Rigid-Body Motions and Rotation Matrices

CS 6341 Robotics

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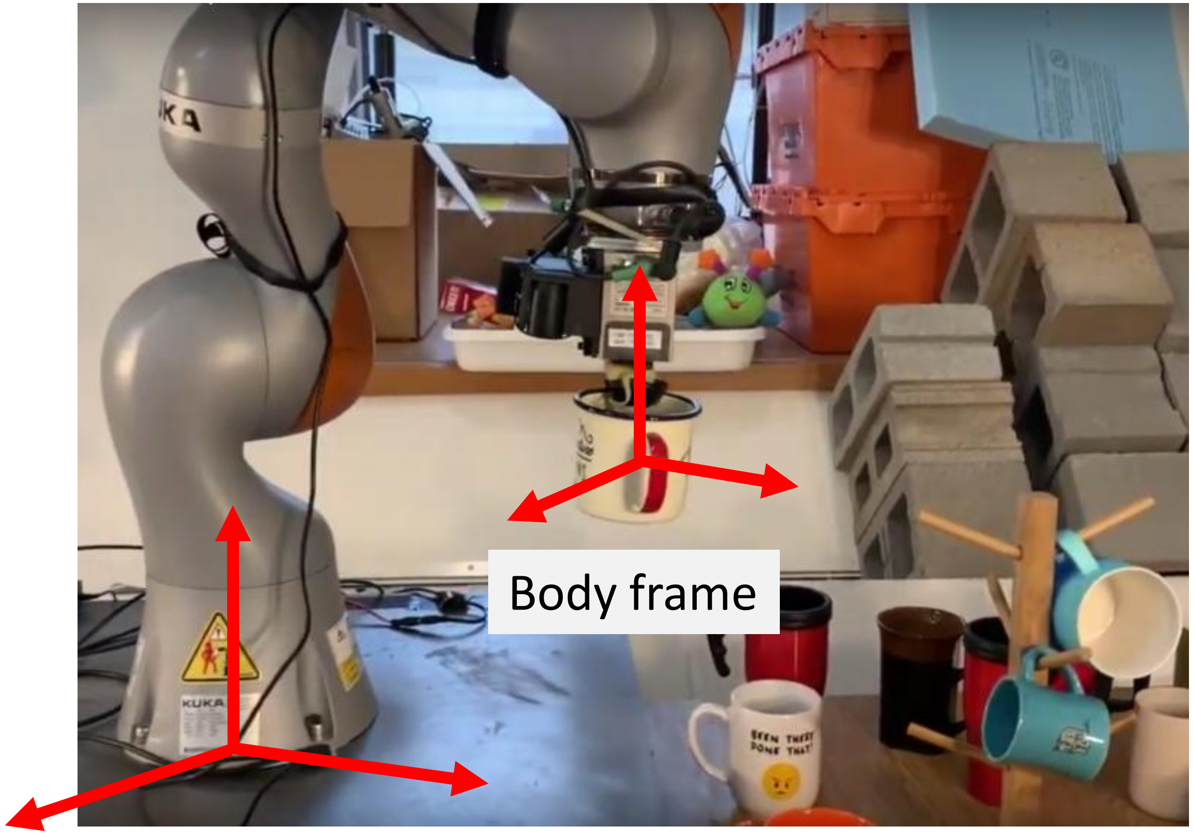
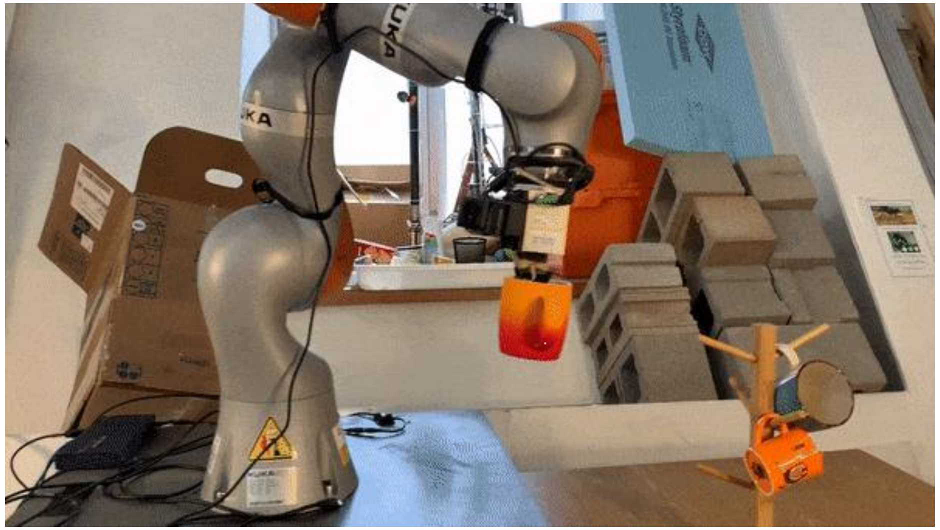
The University of Texas at Dallas

