1. Prove that $\mathrm{rank}(X) = \mathrm{rank}(X^TX)$

Suppose X's dimension is $n \times m$ and its rank is r.

There exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{m\times m}$ such that

$$PXQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow X = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Partition $P^{-1} = [K\ W]$, then $X = [K\ W]\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$ and

$$X^TX = (Q^{-1})^T\begin{bmatrix} K^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = (Q^{-1})^T\begin{bmatrix} K^TK & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

where $K \in \mathbb{R}^{n\times r}$, $W \in \mathbb{R}^{n\times(n-r)}$, and both have full column rank.

so $\mathrm{rank}(K^TK) = r$

Since Q is invertable, then

$$\begin{aligned} r = \mathrm{rank}(K^TK) &= \mathrm{rank}\left(\begin{bmatrix} K^TK & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \mathrm{rank}\left((Q^{-1})^T\begin{bmatrix} K^TK & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}\right) \\ &= \mathrm{rank}(X^TX) \quad // \end{aligned}$$

2. Prove or disprove:

(a) If $PXX^TP^T = QXX^TQ^T$, then $PX = QX$

Disprove (a) by finding a counter example.

$$P = I_n,\quad X = I_n,\quad Q = -I_n$$

we have $PXX^TP^T = I_n$, $QXX^TQ^T = I_n$, but

$$PX = I_n \neq -I_n = QX$$

(b) If $PXX^T = QXX^T$, then $PX = QX$

Prove (b) by follows.

$$PXX^T = QXX^T \Rightarrow (P-Q)XX^T = 0 \Rightarrow (P-Q)XX^T(P-Q)^T = 0$$

$$\Rightarrow [(P-Q)X][(P-Q)X]^T = 0 \Rightarrow (P-Q)X = 0 \Rightarrow PX = QX \quad //$$

3. Suppose A is $n\times p$ of rank r, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} is $r\times r$ of rank r.

Prove a generalized inverse of A is given by $A^- = \begin{bmatrix} A_{11}^- & 0 \\ 0 & 0 \end{bmatrix}$

$$AA^-A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} A_{11}^- & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^- & 0 \\ A_{21}A_{11}^- & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^-A_{11} & A_{11}A_{11}^-A_{12} \\ A_{21}A_{11}^-A_{11} & A_{21}A_{11}^-A_{12} \end{bmatrix} \quad \cdots (+)$$

Since A_{11} is $r\times r$ of rank r and $\mathrm{rank}(A) = r$, then by primary operations.

A_{11} is invertible $A_{11}^- = A_{11}^{-1} \quad \cdots (*)$

$[A_{21}\ A_{22}] - A_{21}A_{11}^{-1}[A_{11}\ A_{12}] = 0$ holds

$\Rightarrow A_{22} = A_{21}A_{11}^{-1}A_{12} \quad \cdots (**)$

Plugin $(*)$ & $(**)$ into $(+)$, we have $AA^-A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A$

and by the definition of generalized inverse.

A^- is a generalized inverse of A. $//$

4. $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{bmatrix}$

(a) find the MP-inverse of A

It is easy to know $\text{rank}(A)=1$ and $A = [2\ 1\ 3\ 5]^T[1\ 3]$

Denote $B = [2\ 1\ 3\ 5]^T$ $C = [1\ 3]$ and MP-inverse of A as A^+

then
$$\begin{aligned} A^+ &= C^T(CC^T)^{-1}(B^TB)^{-1}B^T \\ &= [1\ 3]^T \cdot \frac{1}{10} \cdot \frac{1}{39} \cdot [2\ 1\ 3\ 5] \\ &= \frac{1}{390}\begin{bmatrix} 2 & 1 & 3 & 5 \\ 6 & 3 & 9 & 15 \end{bmatrix} \\ &= \frac{1}{390}A^T \end{aligned}$$

(b) find a G-inverse different with MP-inverse of A.

