Math 5311 Homework 3

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This home work covers continuous time Markov Process particularly birth-death process.

Exercise 1

Repairs are carried sequentially: T (tune-up), A (air-conditioning), B (breaks) with the mean duration time of $1, 2, 3$ hours respectively. Assume the repair times $R_i, i = 1, 2, 3$ are independent and exponentially distributed.
What is the probability that 5 hours later the car is in the break repair stage? That is, find $P(X(5) = 3)$.
Hint: Use a 4-state Birth-Death process such that $\lambda_4 + \mu_4 = 0$. This means that the 4th state (the state of repair completion) is an absorbing state for which:

$$E(W) = \frac{1}{\lambda_4 + \mu_4} = \infty,$$

where W = waiting time to exit state 4 (the entries of the infinitesimal rates matrix A are reciprocal of the corresponding mean values of exponential distributions).

Proof. Assume repair times $R_i \sim^{\text{iid}}$ exponential for $i = 1, 2, 3$. The infinitesimal generator matrix Q is given as:

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & \lambda_1 & 0 \\ 0 & 0 & -\lambda_2 & \lambda_2 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix},$$

where

$$\mathbb{E}[T_1] = \frac{1}{\lambda_1} = 1 \implies \lambda_1 = 1, \quad \mathbb{E}[T_2] = \frac{1}{\lambda_2} = 2 \implies \lambda_2 = \frac{1}{2}, \quad \mathbb{E}[T_3] = \frac{1}{\lambda_3} = 3 \implies \lambda_3 = \frac{1}{3}.$$

Thus:

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using the result:

$$e^{Qt} = Pe^{\Lambda t}P^{-1},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Substitute:

$$e^{Qt} = P \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-\frac{t}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{t}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}.$$

Given initial condition:

$$p_0 = (P(X(0) = 1) = 1, P(X(0) = 2) = 0, P(X(0) = 3) = 0, P(X(0) = 4) = 0)\,.$$

The transition probability matrix:

$$e^{Qt} = \begin{bmatrix} e^{-t} & 1 - e^{-t} & 0 & 0 \\ 0 & e^{-\frac{t}{2}} & 1 - e^{-\frac{t}{2}} & 0 \\ 0 & 0 & e^{-\frac{t}{3}} & 1 - e^{-\frac{t}{3}} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

After doing some algebra, we have:

$$P(X(t)) = p_0 e^{Q5} = (e^{-t}, 2e^{-t/2}, \frac{9}{2}e^{-\frac{t}{3}} - 6e^{-\frac{t}{2}} + \frac{3}{2}e^{-t}, -\frac{9}{2}e^{-\frac{t}{3}} + 4e^{-\frac{t}{2}} - \frac{1}{2}e^{-t})$$

Thus,

$$P(X(5) = 3) = \frac{9}{2}e^{-\frac{5}{3}} - 6e^{-\frac{5}{2}} + \frac{3}{2}e^{-5}$$

□

Exercise 2

Consider a Birth-Death process with rates $\lambda = \lambda + an$ and $\mu = \mu n$ for $\lambda, \mu > 0$, given $X(0) = n, n = 0, 1, 2, \ldots$, and $a =$ immigration rate.

(i) For $\mathbb{E}[X(t)] = M(t) = \sum_{j=0}^{N} jP_{ij}(t)$, show

$$M'(t) = \lambda + (a - \mu)M(t), \quad M(0) = X(0) = i.$$

(ii) Solve for $M(t)$ in (i).

(iii) Analyze $\lim_{t\to\infty} M(t)$ in (ii).

(iv) Assuming $\lambda < \mu$, find the stationary distribution for $X(t)$.

Proof.