Rigid-Body Motions



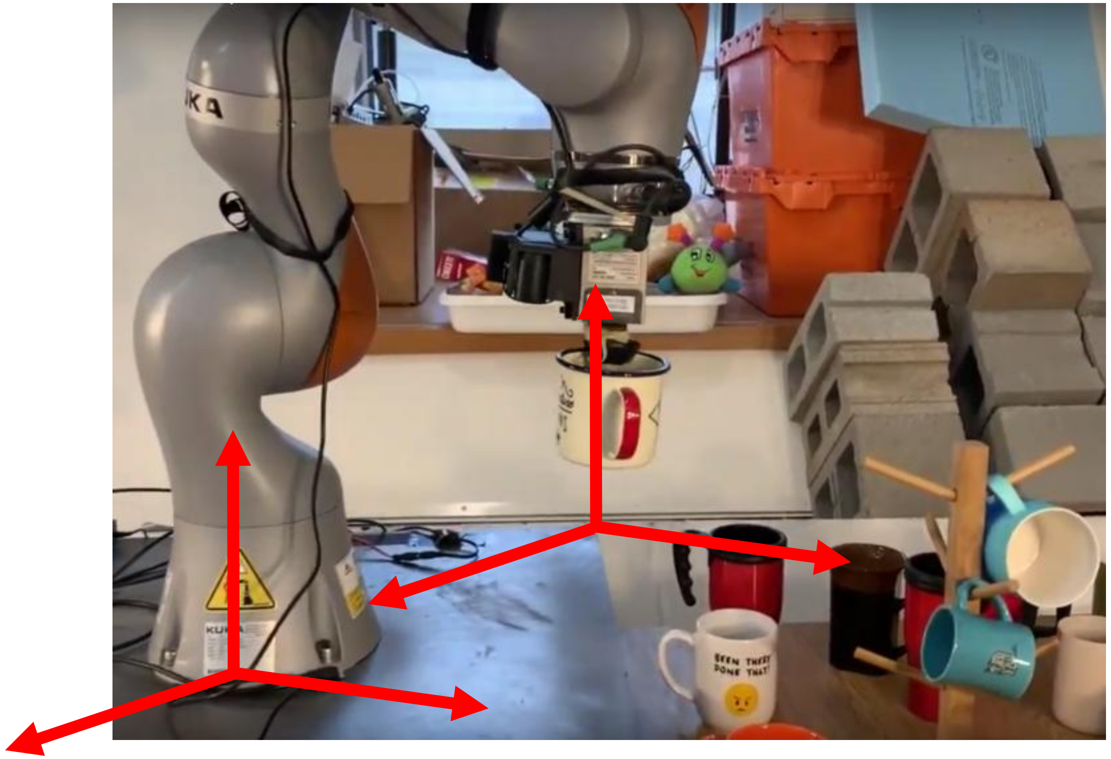
Space frame (fixed frame)



https://venturebeat.com/ai/mit-csail-refines-picker-robots-ability-to-handle-new-objects/

Reference Frames

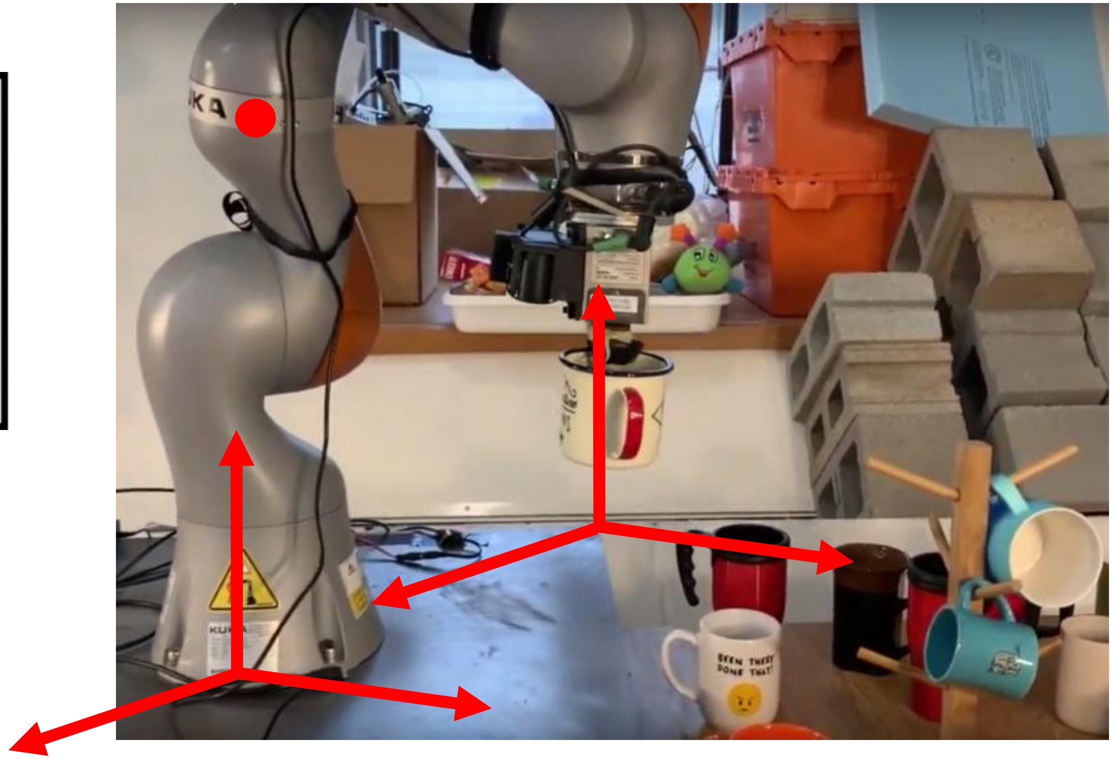
- A reference frame can be attached anywhere



More about Reference Frames

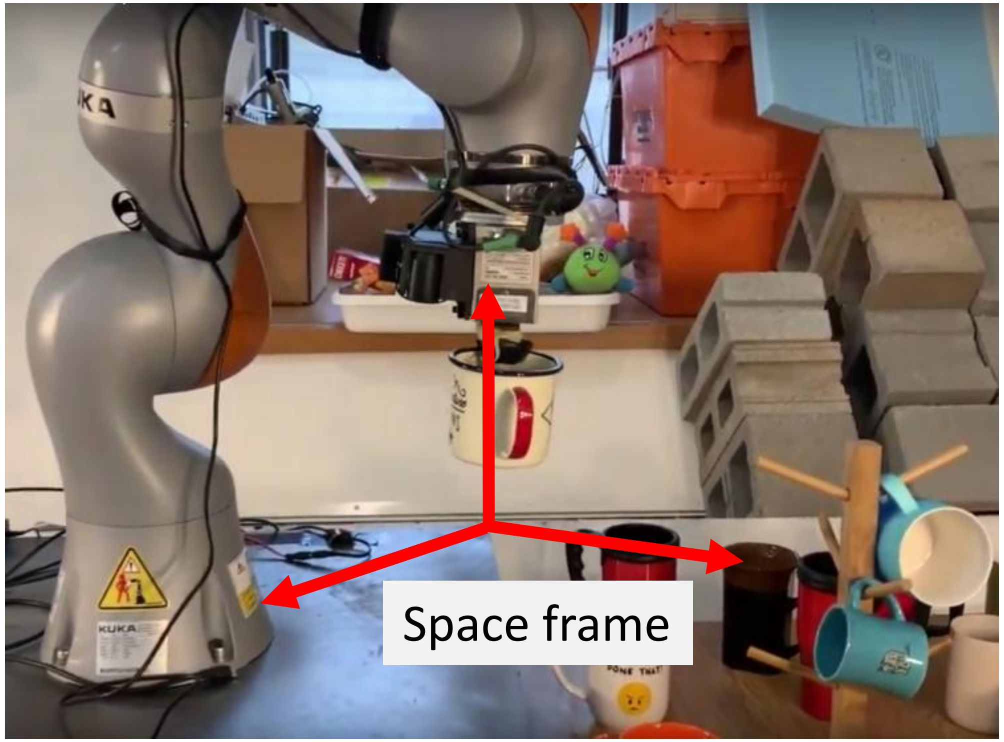
- Different reference frames result in different representations of the space and objects, but the underlying geometry is the same

