1. Let $X_1, \dots, X_n$ be iid from Gamma(a, b) where a is known.

   (a) Express the likelihood function $f(X_1, \dots, X_n|b)$ in terms of $\eta = -\frac{1}{b}$ and find the conjugate prior for η.

The density of a Gamma(a, b) distribution is

$$f(x|\eta) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} = \frac{1}{\Gamma(a)} x^{a-1} (-\eta)^a e^{\eta x} = \frac{1}{\Gamma(a)} x^{a-1} e^{\eta x - (-a)\log(-\eta)}.$$

The likelihood function is thus

$$f(\mathbf{X}|\eta) = \prod_{i=1}^{n} f(X_i|\eta) = \left(\frac{1}{\Gamma(a)}\right)^n \left(\prod_{i=1}^{n} X_i\right)^{a-1} \exp\left(\eta \sum_{i=1}^{n} X_i - n(-a)\log(-\eta)\right).$$

The conjugate prior family is

$$\pi(\eta|k, \mu) = c(k, \mu) \exp\left(k\eta\mu - k(-a)\log(-\eta)\right)$$

where μ can be thought of as a prior mean and k is proportional to a prior variance.

   (b) Using the prior for η in (a), find the Bayes estimator under the losses (i) $L(b, \delta) = (b - \delta)^2$ and (ii) $L(b, \delta) = (1 - \delta/b)^2$.

The posterior distribution is

$$\begin{aligned}\pi(\eta|\mathbf{x}, k, \mu) &\propto \exp\left(\eta \sum_{i=1}^{n} x_i - n(-a)\log(-\eta)\right) \exp\left(k\eta\mu - k(-a)\log(-\eta)\right) \\ &= \exp\left(\eta(n\overline{x} + k\mu) - (n+k)(-a)\log(-\eta)\right),\end{aligned}$$

(i) Using the conclusion of Problem 3.9 in Theory of Point Estimation, we know that

$$\mathbb{E}\left(\frac{\partial(-a)\log(-\eta)}{\partial\eta}|\mathbf{x}, k, \mu\right) = \mathbb{E}\left(-\frac{a}{\eta}\right) = \frac{n\overline{x} + k\mu}{n+k}.$$

Under $L(b, \delta) = (b - \delta)^2$, using Corollary 1.2, we know the Bayes estimator of b is

$$\delta(\mathbf{x}) = \mathbb{E}(b|\mathbf{x}) = \mathbb{E}\left(-\frac{1}{\eta}|\mathbf{x}\right) = \frac{n\overline{x} + k\mu}{a(n+k)}.$$

(ii) Under $L(b, \delta) = (1 - \delta/b)^2 = (b - \delta)^2/b^2$, using Corollary 1.2, we know the Bayes estimator of b is

$$\delta(\mathbf{x}) = \frac{\mathbb{E}(1/b|\mathbf{x})}{\mathbb{E}(1/b^2|\mathbf{x})} = \frac{\mathbb{E}(-\eta|\mathbf{x})}{\mathbb{E}(\eta^2|\mathbf{x})}.$$

Since the posterior distribution of η is proportional to

$$\exp\left(\eta(n\overline{x} + k\mu) - (n+k)(-a)\log(-\eta)\right),$$

we know $-\eta|\mathbf{x}$ follows Gamma distribution with shape $a(n+k)+1$ and scale $1/(n\overline{x} + k\mu)$ so

$$\mathbb{E}(-\eta|\mathbf{x}) = \frac{a(n+k)+1}{n\overline{x} + k\mu},$$

and

$$\text{Var}(-\eta|\mathbf{x}) = \frac{a(n+k)+1}{(n\overline{x} + k\mu)^2} \qquad \Longrightarrow \qquad \mathbb{E}(\eta^2|\mathbf{x}) = \frac{(a(n+k)+1) + (a(n+k)+1)^2}{(n\overline{x} + k\mu)^2},$$

so

$$\delta(\mathbf{x}) = \frac{\mathbb{E}(-\eta|\mathbf{x})}{\mathbb{E}(\eta^2|\mathbf{x})} = \frac{a(n+k)+1}{n\overline{x} + k\mu} \frac{(n\overline{x} + k\mu)^2}{(a(n+k)+1) + (a(n+k)+1)^2} = \frac{n\overline{x} + k\mu}{a(n+k)+2}.$$

   (c) The problem is misleading, in fact it lies in the Chapter 4 (3.9) of the indicated book.

