Plotting eccentric Kepler orbits from instantaneous position and velocity

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Abstract

When calculating trajectories of celestial bodies, the eccentricity and semi-major axis may not always be known. The aim of this paper is to generalise the plotting methods developed by French [2] to cover Kepler orbits defined by velocity, position, and mass of the satellite, as well as systems with two bodies. We finally consider eccentric trajectories that are non-elliptical.

1 Introduction

In the British Physics Olympiad Computational Challenge 2023 Challenge Presentation document [2], methods were explained describing how to plot Keplerian orbits of satellites around a primary body, given the eccentricity and semi-major axis of orbit. However, given an arbitrary set of orbiting bodies, such quantities may not always be easily deduced, and it may be easier instead to measure quantities associated with the relative motion of the secondary body itself, such as velocity, position and mass. Furthermore, it is not always true that an orbiting system has one primary mass much greater in magnitude than any secondary masses.

We apply concepts from orbital dynamics to calculate the eccentricity and semi-major axis of a satellite given an instantaneous velocity vector, relative position vector and the masses of bodies in the system, first for the simpler central-force problem, and then for systems with two bodies. Finally, we rewrite in Cartesian coordinates eccentric Kepler orbits that yield parabolic and hyperbolic trajectories.

2 The central-force problem

2.1 Preliminary definitions

In this instance of the central-force problem, we have a primary body with mass M and a secondary orbiting body with mass $m \ll M$. Since we are concerned with the relative motion of the body m around the body M, we can first choose a coordinate system giving M and m the position vectors $\mathbf{r_M}$ and $\mathbf{r_m}$ respectively. Now, we translate the axes such that M is located the origin and m has displacement vector $\mathbf{r}$, so that $\mathbf{r} = \mathbf{r_m} - \mathbf{r_M}$ is the displacement between m and M. We write the orbital velocity of m as $\mathbf{v}$ or $\dot{\mathbf{r}}$ and the centripetal acceleration as $\dot{\mathbf{v}}$ or $\ddot{\mathbf{r}}$ .We also introduce the standard gravitational parameter $\mu = G(M + m)$ where G is the gravitational constant.

Recall an important equation from Newton [3]. The *law of universal gravitation* states that, concerning a gravitational force between two bodies,

$$\mathbf{F} = \frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$

The remainder of this subsection closely follows the derivation provided in the book Fundamentals of Astrodynamics by Muller, White, and Bate [1].

Applying Newton's second law to the law of universal gravitation, we get

$$M\ddot{\mathbf{r}}_\mathbf{M} = -m\ddot{\mathbf{r}}_\mathbf{m} = \frac{GMm}{r^2}\frac{\mathbf{r}}{r}$$

which yields

$$\ddot{\mathbf{r}}_\mathbf{M} = \frac{Gm}{r^3}\mathbf{r}$$

and

$$\ddot{\mathbf{r}}_{\mathbf{m}} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r} = \mathbf{r_m} - \mathbf{r_M}$, we can take the second derivative to yield $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{\mathbf{m}} - \ddot{\mathbf{r}}_{\mathbf{M}}$, finally giving us

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} \tag{1}$$

2.2 Specific orbital energy

Let us begin by considering the energy of the system. In particular, we are concerned with the mechanical energy in the orbiting body per unit mass, also known as the specific orbital energy ε. We define the specific orbital energy of the satellite to be equal to the sum of its kinetic energy and potential energy per unit mass. This can be written simply as

$$\varepsilon = \varepsilon_k + \varepsilon_p$$

It is not difficult to see that $\varepsilon_k = v^2/2$, and we can express ε_p as follows:

$$\begin{aligned}\varepsilon_p = \frac{W}{m} &= \frac{Fd}{m} \\ &= \ddot{r}r \\ &= -\frac{\mu}{r^2}r \\ &= -\frac{\mu}{r}\end{aligned}$$

Thus, we can write down the formula for specific orbital energy as follows, as derived by [1]:

$$\varepsilon = \frac{v^2}{2} - \left(\frac{\mu}{r} + c\right)$$

where c represents a constant value depending on where we choose $\varepsilon_p = 0$ to be. In fact, it is helpful in this case to choose $c = 0$, allowing us to write

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

As it turns out, this value of specific orbital energy is exactly the negative of the amount of additional energy required for 1 kg of the orbiting body to reach escape velocity.