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



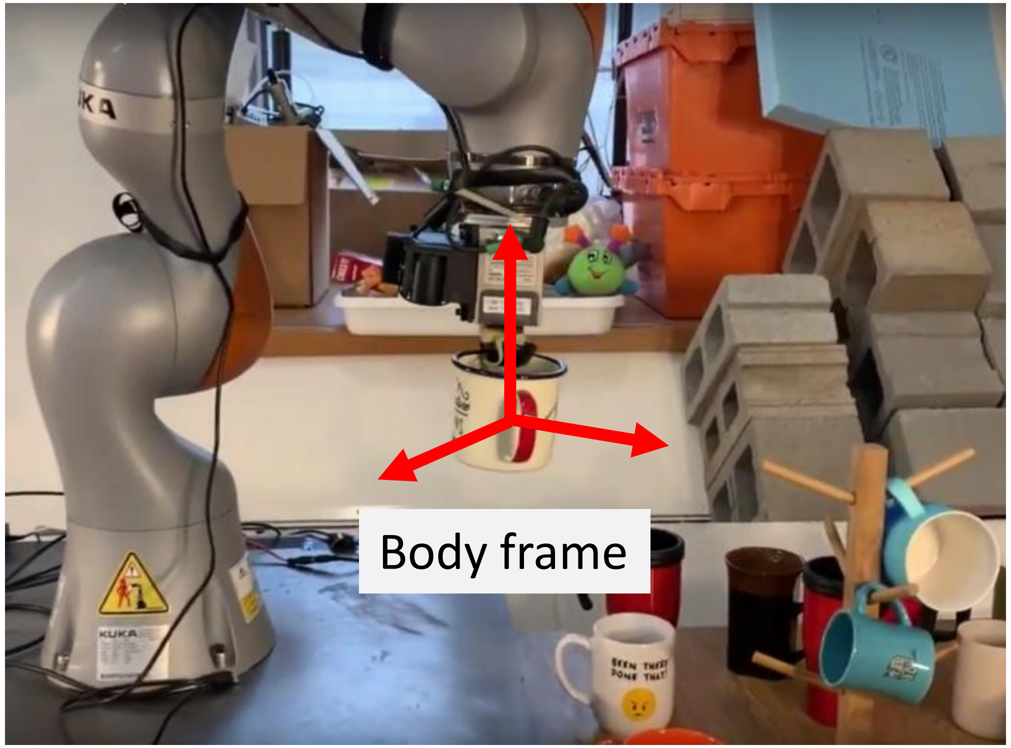
More about Reference Frames

- Always assume one stationary **fixed frame** or **space frame** {s}
    - E.g., a corner of a room



More about Reference Frames

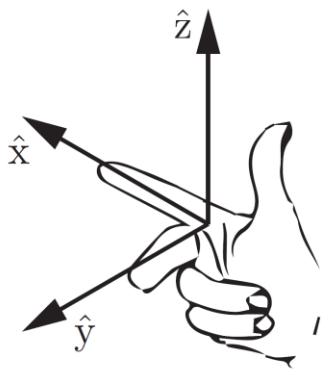
- **Body frame** {b} has been attached to some moving rigid body
  - E.g., origin on the center of mass of the body
  - No need to be on the physical body!

