# Math 5311 Homework 1

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This HW mainly covers Poisson processes. I assume that some theorems regarding Poisson processes such as "conditional expectation of arrival time for i-th shock $S_i$ given n number of events occur by time t (i.e., $N(t) = n$) is uniformly iid" are familiar to the readers.

## Exercise 1

Show that for nonnegative integer valued random variable $X$,

$$\mathbb{E}X = \sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=1}^{\infty} P(X > n)$$

*Proof.* Notice that

$$\begin{aligned} P(X \geq n) &= P(n \leq X < n+1) + P(n+1 \leq X < n+2) + ... \\ &= P(n-1 < X \leq n) + P(n < X \leq n+1) + ... \\ &= P(X = n) + P(X = n+1) + ... \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{n=1}^{\infty} P(X \geq n) = 1P(1 \leq X < 2) + 2P(2 \leq X < 3) + ... \\ &= \sum_{n=1}^{\infty} P(X > n) = 1P(0 < X \leq 1) + 2P(1 < X \leq 2) + ... \\ &= P(X = 1) + 2P(X = 2) + ... = \sum_{n=1}^{\infty} nP(X = n) = \mathbb{E}X \end{aligned}$$

.

$\square$

## Exercise 2

If $X \geq 0$ has distribution $F(x)$ then $\mathbb{E}X^n = \int_0^\infty nx^{n-1}(1 - F(x))dx$.

*Proof.* Using a integration by parts, we have

$$\mathbb{E}X^n = \int_0^\infty x^n f(x)dx = \lim_{t\to\infty} \int_0^t x^n f(x)dx = \lim_{t\to\infty} [t^n F(t) - \int_0^t nx^{n-1} F(x)dx]$$

$$= \lim_{t\to\infty} [\int_0^t nx^{n-1} F(t) - \int_0^t nx^{n-1} F(x)dx] = \lim_{t\to\infty} \int_0^t nx^{n-1}(F(t) - F(x))$$

$$= \int_0^\infty nx^{n-1}(1 - F(x))dx$$

□

## Exercise 3

Let X be exponentially distributed with density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Using the moment generating function find the expected value $\mathbb{E}X$ and the variance $VarX$.

*Proof.*

$$\mathbb{E}(X) = \frac{d}{dt}|_{t=0}\mathbb{E}(e^{Xt}) = \frac{d}{dt}|_{t=0} \lim_{z\to\infty} \int_0^z e^{xt}\lambda e^{-\lambda x}dx = \frac{d}{dt}|_{t=0}\lambda \lim_{z\to\infty} \int_0^z e^{(t-\lambda)x}dx$$

$$= \frac{d}{dt}|_{t=0}\frac{\lambda}{\lambda - t} \lim_{z\to\infty} e^{(t-\lambda)z} - 1 = \lim_{z\to\infty} \frac{d}{dt}|_{t=0}\frac{\lambda}{\lambda - t}(e^{(t-\lambda)z} - 1) = \frac{1}{\lambda}$$

In similar fashion, one can show that $\mathbb{E}(X^2) = \frac{2}{\lambda^2}$. This gives

$$VarX = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

□

## Exercise 4

Let $X, Y$ be independently exponentially distributed with parameters $\lambda_1, \lambda_2$. Find (a) distribution of $Z = \min(X, Y)$ (b) conditional distribution of $Z$ given $Z = X$.

*Proof.* (a) By independence, we have

$$F(t) = P(\min(X, Y) \leq t) = 1 - P(\min(X, Y) > t)$$

$$= 1 - P(X > t, Y > t) = 1 - P(X > t)P(Y > t) = 1 - (1 - (1 - e^{-\lambda_1 t})(1 - (1 - e^{-\lambda_1 t}))$$

$$= 1 - e^{-(\lambda_1 + \lambda_2)t}$$

(b)

$$P(Z \leq t | Z = X) = 1 - P(X \geq t, Y \geq t | Y \geq X) = 1 - \frac{P(X \geq t, Y \geq t, Y \geq X)}{P(Y \geq X)}$$

$$= 1 - \frac{P(t \leq X \leq Y)}{P(Y \geq X)} = 1 - e^{-(\lambda_1 + \lambda_2)t}$$

where $P(t \leq X \leq Y) = \int_t^\infty P(t \leq X \leq y | Y = y) f_Y(y) dy = \int_t^\infty P(t \leq X \leq y) f_Y(y) dy = \int_t^\infty (\int_t^y \lambda_1 e^{-\lambda_1} dx) \lambda_2 e^{-\lambda_2 y} dy = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}$, and $P(Y \geq X) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

