The Geometry of Virtual Worlds

CS 6334 Virtual Reality

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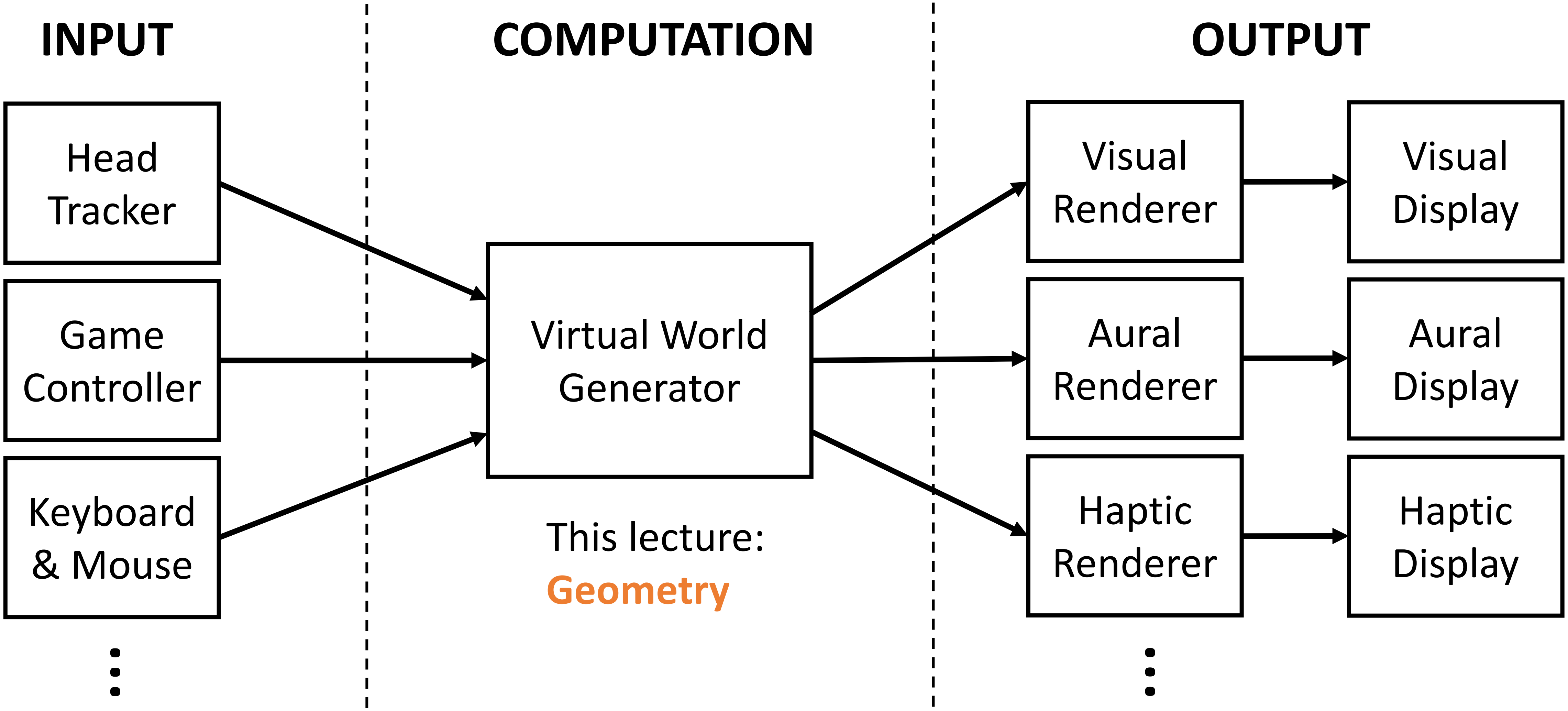
The University of Texas at Dallas

Some slides of this lecture are courtesy Dr. Steven LaValle

Review of VR Systems



This lecture:
Geometry

How to Build the Virtual World?

- Computer games

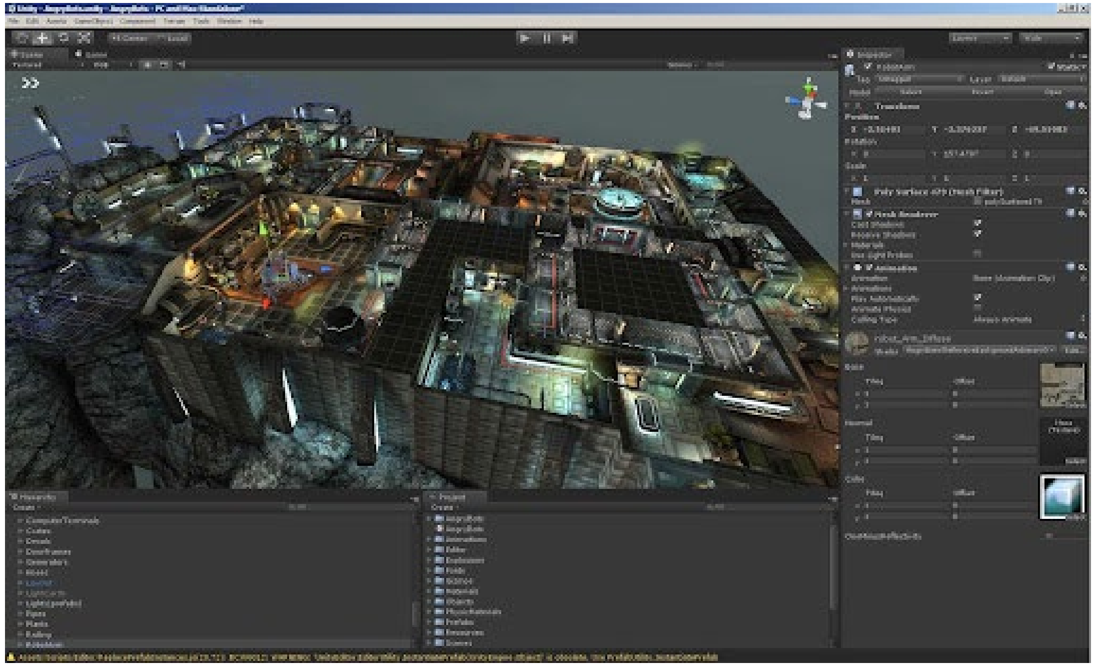
2D Virtual World

3D Virtual World

3D Virtual World, first person

How to Build the Virtual World?