2.3 The orbital-energy-invariance law and semi-major axis

The orbital-energy-invariance law [1] states that

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

where $2a$ represents the major axis of the orbiting body. This means that the semi-major axis of the orbital path depends only on the masses of the two bodies (μ), the instantaneous velocity vector ($\mathbf{v}$), and the instantaneous position vector ($\mathbf{r}$). Solving for a, we get

$$\boxed{a = \frac{\mu r}{2\mu - v^2 r}} \tag{2}$$

2.4 Semi-latus rectum and eccentricity

The specific angular momentum $\mathbf{h}$ of the orbiting body is defined as

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \frac{\mathbf{L}}{m}$$

where $\mathbf{L}$ represents the total angular momentum vector of the body. Due to conservation of angular momentum, the vector $\mathbf{h}$ is always constant, and thus we can consider the constant $h = |\mathbf{r} \times \mathbf{v}| = rv\sin\theta$ where θ is the angle between the position and velocity vectors.

Take the right cross product of equation (1) with $\mathbf{h}$ and integrate once with respect to time [1]. After simplification, we obtain

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos\nu}$$

where $\mathbf{B}$ is the vector constant of integration and ν is the angle between $\mathbf{B}$ and $\mathbf{r}$.
We can compare this equation with the polar equation of a general conic section

$$r = \frac{p}{1 + e\cos\nu}$$

where e is our desired eccentricity and p is the semi-latus rectum of the conic section (half the length of the chord passing through one focus, that is perpendicular to the major axis). By directly comparing the two, we find that $p = h^2/\mu$. Now, we want to find the relationship linking semi-major axis a, eccentricity e, and semi-latus rectum p.
Consider the equation of a standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is the semi-major axis and b is the semi-minor axis. Then, the foci are located at $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$, and the eccentricity is defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Now, consider the right-angled triangle with the line segment between the two foci as the base and the semi-latus rectum as the height, with its right angle at a focus. The length of the base of the triangle is

$$\begin{aligned} l &= \sqrt{a^2 - b^2} - (-\sqrt{a^2 - b^2}) \\ &= 2\sqrt{a^2 - b^2} \\ &= 2a\sqrt{1 - \frac{b^2}{a^2}} \\ &= 2ae \end{aligned}$$

Using the definition of an ellipse as "the set of all points around two foci such that the sum of the distances to each focus is $2a$", it follows that since the height of the triangle is p, the hypotenuse must be $2a - p$. We can use Pythagoras to write

$$\begin{aligned} (2a - p)^2 &= p^2 + (2ae)^2 \\ 4a^2 - 4ap &= 4a^2e^2 \\ a - p &= ae^2 \\ e &= \sqrt{1 - \frac{p}{a}} \end{aligned} \tag{3}$$

Using $p = h^2/\mu$ and (2), we can write down a formula for the eccentricity of the orbit:

$$e = \sqrt{1 - \frac{h^2(2\mu - v^2 r)}{\mu^2 r}}$$

or, using only v, r and μ:

$$\boxed{e = \sqrt{1 - \frac{|\mathbf{r} \times \mathbf{v}|^2(2\mu - v^2 r)}{\mu^2 r}}} \tag{4}$$

2.5 Plotting the single-body orbit

Now that we can express both a and e in terms of the desired quantities, we can use the same methods outlined by French [2] to plot the elliptical orbit.
We are given the vectors $\mathbf{r}$ and $\mathbf{v}$ associated with the satellite at an arbitrary point in its orbit,

as well as the values for m and M. From this, we can calculate μ and then a and e, and create a parametric equation

$$r(\theta) = \frac{a(1-e^2)}{1-e\cos\theta}$$

to calculate r as θ varies. We evaluate the period of the orbit by Kepler's Third Law [1]:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Finally, we evaluate $r(\theta)$ as θ varies between 0 and 2π and obtain the elliptical orbit of the secondary body.

