STAT5020 Assignment1

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Question 1

a) The measurement equation is

$$\begin{bmatrix} \text{PVD} \\ \text{IHD} \\ \text{CVA} \\ \text{BMI} \\ \text{WHR} \\ \text{SBP} \\ \text{DBP} \\ \ln\text{TG} \\ \text{LDL-C} \\ \text{PON11} \\ \text{PON12} \\ \text{FBG} \\ \text{SELE2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda_{21} & 0 & 0 & 0 & 0 & 0 \\ \lambda_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{73} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{94} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{11.5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{12.6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix}, \tag{1}$$

and the structural equation is

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_6\xi_2\xi_4 + \gamma_7\xi_3\xi_5 + \gamma_8\xi_4\xi_5 + \delta_i. \tag{2}$$

b) BMI and WHR are measured to form explanatory latent variable Body Shape ξ_1.

SBP and DBP are measured to form explanatory latent variable Blood Pressure ξ_2.

LDL-C and lnTG are measured to form explanatory latent variable Lipid Control ξ_3.

PON11 and PON12 are measured to form explanatory latent variable Gene-Lipid Control ξ_4.

FGB and SELE2 are measured to form explanatory latent variable Gene-Lipid Control ξ_5.

The outcome latent variable (η), measuring cardiovascular heart disease, is affected by all the 5 explanatory latent variables above and 3 interaction terms $\xi_2\xi_4$, $\xi_3\xi_5$ and $\xi_4\xi_5$.

PVD, IHD and CVA are directly related to the outcome latent variable.

c) The model is identifiable, since each column of loading matrix contains a fixed entry 1.

d) In the structural equation, the effects of explanatory latent variables Blood Pressure (ξ_1) and Body Shape (ξ_2) and the interaction term between Blood Pressure and Gene-inflammatory ($\xi_2\xi_4$) on the outcome latent variable (η) are weak. Except these three terms, all the other terms are significantly associated with the outcome latent variable.

In the measurement equation, only the factor loading $\lambda_{11.5}$ and $\lambda_{12.8}$ are not significant, and the rest relationship between the observed data and the latent variables are strong.

For the fixed parameters, the observed data (e.g. SBP) is directly associated with the corresponding latent variables (e.g. Blood Pressure).

e) M1:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5. \tag{3}$$

M2:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_6\xi_2\xi_4. \tag{4}$$

M3:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_7\xi_3\xi_5. \tag{5}$$

M4:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_8\xi_4\xi_5. \tag{6}$$

M5:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_6\xi_2\xi_4 + \gamma_7\xi_3\xi_5. \tag{7}$$

M6:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_6\xi_2\xi_4 + \gamma_8\xi_4\xi_5. \tag{8}$$

M7:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_4 + \gamma_5\xi_5 + \gamma_7\xi_3\xi_5 + \gamma_8\xi_4\xi_5. \tag{9}$$

f) The main difference between SEMs and conventional regression models is the presence of latent variables in SEMs, which greatly improves the flexibility of SEMs.

Question 2

a) The matrix form of the measurement equation is

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \\ y_{6i} \\ y_{7i} \\ y_{8i} \\ y_{9i} \end{bmatrix} = \begin{bmatrix} \mu_1 & a_1 \\ \mu_2 & a_2 \\ \mu_3 & a_3 \\ \mu_4 & a_4 \\ \mu_5 & a_5 \\ \mu_6 & a_6 \\ \mu_7 & a_7 \\ \mu_8 & a_8 \\ \mu_9 & a_9 \end{bmatrix} \begin{bmatrix} 1 \\ c_i \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ \lambda_{31} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_{52} & 0 \\ 0 & \lambda_{62} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda_{83} \\ 0 & 0 & \lambda_{93} \end{bmatrix} \begin{bmatrix} \eta_i \\ \xi_{1i} \\ \xi_{2i} \end{bmatrix} + \begin{bmatrix} \epsilon_{1i} \\ \epsilon_{2i} \\ \epsilon_{3i} \\ \epsilon_{4i} \\ \epsilon_{5i} \\ \epsilon_{6i} \\ \epsilon_{7i} \\ \epsilon_{8i} \\ \epsilon_{9i} \end{bmatrix}, \tag{10}$$

