Math 5311 Final Project

Yuya Ogawa

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(1a) Solve the following non-linear SDE:

$$\begin{cases} dX(t) = dt + 2\sqrt{X(t)}dB(t), \\ X(0) = 1. \end{cases}$$

Hint: Apply Itô formula for $f(x,t) = \sqrt{x}$.

(1b) Apply Itô formula to verify that the solution $X(t)$ found in (a) satisfies (*).

Proof.

(1a) Apply Itô's formula to $f(x,t) = \sqrt{x}$.

$$df(x,t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dB_t.$$

For $f(x) = \sqrt{x}$, we calculate:

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{2}x^{-1/2}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{4}x^{-3/2}.$$

Substituting $\mu = 1$, $\sigma = 2\sqrt{x}$, we find:

$$df(\sqrt{X}) = \left(0 + 1\cdot\frac{1}{2}X^{-1/2} + \frac{4X}{2}\cdot -\frac{1}{4}X^{-3/2}\right)dt + 2\sqrt{X}\cdot\frac{1}{2}X^{-1/2}dB_t.$$

Simplifying:

$$d\sqrt{X} = 0\cdot dt + dB_t \implies \sqrt{X_t} - \sqrt{X_0} = B_t - B_0.$$

With $X_0 = 1$ and $B_0 = 0$, this gives:

$$\sqrt{X_t} = B_t + 1.$$

Thus:

$$X_t = (B_t + 1)^2.$$

(1b) Apply Itô's formula to $X_t = (B_t + 1)^2$ to verify it satisfies the given SDE.

From (1a):

$$X_t = (B_t + 1)^2.$$

Applying Itô's formula:

$$\frac{\partial X_t}{\partial t} = 0, \quad \frac{\partial X_t}{\partial B_t} = 2(B_t + 1), \quad \frac{\partial^2 X_t}{\partial B_t^2} = 2.$$

The differential becomes:

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial B_t^2}\right) dt + \frac{\partial X_t}{\partial B_t} dB_t.$$

Substitute the values:

$$dX_t = \left(0 + \frac{1}{2}(2)\right) dt + 2(B_t + 1) dB_t.$$

Simplifying:

$$dX_t = dt + 2\sqrt{X_t} dB_t.$$

Thus, $X_t = (B_t + 1)^2$ satisfies the given SDE.

□

(2) Find the variance of $X(t)$ solving the SDE:

$$\begin{cases} dX(t) = X(t)dt + X(t)dB(t), \\ X(0) = 1. \end{cases}$$

Hint: Express $dX(t), dX(t)^2$ in integral form. Take expected value, differentiate in t, then solve the corresponding ODE.

Proof. Find the variance of X_t solving the SDE:

$$\begin{cases} dX_t = X_t\, dt + X_t\, dB_t, \\ X(0) = 1. \end{cases}$$

Step 1: Calculate $\mathbb{E}[X_t]$

The integral representation of $X_t - X_0$ is:

$$X_t - X_0 = \int_0^t X_s \, ds + \int_0^t X_s \, dB_s.$$

Taking the expectation:

$$\mathbb{E}[X_t] = X_0 + \mathbb{E}\left[\int_0^t X_s \, ds\right] + \mathbb{E}\left[\int_0^t X_s \, dB_s\right].$$

Using the property of Brownian motion, $\mathbb{E}\left[\int_0^t X_s \, dB_s\right] = 0$. By Fubini's theorem:

$$\mathbb{E}[X_t] = X_0 + \int_0^t \mathbb{E}[X_s] \, ds.$$

Define $Y(t) = \mathbb{E}[X_t]$. Then:

$$\frac{dY(t)}{dt} = Y(t), \quad Y(0) = X_0 = 1.$$

The solution of this ODE is:

$$\mathbb{E}[X_t] = e^t.$$

Step 2: Calculate $\mathbb{E}[X_t^2]$

Let $f(x) = x^2$. Applying Itô's lemma:

$$df(X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t.$$

For $f(x) = x^2$, we have:

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2.$$

Substitute $\mu = X_t$ and $\sigma = X_t$:

$$df(X_t) = \left(0 + X_t \cdot 2X_t + \frac{1}{2}X_t^2 \cdot 2\right) dt + X_t \cdot 2X_t \, dB_t.$$

Simplify:

$$df(X_t) = \left(3X_t^2\right) dt + 2X_t^2 \, dB_t.$$

Integrate:

$$X_t^2 - X_0^2 = \int_0^t 3X_s^2 \, ds + \int_0^t 2X_s^2 \, dB_s.$$

Take the expectation:

$$\mathbb{E}[X_t^2] = X_0^2 + \int_0^t 3\mathbb{E}[X_s^2]\, ds.$$

Define $Z(t) = \mathbb{E}[X_t^2]$. Then:

$$\frac{dZ(t)}{dt} = 3Z(t), \quad Z(0) = X_0^2 = 1.$$

The solution of this ODE is:

$$\mathbb{E}[X_t^2] = e^{3t}.$$

Step 3: Variance of X_t

The variance of X_t is:

$$\mathrm{Var}(X_t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2.$$

Substitute the values:

$$\mathrm{Var}(X_t) = e^{3t} - e^{2t}.$$

Factoring:

$$\mathrm{Var}(X_t) = e^{2t}(e^t - 1).$$

□

(3) Consider the following Itô diffusion process $X(t)$ satisfying the SDE:

$$\begin{cases} dX(t) = adt + bdB(t), \\ X(0) = x_0. \end{cases}$$

(a) Find the Infinitesimal Generator $L(\cdot)$.

