Velocity Kinematics: Jacobian

CS 6341 Robotics

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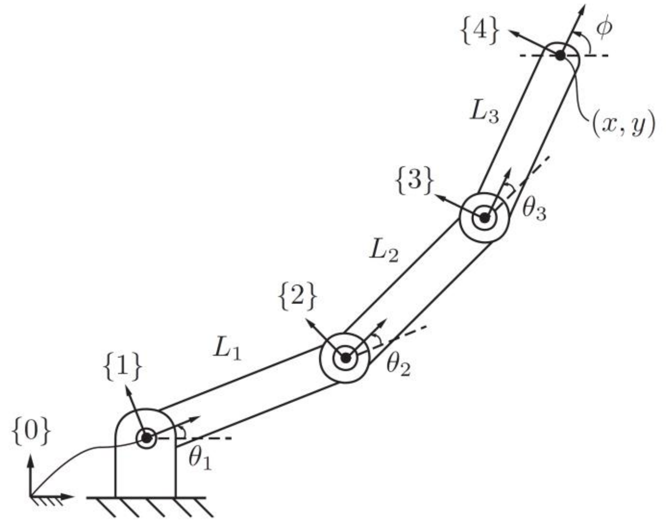
Recall Forward Kinematics

- Forward kinematics of a robot: calculation of the position and orientation of its end-effector from its joint coordinates

End-effector transformation

Joint coordinates θ $\Rightarrow$ $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

Recall Forward Kinematics



- Method 1: uses homogeneous transformations
  - Need to define the coordinates of frames
  - Denavit-Hartenberg Parameters

$$T_{04} = T_{01} T_{12} T_{23} T_{34}$$

- Method 2: uses screw-axis representations of transformations
  - No need to define frame references

Space form
$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Screw axis when all the thetas are 0s

Body form
$$T_{04} = M e^{[\mathcal{B}]_1\theta_1} e^{[\mathcal{B}]_2\theta_2} e^{[\mathcal{B}]_3\theta_3}$$

Velocity Kinematics

- Assume end-effector configuration $x \in \mathbb{R}^m$
- End-effector velocity $\dot{x} = dx/dt \in \mathbb{R}^m$
- Forward kinematics $x(t) = f(\theta(t))$ $\theta \in \mathbb{R}^n$ Joint variable
- Chain rule

$$\dot{x} = \frac{\partial f(\theta)}{\partial \theta} \frac{d\theta(t)}{dt} = \frac{\partial f(\theta)}{\partial \theta} \dot{\theta}$$
$$= J(\theta)\dot{\theta},$$

$J(\theta) \in \mathbb{R}^{m \times n}$ Jacobian

$\dot{\theta}$ Joint velocity

Gradients

How to compute gradient?

$L(\mathbf{y})$ scalar $\mathbf{y} : m \times 1$

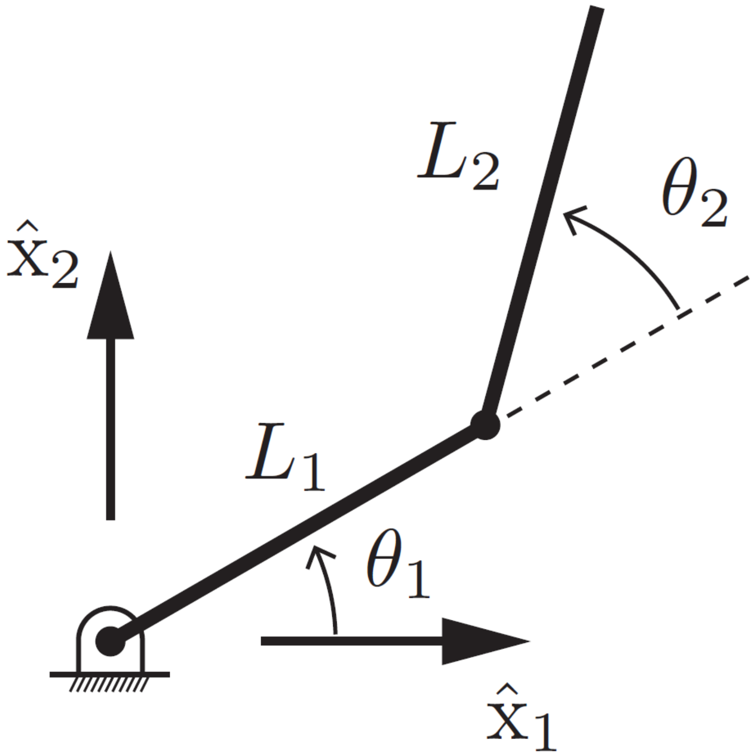
$$\frac{\partial L}{\partial \mathbf{y}} \begin{bmatrix} \frac{\partial L}{y_1} & \frac{\partial L}{y_2} & \cdots & \frac{\partial L}{y_m} \end{bmatrix}$$