3 The two-body problem

3.1 Preliminary definitions

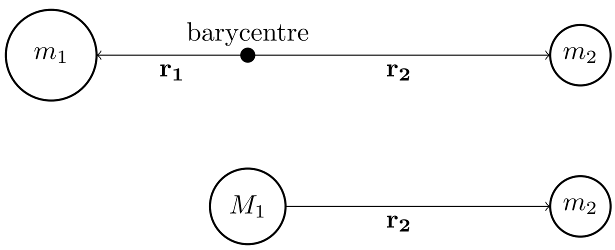
The two-body problem describes a system of two bodies of similar masses m_1 and m_2 which orbit a common centre of mass known as the barycentre. We assume no external perturbation forces, so the acceleration of the barycentre must be zero. This allows us to pick our inertial reference frame to have the barycentre as the origin, and we write the positions of m_1 and m_2 as $\mathbf{r_1}$ and $\mathbf{r_2}$, and the velocities to be $\mathbf{v_1}$ and $\mathbf{v_2}$ respectively. We may approach this problem in a similar manner as the central-force problem, since it is simply a special case of the two-body problem in which $m_1 \gg m_2$ and the barycentre is taken to be the centre of m_1.

3.2 The barycentre

The barycentre is defined as the centre of mass of bodies m_1 and m_2, and both bodies follow eccentric orbits around it. However, since the barycentre is no longer coincident with either body, it is not possible to use the results obtained in 2 directly. One method of calculating the desired eccentricities and semi-major axes is by assigning the barycentre an "effective mass" such that if one body is removed, the other's orbit is unchanged.
We begin by calculating the relative position of the barycentre. Since it is the centre of mass of the system, we can convert this problem into a lever in equilibrium, with masses m_1 and m_2 at either end and the barycentre acting as the fulcrum. Now,

$$\begin{aligned}
m_1 r_1 &= m_2 r_2 \\
m_1 r_1 + m_1 r_2 &= m_2 r_2 + m_1 r_2 \\
m_1(r_1 + r_2) &= r_2(m_1 + m_2) \\
r_2 &= \frac{m_1}{m_1 + m_2}(r_1 + r_2)
\end{aligned}$$

Let us return to the orbiting system. Consider the following diagrams, in which we replace m_1 with the barycentre's "effective mass" M_1 without loss of generality:



We may write $r = r_1 + r_2$, and we wish to find M_1 in terms of m_1 and m_2 such that m_2 experiences the same force due to gravity.

$$\frac{Gm_1m_2}{r^2} = \frac{GM_1m_2}{r_2^2}$$

$$\frac{m_1}{r^2} = \frac{M_1}{r_2^2}$$

From above, we may use the substitution $r_2 = \dfrac{m_1 r}{m_1 + m_2}$, giving us

$$\frac{m_1}{r^2} = \frac{M_1(m_1+m_2)^2}{m_1^2 r^2}$$

$$\boxed{M_1 = \frac{m_1^3}{(m_1+m_2)^2}} \tag{5.1}$$

and, by symmetry,

$$\boxed{M_2 = \frac{m_2^3}{(m_1+m_2)^2}} \tag{5.2}$$

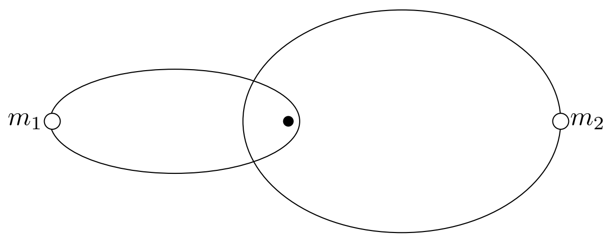
The problem is now reduced to two one-body problems with the primary masses given by (5.1) and (5.2). We may now separately define two standard gravitational parameters μ_1 and μ_2 such that $\mu_1 = G(M_1 + m_2)$ and $\mu_2 = G(m_1 + M_2)$, and evaluate the semi-major axis and eccentricity of each orbit according to (2) and (4) above.