and the structural equation is

$$\eta_i = b \times d_i + \gamma_1 \times \xi_{1i} + \gamma_2 \times \xi_{2i} + \gamma_3 \times \xi_{1i} \times \xi_{2i} + \gamma_4 \times \xi_{1i}^2 + \delta_i. \tag{11}$$

The following Figure 1 is path diagram.

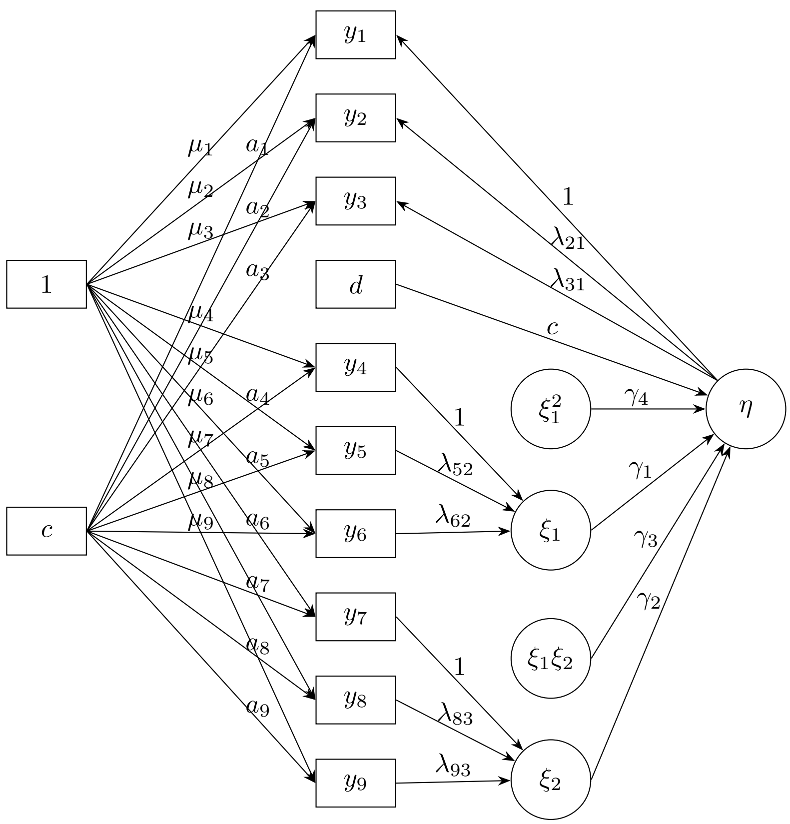

Figure 1: Path Diagram

Denote the effect coefficient matrices of measurement equation and structural equation as

$$\mathbf{\Lambda}_y = \begin{bmatrix} \vdots \\ \mathbf{\Lambda}_{yk}^\top \\ \vdots \end{bmatrix} = \begin{bmatrix} \mu_1 & a_1 & 1 & 0 & 0 \\ \mu_2 & a_2 & \lambda_{21} & 0 & 0 \\ \mu_3 & a_3 & \lambda_{31} & 0 & 0 \\ \mu_4 & a_4 & 0 & 1 & 0 \\ \mu_5 & a_5 & 0 & \lambda_{52} & 0 \\ \mu_6 & a_6 & 0 & \lambda_{62} & 0 \\ \mu_7 & a_7 & 0 & 0 & 1 \\ \mu_8 & a_8 & 0 & 0 & \lambda_{83} \\ \mu_9 & a_9 & 0 & 0 & \lambda_{93} \end{bmatrix} \tag{12}$$

and

$$\mathbf{\Lambda}_\omega^\top = \begin{bmatrix} b & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}. \tag{13}$$