Since $\text{rank}(A) = 1$, A can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ where } \begin{matrix} A_{11} = [2] & A_{12} = [6] \\ A_{21} = [1\ 3\ 5]^T & A_{22} = [3\ 9\ 15]^T \end{matrix}$$

By question 3, $A^- = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

To check A^- is the G-inverse, we can show that:

$$\begin{aligned} AA^-A &= \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 3/2 & 0 & 0 & 0 \\ 5/2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{bmatrix} = A \qquad // \end{aligned}$$

5. Suppose $X \in \mathbb{R}^{n \times p}$, $\text{rank}(A) = k$, $X_{\cdot 1} = \mathbb{1}_n$, $J = [1]_{n \times n}$

$A = X(X^TX)^-X^T$; $B = I_n - X(X^TX)^-X^T$

$C = X(X^TX)^-X^T - \frac{1}{n}J$; $D = I_n - \frac{1}{n}J$

(a) Prove that A, B, C, and D are symmetric and idempotent.

Denote $G := (X^TX)^-$ is the generalized inverse of X^TX

then we have $X^TXGX^TX = X^TX$

By transposing it, we have $X^TXG^TX^TX = X^TX$.

so G^T is also a generalized inverse of X^TX ... (1)

And by Q2(b): $PXX^T = QXX^T \Rightarrow PX = QX$

$X^TXGX^T = X^T$ ... (2)

Let $T = GX^TXG^T$, which is symmetric, and by (1) & (2), we have

$XTX^T = XGX^TXG^TX^T = XGX^T = A$ is symmetric

$$A^2 = AA = XGX^TXGX^T$$
$$= XGX^T \quad (\text{by } (2))$$
$$= A,$$

so A is idempotent.

$B = I_n - X(X^TX)^-X^T = I_n - A$ both of I_n and A is symmetric,
so B is symmetric.

$B^2 = (I_n - A)(I_n - A) = I_n - 2A + A^2 = I_n - A = B,$
so B is idempotent.

$C = A - \frac{1}{n}J$ both of A and $\frac{1}{n}J$ is symmetric,
so C is symmetric

Notice that $AX = A^TX = (X^TA)^T = [X^TXGX^T]^T$
$$= [X^TJ]^T \quad (\text{by } (2))$$
$$= X$$

Since $X_{\cdot 1} = \mathbb{1}_n$, then partition $X = [\mathbb{1}_n \; \tilde{X}]$. $\hat{X} \in R^{n \times (p-1)}$

$AX = A[\mathbb{1}_n \;\; \tilde{X}] = [A\mathbb{1}_n \;\; A\tilde{X}] = X = [\mathbb{1}_n \; \tilde{X}]$

$\Rightarrow A\mathbb{1}_n = \mathbb{1}_n \Rightarrow AJ = A[\mathbb{1}_n \cdots \mathbb{1}_n] = J$

$$C^2 = (A - \tfrac{1}{n}J)(A - \tfrac{1}{n}J) = A^2 - \tfrac{1}{n}(AJ + JA) + \tfrac{1}{n^2}J^2$$
$$= A - \tfrac{1}{n}(AJ + (AJ)^T) + \tfrac{1}{n}J$$
$$= A - \tfrac{1}{n}J = C,$$

so C is idempotent.

$D = I_n - \frac{1}{n}J$, both of I_n and $\frac{1}{n}J$ is symmetric
so D is symmetric

$$D^2 = (I_n - \tfrac{1}{n}J)(I_n - \tfrac{1}{n}J) = I_n^2 - \tfrac{1}{n}J - \tfrac{1}{n}J - \tfrac{1}{n^2}J^2$$
$$= I_n - \tfrac{1}{n}J = D$$

so D is idempotent.

(b) Find the rank of A, B, C, and D.

$$\text{rank}(A) \geq \text{rank}(X^TAX) = \text{rank}(X^TXGX^TX) = \text{rank}(X^TX)$$
$$= \text{rank}(X) = k$$
$$\text{rank}(A) = \text{rank}(XGX^T) \leq \text{rank}(X) = k$$

so $\text{rank}(A) = k$.

Since B is idempotent,
$$\text{rank}(B) = tr(B) = tr(I_n - A) = tr(I_n) - tr(A)$$
$$= n - \text{rank}(A) = n - k \quad (\text{by } A \text{ is idempotent})$$

Since C is idempotent,

$$\text{rank}(C) = tr(C) = tr(A - \tfrac{1}{n}J) = tr(A) - tr(\tfrac{1}{n}J)$$
$$= k - 1$$

Since D is idempotent,

$$\text{rank}(D) = tr(D) = tr(I_n - \tfrac{1}{n}J) = tr(I_n) - tr(\tfrac{1}{n}J) \quad //$$
$$= n - 1$$

