Math 5311 Homework 1

Yuya Ogawa

December 9, 2024

This HW mainly covers Poisson processes. I assume that some theorems regarding Poisson processes such as "conditional expectation of arrival time for i-th shock S_i given n number of events occur by time t (i.e., $N(t) = n$) is uniformly iid" are familiar to the readers.

Exercise 1

Show that for nonnegative integer valued random variable X,

$$\mathbb{E}X = \sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=1}^{\infty} P(X > n)$$

Proof. Notice that

$$\begin{aligned} P(X \geq n) &= P(n \leq X < n+1) + P(n+1 \leq X < n+2) + ... \\ &= P(n-1 < X \leq n) + P(n < X \leq n+1) + ... \\ &= P(X = n) + P(X = n+1) + ... \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} P(X \geq n) = 1P(1 \leq X < 2) + 2P(2 \leq X < 3) + ...$$

$$= \sum_{n=1}^{\infty} P(X > n) = 1P(0 < X \leq 1) + 2P(1 < X \leq 2) + ...$$

$$= P(X = 1) + 2P(X = 2) + ... = \sum_{n=1}^{\infty} nP(X = n) = \mathbb{E}X$$

.

□

Exercise 2

If $X \geq 0$ has distribution $F(x)$ then $\mathbb{E}X^n = \int_0^\infty nx^{n-1}(1 - F(x))dx$.

Proof. Using a integration by parts, we have

$$\mathbb{E}X^n = \int_0^\infty x^n f(x)dx = \lim_{t\to\infty} \int_0^t x^n f(x)dx = \lim_{t\to\infty} [t^n F(t) - \int_0^t nx^{n-1} F(x)dx]$$

$$= \lim_{t\to\infty} [\int_0^t nx^{n-1} F(t) - \int_0^t nx^{n-1} F(x)dx] = \lim_{t\to\infty} \int_0^t nx^{n-1}(F(t) - F(x))$$

$$= \int_0^\infty nx^{n-1}(1 - F(x))dx$$

□

Exercise 3

Let X be exponentially distributed with density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Using the moment generating function find the expected value $\mathbb{E}X$ and the variance $VarX$.

Proof.

$$\mathbb{E}(X) = \frac{d}{dt}|_{t=0}\mathbb{E}(e^{Xt}) = \frac{d}{dt}|_{t=0} \lim_{z\to\infty} \int_0^z e^{xt}\lambda e^{-\lambda x}dx = \frac{d}{dt}|_{t=0}\lambda \lim_{z\to\infty} \int_0^z e^{(t-\lambda)x}dx$$

$$= \frac{d}{dt}|_{t=0}\frac{\lambda}{\lambda - t} \lim_{z\to\infty} e^{(t-\lambda)z} - 1 = \lim_{z\to\infty} \frac{d}{dt}|_{t=0}\frac{\lambda}{\lambda - t}(e^{(t-\lambda)z} - 1) = \frac{1}{\lambda}$$

In similar fashion, one can show that $\mathbb{E}(X^2) = \frac{2}{\lambda^2}$. This gives

$$VarX = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

□

Exercise 4

Let X, Y be independently exponentially distributed with parameters λ_1, λ_2. Find (a) distribution of $Z = \min(X, Y)$ (b) conditional distribution of Z given $Z = X$.

Proof. (a) By independence, we have

$$F(t) = P(\min(X,Y) \le t) = 1 - P(\min(X,Y) > t)$$
$$= 1 - P(X > t, Y > t) = 1 - P(X > t)P(Y > t) = 1 - (1 - (1 - e^{-\lambda_1 t})(1 - (1 - e^{-\lambda_1 t}))$$
$$= 1 - e^{-(\lambda_1+\lambda_2)t}$$