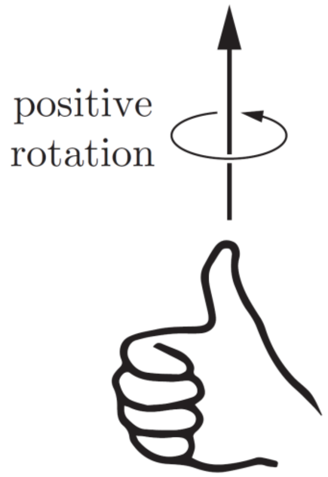


More about Reference Frames

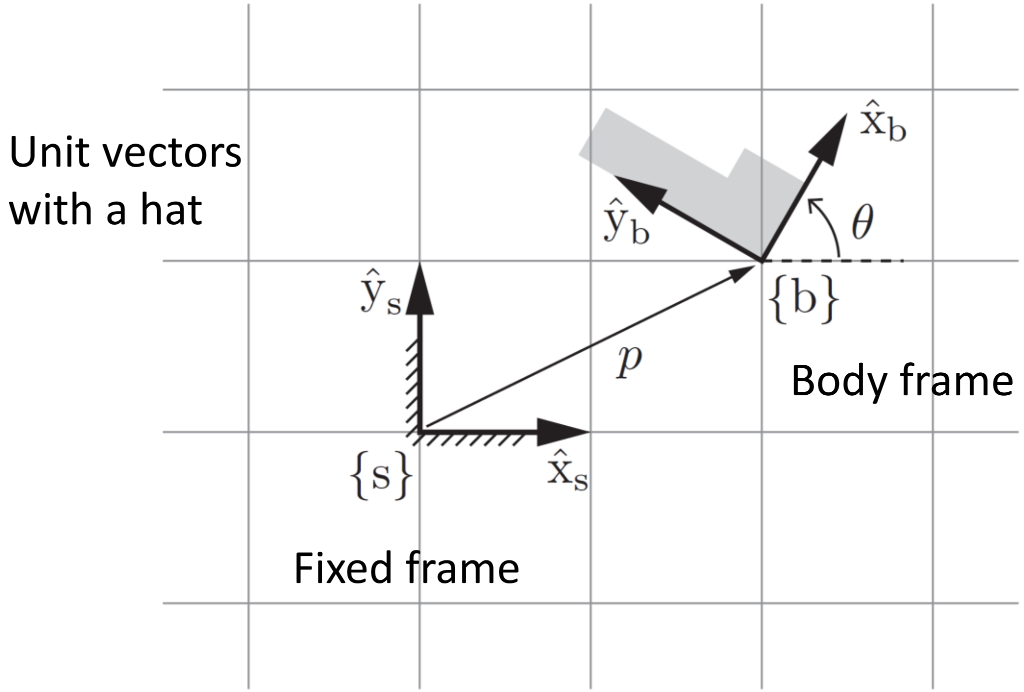
- All frames in this course are stationary, inertial frames
  - Body frame is a motionless frame that is instantaneously coincident with a frame that is fixed to (possibly moving) body

- All frames in this course are right-handed





Rigid-Body in the Plane



- Configuration of the planer body
  - Position and orientation with respect to the fixed frame
- Body frame origin in the fixed frame

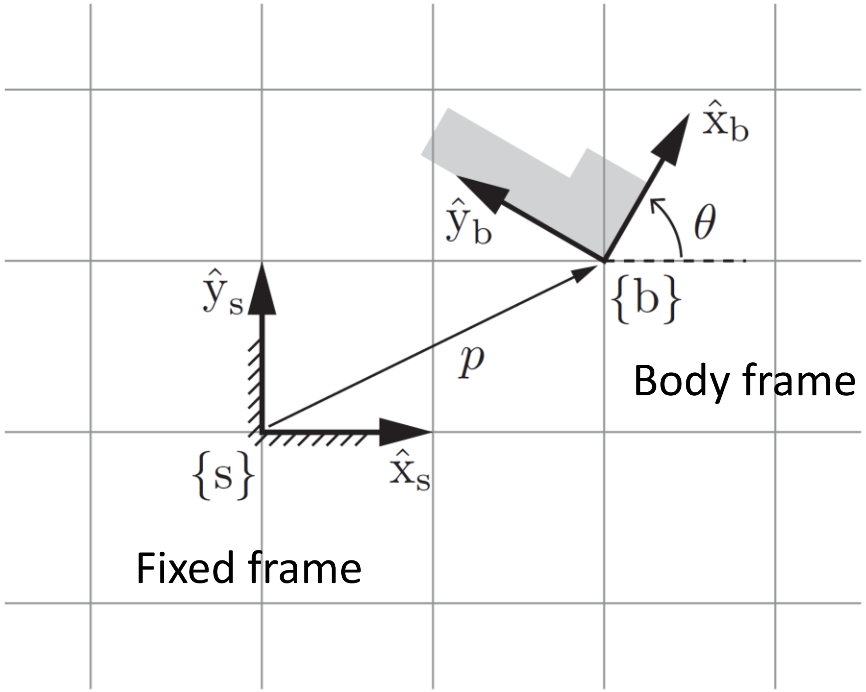
$$p = p_x \hat{x}_s + p_y \hat{y}_s$$

$$p = (p_x, p_y)$$ Vector form

- Rotation angle θ
- Directions of the body frame

$$\begin{aligned} \hat{x}_b &= \cos\theta\, \hat{x}_s + \sin\theta\, \hat{y}_s, \\ \hat{y}_b &= -\sin\theta\, \hat{x}_s + \cos\theta\, \hat{y}_s \end{aligned}$$

Rigid-Body in the Plane



- The two axes of the body frame in {s}

$$R = [\hat{x}_b \;\; \hat{y}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Write as column vectors

Rotation matrix

1DOF

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

Translation

$$(R, p)$$

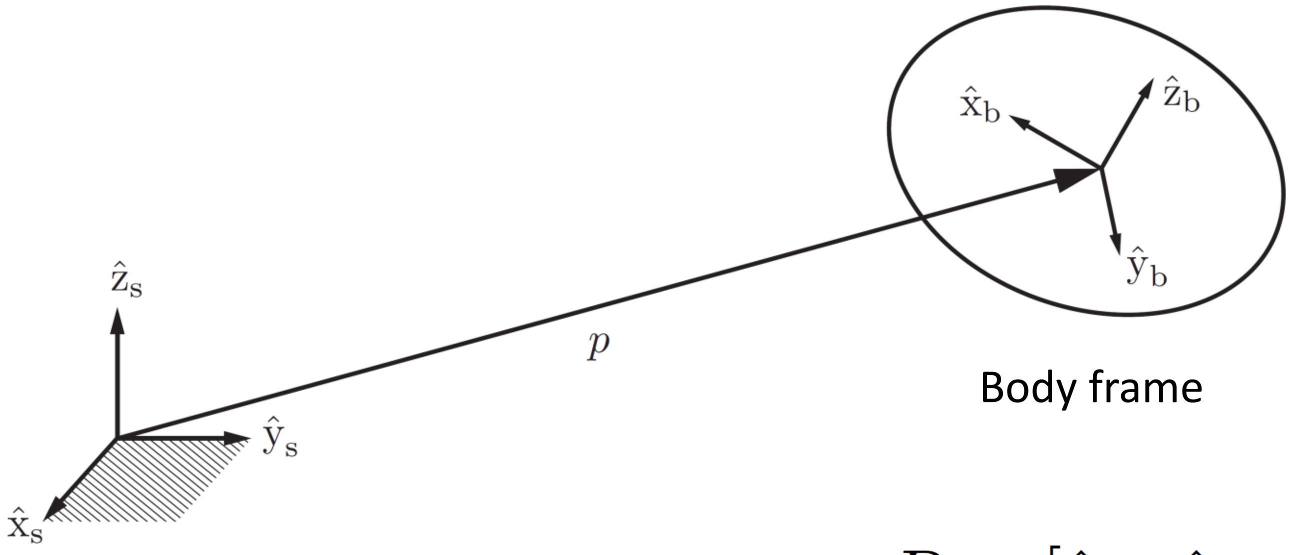
specifies the orientation and position of {b} relative to {s}

