Math 5311 Homework 1

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December 8, 2024

Exercise 1

Consider a general Markov chain on 2-states $\{0, 1\}$ with

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where $0 < a, b < 1$.

(i) Show that

$$P^n = \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{bmatrix} = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} -a & a \\ b & -b \end{bmatrix}, \quad \text{for } |1-a-b| < 1.$$

Show that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

This means that in the long run, the system will be in state 0 with probability $\frac{b}{a+b}$ and in state 1 with probability $\frac{a}{a+b}$, irrespective of the initial state.

(ii) Verify that $\pi = (\pi_0, \pi_1) = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$ is a stationary distribution.

(iii) Show that the first return distribution to 0 is given by

$$f_{00}^{(n)} = ab(1-b)^{n-2}, \quad n \geq 2.$$

(iv) Calculate the mean return time to 0, $\mu_{00} = \sum_{n=1}^{\infty} n f_{00}^{(n)}$, and verify $\pi_0 = \frac{1}{\mu_{00}}$.

Hint: $P = SDS^{-1}$, where D is the diagonal matrix of eigenvalues, and columns of S are eigenvectors.

Proof.

(i) Let

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}.$$

After diagonalizing this matrix, we have

$$P = S\Lambda S^{-1},$$

where

$$S = \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix}, \quad S^{-1} = \frac{1}{a+b}\begin{bmatrix} b & a \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix}.$$

Using the diagonalization:

$$P^n = S\Lambda^n S^{-1}.$$

Thus

$$P^n = \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} \frac{1}{a+b}\begin{bmatrix} b & a \\ 1 & 1 \end{bmatrix}.$$

Moreover, (since $0 \leq 1-a-b < 1$)

$$\lim_{n\to\infty} \Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus:

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{a+b}\begin{bmatrix} b & a \\ 1 & 1 \end{bmatrix} = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}$$

(ii) From (i), the rows of P^n as $n \to \infty$ converge to the stationary distribution:

$$\pi = \begin{bmatrix} \frac{b}{a+b} \\ \frac{a}{a+b} \end{bmatrix}.$$

Indeed it is easy to see that above distribution satisfies:

$$\pi = \pi P$$

(iii) To compute f_{00}^n, we start with:

$$f_{00}^1 = P(X_1 = 0 \mid X_0 = 0) = p_{00} = 1-a.$$

For f_{00}^n, we have:

$$f_{00}^n = P(X_n = 0, X_{n-1} = 1, \ldots, X_1 = 1 \mid X_0 = 0).$$

That is:

$$f_{00}^n = P(X_n = 0 \mid X_{n-1} = 1, X_{n-2} = 1, \ldots, X_1 = 1, X_0 = 0) \cdot P(X_{n-1} = 1, \ldots, X_1 = 1 \mid X_0 = 0).$$

Expanding in this manner and applying the property of Markov chain (i.e., $P(X_k = n_k \mid X_{k-1} = n_{k-1}, X_{k-2} = n_{k-2}, ...X_o = n_0) = P(X_k = n \mid X_{k-1} = n_1)$, we have

$$f_{00}^n = P(X_n = 0 \mid X_{n-1} = 1)P(X_{n-1} = 1 \mid X_{n-2} = 1)\ldots P(X_2 = 1 \mid X_1)P(X_1 = 1 \mid X_0 = 0)$$

This simplifies to:

$$f_{00}^n = a(1-b)^{n-2}b, \quad n \geq 2.$$

(iv) To compute μ_{00}:

$$\mu_{00} = \sum_{n=1}^{\infty} n f_{00}^{n} = (1-a) + \sum_{n=2}^{\infty} nab(1-b)^{n-2}.$$

$$= \mu_{00} = (1-a) + \sum_{n=0}^{\infty} (n+2)ab(1-b)^{n}$$

$$= (1-a) + 2ab\sum_{n=0}^{\infty}(1-b)^{n} + ab\sum_{n=0}^{\infty} n(1-b)^{n}.$$

Using the sum of a geometric series:

$$\sum_{n=0}^{\infty} nx^{n} = \frac{x}{(1-x)^2} \text{ when } |x| < 1,$$

we simplify:

$$\mu_{00} = (1-a) + 2a + \frac{ab(1-b)}{b^2} = \frac{a+b}{b} = \frac{1}{\pi_0}.$$

□

Exercise 2

Let Markov transition matrix P on $\{0, 1, \ldots, N-1\}$ be doubly stochastic, i.e., the sum over each column is 1 (for any Markov chain, the sum over each row is 1). Show that the uniform distribution $\pi = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$ is always a stationary distribution for P, and if in addition P is regular, then it is the only stationary distribution for P.

