{2}t^2\mathbb{E}[\xi_{n,i}^2]$, ex3.1.1 -> lem3.4.3 -> lem3.3.7. -> ϵ.

Remark: $\sum_i \mathbb{E}[|\xi_{n,i}|^p] \to 0$, $p > 2$ => L's cond- => $\max_i \mathbb{E}[\xi_{n,i}^2] \to 0$.

Thm 2.5.4. (converse of Three-Series Theorem)
indep-, if $\sum_n X_n$ converges a.s., then $\forall\, A$, $Y_i = X_i 1_{\{|X_i| \leq A\}}$ i) ii) iii).

Hint: i) contra- -> iii) contra-, $\xi_{n,m}$, L-F -> ii).



Thm MCT $X_n \geq 0$, $X_n \uparrow X \Rightarrow \mathbb{E}X_n \uparrow \mathbb{E}X$

Proof By monotonicity. $\lim_{n\to\infty} X_n =: a \in [0, \infty]$

if $a = \infty$, then $\mathbb{E}X \geq \mathbb{E}X_n \to \infty$ then $\mathbb{E}X = \infty$
if $a < \infty$, then still $\mathbb{E}X \geq \mathbb{E}X_n \to a$ then $\mathbb{E}X \geq a$
NTS $\mathbb{E}X \leq a$.
By def ③ $\mathbb{E}X = \sup_{\substack{0 \leq Y \leq X \\ Y \text{ simple}}} \mathbb{E}Y$. it is sufficient to show $\mathbb{E}Y \leq a$.
Let $Y = \sum_{i=1}^m a_i 1_{A_i}$ and $Y_\varepsilon = \sum_{i=1}^m (a_i - \varepsilon) 1_{A_i}$ A_i disjoint.

then $\mathbb{E}X_n = \mathbb{E}X_n 1\{X_n \geq Y_\varepsilon\} + \mathbb{E}X_n 1\{X_n < Y_\varepsilon\}$
$\geq \mathbb{E}Y_\varepsilon 1\{X_n \geq Y_\varepsilon\} + \mathbb{E}X_n 1\{X_n < Y_\varepsilon\}$
$\geq \mathbb{E}Y_\varepsilon 1\{X_n \geq Y_\varepsilon\}$
$= \mathbb{E}Y_\varepsilon - \mathbb{E}Y_\varepsilon 1\{X_n < Y_\varepsilon\}$
$\geq \mathbb{E}Y_\varepsilon - MP(X_n < Y_\varepsilon)$ M large sufficently. $< \infty$.
$\geq \mathbb{E}Y_\varepsilon$ $\{X_n < Y_\varepsilon\} \to \emptyset$.
Therefore $\mathbb{E}Y = \mathbb{E}Y_\varepsilon + \varepsilon \leq \mathbb{E}X_n + \varepsilon \leq a + \varepsilon$.
thus $\mathbb{E}Y \leq a$ as $\varepsilon \to 0$.

Thm Fatou's lemma. $X_n \geq 0$ $\liminf_{n\to\infty} \mathbb{E}X_n \geq \mathbb{E}\liminf_{n\to\infty} X_n$

Proof $\liminf_{n\to\infty} \mathbb{E}X_n = \lim_{n\to\infty}\inf_{m\geq n} \mathbb{E}X_m$
$\geq \lim_{n\to\infty}\inf_{m\geq n} \mathbb{E}\inf_{k\geq n} X_k$
$= \lim_{n\to\infty} \mathbb{E}\inf_{k\geq n} X_k$
$\stackrel{MCT}{=} \mathbb{E}\lim_{n\to\infty}\inf_{k\geq n} X_k$
$= \mathbb{E}\liminf_{n\to\infty} X_n$

Random Walks

Sec1. Random Walks

Def. if r-v-s X_i i.i.d. $S_n = \sum_i X_i$, then $\{S_n\}$ is random walk with $S_0 = 0$.

Thm 4.1.1. (Hewitt-Savage 0-1 Law)
if A is permutable (Not change under finite perm-), then $P(A) = 0$ or 1.

