

# Plotting eccentric orbits for stapmonic systems

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## Abstract

When calculating trajectories of celestial bodies, the eccentricity and semi-major axis may not always be known. The aim of this paper is to generalise the plotting methods developed by French [2] to cover Kepler orbits defined by velocity, position, and mass of the satellite, as well as systems with multiple bodies and variable-mass systems.

## 1 Introduction

In the British Physics Olympiad Computational Challenge 2023 Challenge Presentation document [2], methods were explained describing how to plot Keplerian orbits of satellites around a primary body, given the eccentricity and semi-major axis of orbit. However, given an arbitrary set of orbiting bodies, such quantities may not always be easily deduced, and it may be easier instead to measure quantities associated with the relative motion of the secondary body itself, such as velocity, position and mass. Furthermore, it is not always true that an orbiting system has one primary mass much greater in magnitude than the secondary masses, nor that the mass of each celestial body is constant.

We apply concepts from orbital dynamics to calculate the eccentricity and semi-major axis of a satellite given the instantaneous velocity vector, relative position vector and the masses of bodies in the system, first for the simpler central-force problem, and then for systems with multiple bodies as well as variable-mass systems.

## 2 The central-force problem

### 2.1 Preliminary definitions

In this instance of the central-force problem, we have a primary body with mass  $M$  and a secondary orbiting body with mass  $m \ll M$ . Since we are concerned with the relative motion of the body  $m$  around the body  $M$ , we can first choose a coordinate system giving  $M$  and  $m$  the position vectors  $\mathbf{r}_M$  and  $\mathbf{r}_m$  respectively. Now, we translate the axes such that  $M$  is located the origin and  $m$  has displacement vector  $\mathbf{r}$ , so that  $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$  is the displacement between  $m$  and  $M$ . We write the orbital velocity of  $m$  as  $\mathbf{v}$  or  $\dot{\mathbf{r}}$  and the centripetal acceleration as  $\dot{\mathbf{v}}$  or  $\ddot{\mathbf{r}}$ . We also introduce the standard gravitational parameter  $\mu = G(M + m)$  where  $G$  is the gravitational constant.

Recall an important equation from Newton [3]. The *law of universal gravitation* states that, concerning a gravitational force between two bodies,

$$\mathbf{F} = \frac{GMm}{r^2} \frac{\mathbf{r}}{r}$$

The remainder of this subsection closely follows the derivation provided in the book *Fundamentals of Astrodynamics* by Muller, White, and Bate [1].

Applying Newton's second law to the law of universal gravitation, we get

$$M\ddot{\mathbf{r}}_M = -m\ddot{\mathbf{r}}_m = \frac{GMm}{r^2} \frac{\mathbf{r}}{r}$$

which yields

$$\ddot{\mathbf{r}}_M = \frac{Gm}{r^3} \mathbf{r}$$

and

$$\ddot{\mathbf{r}}_m = -\frac{GM}{r^3} \mathbf{r}$$

Since  $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$ , we can take the second derivative to yield  $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_m - \ddot{\mathbf{r}}_M$ , finally giving us

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} \quad (1)$$

## 2.2 Specific orbital energy

Let us begin by considering the energy of the system. In particular, we are concerned with the mechanical energy in the orbiting body per unit mass, also known as the specific orbital energy  $\varepsilon$ . We define the specific orbital energy of the satellite to be equal to the sum of its kinetic energy and potential energy per unit mass. This can be written simply as

$$\varepsilon = \varepsilon_k + \varepsilon_p$$

It is not difficult to see that  $\varepsilon_k = v^2/2$ , and we can express  $\varepsilon_p$  as follows:

$$\begin{aligned} \varepsilon_p &= \frac{W}{m} = \frac{Fd}{m} \\ &= \ddot{r}r \\ &= -\frac{\mu}{r^2}r \\ &= -\frac{\mu}{r} \end{aligned}$$

Thus, we can write down the formula for specific orbital energy as follows, as derived by [1]:

$$\varepsilon = \frac{v^2}{2} - \left(\frac{\mu}{r} + c\right)$$

where  $c$  represents a constant value depending on where we choose  $\varepsilon_p = 0$  to be. In fact, it is helpful in this case to choose  $c = 0$ , allowing us to write

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

## 2.3 The orbital-energy-invariance law and semi-major axis

The orbital-energy-invariance law [1] states that

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

where  $2a$  represents the major axis of the orbiting body. This means that the semi-major axis of the orbital path depends only on the masses of the two bodies ( $\mu$ ), the instantaneous velocity vector ( $\mathbf{v}$ ), and the instantaneous position vector ( $\mathbf{r}$ ). Solving for  $a$ , we get

$$a = \frac{\mu r}{2\mu - v^2 r} \quad (2)$$

## 2.4 Semi-latus rectum and eccentricity

The specific angular momentum  $\mathbf{h}$  of the orbiting body is defined as

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \frac{\mathbf{L}}{m}$$

where  $\mathbf{L}$  represents the total angular momentum vector of the body. Due to conservation of angular momentum, the vector  $\mathbf{h}$  is always constant, and thus we can consider the constant  $h = |\mathbf{r} \times \mathbf{v}| = rv \sin \theta$  where  $\theta$  is the angle between the position and velocity vectors.