$1 \times m$

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \nabla f_1(\mathbf{x}) \\ \nabla f_2(\mathbf{x}) \\ \cdots \\ \nabla f_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \mathbf{x}} f_1(\mathbf{x}) \\ \frac{\partial}{\partial \mathbf{x}} f_2(\mathbf{x}) \\ \cdots \\ \frac{\partial}{\partial \mathbf{x}} f_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(\mathbf{x}) & \frac{\partial}{\partial x_2} f_1(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n} f_1(\mathbf{x}) \\ \frac{\partial}{\partial x_1} f_2(\mathbf{x}) & \frac{\partial}{\partial x_2} f_2(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n} f_2(\mathbf{x}) \\ & & \cdots & \\ \frac{\partial}{\partial x_1} f_m(\mathbf{x}) & \frac{\partial}{\partial x_2} f_m(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n} f_m(\mathbf{x}) \end{bmatrix}$$

Jacobian matrix

Jacobian



a 2R planar open chain

Forward kinematics

$$\begin{aligned} x_1 &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ x_2 &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2). \end{aligned}$$

Differentiate with respect to time

$$\begin{aligned} \dot{x}_1 &= -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \\ \dot{x}_2 &= L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2), \end{aligned}$$

$$\dot{x} = J(\theta)\dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

Velocity Kinematics

- Given joint positions and velocities $\theta \in \mathbb{R}^n$ $\dot{\theta}$

- Compute the velocity of the end-effector

End-effector configuration

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \xrightarrow[\dot{p}(t) = \frac{d}{dt}p(t)]{\dot{R}(t) = \frac{d}{dt}R(t)} \dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

What is this?

Twists: Angular Velocity and Linear Velocity

Spatial twist $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$

$$\dot{T} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$[\omega_s] = \dot{R}R^{-1} \qquad v_s = \dot{p} + \omega_s \times (-p)$$

Body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$

$$[\omega_b] = R^{-1}\dot{R} \qquad v_b = R^T\dot{p}$$

Twists: Angular Velocity and Linear Velocity

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Lynch & Park 3.3.2

Adjoint mapping

$$[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3)$$

$$[\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1}$$

$$[\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} = T^{-1}\dot{T}$$

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}\,[\mathcal{V}_s]\,T \end{aligned}$$

Manipulator Jacobian

- Forward kinematics

$$T(\theta_1, \ldots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M$$

$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$\begin{aligned}\dot{T} &= \left(\frac{d}{dt} e^{[\mathcal{S}_1]\theta_1}\right) \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} \left(\frac{d}{dt} e^{[\mathcal{S}_2]\theta_2}\right) \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots \\ &= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2]\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots\end{aligned}$$

$\dot{\theta}_i$ is a scalar

$$T^{-1} = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \cdots e^{-[\mathcal{S}_1]\theta_1}$$

$$d(e^{A\theta})/dt = Ae^{A\theta}\dot{\theta} = e^{A\theta}A\dot{\theta}$$

Proposition 3.10

Space Jacobian

Adjoint map associated with T

$$\mathcal{V}' = \mathrm{Ad}_T(\mathcal{V})$$

$$[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$[\mathcal{V}_s] = \dot{T}T^{-1}$$

$$[\mathcal{V}_s] = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots$$

Adjoint mapping

$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}}\dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}}\dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}}\dot{\theta}_3 + \cdots$$

$$\mathcal{V}_s = J_{s1}\dot{\theta}_1 + J_{s2}(\theta)\dot{\theta}_2 + \cdots + J_{sn}(\theta)\dot{\theta}_n$$