(i) Consider a Birth-Death process with $\lambda_n = \lambda + an$, $\mu_n = \mu n$. The expectation of $X(t)$, denoted $M(t)$, is:

$$M(t) = \mathbb{E}[X(t)] = \sum_{j=1}^{\infty} jP_{ij}(t).$$

Differentiating:

$$M'(t) = \sum_{j=1}^{\infty} jP'_{ij}(t).$$

Using Kolmogorov's Forward Equation:

$$P'_{ij}(t) = \sum_{k=1}^{\infty} P_{ik}(t)Q_{kj}.$$

Substitute:

$$M'(t) = \sum_{j=1}^{\infty} j \sum_{k=1}^{\infty} P_{ik}(t)Q_{kj}.$$

Reorganize:

$$M'(t) = \sum_{k=1}^{\infty} P_{ik}(t) \sum_{j=1}^{\infty} jQ_{kj}.$$

Simplify $\sum_{j=1}^{\infty} jQ_{kj}$:

$$\sum_{j=1}^{\infty} jQ_{kj} = \mu k(k-1) + k(-\lambda - a - \mu k) + (k+1)(\lambda + ak) = \lambda k - \mu k + a.$$

Thus:

$$M'(t) = \sum_{k=1}^{\infty} P_{ik}(t)(\lambda k - \mu k + a).$$

Reorganizing terms:

$$M'(t) = (\lambda - \mu)M(t) + a.$$

(ii) Case 1: when $\mu = \lambda$

$$M'(t) = a \implies M(t) = \int a\, dt \implies M(t) = M(0) + at.$$

Case 2: when $\mu \neq \lambda$ When $\lambda \neq \mu$:

$$M'(t) = a + (\lambda - \mu)M(t).$$

Let $c = \lambda - \mu$:

$$M'(t) - cM(t) = a.$$

Use integrating factor e^{-ct}:

$$e^{-ct}M'(t) - ce^{-ct}M(t) = ae^{-ct}.$$

Integrate:

$$\frac{d}{dt}\big(e^{-ct}M(t)\big) = ae^{-ct}.$$

$$e^{-ct}M(t) - M(0) = \int_0^t ae^{-cs}\,ds = \frac{a}{c} - \frac{a}{c}e^{-ct}.$$

Thus:

$$M(t) = M(0)e^{ct} + \frac{a}{c}(e^{ct} - 1).$$

Final result:

$$M(t) = M(0)e^{(\lambda-\mu)t} + \frac{a}{\lambda - \mu}(e^{(\lambda-\mu)t} - 1).$$

(iii) Taking the limit of $M(t)$ as $t \to \infty$:

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} \frac{a}{\lambda-\mu}\left[e^{(\lambda-\mu)t} - 1\right] + ie^{(\lambda-\mu)t}$$

$$= \begin{cases} \infty & \text{if } \lambda \geq \mu, \\ \frac{a}{\mu-\lambda} & \text{if } \lambda < \mu. \end{cases}$$

(iv) If $\lambda < \mu$, the stationary distribution is found using the formula (details about this formula can be found in Stochastic Process by Ross):

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda+a(i-1)}{\mu i}}.$$

For general P_n:

$$P_n = \prod_{i=1}^{n}\frac{\lambda + a(i-1)}{\mu i}\cdot P_0.$$

Substitute P_0 into P_n:

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda+a(i-1)}{\mu i}}.$$

Rewriting:

$$P_n = \frac{\prod_{i=1}^{n}\frac{\lambda+a(i-1)}{\mu i}}{1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda+a(i-1)}{\mu i}}.$$

The ratio test is used to check convergence:

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda + a(i-1)}{\mu i} < \infty.$$

□

Exercise 3

Population $X(t)$ is assumed to satisfy $N \leq X(t) \leq M$ ($N < M$, fixed integers). Let the birth and death rates per individual be $\lambda = a(M - X(t))$, $\mu = \beta(X(t) - N)$, and assume that individual members of the population act independently. Then, given $X(t) = n$, population rates are:

$$\lambda_n = an(M-n), \quad \mu_n = \beta n(n-N).$$

Observe that if $X(t)$ reaches M, then the birth rate drops to 0 while the death rates reach max (i.e., overcrowding causes a high death rate). Find the stationary distribution.