Thm 4.1.2. For random walk on $\mathbb{R}$, one of follow- has prob- 1:
i) $S_n = 0$ ii) $S_n \to \infty$ iii) $S_n \to -\infty$ iv) $-\infty = \liminf S_n < \limsup S_n = \infty$.
Hint: consider $P(\limsup S_n > c)$.

Def. $\mathcal{F}_0 = \{\varnothing, \Omega\}, \mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ seq- of incr- σ-fields is filtration.

Def. rand-time $\tau \in \mathbb{R}^+ \cup \{\infty\}$ is stopping time wrt $\{\mathcal{F}_n\}$ if $\{\tau = n\} \in \mathcal{F}_n$.

Fact. i) $\{\tau = n\} \in \mathcal{F}_n$ ii) $\{\tau \le n\} \in \mathcal{F}_n$ iii) $\{\tau \ge n+1\} \in \mathcal{F}_n$ are equival-.

Fact. i) $\tau_1 \vee \tau_2$ ii) $\tau_1 \wedge \tau_2$ iii) $\tau_1 + \tau_2$ iv) $\tau_1 \times \tau_2$ are stopping times.

Thm 4.1.5. (Wald's eq) i.i.d. if $\mathbb{E}[X_1], \mathbb{E}[\tau] \le \infty$, then $\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\mathbb{E}[\tau]$.

Thm 4.1.6. (W's 2eq) i.i.d.$\sim(0, \sigma^2)$, if $\mathbb{E}[\tau] < \infty$, then $\mathbb{E}[S_\tau^2] = \sigma^2 \mathbb{E}[\tau]$.
Hint: $\mathbb{E}[S^2_{\tau\wedge n}] = \mathbb{E}[S^2_{\tau\wedge(n-1)}] + \sigma^2 P(\tau \ge n)$, consider $\mathbb{E}[(S_{\tau\wedge n} - S_{\tau\wedge m})^2]$.

Sec2. Recurrence v.s. Transience

Def. RW, $\tau = \inf\{m \ge 1 : S_m = 0\}$, $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$.

RW is recurrent, if $P(\tau < \infty) = 1$; transient, if $P(\tau < \infty) < 1$.

Thm 4.2.2. equivalent i) $P(\tau < \infty) = 1$; ii) $P(\tau_n < \infty) = 1, \forall n$;
iii) $P(S_n = 0,\ \text{i.o.}) = 1$; iv) $\sum_m P(S_m = 0) = \infty$.

Thm 4.2.3. SRW is recurrent in $d \le 2$, but is transient in $\mathbb{R}^d$ for $d \ge 3$.

Ex X_i ind $\mathbb{E}X_i = \mu$ $\Rightarrow$ $\frac{S_n}{n} \xrightarrow{a.s.} \mu$ as $n \to \infty$.

pf. let $Y_i = X_i \mathbb{1}\{|X_i| \le i\}$ then

(i) $\sum_{i=1}^\infty P(|\frac{X_i}{i}| > 1) = \sum_{i=1}^\infty P(|X_i| > i) \le \mathbb{E}|X_i| < \infty$.

(ii) $\sum_{i=1}^\infty \mathbb{E}\frac{Y_i - \mathbb{E}Y_i}{i} = 0$ converges.

(iii) $\sum_{i=1}^\infty \frac{\mathrm{Var} Y_i}{i^2} \le \sum_{i=1}^\infty \frac{\mathbb{E}Y_i^2}{i^2} = \sum_{i=1}^\infty \frac{\mathbb{E}X_i^2 \mathbb{1}\{|X_i| \le i\}}{i^2}$

$\Rightarrow$ $\le \sum_{i=1}^\infty \sum_{j=1}^i \frac{\mathbb{E}X_i^2 \mathbb{1}\{j-1 < |X_i| \le j\}}{i^2}$

$= \sum_{j=1}^\infty \sum_{i=j}^\infty i^{-2} \mathbb{E}X_i^2 \mathbb{1}\{j-1 < |X_i| \le j\}$

$\le \sum_{j=1}^\infty \sum_{i=j}^\infty \frac{j}{i^2} \mathbb{E}|X_i| \mathbb{1}\{j-1 < |X_i| \le j\}$

$= \lambda \sum_{j=1}^\infty \mathbb{E}|X_i| \mathbb{1}\{j-1 < |X_i| \le j\}$.

