Math 5311 Homework 4

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This homework covers the property of Brownian motions and stochastic differential equations.

Exercise 1

Show:

$$X(t) = tB(1/t), \quad X(0) = 0$$

is stanadrd Brownian motion.

Proof. Let $X(t) = tB\left(\frac{1}{t}\right)$, $X(0) = 0$.
We show that the covariance of $X(t)$ is the same as $B(t)$ since Gaussian processes are characterized by its covariance:

$$\mathrm{Cov}(X(t), X(s)) = \mathbb{E}\left[tB\left(\frac{1}{t}\right) sB\left(\frac{1}{s}\right)\right].$$

Since $\mathbb{E}[B\left(\frac{1}{t}\right) B\left(\frac{1}{s}\right)] = \min\left(\frac{1}{t}, \frac{1}{s}\right)$, we have:

$$\mathrm{Cov}(X(t), X(s)) = ts \min\left(\frac{1}{t}, \frac{1}{s}\right) = \min(t, s).$$

Thus, $\mathrm{Cov}(X(t), X(s)) = \mathrm{Cov}(B(t), B(s))$.

$\square$

Exercise 2

Let $X(t) = B(t) - tB(1)$, $0 \leq t \leq 1$. Notice that $X(0) = X(1) = 0$, i.e., $X(t)$ connects the starting point and the endpoint by a continuous path (bridge). Show:

$$\text{(a)} \quad \mathrm{Cov}(X(s), X(t)) = s(1-t), \quad \text{(b)} \quad X(t) \text{ has density } \frac{1}{\sqrt{2\pi t(1-t)}} e^{-\frac{x^2}{2t(1-t)}}.$$

Proof. Let $X(t) = B(t) - tB(1)$, where $0 \le t \le 1$ and $X(0) = X(1) = 0$.

(a)

$$\mathrm{Cov}(X(s), X(t)) = \mathbb{E}[(B(s) - sB(1))(B(t) - tB(1))].$$

Expanding:

$$= \mathbb{E}[B(s)B(t) - sB(1)B(t) - tB(1)B(s) + stB(1)^2].$$

Using independence and properties of Brownian motion:

$$= \mathbb{E}[B(s)B(t)] - s\mathbb{E}[B(1)B(t)] - t\mathbb{E}[B(1)B(s)] + st\mathbb{E}[B(1)^2].$$

Simplify:

$$= s - ts - st + st = s(1-t).$$

Thus:

$$\mathrm{Cov}(X(s), X(t)) = s(1-t).$$

(b) Rewrite $X(t)$:

$$X(t) = B(t) - tB(1) = B(t) - t(B(1) - B(t)) - tB(t).$$

This simplifies to:

$$X(t) = (1-t)B(t) - t(B(1) - B(t)).$$

Since $B(t)$ is independent of $B(1) - B(t)$, and sums of two independent normals are normal:

$$X(t) \sim N(0, t(1-t)).$$

The density is:

$$f(x) = \frac{1}{\sqrt{2\pi t(1-t)}} e^{-\frac{x^2}{2t(1-t)}}.$$

□

Exercise 3

Define:

$$g(\theta, x, t) = e^{\theta B(t) - \frac{\theta^2 t}{2}}, \quad \text{and its } k\text{-th derivative } g_k(\theta, x, t) = \frac{\partial^k}{\partial x^k} g(\theta, x, t).$$

Let $B(t)$ be the standard Brownian motion and $\mathcal{F}_t = \sigma(B(s), 0 \le s \le t)$. Show that for any real θ:

$$M(t) = g(\theta, B(t), t) = e^{\theta B(t) - \frac{\theta^2 t}{2}}$$

is a martingale.

By differentiating the relevant conditional expectation associated with $M(t)$ and assuming interchange of differentiation with conditional expectation, show that:

$$M_1(t) = g_1(\theta, B(t), t)$$

is a martingale.

Note: $M(t)$ is called an exponential martingale, while $M_k(t)$ are polynomial martingales.