Space Jacobian

Spatial twist
$$\mathcal{V}_s = \begin{bmatrix} J_{s1} & J_{s2}(\theta) & \cdots & J_{sn}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$
$$= J_s(\theta)\dot{\theta}.$$

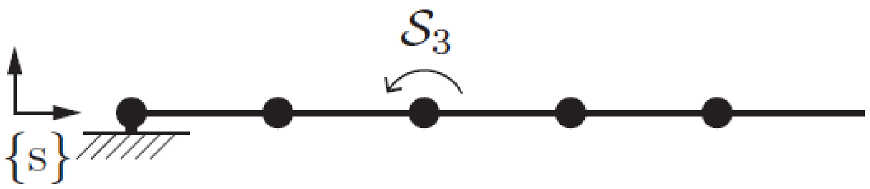
$$J_s(\theta) \in \mathbb{R}^{6\times n} \qquad \dot{\theta} \in \mathbb{R}^n$$

$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$ ith column $i = 2, \ldots, n,$

$$J_{s1} = \mathcal{S}_1$$

Visualizing the Space Jacobian

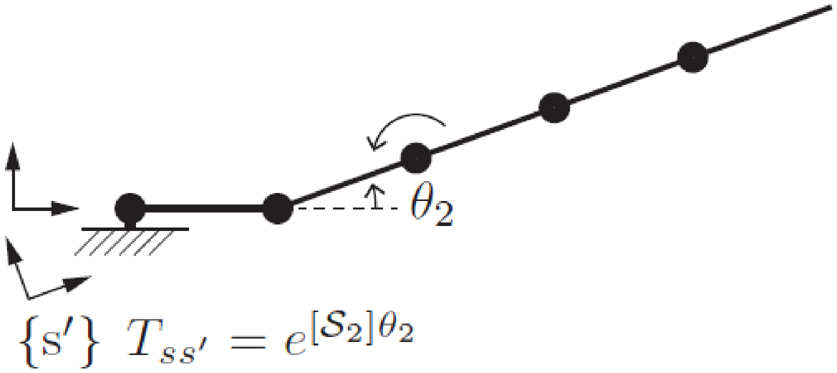
$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}} \dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}} \dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}} \dot{\theta}_3 + \cdots$$

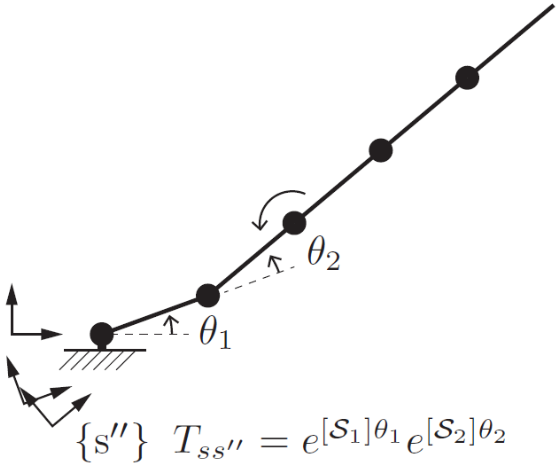


Consider some input $\dot{\theta}_3$ on $\mathcal{S}_3$

$\theta_3, \theta_4, \theta_5$ won't change $\mathcal{S}_3$ in {s}

No contribution to the twist





$\mathcal{S}_3$ represents the screw relative to $\{s''\}$

for arbitrary θ_1, θ_2

$$[\mathrm{Ad}_{T_{ss''}}] = [\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}]$$

