STAT 5010 Tutorial 1[*]

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Definitions

1. For X, $\{X_n\}$ random variables, F, $\{F_n\}$ the corresponding distribution functions,
   - $X_n \xrightarrow{a.s.} X$: $\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1$
   - $X_n \xrightarrow{L^r} X$: $\lim_{n\to\infty} \mathbf{E}|X_n - X|^r = 0$
   - $X_n \xrightarrow{p} X$: $\forall\, \delta > 0, \quad \lim_{n\to\infty} \mathbf{P}(|X_n - X| > \delta) = 0$
   - $F_n \xrightarrow{w} F$: $\forall\, x$ continuity point of F, $\lim_{n\to\infty} F_n(x) = F(x)$
   - $X_n \xrightarrow{d} X$: $F_n \xrightarrow{w} F$
2. Let $X_1, X_2, \ldots$ be random vectors and $Y_1, Y_2, \ldots$ be random variables defined on a common probability space.
   - $X_n = O_p(Y_n)$ if and only if, $\forall\, \epsilon > 0$, $\exists\, C_\epsilon > 0$ such that $\sup_n \mathbf{P}(||X_n|| \geq C_\epsilon |Y_n|) < \epsilon$.
   - $X_n = o_p(Y_n)$ if and only if $X_n/Y_n \xrightarrow{p} 0$

Propositions and Theorems

1. Chebyshev's Inequality: $\forall\, t > 0$, $\mathbf{P}(|X - EX| \geq t) \leq \frac{Var(X)}{t^2}$

   Proof. $t^2\mathbf{P}(|X - EX| \geq t) = t^2\mathbf{P}(|X - EX|^2 \geq t^2) = t^2\mathbf{E}[I_{|X-EX|^2 \geq t^2}] \leq E[|X - EX|^2] = Var(X)$ ■

2. $X_n \xrightarrow{a.s.} X \Leftrightarrow \forall\, \epsilon > 0, \quad \lim\limits_{k\to\infty} \mathbf{P}(\bigcup\limits_{n=k}^{\infty} \{||X_n - X|| > \epsilon\}) = 0$

   Proof. $\mathbf{P}(\lim_{n\to\infty} X_n = X) = \mathbf{P}(\bigcap\limits_{k=1}^{\infty} \bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = \lim\limits_{k\to\infty} \mathbf{P}(\bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = 1.$
   Note that $A_k \triangleq \bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}$ are nonincreasing in k. The equations above hold if and only if $\mathbf{P}(A_k) = 1$ for all k, which is equivalent to the statement in the right hand side. ■

3. $X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{p} X$

   Proof. Similar to the proof of Chebyshev's Inequality. ■

4. $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$

   Proof. The result follows directly from the proposition 2 above. ■

5. $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

   Proof. Let x be a continuity point of F, $\epsilon > 0$ given,

   $$\begin{aligned} F(x-\epsilon) &= \mathbf{P}(X \in (-\infty, x-\epsilon]) \\ &\leq \mathbf{P}(X \in (-\infty, x-\epsilon],\ X_n \notin (-\infty, x]) + \mathbf{P}(X_n \in (-\infty, x]) \\ &\leq F_n(x) + \mathbf{P}(|X_n - X| > \epsilon) \end{aligned}$$

   Letting $n \to \infty$, we obtain that $F(x-\epsilon) \leq \liminf_n F_n(x)$.
   Similarly, we have $F(x+\epsilon) \geq \limsup_n F_n(x)$. Since ϵ is arbitrary and F is continuous at x,

   $$F(x) = \lim_{n\to\infty} F_n(x)$$

   ■

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6. The corresponding converses of propositions 3-5 are all incorrect.

   Proof. Construction of counter examples. ■

7. $X_n \xrightarrow{p} c \Leftrightarrow X_n \xrightarrow{d} c$, here c is a constant.

8. Slutsky's Theorem: For random variables X_n, Y_n on a probability space, $X_n \xrightarrow{d} c, \quad Y_n \xrightarrow{d} Y$, where c is a fixed real number. Then $X_n + Y_n \xrightarrow{d} c + Y$.

Question 1

1. If random variables $X_n = O_p(1)$, $Y_n = o_p(1)$, prove that $X_n Y_n = o_p(1)$.

   Solution: See the handout later.

Question 2

1. $X_1, X_2, \ldots, X_n, \ldots$ are a series of i.i.d. random variables following $Uniform[0, 1]$ and defined on the same probability space. Give the asymptotic distribution of $n(1 - X_{(n)})$.

   Solution: See the handout later

2. Is it possible to have a random variable Y s.t. $n(1 - X_{(n)}) \xrightarrow{a.s.} Y$?

   Solution: See the handout later