Rigid-Body in 3D

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

- Origin of the body frame

$$p = p_1 \hat{\mathrm{x}}_{\mathrm{s}} + p_2 \hat{\mathrm{y}}_{\mathrm{s}} + p_3 \hat{\mathrm{z}}_{\mathrm{s}}$$



Body frame

Fixed frame

- Axes of the body frame

$$\begin{aligned}
\hat{\mathrm{x}}_{\mathrm{b}} &= r_{11}\hat{\mathrm{x}}_{\mathrm{s}} + r_{21}\hat{\mathrm{y}}_{\mathrm{s}} + r_{31}\hat{\mathrm{z}}_{\mathrm{s}}, \\
\hat{\mathrm{y}}_{\mathrm{b}} &= r_{12}\hat{\mathrm{x}}_{\mathrm{s}} + r_{22}\hat{\mathrm{y}}_{\mathrm{s}} + r_{32}\hat{\mathrm{z}}_{\mathrm{s}}, \\
\hat{\mathrm{z}}_{\mathrm{b}} &= r_{13}\hat{\mathrm{x}}_{\mathrm{s}} + r_{23}\hat{\mathrm{y}}_{\mathrm{s}} + r_{33}\hat{\mathrm{z}}_{\mathrm{s}}.
\end{aligned}$$

$$R = [\hat{\mathrm{x}}_{\mathrm{b}} \;\; \hat{\mathrm{y}}_{\mathrm{b}} \;\; \hat{\mathrm{z}}_{\mathrm{b}}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Rotation matrix

Write as column vectors

Rotation Matrix

- Unit norm condition

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1, \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1, \\ r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1. \end{aligned}$$

$$R = [\hat{\mathrm{x}}_{\mathrm{b}} \ \ \hat{\mathrm{y}}_{\mathrm{b}} \ \ \hat{\mathrm{z}}_{\mathrm{b}}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Orthogonality condition $\hat{\mathrm{x}}_{\mathrm{b}} \cdot \hat{\mathrm{y}}_{\mathrm{b}} = \hat{\mathrm{x}}_{\mathrm{b}} \cdot \hat{\mathrm{z}}_{\mathrm{b}} = \hat{\mathrm{y}}_{\mathrm{b}} \cdot \hat{\mathrm{z}}_{\mathrm{b}} = 0$

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0, \\ r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} &= 0, \\ r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} &= 0. \end{aligned}$$

Rotation Matrix

- Orthogonal matrix $R^{\mathrm{T}} R = I$
- Right-handed $\hat{\mathrm{x}}_{\mathrm{b}} \times \hat{\mathrm{y}}_{\mathrm{b}} = \hat{\mathrm{z}}_{\mathrm{b}}$
- Left-handed $\hat{\mathrm{x}}_{\mathrm{b}} \times \hat{\mathrm{y}}_{\mathrm{b}} = -\hat{\mathrm{z}}_{\mathrm{b}}$

$$R = [\hat{\mathrm{x}}_{\mathrm{b}} \;\; \hat{\mathrm{y}}_{\mathrm{b}} \;\; \hat{\mathrm{z}}_{\mathrm{b}}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Determinant of a 3x3 matrix M

$$\det M = a^{\mathrm{T}}(b \times c) = c^{\mathrm{T}}(a \times b) = b^{\mathrm{T}}(c \times a)$$

$$\det R = \pm 1$$ does not change the number of independent continuous variables

$$\det R = 1$$ Right-handed frames only

Properties of Rotation Matrices

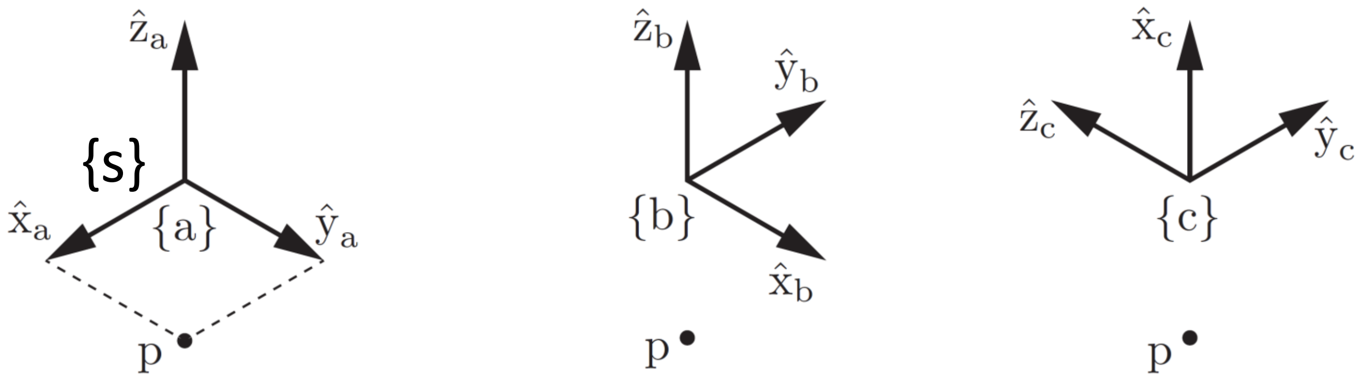
- Closure $R_1 R_2$
- Associativity $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
- Identity element: identity matrix I
- Inverse element $R^{-1} = R^{\mathrm{T}}$
- Not commutative $R_1 R_2 \neq R_2 R_1$

Uses of Rotation Matrices

- Represent an orientation
- Change the reference frame
- Rotate a vector or a frame