$$J_{s3} = [\mathrm{Ad}_{T_{ss''}}]\mathcal{S}_3$$

Space Jacobian

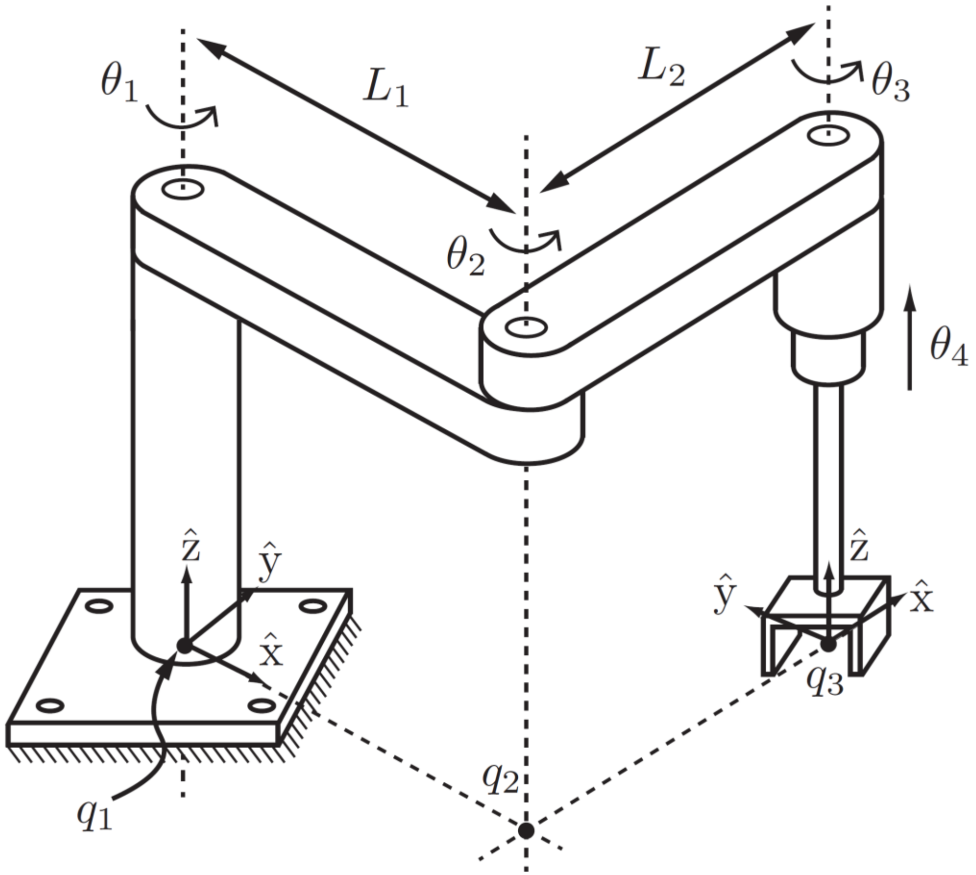
- The ith column of the space Jacobian

$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$

$$\mathrm{Ad}_{T_{i-1}}(\mathcal{S}_i) \qquad T_{i-1} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}$$

$J_{si}(\theta)$ is simply the screw vector describing joint axis i, expressed in fixed-frame coordinates, as a function of the joint variables $\theta_1, \ldots, \theta_{i-1}$.

