1. Assume u is the null space of X^TX.

$X^TXu=0 \Rightarrow u^TX^TXu=0$, which means $Xu=0$.

$null(X^TX) \subset null(X)$.

Assume u is the null space of X.

$Xu=0 \Rightarrow X^TXu=0$, $null(X) \subset null(X^TX)$

Hence $null(X^TX) = null(X)$

$rank(X) = rank(X^TX)$

2. (a). False. Let $P=-Q$, $PXX^TP^T = QXX^TQ^T$. but $PX=-QX$.

(b). Since $(P-Q)XX^T=0$, $(X^TX)(P-Q)^T=0$, $(P-Q)^T \in null(X^TX) \Rightarrow$

$(P-Q)^T \in null(X^T)$

Hence $PX=QX$.

3. Since $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^-A_{12} \end{pmatrix}$

and the row rank of $(A_{11}\ A_{12})$ and A are r,

$\exists\ P_{(n-r)\times r}$ s.t. $P(A_{11}\ A_{12}) = (A_{21}\ A_{22})$,

$P = A_{21}A_{11}^-$, $PA_{12} = A_{22}$.

Hence $A_{21}A_{11}^-A_{12} = A_{22}$. $A^- = \begin{pmatrix} A_{11}^- & 0 \\ 0 & 0 \end{pmatrix}$

4. (a). $A = (2\ 1\ 3\ 5)^T(1\ 3)$.

Let $B=(2\ 1\ 3\ 5)^T$, $C=(1\ 3)$

$A^+ = C^T(CC^T)^{-1}(B^TB)^{-1}B^T$

$= \frac{1}{390}\begin{pmatrix} 2 & 1 & 3 & 5 \\ 6 & 3 & 9 & 5 \end{pmatrix}$

(b). Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{11} = (2)$ $A_{12} = (6)$

$A_{21} = (1\ 3\ 5)^T$ $A_{22} = (3\ 9\ 15)^T$

$$A^- = \begin{pmatrix} A_{11}^- & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $AA^-A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{pmatrix} = A.$

A^- is the G-inverse.

5.(a) Since $(X^TX)^-$, I_n, J are sysmetric,

A, B, C, D are all sysmetric.

Let $G = (X^TX)^-$, $A^2 = XGX^TXGX^T = XGX^T = A$. ①

$B^2 = (I_n - A)(I_n - A) = I_n - 2A + A^2 = I_n - A = B$. ②.

$C^2 = (A - \frac{1}{n}J)(A - \frac{1}{n}J) = A - \frac{1}{n}(AJ + (AJ)^T) + \frac{1}{n}J$

$= A - \frac{1}{n} = C$. ③

$D^2 = (I_n - \frac{1}{n}J)(I_n - \frac{1}{n}J) = I_n^2 - \frac{1}{n}J - \frac{1}{n}J - \frac{1}{n^2}J^2$

$= I_n - \frac{1}{n}J = D$. ④

By ①②③④, A, B, C, D are idempotent.

(b). $\text{rank}(A) = tr(A) = \text{rank}(X) = k$.

$\text{rank}(B) = tr(B) = tr(I_n - A) = tr(I_n) - tr(A)$

$= n - k$.

$\text{rank}(C) = tr(C) = tr(A - \frac{1}{n}J) = tr(A) - tr(\frac{1}{n}J)$

$= k - 1$.

$\text{rank}(D) = tr(D) = tr(I_n - \frac{1}{n}J) = tr(I_n) - \frac{1}{n}tr(J) = n-1$

6. (a). Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{11} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$.

$$\begin{pmatrix} A_{11}^- & 0 \\ 0 & 0 \end{pmatrix} \text{ is a g-inverse of } A.$$

where $A_{11}^- = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ is sysmetric.

Hence $A^- = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b). Let $G = \begin{pmatrix} A_{11}^- & g \\ & g \\ 0 \quad 0 & g \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & g \\ -\frac{1}{2} & 1 & g \\ 0 & 0 & g \end{pmatrix}$,

$$AGA = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & g \\ -\frac{1}{2} & 1 & g \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} = A.$$

$g = -1$. i.e. $G = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ is the non-symmetric g-inverse.

