Forward Kinematics and Denavit-Hartenberg Parameters

CS 6341: Robotics

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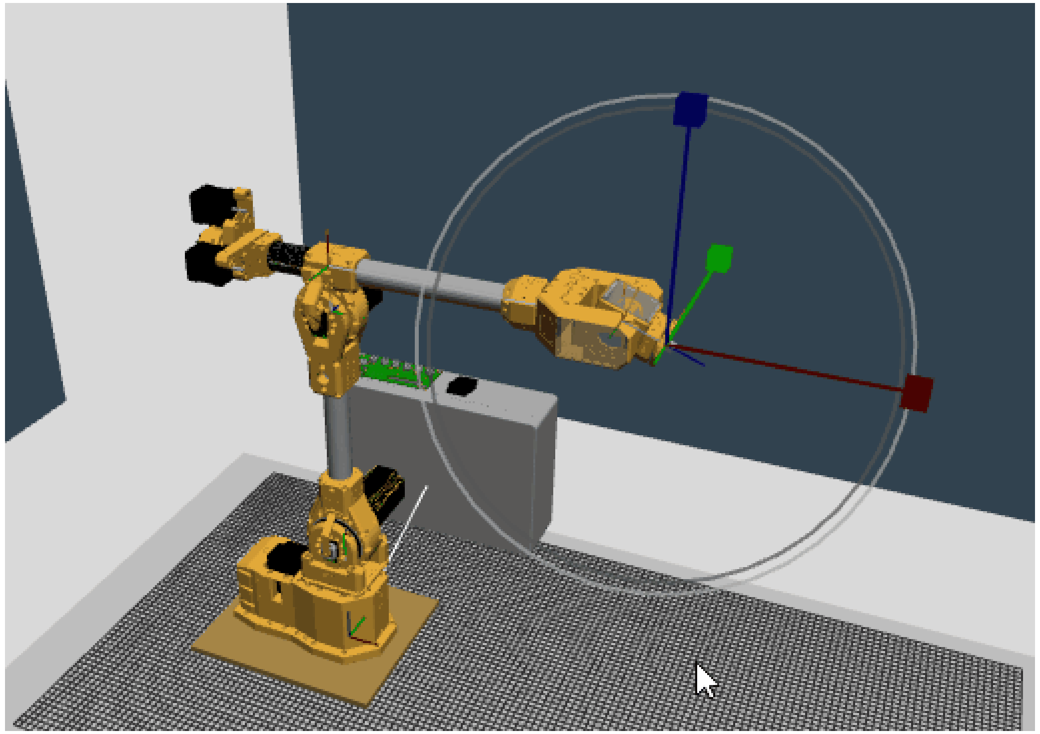
The University of Texas at Dallas

Robot Kinematics

- The relationship between a robot's joint coordinates and its spatial layout

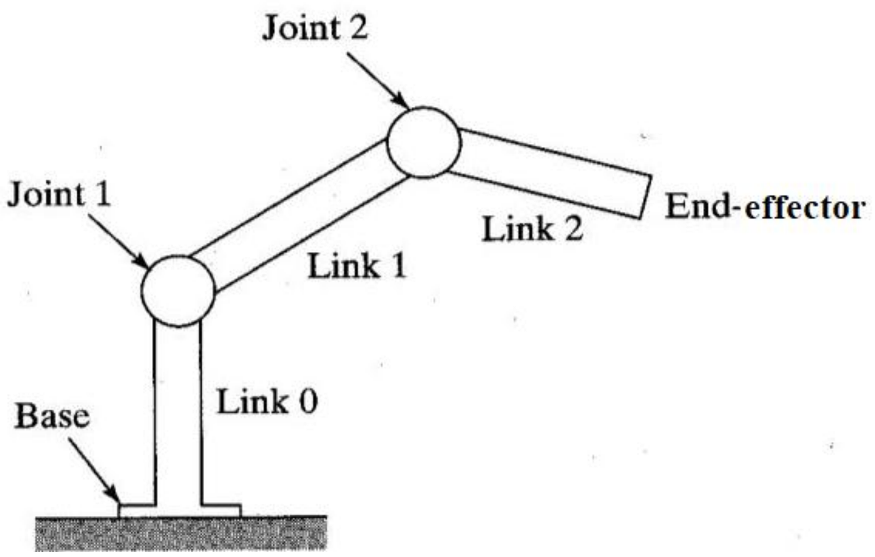


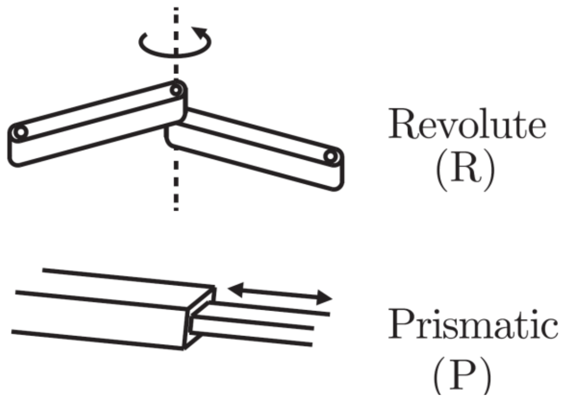
https://www.marginallyclever.com/2020/04/gradient-descent-inverse-kinematics-for-6dof-robot-arms/