- Examples of game engines

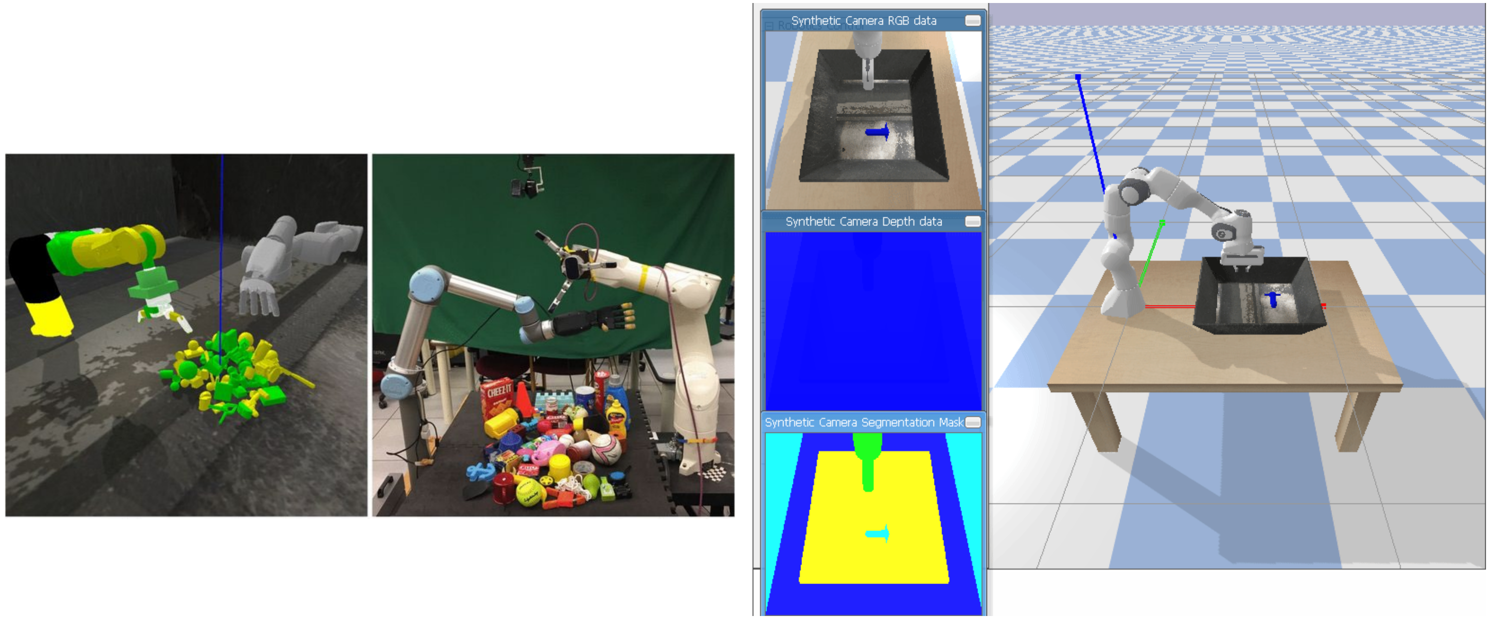


Unity

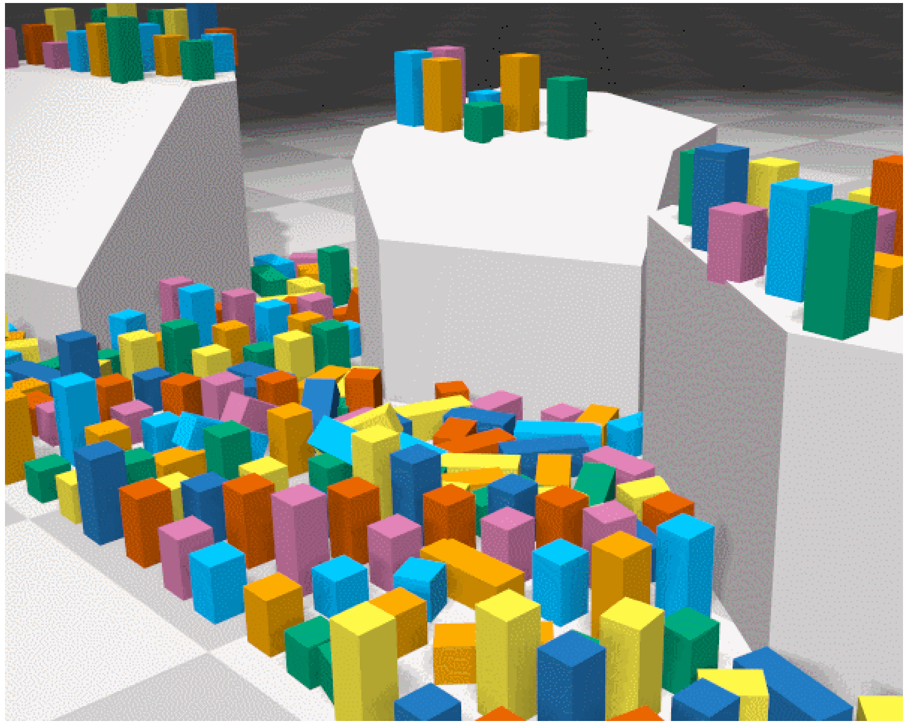
Unreal

How to Build the 3D World?

- Physics simulation



PyBullet Simulation



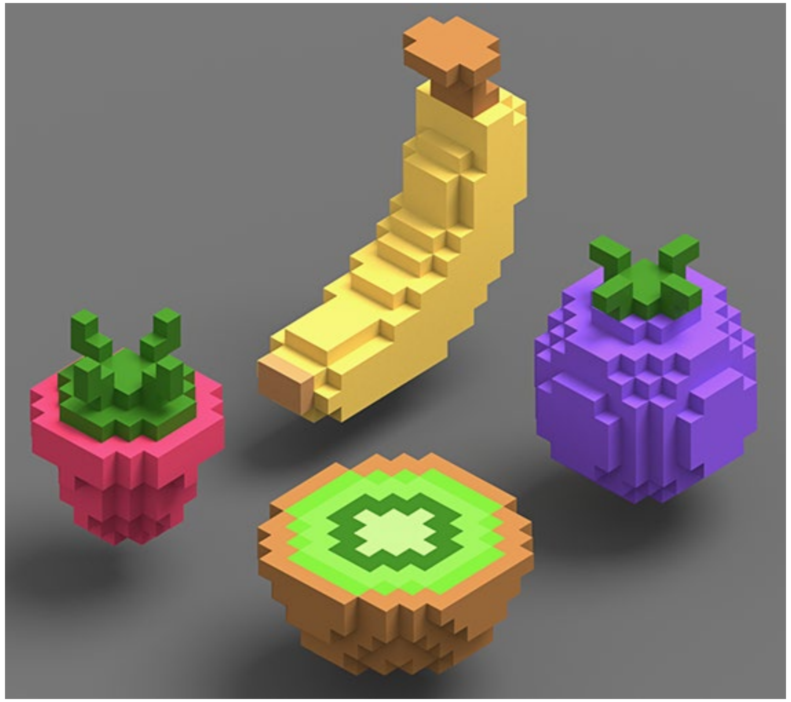
NVIDIA FleX Simulation

How to Build the 3D World?

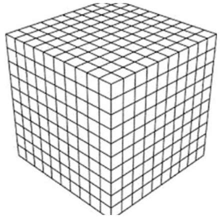
- Game engines
  - Photo-realistic rendering
  - Built in physics simulation, e.g., Unity uses the NVIDIA PhysX engine
  - Need more experience

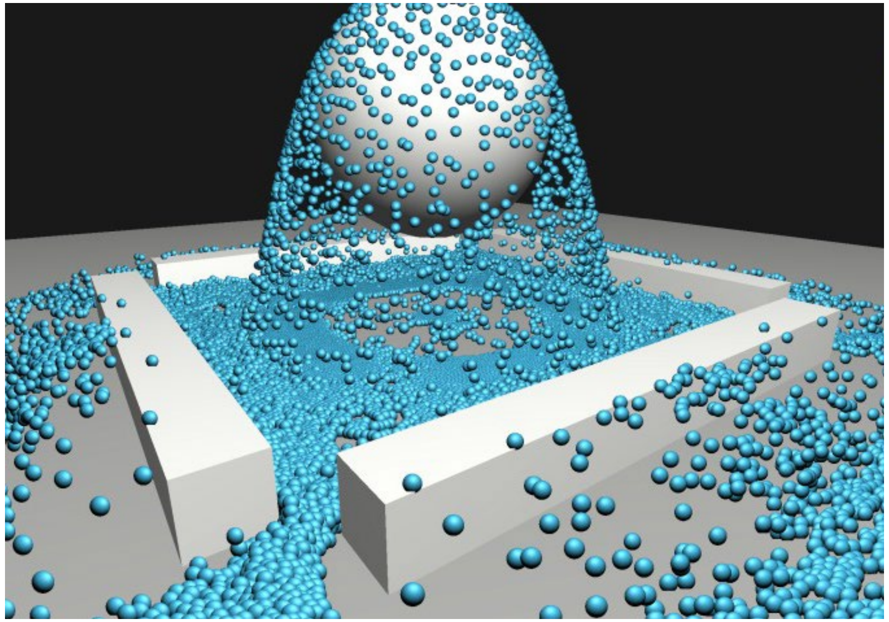
- Physics simulators
  - Usually non-photo-realistic rendering
  - Usually easy to program, e.g., PyBullet
  - Good for learning the concepts in VR