Space Jacobian



a spatial RRRP chain

$$J_s(\theta) \text{ by } J_{si} = (\omega_{si}, v_{si})$$

$$\omega_{s1} = (0,0,1) \quad v_{s1} = (0,0,0)$$

$$\omega_{s2} = (0,0,1) \quad q_2 \; (L_1 c_1, L_1 s_1, 0)$$

$$v_{s2} = -\omega_2 \times q_2 = (L_1 s_1, -L_1 c_1, 0)$$

$$c_1 = \cos\theta_1, \; s_1 = \sin\theta_1$$

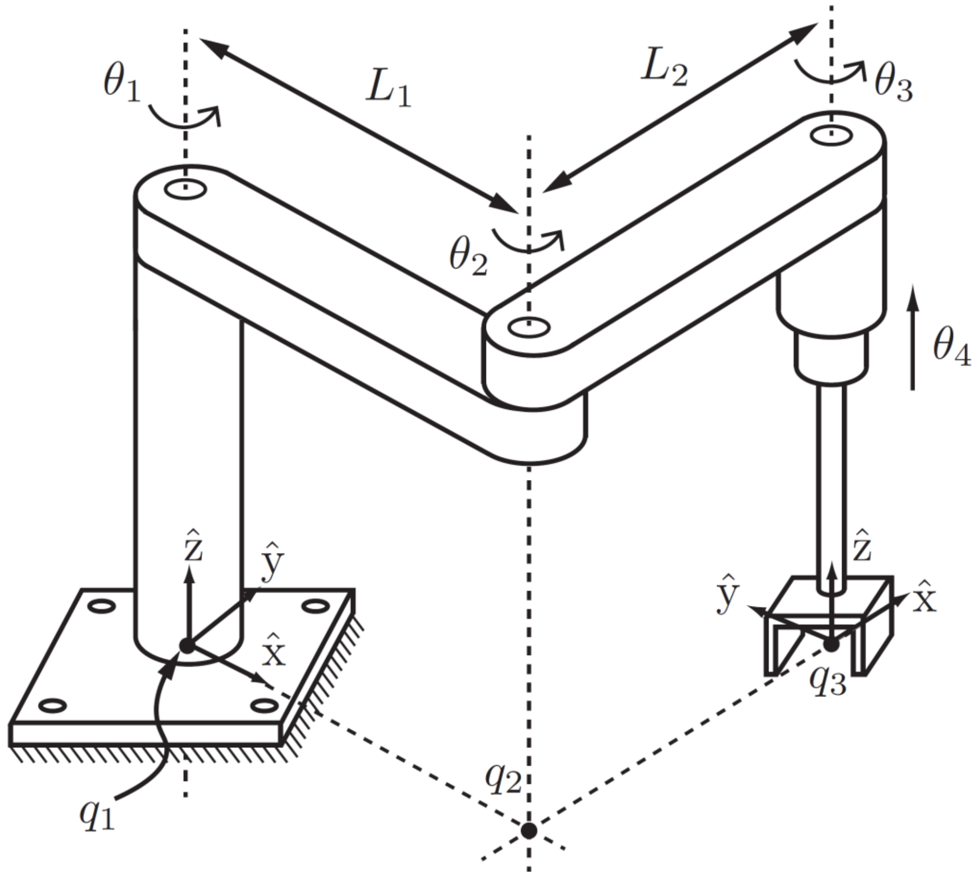
$$\omega_{s3} = (0,0,1) \quad q_3 = (L_1 c_1 + L_2 c_{12}, L_1 s_1 + L_2 s_{12}, 0)$$

$$c_{12} = \cos(\theta_1 + \theta_2), \; s_{12} = \sin(\theta_1 + \theta_2)$$

$$v_{s3} = (L_1 s_1 + L_2 s_{12}, -L_1 c_1 - L_2 c_{12}, 0)$$

$$\omega_{s4} = (0,0,0) \quad v_{s4} = (0,0,1)$$

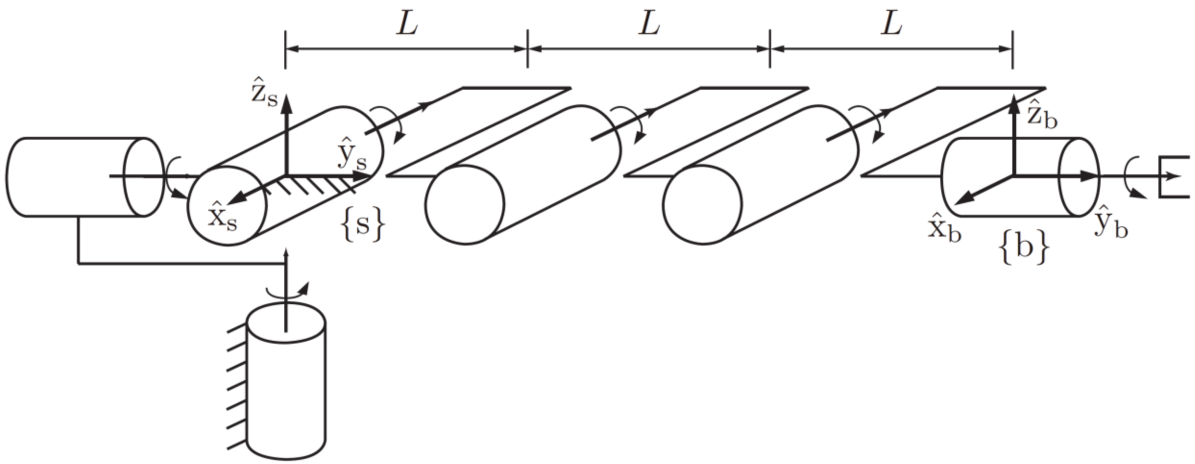
Space Jacobian



a spatial RRRP chain

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1 \mathrm{s}_1 & L_1 \mathrm{s}_1 + L_2 \mathrm{s}_{12} & 0 \\ 0 & -L_1 \mathrm{c}_1 & -L_1 \mathrm{c}_1 - L_2 \mathrm{c}_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall Screw Axes in the End-Effector Frame