Forward Kinematics

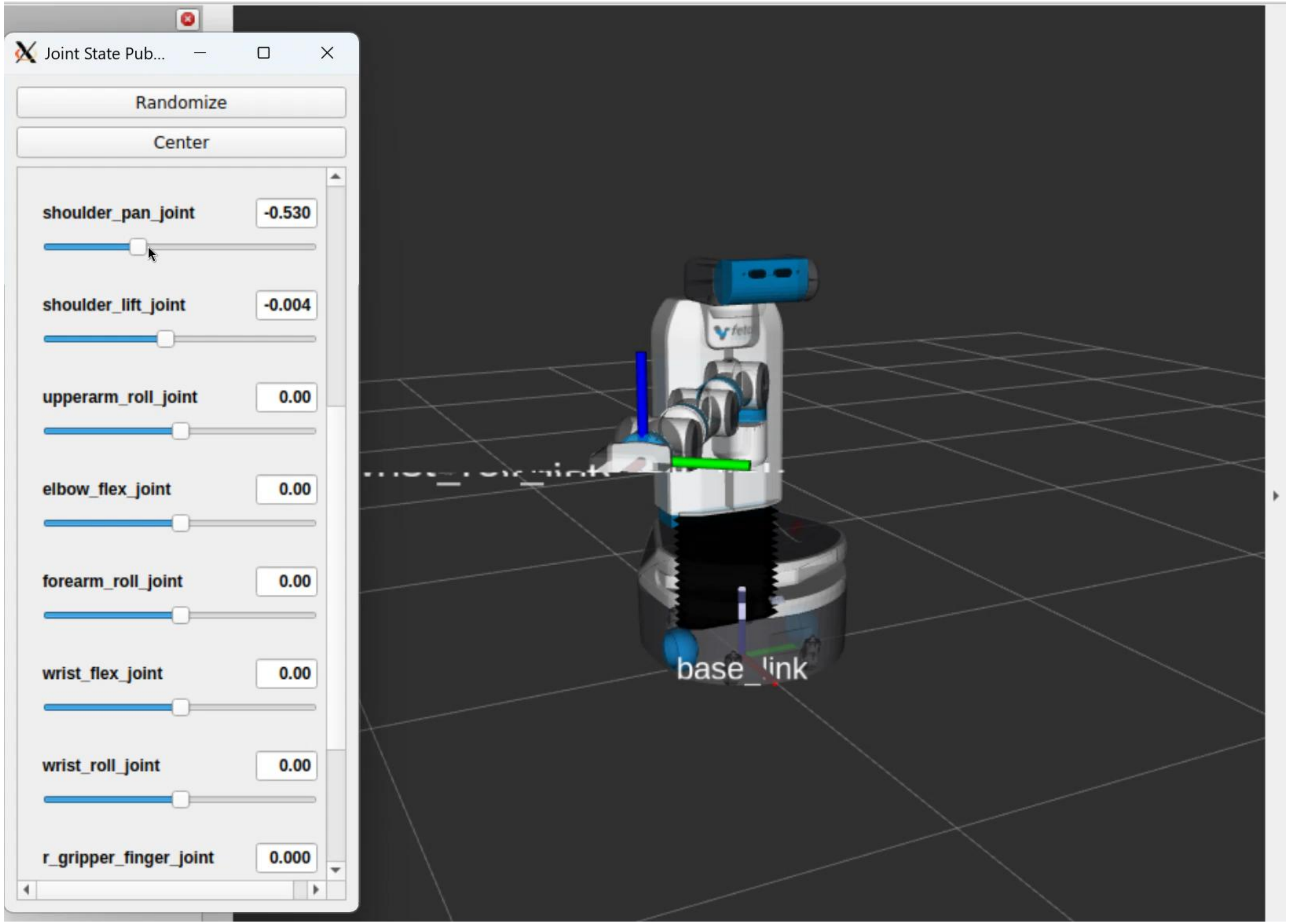
- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates θ