Representations of the 3D World

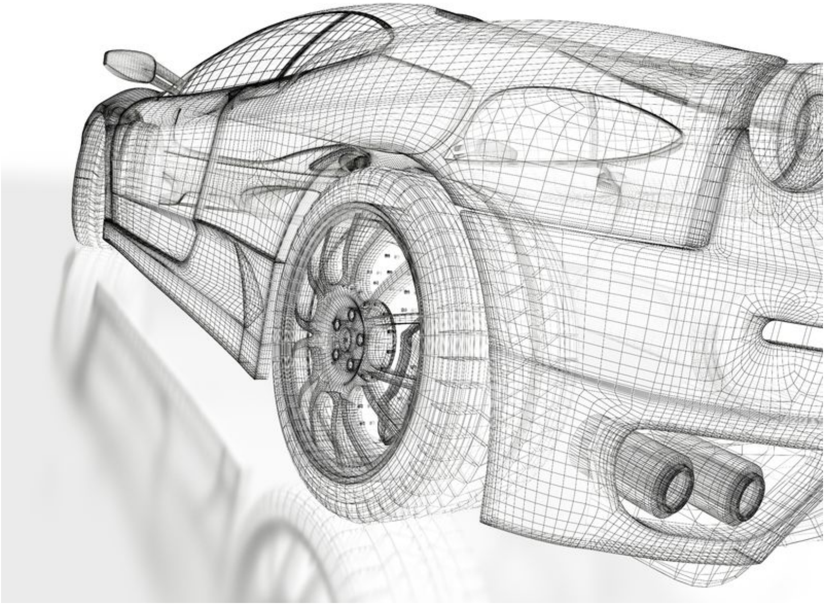


3D Voxels



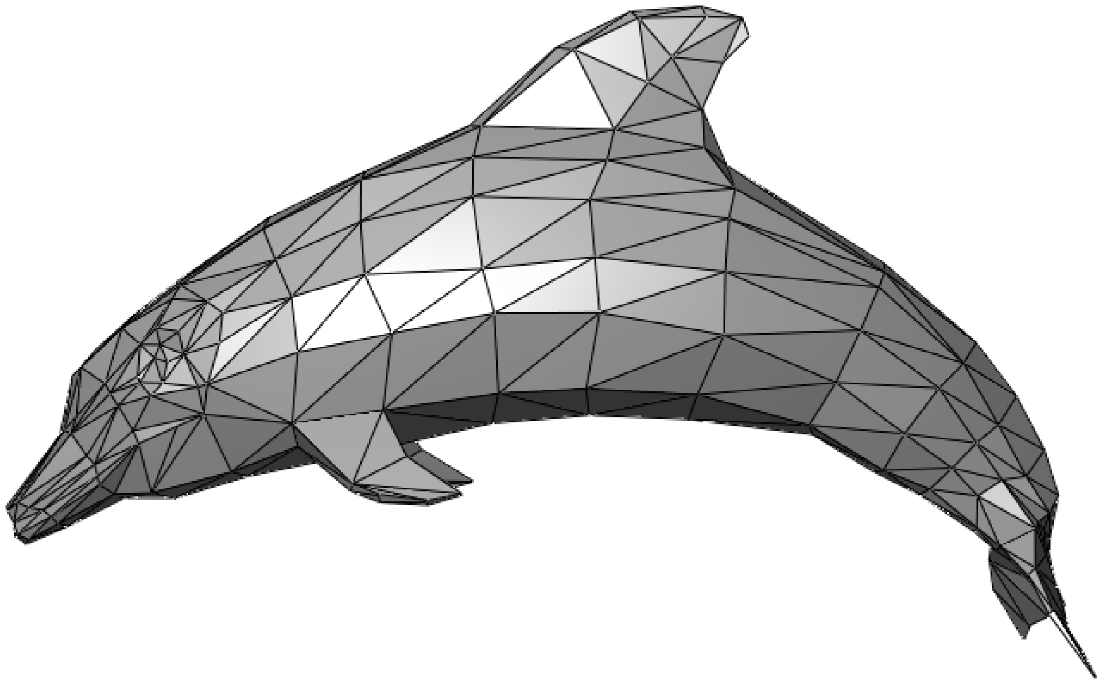


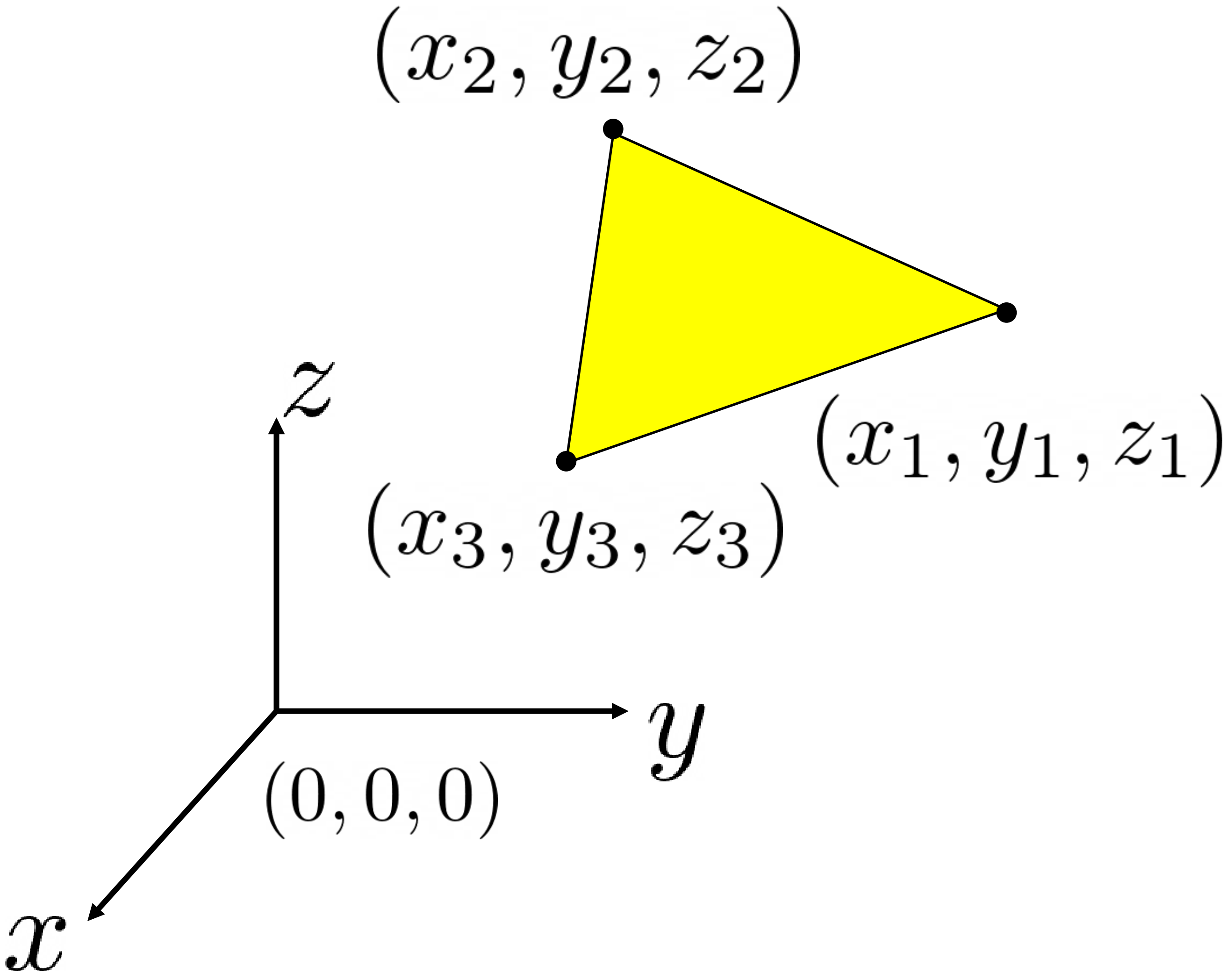
3D Particles



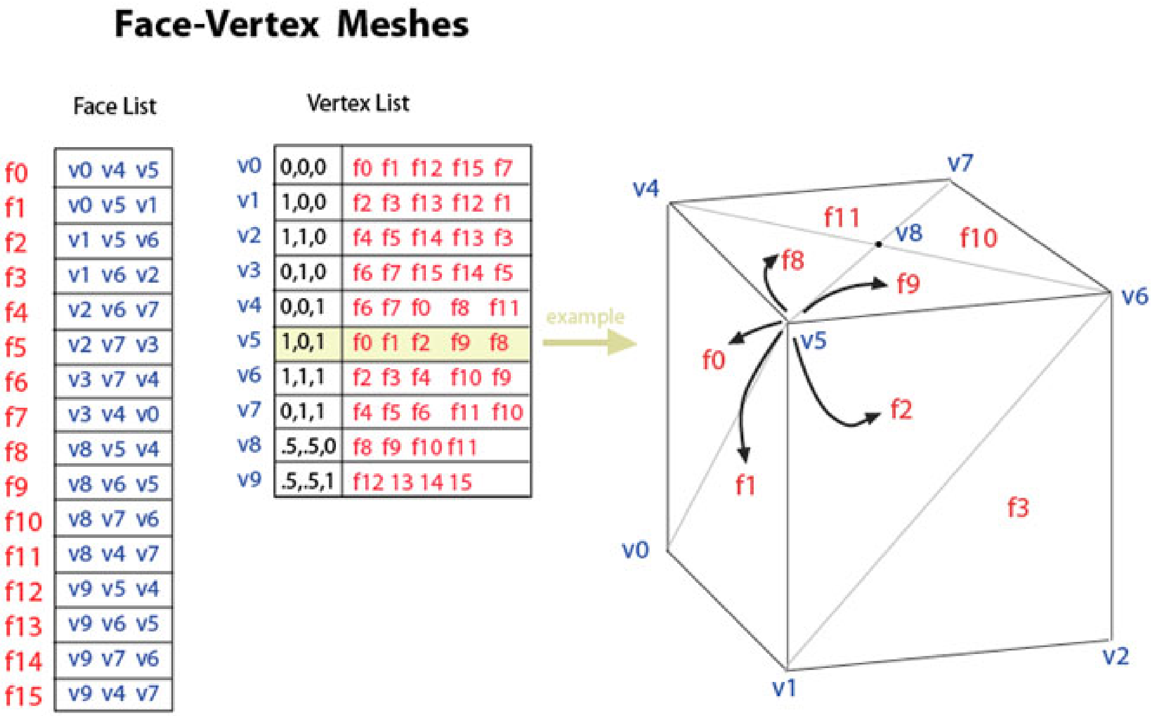
3D Meshes

3D Triangle Mesh



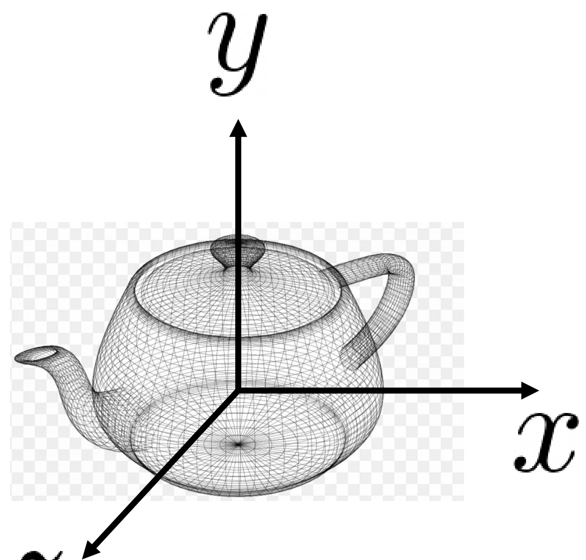


The Virtual World as 3D Triangle Meshes

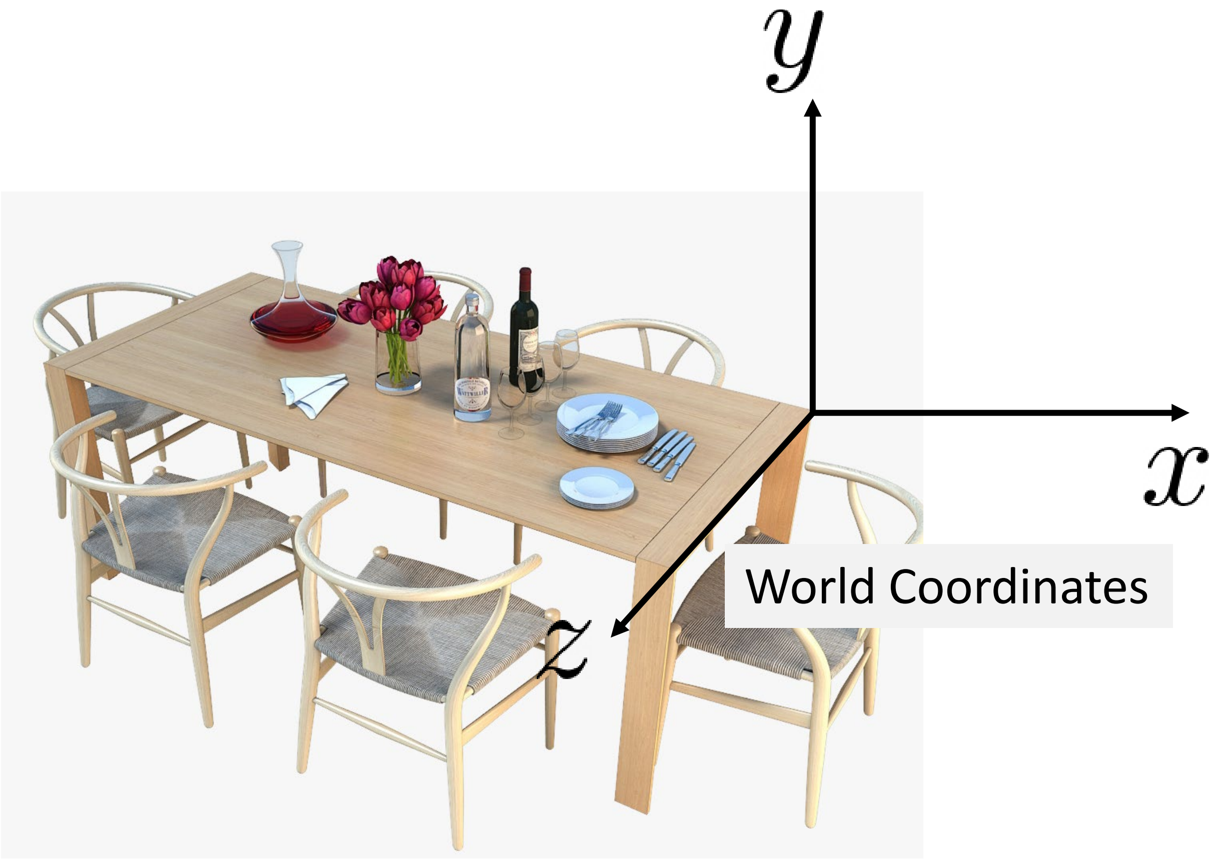