$= \lambda \mathbb{E}|X_i| < \infty$.

thus $\sum_{i=1}^\infty \frac{X_i - \mathbb{E}X_i}{i}$ converges a.s. by Kolmogrov's three-series thm.

then $\frac{1}{n}\sum_{i=1}^n (X_i - \mathbb{E}X_i) \xrightarrow{a.s.} 0$ i.e. $\frac{S_n}{n} \xrightarrow{a.s.} \mu$ as $n \to \infty$

Ex. Marchkiewicz – Zygmund SLLN.

X_i iid $\mathbb{E}X_1 = 0$ Assume $\mathbb{E}|X_1|^p < \infty$ for some $1 < p < 2$ then $\frac{S_n}{n^{1/p}} \xrightarrow{a.s.} 0$.

Proof. NTS $\sum_{i=1}^n X_i / n^{1/p} \xrightarrow{a.s.} 0$.

By the prove above it is sufficient to show $\sum_{i=1}^\infty \frac{X_i}{i^{1/p}}$ converges a.s.

then let $Y_i = \frac{X_i}{i^{1/p}} \mathbb{1}\{|X_i| \le i^{1/p}\}$

(i) $\sum_{i=1}^\infty P(|\frac{X_i}{i^{1/p}}| > 1) = \sum_{i=1}^\infty P(|X_i| > i^{1/p}) = \sum_{i=1}^\infty P(|X_i|^p > i)$

$\le \mathbb{E}|X_1|^p < \infty$.

(ii) $\sum_{i=1}^\infty |\mathbb{E}Y_i| \le \sum_{i=1}^\infty \frac{|\mathbb{E}X_1 \mathbb{1}\{|X_1|^p \le i\}|}{i^{1/p}} \le \sum_{i=1}^\infty \frac{\mathbb{E}|X_1 \mathbb{1}\{|X_1|^p > i\}|}{i^{1/p}}$

$\le \sum_{i=1}^\infty \sum_{j=i}^\infty \frac{\mathbb{E}|X_1| \mathbb{1}\{j < |X_1|^p \le j+1\}}{i^{1/p}}$

$= \sum_{j=1}^\infty \mathbb{E}|X_1| \mathbb{1}\{j < |X_1|^p \le j+1\} \sum_{i=1}^j i^{-1/p}$ $\le c j^{1-\frac{1}{p}}$

$\le c \sum_{j=1}^\infty \mathbb{E}|X_1| [|X_1|^p]^{1-\frac{1}{p}} \mathbb{1}\{j < |X_1|^p \le j+1\}$

$\le c \mathbb{E}|X_1|^p < \infty$.

(iii) $\sum_{i=1}^\infty \mathrm{Var} Y_i \le \sum_{i=1}^\infty \frac{\mathbb{E}X_i^2 \mathbb{1}\{|X_1|^p \le i\}}{i^{2/p}} \le \sum_{i=1}^\infty \sum_{j=1}^i i^{-2/p} \mathbb{E}X_i^2 \mathbb{1}\{j-1 < |X_1|^p \le j\}$

$= \sum_{j=1}^\infty \sum_{i=j}^\infty i^{-2/p} \mathbb{E}X_i^2 \mathbb{1}\{j-1 < |X_1|^p \le j\}$ $\le c j^{1-\frac{2}{p}}$

$\le c \sum_{j=1}^\infty j^{1-\frac{2}{p}} \mathbb{E}X_1^2 \mathbb{1}\{j-1 < |X_1|^p \le j\}$

$\le c \sum_{j=1}^\infty \mathbb{E}\, X_1^2 \cdot (1 + |X_1|^p)^{1-\frac{2}{p}} \mathbb{1}\{j-1 < |X_1|^p \le j\}$

$\le c \sum_{j=1}^\infty \mathbb{E}(|X_1|^p + 1)^{\frac{2}{p}} (1 + |X_1|^p)^{1-\frac{2}{p}} \mathbb{1}\{j-1 < |X_1|^p \le j\}$

