1 We want to prove $\int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\,dx = 1$

Let $I = \int_{-\infty}^{\infty} \exp(-\frac{1}{2\sigma^2}x^2)\,dx$

$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\frac{1}{2\sigma^2}(x^2+y^2))\,dx\,dy$

Let $x = r\cos\theta$, $y = r\sin\theta$

Rewrite I^2 as

$$I^2 = \int_{-\infty}^{\infty}\int_{0}^{\infty} \exp(-\frac{1}{2\sigma^2}r^2)\,r\,dr\,d\theta$$

$$= 2\pi \int_{-\infty}^{\infty} \exp(-\frac{1}{2\sigma^2}r^2)\,r\,dr$$

$$= 2\pi \int_{-\infty}^{\infty} \exp(-\frac{1}{2\sigma^2}r^2)\,\frac{1}{2}\,dr^2$$

Let $u = r^2$

$$= \frac{2\pi}{2}\int_{0}^{\infty} \exp(-\frac{u}{2\sigma^2})\,du$$

$$= \pi\left[\exp(-\frac{u}{2\sigma^2})(-2\sigma^2)\right]\Big|_0^{\infty}$$

$$= 2\pi\sigma^2$$

$\because I^2 = 2\pi\sigma^2 \quad \therefore I = \sqrt{2\pi\sigma^2}$

$$\int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp(-\frac{(x-\mu)^2}{2\sigma^2})\,dx \xlongequal{\text{Let } y=x-\mu} \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp(-\frac{y^2}{2\sigma^2})\,dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\,I$$

$$= 1$$