From Wikipedia

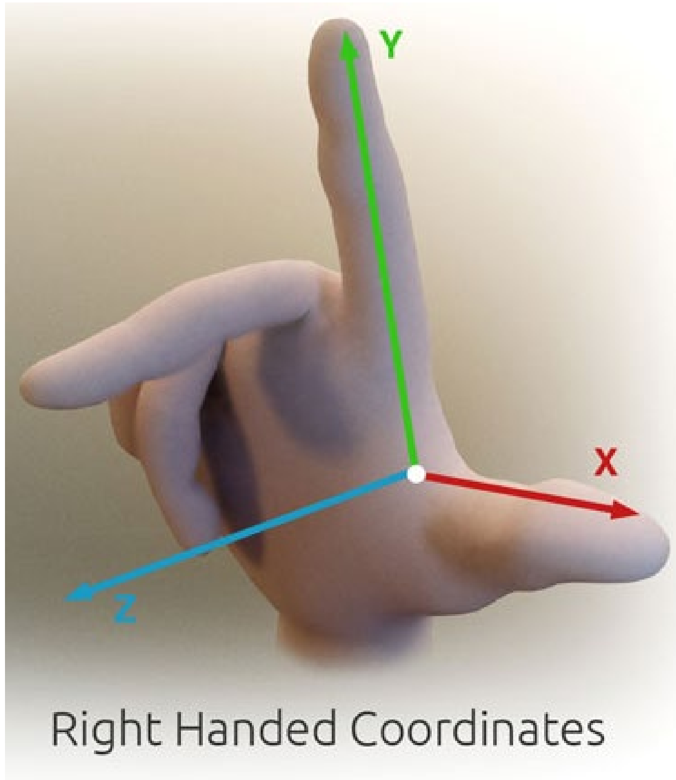
Coordinate Systems



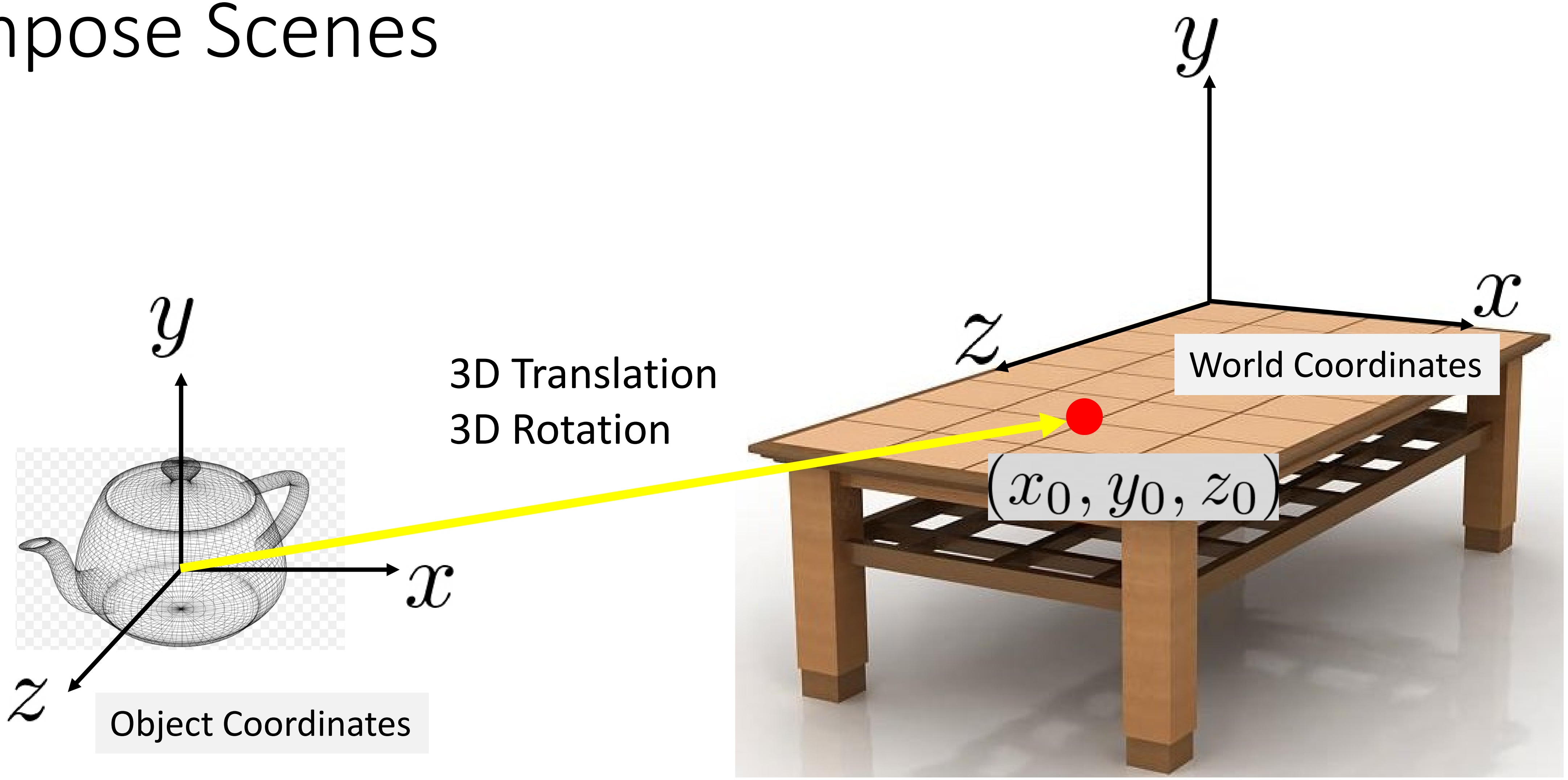
Object Coordinates



World Coordinates

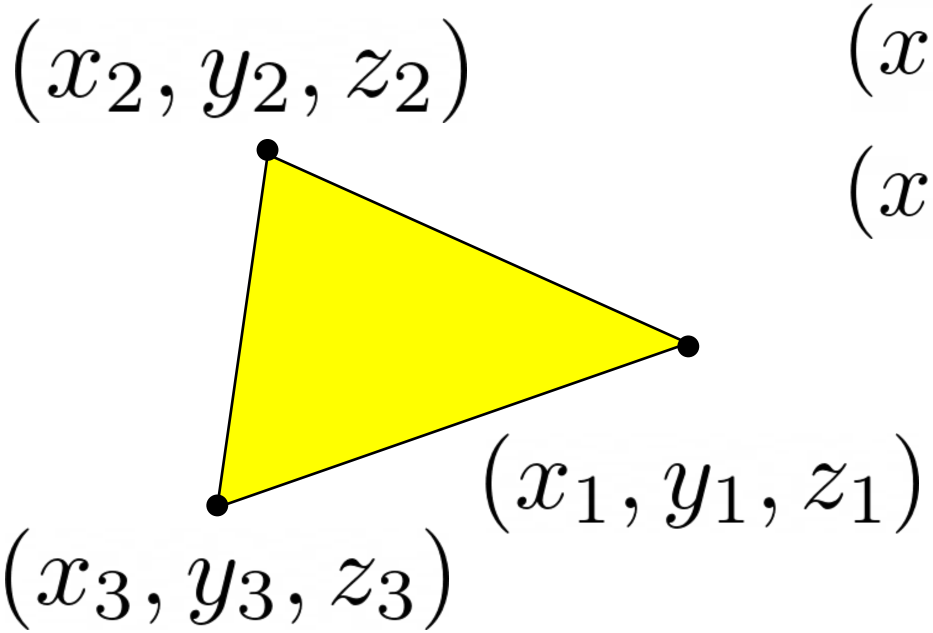


Compose Scenes



3D Translation

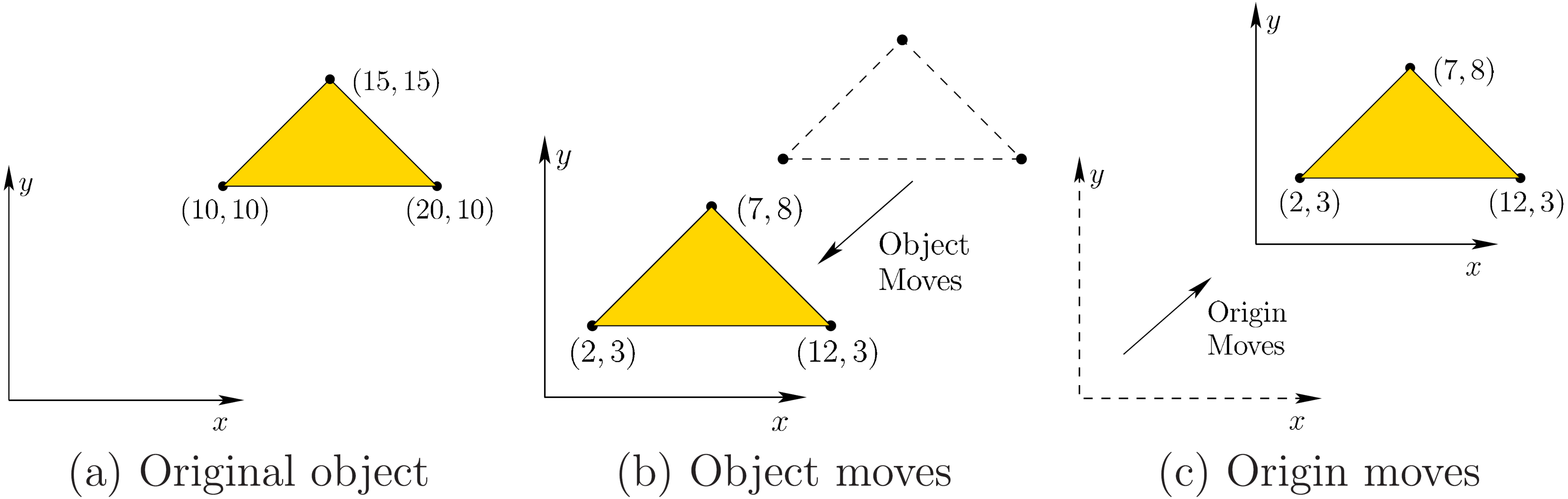
$$(x_1, y_1, z_1) \mapsto (x_1 + x_t, y_1 + y_t, z_1 + z_t)$$
$$(x_2, y_2, z_2) \mapsto (x_2 + x_t, y_2 + y_t, z_2 + z_t)$$