Take the right cross product of equation (1) with  $\mathbf{h}$  and integrate once with respect to time [1]. After simplification, we obtain

$$r = \frac{h^2/\mu}{1 + (B/\mu) \cos \nu}$$

where  $\mathbf{B}$  is the vector constant of integration and  $\nu$  is the angle between  $\mathbf{B}$  and  $\mathbf{r}$ . We can compare this equation with the polar equation of a general conic section

$$r = \frac{p}{1 + e \cos \nu}$$

where  $e$  is our desired eccentricity and  $p$  is the semi-latus rectum of the conic section (half the length of the chord passing through one focus, that is perpendicular to the major axis). By directly comparing the two, we find that  $p = h^2/\mu$ . Now, we want to find the relationship linking semi-major axis  $a$ , eccentricity  $e$ , and semi-latus rectum  $p$ .

Consider the equation of a standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where  $a$  is the semi-major axis and  $b$  is the semi-minor axis. Then, the foci are located at  $(\sqrt{a^2 - b^2}, 0)$  and  $(-\sqrt{a^2 - b^2}, 0)$ , and the eccentricity is defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Now, consider the right-angled triangle with the line segment between the two foci as the base and the semi-latus rectum as the height, with its right angle at a focus. The length of the base of the triangle is

$$\begin{aligned} l &= \sqrt{a^2 - b^2} - (-\sqrt{a^2 - b^2}) \\ &= 2\sqrt{a^2 - b^2} \\ &= 2a\sqrt{1 - \frac{b^2}{a^2}} \\ &= 2ae \end{aligned}$$

Using the definition of an ellipse as “the set of all points around two foci such that the sum of the distances to each focus is  $2a$ ”, it follows that since the height of the triangle is  $p$ , the hypotenuse must be  $2a - p$ . We can use Pythagoras to write

$$\begin{aligned} (2a - p)^2 &= p^2 + (2ae)^2 \\ 4a^2 - 4ap &= 4a^2e^2 \\ a - p &= ae^2 \\ e &= \sqrt{1 - \frac{p}{a}} \end{aligned} \tag{3}$$

Using  $p = h^2/\mu$  and (2), we can write down a formula for the eccentricity of the orbit:

$$e = \sqrt{1 - \frac{h^2(2\mu - v^2r)}{\mu^2r}}$$

or, using only  $v$ ,  $r$  and  $\mu$ :

$$e = \sqrt{1 - \frac{|\mathbf{r} \times \mathbf{v}|^2(2\mu - v^2r)}{\mu^2r}} \tag{4}$$

## 2.5 Plotting the single-body orbit

Now that we can express both  $a$  and  $e$  in terms of the desired quantities, we can use the same methods outlined by French [2] to plot the elliptical orbit.

We are given the vectors  $\mathbf{r}$  and  $\mathbf{v}$  associated with the satellite at an arbitrary point in its orbit, as well as the values for  $m$  and  $M$ . From this, we can calculate  $\mu$  and then  $a$  and  $e$ , and create a parametric equation

$$r(\theta) = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

to calculate  $r$  as  $\theta$  varies. We evaluate the period of the orbit by Kepler's Third Law [1]:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Finally, we evaluate  $r(\theta)$  as  $\theta$  varies between 0 and  $2\pi$  and obtain the elliptical orbit of the secondary body.

### 3 The two-body problem

#### 3.1 Preliminary definitions

The two-body problem describes a system of two bodies of similar masses  $m_1$  and  $m_2$  which orbit a common centre of mass known as the barycentre. We assume no external perturbation forces, so the acceleration of the barycentre must be zero. This allows us to pick our inertial reference frame to have the barycentre as the origin, and we write the positions of  $m_1$  and  $m_2$  as  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , and the velocities to be  $\mathbf{v}_1$  and  $\mathbf{v}_2$  respectively. We may approach this problem in a similar manner as the central-force problem, since it is simply a special case of the two-body problem in which  $m_1 \gg m_2$  and the barycentre is taken to be the centre of  $m_1$ .

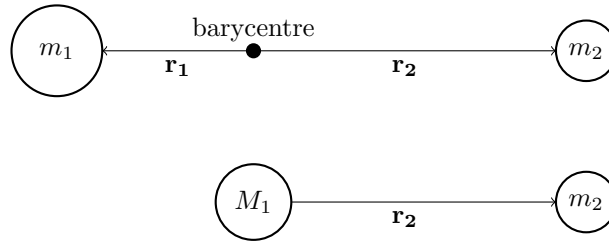
#### 3.2 The barycentre

The barycentre is defined as the centre of mass of bodies  $m_1$  and  $m_2$ , and both bodies follow eccentric orbits around it. However, since the barycentre is no longer coincident with either body, it is not possible to use the results obtained in 2 directly. One method of calculating the desired eccentricities and semi-major axes is by assigning the barycentre an "effective mass" such that if one body is removed, the other's orbit is unchanged.