$= c \sum_{j=1}^\infty \mathbb{E}(1 + |X_1|^p) \mathbb{1}\{j-1 \le |X_1|^p \le j\}$

Martingale

Sec1. Conditional Expectation

Def. $(\Omega, \mathcal{F}, \mathcal{P})$, $\mathbb{E}[|X|] < \infty$, σ-field $\mathcal{A} \subset \mathcal{F}$, $\mathbb{E}[X|\mathcal{A}]$ cond-expectation,
if i) $\mathbb{E}[X|\mathcal{A}]$ is $\mathcal{A}$-measurable, ii) $\forall A \in \mathcal{A}$, $\mathbb{E}[X 1_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{A}] 1_A]$.
uniqueness. Hint: consider $A = \{Y_1 - Y_2 \ge \epsilon > 0\}$.

Prop. (a) $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\mathcal{A}]]$.

(b) $|\mathbb{E}[X|\mathcal{A}]| \le \mathbb{E}[|X||\mathcal{A}]$.

(c) if $X \in \mathcal{A}$, then $\mathbb{E}[X|\mathcal{A}] = X$.

(d) if X is independent of $\mathcal{A}$, then $\mathbb{E}[X|\mathcal{A}] = \mathbb{E}[X]$.

(e) (linearity) if $\mathbb{E}[X], \mathbb{E}[Y] < \infty$, then $\mathbb{E}[aX + Y|\mathcal{A}] = a\mathbb{E}[X|\mathcal{A}] + \mathbb{E}[Y|\mathcal{A}]$.

(f) (monotonicity) if $X \le Y$, then $\mathbb{E}[X|\mathcal{A}] \le \mathbb{E}[Y|\mathcal{A}]$.
Hint: consider $A = \{\mathbb{E}[X|\mathcal{A}] - \mathbb{E}[Y|\mathcal{A}] \ge \epsilon > 0\}$.

(g) (cmct) if $X_n \ge 0, X_n \uparrow X, \mathbb{E}[X] < \infty$, then $\mathbb{E}[X_n|\mathcal{A}] \uparrow \mathbb{E}[X|\mathcal{A}]$.

(h) (cFatou) if $X_n \ge 0$, $\mathbb{E}[\underline{\lim} X_n] < \infty$, then $\underline{\lim}\mathbb{E}[X_n|\mathcal{A}] \ge \mathbb{E}[\underline{\lim} X_n|\mathcal{A}]$.

(i) (cdct) if $|X_n| \le Y, \mathbb{E}[Y] < \infty, X_n \to X$a.s, then $\mathbb{E}[X_n|\mathcal{A}] \to \mathbb{E}[X|\mathcal{A}]$ a.s.

(j) (cJen) if conv-φ, $\mathbb{E}[|X|], \mathbb{E}[|\varphi(X)|] < \infty$, then $\varphi(\mathbb{E}[X|\mathcal{A}]) \le \mathbb{E}[\varphi(X)|\mathcal{A}]$.

(k) (cHol) if $p, q \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\mathbb{E}[|X|^p], \mathbb{E}[|Y|^q] < \infty$,
$\mathbb{E}[|XY||\mathcal{A}] \le (\mathbb{E}[|X|^p|\mathcal{A}])^{1/p} (\mathbb{E}[|Y|^q|\mathcal{A}])^{1/q}$.
Hint: suppose $\mathbb{E}[|X|^p|\mathcal{A}] \ge \epsilon > 0$, take $|X'| = (|X|^p + \epsilon^p)^{1/p}$, DCT.

(l) (cMin) if $p \ge 1, \mathbb{E}[|X|^p], \mathbb{E}[|Y|^p] < \infty$,
then $(\mathbb{E}[|X + Y|^p|\mathcal{A}])^{1/p} \le (\mathbb{E}[|X|^p|\mathcal{A}])^{1/p} + (\mathbb{E}[|Y|^p|\mathcal{A}])^{1/p}$

(m) (cMar) if $X \ge 0, a > 0$, then $P(X \ge a|\mathcal{A}) \le \mathbb{E}[X|\mathcal{A}]/a$.