PoE forward kinematics for the 6R open chain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,0)$
4	$(-1,0,0)$	$(0,0,L)$
5	$(-1,0,0)$	$(0,0,2L)$
6	$(0,1,0)$	$(0,0,0)$

Space form

i	ω_i	v_i
1	$(0,0,1)$	$(-3L,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,-3L)$
4	$(-1,0,0)$	$(0,0,-2L)$
5	$(-1,0,0)$	$(0,0,-L)$
6	$(0,1,0)$	$(0,0,0)$

Body form

Body Jacobian

- End-effect twist in the end-effector frame $[\mathcal{V}_b] = T^{-1}\dot{T}$
- Forward kinematics

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}$$

$$\begin{aligned}
\dot{T} =& Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}\left(\frac{d}{dt}e^{[\mathcal{B}_n]\theta_n}\right) \\
&+ Me^{[\mathcal{B}_1]\theta_1}\cdots\left(\frac{d}{dt}e^{[\mathcal{B}_{n-1}]\theta_{n-1}}\right)e^{[\mathcal{B}_n]\theta_n} + \cdots \\
=& Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_n]\theta_n}[\mathcal{B}_n]\dot{\theta}_n \\
&+ Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \cdots \\
&+ Me^{[\mathcal{B}_1]\theta_1}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1.
\end{aligned}$$

$$d(e^{A\theta})/dt = Ae^{A\theta}\dot{\theta} = e^{A\theta}A\dot{\theta}$$

$$T^{-1} = e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_1]\theta_1}M^{-1}$$

Body Jacobian

$$[\mathcal{V}_b] = T^{-1}\dot{T}$$

$$\begin{aligned}[\mathcal{V}_b] \quad &= \quad [\mathcal{B}_n]\dot{\theta}_n + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \cdots \\ &\qquad + e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_2]\theta_2}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1\end{aligned}$$

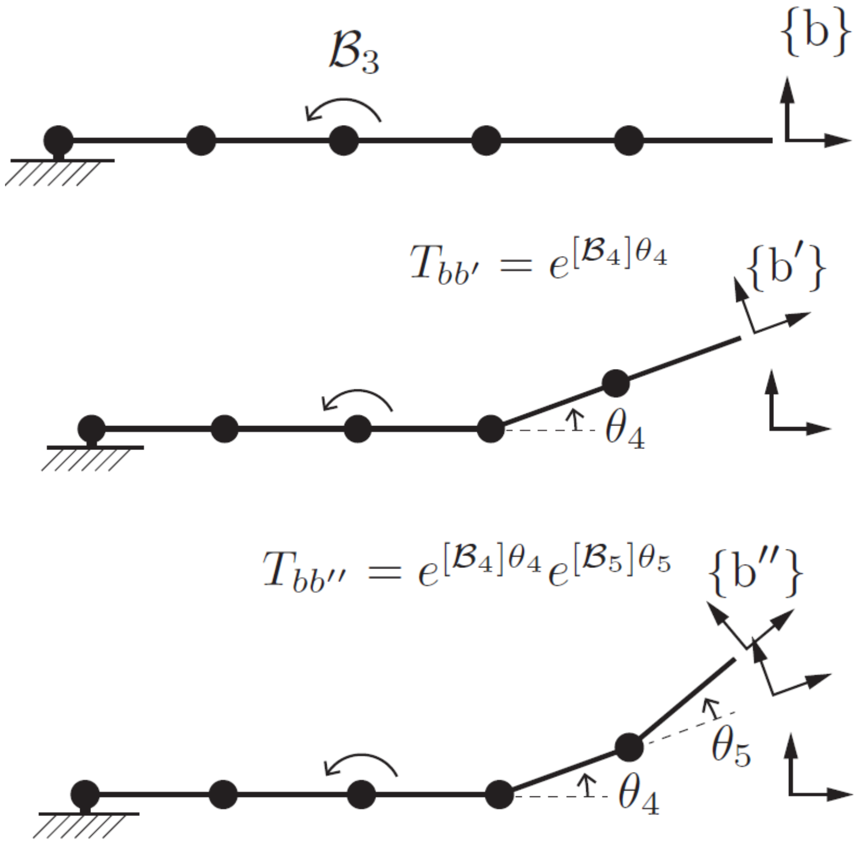
$$\mathcal{V}_b = \underbrace{\mathcal{B}_n}_{J_{bn}}\dot{\theta}_n + \underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})}_{J_{b,n-1}}\dot{\theta}_{n-1} + \cdots + \underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)}_{J_{b1}}\dot{\theta}_1$$