(b)

$$P(Z \le t | Z = X) = 1 - P(X \ge t, Y \ge t | Y \ge X) = 1 - \frac{P(X \ge t, Y \ge t, Y \ge X)}{P(Y \ge X)}$$
$$= 1 - \frac{P(t \le X \le Y)}{P(Y \ge X)} = 1 - e^{-(\lambda_1+\lambda_2)t}$$

where $P(t \le X \le Y) = \int_t^\infty P(t \le X \le y | Y = y) f_Y(y)dy = \int_t^\infty P(t \le X \le y) f_Y(y)dy = \int_t^\infty (\int_t^y \lambda_1 e^{-\lambda_1} dx) \lambda_2 e^{-\lambda_2 y} dy = \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$, and $P(Y \ge X) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

□

Exercise 5

Consider a device subject to shocks arriving as Poisson Process with rate λ. Assume that i-th shock produces damage D_i and D_i are i.i.d. and independent of the process $N(t) =$ # of shocks in $[0, t]$. The damage due to a shock decreases exponentially in time, i.e., if initial damage is D then its damage is $De^{\alpha t}$ after time t with $\alpha > 0$ being a fixed parameter. Then the total damage at time t has the form

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-\alpha(t-S_i)}, \text{ where } S_i = \text{arrival time of i-th shock}$$

Evaluate $\mathbb{E}D(t)$ in terms of $\lambda, \alpha, t, \mathbb{E}D$. Here $\mathbb{E}D = \mathbb{E}D_i$.

Proof. First calculate conditional probability. Since D_i is i.i.d. and S_i given $N(t) = n$ is uniformly distributed (see Page 71 of Stochastic Process by Ross),

$$\mathbb{E}[D(t)|N(t) = n] = \mathbb{E}[\sum_{i=1}^{n} D_i e^{-\alpha(t-S_i)}|N(t) = n] = \sum_{i=1}^{n} \mathbb{E}[D]e^{-\alpha t}\mathbb{E}[e^{-\alpha S_i}|N(t) = n]$$

$$= \mathbb{E}[D]e^{-\alpha t}\sum_{i=1}^{n} \mathbb{E}[e^{\alpha U}] = \mathbb{E}[D]e^{-\alpha t}n\mathbb{E}[e^{\alpha U}] = \mathbb{E}[D]e^{-\alpha t}n\frac{1}{t}\int_0^t e^{\alpha x}dx = \mathbb{E}[D]e^{-\alpha t}\frac{n}{\alpha t}(1 - e^{\alpha t})$$

$$= \mathbb{E}[D]\frac{n}{\alpha t}(1 - e^{-\alpha t})$$

This gives us,

$$\mathbb{E}[D] = \sum_{i=1}^{n} \mathbb{E}[D(t)|N(t) = i]P(N(t) = i) = \sum_{i=1}^{n} \mathbb{E}[D]\frac{i}{\alpha t}(1 - e^{-\alpha t})P(N(t) = i)$$

$$= E[D]\frac{1}{\alpha t}(1 - e^{-\alpha t})\sum_{i=1}^{n} iP(N(t) = i) = E[D]\frac{1}{\alpha t}(1 - e^{-\alpha t})\lambda t = \frac{\lambda E(D)}{\alpha}(1 - e^{-\alpha t})$$

□

Exercise 6

Let $N(t)$ be Poisson with rate λ. Show that for $s < t$

$$P(N(t) = k|N(t) = n) = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k}$$

Proof.

$$P(N(s) = k|N(t) = n) = \frac{P(N(s) = k, N(t) = n)}{P(N(t) = n)} = \frac{P(k \text{ events in } [0, s], n - k \text{ events in } [s, t])}{P(N(t) = n}$$

$$= \frac{P(k \text{ events in } [0, s])P(n - k \text{ events in } [s, t])}{P(N(t) = n} = \frac{P(N(s) = k)P(N(t - s) = n - k)}{P(N(t) = n)}$$

We know that $P(N(s) = k) = e^{-\lambda s}\frac{(\lambda s)^k}{k!}$, $P(N(t - s) = n - k) = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!}$, and $P(N(s) = n) = e^{-\lambda s}\frac{(\lambda s)^n}{n!}$. After algebra, we have

$$P(N(s) = k|N(t) = n) = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k}$$

□

Exercise 7

Let $N(t)$ be Poisson with rate λ. Evaluate $\mathbb{E}N(t)N(t+s)$.