3.3 Plotting the two-body orbit

We must be careful when plotting the orbits for the two-body problem. Like the central-force problem, both orbits are elliptical with one focus at the barycentre, but this allows for many configurations of the two ellipses based on rotation and inclination angles. The first thing to notice is that due to conservation of (specific) angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, we find that the position and velocity vectors are always perpendicular to the constant vector $\mathbf{h}$ and thus both orbits are coplanar.

By the definition of the barycentre as the centre of mass, we know that m_1 and m_2 must always be collinear with the barycentre, and on opposite sides. This implies that m_1 and m_2 have the same orbital period: there is a bijection between points on the orbit of m_1 and points on the orbit of m_2 such that they are on opposite sides and collinear with the barycentre.

Finally, let us consider the motion of m_2 as m_1 moves from *apoapsis* (farthest point from the barycentre) to *periapsis* (closest point to the barycentre) and back. Clearly, both "semi-orbits" take half of the period, and thus by Kepler's second law m_2 must sweep out two equal areas in each "semi-orbit" of m_1. However, noting that the apoapsis and periapsis of m_1 are collinear with the barycentre, the start and end positions of m_2 must also be collinear with the barycentre. The only possible configuration under such conditions is for m_1 and m_2 to have their periapses and apoapses coincide, such as in the diagram below.



4 Non-elliptical trajectories

So far, we have dealt only with Kepler orbits that happen to be elliptical. However, it is not always true that an orbiting body will follow an ellipse as its trajectory. In particular, by varying the value of the eccentricity e, the locus of the conic section equation $r = \dfrac{p}{1 + e\cos\theta}$ may take one of four different forms.

4.1 Circular orbit

Let us consider the trajectory of a planet with $e = 0$. By substituting into the polar equation for conic sections, we get

$$r = p$$

which describes the set of points with distance p from the centre, or equivalently a circle with radius p. Let us also consider the semi-major axis of this orbit. Using (4) from above, we have

$$\begin{aligned} \sqrt{1 - \frac{|\mathbf{r} \times \mathbf{v}|^2 (2\mu - v^2 r)}{\mu^2 r}} &= 0 \\ \frac{h^2 (2\mu - v^2 r)}{\mu^2 r} &= 1 \\ 2\mu - v^2 r &= \frac{\mu^2 r}{h^2} \end{aligned}$$

which we can plug into the semi-major axis equation (2) to get

$$\begin{aligned} a &= \frac{\mu r}{2\mu - v^2 r} \\ &= \mu r \times \frac{h^2}{\mu^2 r} \\ &= \frac{h^2}{\mu} \\ &= p \end{aligned}$$

This means that our circular orbit has its radius equal to its semi-latus rectum and semi-major axis, which is consistent with the description of the locus as we consider the circle's foci to be coincident at the centre of the circle. In fact, the circular orbit can be considered to be a special case of the elliptical orbit in which the two foci overlap.

4.2 Parabolic trajectory

It is perhaps more interesting if we take $e = 1$. The polar equation becomes:

$$r = \frac{p}{1 + \cos \nu}$$

It may not be obvious at first what shape this represents, so we may try to gain a better picture by calculating a as well.

$$\begin{aligned} \sqrt{1 - \frac{|\mathbf{r} \times \mathbf{v}|^2 (2\mu - v^2 r)}{\mu^2 r}} &= 1 \\ \frac{h^2 (2\mu - v^2 r)}{\mu^2 r} &= 0 \\ h &= 0 \quad \text{or} \\ 2\mu - v^2 r &= 0 \end{aligned}$$