(n) if $\mathbb{E}[|X|], \mathbb{E}[|XY|] < \infty$ and $X \in \mathcal{A}$, then $\mathbb{E}[XY|\mathcal{A}] = X\mathbb{E}[Y|\mathcal{A}]$.
Hint: four cases of X.

(o) (Tower Property) if $\mathbb{E}[|X|] < \infty, \mathcal{A}_1 \subset \mathcal{A}_2$, then
i) $\mathbb{E}[\mathbb{E}[X|\mathcal{A}_1]|\mathcal{A}_2] = \mathbb{E}[X|\mathcal{A}_1]$ ii) $\mathbb{E}[\mathbb{E}[X|\mathcal{A}_2]|\mathcal{A}_1] = \mathbb{E}[X|\mathcal{A}_1]$
Remark: small σ-field always wins.

(p) (triangular eq) if $\mathbb{E}[X^2] < \infty$, then for $\forall Y \in \mathcal{A}$ with $\mathbb{E}[Y^2] < \infty$,
$\mathbb{E}[(X - \mathbb{E}[X|\mathcal{A}])^2] \le \mathbb{E}[(X - Y)^2]$.

(q) $\mathrm{Var}(X) \ge \mathbb{E}[\mathrm{Var}(X|\mathcal{A})]$.

(r) if $Z \perp\!\!\!\perp (X, Y)$, then $\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y]$.
Hint: $\mathcal{P} = \{B \cap C\}$, $\mathcal{L} = \{A \in \sigma(Y, Z) : \mathbb{E}[X 1_A] = \mathbb{E}[\mathbb{E}[X|Y] 1_A]\}$.

Sec2. Martingale

Def. $\{S_n\}$ is martingale w.r.t. filtration $\{\mathcal{F}_n\}$ if
i) $\mathbb{E}[|S_n|] < \infty$, ii) $S_n \in \mathcal{F}_n$, iii) $\mathbb{E}[S_{n+1}|\mathcal{F}_n] = S_n$.

Fact. i) $\mathbb{E}[S_{n+m}|\mathcal{F}_n] = S_n$, ii) $\mathbb{E}[S_1] = \mathbb{E}[S_2] =$, iii) $\mathbb{E}[|S_1|] \le \mathbb{E}[|S_2|] \le$.

Def. martingale difference: $X_n = S_n - S_{n-1}$, and $S_0 = 0$.

Fact. i) $\mathbb{E}[X_i] = 0$ for $i \ge 2$. ii) $\mathbb{E}[S_n^2] = \mathbb{E}[X_1^2] + \cdots + \mathbb{E}[X_n^2]$.

Def. super-(sub-)martingale: iii) $\mathbb{E}[S_n|\mathcal{F}_{n-1}] \le (\ge) S_{n-1}$.

Remark: martingale is both super-/sub-martingale.

Thm 5.2.3-4. (martingale transformation)
i) if $\{S_n, \mathcal{F}_n\}$ mt, φ convex, $\mathbb{E}[|\varphi(S_n)|] < \infty$, then $\{\varphi(S_n)\}$ is sub-mt.
ii) if $\{S_n\}$ sub-mt, φ convex $\uparrow$, $\mathbb{E}[|\varphi(S_n)|] < \infty$, then $\{\varphi(S_n)\}$ is sub-mt.

Thm 5.2.8. (martingale convergence theorem)
if $\{S_n, \mathcal{F}_n\}$ sub-mt, $\liminf \mathbb{E}[S_n^+] < \infty$, then $S_n \to S$ a.s. and $\mathbb{E}[|S|] < \infty$.
Remark: by Cauchy-Schwarz ineq, $\mathbb{E}[S_n^+] \le \sqrt{\mathbb{E}[S_n^2]}$.

Thm 5.2.9. if $\{S_n \ge 0\}$ super-mt, then $S_n \to S$ a.s. and $\mathbb{E}[|S|] < \infty$.

Thm 5.2.7. if S_n sub-mt, then $\mathbb{E}[U_n] \le \frac{1}{b-a}[\mathbb{E}[(S_n - a)^+] - \mathbb{E}[(S_1 - a)^+]]$.
$N_{2k-1} = \inf\{m > N_{2k-2} : X_m \le a\}$, $N_{2k} = \inf\{m > N_{2k-1} : X_m \ge b\}$,
$N_0 = -1, H_m = \sum_k 1_{\{N_{2k-1} < m \le N_{2k}\}}, U_n = \sup\{k : N_{2k} \le n\}$.