$$(x_3, y_3, z_3) \mapsto (x_3 + x_t, y_3 + y_t, z_3 + z_t)$$



$$\mathbf{v_1} \mapsto \mathbf{v_1} + \mathbf{t}$$
$$\mathbf{v_2} \mapsto \mathbf{v_2} + \mathbf{t}$$
$$\mathbf{v_3} \mapsto \mathbf{v_3} + \mathbf{t}$$

3D Translation $\mathbf{t} = (x_t, y_t, z_t)$

Relativity



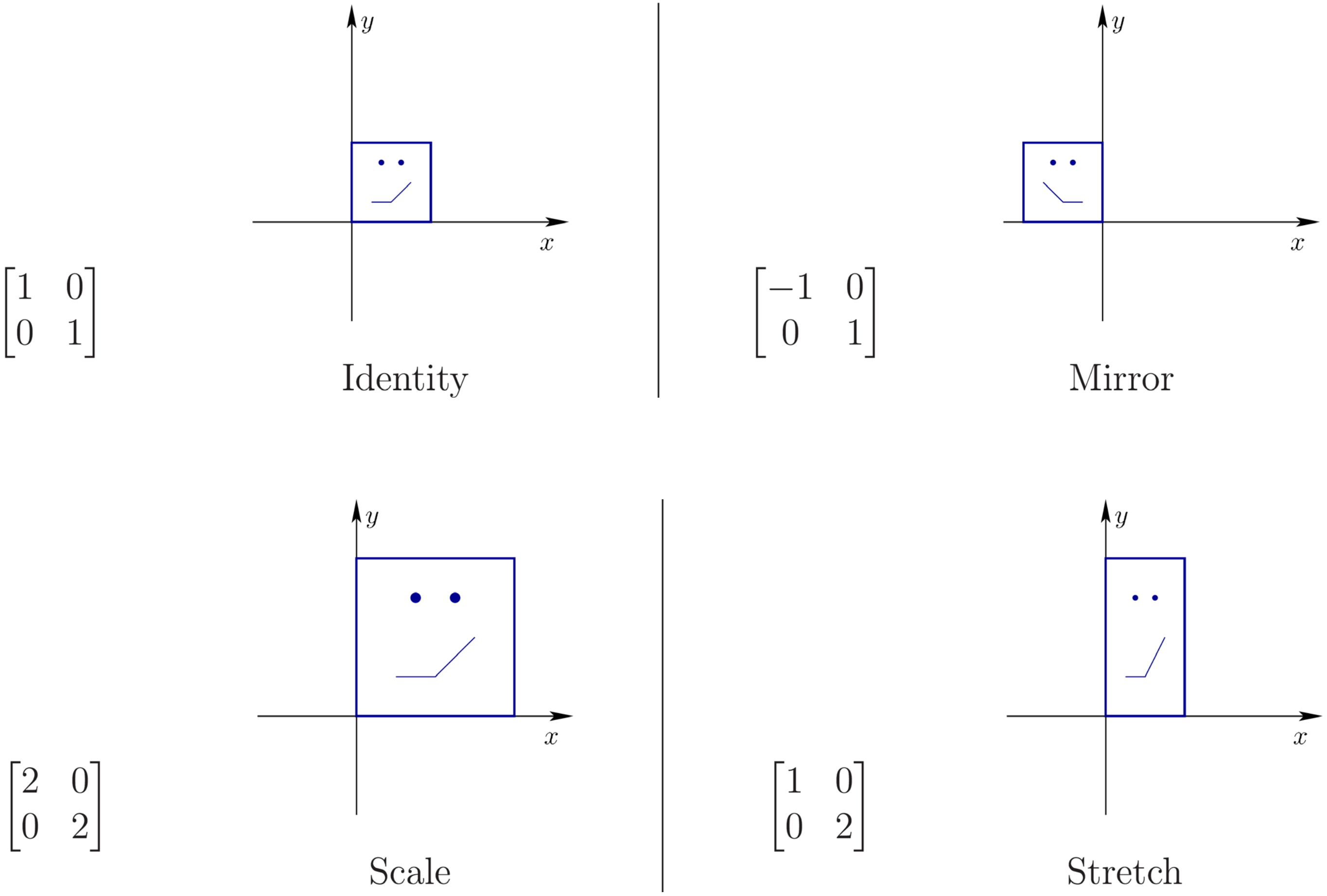
Both result in the same coordinates of the triangle

