Solutions Assignment #1: Due Friday February 13, 2015 at 2:30 pm

Reading: Jackson Chapter 1 Sections 1.1-1.11 only.

Problems

Problem 1-1: Vector Identities Involving Cross Products

Why? On airplanes it is nice to be able to be able to derive vector identities without reference to a text, and the Levi-Civita symbol is really useful to that end.

In manipulating cross products, it is useful to define ε_{ijk} (the Levi-Civita antisymmetric symbol) to be:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if ijk} = (123,\ 231,\ 312) \\ -1 & \text{if ijk} = (213,\ 321,\ 132) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that $\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}$. With this definition, the *i-th* component of the cross product of two vectors **A** and **B** can be written as

$$(\mathbf{A} \text{ x } \mathbf{B})_{\mathrm{i}} = \varepsilon_{ijk} A_j B_k \tag{2}$$

where we have again used the summation convention that indices *repeated twice are summed over* (that is, $\varepsilon_{ijk} A_{jl} B_{km} = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_{jl} B_{km}$). In the future, we will always assume that this summation convention is implied, unless explicitly stated otherwise. From the definition in (1), it can be shown that

$$\varepsilon_{ijk}\, \varepsilon_{inm} = \delta_{jn}\, \delta_{km} - \delta_{jm}\, \delta_{kn} \tag{3}$$

where there is an implied sum over the *i* index in (3), but the indices *j*, *k*, *n*, and *m* are free.

(a) Using (2) and (3), show that for any vectors **A**, **B**, and **C**,

$$\mathbf{A} \text{ x } (\mathbf{B} \text{ x } \mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$$

$$\left(\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)\right)_i = \varepsilon_{ijk} A_j \left(\mathbf{B}\times\mathbf{C}\right)_k \quad \text{but} \quad \left(\mathbf{B}\times\mathbf{C}\right)_k = \varepsilon_{kmn} B_m C_n$$

So

$$\left(\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)\right)_i = \varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n = \varepsilon_{ijk} \varepsilon_{kmn} A_j B_m C_n$$

But $\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}$, so $\varepsilon_{ijk}\varepsilon_{kmn} = \varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$, and thus we have

$$\left(\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)\right)_i=\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)A_jB_mC_n=\delta_{im}\delta_{jn}A_jB_mC_n-\delta_{in}\delta_{jm}A_jB_mC_n$$

$$\left(\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)\right)_i=A_nB_iC_n-A_jB_jC_i=(\mathbf{A}\cdot\mathbf{C})B_i-(\mathbf{A}\cdot\mathbf{B})C_i$$

This is what we want, that is,

$$\mathbf{A\ x\ (B\ x\ C)}=\mathbf{B\ (A\cdot C)}-\mathbf{C\ (A\cdot B)}$$

(b) Using (2) and (3), show that for any vector **A**,

$$\nabla\times\nabla\times\mathbf{A}=\frac{1}{2}\nabla A^2-\left(\mathbf{A}\cdot\nabla\right)\mathbf{A}$$

$$\left(\nabla\times\mathbf{A}\right)_i=\varepsilon_{ijk}\partial_jA_k\qquad\text{where }\partial_j=\frac{\partial}{\partial x_j}$$

So

$$\left[\mathbf{A}\times\left(\nabla\times\mathbf{A}\right)\right]_i=\varepsilon_{ijk}A_j\left(\nabla\times\mathbf{A}\right)_k=\varepsilon_{ijk}A_j\left(\varepsilon_{kmn}\partial_mA_n\right)=\varepsilon_{ijk}\varepsilon_{kmn}A_j\partial_mA_n$$

$$\left[\mathbf{A}\times\left(\nabla\times\mathbf{A}\right)\right]_i=\varepsilon_{kij}\varepsilon_{kmn}A_j\partial_mA_n=\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)A_j\partial_mA_n$$

$$\left[\mathbf{A}\times\left(\nabla\times\mathbf{A}\right)\right]_i=\left(A_n\partial_iA_n-A_m\partial_mA_i\right)=\frac{1}{2}\partial_i\left(A_nA_n\right)-\left(A_m\partial_m\right)A_i$$