$$\mathcal{V}_b = J_{b1}(\theta)\dot{\theta}_1 + \cdots + J_{bn-1}(\theta)\dot{\theta}_{n-1} + J_{bn}\dot{\theta}_n$$

$$J_{bi}(\theta) = (\omega_{bi}(\theta), v_{bi}(\theta))$$

Visualizing the Body Jacobian

$$\mathcal{V}_b = \underbrace{\mathcal{B}_n}_{J_{bn}} \dot{\theta}_n + \underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})}_{J_{b,n-1}} \dot{\theta}_{n-1} + \cdots + \underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)}_{J_{b1}} \dot{\theta}_1$$



Consider some input $\dot{\theta}_3$ on $\mathcal{B}_3$

$\theta_1, \theta_2, \theta_3$ won't change $\mathcal{B}_3$ in {b}

No contribution to the twist

$\mathcal{B}_3$ is expressed in $\{b\}$

$$\begin{aligned} J_{b3} &= [\mathrm{Ad}_{T_{b''b}}]\mathcal{B}_3 \\ &= [\mathrm{Ad}_{T^{-1}_{bb''}}]\mathcal{B}_3 \\ &= [\mathrm{Ad}_{e^{-[\mathcal{B}_5]\theta_5} e^{-[\mathcal{B}_4]\theta_4}}]\mathcal{B}_3 \end{aligned}$$

Body Jacobian

$$\mathcal{V}_b = \begin{bmatrix} J_{b1}(\theta) & \cdots & J_{bn-1}(\theta) & J_{bn} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = J_b(\theta)\dot{\theta}$$

body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ $\quad \dot{\theta} \in \mathbb{R}^n$

$$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i) \qquad i = n-1, \ldots, 1$$

$J_{bn} = \mathcal{B}_n$ The screw vector for joint axis i, expressed in the coordinates of the end-effector frame rather than those of the fixed frame

Relationship between the Space and Body Jacobian

- Fixed frame {s}, body frame {b}
- Forward kinematics $T_{sb}(\theta)$
- Twist of the end-effector frame

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\begin{aligned} [\mathcal{V}_s] &= \dot{T}_{sb}T_{sb}^{-1}, & \mathcal{V}_s &= J_s(\theta)\dot{\theta}, \\ [\mathcal{V}_b] &= T_{sb}^{-1}\dot{T}_{sb}, & \mathcal{V}_b &= J_b(\theta)\dot{\theta}. \end{aligned}$$

$$\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$$

$$\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) = J_s(\theta)\dot{\theta}$$

Applying $[\mathrm{Ad}_{T_{bs}}]$ to both sides

$$\mathrm{Ad}_{T_{bs}}(\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)) = \mathrm{Ad}_{T_{bs}T_{sb}}(\mathcal{V}_b) = \mathcal{V}_b = Ad_{T_{bs}}(J_s(\theta)\dot{\theta})$$

$$J_b(\theta) = \mathrm{Ad}_{T_{bs}}(J_s(\theta)) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)$$

$$J_s(\theta) = \mathrm{Ad}_{T_{sb}}(J_b(\theta)) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta)$$

Summary

- Velocity kinematics
- Jacobian
  - Space Jacobian
  - Body Jacobian

Further Reading

- Chapter 5 in Kevin M. Lynch and Frank C. Park. Modern Robotics: Mechanics, Planning, and Control. 1st Edition, 2017.

- T. Yoshikawa. Manipulability of robotic mechanisms. International Journal of Robotics Research, 4(2):3-9, 1985.