7. ($\Rightarrow$). $AX = C$ has solution(s) means C is a linear combination of column vectors of A. Hence $\text{rank}(A) = \text{rank}(A, C)$.

($\Leftarrow$). If $\text{rank}(A) = \text{rank}(A, C)$, then c is the linear combination of column vectors of A, or $c = 0$. Hence $AX = C$ has at least one solution.

8. ($\Rightarrow$). If $AX=c$ has solution X, then for $\forall A^-$,

$C = AX = AA^-AX = AA^-C$.

($\Leftarrow$). If $AA^-C = C$ holds, let $X = A^-C$, X is a solution to $AX=C$.

9. Assume $\text{rank}(A) = p < n$, then $\text{rank}(A^TA) = p$

A^TA is invertible,

$I = (A^TA)^{-1}(A^TA) = (A^TA)^{-1}A^T(AA^-A) = A^-A$.

10. (a) $k = \text{rank}(X) = \text{rank}(XX^-X) \leq \text{rank}(X^-)$.

(b). $(X^-X)^2 = X^-(XX^-X) = X^-X$.

$(XX^-)^2 = (XX^-X)X^- = XX^-$.

(c). $k = \text{rank}(X) = \text{rank}(XX^-X) \leq \text{rank}(X^-X) \leq \text{rank}(X) = k$

$\leq \text{rank}(XX^-)$

Hence, $\text{rank}(XX^-) = \text{rank}(X^-X) = k$.

(d). ($\Rightarrow$) $n = \text{rank}(I) = \text{rank}(X^-X) = \text{rank}(X)$

($\Leftarrow$) Since X is full column rank, $X^-X = I$.

(e) $\text{rank}(X) = k = \text{rank}(X^-X) = \text{rank}(XX^-)$

$\text{rank}(X^-X) = \text{tr}(X^-X)$ and $\text{rank}(XX^-) = \text{tr}(XX^-)$

Thus, $\text{tr}(X^-X) = \text{tr}(XX^-) = k = \text{rank}(X)$.

(f) $XX^-X = X \Rightarrow X^T(X^-)^TX^T = X^T$. Hence $(X^-)^T$ is a g-inverse of X^T.

11. (a) $K = K^T$ by 10(f).

Hence $K^2 = KK^T = X(X^TX)^- X^T X (X^TX)^- X^T$

$= X(X^TX)^- X^T$ by 2(b)

$= K$.

(b). $\text{rank}(K) = \text{rank}(X(X^TX)^- X^T) \geq \text{rank}(X^TX(X^TX)^- X^TX)$

$= \text{rank}(X^TX) = \text{rank}(X) = r$.

$\text{rank}(K) = \text{rank}(X(X^TX)^- X^T) \leq \text{rank}(X) = r$.

Hence $\text{rank}(K) = \text{rank}(X) = r$.

(c) $KX = X(X^TX)^- X^T X = ((X^TX)(X^TX)^T X^T)^T = X$

(d). $X((X^TX)^- X^T) X = X$. by (c).

i.e. $(X^TX)^- X^T$ is a g-inverse of X.

12. (a) Let A_1^+ and A_2^+ be two M-P inverse of A.

$AA_1^+ = (AA_2^+A)A_1^+ = (AA_2^+)(AA_1^+) = (AA_2^+)^T(AA_1^+)^T$

$= (A_2^+)^T (AA_1^+A)^T = (A_2^+)^T A^T = AA_2^+$

Hence $A_1^+ = A_1^+AA_1^+ = A_1^+AA_2^+ = A_2^+AA_2^+ = A_2^+$.

(b). $\text{rank}(A^+) = \text{rank}(A^+AA^+) \leq \text{rank}(A) = \text{rank}(AA^+A)$

$\leq \text{rank}(A^+)$, Hence $\text{rank}(A^+) = \text{rank}(A)$.

(c). $AA = A = A^T = (AA)^T$.

$AAA = AA = A$.

Hence $A^+ = A$.