Let us first consider the case when $h = 0$. This means that $rv \sin \theta = 0$, which gives us three possibilities of degenerate orbits. If $r = 0$, the "orbiting" body is located inside the primary body. If $v = 0$, the "orbiting" body is stationary and will begin to fall towards the centre. Finally, if $\theta = 0$, the velocity and position vectors are parallel, so the secondary body is either falling directly towards or away from the primary body in a straight line instead of orbiting.
Now, let us consider the case when $2\mu - v^2 r = 0$. Using (2) as above, we get

$$a = \frac{\mu r}{0}$$

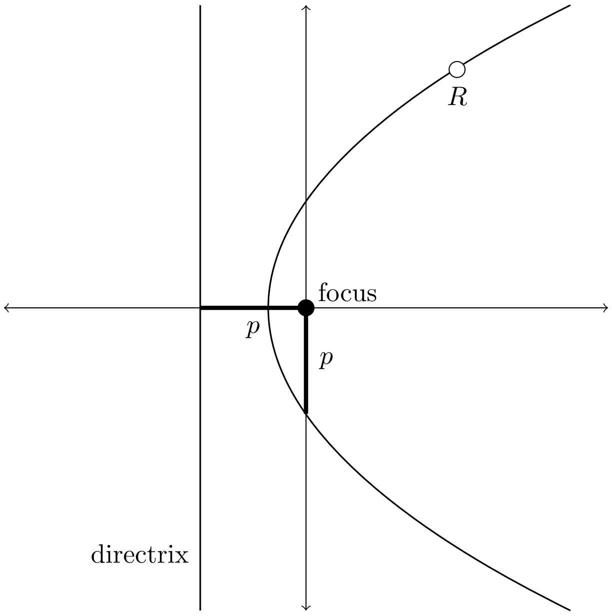
It seems like setting $e = 1$ results in division by zero. Luckily, we can make use of the fact that the locus is a conic section, and it turns out that the shape is a *parabola*, which we say has eccentricity 1 and "infinite" semi-major axis. Now, using the fact that $p = h^2/\mu$, we can plot the trajectory of the body (up to reorientation of the system) according to the polar equation

$$r(\theta) = \frac{h^2}{\mu(1 + \cos \theta)}$$

To understand this better, we can try to express this parabolic trajectory in another form that we are more familiar with, and we introduce the focus-directrix definition of a parabola to do so.

A *parabola* is defined as the set of all points equidistant from a given point (focus) to a given line (directrix). In particular, the perpendicular distance from the focus to the directrix is equal to the semi-latus rectum p of the trajectory, with

$$p = \frac{h^2}{\mu} = \frac{|\mathbf{r} \times \mathbf{v}|^2}{G(m_1 + m_2)}$$



By setting the focus to be the origin, we can work out the equation of the directrix and therefore the equation of the parabola as a function of x. We can rotate the graph so that the parabola faces upwards, giving us the equation of the directrix d to be

$$y = -p$$

Now, we introduce an arbitrary point on the parabola $R = (x, y)$. We use the fact that all points on the parabola are equidistant from the focus and the directrix to write

$$\begin{aligned}
|RF| = |RO| &= |Rd| \\
x^2 + y^2 &= (y + p)^2 \\
x^2 - p^2 &= 2py \\
&\boxed{y = \frac{1}{2p}x^2 - \frac{p}{2}}
\end{aligned}$$

This equation represents the trajectory of the orbiting body in Cartesian coordinates, rotated and translated such that the focus is at the origin and the parabola points upwards.
Let us consider the specific orbital energy associated with a parabolic orbit. As stated above, the specific orbital energy of a Kepler orbit is the negative of the amount of additional energy required for an object to reach escape velocity. Using the result from above that $2\mu - v^2 r = 0$, we have

$$\begin{aligned}
\varepsilon_{parabola} &= \frac{v^2}{2} - \frac{\mu}{r} \\
&= \frac{v^2 r - 2\mu}{2r} \\
&= 0
\end{aligned}$$