6. $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

(a) find a symmetric generalized inverse for A

It is easy to know $\text{rank}(A) = 2$, and partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ where } A_{11} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } \text{rank}(A_{11}) = 2 = \text{rank}(A)$$

By Q3, $\begin{bmatrix} A_{11}^- & 0 \\ 0 & 0 \end{bmatrix}$ is a generalized inverse of A

where $A_{11}^- = A_{11}^{-1} = \frac{1}{4\times2-2\times2}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$ is symmetric

Thus a symmetric generalized inverse for A is $\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: A^-$

(b) find a nonsymmetric generalized inverse for A

Construct a matrix $B := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix}$ satisfying that

$ABA = 0$ such that $A(A^- + B)A = AA^-A + ABA = 0$

so $A^- + B$ is a nonsymmetric generalized inverse of A if $a \neq 0$ or $b \neq 0$.

$$\text{Since } ABA = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2a & 2b & 2c \\ 0 & 0 & 0 \\ 2a & 2b & 2c \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 8a+4b+4c & 4a+4b & 4a+4c \\ 0 & 0 & 0 \\ 8a+4b+4c & 4a+4b & 4a+4c \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 8a+4b+4c = 0 \\ 4a+4b = 0 \\ 4a + 4c = 0 \end{cases} \Rightarrow \begin{cases} a \neq 0 \\ b = -a \\ c = -a \end{cases} \Rightarrow B = a\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix} \quad \forall a \neq 0.$$

the nonsymmetric inverse of A can be written as $\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ a & -a & -a \end{bmatrix} \quad \forall a \neq 0.$ //

7. Prove that $Ax=c$ has at least one solution vector x iff $\text{rank}(A)=\text{rank}(A,c)$

"$\Rightarrow$" Assume $Ax=c$ has at least one solution vector x
then c is the linear combination of column vectors of A
or $c=0$ at the time $x=0$.
thus $\text{rank}(A) = \text{rank}(A,c)$

"$\Leftarrow$" Assume $\text{rank}(A) = \text{rank}(A,c)$
then c is the linear combination of column vectors of A
or $c=0$
thus there exists a vector x such that $Ax=c$ //

8. Prove that $Ax=c$ has a solution $\Leftrightarrow$ $AA^-c=c$ $\forall$ generalized inverse A^- of A

"$\Rightarrow$" Assume $Ax=c$ has a solution vector x
then for any generialized inverse of A : A^-, we have
$c = Ax = AA^-Ax = AA^-c$

"$\Leftarrow$" Assume $AA^-c=c$ holds for any A^-
then we can take $x=A^-c$, which satisfies that
$Ax = AA^-c = c$ //

9. Prove that if $A\in\mathbb{R}^{n\times p}$ and $\text{rank}(A)=p<n$, then A^- is a left inverse of A, i.e. $A^-A=I$

Assume $\text{rank}(A)=p<n$
then $\text{rank}(A^TA)=p$ and A^TA invertible.
then $I = (A^TA)^{-1}(A^TA) = (A^TA)^{-1}A^TAA^-A = A^-A$ //

10. $X\in\mathbb{R}^{m\times n}$, X^- is a generalized inverse of X, $\text{rank}(X)=k>0$. Prove

(a) $\text{rank}(X^-)\geq k$

$k = \text{rank}(X) = \text{rank}(XX^-X) \leq \text{rank}(XX^-) \leq \text{rank}(X^-)$

(b) X^-X and XX^- are idempotent

$(X^-X)^2 = X^-XX^-X = X^-(XX^-X) = X^-X \Rightarrow X^-X$ is idempotent
$(XX^-)^2 = XX^-XX^- = (XX^-X)X^- = XX^- \Rightarrow XX^-$ is idempotent

(c) $\text{rank}(X^-X) = \text{rank}(XX^-) = k$

$$\begin{cases} k = \text{rank}(X) = \text{rank}(XX^-X) \leq \text{rank}(X^-X) \leq \text{rank}(X) = k \Rightarrow \text{rank}(X^-X)=k \\ k = \text{rank}(X) = \text{rank}(XX^-X) \leq \text{rank}(XX^-) \leq \text{rank}(X) = k \Rightarrow \text{rank}(XX^-)=k. \end{cases}$$
$\Rightarrow \text{rank}(X^-X) = \text{rank}(XX^-) = \text{rank}(X) = k$