Proof. Define:

$$g(\theta, x, t) = e^{\frac{\theta x}{2}}, \quad M(t) = g(\theta, B(t), t) = e^{\theta B(t) - \frac{\theta^2 t}{2}}.$$

To show $M(t)$ is a martingale:

$$\mathbb{E}[M(t) \mid \mathcal{F}_s] = \mathbb{E}\left[e^{\theta B(t) - \frac{\theta^2 t}{2}} \mid \mathcal{F}_s\right].$$

Expanding:

$$= e^{-\frac{\theta^2 t}{2}} \mathbb{E}\left[e^{\theta B(s) + \theta(B(t) - B(s))} \mid \mathcal{F}_s\right].$$

Using independence of $B(t) - B(s)$:

$$= e^{\theta B(s) - \frac{\theta^2 s}{2}} \cdot \mathbb{E}\left[e^{\theta(B(t) - B(s))}\right].$$

Since $B(t) - B(s) \sim N(0, t-s)$:

$$\mathbb{E}\left[e^{\theta(B(t) - B(s))}\right] = e^{\frac{\theta^2(t-s)}{2}}.$$

Thus:

$$\mathbb{E}[M(t) \mid \mathcal{F}_s] = e^{\theta B(s) - \frac{\theta^2 s}{2}} = M(s).$$

Therefore, $M(t)$ is a martingale.

(a) Let $g(\theta.B(t), t) = e^{\theta B(t) - \frac{\theta^2 t}{2}}$

$$M(t) = g_1(\theta.B(t), t) = \frac{d}{d\theta} g(\theta.B(t), t) = (B_t - \theta_t)\, g(\theta.B(t), t)$$

Thus:

$$\mathbb{E}\left[M(t) \mid \mathcal{F}_s\right] = \mathbb{E}\left[B_t g(t) \mid \mathcal{F}_s\right] - \theta t \mathbb{E}\left[g(t) \mid \mathcal{F}_s\right]$$

$$= e^{\theta B(s) - \frac{\theta^2 t}{2}} \left[\mathbb{E}[B_{t-s} e^{\theta B_{t-s}}] + \mathbb{E}[B_s e^{\theta B_{t-s}}]\right] - \theta t g(s)$$

where $\mathbb{E}[e^{B_{t-s}}] = e^{\frac{\theta^2(t-s)}{2}}$ and $\mathbb{E}[B_{t-s} e^{B_{t-s}}] = \int_{-\infty}^{\infty} x e^{\theta x} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}} = e^{\frac{\theta^2(t-s)}{2}} \theta(t-s)$.
After substitution:

$$\mathbb{E}\left[M(t) \mid \mathcal{F}_s\right] = (B_s - \theta s) g(s) = M(s)$$

Therefore, $g_1(\theta.B(t), t)$ is a martingale.

$\square$

Exercise 4

Show that:

$$I = \int_0^t B^2(s)dB(s) = B^3(t) - 3\int_0^t B(s)ds,$$

and compute $\text{Var}(I)$.
Find the mean, variance, and covariance of the process:

$$\int_0^t sdB(s).$$

Proof.

(i)

$$I = \int_0^t B^2(s)dB(s) = \frac{1}{3}B^3(t) - \int_0^t B(s)ds.$$

Let $f(x) = \frac{x^3}{3}$, where $x = B(t)$. Then Itô's formula gives:

$$d\left(\frac{1}{3}B^3(t)\right) = B^2(t)dB(t) + \frac{1}{2}(2B(t))dt.$$

Substituting:

$$\int_0^t B^2(s)dB(s) = \frac{1}{3}B^3(t) - \int_0^t B(s)ds.$$

To find its variance, we can first calculate:

$$\mathbb{E}[I] = \mathbb{E}\left[\int_0^t B^2(s)dB(s)\right] = \frac{1}{3}\mathbb{E}\left[B^3(s)\right] - \mathbb{E}\left[\int_0^t B(s)ds\right] = 0$$

where third equality holds due to Fubini theorem, $\mathbb{E}[B(t)] = 0$, and $\mathbb{E}[B^3(t)] = 0$.
Now, we can calculate:

$$\mathbb{E}[I^2] = \mathbb{E}\left[\left(\int_0^t B^2(s)dB(s)\right)^2\right] = \int_0^t \mathbb{E}[B^4(s)]ds = \int_0^t 3s^2ds = t^3$$

where second equality holds due to Ito Isometry, and third equality comes from the fourth moment of $N(0, s)$.
Variance:

$$\text{Var}(I) = \mathbb{E}[I^2] - (\mathbb{E}[I])^2 = t^3$$

(ii) Let:

$$X(s,t) = \int_0^t s dB(s).$$

Let $f(x,t) = tx$ and using Itô's lemma:

$$\int_0^t s dB(s) = tB(t) - \int_0^t B(s) ds.$$

It is obvious that $\mathbb{E}\left[\int_0^t s dB(s)\right] = 0$, and by Ito isometry