2. Page 242, Corollary 3.3 in Chapter 4, Lehmann and Casella (1998): If $\mathbf{X} = (X_1, \dots, X_p)$ has the density

$$p_{\boldsymbol{\eta}}(\mathbf{x}) = \exp\left(\sum_{i=1}^{p} \eta_i x_i - A(\boldsymbol{\eta})\right) h(\mathbf{x})$$

and $\boldsymbol{\eta}$ has prior density $\pi(\boldsymbol{\eta})$, the Bayes estimator of $\boldsymbol{\eta}$ under the loss $L(\boldsymbol{\eta}, \delta) = \sum(\eta_i - \delta_i)^2$ is given by

$$\mathbb{E}(\eta_i|\mathbf{x}) = \frac{\partial}{\partial x_i} \log m(\mathbf{x}) - \frac{\partial}{\partial x_i} \log h(\mathbf{x}).$$

We prove Theorem 3.2 first: If $\mathbf{X}$ has density

$$p_{\boldsymbol{\eta}}(\mathbf{x}) = \exp\left(\sum_{i=1}^{p} \eta_i T_i(\mathbf{x}) - A(\boldsymbol{\eta})\right) h(\mathbf{x})$$

and $\boldsymbol{\eta}$ has prior density $\pi(\boldsymbol{\eta})$, then for $j = 1, \dots, n$,

$$\mathbb{E}\left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j} |\mathbf{x}\right) = \frac{\partial}{\partial x_j} \log m(\mathbf{x}) - \frac{\partial}{\partial x_j} \log h(\mathbf{x})$$

where $m(\mathbf{x}) = \int p_{\boldsymbol{\eta}}(\mathbf{x})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}$ is the marginal distribution of $\mathbf{X}$.

Proof. Note that

$$\frac{\partial}{\partial x_j} \exp\left(\sum_{i=1}^{s} \eta_i T_i\right) = \left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i}{\partial x_j}\right) \exp\left(\sum_{i=1}^{s} \eta_i T_i\right),$$

we have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j} |\mathbf{x}\right) \\
=&\int \left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j}\right) \pi(\boldsymbol{\eta}|\mathbf{x}) d\boldsymbol{\eta} \\
=&\frac{1}{m(\mathbf{x})} \int \left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j}\right) \exp\left(\sum_{i=1}^{p} \eta_i T_i(\mathbf{x}) - A(\boldsymbol{\eta})\right) h(\mathbf{x})\pi(\boldsymbol{\eta})d\boldsymbol{\eta} \\
=&\frac{1}{m(\mathbf{x})} \int \left[h(\mathbf{x}) \frac{\partial}{\partial x_j} \exp\left(\sum_{i=1}^{s} \eta_i T_i\right)\right] \exp(-A(\boldsymbol{\eta}))\pi(\boldsymbol{\eta})d\boldsymbol{\eta} \\
=&\frac{1}{m(\mathbf{x})} \int \left[\frac{\partial}{\partial x_j} \exp\left(\sum_{i=1}^{s} \eta_i T_i\right) h(\mathbf{x}) - \exp\left(\sum_{i=1}^{s} \eta_i T_i\right) \frac{\partial}{\partial x_j} h(\mathbf{x})\right] \exp(-A(\boldsymbol{\eta}))\pi(\boldsymbol{\eta})d\boldsymbol{\eta} \\
=&\frac{1}{m(\mathbf{x})} \frac{\partial}{\partial x_j} \int p_{\boldsymbol{\eta}}(\mathbf{x})\pi(\boldsymbol{\eta})d\boldsymbol{\eta} - \frac{1}{m(\mathbf{x})h(\mathbf{x})} \frac{\partial}{\partial x_j} h(\mathbf{x}) \int p_{\boldsymbol{\eta}}(\mathbf{x})\pi(\boldsymbol{\eta})d\boldsymbol{\eta} \\
=&\frac{\partial}{\partial x_j} \log m(\mathbf{x}) - \frac{\partial}{\partial x_j} \log h(\mathbf{x}).
\end{aligned}$$

