STAT5030 Assignment1 Solution

1. It suffices to prove the null space of $\boldsymbol{x}$ and $\boldsymbol{x}^\top\boldsymbol{x}$ (as linear transformation) are equal, that is for any vector $\boldsymbol{v}$, $\boldsymbol{x}\boldsymbol{v}=0$ if and only if $\boldsymbol{v}^\top\boldsymbol{x}^\top\boldsymbol{x}\boldsymbol{v}=\|\boldsymbol{x}\boldsymbol{v}\|_2^2=0$.

2. (a) If $\boldsymbol{P}=-\boldsymbol{Q}$, then $\boldsymbol{P}\boldsymbol{X}\boldsymbol{X}^\top\boldsymbol{P}^\top=\boldsymbol{Q}\boldsymbol{X}\boldsymbol{X}^\top\boldsymbol{Q}^\top$.

   (b) If $\boldsymbol{P}\boldsymbol{X}\boldsymbol{X}^\top=\boldsymbol{Q}\boldsymbol{X}\boldsymbol{X}^\top$, by question 1, $(\boldsymbol{P}-\boldsymbol{Q})^\top$ is in the null space of $\boldsymbol{X}\boldsymbol{X}^\top$ as well as the null space of $\boldsymbol{X}^\top$, thus $\boldsymbol{P}\boldsymbol{X}=\boldsymbol{Q}\boldsymbol{X}$.

3. Note

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{A}_{11}^- & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}=\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{21}\boldsymbol{A}_{11}^-\boldsymbol{A}_{12} \end{bmatrix}.$$

   By the definition of generalized inverse, it suffices to verify $\boldsymbol{A}_{21}\boldsymbol{A}_{11}^-\boldsymbol{A}_{12}=\boldsymbol{A}_{22}$. Since the row rank of $[\boldsymbol{A}_{11},\boldsymbol{A}_{12}]$ and $\boldsymbol{A}$ are both r, then there exists a $(n-r)\times r$-matrix $\boldsymbol{P}$ such that $\boldsymbol{P}[\boldsymbol{A}_{11},\boldsymbol{A}_{12}]=[\boldsymbol{A}_{21},\boldsymbol{A}_{22}]$. And it can be found that $\boldsymbol{P}=\boldsymbol{A}_{21}\boldsymbol{A}_{11}^-$, and $\boldsymbol{P}\boldsymbol{A}_{12}=\boldsymbol{A}_{22}$.

4. (a) Find the Moore-Penrose inverse.

   (b) Find a generalized inverse different from Moore-penrose inverse.

5. (a) Since $(\boldsymbol{x}^\top\boldsymbol{x})^-$, $\boldsymbol{I_n}$ and $\boldsymbol{J}$ are symmetric, then $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ and $\boldsymbol{D}$ are symmetric. And by the definition of matrix $\boldsymbol{x},\boldsymbol{J}$ and generalized inverse matrices, it is not hard to verify $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ and $\boldsymbol{D}$ are idempotent.

   (b) $\operatorname{rank}(\boldsymbol{A})=\operatorname{tr}(\boldsymbol{A})=\operatorname{rank}(\boldsymbol{x})=k$;

   $\operatorname{rank}(\boldsymbol{B})=\operatorname{tr}(\boldsymbol{B})=\operatorname{tr}(\boldsymbol{I_n}-\boldsymbol{A})=\operatorname{tr}(\boldsymbol{I_n})-\operatorname{tr}(\boldsymbol{A})=n-k$;

   $\operatorname{rank}(\boldsymbol{C})=\operatorname{tr}(\boldsymbol{C})=\operatorname{tr}(\boldsymbol{A}-\frac{\boldsymbol{1}}{\boldsymbol{n}}\boldsymbol{J})=\operatorname{tr}(\boldsymbol{A})-\operatorname{tr}(\frac{\boldsymbol{1}}{\boldsymbol{n}}\boldsymbol{J})=k-1$;

   $\operatorname{rank}(\boldsymbol{D})=\operatorname{tr}(\boldsymbol{D})=\operatorname{tr}(\boldsymbol{I_n}-\frac{\boldsymbol{1}}{\boldsymbol{n}}\boldsymbol{J})=\operatorname{tr}(\boldsymbol{I_n})-\operatorname{tr}(\frac{\boldsymbol{1}}{\boldsymbol{n}}\boldsymbol{J})=n-1$;

6. (a) A symmetric generalized inverse for $\boldsymbol{A}$ could be

$$\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (b) A nonsymmetric generalized inverse for $\boldsymbol{A}$ could be

$$\begin{bmatrix} 1/2 & -1/2 & -1 \\ -1/2 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

7. If $\boldsymbol{x}$ is a vector such that $\boldsymbol{Ax} = \boldsymbol{c}$, then $[\boldsymbol{A}, \boldsymbol{c}] = \boldsymbol{A}[\boldsymbol{I}, \boldsymbol{x}]$ and $\text{rank}(\boldsymbol{A}) \leq \text{rank}([\boldsymbol{A}, \boldsymbol{c}]) = \text{rank}(\boldsymbol{A}[\boldsymbol{I}, \boldsymbol{x}]) \leq \text{rank}(\boldsymbol{A})$. If $\text{rank}(\boldsymbol{A}) = \text{rank}([\boldsymbol{A}, \boldsymbol{c}])$, then the vector $\boldsymbol{c}$ can be linearly represented by the column vectors of $\boldsymbol{A}$, that is there exists a vector $\boldsymbol{x}$ such that $\boldsymbol{Ax} = \boldsymbol{c}$.