This is as desired, that is

$$\mathbf{A}\times\left(\nabla\times\mathbf{A}\right)=\frac{1}{2}\nabla A^2-\left(\mathbf{A}\cdot\nabla\right)\mathbf{A}$$

(c) Show that for any vectors **A** and **B**,

$$\nabla\times\left(\mathbf{A}\times\mathbf{B}\right)=\left(\mathbf{B}\cdot\nabla\right)\mathbf{A}-\left(\mathbf{A}\cdot\nabla\right)\mathbf{B}+\mathbf{A}\left(\nabla\cdot\mathbf{B}\right)-\mathbf{B}\left(\nabla\cdot\mathbf{A}\right)$$

Using $\left(\mathbf{A}\times\mathbf{B}\right)_i=\varepsilon_{ijk}A_jB_k$, for any vectors **A** and **B**,

$$\left[\nabla\times\left(\mathbf{A}\times\mathbf{B}\right)\right]_i=\varepsilon_{ijk}\partial_j\left(\mathbf{A}\times\mathbf{B}\right)_k=\varepsilon_{ijk}\partial_j\varepsilon_{kmn}A_mB_n=\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\partial_j\left(A_mB_n\right)$$

$$\left[\nabla\times\left(\mathbf{A}\times\mathbf{B}\right)\right]_i=\partial_n\left(A_iB_n\right)-\partial_m\left(A_mB_i\right)=A_i\left(\partial_nB_n\right)+\left(B_n\partial_n\right)A_i-B_i\left(\partial_mA_m\right)-\left(A_m\partial_m\right)B_i$$

or

$$\nabla\times\left(\mathbf{A}\times\mathbf{B}\right)=\mathbf{A}\left(\nabla\cdot\mathbf{B}\right)+\left(\mathbf{B}\cdot\nabla\right)\mathbf{A}-\mathbf{B}\left(\nabla\cdot\mathbf{A}\right)-\left(\mathbf{A}\cdot\nabla\right)\mathbf{B}$$

Problem 1-2: The Dirac Delta Function

Why? Delta functions arise in unexpected places, and have many useful forms.

(a) The Heaviside step function is defined to be $\Theta(t)=1$ if $t>0$ and zero if $t<0$. Show that $d\Theta/dt=\delta\left(t\right)$ using the definitions Jackson p 26.

Solution: Clearly $d\Theta/dt = 0$ for $t \neq 0$ and $\int_{-\varepsilon}^{\varepsilon} \left(d\Theta/dt\right)dt = \Theta(\varepsilon) - \Theta(-\varepsilon) = 1$.

(b) The sign function is defined to be $\operatorname{sgn}(t) = 1$ if $t > 0$ and -1 if $t > 0$. What is $d\operatorname{sgn}(t)/dt$?

Solution: $\operatorname{sgn}(t) = 2\Theta(t) - 1$ so $d\operatorname{sgn}(t)/dt = 2\delta(t)$

Using delta functions and the Heaviside step function in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(c) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

Solution:

$$\rho(\mathbf{x}) = \frac{Q}{4\pi R^2}\delta(r-R) \text{ since} \int \rho(\mathbf{x})d^3x = \frac{Q}{4\pi R^2}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} \delta(r-R)r^2 dr = Q$$

(d) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b whose axis lies along the z-axis.

Solution: $$\rho(\mathbf{x}) = \frac{\lambda}{2\pi b}\delta(\rho-b) \text{ since} \int \rho(\mathbf{x})d^3x = \frac{\lambda}{2\pi b}\int_0^{L} dz \int_0^{2\pi} d\phi \int_0^{\infty} \delta(\rho-b)\rho d\rho = \lambda L$$

(e) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius b.

Solution:

$$\rho(\mathbf{x}) = \frac{Q}{\pi b^2}\delta(z)\Theta(b-\rho) \text{ since} \int \rho(\mathbf{x})d^3x = \frac{Q}{\pi b^2}\int_{-\infty}^{\infty} dz\delta(z) \int_0^{2\pi} d\phi \int_0^{\infty} \Theta(b-\rho)\rho d\rho = Q$$

Problem 1-3: The Dirac Delta Function Again

Why? One of the most fundamental identities in this course is $\nabla^2(1/r) = -4\pi\,\delta^3(\mathbf{x})$.