Denote the matrices containing covariates and factors in measurement equation and structural equation as

$$\mathbf{U} = \begin{bmatrix} \mathbf{1}^\top \\ \boldsymbol{\mu}^\top \\ \mathbf{\Omega} \end{bmatrix} = \begin{bmatrix} \mathbf{1}^\top \\ \boldsymbol{\mu}^\top \\ \boldsymbol{\eta}^\top \\ \mathbf{\Xi} \end{bmatrix} = \begin{bmatrix} \mathbf{1}^\top \\ \boldsymbol{\mu}^\top \\ \boldsymbol{\eta}^\top \\ \boldsymbol{\xi}_1^\top \\ \boldsymbol{\xi}_2^\top \end{bmatrix}, \quad \mathbf{\Xi} = \begin{bmatrix} \cdots & \mathbf{\Xi}_i & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \xi_{1i} & \cdots \\ \cdots & \xi_{2i} & \cdots \end{bmatrix}, \tag{14}$$

and

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{d}^\top \\ \boldsymbol{\xi}_1^\top \\ \boldsymbol{\xi}_2^\top \\ \boldsymbol{\xi}_{12}^\top \\ \boldsymbol{\xi}_{11}^\top \end{bmatrix}, \quad \boldsymbol{\xi}_{12} = \begin{bmatrix} \xi_{1i}\xi_{2i} \end{bmatrix}, \quad \boldsymbol{\xi}_{11} = \begin{bmatrix} \xi_{1i}^2 \end{bmatrix}. \tag{15}$$

Denote the covariance matrices and variance of explanatory latent variable, $[\boldsymbol{\epsilon}_i]$ and δ_i as

$$\mathbf{\Phi} = \mathrm{Cov}(\begin{bmatrix} \xi_{1i} \\ \xi_{2i} \end{bmatrix}), \quad \mathbf{\Psi}_\epsilon = \mathrm{Cov}(\boldsymbol{\epsilon}_i), \quad \psi_\delta = \mathrm{Var}(\delta_i). \tag{16}$$

Set the prior for $\mathbf{\Phi}$ as

$$\mathbf{\Phi} \sim \mathrm{IW}_q[\mathbf{R}_0^{-1}, \rho_0], \quad q = 2, \tag{17}$$

the priors for $\mathrm{diag}(\mathbf{\Psi}_\epsilon) = [\psi_{\epsilon k}]$ and $\mathbf{\Lambda}_{yk}$ as

$$\psi_{\epsilon k} \sim \mathrm{IG}[\alpha_{0k}, \beta_{0k}], \quad \mathbf{\Lambda}_{yk} \,|\, \psi_{\epsilon k} \sim \mathrm{N}(\mathbf{\Lambda}_{0yk}, \psi_{\epsilon k}\mathbf{H}_{0yk}), \tag{18}$$

and the priors for ψ_δ and $\mathbf{\Lambda}_\omega$ as

$$\psi_\delta \sim \mathrm{IG}[\alpha_{0\delta}, \beta_{0\delta}], \quad \mathbf{\Lambda}_\omega \,|\, \psi_\delta \sim \mathrm{N}(\mathbf{\Lambda}_{0\omega}, \psi_\delta \mathbf{H}_{0\omega}). \tag{19}$$

Denote the vector of all the parameters as

$$\boldsymbol{\theta} = (\mathbf{\Phi}, \mathbf{\Lambda}_y, \mathbf{\Psi}_\epsilon, \mathbf{\Lambda}_\omega, \psi_\delta). \tag{20}$$

We can separate these parameters into three parts $(\mathbf{\Phi})$, $(\mathbf{\Lambda}_y, \mathbf{\Psi}_\epsilon)$, $(\mathbf{\Lambda}_\omega, \psi_\delta)$ to consider the full conditional distribution.