8. If $\boldsymbol{x}$ is a vector such that $\boldsymbol{Ax} = \boldsymbol{c}$, then $\boldsymbol{AA^-c} = \boldsymbol{AA^-Ax} = \boldsymbol{Ax} = \boldsymbol{c}$ by the definition of $\boldsymbol{A}^-$. If $\boldsymbol{AA^-c} = \boldsymbol{c}$ holds for any generalized inverse $\boldsymbol{A}^-$ of $\boldsymbol{A}$, let $\boldsymbol{x} = \boldsymbol{A^-c}$, then $\boldsymbol{x}$ is a solution to the system of equations $\boldsymbol{Ax} = \boldsymbol{c}$.

9. By the definition of generalized inverse of $\boldsymbol{A}$, we have $\boldsymbol{AA^-A} = \boldsymbol{A}$, which is $\boldsymbol{A(A^-A - I_p) = 0}$. If $\boldsymbol{A}$ is $n \times p$ of rank $p < n$, then the column vectors of $\boldsymbol{A}$ have full rank, and the system of equations $\boldsymbol{Ax} = \boldsymbol{0}$ has only zero solution, implying $\boldsymbol{A^-A - I_p} = 0$.

10. Let $\boldsymbol{X}$ be $m \times n$, $\boldsymbol{X}^-$is the corresponding generalized inverse, and $r(\boldsymbol{X}) = k > 0$. Then:

    (a) $r(\boldsymbol{X}^-) \geq r(\boldsymbol{XX^-X}) = r(\boldsymbol{X}) = k$.

    (b) $\boldsymbol{X^-X}$ and $\boldsymbol{XX^-}$ are idempotent.

    (c) $k = r(\boldsymbol{X}) \geq r(\boldsymbol{X^-X}) = r(\boldsymbol{XX^-}) \geq r(\boldsymbol{XX^-X}) = k$.

    (d) $\boldsymbol{X^-X} = \boldsymbol{I}$ if and only if $r(\boldsymbol{X}) = n$. (See Question 9)

    (e) $tr(\boldsymbol{X^-X}) = tr(\boldsymbol{XX^-}) = r(\boldsymbol{XX^-}) = r(\boldsymbol{X}) = k$.

    (f) $\boldsymbol{X}^-$ is any G-inverse of $\boldsymbol{X}$, then $(\boldsymbol{XX^-X})^\top = \boldsymbol{X}^\top$ implying $(\boldsymbol{X}^-)^\top$ is a G-inverse of $\boldsymbol{X}^\top$.

11. For $\boldsymbol{K} = \boldsymbol{X(X^\top X)^-X^\top}$, then:

    (a) $\boldsymbol{K} = \boldsymbol{K}^\top$ by question 10 (f), $\boldsymbol{K} = \boldsymbol{K}^2$(Symmetric Idempotent).

    (b) $r(\boldsymbol{X}) \geq r(\boldsymbol{K}) \geq r(\boldsymbol{X^\top KX}) = r(\boldsymbol{X^\top X}) = r(\boldsymbol{X}) = r$.

    (c) It can be verified that $(\boldsymbol{KX} - \boldsymbol{X})^\top(\boldsymbol{KX} - \boldsymbol{X}) = 0$.

    (d) $(\boldsymbol{X^\top X})^-\boldsymbol{X}^\top$ is a G-inverse of $\boldsymbol{X}$ for any G-inverse of $\boldsymbol{X^\top X}$ by 11 (c).

12. (a) Let $\boldsymbol{A}_1^+$ and $\boldsymbol{A}_2^+$ be two Moore-Penrose inverse of $\boldsymbol{A}$. By the definition and properties of Moore-Penrose inverse, $\boldsymbol{AA_1^+} = (\boldsymbol{AA_2^+A})\boldsymbol{A_1^+} = (\boldsymbol{AA_2^+})(\boldsymbol{AA_1^+}) = (\boldsymbol{AA_2^+})^\top(\boldsymbol{AA_1^+})^\top = (\boldsymbol{A_2^+})^\top\boldsymbol{A}^\top(\boldsymbol{A_1^+})^\top\boldsymbol{A}^\top = (\boldsymbol{A_2^+})^\top(\boldsymbol{AA_1^+A})^\top = (\boldsymbol{A_2^+})^\top\boldsymbol{A}^\top = (\boldsymbol{AA_2^+})^\top = \boldsymbol{AA_2^+}$. And $\boldsymbol{A_1^+A} = \boldsymbol{A_2^+A}$ by the similar argument. Then $\boldsymbol{A_1^+} = \boldsymbol{A_1^+AA_1^+} = \boldsymbol{A_1^+AA_2^+} = \boldsymbol{A_2^+AA_2^+} = \boldsymbol{A_2^+}$.

    (b) $r(\boldsymbol{A}^+) \leq r(\boldsymbol{A^+AA^+}) \leq r(\boldsymbol{A}) = r(\boldsymbol{AA^+A}) \leq r(\boldsymbol{A}^+)$.

    (c) If $\boldsymbol{A}$ is symmetric idempotent, it is easy to verify that $\boldsymbol{A}$ itself satisfy the definition of Moore-Penrose inverse. By the uniqueness, $\boldsymbol{A}^+ = \boldsymbol{A}$.