To get a better feel for this, let's look at a different function which approaches -(*1/4π r*) in some limit, but which is well behaved everywhere. The function is

$f_a(r) = -\frac{1}{4\pi}\frac{1}{\sqrt{r^2+a^2}}$. For a non-zero, $f_a(r)$ is well-behaved everywhere, and

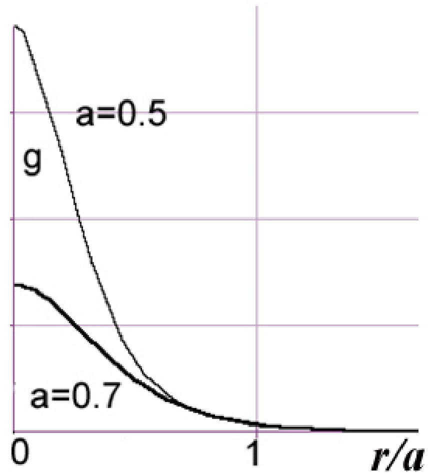
$\lim_{a\to 0} f_a(r) = -\frac{1}{4\pi r}$.

(a) Calculate $g_a(r) = \nabla^2 f_a(r)$ and show that it is also well behaved for all r. Sketch $g_a(r)$ for two values of a as a function of *r/a*.

$$g_a(r) = \nabla^2 f_a(r) = -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+a^2}}\right) = \frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{\left(r^2+a^2\right)^{3/2}}\right)$$

$$g_a(r) = \frac{1}{4\pi}\frac{1}{r^2}\left\{\left(\frac{3r^2}{\left(r^2+a^2\right)^{3/2}}\right) - \frac{3r^4}{\left(r^2+a^2\right)^{5/2}}\right\} = \frac{1}{4\pi}\frac{3}{\left(r^2+a^2\right)^{5/2}}\left\{\left(r^2+a^2\right)-r^2\right\}$$

$$g_a(r) = \frac{3}{4\pi}\frac{a^2}{\left(r^2+a^2\right)^{5/2}} = \frac{3}{4\pi a^3}\frac{1}{\left(1+\left(r/a\right)^2\right)^{5/2}}$$



(b) Show that $\int\limits_{all\ space} g_a(r)\, d^3x = 1$.

$$\int\limits_{all\ space} g_a(r)\, d^3x = 4\pi\int_0^\infty r^2\frac{3}{4\pi}\frac{a^2}{\left(r^2+a^2\right)^{5/2}}dr = \int_0^\infty \frac{3a^2r^2}{\left(r^2+a^2\right)^{5/2}}dr = 3\int_0^\infty \frac{\eta^2 d\eta}{\left(1+\eta^2\right)^{5/2}}$$

Let $u=\eta$ and $dv = \frac{3\eta d\eta}{\left(1+\eta^2\right)^{5/2}}$, so that $du = d\eta$ and $v = \frac{1}{\left(1+\eta^2\right)^{3/2}}$. Then by parts

$$\int\limits_{all\ space} g_a(r)\, d^3x = \int_0^\infty \frac{d\eta}{\left(1+\eta^2\right)^{3/2}} = \int_0^{\pi/2}\frac{d\psi}{\left(1+\tan^2\psi\right)^{3/2}}\frac{1}{\cos^2\psi}$$

where to get the last form, we have put $\eta = \tan\psi$ and therefore $d\eta = \frac{d\psi}{\cos^2\psi}$.

Thus $\int\limits_{all\ space} g_a(r)\, d^3x = \int_0^{\pi/2}\frac{\cos^3\psi d\psi}{\cos^2\psi} = \int_0^{\pi/2}\cos\psi d\theta = \sin\psi\Big|_0^{\pi/2} = 1$ QED

(c) Show that $\lim_{a\to 0} g_a(r) = 0 \;\; if \;\; r \neq 0$.