Variance:

$$\text{Var}\left[\int_0^t s dB(s)\right] = \frac{t^3}{3}$$

Covariance:

$$\text{Cov}\left(\int_0^t s dB(s), \int_0^s s dB(s)\right) = \mathbb{E}\left[\left(\int_0^t s dB(s)\right)\left(\int_0^s s dB(s)\right)\right]$$

$$= \mathbb{E}\left[\left(\int_0^s s dB(s) + \int_s^t s dB(s)\right)\left(\int_0^s s dB(s)\right)\right]$$

$$= \mathbb{E}\left[\left(\int_s^t s dB(s)\right)\left(\int_0^s s dB(s)\right)\right] + \mathbb{E}\left[\left(\int_0^s s dB(s)\right)\left(\int_0^s s dB(s)\right)\right]$$

$$= \mathbb{E}\left[\int_s^t s dB(s)\right]\mathbb{E}\left[\left(\int_0^s s dB(s)\right)\right] + \int_0^s s^2 ds = \frac{t^3}{3}$$

where fourth equality uses independence of $\left(\int_s^t s dB(s)\right)\left(\int_0^s s dB(s)\right)$ and Ito isometry.

□

Exercise 5

Evaluate $\mathbb{E}\left[e^{-rT}(X_T - K)^+\right]$ to verify the Call Option price $C(0)$ at $t = 0$ for the strike price $K > 0$ and Call option expiration time $T > 0$:

$$C(0) = \mathbb{E}\left[e^{-rT}(X_T - K)^+\right] = S_0 \Phi\left(\frac{r + \frac{\sigma^2}{2}T + \ln\frac{S_0}{K}}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{r - \frac{\sigma^2}{2}T + \ln\frac{S_0}{K}}{\sigma\sqrt{T}}\right),$$

where:

$$X_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T},$$

S_0 is the stock price at $t = 0$, σ is the stock price volatility, B_T is the standard Brownian motion at $t = T$, r is the risk-free return rate, and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.$$

Proof. Evaluate

$$\mathbb{E}\left[e^{-rT}(X_T - K)^+\right]$$

to verify the call option $C(0)$ for $K > 0$ and $T > 0$.

$$\begin{aligned} C(0) &= \mathbb{E}\left[e^{-rT}(X_T - K)^+\right] \\ &= \mathbb{E}\left[\max(e^{-rT}X_T - e^{-rT}K, 0)\right] \\ &= \mathbb{E}\left[e^{-rT}X_T\mathbf{1}_{\{X_T>K\}}\right] - \mathbb{E}\left[e^{-rT}K\mathbf{1}_{\{X_T>K\}}\right]. \end{aligned}$$

For the first term:

$$\begin{aligned} \mathbb{E}\left[e^{-rT}X_T\mathbf{1}_{\{X_T>K\}}\right] &= \int_{\{S_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B_T}>K\}} e^{-rT}S_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma x}\frac{1}{\sqrt{2\pi T}}e^{-\frac{x^2}{2T}}\,dx \\ &= S_0\int_{\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi T}}e^{-\frac{(x-\sigma t)^2}{2T}}\,dx \end{aligned}$$

Change of variable:

$$y = \frac{x-\sigma T}{\sqrt{T}}, \quad dy = \frac{1}{\sqrt{T}}dx.$$

After substitution:

$$\mathbb{E}\left[e^{-rT}X_T\mathbf{1}_{\{X_T>K\}}\right] = S_0\int_{\frac{\ln(K/S_0)-(r+\sigma^2/2)T}{\sigma\sqrt{T}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$$

$$= S_0\left[1-\Phi\left(\frac{\ln(K/S_0)-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right] = S_0\left[\Phi\left(\frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right].$$

For the second term:

$$\mathbb{E}\left[e^{-rT}K\mathbf{1}_{X_T>K}\right] = Ke^{-rT}\int_{\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi T}}e^{-\frac{y^2}{2T}}\,dy$$

Letting $y = \frac{x}{\sqrt{T}}$ by change of variable,

$$\begin{aligned} &= Ke^{-rT}\left[1-\Phi\left(\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right] \\ &= Ke^{-rT}\Phi\left(\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

Combining the results:

$$C(0) = S_0\left[\Phi\left(\frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right] - Ke^{-rT}\Phi\left(\frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

$\square$