

1.

(a)

$$\begin{aligned}\nabla \times \vec{E} &= i\vec{k} \times E \\ \frac{\partial \vec{B}}{\partial t} &= -i\omega \vec{B} \\ \omega \vec{B} &= \vec{k} \times E \\ \vec{B} &= \frac{\vec{k}}{\omega} \times E\end{aligned}$$

when $\vec{k} \parallel \hat{z}$

$$\begin{aligned}\vec{B} &= \frac{k}{\omega} \hat{z} \times E \\ &= \frac{n}{c} \hat{z} \times E\end{aligned}$$

Since the material is non-magnetic, both \vec{E} and \vec{B} are continuous across the surface

$$\begin{aligned}E_{0i} + E_{0r} &= E_{0t} \\ E_{0i} - E_{0r} &= nE_{0t} \\ E_{0t} &= \frac{2}{n+1} E_{0i} \\ E_{0r} &= -\frac{n-1}{n+1} E_{0i}\end{aligned}$$

(b)

Define

$$\begin{aligned}\tilde{a} &= a_0(z) e^{i\phi_a(z)} \\ \tilde{b} &= b_0(z) e^{i\phi_b(z)} \\ a(z, t) b^*(z, t) &= \tilde{a} \tilde{b}^* \\ \langle a(z, t) b^*(z, t) \rangle &= \tilde{a} \tilde{b}^* \\ \langle a^*(z, t) b(z, t) \rangle &= \tilde{a}^* \tilde{b} \\ \langle a(z, t) b(z, t) \rangle &= \tilde{a} \tilde{b} \langle e^{-2i\omega t} \rangle = 0 \\ \langle a^*(z, t) b^*(z, t) \rangle &= \tilde{a}^* \tilde{b}^* \langle e^{-2i\omega t} \rangle = 0 \\ \langle \Re(a(z, t)) \Re(b(z, t)) \rangle &= \frac{1}{4} (\langle a(z, t) b^*(z, t) \rangle + \langle a^*(z, t) b(z, t) \rangle + \langle a(z, t) b(z, t) \rangle + \langle a^*(z, t) b^*(z, t) \rangle) \\ &= \frac{1}{4} (\tilde{a}^* \tilde{b} + \tilde{a} \tilde{b}^*) \\ &= \frac{1}{2} \Re(\tilde{a}^* \tilde{b}) \\ &= \frac{1}{2} \Re(a(z, t) b^*(z, t))\end{aligned}$$

2.

(a)

$$\begin{aligned}
 |n|^2 &= \left| \frac{kc}{\omega} \right|^2 \\
 &= \left| \frac{k^2}{\omega^2} \right| c^2 \\
 &\approx \frac{\mu_0 \sigma}{\omega} c^2 \\
 &= \frac{\sigma}{\varepsilon_0 \omega} \\
 &\gg 1 \\
 \frac{E_{0t}}{E_{0i}} &= \frac{2}{n+1} \\
 &\approx \frac{2}{n} \\
 \frac{E_{0r}}{E_{0i}} &= -\frac{n-1}{n+1} \\
 &\approx -1 + \frac{2}{n}
 \end{aligned}$$

(b)

Incident

$$\begin{aligned}
 \vec{E}_i &= \hat{x} E_{0i} e^{-i\omega t + ik_0 z} \\
 \vec{B}_i &= \hat{y} \frac{E_{0i}}{c} e^{-i\omega t + ik_0 z}
 \end{aligned}$$

Transmitted

$$\begin{aligned}
 \vec{E}_t &= \hat{x} E_{0t} e^{-i\omega t + ikz} \\
 &= \hat{x} E_{0i} \frac{\sqrt{2}\delta\omega}{c} e^{-i\omega t + i(1+i)z/\delta - i\pi/4} \\
 \vec{B}_t &= \hat{y} \frac{nE_{0t}}{c} e^{-i\omega t + ikz} \\
 &= \hat{y} \frac{2E_{0i}}{c} e^{-i\omega t + i(1+i)z/\delta}
 \end{aligned}$$