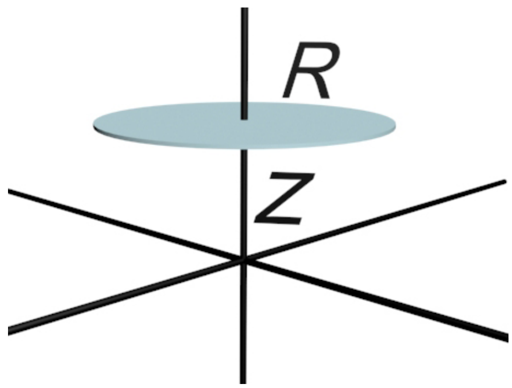
Thus in the limit that *a* goes to zero, our well-behaved function $g_a(r)$ exhibits the properties we expect of a three-dimensional delta function.

It is obvious that $\lim_{a\to 0} g_a(r) = \lim_{a\to 0} \frac{3}{4\pi}\frac{a^2}{\left(r^2+a^2\right)^{5/2}} = 0 \;\; if \;\; r \neq 0$

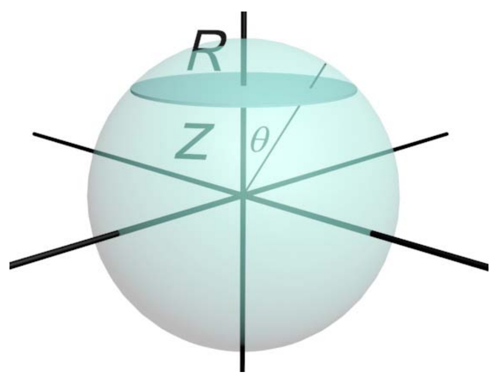
Problem 1-4: The solid angle subtended by a disk

Why? Many students are not familiar with the mathematical definition of solid angle given on Jackson p 33 just after eq 1.25, and it is an important concept.

A disk of radius *R* is parallel to the *xy* plane and its center is located a distance *z* up the *z*-axis (see figure). The normal to the disk is in the positive *z* direction. An observer is located at the origin. What is the solid angle subtended by this disk for that observer? Be careful of the sign of the solid angle. Make sure your answer is appropriate for both $z > 0$ and $z < 0$. Check your answer in the limit that $|z| << R$ for both $z>0$ and $z<0$, and explain why your answer makes sense in both these cases.



Solution: The solid angle subtended by the disk is 4π times the fraction subtended by the disk of the solid sphere of radius $\sqrt{R^2+z^2}$ centered at the origin (see figure). If $z > 0$ then the subtended solid angle is positive because the angle between the normal to the disk $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and the direction from the observer to the disk is always less than 90 degrees. Let $\theta_o = \arctan\left(R/z\right)$. Then



$$\Omega = \int_0^{2\pi} d\phi \int_0^{\theta_o} \sin\theta d\theta = 2\pi\left(1-\cos\theta_o\right) = 2\pi\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)$$

If $z < 0$, the solid angle is negative with the same absolute value (because the angle between the normal to the disk and the direction from the observer to the disk is always greater than 90 degrees). Thus in general our expression for the solid angle is

$$\Omega = 2\pi\left(1-\frac{|z|}{\sqrt{R^2+z^2}}\right)\text{sgn}(z) \quad \text{where} \quad \text{sgn}(z) = z/|z|$$

This can be written as $\Omega = 2\pi\left(\text{sgn}(z)-\frac{z}{\sqrt{R^2+z^2}}\right)$.

In the limit that $|z| << R$ $\Omega = 2\pi\,\mathrm{sgn}(z)$, which makes sense because for either sign of z the disk takes up "half" the sky, or 2π steradians, with the sign of the solid angle depending on whether the disk is just below the origin or just above it.

Problem 1-5: The potential and field on the axis of a disk with a dipole layer

Why? Electric dipole layers arise naturally in many circumstances, usually when mobile charges exist near boundary layers or surfaces. This problem looks at some of the mathematical properties of dipole layers.

A disk of radius a lies in the xy plane with its center at the origin. The disk has a constant dipole layer $D_o\hat{\mathbf{z}}$.

(a) Using J p 34, eq 1.26, and the results of the above problem, calculate the electric potential due to this dipole layer for an observer located on the z-axis a distance z from the origin. Be careful of the sign of the solid angle subtended by the disk for the observer, and make sure your answer is appropriate for both $z > 0$ and $z < 0$. Also, show that Jackson p 34 eq 1.27 holds for the change in potential across the disc?