Proof.

$$\mathbb{E}(N(t+s)-N(t))^2 = \mathbb{E}N(t)^2 - 2\mathbb{E}N(t)N(t+s) + \mathbb{E}N(t+s)^2$$

. This implies

$$\mathbb{E}N(t)N(t+s) = \frac{\mathbb{E}N(t)^2 + \mathbb{E}N^2(t+s) - \mathbb{E}(N(t+s)-N(t))^2}{2}$$

Find

$$\mathbb{E}N(t)^2 = \sum_{n=1}^{\infty} nP(N^2(t)=n) = \sum_{n=1}^{\infty} n^2 P(N(t)=n) = \sum_{n=1}^{\infty} n^2 e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

After some algebra, one can find that $\mathbb{E}N(t)^2 = \lambda t(\lambda t+1)$. Likewise,

$$\mathbb{E}(N(t+s)-N(t))^2 = \sum_{n=1}^{\infty} n^2 P(N(t+s)-N(t)=n) = \sum_{n=1}^{\infty} n^2 P(N(s)=n) = \lambda s(\lambda s+1)$$

By combining and after some algebra, we will finally have

$$\mathbb{E}N(t)N(t+s) = \lambda t(\lambda t + \lambda s + 1)$$

.

□

Exercise 8

Find the variance of the total waiting time $W = \sum_{i=1}^{N(t)}(t-S_i)$. Hint: $EW = \lambda t^2/2$.

Proof. First, notice that:

$$\left(\sum_{n=1}^{N}(t-S_n)\right)^2 = \sum_{n=1}^{N}(t-S_n)^2 + 2\sum_{j=1}^{N}\sum_{i=1}^{j-1}(t-S_i)(t-S_j)$$

Expanding this and taking a conditional expectation $\mathbb{E}[\cdot|N(t)=N]$ gives:

$$\mathbb{E}\left[\left(\sum_{n=1}^{N}(t-S_n)\right)^2|N(t)=N\right]=Nt^2-2t\mathbb{E}\left[\sum_{n=1}^{N}S_n|N(t)=N\right]+\mathbb{E}\left[\sum_{n=1}^{N}S_n^2|N(t)=N\right]$$

$$+2\sum_{j=1}^{N}\sum_{i=1}^{j-1}t^2-\mathbb{E}[S_i|N(t)=n]-\mathbb{E}[S_j|N(t)=n]+\mathbb{E}[S_iS_j|N(t)=n]$$

Since conditional expectation of S_i given $N(t)=N$ is uniformly iid, $\mathbb{E}[S_n|N(t)=n]=\mathbb{E}[U]$, $\mathbb{E}[S_n^2|N(t)=n]=\mathbb{E}[U^2]$, and $\mathbb{E}[S_nS_k|N(t)=n]=\mathbb{E}[U]^2$. Applying this in formula above and solving some algebra, we will have:

$$\mathbb{E}[W^2|N(t)=N]=N^2\frac{t^2}{4}+N\frac{t^2}{12}$$

Now law of total probability gives:

$$\mathbb{E}[W^2]=\sum_{n=1}^{\infty}\mathbb{E}[W^2|N(t)=n]P(N(t)=n)=\sum_{n=1}^{\infty}(n^2\frac{t^2}{4}+n\frac{t^2}{12})P(N(t)=n)$$

$$=\frac{t^2}{4}\sum_{n=1}^{\infty}n^2P(N(t)=n)+\frac{t^2}{12}\sum_{n=1}^{\infty}nP(N(t)=n)=\frac{t^2}{4}\lambda t(\lambda t+1)+\frac{t^2}{12}\lambda t$$

Finally, using the formula for variance: $Var(W)=\mathbb{E}[W^2]-(\mathbb{E}W)^2=\frac{\lambda t^3}{3}$