$\square$

## Exercise 5

Consider a device subject to shocks arriving as Poisson Process with rate $\lambda$. Assume that i-th shock produces damage $D_i$ and $D_i$ are i.i.d. and independent of the process $N(t) =$ # of shocks in $[0, t]$. The damage due to a shock decreases exponentially in time, i.e., if initial damage is $D$ then its damage is $De^{\alpha t}$ after time $t$ with $\alpha > 0$ being a fixed parameter. Then the total damage at time t has the form

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-\alpha(t - S_i)}, \text{ where } S_i = \text{arrival time of i-th shock}$$

Evaluate $\mathbb{E}D(t)$ in terms of $\lambda, \alpha, t, \mathbb{E}D$. Here $\mathbb{E}D = \mathbb{E}D_i$.

*Proof.* First calculate conditional probability. Since $D_i$ is i.i.d. and $S_i$ given $N(t) = n$ is uniformly distributed (see Page 71 of Stochastic Process by Ross),

$$\mathbb{E}[D(t)|N(t) = n] = \mathbb{E}[\sum_{i=1}^{n} D_i e^{-\alpha(t-S_i)}|N(t) = n] = \sum_{i=1}^{n} \mathbb{E}[D]e^{-\alpha t}\mathbb{E}[e^{-\alpha S_i}|N(t) = n]$$

$$= \mathbb{E}[D]e^{-\alpha t}\sum_{i=1}^{n} \mathbb{E}[e^{\alpha U}] = \mathbb{E}[D]e^{-\alpha t}n\mathbb{E}[e^{\alpha U}] = \mathbb{E}[D]e^{-\alpha t}n\frac{1}{t}\int_0^t e^{\alpha x}dx = \mathbb{E}[D]e^{-\alpha t}\frac{n}{\alpha t}(1 - e^{\alpha t})$$

$$= \mathbb{E}[D]\frac{n}{\alpha t}(1 - e^{-\alpha t})$$

This gives us,

$$\mathbb{E}[D] = \sum_{i=1}^{n} \mathbb{E}[D(t)|N(t) = i]P(N(t) = i) = \sum_{i=1}^{n} \mathbb{E}[D]\frac{i}{\alpha t}(1 - e^{-\alpha t})P(N(t) = i)$$

$$= E[D]\frac{1}{\alpha t}(1 - e^{-\alpha t})\sum_{i=1}^{n} iP(N(t) = i) = E[D]\frac{1}{\alpha t}(1 - e^{-\alpha t})\lambda t = \frac{\lambda E(D)}{\alpha}(1 - e^{-\alpha t})$$

□

## Exercise 6

Let $N(t)$ be Poisson with rate $\lambda$. Show that for $s < t$

$$P(N(t) = k|N(t) = n) = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k}$$

*Proof.*

$$P(N(s) = k|N(t) = n) = \frac{P(N(s) = k, N(t) = n)}{P(N(t) = n)} = \frac{P(k \text{ events in } [0, s], n - k \text{ events in } [s, t])}{P(N(t) = n}$$

$$= \frac{P(k \text{ events in } [0, s])P(n - k \text{ events in } [s, t])}{P(N(t) = n} = \frac{P(N(s) = k)P(N(t - s) = n - k)}{P(N(t) = n)}$$

We know that $P(N(s) = k) = e^{-\lambda s}\frac{(\lambda s)^k}{k!}$, $P(N(t - s) = n - k) = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!}$, and $P(N(s) = n) = e^{-\lambda s}\frac{(\lambda s)^n}{n!}$. After algebra, we have

$$P(N(s) = k|N(t) = n) = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k}$$

□

## Exercise 7

Let $N(t)$ be Poisson with rate $\lambda$. Evaluate $\mathbb{E}N(t)N(t+s)$.