For the first part $\mathbf{\Phi}$, since

$$
\begin{aligned}
p(\mathbf{\Phi} \mid \mathbf{Y}, \mathbf{\Omega}) &\propto p(\mathbf{\Omega} \mid \mathbf{\Phi}, \mathbf{Y}) \times p(\mathbf{\Phi} \mid \mathbf{Y}) \\
&= p(\mathbf{\Omega} \mid \mathbf{\Phi}) \times p(\mathbf{\Phi}) \\
&\propto p(\mathbf{\Xi} \mid \mathbf{\Phi}) \times p(\mathbf{\Phi}) \\
&\propto \left\{ |\mathbf{\Phi}|^{-n/2} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \mathbf{\Xi}_i^\top \mathbf{\Phi}^{-1} \mathbf{\Xi}_i \right] \right\} \\
&\qquad \times \left[|\mathbf{\Phi}|^{-(\rho_0+q+1)/2} \exp\left(-\frac{1}{2} \mathrm{tr}\left[\mathbf{\Phi}^{-1} \mathbf{R}_0^{-1} \right] \right) \right] \\
&= |\mathbf{\Phi}|^{-(n+\rho_0+q+1)/2} \exp\left\{ -\frac{1}{2} \mathrm{tr}\left[\mathbf{\Phi}^{-1} \left(\mathbf{\Xi} \cdot \mathbf{\Xi}^\top + \mathbf{R}_0^{-1} \right) \right] \right\},
\end{aligned}
\tag{21}
$$

then the full conditional distribution of $\mathbf{\Phi}$ is

$$
\mathbf{\Phi} \mid \mathbf{Y}, \mathbf{\Omega} \sim \mathrm{IW}_q \left[\left(\mathbf{\Xi} \cdot \mathbf{\Xi}^\top + \mathbf{R}_0^{-1} \right), (n + \rho_0) \right]. \tag{22}
$$

For the second part $(\psi_{\epsilon k}, \mathbf{\Lambda}_{yk})$, since

$$
\begin{aligned}
p(\psi_{\epsilon k}, \mathbf{\Lambda}_{yk} \mid \mathbf{Y}, \mathbf{\Omega}) &\propto p(\mathbf{Y} \mid \psi_{\epsilon k}, \mathbf{\Lambda}_{yk}, \mathbf{\Omega}) \times p(\psi_{\epsilon k}, \mathbf{\Lambda}_{yk} \mid \mathbf{\Omega}) \\
&\propto p(\mathbf{Y} \mid \psi_{\epsilon k}, \mathbf{\Lambda}_{yk}, \mathbf{U}) \times p(\mathbf{\Lambda}_{yk} \mid \psi_{\epsilon k}) \times p(\psi_{\epsilon k}) \\
&\propto (\psi_{\epsilon k})^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\psi_{\epsilon k}} \left(\mathbf{Y}_k^\top - \mathbf{\Lambda}_{yk}^\top \mathbf{U} \right) \left(\mathbf{Y}_k - \mathbf{U}^\top \mathbf{\Lambda}_{yk} \right) \right\} \\
&\qquad \times |\psi_{\epsilon k} \mathbf{H}_{0yk}|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\mathbf{\Lambda}_{yk}^\top - \mathbf{\Lambda}_{0yk}^\top) \left(\psi_{\epsilon k} \mathbf{H}_{0yk} \right)^{-1} \left(\mathbf{\Lambda}_{yk} - \mathbf{\Lambda}_{0yk} \right) \right\} \\
&\qquad \times \psi_{\epsilon k}^{-(\alpha_{0k}+1)} \exp\left\{ -\frac{1}{\psi_{\epsilon k}} \beta_{0k} \right\} \\
&\propto (\psi_{\epsilon k})^{-(\frac{n}{2}+\alpha_{0k}+1)} \exp\left\{ -\frac{1}{\psi_{\epsilon k}} \left[\beta_{0k} + \frac{1}{2} \left(\mathbf{Y}_k^\top \mathbf{Y}_k + \mathbf{\Lambda}_{0yk}^\top \mathbf{H}_{0yk}^{-1} \mathbf{\Lambda}_{0yk} - \mathbf{x}_k^\top \mathbf{A}_k^{-1} \mathbf{x}_k \right) \right] \right\} \\
&\qquad \times |\psi_{\epsilon k} \mathbf{A}_k|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\mathbf{\Lambda}_{yk}^\top - \mathbf{x}_k^\top) \left(\psi_{\epsilon k} \mathbf{A}_k \right)^{-1} \left(\mathbf{\Lambda}_{yk} - \mathbf{x}_k \right) \right\},
\end{aligned}
\tag{23}
$$

