STAT5030 Assignment2 solution

1. Suppose $A = I_n - \frac{1}{n}J_n$, $B = \frac{1}{n}\mathbf{1}^\top\mathbf{1}$, and $Z = Y - \alpha$, where $J_n = \mathbf{1}^\top\mathbf{1}$. Then $U = \frac{1}{\sigma^2}Z^\top A^\top AZ$, $\bar{Y} = BY$, and $V = \frac{1}{\sigma^2}BZ^2$. Since $AB = (I_n - \frac{1}{n}J_n) = \frac{1}{n}\mathbf{1}^\top\mathbf{1} = 0$, therefore U and V are independent.

2. Denote $\bar{Y} = \frac{1}{n}\mathbf{1}^\top Y$, $Q_1 = \frac{1}{n}Y^\top\mathbf{1}^\top\mathbf{1}Y$ and $Q_2 = Y^\top(I - \frac{1}{n}J)Y$.
   (a) Since $\frac{1}{n}\mathbf{1}^\top(I - \frac{1}{n}J) = 0$, $\bar{Y}$ and Q_2 are independent.
   (b) Since $\mathbf{1}^\top\mathbf{1}(I - \frac{1}{n}J) = 0$, Q_1 and Q_2 are independent.
   (c) $Q_1 \sim \chi^2_{1,\frac{\mu^2 n}{2}}$, and $Q_2 \sim \chi^2_{n-1}$.

3. Refer to Theorem 1 in Chapter 2. Distributions and Quadratic Forms.

4. (a) $\frac{y^\top Ay}{\sigma^2} = \chi^2_{2,\frac{19}{3\sigma^2}}$.
   (b) Since $BA \neq 0$, then $y^\top Ay$ and By are not independent.
   (c) Let $C = (1,1,1)^\top$, then $y_1 + y_2 + y_3 = Cy$. Since $AC = 0$, therefore $y^\top Ay$ and $y_1 + y_2 + y_3$ are independent.

5. $X_1X_2 - X_3X_4$ can be negative, but χ^2 distribution will not take negtive value.

6. Denote that
$$\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2(1-\rho)} = y^\top \left\{\frac{I - \frac{1}{n}J}{\sigma^2(1-\rho)}\right\} y = y^\top Ay.$$
   Since $AA = A$ is idempotent, then $y^\top Ay$ follows χ^2 distribution. $rank\left(\frac{I-\frac{1}{n}J}{\sigma^2(1-\rho)}\right) = n-1$. $(\mu\mathbf{1})^\top(I - \frac{1}{n}J)(\mu\mathbf{1}) = 0$. Therefore, $y^\top Ay \sim \chi^2_{n-1}$.

7. $E(U) = E(Y^\top(I - \frac{J}{n})Y) = tr((I - \frac{J}{n})\Sigma) + \mu^\top(I - \frac{J}{n})\mu = 6$.

8. (a) $E(U) = E(\sum_{i<j}(Y_i - Y_j)^2) = \sum_{i<j} E(Y_i - Y_j)^2 = \sum_{i<j} E(Y_i^2 - 2Y_iY_j + Y_j^2) = \sum_{i<j} 2\sigma^2 = n(n-1)\sigma^2$.
   (b) $k = \frac{1}{n(n-1)}$.

9. Since $E(U) = E[\sum_{i=1}^n (Y_i - \bar{Y})^2] = (n-1)(1-\rho)\sigma^2$. Therefore $k = \frac{1}{(n-1)(1-\rho)}$.