Optimal Stopping Time: Algorithm Benchmarking

Comparative Analysis of Discrete VFI (Howard) and Continuous HJB (Newton)

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Outline

The Research Question

Objective

To implement and benchmark two distinct numerical methods for solving **Optimal Stopping Time** problems in macroeconomics.

Key Trade-off to Investigate:

- **Discrete Time VFI:** Known for robustness and intuitive probability interpretation.
- **Continuous Time HJB:** Known for analytical elegance and potential speed.

Question: Can the HJB method achieve superior efficiency without sacrificing stability in high-volatility regimes?

The Economic Model: Hopenhayn-style Exit

We model a firm with stochastic profitability facing a fixed operating cost.

- **State Variable:** Profitability x_t follows a Geometric Brownian Motion (GBM):

$$dx_t = \mu x_t dt + \sigma x_t dZ_t$$

- **Decisions:**
  1. Continue operating: Flow payoff $\pi(x_t) = x_t - c_f$.
  2. Exit: Receive scrap value $S = 0$.
- **Bellman Equation (Continuous):**

$$\rho V(x) = \max \left\{ 0,\ (x - c_f) + \frac{1}{dt}\mathbb{E}[dV] \right\}$$

Analytical Benchmark (Ground Truth)

Calibration (Baseline):

- Drift $\mu = -0.01$ (Declining industry)
- Volatility $\sigma = 0.20$
- Discount $\rho = 0.05$
- Fixed Cost $c_f = 1.0$

Exact Solution: Using the smooth pasting condition, the optimal exit threshold x^* is derived from the characteristic roots:

$$x^* = \frac{\beta_2}{\beta_2 - 1} c_f \frac{\rho - \mu}{\rho}$$

Target Result

For baseline parameters: $\mathbf{x^* = 0.6000}$

Method 1: Discrete Time VFI

Approach: Discretize the state space (Log-grid) and time step.

1. **Grid Construction:** Used a **Log-spaced grid** (N points) to capture the GBM skewness.
2. **Recursive Formulation:**

$$V(x) = \max \left\{ S, \ (x - c_f)\Delta t + \beta \sum_{x'} P(x'|x)V(x') \right\}$$

3. **Solver: Howard Improvement (Policy Iteration)**
   - *Innovation:* Instead of relying solely on value function contraction (slow), we solve the linear system $(I - \beta T_{pol})V = u$ whenever the policy stabilizes.
   - **Result:** Reduces iterations from thousands to < 50.

Method 2: Continuous Time HJB

Approach: Finite Difference Method (FDM) with Newton Solver.

1. **HJB Variational Inequality:**

$$\min \left\{ \underbrace{\rho V - (\mu x V_x + \frac{1}{2}\sigma^2 x^2 V_{xx} + x - c_f)}_{\text{Continuation Residual}}, \underbrace{V - S}_{\text{Exit Residual}} \right\} = 0$$

2. **Linear System:** $(\rho I - A)V = u$, where A is a tridiagonal matrix constructed using an **Upwind Scheme** to ensure stability.
3. **Solver: Semi-Smooth Newton Method**
   - Treats the problem as a Linear Complementarity Problem (LCP).
   - Updates the "Active Set" (Exit nodes vs Stay nodes) iteratively.
   - **Result:** Extremely fast convergence (quadratic).

Discrete VFI: Convergence to Truth

Time Discretization Bias: Discrete VFI cannot capture exits "between" time steps.

As $dt \to 0$, the numerical threshold converges to the analytical truth (0.60).

dt	Discrete x*	Error
0.100	0.6220	3.67%
0.010	0.6050	0.84%
0.001	**0.6006**	**0.09%**

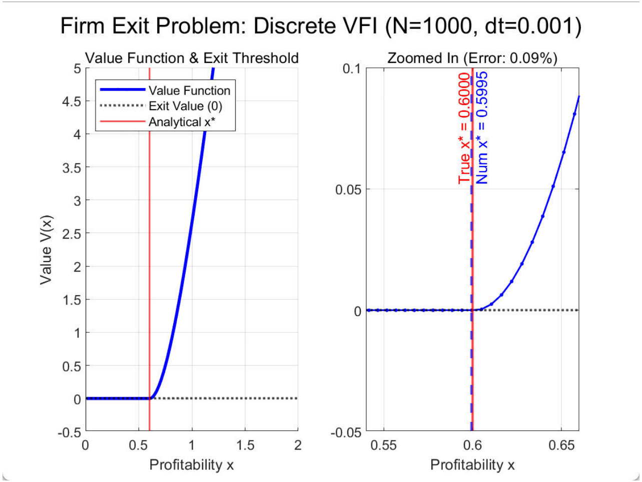


Figure: Discrete Solution ($N = 1000$)