- Recall robot links and joints

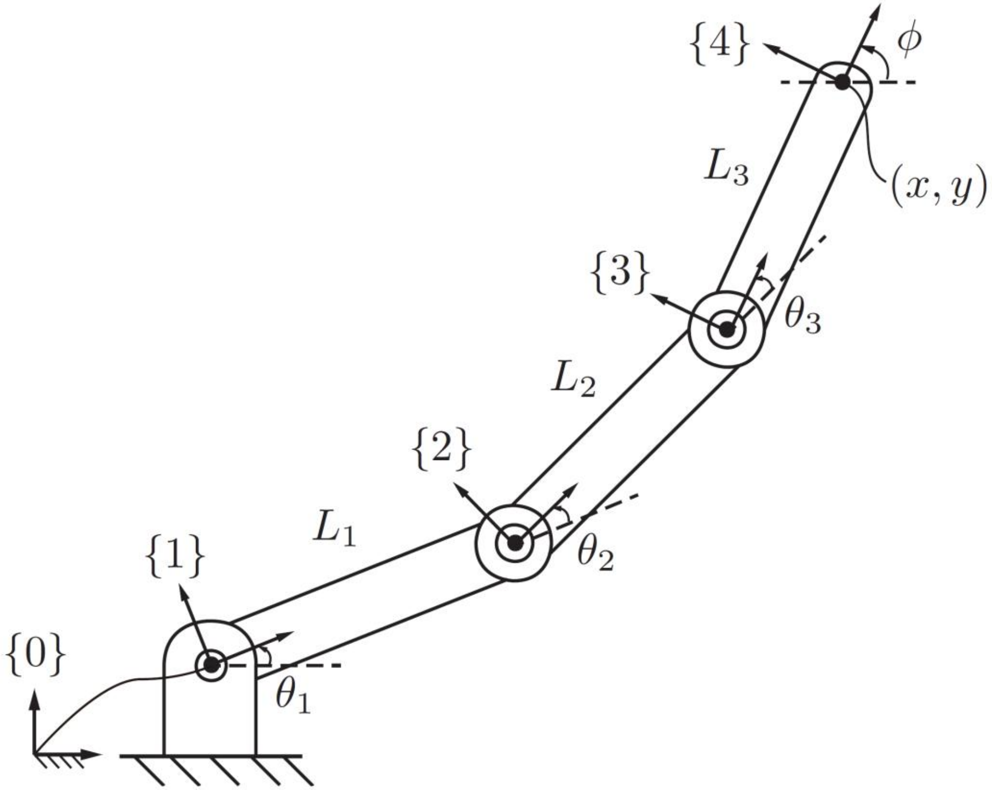




Forward Kinematics



Forward Kinematics

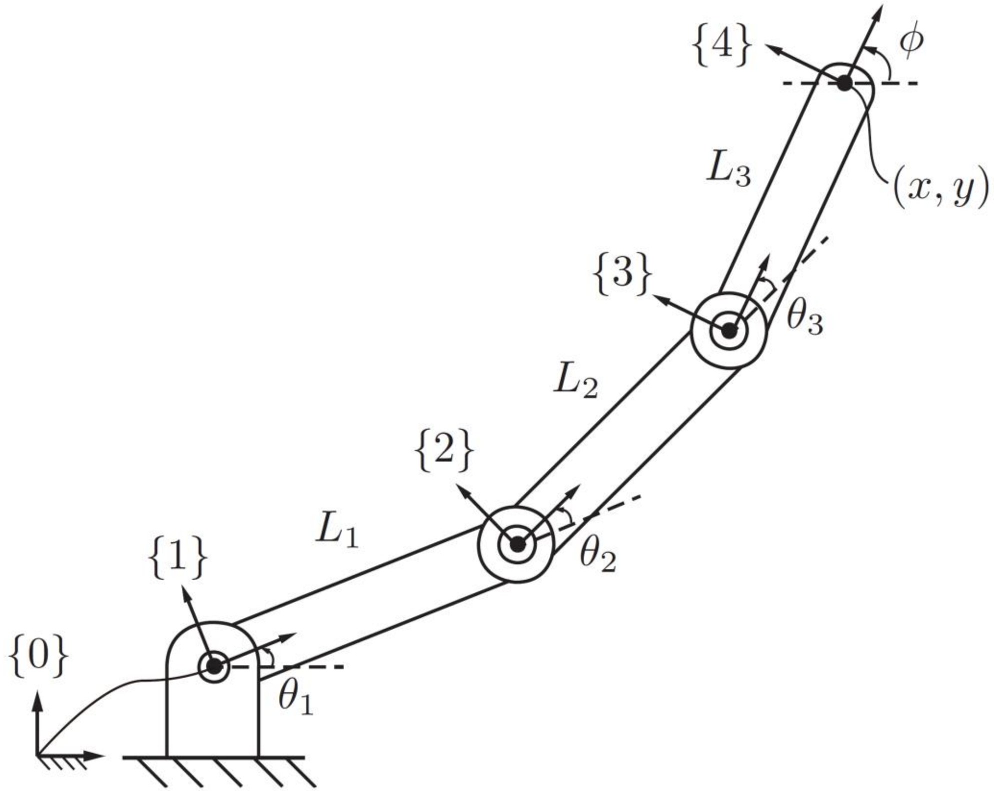


- End-effector frame {4}
- Joint angles $(\theta_1, \theta_2, \theta_3)$
- Position and orientation of the end-effector frame in base frame {0}

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \\
\phi &= \theta_1 + \theta_2 + \theta_3.
\end{aligned}
$$

Forward kinematics of a 3R planar open chain.

Forward Kinematics



Forward kinematics of a 3R planar open chain.

- General cases
  - Attaching frames to links
  - Using homogeneous transformations

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T_{i-1,i}$ Depends only on the joint variable θ_i

Forward Kinematics

- Using homogeneous transformations
  - Need to define the coordinates of frames
  - Denavit-Hartenberg Parameters

Northwestern Engineering’s legacy in robotics started in the 1950s when Dick Hartenberg, a professor, and Jacques Denavit, a PhD student, developed a way to represent mathematically how mechanisms move https://robotics.northwestern.edu/history.html

Denavit-Hartenberg Parameters

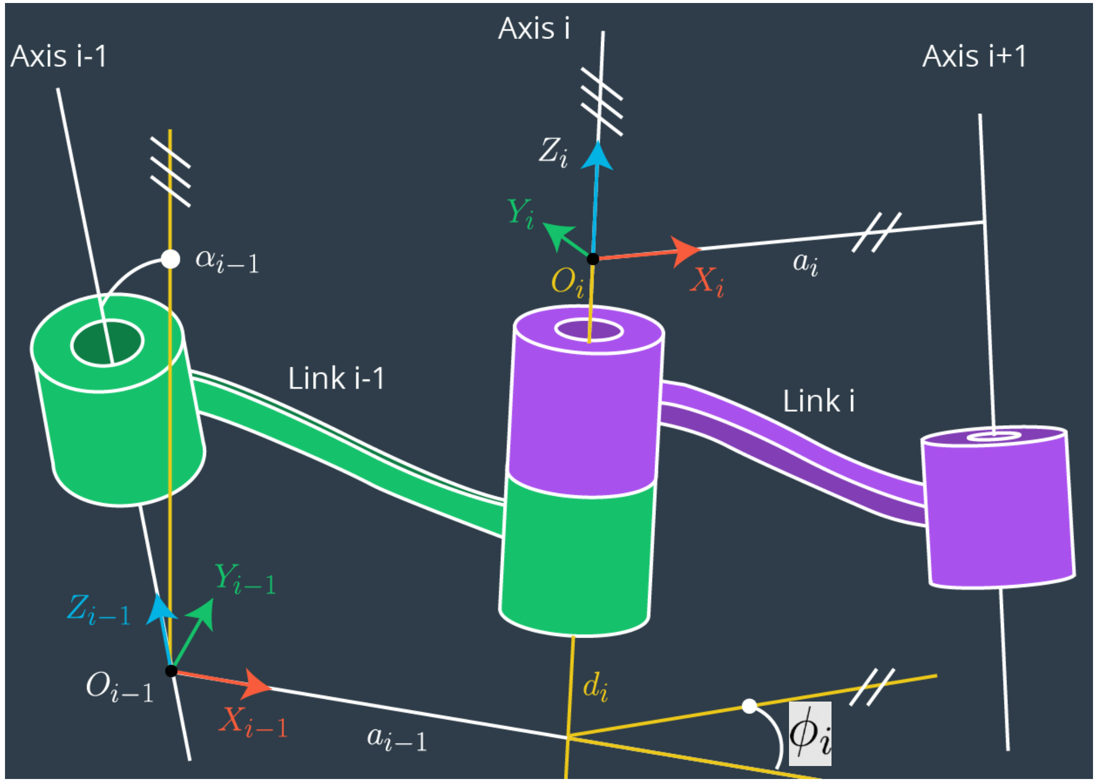
- Attach reference frames to each link of an open chain
- Derive forward kinematics using the relative displacements between adjacent line frames
- For a chain with n 1DOF joints, 0,…,n
  - The ground link is 0
  - The end-effect frame is attached to link n