□

Under the sum of square loss, the Bayesian estimator of $\boldsymbol{\eta}$ is given by the posterior means $\mathbb{E}(\eta_i|\mathbf{x})$ for $i = 1, \dots, s$. To see this, note that the value $\delta(\mathbf{x})$ minimizing

$$\mathbb{E}[L(\boldsymbol{\eta}, \delta(\mathbf{x}))|\mathbf{x}] = \mathbb{E}[\mathbb{E}[L(\boldsymbol{\eta}, \delta(\mathbf{x}))|\eta_{[-1]}]|\mathbf{x}]$$

must have $\mathbb{E}(\eta_1|\mathbf{x})$ as its first component since the squared loss is used. Similar arguments can be applied on

$$\mathbb{E}[\mathbb{E}[L(\boldsymbol{\eta}, \delta(\mathbf{x}))|\eta_{[-1]}])|\mathbf{x}] = \mathbb{E}[\mathbb{E}[\mathbb{E}[L(\boldsymbol{\eta}, \delta(\mathbf{x}))||\eta_{[-2]}]|\eta_{[-1]}])|\mathbf{x}]$$

and so on to reach the conclusion.
Use Theorem 3.2, we know

$$\mathbb{E}\left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j} |\mathbf{x}\right) = \frac{\partial}{\partial x_j} \log m(\mathbf{x}) - \frac{\partial}{\partial x_j} \log h(\mathbf{x}),$$

and since $T_i(\mathbf{x}) = x_i$, we have $\partial T_i/\partial x_j = x_i \mathbb{1}(j = i)$ so

$$\mathbb{E}\left(\eta_i|\mathbf{x}\right) = \frac{\partial}{\partial x_j} \log m(\mathbf{x}) - \frac{\partial}{\partial x_j} \log h(\mathbf{x}).$$

3. Example 3.4 in Chapter 4, Lehmann and Casella (1998): **Multiple normal model**. For

$$X_i|\theta_i \sim \mathcal{N}(\theta_i, \sigma^2), \qquad i = 1, \ldots, p, \text{ independent},$$

$$\Theta_i \sim \mathcal{N}(\mu, \tau^2), \qquad i = 1, \ldots, p, \text{ independent},$$

where σ^2, τ^2, and μ are known, $\eta_i = \theta_i/\sigma^2$ and the Bayes estimator of θ_i is

$$\mathbb{E}(\Theta_i|x) = \sigma^2\,\mathbb{E}(\eta_i|x) = \sigma^2\left[\frac{\partial}{\partial x_i}\log m(\mathbf{x}) - \frac{\partial}{\partial x_i}\log h(\mathbf{x})\right] = \frac{\tau^2}{\sigma^2+\tau^2}x_i + \frac{\sigma^2}{\sigma^2+\tau^2}\mu.$$

Example 3.6 in Chapter 4, Lehmann and Casella (1998): **Continuation of Example 3.4**. To evaluate the risk of the Bayes estimator, we also calculate

$$\frac{\partial^2}{\partial x_i^2}\log m(x) = -\frac{1}{\sigma^2+\tau^2},$$

and hence by Theorem 3.5,

$$R[\boldsymbol{\eta}, \mathbb{E}(\boldsymbol{\eta}|\mathbf{X})] = R[\boldsymbol{\eta}, -\nabla\log h(\mathbf{X})] - \frac{2p}{\sigma^2+\tau^2} + \sum_i \mathbb{E}_{\boldsymbol{\eta}}\left(\frac{X_i-\mu}{\sigma^2+\tau^2}\right)^2.$$

The best unbiased estimator of $\eta_i = \theta_i/\sigma^2$ is

$$-\frac{\partial}{\partial X_i}\log h(\mathbf{X}) = \frac{X_i}{\sigma^2}$$

with risk $R[\boldsymbol{\eta}, -\nabla\log h(\mathbf{X})] = p/\sigma^2$.