Representing an Orientation

- R_{sc} frame {c} relative to frame {s}



$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

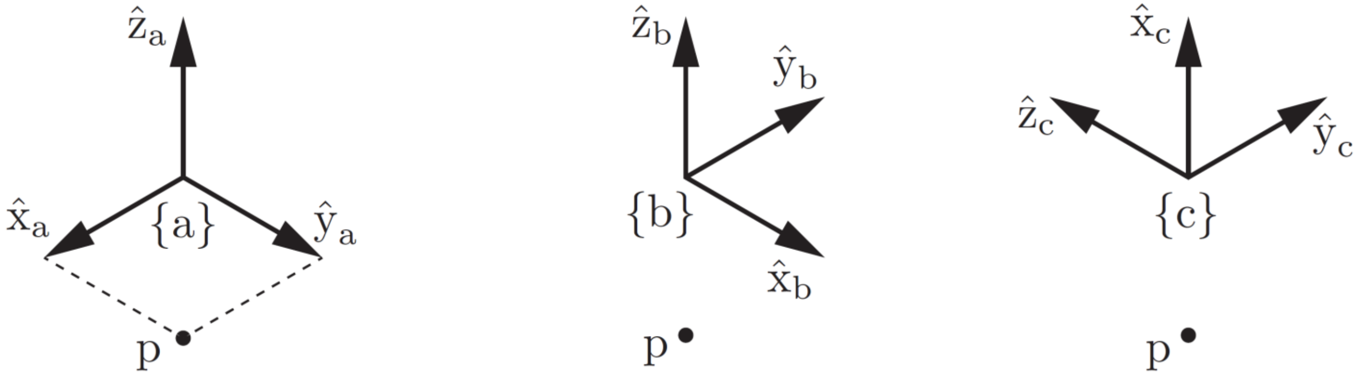
Imagine the three frames have the same origin

$$p_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad p_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Representing an Orientation

- R_{sc} frame {c} relative to frame {s}

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$



$$R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Imagine the three frames have the same origin

$$R_{ac}R_{ca} = I \qquad R_{ac} = R_{ca}^{-1} \qquad R_{ac} = R_{ca}^{\mathrm{T}}$$

Changing the Reference Frame

- Orientation of {b} in {a} R_{ab}
- Orientation of {c} in {b} R_{bc}
- Orientation of {c} in {a}

Representation of orientation of {c}

$$R_{ac} = R_{ab}R_{bc}$$

$$= \text{change\_reference\_frame\_from\_\{b\}\_to\_\{a\}}\,(R_{bc})$$

- Subscript cancel rule

$$R_{ab}R_{bc} = R_{a\cancel{b}}R_{\cancel{b}c} = R_{ac} \qquad R_{ab}p_b = R_{a\cancel{b}}p_{\cancel{b}} = p_a$$

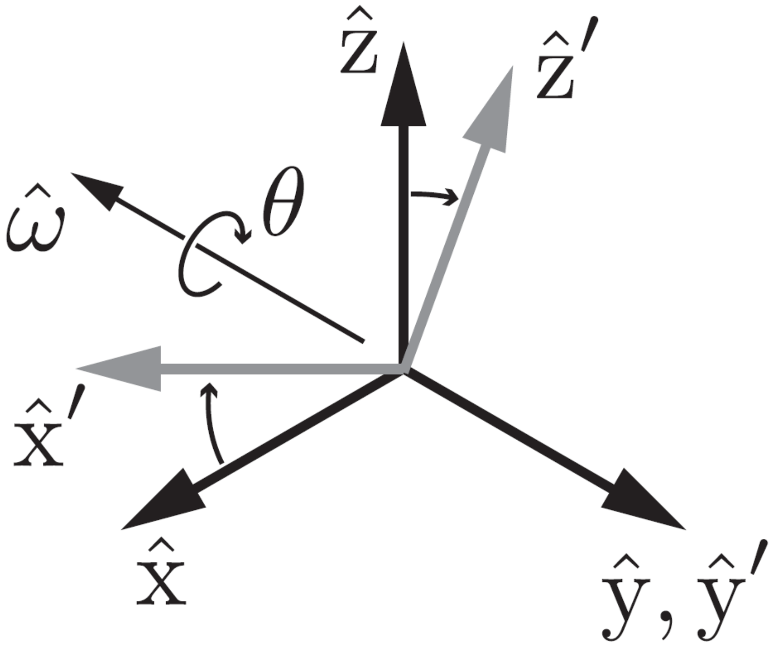
Rotating a Vector or a Frame

- Rotate frame {c} about a unit axis $\hat{\omega}$ by θ to get frame {c'}

$$R = R_{sc'}$$

- Rotation operation

$$R = \mathrm{Rot}(\hat{\omega}, \theta)$$

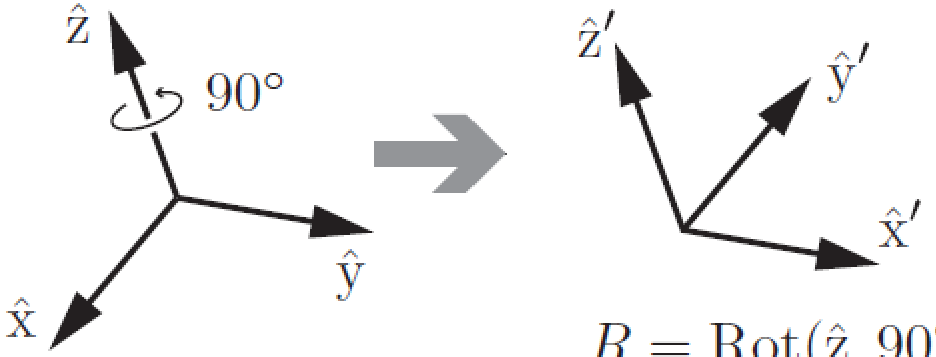


{c} → {c'}

Rotating a Vector or a Frame

$$\mathrm{Rot}(\hat{\mathrm{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \mathrm{Rot}(\hat{\mathrm{y}}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$R = \mathrm{Rot}(\hat{\mathrm{z}}, 90°)$

Rotating a Vector or a Frame

$$\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$$

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta \qquad c_\theta = \cos\theta$$