Proof. Suppose $N \leq X(t) \leq M$ with:

$$\lambda = a(M - X(t)), \quad \mu = \beta(X(t) - N).$$

The infinitesimal generator matrix Q is:

$$Q = \begin{bmatrix} -\lambda_N & \lambda_N & 0 & 0 & \cdots & 0 & 0 \\ \mu_{N+1} & -(\mu_{N+1} + \lambda_{N+1}) & \lambda_{N+1} & 0 & \cdots & 0 & 0 \\ 0 & \mu_{N+2} & -(\mu_{N+2} + \lambda_{N+2}) & \lambda_{N+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \mu_M & -\mu_M & \lambda_M \end{bmatrix},$$

where:

$$\lambda_n = an(M-n), \quad \mu_n = \beta n(n-N).$$

For the stationary distribution π, the condition $\pi Q = 0$ must be satisfied:

$$\pi = [\pi_N, \pi_{N+1}, \ldots, \pi_M].$$

Using induction:

$$\pi_{N+k} = \frac{\lambda_N \cdots \lambda_{N+k-1}}{\mu_{N+1} \cdots \mu_{N+k}} \pi_N \text{ and } \sum_{k=0}^{M-N} \pi_{N+k} = 1$$

We have the formula for π_N:

$$\pi_N = \frac{1}{1 + \sum_{k=1}^{M-N} \prod_{j=1}^{k} \frac{\lambda_{N+j-1}}{\mu_{N+j}}}.$$

Substitute:

$$\pi_{N+k} = \frac{\prod_{j=1}^{k} \lambda_{N+j-1}}{\prod_{j=1}^{k} \mu_{N+j}} \cdot \pi_N.$$

For general π_{N+k}:

$$\pi_{N+k} = \frac{\prod_{j=1}^{k} \frac{\lambda_{N+j-1}}{\mu_{N+j}}}{1 + \sum_{k=1}^{M-N} \prod_{j=1}^{k} \frac{\lambda_{N+j-1}}{\mu_{N+j}}}.$$

Explicitly, substituting:

$$\lambda_{N+j-1} = a(M-(N+j-1)), \quad \mu_{N+j} = \beta((N+j)-N),$$

yields:

$$\pi_N = \frac{1}{1+\sum_{k=1}^{M-N} \frac{N}{N+1} \left(\frac{a}{b}\right)^k \binom{M-N}{k}}.$$

Thus:

$$\pi_{N+i} = \frac{\frac{N}{N+1} \left(\frac{a}{b}\right)^i \binom{M-N}{i}}{1+\sum_{k=1}^{M-N} \frac{N}{N+1} \left(\frac{a}{b}\right)^k \binom{M-N}{k}}$$

This satisfies the stationary condition $\pi Q = 0$.

□

Exercise 4

There are 2 waiting chairs and 2 barbers. Customers arriving to a fully occupied shop are turned away. Assume customers arrive with Poisson rate 5/hour and service rate is exponential with rate 2/hour. Let $X(t) = \#$customers in the shop $\in \{0,1,2,3,4\}$.
The transition rates diagram is as follows:

$$0 \xrightarrow{\frac{5}{2}} 1 \xrightarrow{\frac{5}{4}} 2 \xrightarrow{\frac{5}{4}} 3 \xrightarrow{\frac{5}{4}} 4$$

Find the stationary distribution.

Proof. The stationary distribution satisfies:

$$\pi Q = 0 \quad \text{and} \quad \sum_{i=0}^{4} \pi_i = 1.$$

Let:

$$\pi = [\pi_0, \pi_1, \pi_2, \pi_3, \pi_4].$$

The infinitesimal generator matrix Q is:

$$Q = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 \\ 2 & -7 & 5 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 4 & -9 & 5 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix}.$$

This leads to the following system of equations:

$$-5\pi_0 + 2\pi_1 = 0,$$

$$5\pi_0 - 7\pi_1 + 4\pi_2 = 0,$$

$$5\pi_1 - 9\pi_2 + 4\pi_3 = 0,$$

$$5\pi_2 - 9\pi_3 + 4\pi_4 = 0,$$

$$5\pi_3 - 4\pi_4 = 0,$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Solving this system, we find:

$$\pi = \frac{500}{1973}\begin{bmatrix} \frac{32}{125} & \frac{16}{25} & \frac{4}{5} & 1 & \frac{5}{4} \end{bmatrix}.$$

$\square$