Thm 5.2.5. if S_n is super-mt, $H_m \in \mathcal{F}_{m-1}$ predict-, and $0 \le H_m \le c_m$,
$T_1 = S_1, T_n = S_1 + \sum_{m=2}^n H_m(S_m - S_{m-1})$, then $\{T_n\}$ is super-mt.

Thm 5.2.6. if S_n sp-mt, τ st, then $S_{n\wedge\tau}$ is sp-mt, and $\mathbb{E}[S_{n\wedge\tau}] \ge \mathbb{E}[S_n]$.
Hint: X_n mt-dif, $T_n = S_{n\wedge\tau} = \sum_{k=1}^n X_k 1_{\{k \le \tau\}}$, $\mathbb{E}[T_n] \ge \mathbb{E}[T_{n-1}] + \mathbb{E}[X_n]$.

Thm 5.4.2. (Doob's inequality) if S_n sub-mt, then $\forall x > 0$,

$$P(\max_{1 \le k \le n} S_k \ge x) \le \frac{1}{x}\mathbb{E}[S_n 1_{\{\max_k S_k \ge x\}}] \le \frac{1}{x}\mathbb{E}[S_n^+]$$

Hint: consider $N = \inf\{k \ge 1 : S_k \ge x\}$ and $1_A \le \frac{S_{n\wedge N}}{x} 1_A$.

Thm 5.4.3. (L^p maximum ineq.) if S_n sub-mt, $p > 1$, then

$$\mathbb{E}[(\max_{1 \le k \le n} S_k^+)^p] \le (\frac{p}{p-1})^p \mathbb{E}[(S_n^+)^p].$$

Thm 5.4.5. (L^p convergene theorem) S_n martingale, $p > 1$.
if $\sup_n \mathbb{E}[|S_n|^p] < \infty$, then $S_n \to S$ a.s. and $\mathbb{E}[|S_n - S|^p] \to 0$.

Def. uniformly integrable, $\lim_{M\to\infty} \sup_n \mathbb{E}[|X_n| 1_{\{|X_n| > M\}}] = 0$.

Thm -. (L^1 convergence) S_n martingale and uniformly integrable, then
$S_n \to S$ a.s. and $\mathbb{E}[|S_n - S|] \to 0$.

Thm -. (LLN) identical X_n with $\mathbb{E}[|X_1|] < \infty$,
$S_1 = X_1$, $S_n = S_{n-1} + X_n - \mathbb{E}[X_n|\mathcal{F}_{n-1}]$, then $S_n/n \to 0$ in prob.

Thm DCT $X_n \xrightarrow{a.s.} X$. $|X_n| \le Y$ for some Y that $\mathbb{E}Y < \infty$
then $\mathbb{E}X_n \to \mathbb{E}X$

Proof $|X_n| \le Y$ $\Rightarrow$ $Y - X_n \ge 0$ and $X_n + Y \ge 0$.

then $\liminf_{n\to\infty} \mathbb{E}(Y - X_n) \ge \mathbb{E}\liminf_{n\to\infty}(Y - X_n)$
$= \mathbb{E}Y - \mathbb{E}\limsup_{n\to\infty} X_n = \mathbb{E}Y - \mathbb{E}X$

Similarly $\liminf_{n\to\infty} \mathbb{E}(X_n + Y) \ge \mathbb{E}\liminf_{n\to\infty} X_n + \mathbb{E}Y = \mathbb{E}X + \mathbb{E}Y$.