Proof. Suppose P is doubly stochastic, then:

$$\sum_{j=0}^{N-1} p_{ij} = 1.$$

For the stationary distribution:

$$\pi_i P = \sum_{j=0}^{N-1} \pi_i p_{ij} = \sum_{j=0}^{N-1} \frac{1}{N} p_{ij} = \frac{1}{N}\sum_{j=0}^{N-1} p_{ij} = \frac{1}{N} \text{ for all } i$$

If P is regular, P has a unique distribution; thus, $\pi = (\frac{1}{N}, \ldots, \frac{1}{N})$ is the only stationary distribution. □

Exercise 3

Let Y_n be a sum of n independent rolls of a fair die. Find with what probability Y_n is a multiple of 7 in the long run, i.e.,

$$\lim_{n\to\infty} P(Y_n \text{ is a multiple of } 7).$$

Proof. Let X_n be the remainder when Y_n is divided by 7. It is easy to show that X_n is a Markov chain on $\{0,1,\dots,6\}$. Transition matrix for X_n is $P = (p_{ij})_{6\times 6}$ where $p_{ij} = \frac{1}{6}$ for $i \neq j$ and 0 for $i = j$. This is indeed doubly stochastic and regular; thus, there is a unique distribution and it is $\pi = \left(\frac{1}{7},\dots,\frac{1}{7}\right)$. It then follows that

$$\lim_{n\to\infty} P(Y_n \text{ is a multiple of } 7) = \frac{1}{7}$$

□

Exercise 4

Consider a branching process where the number N of offspring per individual is binomial, i.e.,

$$P_j = P(N=j) = \binom{2}{j} p^j (1-p)^{2-j}, \quad j = 0,1,2, \quad 0 < p < 1.$$

(i) Show that $P(\text{population ever dies out}) = \min\left[1, \left(\frac{1-p}{p}\right)^2\right]$.

(ii) Find the probability that the population becomes extinct in the 3rd generation.

Remark: If X_n is the size of the n-th generation, then $P(\text{extinction by time } n) = P(\bigcup_{j=1}^{n}\{X_j = 0\}) \neq P(X_n = 0, X_{n-1} \neq 0, \dots, X_2 \neq 0)$.

Proof.

(i) Calculate $\pi_0 = P(\text{population dies out})$:

$$\pi_0 = \sum_{i=0}^{\infty} P(\text{population dies out} \mid X_0 = i) P_i.$$

$$= \sum_{i=0}^{\infty} \pi_0^i \binom{2}{i} p^i (1-p)^{2-i}.$$

$$= (1-p)^2 + \pi_0 \cdot 2p(1-p) + \pi_0^2 p^2.$$

$$= p^2\pi_0^2 + 2p(1-p)\pi_0 + (1-p)^2$$

We now have:

$$p^2\pi_0^2 + 2p(1-p)\pi_0 + (1-p)^2 = \pi_0$$

Using the following Theorem from Stochastic Process by Ross:

Theorem 1. *Suppose that $P_0 > 0$ and $P_0 + P_1 < 1$. Then:*
1) π_0 is the smallest positive number satisfying $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$.
2) $\pi_0 = 1$ if, and only if, $\mu \leq 1$.

Smallest solution to the following polynomial is the answer:

$$p^2\pi_0^2 + (2p(1-p) - 1)\pi_0 + (1-p)^2 = 0$$

Solving for π_0 gives:

$$\pi_0 = \min\left\{1, \left(\frac{1}{p} - 1\right)^2\right\}.$$

(ii) Find P_{j+1} (induction):

$$P(X_{n+1} = 0) = \sum_{j=0}^{\infty} P(X_{j+1} = 0 \mid X_1 = j)P_j = \sum_{j=0}^{2} P(X_n = 0)^j P_j.$$

Applying this formula for X_2:

$$P(X_2 = 0) = P_0 + P(X_1 = 0)P_1 + P(X_1 = 0)^2 P_2.$$

Applying this formula for X_3:

$$P(X_3 = 0) = P_0 + P(X_2 = 0)P_1 + P(X_1 = 0)^2 P_2.$$

Solving the expression below would yield the answer:

$$P(\text{extinction in 3rd gen}) = P(X_3 = 0) - P(X_2 = 0).$$

where:

$$P_0 = (1-p)^2, \quad P_1 = 2p(1-p), \quad P_2 = p^2.$$

□