This problem further discusses Example 3.6 in Chapter 4, Lehmann and Casella (1998).

(a) Show that if δ is a Bayes estimator of θ, then $\delta' = \delta/\sigma^2$ is a Bayes estimator of η, and hence $R(\theta, \delta) = \sigma^4 R(\eta, \delta')$.

If δ is a Bayes estimator of θ for the loss function $L(\theta, d) = \sum_{i=1}^p(\theta_i - d_i)^2$, then

$$\delta_i(\mathbf{x}) = \mathbb{E}[\theta_i|\mathbf{x}], \qquad i = 1, 2, \ldots, p.$$

Divide the equalities by σ^2, we have

$$\frac{\delta_i(\mathbf{x})}{\sigma^2} = \mathbb{E}[\eta_i|\mathbf{x}], \qquad i = 1, 2, \ldots, p.$$

Hence under the loss function $L(\eta, d) = \sum_{i=1}^p(\eta_i - d_i)^2$, the Bayes estimator of η is given by $\delta' = \delta/\sigma^2$. Knowing that $\delta' = \delta/\sigma^2$, we have

$$\mathbb{E}[L(\theta, \delta(X))|\theta] = \mathbb{E}[\sum_{i=1}^p(\theta_i - \delta_i(X))^2|\theta] = \sigma^4\,\mathbb{E}\left[\sum_{i=1}^p\left(\eta_i - \frac{\delta_i(X)}{\sigma^2}\right)^2|\theta\right] = \sigma^4\,\mathbb{E}[L(\eta, \delta'(X))|\theta],$$

which is equivalent to

$$R(\theta, \delta) = \sigma^4 R(\eta, \delta').$$

(b) Show that the risk of the Bayes estimator of η is given by

$$\frac{p\tau^4}{\sigma^2\left(\sigma^2+\tau^2\right)^2} + \left(\frac{\sigma^2}{\sigma^2+\tau^2}\right)^2\sum_i a_i^2,$$

with $a_i = \eta_i - \mu/\sigma^2$.

Using the results of Example 3.6,

$$\begin{aligned} R[\boldsymbol{\eta}, \mathbb{E}(\boldsymbol{\eta}|\mathbf{X})] &= R[\boldsymbol{\eta}, -\nabla\log h(\mathbf{X})] - \frac{2p}{\sigma^2+\tau^2} + \sum_i \mathbb{E}_{\boldsymbol{\eta}}\left(\frac{X_i-\mu}{\sigma^2+\tau^2}\right)^2 \\ &= \frac{p}{\sigma^2} - \frac{2p}{\sigma^2+\tau^2} + \sum_i \mathbb{E}_{\boldsymbol{\eta}}\left(\frac{X_i-\mu}{\sigma^2+\tau^2}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{p(\tau^2-\sigma^2)}{\sigma^2(\sigma^2+\tau^2)} + \frac{1}{(\sigma^2+\tau^2)^2}\sum_i \mathbb{E}_{\boldsymbol{\eta}}(X_i^2 - 2\mu X_i + \mu^2) \\
&= \frac{p(\tau^4-\sigma^4)}{\sigma^2(\sigma^2+\tau^2)^2} + \frac{1}{(\sigma^2+\tau^2)^2}\sum_i \left[\sigma^4\left(\eta_i - \frac{\mu}{\sigma^2}\right)^2 + \sigma^2\right] \\
&= \frac{p(\tau^4-\sigma^4)}{\sigma^2(\sigma^2+\tau^2)^2} + \frac{p\sigma^2}{(\sigma^2+\tau^2)^2} + \frac{\sigma^4}{(\sigma^2+\tau^2)^2}\sum_i \left(\eta_i - \frac{\mu}{\sigma^2}\right)^2 \\
&= \frac{p\tau^4}{\sigma^2(\sigma^2+\tau^2)^2} + \left(\frac{\sigma^2}{\sigma^2+\tau^2}\right)^2 \sum_i a_i^2,
\end{aligned}
$$

with $a_i = \eta_i - \mu/\sigma^2$.