$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1) T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n)$$

$T_{i,i-1} \in SE(3)$ $\quad$ $T_{n,n+1}$ is constant for end-effector frame {n+1}

Denavit-Hartenberg Parameters

- Assigning link frames



- $\hat{\mathrm{z}}_i$-axis coincides with joint axis i
- $\hat{\mathrm{z}}_{i-1}$-axis coincides with joint axis i-1
- Origin of the link frame
  - Find the line segment that orthogonally intersects both the joint axes
  - Origin of frame {i-1} is the intersection of the line and the joint axis i-1
- $\hat{\mathrm{x}}$-axis in the direction of the mutual perpendicular line pointing from (i-1)-axis to i-axis
- $\hat{\mathrm{y}}$-axis given by $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$

Denavit-Hartenberg (D-H) Parameters

- Link length: the length of the mutual perpendicular line a_{i-1}
  - Not the actual length of the physical link

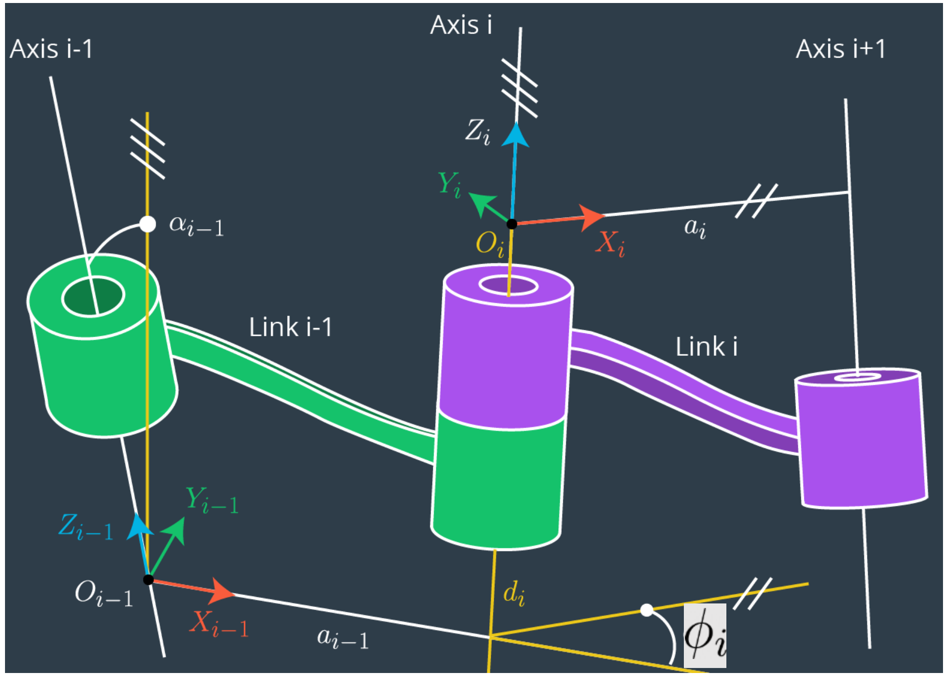
- Line twist α_{i-1}

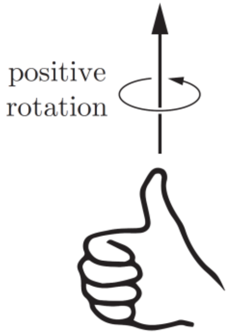
the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, $\boxed{\text{measured about } \hat{x}_{i-1}}$

- Line offset d_i
  - Distance from the intersection to the origin of the link-i frame

- Joint angle ϕ_i

the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, $\boxed{\text{measured about the } \hat{z}_i\text{-axis}}$