Apply a 2D Matrix to a 2D point

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x' = m_{11}x + m_{12}y$$
$$y' = m_{21}x + m_{22}y$$



Apply a 2D Matrix to a 2D point

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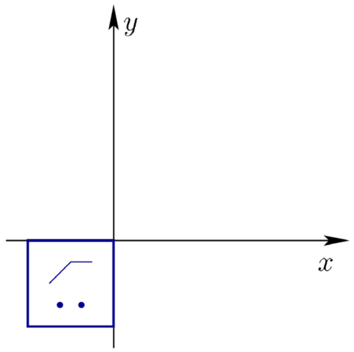
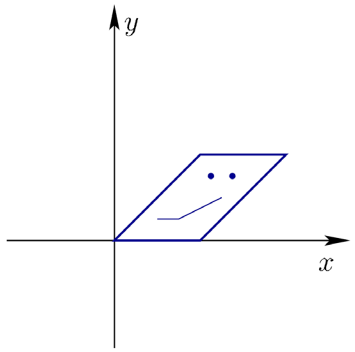
$$x' = m_{11}x + m_{12}y$$
$$y' = m_{21}x + m_{22}y$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



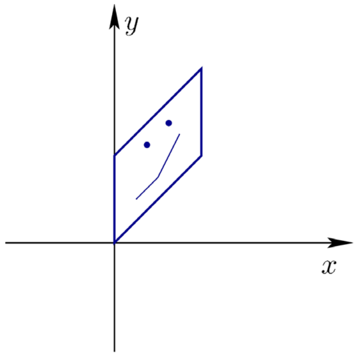
Rotate 180

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



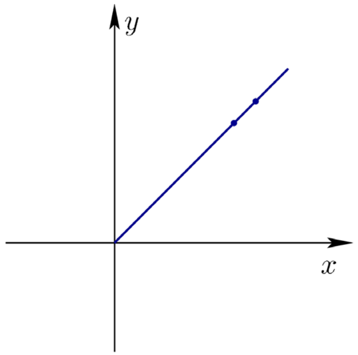
x-shear

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



y-shear

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



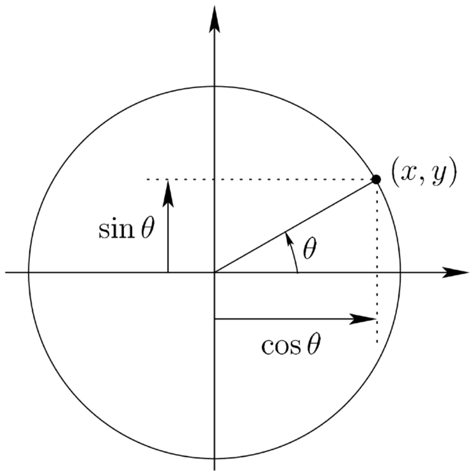
Singular

2D Rotations

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

- No stretching of axes $m_{11}^2 + m_{21}^2 = 1 \text{ and } m_{12}^2 + m_{22}^2 = 1$
- No shearing Dot product $m_{11}m_{12} + m_{21}m_{22} = 0$
- No mirror images $\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = m_{11}m_{22} - m_{12}m_{21} = 1$

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

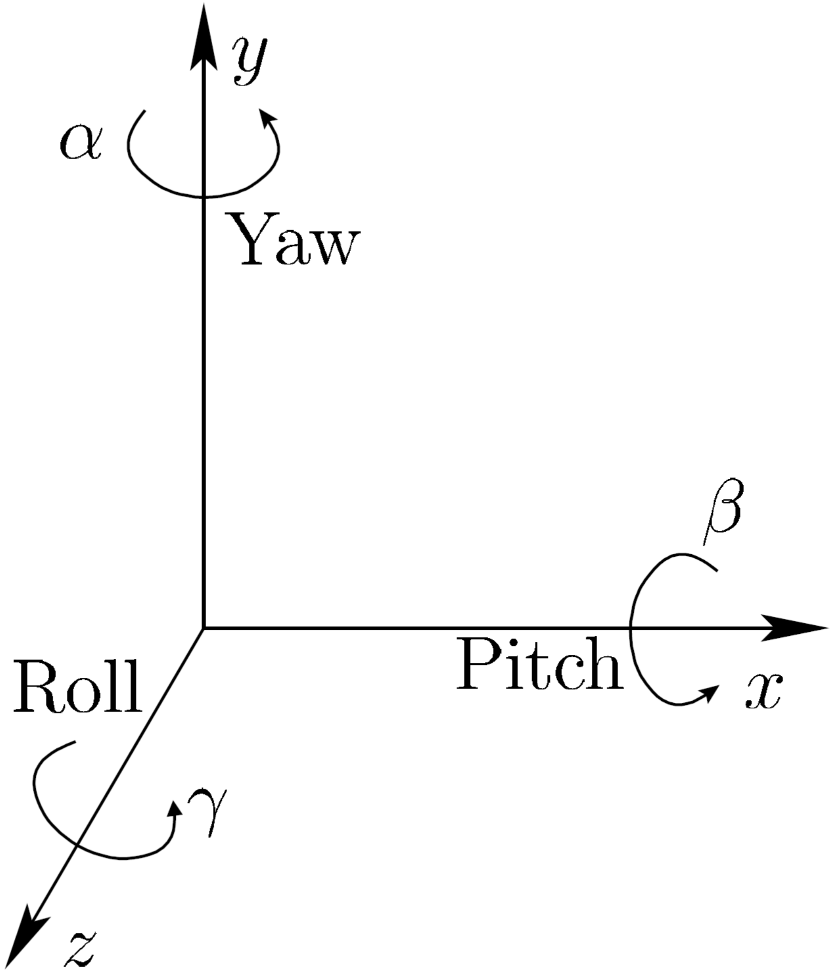


1 Degree of Freedom
Rotate by θ

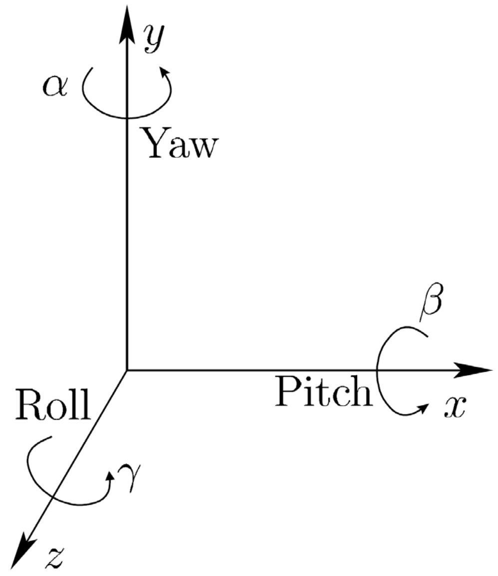
3D Rotations

- Unit-length columns
- Perpendicular columns
- $\det M = 1$
- 3 DOFs

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$



Euler Angles: Yaw, Pitch, Roll



- Counterclockwise rotation

Roll

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pitch

$$R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

Yaw

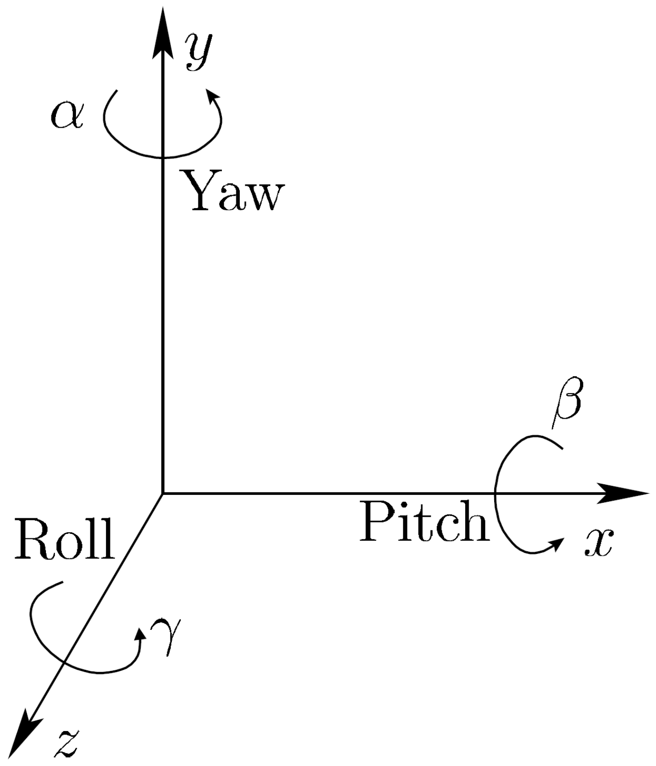
$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Combining Rotations

- Matrix multiplications are “backwards”