(c) If $\sum_i a_i^2 = k$, a fixed constant, then the minimum risk is attained at $\eta_i = \mu/\sigma^2 + \sqrt{k/p}$.

If $\sum_i a_i^2 = k$, a fixed constant, then $R[\boldsymbol{\eta}, \mathbb{E}(\boldsymbol{\eta}|\mathbf{X})]$ is fixed. Hence the minimum risk is attained at $\eta_i = \mu/\sigma^2 + \sqrt{k/p}$ by the usual math trick.

4. If $X_1, \ldots, X_n$ are iid from a one-parameter exponential family, the Bayes estimator of the mean, under squared error loss using a conjugate prior, is of the form $a\overline{X} + b$ for constants a and b. Prove the followings:

(a) If $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, then no matter what the distribution of the X_i's, the mean squared error is $\mathbb{E}\{(a\overline{X}+b) - \mu\}^2 = a^2\,\mathrm{Var}(\overline{X}) + \{(a-1)\mu + b\}^2$.

We have

$$
\begin{aligned}
\mathbb{E}\{(a\overline{X}+b) - \mu\}^2 &= \mathbb{E}((a\overline{X}+b)^2 - 2\mu(a\overline{X}+b) + \mu^2) \\
&= \mathrm{Var}(a\overline{X}+b) + [\mathbb{E}(a\overline{X}+b)]^2 - 2\mu\,\mathbb{E}(a\overline{X}+b) + \mu^2 \\
&= a^2\,\mathrm{Var}(\overline{X}) + (a\mu+b)^2 - 2\mu(a\mu+b) + \mu^2 \\
&= a^2\,\mathrm{Var}(\overline{X}) + (a^2-2a+1)\mu^2 + 2(a-1)b\mu + b^2 \\
&= a^2\,\mathrm{Var}(\overline{X}) + \{(a-1)\mu+b\}^2,
\end{aligned}
$$

regardless of the distribution of $\mathbf{X}$.

(b) If μ is unbounded, then no estimator of the form $a\overline{X} + b$ can have finite squared error for $a \neq 1$.

Since $\mathbb{E}\{(a\overline{X}+b) - \mu\}^2 = a^2\,\mathrm{Var}(\overline{X}) + \{(a-1)\mu + b\}^2$, if μ is unbounded, the squared error of $a\overline{X}+b$ will be unbounded for $a \neq 1$, so no estimator of the form $a\overline{X}+b$ can have finite squared error for $a \neq 1$.

(c) Can a conjugate-prior Bayes estimator in an exponential family have finite squared error?

Yes. Consider $X_1, \ldots, X_n \overset{\text{IID}}{\sim} \mathit{Binomial}(n,p)$ with p unknown and of interest, then the Bayes estimator of p, under squared error loss using a conjugate prior $\mathit{Beta}(a_0, b_0)$, is of the form $a\overline{X}+b$ for constants a and b and the mean squared error is finite.

5. Suppose that Θ follows a log-normal distribution with known hyperparameters $\mu_0 \in \mathbb{R}$ and $\sigma_0^2 > 0$ and that, given $\Theta = \theta$, $(X_1, \ldots, X_n)$ is an iid sample from $\mathrm{Uniform}(0, \theta)$.

(a) What is the posterior distribution of $\log(\Theta)$?

We are given that

$$\eta := \log(\Theta) \sim \mathcal{N}(\mu_0, \sigma_0^2)$$

and

$$
\begin{aligned}
f(\mathbf{X}|\eta) &= f(\mathbf{X}|\theta) \\
&= \frac{1}{\theta^n}\,\mathbb{1}(X_{(0)} \geq 0)\,\mathbb{1}(X_{(n)} \leq \theta) \\
&= e^{-n\log\theta}\,\mathbb{1}(\log X_{(n)} \leq \log\theta)\,\mathbb{1}(X_{(0)} \geq 0) \\
&= e^{-n\eta}\,\mathbb{1}(\log X_{(n)} \leq \eta)\,\mathbb{1}(X_{(0)} \geq 0).
\end{aligned}
$$