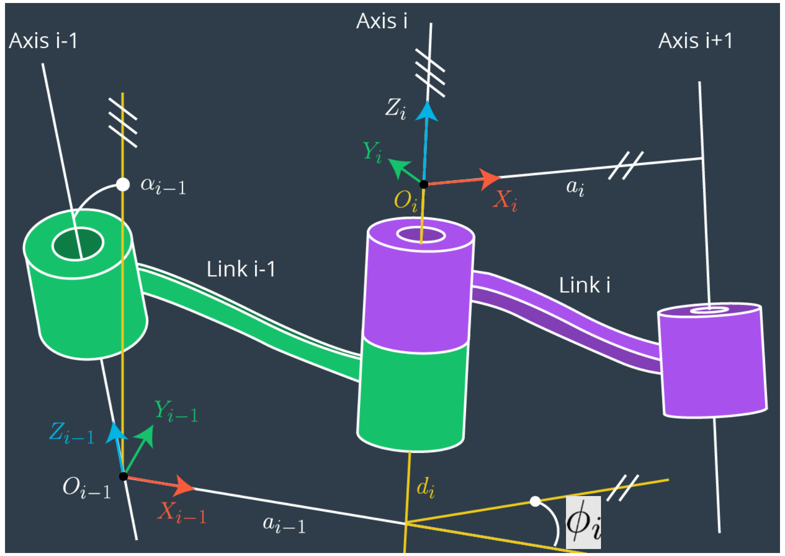
D-H Parameters

- For an open chain with n 1DOF joints, 4n D-H parameters

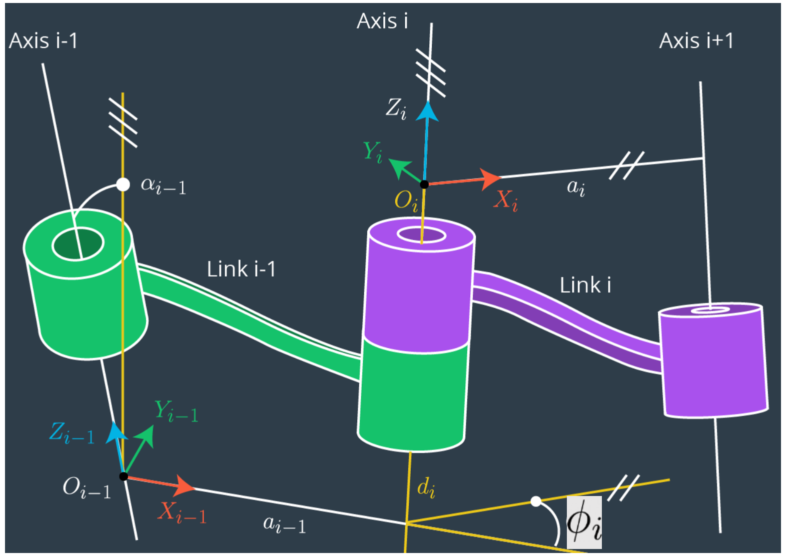
- For an open chain with all joints revolute
  - Link lengths a_{i-1}
  - Line twists α_{i-1}
  - Line offsets d_i

  Constants

  - Joint angle parameters are the joint variables ϕ_i (rotation around z-axis: joint axis)



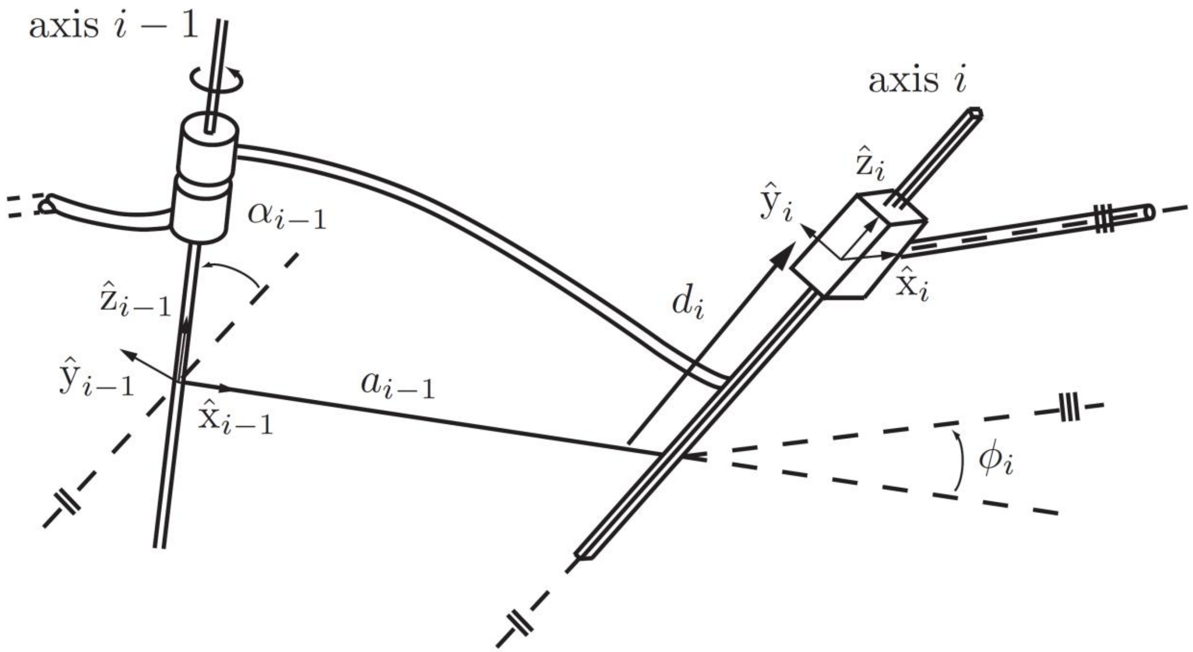
D-H Parameters



- When adjacent revolute joint axes intersect
  - No mutual perpendicular line
  - Link length 0
  - $\hat{x}_{i-1}$ perpendicular to the plane spanned by $\hat{z}_{i-1}$ and $\hat{z}_i$
- When adjacent revolute joint axes are parallel
  - Many possibilities for a mutually perpendicular line
  - Choose the one that is most physically intuitive and results in many zero parameters as possible
- Ground frame {0} and End-Effector frame {n+1}
  - Choose the one that is most physically intuitive and results in many zero parameters as possible