where

$$
\begin{cases}
\mathbf{A}_k = \left(\mathbf{H}_{0yk}^{-1} + \mathbf{U}\mathbf{U}^\top \right)^{-1} \\
\mathbf{x}_k = \mathbf{A}_k \left(\mathbf{H}_{0yk}^{-1} \mathbf{\Lambda}_{0yk} + \mathbf{U}\mathbf{Y}_k \right)
\end{cases},
\tag{24}
$$

then the full conditional distribution of $(\psi_{\epsilon k}, \mathbf{\Lambda}_{yk})$ is

$$
\begin{cases}
\psi_{\epsilon k} \mid \mathbf{Y}, \mathbf{\Omega} \sim \mathrm{IG}\left[\left(\alpha_{0k} + \frac{n}{2} \right), \beta_{0k} + \frac{1}{2} \left(\mathbf{Y}_k^\top \mathbf{Y}_k + \mathbf{\Lambda}_{0yk}^\top \mathbf{H}_{0yk}^{-1} \mathbf{\Lambda}_{0yk} - \mathbf{x}_k^\top \mathbf{A}_k^{-1} \mathbf{x}_k \right) \right] \\
\mathbf{\Lambda}_{yk} \mid \psi_{\epsilon k}, \mathbf{Y}, \mathbf{\Omega} \sim \mathrm{N}\left(\mathbf{x}_k, \psi_{\epsilon k} \mathbf{A}_k \right).
\end{cases}
\tag{25}
$$

For the last part $(\boldsymbol{\Lambda}_\omega, \psi_\delta)$, which is similar to the previous part, since

$$
\begin{aligned}
p(\psi_\delta, \boldsymbol{\Lambda}_\omega \mid \mathbf{Y}, \boldsymbol{\Omega}) &\propto p(\boldsymbol{\eta} \mid \psi_\delta, \boldsymbol{\Lambda}_\omega, \boldsymbol{\Xi}) \times p(\boldsymbol{\Lambda}_\omega \mid \psi_\delta) \times p(\psi_\delta) \\
&\propto (\psi_\delta)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\psi_\delta}\left(\boldsymbol{\eta}^\top - \boldsymbol{\Lambda}_\omega^\top \boldsymbol{V}\right)\left(\boldsymbol{\eta} - \boldsymbol{V}^\top \boldsymbol{\Lambda}_\omega\right)\right\} \\
&\quad \times |\psi_\delta \mathbf{H}_{0\omega}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{\Lambda}_\omega^\top - \boldsymbol{\Lambda}_{0\omega}^\top)\left(\psi_\delta \mathbf{H}_{0\omega}\right)^{-1}\left(\boldsymbol{\Lambda}_\omega - \boldsymbol{\Lambda}_{0\omega}\right)\right\} \\
&\quad \times \psi_\delta^{-(\alpha_{0\delta}+1)} \exp\left\{-\frac{1}{\psi_\delta}\beta_{0\delta}\right\} \\
&\propto (\psi_\delta)^{-(\frac{n}{2}+\alpha_{0\delta}+1)} \exp\left\{-\frac{1}{\psi_\delta}\left[\beta_{0\delta} + \frac{1}{2}\left(\boldsymbol{\eta}^\top \boldsymbol{\eta} + \boldsymbol{\Lambda}_{0\omega}^\top \mathbf{H}_{0\omega}^{-1} \boldsymbol{\Lambda}_{0\omega} - \mathbf{z}^\top \mathbf{B}^{-1} \mathbf{z}\right)\right]\right\} \\
&\quad \times |\psi_\delta \mathbf{B}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{\Lambda}_\omega^\top - \mathbf{z}^\top)\left(\psi_\delta \mathbf{B}\right)^{-1}\left(\boldsymbol{\Lambda}_\omega - \mathbf{z}\right)\right\},
\end{aligned} \tag{26}
$$