(d) $X^-X = I \iff \text{rank}(X) = n$

"$\Rightarrow$" by (c) $n = \text{rank}(I) = \text{rank}(X^-X) = \text{rank}(X)$

"$\Leftarrow$" Since X is full column-full rank,

by Q9, $X^-X = I$

(e) $tr(X^-X) = tr(XX^-) = k = \text{rank}(X)$

By (c), $\text{rank}(X) = k = \text{rank}(X^-X) = \text{rank}(XX^-)$

and, by (b), $\text{rank}(X^-X) = tr(X^-X)$ and $\text{rank}(XX^-) = tr(XX^-)$

Thus $tr(X^-X) = tr(XX^-) = k = \text{rank}(X)$.

(f) X^- is any generalized inverse of $X \Rightarrow (X^-)^T$ is a generalized inverse of X^T

$XX^-X = X \xRightarrow{\text{transpose}} X^T(X^-)^TX^T = X^T$, thus $(X^-)^T$ is a G-inverse of X^T. //

11. $K = X(X^TX)^-X^T$, prove that.

(a) $K = K^T$, $K = K^2$

(Same with Q5(a) A)

By Q2(b) $X^TX(X^TX)^-X^TX = X^TX \Rightarrow X^TX(X^TX)^-X^T = X^T$ ····(1)

$X^TX(X^TX)^-X^TX = X^TX \xRightarrow{\text{transpose}} X^TX[(X^TX)^-]^TX^TX = X^TX$

$\Rightarrow [(X^TX)^-]^T$ is G-inverse of X^TX ····(2)

Construct a symmetrix matrix

$$KK^T = X(X^TX)^-X^TX[(X^TX)^-]^TX^T$$
$$= X(X^TX)^-X^T \quad \text{by (1)(2)}$$
$$= K$$
$$\Rightarrow K^T = K$$

$$K^2 = X(X^TX)^-X^TX(X^TX)^-X^T$$
$$= X(X^TX)^-X^T \quad \text{by (1)}$$
$$= K$$

(b) $\text{rank}(K) = \text{rank}(X) = r$

$$\begin{cases} \text{rank}(K) = \text{rank}(X(X^TX)^-X^T) \geq \text{rank}(X^TX(X^TX)^-X^TX) = \text{rank}(X^TX) = \text{rank}(X) = r \\ \text{rank}(K) = \text{rank}(X(X^TX)^-X^T) \leq \text{rank}(X) = r \end{cases}$$

$\Rightarrow \text{rank}(K) = \text{rank}(X) = r$

(c) $KX = X$

$$KX = X(X^TX)^-X^TX = [X^TX[(X^TX)]^TX^T]^T$$
$$= [X^T]^T = X \quad (\text{by (a)(1)(2)})$$

(d) $(X^TX)^-X^T$ is a G-inverse of X for any G-inverse of X^TX

$X[(X^TX)^-X^T]X = X$ by (c)

then $(X^TX)^-X^T$ is a G-inverse of X //

12 Prove the properties

(a) M-P inverse is unique

Assum there are two distinct M-P inverses of X. M and F.

then $\begin{cases} XM = (XM)^T, & XF = (XF)^T \\ MX = (MX)^T, & FX = (FX)^T \\ XMX = X, & XFX = X \\ MXM = M, & FXF = F \end{cases}$ by the definition of M-P inverses.

$\Rightarrow \begin{cases} XM = (XM)^T = (XFXM)^T = (XM)^T(XF)^T = XMXF = XF \\ MX = (MX)^T = (MXFX)^T = (FX)^T(MX)^T = FXMX = FX \end{cases}$

$\Rightarrow M = MXM = FXM = FXF = F$ contradicts with assumption,

thus M-P inverse is unique.

(b) $rank(X^+) = rank(X)$

$rank(X^+) = rank(X^+XX^+) \leq rank(X) = rank(XX^+X) \leq rank(X^+)$

$\Rightarrow rank(X^+) = rank(X)$

(c) If $A = A^T$ and $A^2 = A$, then $A^+ = A$

Assume $A = A^T$ and $A^2 = A$, check the definition of M-P inverse.

$\begin{cases} AA = A = A^T = (AA)^T & \text{(i) \& (ii) hold} \\ AAA = AA = A & \text{(iii) \& (iv) hold} \end{cases}$

$\Rightarrow A^+ = A$