D-H Parameters

- Prismatic joints

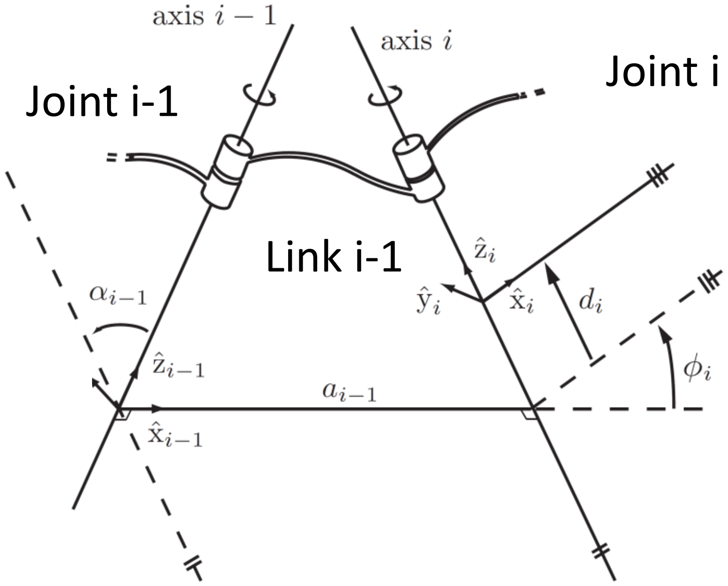


- $\hat{z}_i$-axis positive direction of translation
- d_i link offset is the joint variable
- ϕ_i joint angle is constant
- $\hat{x}$-axis in the direction of the mutual perpendicular line pointing from (i-1)-axis to i-axis
- $\hat{y}$-axis given by $\hat{x} \times \hat{y} = \hat{z}$

Forward Kinematics with D-H Parameters

- Link frame transformation

$$T_{i-1,i} \quad = \quad \text{Rot}(\hat{\text{x}}, \alpha_{i-1})\text{Trans}(\hat{\text{x}}, a_{i-1})\text{Trans}(\hat{\text{z}}, d_i)\text{Rot}(\hat{\text{z}}, \phi_i)$$



(a) A rotation of frame $\{i-1\}$ about its $\hat{\text{x}}$-axis by an angle α_{i-1}.

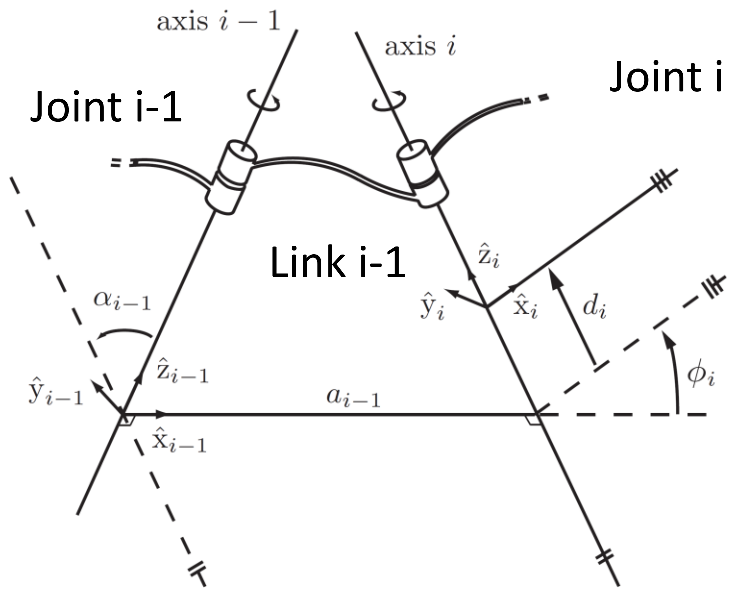
(b) A translation of this new frame along its $\hat{\text{x}}$-axis by a distance a_{i-1}.

(c) A translation of the new frame formed by (b) along its $\hat{\text{z}}$-axis by a distance d_i.

(d) A rotation of the new frame formed by (c) about its $\hat{\text{z}}$-axis by an angle ϕ_i.