Reflected

$$\begin{aligned}
 \vec{E}_r &= \hat{x} E_{0r} e^{-i\omega t - ik_0 z} \\
 &= \hat{x} \left(1 - \frac{2\omega\delta}{(1+i)c} \right) E_{0i} e^{-i\omega t - ik_0 z} \\
 \vec{B}_r &= \hat{y} \frac{E_{0r}}{c} e^{-i\omega t - ik_0 z} \\
 &= \hat{y} \left(1 - \frac{2\omega\delta}{(1+i)c} \right) \frac{E_{0i}}{c} e^{-i\omega t - ik_0 z}
 \end{aligned}$$

Current density

$$\begin{aligned}
 \vec{j} &= \sigma \vec{E}_t \\
 &= \hat{x} \sigma E_{0i} \frac{\sqrt{2}\delta\omega}{c} e^{-i\omega t + i(1+i)z/\delta - i\pi/4} \\
 &= \hat{x} \frac{2E_{0i}}{\mu_0\omega\delta^2} \frac{\sqrt{2}\delta\omega}{c} e^{-i\omega t + i(1+i)z/\delta - i\pi/4} \\
 &= \hat{x} \frac{2E_{0i}}{\mu_0 c} \frac{\sqrt{2}}{\delta} e^{-i\omega t + i(1+i)z/\delta - i\pi/4}
 \end{aligned}$$

(c)

Incident

$$\langle S_i \rangle = \bar{z} \frac{E_{0i}^2}{2c\mu_0}$$

Reflected

$$\begin{aligned}
 \langle S_r \rangle &= -\bar{z} \frac{\Re(E_{0r})^2}{2c\mu_0} \\
 &= -\bar{z} \Re\left(1 - \frac{4\omega}{ck}\right) \frac{E_{0i}^2}{2c\mu_0} \\
 &= -\langle S_i \rangle \left(1 - \frac{2\omega\delta}{c}\right)
 \end{aligned}$$

Transmitted

$$\begin{aligned}
 \langle S_t \rangle &= \bar{z} \frac{|nE_{0t}|^2}{2c\mu_0} \\
 &= \bar{z} \Re\left(\frac{4}{n}\right) \frac{E_{0i}^2}{2c\mu_0} \\
 &= \langle S_i \rangle \frac{2\omega\delta}{c}
 \end{aligned}$$

(d)

$$\begin{aligned}
 R &= \left(1 - \frac{2\omega\delta}{c}\right) \\
 T &= \frac{2\omega\delta}{c}
 \end{aligned}$$

(e)

$$\begin{aligned}
 P &= \int_0^\infty \left\langle \frac{j^2}{\sigma} \right\rangle dz \\
 &= \frac{1}{2\sigma} \int_0^\infty e^{-2z/\delta} dz \left(\frac{2E_{0i}}{\mu_0 c} \frac{\sqrt{2}}{\delta} \right)^2 \\
 &= \omega \delta \frac{E_{0i}^2}{\mu_0 c^2} \\
 &= \langle S_i \rangle \frac{2\omega \delta}{c}
 \end{aligned}$$

(f)

$$\begin{aligned}
 \vec{p} &= \int_0^\infty \langle \vec{j} \times \vec{B} \rangle dz \\
 &= \frac{\hat{z}}{2} \int_0^\infty \Re \left(\frac{2E_{0i}}{\mu_0 c} \frac{\sqrt{2}}{\delta} e^{-2z/\delta - i\pi/4} \frac{2E_{0i}}{c} \right) dz \\
 &= \hat{z} \frac{E_{0i}^2}{\mu_0 c^2}
 \end{aligned}$$

(g)

$$\begin{aligned}
 \lim_{\delta \rightarrow 0} \vec{j} &= \vec{x} \frac{2B_{0i}}{\mu_0} \frac{e^{-z/\delta}}{\delta} \cos \left(\omega t - \frac{z}{\delta} \right) \\
 &= \vec{x} \frac{2B_{0i}}{\mu_0} \delta(z) \cos \left(\omega t - \frac{z}{\delta} \right) \\
 &= \vec{x} \frac{2B_{0i}}{\mu_0} \delta(z) \cos \omega t
 \end{aligned}$$

(h)

The field generated near the surface is $-\vec{y}B_{0i} \cos \omega t$, which is just enough to cancel the incident field.