where

$$
\begin{cases}
\mathbf{B} = \left(\mathbf{H}_{0\omega}^{-1} + \boldsymbol{V}\boldsymbol{V}^\top\right)^{-1} \\
\mathbf{z} = \mathbf{B}\left(\mathbf{H}_{0\omega}^{-1} \boldsymbol{\Lambda}_{0\omega} + \boldsymbol{V}\boldsymbol{\eta}\right),
\end{cases} \tag{27}
$$

then the full conditional distribution of $(\psi_\delta, \boldsymbol{\Lambda}_\omega)$ is

$$
\begin{cases}
\psi_\delta \mid \mathbf{Y}, \boldsymbol{\Omega} \sim \mathrm{IG}\left[\left(\alpha_{0\delta} + \frac{n}{2}\right), \beta_{0\delta} + \frac{1}{2}\left(\boldsymbol{\eta}^\top \boldsymbol{\eta} + \boldsymbol{\Lambda}_{0\omega}^\top \mathbf{H}_{0\omega}^{-1} \boldsymbol{\Lambda}_{0\omega} - \mathbf{z}^\top \mathbf{B}^{-1} \mathbf{z}\right)\right] \\
\boldsymbol{\Lambda}_\omega \mid \psi_\delta, \mathbf{Y}, \boldsymbol{\Omega} \sim \mathrm{N}\left(\mathbf{z}, \psi_\delta \mathbf{B}\right).
\end{cases} \tag{28}
$$

b) The true values for the model parameters in the model were taken to be:

$$
\begin{cases}
\mu_1 = \mu_2 = \cdots = \mu_9 = 0, \\
a_1 = a_2 = a_3 = 0.5, \\
a_4 = a_5 = a_6 = -0.5, \\
a_7 = a_8 = a_9 = 0.8, \\
\lambda_{21} = \lambda_{52} = \lambda_{83} = 0.8, \\
\lambda_{31} = \lambda_{62} = \lambda_{93} = 0.5, \\
\psi_{\epsilon 1} = \psi_{\epsilon 2} = \psi_{\epsilon 3} = 0.3, \\
\psi_{\epsilon 4} = \psi_{\epsilon 5} = \psi_{\epsilon 6} = 0.5, \\
\psi_{\epsilon 7} = \psi_{\epsilon 8} = \psi_{\epsilon 9} = 0.4, \\
b = -1, \\
\gamma_1 = \gamma_2 = 0.5, \\
\gamma_3 = \gamma_4 = 0.3, \\
\phi_{11} = \phi_{22} = 1, \\
\psi_\delta = 0.36.
\end{cases} \tag{29}
$$

and replicate 10 times with sample size 500.

c) Draw the trace plots of all sampled parameters in the program.