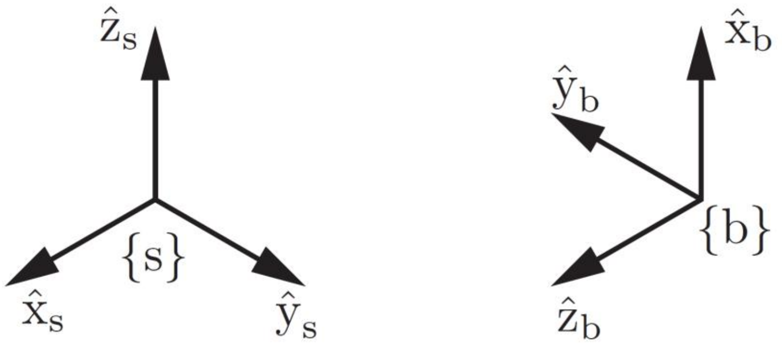
Rodrigues' formula

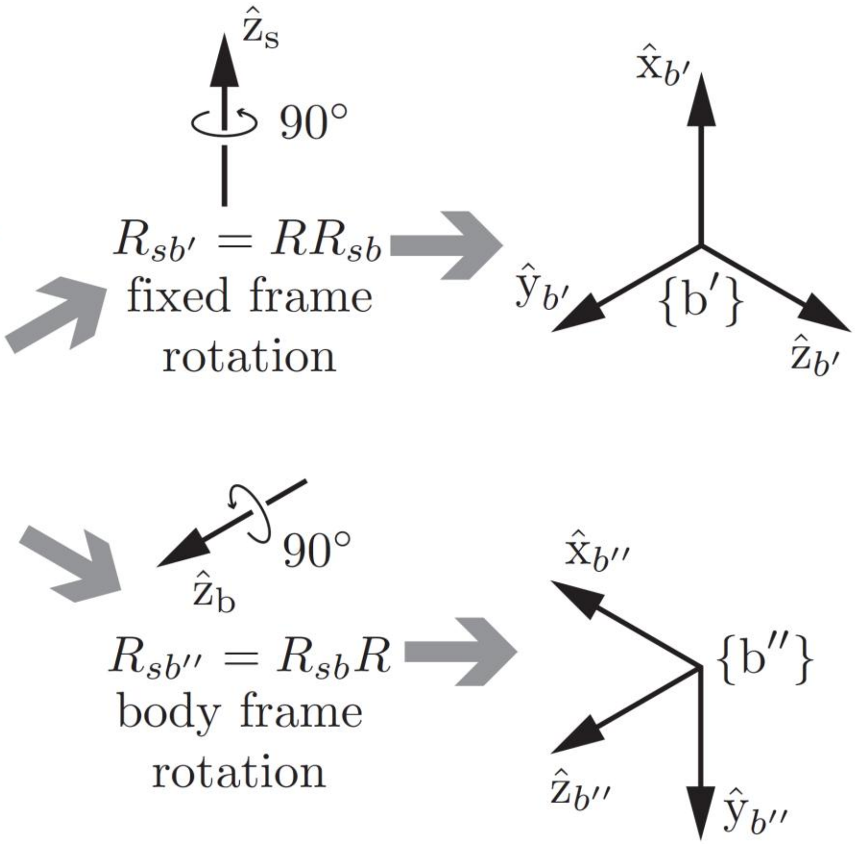
$$\mathrm{Rot}(\hat{\omega}, \theta) = \mathrm{Rot}(-\hat{\omega}, -\theta)$$

Rotating a Vector or a Frame

- {b} in {s} R_{sb}
- Rotate {b} with $\mathrm{Rot}(\hat{\omega}, \theta)$

$\hat{\omega}$ represented in {s} or {b}?





Rotating a Vector or a Frame

- {b} in {s} R_{sb} $\hat{\omega}$ represented in {s} or {b}?
- Rotate {b} with $\mathrm{Rot}(\hat{\omega}, \theta)$

$$R_{sb'} = \mathrm{rotate\_by\_}R\mathrm{\_in\_\{s\}\_frame}\ (R_{sb}) = RR_{sb}$$

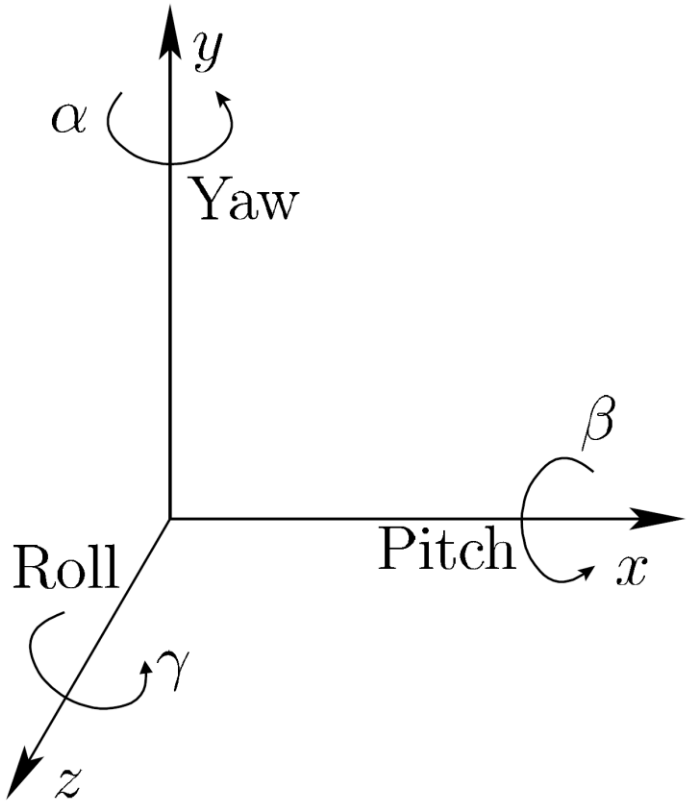
$$R_{sb''} = \mathrm{rotate\_by\_}R\mathrm{\_in\_\{b\}\_frame}\ (R_{sb}) = R_{sb}R$$