3.

(a)

$$\begin{aligned}
 \vec{a} &= -\frac{e}{m}\vec{E} \\
 \vec{v} &= -\frac{e}{-im\omega}\vec{E} \\
 &= -i\frac{e}{m\omega}\vec{E} \\
 \vec{j} &= i\frac{n_e e^2}{m\omega}\vec{E} \\
 -\vec{k} \times (\vec{k} \times \vec{E}) &= i\omega\mu_0 \left(i\frac{n_e e^2}{m\omega}\vec{E} - i\omega\epsilon_0\vec{E} \right) \\
 k^2 &= \frac{1}{c^2} \left(\omega^2 - \frac{n_e e^2}{m\epsilon_0} \right)
 \end{aligned}$$

(b)

$$n = \frac{kc}{\omega} = \frac{ic}{\omega\delta}$$

Incident

$$\begin{aligned}
 \vec{E}_i &= \hat{x}E_{0i}e^{-i\omega t + ik_0 z} \\
 \vec{B}_i &= \hat{y}\frac{E_{0i}}{c}e^{-i\omega t + ik_0 z}
 \end{aligned}$$

Transmitted

$$\begin{aligned}
 \vec{E}_t &= \hat{x}E_{0t}e^{-i\omega t - z/\delta} \\
 &= \hat{x}\frac{2}{n+1}E_{0i}e^{-i\omega t - z/\delta} \\
 \vec{B}_t &= \hat{y}\frac{nE_{0t}}{c}e^{-i\omega t - z/\delta} \\
 &= \hat{y}\frac{2n}{(n+1)}\frac{E_{0i}}{c}e^{-i\omega t - z/\delta}
 \end{aligned}$$

Reflected

$$\begin{aligned}
 \vec{E}_r &= \hat{x}E_{0r}e^{-i\omega t - ik_0 z} \\
 &= -\hat{x}\frac{n-1}{n+1}E_{0i}e^{-i\omega t - ik_0 z} \\
 \vec{B}_r &= -\hat{y}\frac{E_{0r}}{c}e^{-i\omega t - ik_0 z} \\
 &= \hat{y}\frac{n-1}{n+1}\frac{E_{0i}}{c}e^{-i\omega t - ik_0 z}
 \end{aligned}$$

Current density

$$\begin{aligned}
 \vec{j} &= i\frac{n_e e^2}{m\omega}\vec{E}_t \\
 &= i\hat{x}\frac{n_e e^2}{m\omega}\frac{2}{n+1}E_{0i}e^{-i\omega t - z/\delta}
 \end{aligned}$$

(c)

Incident

$$\langle S_i \rangle = \hat{z} \frac{E_{0i}^2}{2c\mu_0}$$

Reflected

$$\begin{aligned} \langle S_r \rangle &= -\hat{z} \frac{|E_{0r}|^2}{2c\mu_0} \\ &= -\hat{z} \left| \frac{n-1}{n+1} \right|^2 \frac{E_{0i}^2}{2c\mu_0} \\ &= -\langle S_i \rangle \end{aligned}$$

Transmitted

$$\langle S_t \rangle = 0$$

(d)

$$R=1$$

$$T=0$$

(e)

There's no ohmic heating since nothing is heated up.

(f)

$$\begin{aligned} \langle \vec{j} \times \vec{B} \rangle &= \frac{1}{2} \Re \left(i \hat{x} \frac{n_e e^2}{m\omega} \frac{2}{n+1} E_{0i} e^{-i\omega t - z/\delta} \times \hat{y} \frac{-2n}{(-n+1)} \frac{E_{0i}}{c} e^{i\omega t - z/\delta} \right) \\ &= \hat{z} \frac{n_e e^2}{m\omega c} \frac{2|n|}{1-n^2} E_{0i}^2 e^{-2z/\delta} \\ \vec{p} &= \hat{z} \frac{n_e e^2}{mc} \frac{1}{\omega_p^2} E_{0i}^2 \\ &= \hat{z} \frac{\varepsilon_0 E_{0i}^2}{c} \end{aligned}$$

4.