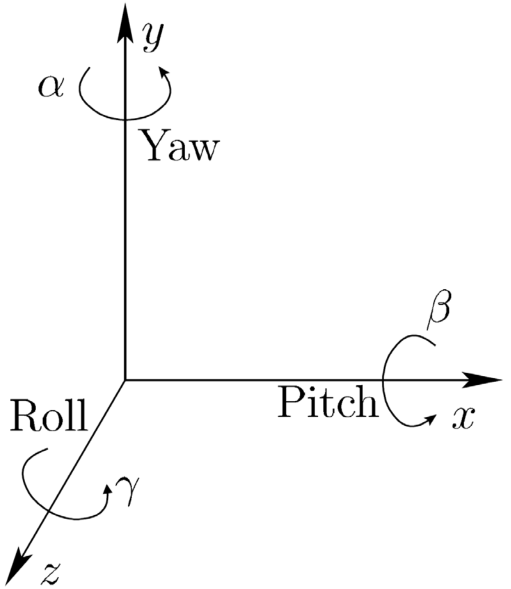
$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\alpha, \gamma \in [0, 2\pi] \qquad \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$



Singularities

- When pitch $\beta = \frac{\pi}{2}$



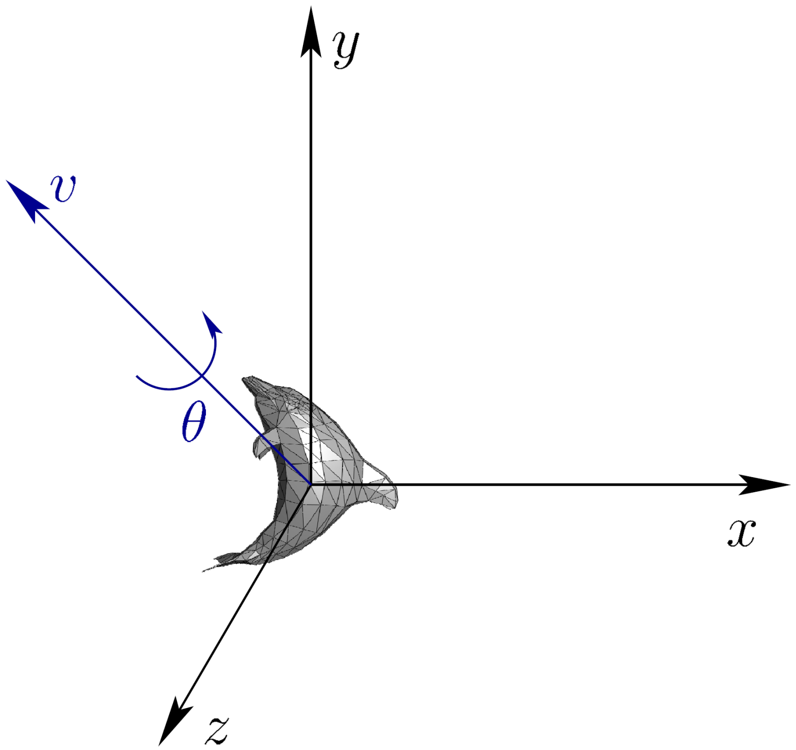
$$R(\alpha, \beta, \gamma) = R_y(\alpha) R_x(\beta) R_z(\gamma)$$

$$\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \\ -\sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \end{bmatrix}$$

Only one DOF

Axis-Angle Representations of Rotation

- Euler's rotation theorem: every 3D rotation can be considered as a rotation by an angle about an axis through the origin

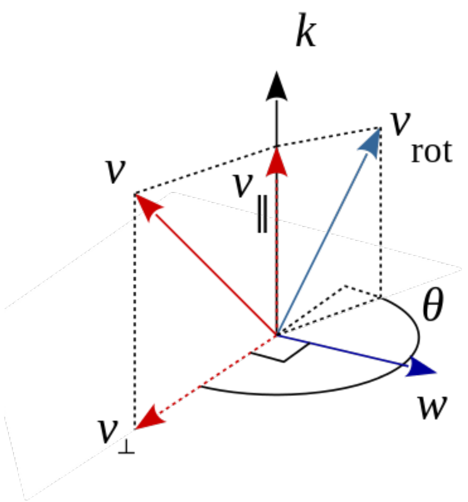


$$\mathbf{v} = (v_1, v_2, v_3)$$

Unit vector

2DOF + 1DOF

Rodrigues' Rotation Formula



- Rotate $\mathbf{v} \in \mathbb{R}^3$ about unit vector $\mathbf{k}$ by angle θ

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}\cos\theta + (\mathbf{k}\times\mathbf{v})\sin\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)$$

Derivation HW1

- Matrix notation $\mathbf{v}_{\mathrm{rot}} = \mathbf{R}\mathbf{v}$

$$\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1-\cos\theta)\mathbf{K}^2$$

Cross product matrix

$$\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

$$\mathbf{k}\times\mathbf{v} = \mathbf{K}\mathbf{v}$$

https://en.wikipedia.org/wiki/Cross_product

SO(n): Special Orthogonal Group

- SO(n): Space of rotation matrices in $\mathbb{R}^n$

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det(R) = 1\}$$

- SO(3): space of 3D rotation matrices
- Group is a set G, with an operation $\bullet$, satisfying the following axioms:
  - Closure: $a \in G, b \in G \Rightarrow a \cdot b \in G$
  - Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in G$
  - Identity element: $\exists e \in G, e \cdot a = a, \forall a \in G$
  - Inverse element: $\forall a \in G, \exists b \in G, a \cdot b = b \cdot a = e$

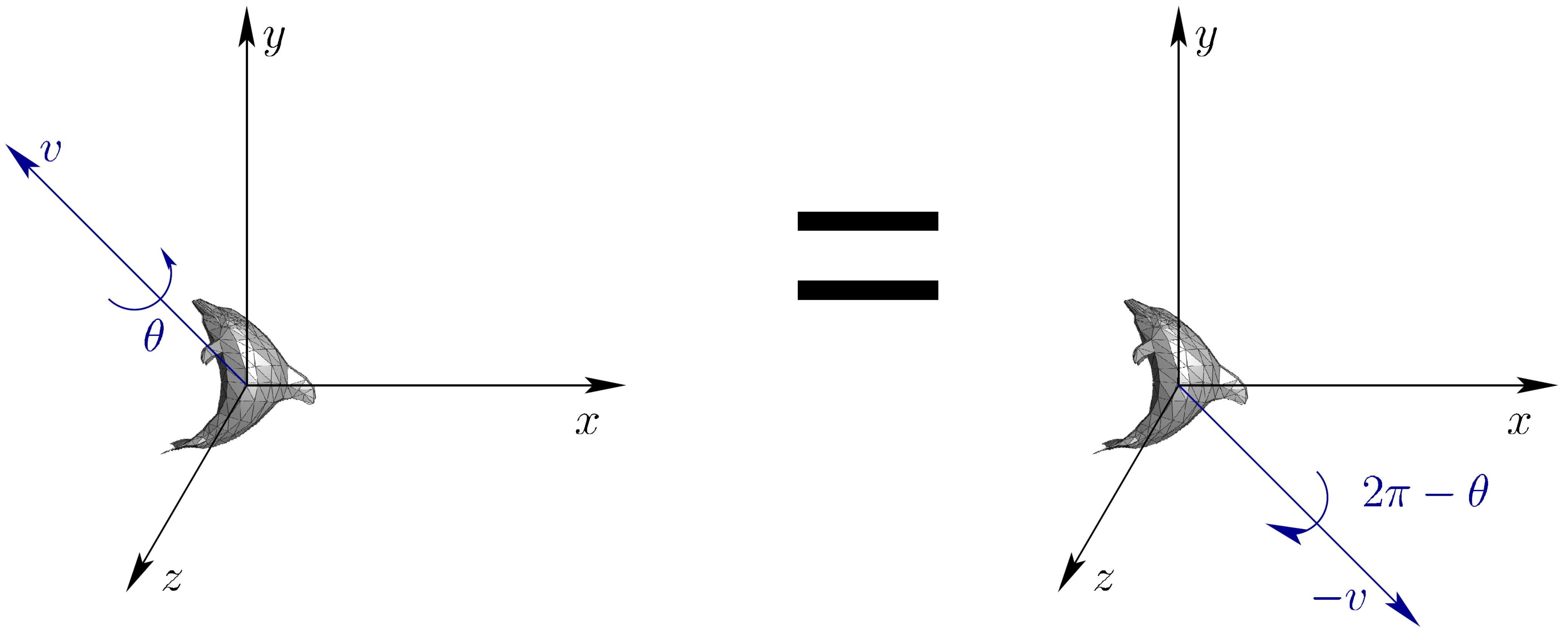
Exponential Map for SO(3)

- Matrix exponential $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$ factorial
- For Lie Group, Hamilton-Cayley theorem $\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k$
  - Coefficients are functions of eigenvalues of X $\quad X : n \times n$
- For SO(3), Rodrigues' Rotation Formula $so(n) = \{K \in \mathbb{R}^{n \times n} : K^T = -K\}$

skew-symmetric matrix $\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$

$$\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1 - \cos\theta)\mathbf{K}^2$$
$$= \exp(\theta K)$$