Hence,

$$\begin{aligned} f(\eta|\mathbf{X}) &\propto f(\mathbf{X}|\eta)\pi(\eta) \\ &\propto e^{-n\eta}\,\mathbb{1}(\log X_{(n)} \leq \eta)\exp\left(-\frac{(\eta-\mu_0)^2}{2\sigma_0^2}\right) \\ &\propto \exp\left(-\frac{(\eta-(\mu_0-n\sigma_0^2))^2}{2\sigma_0^2}\right)\mathbb{1}(\eta \geq \log X_{(n)}). \end{aligned}$$

The posterior distribution of $\log(\Theta)$ is, therefore, a truncated normal distribution with mean $\mu_0 - n\sigma_0^2$ and variance σ_0^2 and truncated below by $\log X_{(n)}$.

(b) Let δ_τ represent the Bayes estimator of θ under the loss

$$L(\theta, d) = \begin{cases} 0 & \text{if } \frac{1}{\tau} \leq \frac{\theta}{d} \leq \tau \\ 1 & \text{otherwise} \end{cases}$$

for fixed $\tau > 1$. Find a simple, closed-form expression for the limit of δ_τ as $\tau \to 1$.

The loss function can be equivalently expressed by

$$L(\theta, d) = \begin{cases} 0 & \text{if } |\log\theta - \log d| \leq \log\tau \\ 1 & \text{if } |\log\theta - \log d| > \log\tau \end{cases}.$$

By Corollary 1.2, the Bayes estimator $\delta_\tau(X)$ of θ is given by

$$\delta_\tau(X) = \arg\max_d \Pr(|\log\theta - \log d| \leq \log\tau).$$

We have the following two situations:

i. If $\log X_{(n)} \leq \mu_0 - n\sigma_0^2 - \log\tau$, then we should choose $\log\delta_\tau(X) = \mu_0 - n\sigma_0^2$;
ii. If $\log X_{(n)} > \mu_0 - n\sigma_0^2 - \log\tau$, then we should choose $\log\delta_\tau(X) = \log X_{(n)} + \log\tau$.

As $\tau \to 1$, we obtain that

$$\delta_\tau(X) \to \delta(X) = \exp\left(\max(\log X_{(n)}, \mu_0 - n\sigma_0^2)\right).$$

6. Consider a Bayesian inference setting in which the prior is continuous and the posterior mean $\mathbb{E}(\Theta|X = x)$ is finite for each x. Show that under the loss function

$$L(\theta, a) = \begin{cases} k_1|\theta - a| & \text{if } a \leq \theta \\ k_2|\theta - a| & \text{otherwise} \end{cases}$$

with $k_1, k_2 > 0$ constant and for p an appropriate function of k_1 and k_2, every p-th quantile of the posterior distribution is a Bayes estimator.

Denote the p-th quantile of the posterior distribution by τ_p, with

$$\Pr(\theta \leq \tau_p|\mathbf{x}) = p.$$

The loss function $L(\theta, a)$ can be rewritten as

$$L(\theta, a) = k_1(\theta - a)\,\mathbb{1}(\theta \geq a) + k_2(a - \theta)\,\mathbb{1}(\theta < a),$$

so

$$\mathbb{E}(L(\theta, a)|\mathbf{x}) = \int_a^\infty k_1(\theta - a)dF(\theta|\mathbf{x}) + \int_{-\infty}^a k_2(a - \theta)dF(\theta|\mathbf{x}).$$

To find the Bayes estimator, take derivatives with respect to a:

$$\frac{\partial}{\partial a}\mathbb{E}(L(\theta, a)|\mathbf{x}) = -k_1\Pr(\theta > a|\mathbf{x}) + k_2\Pr(\theta \leq a|\mathbf{x}) = (k_1 + k_2)\Pr(\theta \leq a|\mathbf{x}) - k_1.$$

For p-th quantile of the posterior distribution to be a Bayes estimator, we must have

$$\left.\frac{\partial}{\partial a}\mathbb{E}(L(\theta, a)|\mathbf{x})\right|_{a=\tau_p} = 0,$$

i.e.,

$$\Pr(\theta \leq \tau_p|\mathbf{x}) = \frac{k_1}{k_1 + k_2},$$

hence $p = k_1/(k_1 + k_2)$.