It turns out that the parabolic trajectory is the lowest-energy Kepler orbit that allows the secondary body to escape orbit. We can also use this result to derive the formula for escape velocity [1]:

$$\begin{aligned}
\frac{v^2}{2} - \frac{\mu}{r} &= 0 \\
v &= \sqrt{\frac{2\mu}{r}} \approx \sqrt{\frac{GM}{r}}
\end{aligned}$$

4.3 Hyperbolic trajectory

Now, consider the case where $e > 1$. Once again, we calculate a as well using equations (2) and (4) from above.

$$\sqrt{1 - \frac{h^2(2\mu - v^2 r)}{\mu^2 r}} > 1$$

$$\frac{h^2(2\mu - v^2 r)}{\mu^2 r} < 0$$

$$\frac{2\mu - v^2 r}{r} < 0$$

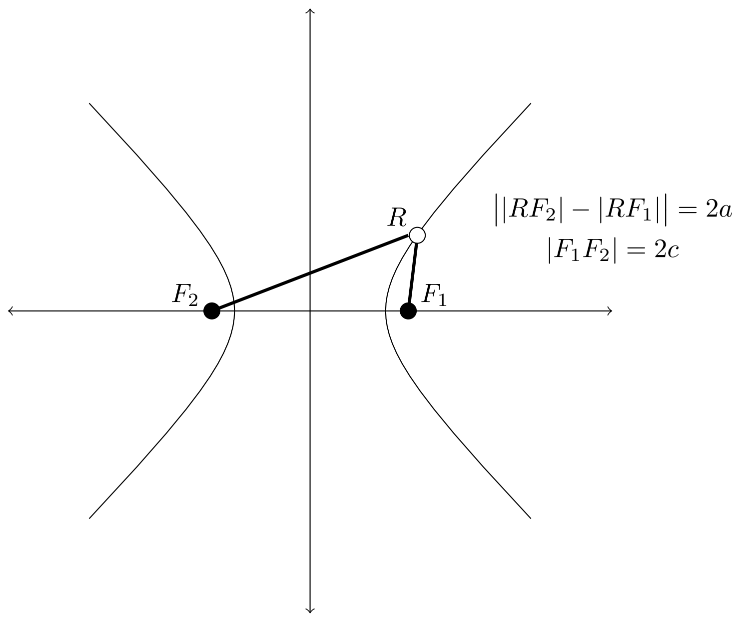
$$\frac{\mu r}{2\mu - v^2 r} < 0$$

$$a < 0$$

We can see that a value of $e > 1$ means that the semi-major axis is negative. This describes the locus of a *hyperbolic* orbit, in which the specific orbital energy is greater than the minimum amount needed to achieve escape velocity.

$$\begin{aligned}
\varepsilon_{hyperbola} &= \frac{v^2}{2} - \frac{\mu}{r} \\
&= \frac{v^2 r - 2\mu}{2r} \\
&= -\frac{1}{2}\left(\frac{2\mu - v^2 r}{r}\right) \\
&> 0
\end{aligned}$$

We may find the equation of such a trajectory in Cartesian coordinates using the fact that for eccentric loci, the focal distance $2c$ satisfies $c = ea$.



Similar to the parabola, we use the definition of the locus of a hyperbola as the set of points such that the magnitude of the difference of the distance from each focus is constant, or

$$\big||RF_2| - |RF_1|\big| = 2a$$

It is clear that the coordinates of the foci are $(0, \pm c)$. Then ,taking $R = (x, y)$, we get

$$\left|\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2}\right| = 2a$$
$$(\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2})^2 = 4a^2$$

Solving for y yields

$$y = \pm \frac{\sqrt{(a^2 - x^2)(a^2 - c^2)}}{a}$$

$$\boxed{y = \pm\sqrt{(a^2 - x^2)(1 - e^2)}}$$

References

[1] Donald D. Mueller, Jerry White, R. R. B. *Fundamentals of Astrodynamics.* Dover Publications, 1971, ch. 1.

[2] French, A. British Physics Olympiad Computational Challenge 2023 Solar System Orbits.

[3] Newton, I. *Philosophiae Naturalis Principia Mathematica.* Edmond Halley, 1687.