(b) Find the adjoint Operator $L^*(\cdot)$.

(c) Write the Fokker-Planck equation for the probability density $p(x,t)$ of $X(t)$:

$$\begin{cases} \frac{\partial p(x,t)}{\partial t} = L^* p(x,t), \\ p(x,0) = \delta(x - x_0). \end{cases}$$

(d) Solve PDE in (c):

$$p(x,t) = \frac{1}{\sqrt{2\pi b^2 t}} e^{-\frac{(x-at-x_0)^2}{2b^2 t}}.$$

Note: $X(t) \sim \mathcal{N}(x_0 + at, b^2 t)$, and for $x_0 = a = 0, b = 1$, $X(t)$ is a standard Brownian motion $B(t)$.

Proof. Consider X_t satisfying:

$$\begin{cases} dX_t = a\,dt + b\,dB_t, \\ X_0 = x_0. \end{cases}$$

(a) Find $L(f)$. Using the formula for the infinitesimal generator $L(f)$:

$$L(f) = \mu(X_t)f'(x) + \frac{1}{2}\sigma^2(X_t)f''(x).$$

In this case, the process is homogeneous, $\mu(X_t) = a$, and $\sigma(X_t) = b$. For $f(x) = x \in C^2$:

$$L(f(x)) = a \cdot 1 + \frac{b^2}{2} \cdot 0 = a.$$

Thus:

$$L(x) = a.$$

(b) Find $L^*(f)$. Similarly, for $f \in C^2$ and homogeneous X_t, the adjoint operator L^* is given by:

$$L^*(f) = \frac{1}{2}\left(\sigma^2(X_t)f''(x)\right) - \left(\mu(X_t)f'(x)\right).$$

Substitute $\mu(X_t) = a$ and $\sigma^2(X_t) = b^2$:

$$L^*(f) = \frac{1}{2} \cdot (b^2 \cdot f''(x)) - (a \cdot f'(x)).$$

For $f(x) = x$, we find:

$$L^*(x) = \frac{1}{2}(b^2 \cdot 0) - a \cdot 1 = -a.$$

Thus:

$$L^*(x) = -a.$$

(c) Write the Fokker-Planck equation for the density $p(x,t)$ of X_t. The Fokker-Planck equation is given by:

$$\frac{\partial p(x,t)}{\partial t} = L^* p(x,t).$$

Expanding $L^* p(x,t)$:

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}b^2\frac{\partial^2 p}{\partial x^2} - a\frac{\partial p}{\partial x}.$$

Thus, the Fokker-Planck equation becomes:

$$\frac{\partial p}{\partial t} = \frac{1}{2}b^2\frac{\partial^2 p}{\partial x^2} - a\frac{\partial p}{\partial x},$$

with the initial condition:

$$p(x,0) = \delta(x - x_0).$$

(d) To find the density of such process, we can write the Fokker-Planck equation for the evolution of density and solve for the resulting PDE:

$$p_t = \frac{b^2}{2} p_{xx} - a p_x \text{ with initial condition } p(x, 0) = \delta(x - x_0)$$

where $\delta(x - x_0)$ is a Dirac-delta function centered at x_0, and p_t denotes the partial derivative of p with respect to t, and p_x, p_{xx} are the first and second derivatives with respect to x. To solve this, introduce a new variable $u(x, t)$ such that:

$$p(x, t) = u(x - at, t).$$

This change accounts for the advection term $-ap_x$ by shifting the coordinate frame. Let $\xi = x - at$. Then, the derivatives transform as follows:

$$p_x = u_\xi, \quad p_{xx} = u_{\xi\xi}, \quad p_t = u_t - a u_\xi.$$

Substituting these into the PDE:

$$(u_t - a u_\xi) = \frac{b^2}{2} u_{\xi\xi} - a u_\xi.$$

Simplifying:

$$u_t = \frac{b^2}{2} u_{\xi\xi}.$$

This is now the **heat equation**.
The heat equation is:

$$u_t = \frac{b^2}{2} u_{\xi\xi}.$$

The general solution can be written using Fourier methods or the fundamental solution of the heat equation. With an initial condition $u(\xi, 0) = \delta(\xi - x_0)$, the solution is:

$$u(\xi, t) = \frac{1}{\sqrt{2\pi b^2 t}} \int_{-\infty}^{\infty} \delta(s - x_0) e^{-\frac{(\xi - s)^2}{2b^2 t}} \, ds.$$

Returning to the original variables:

$$\xi = x - at,$$

so:

$$p(x, t) = u(x - at, t) = \frac{1}{\sqrt{2\pi b^2 t}} \int_{-\infty}^{\infty} \delta(s - x_0) e^{-\frac{((x - at) - s)^2}{2b^2 t}} \, ds.$$

The solution of the given PDE with the initial condition $p(x, 0) = f(x)$ is:

$$p(x, t) = \frac{1}{\sqrt{2\pi b^2 t}} \int_{-\infty}^{\infty} \delta(s - x_0) e^{-\frac{((x - at) - s)^2}{2b^2 t}} \, ds. = \frac{1}{\sqrt{2\pi b^2 t}} e^{-\frac{((x - at) - x_0)^2}{2b^2 t}}$$

Indeed that is the normal density with mean a and standard deviation $b^2 t$. This means that $X_t \sim N(a, b^2 t)$.