Forward Kinematics with D-H Parameters

- Link frame transformation

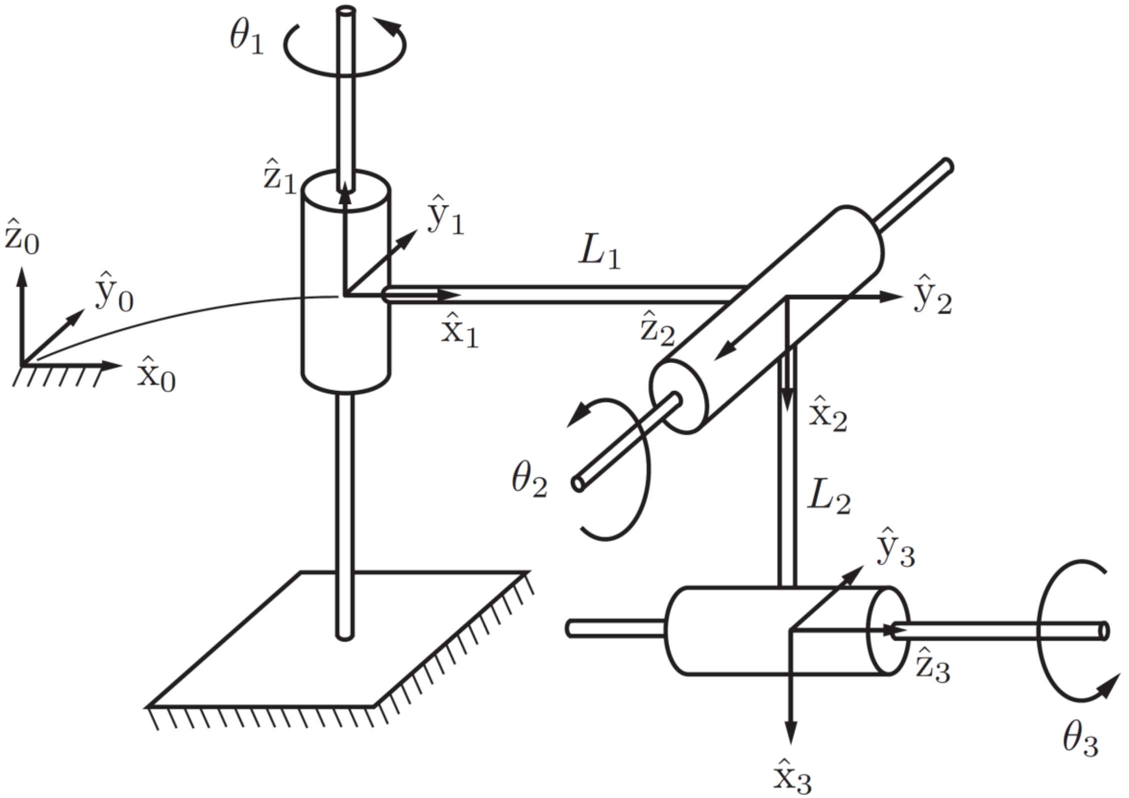


$$
\begin{aligned}
T_{i-1,i} &= \mathrm{Rot}(\hat{x}, \alpha_{i-1})\mathrm{Trans}(\hat{x}, a_{i-1})\mathrm{Trans}(\hat{z}, d_i)\mathrm{Rot}(\hat{z}, \phi_i) \\
&= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i\cos\alpha_{i-1} & \cos\phi_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\phi_i\sin\alpha_{i-1} & \cos\phi_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\mathrm{Rot}(\hat{z}, \phi_i) = \begin{bmatrix} \cos\phi_{i-1} & -\sin\phi_{i-1} & 0 & 0 \\ \sin\phi_{i-1} & \cos\phi_{i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Trans}(\hat{z}, d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathrm{Trans}(\hat{x}, a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}(\hat{x}, \alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

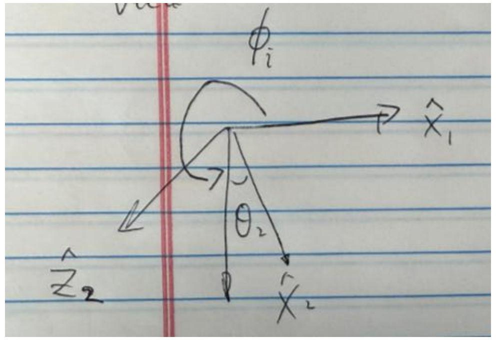
Forward Kinematics with D-H Parameters

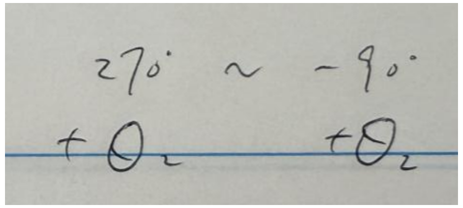


A 3R spatial open chain in its zero position

D-H Parameters

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	L_1	0	$\theta_2 - 90°$
3	$-90°$	L_2	0	θ_3