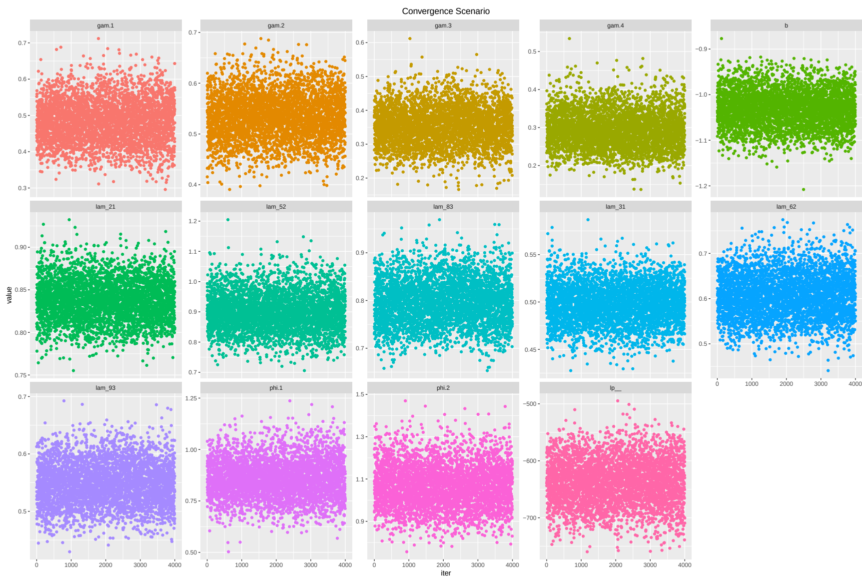

Figure 2: Convergence Plot

d) The following table contains the Bias and the RMS of the Bayes estimation with 10 replicates,

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
BIAS	0.00822	0.0091	0.00788	0.00099	0.01099	0.01103	0.00594	0.01527	0.01101
RMS	0.06954	0.06605	0.03654	0.04821	0.04776	0.05415	0.04178	0.03081	0.04105
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
BIAS	-0.00145	-0.02122	-0.00363	0.00632	-0.00914	-0.00773	-0.03108	-0.01178	-0.009
RMS	0.03911	0.0422	0.03261	0.04021	0.03029	0.02136	0.06162	0.04252	0.0368
	$\psi_{\epsilon 1}$	$\psi_{\epsilon 2}$	$\psi_{\epsilon 3}$	$\psi_{\epsilon 4}$	$\psi_{\epsilon 5}$	$\psi_{\epsilon 6}$	$\psi_{\epsilon 7}$	$\psi_{\epsilon 8}$	$\psi_{\epsilon 9}$
BIAS	0.01088	-0.01552	0.00464	0.00275	-0.01114	-0.00792	0.02079	-0.01124	-0.00012
RMS	0.02744	0.02033	0.02164	0.0501	0.03436	0.04796	0.04853	0.03722	0.03118
	λ_{21}	λ_{52}	λ_{83}	λ_{31}	λ_{62}	λ_{93}	phi_1	phi_2	
BIAS	0.01744	0.0057	0.00947	-0.00154	0.01413	0.00318	0.05295	-0.01285	
RMS	0.02756	0.03619	0.03464	0.01497	0.0435	0.0313	0.12923	0.10044	
	b	γ_1	γ_2	γ_3	γ_4	ψ_δ			
BIAS	0.01331	0.02974	0.00195	0.00389	0.00804	-0.01396			
RMS	0.04137	0.08019	0.03984	0.05016	0.05678	0.03012			

Table 1: Bias and RMS of Bayes Estimation

e) Previous prior inputs:

$$\alpha_{0k} = \alpha_{0\delta} = 9, \beta_{0k} = \beta_{0\delta} = 4, \mathbf{H}_{0yk} = \boldsymbol{I}, \mathbf{H}_{0\omega} = \boldsymbol{I}, \tag{30}$$

and other location parameters, like $\boldsymbol{\Lambda}_{yk}$ and $\boldsymbol{\Lambda}_\omega$, equal to their true value.

Change partial prior inputs

$$\alpha_{0k} = \alpha_{0\delta} = 6, \beta_{0k} = \beta_{0\delta} = 10, \tag{31}$$

and keep the rest unchanged.

The following Table is the comparison of Bayes estimation results based on these two different priors.

Notice that the difference between the estimates from these two priors are negligible, thus the Bayesian analysis is not sensitive to the inputs.