Two-to-one Problem of Axis-Angle Representations



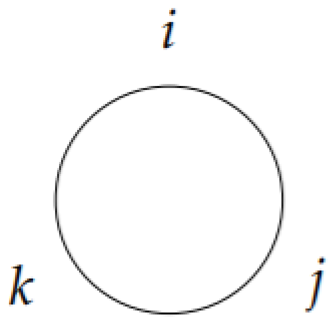
Quaternions for 3D Rotations

- Quaternions generalize complex numbers and can be used to represents 3D rotations

$$q = \underset{\substack{\uparrow \\ \text{Scale (real part)}}}{w} + \underbrace{xi + yj + zk}_{\text{Vector (imaginary part)}}$$

- Properties $i^2 = j^2 = k^2 = -1$

$$ij = k, ji = -k$$
$$jk = i, kj = -i$$
$$ki = j, ik = -j$$



Quaternion Addition and Multiplication

- Addition

$$p + q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k}$$

- Multiplication

$$\begin{aligned} pq &= (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) + q_0(p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\ &\qquad + (p_2q_3 - p_3q_2)\boldsymbol{i} + (p_3q_1 - p_1q_3)\boldsymbol{j} + (p_1q_2 - p_2q_1)\boldsymbol{k}. \end{aligned}$$

$$pq = p_0q_0 - \boldsymbol{p} \cdot \boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q}$$

$$\boldsymbol{p} = (p_1, p_2, p_3) \quad \boldsymbol{q} = (q_1, q_2, q_3)$$

Complex Conjugate, Norm and Inverse

- Conjugate $$q = q_0 + \boldsymbol{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$
$$q^* = q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$$

- Norm $$|q| = \sqrt{q^*q}$$

$$\begin{aligned} q^*q &= (q_0 - \boldsymbol{q})(q_0 + \boldsymbol{q}) \\ &= q_0q_0 - (-\boldsymbol{q}) \cdot \boldsymbol{q} + q_0\boldsymbol{q} + (-\boldsymbol{q})q_0 + (-\boldsymbol{q}) \times \boldsymbol{q} \\ &= q_0^2 + \boldsymbol{q} \cdot \boldsymbol{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ &= qq^*. \end{aligned}$$

- Inverse $$q^{-1} = \frac{q^*}{|q|^2} \qquad q^{-1}q = qq^{-1} = 1$$

Unit Quaternions as 3D Rotations

- For $\boldsymbol{v} \in \mathbb{R}^3$, rotation according to a unit quaternion $q = q_0 + \boldsymbol{q}$

$$\begin{aligned} L_q(\boldsymbol{v}) &= q\boldsymbol{v}q^* \\ &= (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q} \cdot \boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}) \end{aligned}$$

The real part of v is 0

- For unit quaternions, axis-angle

$$(v, \theta) \Longleftrightarrow q = \left(\cos\frac{\theta}{2}\,,\ v_1 \sin\frac{\theta}{2}\,,\ v_2 \sin\frac{\theta}{2}\,,\ v_3 \sin\frac{\theta}{2}\right)$$

Two Equivalent Quaternions for 3D Rotation

- Multiply -1 to a quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\frac{\vec{u}}{||\vec{u}||}$$

$$-q = \cos\left(\frac{\theta}{2} + \pi\right) + \sin\left(\frac{\theta}{2} + \pi\right)\frac{\vec{u}}{||\vec{u}||}$$

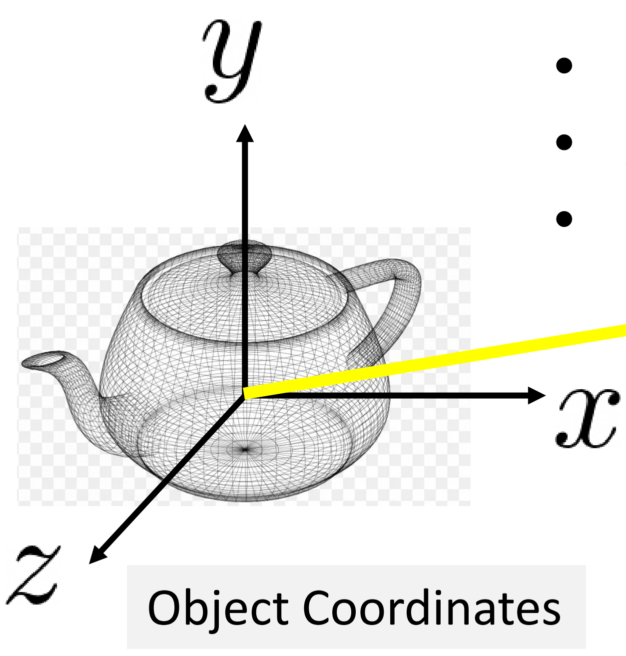
- q rotates θ , $-q$ rotates $\theta + 2\pi$

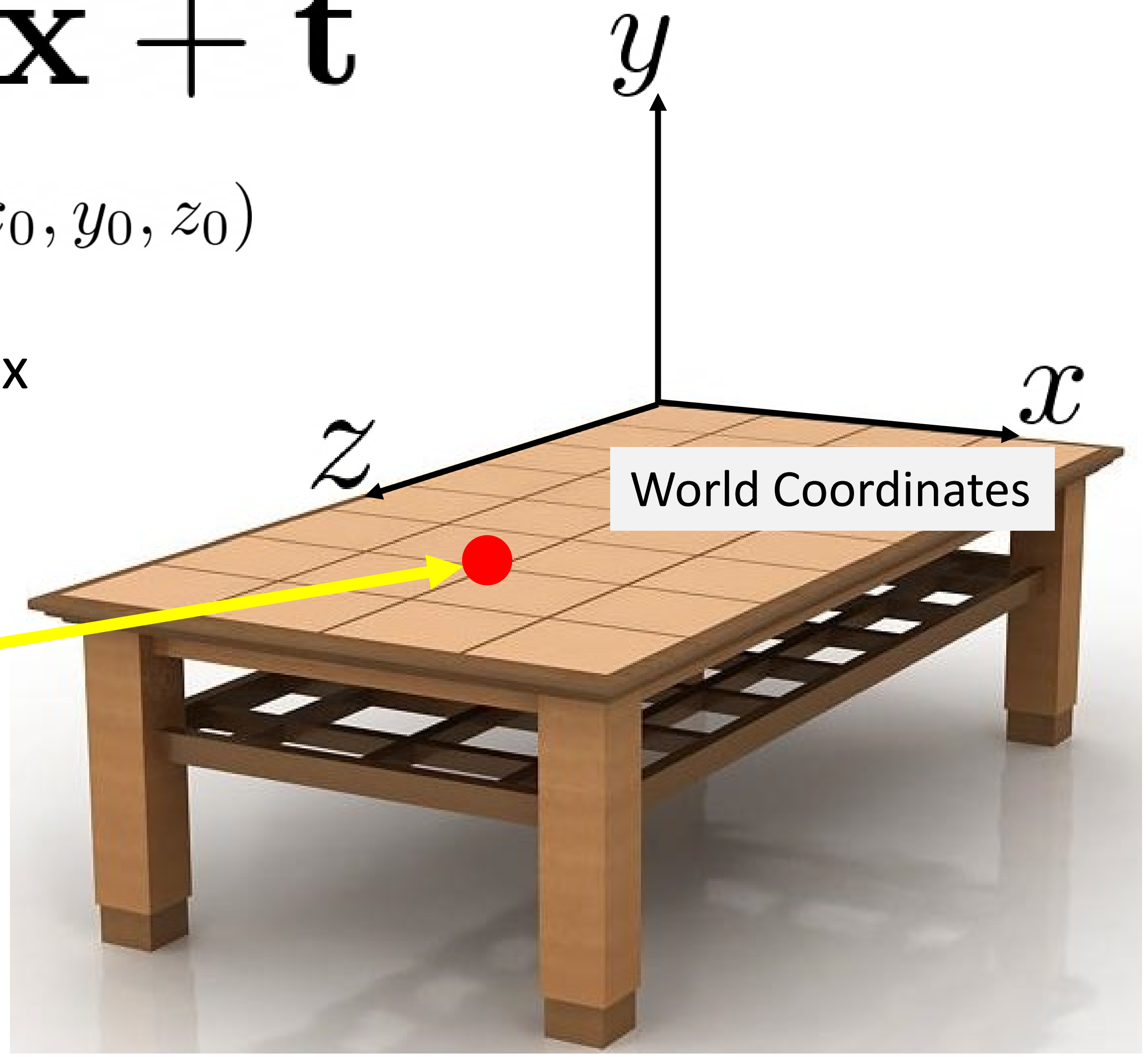
Transform 3D Models $R\mathbf{x} + \mathbf{t}$

3D Translation (x_0, y_0, z_0)

3D Rotation

- Rotation matrix
- Euler angles
- Axis-angle
- Quaternion





Further Reading

- Chapter 3, Virtual Reality, Steven LaValle
- Quaternion and Rotations, Yan-Bin Jia, https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf
- Introduction to Robotics, Prof. Wei Zhang, OSU, Lecture 3, Rotational Motion, http://www2.ece.ohio-state.edu/~zhang/RoboticsClass/index.html