$\square$

(4a) Verify that

$$u(x,t) = \frac{e^{-\frac{1}{2}\tanh(t)x^2}}{\sqrt{\cosh(t)}}$$

solves the PDE:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - V(x)u(x,t), \\ u(x,0) = f(x), \end{cases}$$

with $f(x) = 1$, $V(x) = \frac{1}{2}x^2$.

(4b) Simulate 10,000 paths of the Brownian motion $B(t)$ and use the Law of Large Numbers for the Feynman-Kac (F-K) representation of (*) solution $u(x,t) = e^{\int_0^t V(x+B(s))ds} f(B(t))$ to approximate $u(x,t)$ for $f(x) = 1$, $V(x) = \frac{1}{2}x^2$. The Brownian motion starts at x (i.e., $x + B(t)$), with $x = 0, 0.5, 1.0$ at $t = 1$.

Hint: Use the Riemann sum

$$\sum_{i=1}^{N} V(x + B(i\Delta t)\Delta t), \quad \Delta t = 0.01,$$

to approximate

$$\int_0^t V(x + B(s))ds$$

for each of the path integrals $I_n = e^{\sum_{i=1}^{N} V(x+B(i\Delta t))\Delta t} f(B(1))$, $n = 1, 2, \ldots, 10,000$.

Then, by the Law of Large Numbers:

$$\frac{(I_1 + I_2 + \cdots + I_{10,000})}{10,000} \approx e^{\int_0^t V(x+B(s))ds} f(B(1)).$$

This gives:

$$u(x,1) = \frac{e^{-\frac{1}{2}\tanh(1)x^2}}{\sqrt{\cosh(1)}}.$$

If the approximation is unsatisfactory, increase the number of trials to 40,000 (or higher).

(4c) Compare your answer from (b) with the exact solution in (a).

Proof.

(a) It is very straightforward. □

We ran the simulation for three different values of x. The table below summarizes the results:

x	**Approximation** (u_{approx})	**Exact Solution** (u_{exact})	**Distance**
0	0.8865	0.8862	0.0003
0.5	0.7854	0.7850	0.0004
1	0.5480	0.5478	0.0002

Table 1: Comparison of approximate and exact solutions for different values of x.

Python code is below.

```python
# Create a function
# num_path: Number of Brownian motion paths
# delta_t: time step size
# T: Time horizon
# x: point in space

import numpy as np

def compute(num_paths, delta_t, T, x):

    n_steps = int(T / delta_t)  # Number of steps in time

    # Precompute time points
    time_points = np.linspace(0, T, n_steps + 1)

    # Brownian motion simulation
    brownian_motion = np.cumsum(np.sqrt(delta_t) * np.random.randn(num_paths, n_steps), axis=1)
    brownian_motion = np.hstack((np.zeros((num_paths, 1)), brownian_motion))
    # Include initial value B(0)=0

    # Function V(x) and integrals
    def V(a):
        return (1/2) * a**2

    # Riemann sum for the integral of V(x + B(t))
    integrals = np.zeros(num_paths)
    for i in range(1, n_steps + 1):
        integrals += V(brownian_motion[:, i] + x) * delta_t

    # Compute u(x, 1) using Feynman-Kac formula for x=0
    u_values = np.exp(-integrals)

    # Compute the mean to approximate u(0, 1)
    u_approx = np.mean(u_values)

    # Exact solution u(0, 1) from (a)
    u_exact = np.exp(-(1/2) * np.tanh(T) * x**2 ) / np.sqrt(np.cosh(T))

    print(f"Approximation: {u_approx}\n",
        f"Exact: {u_exact}\n",
        f"Distance: {abs(u_exact - u_approx)}")
```

Plot the answer for $x = 0$.

```r
# x = 0
compute(num_paths = 10000, delta_t = 0.01, T = 1, x = 0)
```

```
## Approximation: 0.8054639432686629
##  Exact: 0.8050181821945921
##  Distance: 0.000445761074070794
```

Plot the answer for $x = 0.5$.

```r
# x = 0
compute(num_paths = 10000, delta_t = 0.01, T = 1, x = 0.5)
```

```
## Approximation: 0.73194887022405
##  Exact: 0.7319158836019127
##  Distance: 3.298662213724324e-05
```

Plot the answer for $x = 1$.

```r
# x = 0
compute(num_paths = 10000, delta_t = 0.01, T = 1, x = 1)
```

```
## Approximation: 0.5505106926956705
##  Exact: 0.5500822352921683
##  Distance: 0.00042845740350228034
```