*Proof.*

$$\mathbb{E}(N(t+s)-N(t))^2 = \mathbb{E}N(t)^2 - 2\mathbb{E}N(t)N(t+s) + \mathbb{E}N(t+s)^2$$

. This implies

$$\mathbb{E}N(t)N(t+s) = \frac{\mathbb{E}N(t)^2 + \mathbb{E}N^2(t+s) - \mathbb{E}(N(t+s)-N(t))^2}{2}$$

Find

$$\mathbb{E}N(t)^2 = \sum_{n=1}^{\infty} nP(N^2(t)=n) = \sum_{n=1}^{\infty} n^2 P(N(t)=n) = \sum_{n=1}^{\infty} n^2 e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

After some algebra, one can find that $\mathbb{E}N(t)^2 = \lambda t(\lambda t+1)$. Likewise,

$$\mathbb{E}(N(t+s)-N(t))^2 = \sum_{n=1}^{\infty} n^2 P(N(t+s)-N(t)=n) = \sum_{n=1}^{\infty} n^2 P(N(s)=n) = \lambda s(\lambda s+1)$$

By combining and after some algebra, we will finally have

$$\mathbb{E}N(t)N(t+s) = \lambda t(\lambda t + \lambda s + 1)$$

.

□

## Exercise 8

Find the variance of the total waiting time $W = \sum_{i=1}^{N(t)}(t-S_i)$. Hint: $EW = \lambda t^2/2$.

*Proof.* First, notice that:

$$\left(\sum_{n=1}^{N}(t-S_n)\right)^2 = \sum_{n=1}^{N}(t-S_n)^2 + 2\sum_{j=1}^{N}\sum_{i=1}^{j-1}(t-S_i)(t-S_j)$$

Expanding this and taking a conditional expectation $\mathbb{E}[\cdot|N(t)=N]$ gives:

$$\mathbb{E}\left[\left(\sum_{n=1}^{N}(t-S_n)\right)^2|N(t)=N\right] = Nt^2 - 2t\mathbb{E}\left[\sum_{n=1}^{N}S_n|N(t)=N\right] + \mathbb{E}\left[\sum_{n=1}^{N}S_n^2|N(t)=N\right]$$

$$+2\sum_{j=1}^{N}\sum_{i=1}^{j-1} t^2 - \mathbb{E}[S_i|N(t)=n] - \mathbb{E}[S_j|N(t)=n] + \mathbb{E}[S_iS_j|N(t)=n]$$

Since conditional expectation of $S_i$ given $N(t) = N$ is uniformly iid, $\mathbb{E}[S_n|N(t)=n] = \mathbb{E}[U]$, $\mathbb{E}[S_n^2|N(t)=n] = \mathbb{E}[U^2]$, and $\mathbb{E}[S_nS_k|N(t)=n] = \mathbb{E}[U]^2$. Applying this in formula above and solving some algebra, we will have:

$$\mathbb{E}[W^2|N(t)=N] = N^2\frac{t^2}{4} + N\frac{t^2}{12}$$

Now law of total probability gives:

$$\mathbb{E}[W^2] = \sum_{n=1}^{\infty}\mathbb{E}[W^2|N(t)=n]P(N(t)=n) = \sum_{n=1}^{\infty}(n^2\frac{t^2}{4} + n\frac{t^2}{12})P(N(t)=n)$$

$$= \frac{t^2}{4}\sum_{n=1}^{\infty}n^2P(N(t)=n) + \frac{t^2}{12}\sum_{n=1}^{\infty}nP(N(t)=n) = \frac{t^2}{4}\lambda t(\lambda t+1) + \frac{t^2}{12}\lambda t$$

Finally, using the formula for variance: $Var(W) = \mathbb{E}[W^2] - (\mathbb{E}W)^2 = \frac{\lambda t^3}{3}$