□

Exercise 9

Consider the Laplace Transform:

$$L[f]=\bar{f}(s)=\int_0^{\infty}e^{-st}f(t)dt$$

and its properties:

$$L\left[\int f(x)dx\right]=\frac{\bar{f}(s)}{s},\quad L[f*g]=L[f]L[g]$$

Let $m(t)=EN(t)$, be the renewal function for $X_1-X_2-F(x)$ with:

$$m(t)=F(t)+\int_0^t m(t-x)dF(x)=\int_0^t F(x)dx+\int_0^t m(t-x)xdF(x)$$

(a) Show that:

$$\bar{m}(s) = \frac{1}{s}\frac{\bar{f}(s)}{1-\bar{f}(s)}$$

(b) Using (a) and $L^{-1}[\bar{m}(s)] = m(t)$, show that for Poisson Process λ:

$$m(t) = \lambda t$$

Proof. (a)

$$\bar{m}(s) = L[m(t)] = L\left[\int_0^t f(x)dx\right] + L[m * f] = \frac{\bar{f}(s)}{s} + \bar{m}(s)\bar{f}(s)$$

$$\implies \bar{m}(s) - \bar{m}(s)\bar{f}(s) = (1-\bar{f}(s))\bar{m}(s) = \frac{\bar{f}(s)}{s}$$

$$\implies \bar{m}(s) = \frac{1}{s}\frac{\bar{f}(s)}{1-\bar{f}(s)}$$

(b)
Use (a) and $L^{-1}[\bar{m}(s)] = m(t)$ to show $m(t) = \lambda t$. First notice that for $X \sim \text{Poisson}(\lambda t)$, poisson proess,

$$f(x) = \lambda e^{\lambda t} \implies \bar{f}(x) = \int_0^\infty \lambda e^{-\lambda t}dt = \frac{\lambda}{\lambda + s}$$

Thus

$$m(t) = L^{-1}\left[\bar{m}(s)\right] = L^{-1}\left[\frac{1}{s}\frac{\hat{f}(s)}{1-\hat{f}(s)}\right] = L^{-1}\left[\frac{\lambda}{s^2}\right] = \lambda L^{-1}\left[\frac{1}{s^2}\right] = \lambda t$$

□

Exercise 10

Consider a revenue process $R(t) = \sum_{i=0}^{N(t)} S_i$, of a department store operating for t hours, given the customers arrive according to Poisson Process $N(t)$ with rate λ, and the i-th customer spends S_i, where S_i are i.i.d. with finite second moment. Show that:

(a) The expected revenue $\mathbb{E}[R(t)] = \lambda t\mathbb{E}[S_1]$

(b) The variance $\text{Var}(R(t)) = \lambda t\mathbb{E}[S_1^2]$

Proof. (a) Note that $\mathbb{E}[R(t)|N(t) = n] = \mathbb{E}[\sum_{i=1}^{n} S_i|N(t) = n] = n\mathbb{E}[S_1]$ since S_i is iid $\Longrightarrow$

$$\mathbb{E}[R(t)] = \sum_{n=1}^{\infty} \mathbb{E}[R(t)|N(t) = n]P(N(t) = n) = \mathbb{E}[S_1] \sum_{n=1}^{\infty} nP(N(t) = n) = \lambda t\mathbb{E}(S_1)$$

(b) Using a similar reasoning as in (a), one can show that $\mathbb{E}[R(t)^2|N(t) = n] = n\mathbb{E}[S_1^2] + n(n-1)\mathbb{E}[S_1]^2$. Thus

$$\mathbb{E}[R(t)^2] = \sum_{n=1}^{\infty} \mathbb{E}[R(t)^2|N(t) = n]P(N(t) = n) = \sum_{n=1}^{\infty}[n\mathbb{E}[S_1^2]+n(n-1)\mathbb{E}[S_1]^2]P(N(t) = n)$$

$$= \lambda t\mathbb{E}[S_1^2] + \lambda^2 t^2 \mathbb{E}[S_1]^2$$

Now using variance formula,

$$Var(R(t)) = \lambda t\mathbb{E}[S_1^2]$$

□