- To rotate a vector $v' = Rv$

R should be in the frame of v

Euler Angles: Yaw, Pitch, Roll

- Counterclockwise rotation



Roll $$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pitch $$R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

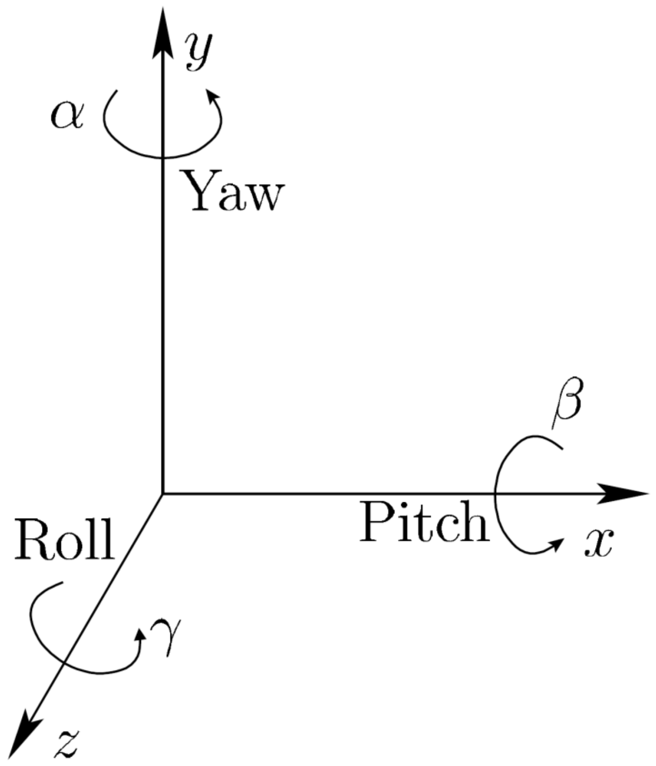
Yaw $$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Combining Rotations

- Matrix multiplications are "backwards"

$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\alpha, \gamma \in [0, 2\pi] \qquad \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

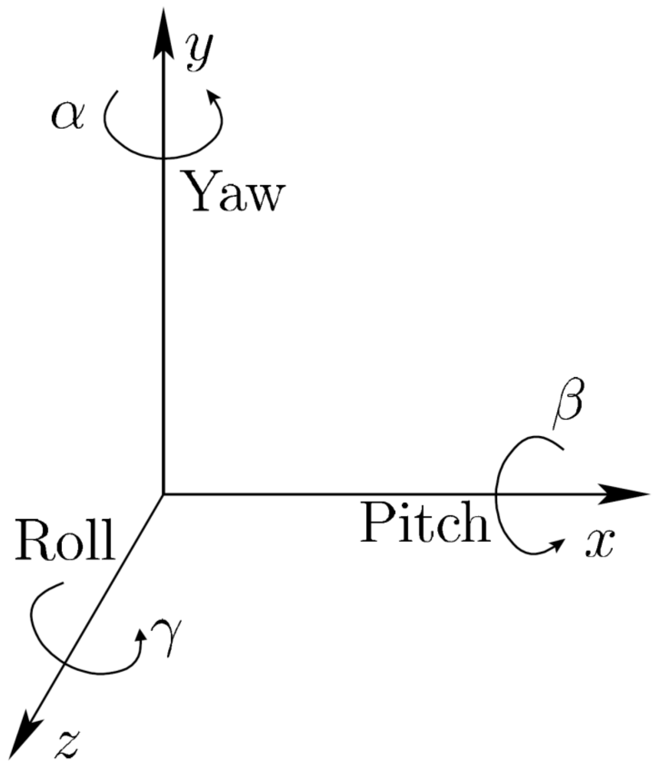


The Order Matters

- 12 possible sequences of rotation axes

Proper Euler angles (z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)

Tait–Bryan angles (x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z)



Quaternions

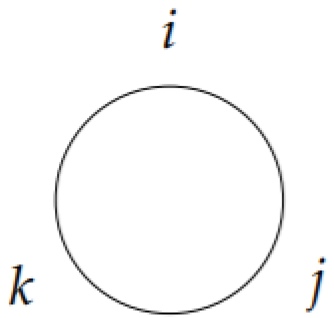
- Quaternions generalize complex numbers and can be used to represents 3D rotations

$$q = w + xi + yj + zk$$

Scale (real part) Vector (imaginary part)

- Properties

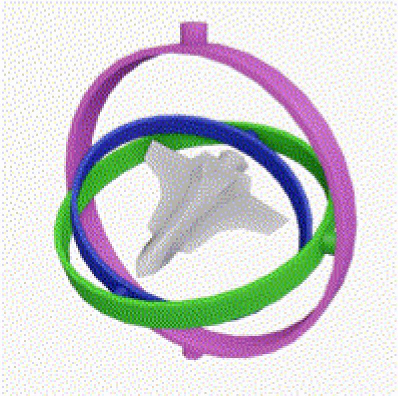
$$i^2 = j^2 = k^2 = -1$$
$$ij = k, ji = -k$$
$$jk = i, kj = -i$$
$$ki = j, ik = -j$$



Unit Quaternions as 3D Rotations

- For unit quaternions, axis-angle

$$q = (w, \mathbf{v}) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{\mathbf{n}})$$



- Why Quaternions Are Better than Matrices/Euler Angles?
  - No gimbal lock (unlike Euler angles).
  - Compact (4 numbers vs 9 in a matrix).
  - Stable interpolation (slerp) for smooth animations/robot trajectories.
  - Numerical stability: Avoids accumulating errors that break orthogonality in rotation matrices.

Summary

- Reference frames
- Rigid-body in 2D
- Rigid-body in 3D
  - Rotation matrices
- Uses of Rotation Matrices
  - Represent an orientation
  - Change the reference frame
  - Rotate a vector or a frame
- Euler Angles and Quaternions

Further Reading

- Chapter 3 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017

- Quaternion and Rotations, Yan-Bin Jia, https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf

- On the Continuity of Rotation Representations in Neural Networks. Zhou et al., CVPR, 2019.