$\liminf_{n\to\infty} \mathbb{E}(Y - X_n) = \mathbb{E}Y - \limsup_{n\to\infty} \mathbb{E}X_n \ge \mathbb{E}Y - \mathbb{E}X$
$\liminf_{n\to\infty} \mathbb{E}(X_n + Y) = \mathbb{E}Y + \liminf_{n\to\infty} \mathbb{E}X_n \ge \mathbb{E}X + \mathbb{E}Y$

thus. $\liminf_{n\to\infty} \mathbb{E}X_n \ge \mathbb{E}X \ge \limsup_{n\to\infty} \mathbb{E}X_n$
thus $\liminf_{n\to\infty} \mathbb{E}X_n = \limsup_{n\to\infty} \mathbb{E}X_n =: \lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

indep. sets, σ-field, r.v.
$P(X^{-1}(B_1) \cap Y^{-1}(B_2)) = P(X^{-1}(B_1)) P(Y^{-1}(B_2))$ $\forall B_1, B_2 \in \mathcal{B}$.
$\Leftrightarrow P(A_1 \cap A_2) = P(A_1) P(A_2)$ $\forall A_1 \in \sigma(X), A_2 \in \sigma(Y)$.

Thm $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$.

Proof Case 1. $X = \mathbb{1}_A$ $Y = \mathbb{1}_B$
$\mathbb{E}XY = \mathbb{E}\mathbb{1}_A\mathbb{1}_B = \mathbb{E}\mathbb{1}_{AB} = P(AB) = P(A)P(B) \overset{\text{def 1}}{=} \mathbb{E}\mathbb{1}_A \mathbb{E}\mathbb{1}_B = \mathbb{E}X\mathbb{E}Y$.

Case 2 $X = \sum_{i=1}^m a_i \mathbb{1}_{A_i}$ $Y = \sum_{j=1}^n b_j \mathbb{1}_{B_j}$ A_i disjoint B_j disjoint.
$\mathbb{E}XY = \mathbb{E}\sum_{i=1}^m a_i \mathbb{1}_{A_i} \sum_{j=1}^n b_j \mathbb{1}_{B_j} = \mathbb{E}\sum_{i=1}^m \sum_{j=1}^n a_i b_j \mathbb{1}_{A_i B_j}$
$\overset{\text{Fubini}}{=} \sum_{i=1}^m \sum_{j=1}^n a_i b_j P(A_i B_j) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j P(A_i) P(A_j)$
$= \sum_{i=1}^m a_i P(A_i) \sum_{j=1}^n b_j P(B_j)$
$\overset{\text{def 2}}{=} \mathbb{E}X\mathbb{E}Y$

Case 3 $X, Y \ge 0$
then let simple r.vs $X_n \uparrow X$, $Y_n \uparrow Y$. (like $X_n = \frac{\lfloor 2^n X \rfloor}{2^n} =: X_n^{(n)}$
then $X_n Y_n \uparrow XY$
By the result of case 2. $\mathbb{E}X_nY_n = \mathbb{E}X_n\mathbb{E}Y_n$
by MCT $\lim_{n\to\infty} \mathbb{E}X_nY_n = \mathbb{E}XY$, $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$ and $\lim_{n\to\infty} \mathbb{E}Y_n = \mathbb{E}Y$
thus $\mathbb{E}XY = \lim_{n\to\infty} \mathbb{E}X_nY_n = \lim_{n\to\infty} \mathbb{E}X_n\mathbb{E}Y_n = \lim_{n\to\infty} \mathbb{E}X_n \lim_{n\to\infty} \mathbb{E}Y_n = \mathbb{E}X\mathbb{E}Y$.

Case 4. $X = X^+ - X^-$, $Y = Y^+ - Y^-$
then $\mathbb{E}XY = \mathbb{E}(X^+ - X^-)(Y^+ - Y^-)$
$= \mathbb{E}X^+Y^+ - \mathbb{E}X^-Y^+ - \mathbb{E}X^+Y^- + \mathbb{E}X^-Y^-$
$= \mathbb{E}X^+\mathbb{E}Y^+ - \mathbb{E}X^-\mathbb{E}Y^+ - \mathbb{E}X^+\mathbb{E}Y^- + \mathbb{E}X^-\mathbb{E}Y^-$
$= (\mathbb{E}X^+ - \mathbb{E}X^-)(\mathbb{E}Y^+ - \mathbb{E}Y^-)$
$= \mathbb{E}(X^+ - X^-)\mathbb{E}(Y^+ - Y^-)$
$= \mathbb{E}X\mathbb{E}Y$.