We begin by calculating the relative position of the barycentre. Since it is the centre of mass of the system, we can convert this problem into a lever in equilibrium, with masses  $m_1$  and  $m_2$  at either end and the barycentre acting as the fulcrum. Now,

$$\begin{aligned} m_1 r_1 &= m_2 r_2 \\ m_1 r_1 + m_1 r_2 &= m_2 r_2 + m_1 r_2 \\ m_1 (r_1 + r_2) &= r_2 (m_1 + m_2) \\ r_2 &= \frac{m_1}{m_1 + m_2} (r_1 + r_2) \end{aligned}$$

Let us return to the orbiting system. Consider the following diagrams, in which we replace  $m_1$  with the barycentre's "effective mass"  $M_1$  without loss of generality:



We may write  $r = r_1 + r_2$ , and we wish to find  $M_1$  in terms of  $m_1$  and  $m_2$  such that  $m_2$  experiences the same force due to gravity.

$$\begin{aligned} \frac{Gm_1m_2}{r^2} &= \frac{GM_1m_2}{r_2^2} \\ \frac{m_1}{r^2} &= \frac{M_1}{r_2^2} \end{aligned}$$

From above, we may use the substitution  $r_2 = \frac{m_1 r}{m_1 + m_2}$ , giving us

$$\frac{m_1}{r^2} = \frac{M_1(m_1 + m_2)^2}{m_1^2 r^2}$$

$$\boxed{M_1 = \frac{m_1^3}{(m_1 + m_2)^2}} \quad (5.1)$$

and, by symmetry,

$$\boxed{M_2 = \frac{m_2^3}{(m_1 + m_2)^2}} \quad (5.2)$$

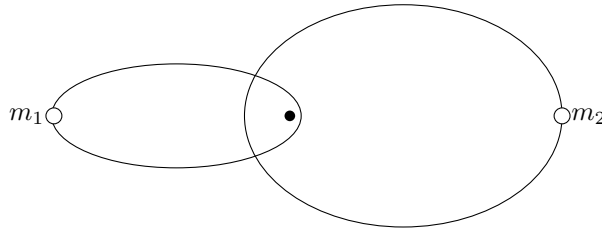
The problem is now reduced to two one-body problems with the primary masses given by (5.1) and (5.2). We may now separately define two standard gravitational parameters  $\mu_1$  and  $\mu_2$  such that  $\mu_1 = G(M_1 + m_2)$  and  $\mu_2 = G(m_1 + M_2)$ , and evaluate the semi-major axis and eccentricity of each orbit according to (2) and (4) above.

### 3.3 Plotting the two-body orbit

We must be careful when plotting the orbits for the two-body problem. Like the central-force problem, both orbits are elliptical with one focus at the barycentre, but this allows for many configurations of the two ellipses based on rotation and inclination angles. The first thing to notice is that due to conservation of (specific) angular momentum  $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ , we find that the position and velocity vectors are always perpendicular to the constant vector  $\mathbf{h}$  and thus both orbits are coplanar.

By the definition of the barycentre as the centre of mass, we know that  $m_1$  and  $m_2$  must always be collinear with the barycentre, and on opposite sides. This implies that  $m_1$  and  $m_2$  have the same orbital period: there is a bijection between points on the orbit of  $m_1$  and points on the orbit of  $m_2$  such that they are on opposite sides and collinear with the barycentre.

Finally, let us consider the motion of  $m_2$  as  $m_1$  moves from *apoapsis* (farthest point from the barycentre) to *periapsis* (closest point to the barycentre) and back. Clearly, both “semi-orbits” take half of the period, and thus by Kepler’s second law  $m_2$  must sweep out two equal areas in each “semi-orbit” of  $m_1$ . However, noting that the apoapsis and periapsis of  $m_1$  are collinear with the barycentre, the start and end positions of  $m_2$  must also be collinear with the barycentre. The only possible configuration under such conditions is for  $m_1$  and  $m_2$  to have their periapses and apoapses coincide, such as in the diagram below.



## 4 The restricted three-body problem

### 4.1 :)

### References

- [1] DONALD D. MUELLER, JERRY WHITE, R. R. B. *Fundamentals of Astrodynamics*. Dover Publications, 1971, ch. 1.
- [2] FRENCH, A. British Physics Olympiad Computational Challenge 2023 Solar System Orbits.
- [3] NEWTON, I. *Philosophiae Naturalis Principia Mathematica*. Edmond Halley, 1687.