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
prior 1	-0.0232	0.0348	-6e-04	-0.0284	0.0301	0.0043	-0.0712	-0.0876	-0.0964
prior 2	-0.0177	0.0423	0.0042	-0.0281	0.0305	0.0055	-0.0701	-0.0872	-0.0951
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
prior 1	0.5508	0.5147	0.5091	-0.4587	-0.4498	-0.4364	0.8178	0.8284	0.814
prior 2	0.5469	0.5124	0.5077	-0.4623	-0.4535	-0.4391	0.8181	0.8276	0.8137
	$\psi_{\epsilon 1}$	$\psi_{\epsilon 2}$	$\psi_{\epsilon 3}$	$\psi_{\epsilon 4}$	$\psi_{\epsilon 5}$	$\psi_{\epsilon 6}$	$\psi_{\epsilon 7}$	$\psi_{\epsilon 8}$	$\psi_{\epsilon 9}$
prior 1	0.3449	0.2984	0.3348	0.5698	0.5282	0.4798	0.3909	0.4135	0.3816
prior 2	0.3836	0.3355	0.3642	0.6142	0.5695	0.5126	0.4541	0.4471	0.4117
	λ_{21}	λ_{52}	λ_{83}	λ_{31}	λ_{62}	λ_{93}	phi_1	phi_2	
prior 1	0.8387	0.8956	0.7993	0.4985	0.6056	0.5496	0.8631	1.0617	
prior 2	0.8267	0.894	0.811	0.4918	0.6048	0.5588	0.8559	1.0245	
	b	γ_1	γ_2	γ_3	γ_4	ψ_δ			
prior 1	-1.0319	0.4811	0.5347	0.3544	0.2917	0.294			
prior 2	-1.0375	0.4759	0.5427	0.3514	0.2863	0.3661			

Table 2: Comparison of Different Prior Inputs

Question 3

Denote the original nonlinear SEM in previous question as $\mathbf{M}_0$:

$$\eta_i = b \times d_i + \gamma_1 \times \xi_{1i} + \gamma_2 \times \xi_{2i} + \gamma_3 \times \xi_{1i} \times \xi_{2i} + \gamma_4 \times \xi_{1i}^2 + \delta_i. \tag{32}$$

a) Considering the linear SEM counterpart $\mathbf{M}_1$:

$$\eta_i = b \times d_i + \gamma_1 \times \xi_{1i} + \gamma_2 \times \xi_{2i} + \delta_i. \tag{33}$$

Construct the link model $\mathbf{M}_t$:

$$\eta_i = b \times d_i + \gamma_1 \times \xi_{1i} + \gamma_2 \times \xi_{2i} + (1-t) \cdot \gamma_3 \times \xi_{1i} \times \xi_{2i} + (1-t) \cdot \gamma_4 \times \xi_{1i}^2 + \delta_i. \tag{34}$$

Since the estimated Bayes Factor is

$$2\,\widehat{\log \mathbf{B}_{01}} = -41.98. \tag{35}$$

then the original non-linear SEM is much better than the linear counterpart.

b) Considering another non-linear SEM $\mathbf{M}_2$:

$$\eta_i = b \times d_i + \gamma_1 \times \xi_{1i} + \gamma_2 \times \xi_{2i} + \gamma_3 \times \xi_{1i} \times \xi_{2i} + \gamma_4 \times \xi_{1i}^2 + \gamma_5 \times \xi_{i2}^2 + \delta_i. \tag{36}$$

Construct the link model $\mathbf{M}_t'$:

$$\eta_i = b \times d_i + \gamma_1 \times \xi_{1i} + \gamma_2 \times \xi_{2i} + \gamma_3 \times \xi_{1i} \times \xi_{2i} + \gamma_4 \times \xi_{1i}^2 + t \times \gamma_5 \times \xi_{i2}^2 + \delta_i. \tag{37}$$

Since the estimated Bayes Factor is

$$2\,\widehat{\log \mathbf{B}_{02}} = -2.12, \tag{38}$$

then the new nonlinear SEM is not better than the original one.