$\square$

## Exercise 9

Consider the Laplace Transform:

$$L[f] = \bar{f}(s) = \int_0^{\infty} e^{-st}f(t)dt$$

and its properties:

$$L\left[\int f(x)dx\right] = \frac{\bar{f}(s)}{s}, \quad L[f*g] = L[f]L[g]$$

Let $m(t) = EN(t)$, be the renewal function for $X_1 - X_2 - F(x)$ with:

$$m(t) = F(t) + \int_0^t m(t-x)dF(x) = \int_0^t F(x)dx + \int_0^t m(t-x)xdF(x)$$

(a) Show that:

$$\bar{m}(s) = \frac{1}{s}\frac{\bar{f}(s)}{1-\bar{f}(s)}$$

(b) Using (a) and $L^{-1}[\bar{m}(s)] = m(t)$, show that for Poisson Process $\lambda$:

$$m(t) = \lambda t$$

*Proof.* (a)

$$\bar{m}(s) = L[m(t)] = L\left[\int_0^t f(x)dx\right] + L[m * f] = \frac{\bar{f}(s)}{s} + \bar{m}(s)\bar{f}(s)$$

$$\implies \bar{m}(s) - \bar{m}(s)\bar{f}(s) = (1 - \bar{f}(s))\bar{m}(s) = \frac{\bar{f}(s)}{s}$$

$$\implies \bar{m}(s) = \frac{1}{s}\frac{\bar{f}(s)}{1 - \bar{f}(s)}$$

(b)
Use (a) and $L^{-1}[\bar{m}(s)] = m(t)$ to show $m(t) = \lambda t$. First notice that for $X \sim \text{Poisson}(\lambda t)$, poisson proess,

$$f(x) = \lambda e^{\lambda t} \implies \bar{f}(x) = \int_0^\infty \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + s}$$

Thus

$$m(t) = L^{-1}[\bar{m}(s)] = L^{-1}\left[\frac{1}{s}\frac{\hat{f}(s)}{1 - \hat{f}(s)}\right] = L^{-1}\left[\frac{\lambda}{s^2}\right] = \lambda L^{-1}\left[\frac{1}{s^2}\right] = \lambda t$$

□

## Exercise 10

Consider a revenue process $R(t) = \sum_{i=0}^{N(t)} S_i$, of a department store operating for $t$ hours, given the customers arrive according to Poisson Process $N(t)$ with rate $\lambda$, and the $i$-th customer spends $S_i$, where $S_i$ are i.i.d. with finite second moment. Show that:

(a) The expected revenue $\mathbb{E}[R(t)] = \lambda t \mathbb{E}[S_1]$

(b) The variance $\text{Var}(R(t)) = \lambda t \mathbb{E}[S_1^2]$

*Proof.* (a) Note that $\mathbb{E}[R(t)|N(t) = n] = \mathbb{E}[\sum_{i=1}^n S_i|N(t) = n] = n\mathbb{E}[S_1]$ since $S_i$ is iid $\implies$

$$\mathbb{E}[R(t)] = \sum_{n=1}^\infty \mathbb{E}[R(t)|N(t) = n]P(N(t) = n) = \mathbb{E}[S_1]\sum_{n=1}^\infty nP(N(t) = n) = \lambda t \mathbb{E}(S_1)$$

(b) Using a similar reasoning as in (a), one can show that $\mathbb{E}[R(t)^2|N(t) = n] = n\mathbb{E}[S_1^2] + n(n-1)\mathbb{E}[S_1]^2$. Thus

$$\mathbb{E}[R(t)^2] = \sum_{n=1}^\infty \mathbb{E}[R(t)^2|N(t) = n]P(N(t) = n) = \sum_{n=1}^\infty [n\mathbb{E}[S_1^2] + n(n-1)\mathbb{E}[S_1]^2]P(N(t) = n)$$

$$= \lambda t \mathbb{E}[S_1^2] + \lambda^2 t^2 \mathbb{E}[S_1]^2$$

Now using variance formula,

$$Var(R(t)) = \lambda t \mathbb{E}[S_1^2]$$

□