Solution: Using eq 1.26, the results above, and a little thought about the geometry, we see that on the z axis,

$$\Phi(z) = \frac{D_o}{2\varepsilon_o}\left(\mathrm{sgn}(z) - \frac{z}{\sqrt{R^2+z^2}}\right)$$

This potential has the jump of D_o / ε_o we expect from Jackson eq 1.27.

(b) What is the electric field on the z-axis? Be careful about the derivative of the potential around $z = 0$, and of the sign of the electric field for $z < 0$ and $z > 0$.

Solution: $E_z = -d\Phi / dz = -\dfrac{D_o}{\varepsilon_o}\delta(z) + \dfrac{D_o R^2}{2\varepsilon_o\left(R^2+z^2\right)^{3/2}}\mathrm{sgn}(z)$

Problem 1-6: Shielding

Why? The equations for the electrostatic potential are second order, which means that if we can introduce arbitrary jumps in the potential and its derivatives we can do all sorts of interesting things, as shown by this problem.

(a) Consider the potential $\Phi_o(\mathbf{x})$ due to a point dipole in free space at the origin, with the dipole moment along the z-axis, that is $\mathbf{p} = p_o\hat{\mathbf{z}}$. Take an imaginary sphere of radius R centered at the origin. The unit normal to the spherical surface points from inside to outside, that is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. What is the value of the above potential $\Phi_o(\mathbf{x})$ and its normal derivative on this imaginary surface?

Solution: $\left.\Phi_o(\mathbf{x})\right|_{r=R} = \frac{p_o}{4\pi\varepsilon_o}\frac{\cos\theta}{R^2}$ and $\left.\frac{\partial\Phi_o(\mathbf{x})}{\partial n}\right|_{r=R} = \frac{\partial}{\partial r}\frac{p_o}{4\pi\varepsilon_o}\frac{\cos\theta}{r^2} = -\frac{p_o}{2\pi\varepsilon_o}\frac{\cos\theta}{R^3}$.

(b) Now we consider a totally different problem. Again we have the same point dipole at the origin, but now we also have a surface charge $\sigma(\theta)$ and a dipole layer $\mathbf{D}(\theta) = D(\theta)\hat{\mathbf{n}}$ on the surface of a sphere of radius *R*. Can we choose the surface charge and dipole layer charge so that the electrostatic potential for this new problem is the same as the potential $\Phi_o(\mathbf{x})$ in (a) for $r < R$, but identically zero for $r > R$? If so, how? To answer this question, consider carefully the meaning of the equation in Jackson on p 36 between eq 1.35 and 1.36.

Solution: We take $\sigma(\theta) = \varepsilon_o \left.\frac{\partial\Phi_o(\mathbf{x})}{\partial n}\right|_{r=R} = -\frac{p_o}{2\pi}\frac{\cos\theta}{R^3}$ and

$D(\theta) = -\varepsilon_o \left.\Phi_o(\mathbf{x})\right|_{r=R} = \frac{p_o}{4\pi}\frac{\cos\theta}{R^2}\hat{\mathbf{n}}$. We justify this choice by looking the Jackson eq on page 36 (divided by 4π) for the potential $\Phi_o(\mathbf{x})$

$$\int_V \left[-\Phi_o(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}') + \frac{\rho(\mathbf{x}')}{4\pi\varepsilon_o|\mathbf{x}-\mathbf{x}'|}\right]d^3x' = \frac{1}{4\pi}\oint_S \left[\Phi_o(\mathbf{x}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial}{\partial n'}\Phi_o(\mathbf{x}')\right]dS'$$
$$= -\frac{1}{4\pi\varepsilon_o}\oint_S \left[D(\mathbf{x}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]dS'$$

If we look at the above equation for $\mathbf{x}$ inside the sphere, we have

$$\Phi_o(\mathbf{x}) = \int_V \left[\frac{\rho(\mathbf{x}')}{4\pi\varepsilon_o|\mathbf{x}-\mathbf{x}'|}\right]d^3x' + \frac{1}{4\pi\varepsilon_o}\oint_S \left[D(\mathbf{x}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]dS'$$