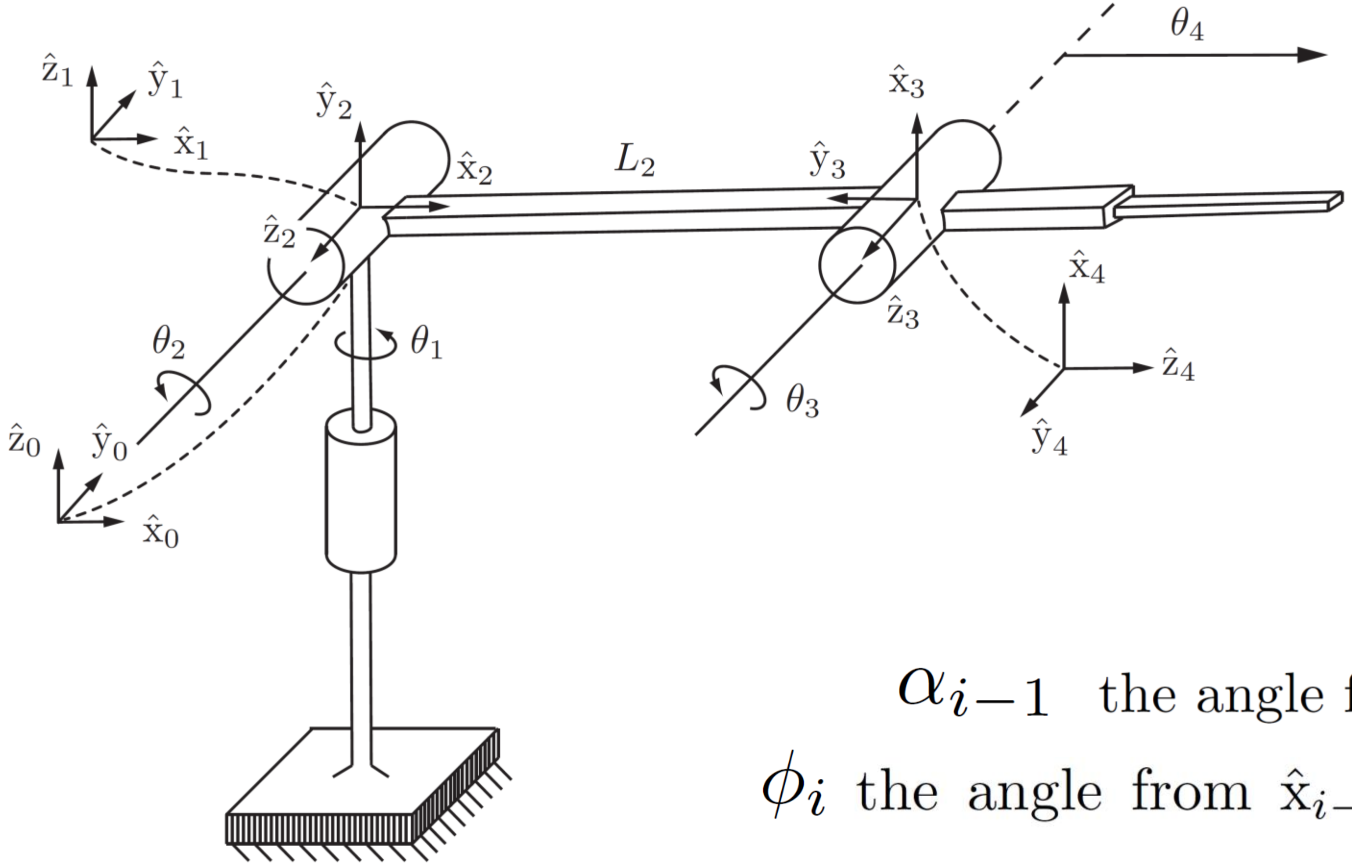




α_{i-1} the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$

ϕ_i the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about the $\hat{z}_i$-axis

Forward Kinematics with D-H Parameters



D-H Parameters

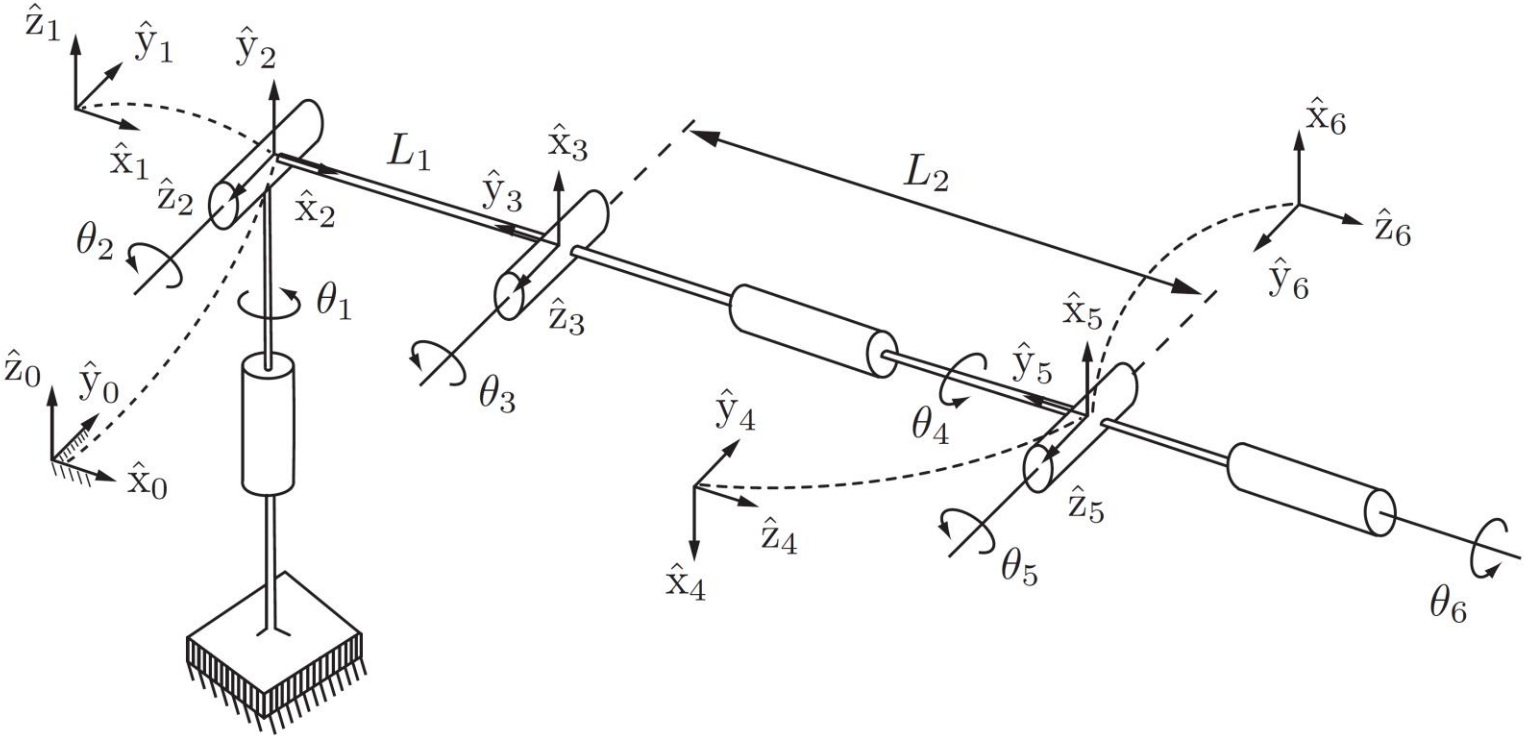
i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	0	0	θ_2
3	0	L_2	0	$\theta_3 + 90°$
4	$90°$	0	θ_4	0

α_{i-1} the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$

ϕ_i the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about the $\hat{z}_i$-axis

An RRRP spatial open chain in its zero position

Forward Kinematics with D-H Parameters



A 6R spatial open chain in its zero position

D-H Parameters

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	0	0	θ_2
3	0	L_1	0	$\theta_3 + 90°$
4	$90°$	0	L_2	$\theta_4 + 180°$
5	$90°$	0	0	$\theta_5 + 180°$
6	$90°$	0	0	θ_6

α_{i-1} the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$

ϕ_i the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about the $\hat{z}_i$-axis

Summary

- Forward kinematics
- Denavit-Hartenberg Parameters

Further Reading

- Chapter 4 and Appendix C in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.

- J. Denavit and R. S. Hartenberg. A kinematic notation for lower-pair mechanisms based on matrices. ASME Journal of Applied Mechanics, 23:215-221, 1955.