Continuous HJB: Speed & Accuracy

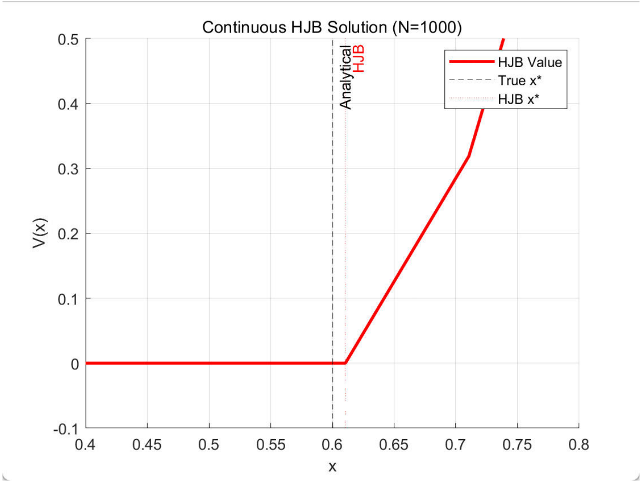


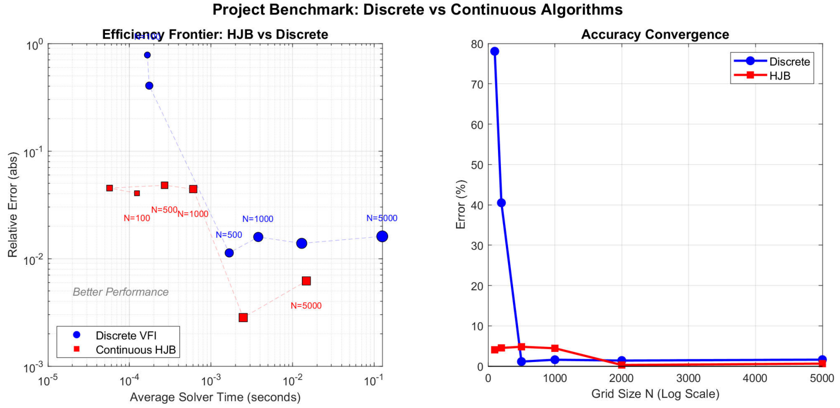
Figure: HJB Solution ($N = 1000$)

Performance Stats:

- **Time:** ~ 0.01 seconds
- **Iterations:** 6 (Newton)
- **Error:** $\sim 1.7\%$

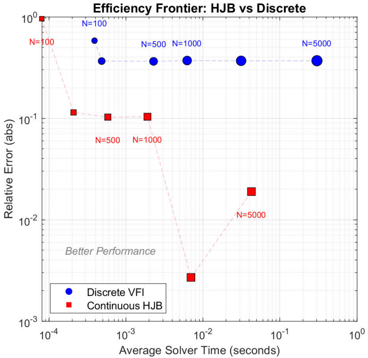
Note: HJB uses a linear grid (finite difference constraint), which is geometrically less efficient than the Log-grid used in VFI, yet it is orders of magnitude faster.

The "Efficiency Frontier": HJB vs. Discrete



- **Efficiency (Left):** HJB (Red) dominates. For 1% error, HJB is $\sim 100\times$ faster.
- **Convergence (Right):** Discrete VFI is monotonically robust. HJB requires careful Newton tuning at very high N.

Scenario A: High Volatility ($\sigma = 0.4$)



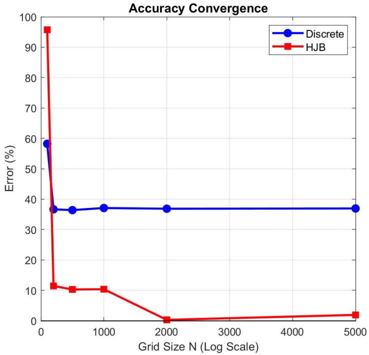


Figure: HJB advantage increases with volatility.

Insight: The diffusion term $\frac{1}{2}\sigma^2 V_{xx}$ acts as a "smoothing agent". HJB becomes *more* accurate as volatility increases, while VFI requires wider grids to capture fat tails.

Scenario B: Positive Drift ($\mu = 0.01$)

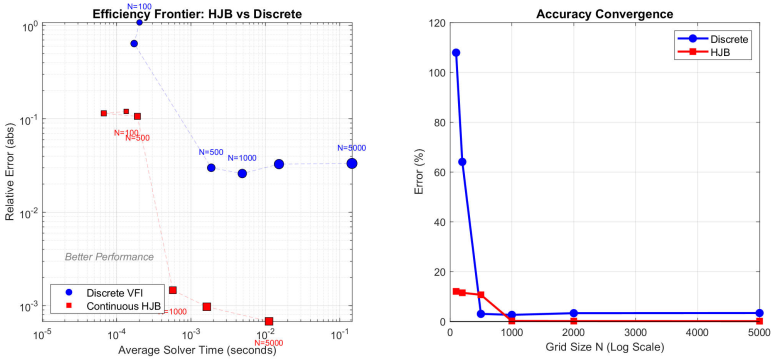


Figure: Structural Robustness under Growth.

Insight: Even when the firm is growing ($\mu > 0$), the **Upwind Scheme** automatically adapts. The threshold x^* drops significantly (option value of waiting increases), but the solver remains stable.

Conclusion

1. **Validation:** Both algorithms successfully recover the analytical solution ($x^* = 0.60$) provided that boundaries are handled correctly.
2. **Efficiency Winner: Continuous HJB (Newton)**.
   - Solving a sparse linear system is fundamentally faster than iterating a transition matrix ($O(N)$ vs $O(N^2)$ operations).
3. **Robustness Winner: Discrete VFI**.
   - Monotonic convergence makes it a "safer" black box, though slower.
4. **Takeaway:** For structural estimation loops where speed is critical, HJB is superior, provided one implements the **Upwind Scheme** and handles **Grid Density** carefully.

Thank You!