If we look at the same equation for $\mathbf{x}$ outside the sphere, we have

$$0 = \int_V \left[\frac{\rho(\mathbf{x}')}{4\pi\varepsilon_o|\mathbf{x}-\mathbf{x}'|}\right]d^3x' + \frac{1}{4\pi\varepsilon_o}\oint_S \left[D(\mathbf{x}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]dS'$$

But in either case we have $\Phi_o(\mathbf{x}) = \int_V \left[\frac{\rho(\mathbf{x}')}{4\pi\varepsilon_o|\mathbf{x}-\mathbf{x}'|}\right]d^3x'$. Therefore we must have

$$\frac{1}{4\pi\varepsilon_o}\oint_S \left[D(\mathbf{x}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]dS' = \begin{cases} 0 \quad \text{for } \mathbf{x} \text{ inside} \\ -\Phi_o(\mathbf{x}) \quad \text{for } \mathbf{x} \text{ outside} \end{cases}$$

The potential due to our charge and dipole layer on the sphere of radius *R* is given by the left hand side of the above equation. This charge and dipole layer thus does exactly what we want, it gives nothing inside the sphere and $-\Phi_o(\mathbf{x})$ outside. Therefore when we consider the full potential, due to the point dipole at the origin and this surface and charge layer on the sphere, we have totally cancelled $\Phi_o(\mathbf{x})$ outside the sphere and left it unchanged inside. QED

Problem 1-7: The Helmholtz Theorem

Why? A vector function is determined by its curl and divergence, up to a constant. We show how to construct that function given its divergence and curl.

Assume that we have an unknown vector field $\mathbf{F}(\mathbf{r},t)$ such that its divergence is $s(\mathbf{r},t)$, a scalar function of space and time, and its curl is $\mathbf{c}(\mathbf{r},t)$, a vector function of space and time, both defined for all $(\mathbf{r},t)$. Construct the vector function $\mathbf{F}(\mathbf{r},t)$ as follows:

$$\mathbf{F}(\mathbf{r},t) = -\nabla \frac{1}{4\pi} \int_{all\,space} \frac{s(\mathbf{r}',t)\, d^3x'}{|\mathbf{r}-\mathbf{r}'|} + \nabla \times \frac{1}{4\pi} \int_{all\,space} \frac{\mathbf{c}(\mathbf{r}',t)\, d^3x'}{|\mathbf{r}-\mathbf{r}'|}$$

We showed in lecture that the Helmholtz construction above is such that $\nabla \cdot \mathbf{F}(\mathbf{r},t) = s(\mathbf{r},t)$. Show that $\nabla \times \mathbf{F}(\mathbf{r},t) = \mathbf{c}(\mathbf{r},t)$, assuming that $\mathbf{c}(\mathbf{r},t)$ falls off sufficiently fast as *r* goes to infinity. How fast does $\mathbf{c}(\mathbf{r},t)$ have to fall off? Note that you can assume that the divergence of $\mathbf{c}(\mathbf{r},t)$ is zero since it is the curl of an unknown vector field.

[Hints: In the course of proving this you will find useful the four identities below

$$\nabla \times [\nabla \times \mathbf{H}] = \nabla[\nabla \cdot \mathbf{H}] - \nabla^2 \mathbf{H} \qquad \nabla \cdot \left(\frac{\mathbf{c}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \right) = \mathbf{c}(\mathbf{r}',t) \cdot \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \qquad \nabla' \cdot \left[\frac{\mathbf{c}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \right] = \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \cdot \mathbf{c}(\mathbf{r}',t) + \mathbf{c}(\mathbf{r}',t) \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

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Solution: The curl of the gradient of any scalar is identically zero, so

$$\nabla \times \mathbf{F}(\mathbf{r},t) = \nabla \times \left[\nabla \times \frac{1}{4\pi} \int_{all\,space} \frac{\mathbf{c}(\mathbf{r}',t)\, d^3x'}{|\mathbf{r}-\mathbf{r}'|} \right]$$

We use the vector identity $\nabla \times [\nabla \times \mathbf{H}] = \nabla[\nabla \cdot \mathbf{H}] - \nabla^2 \mathbf{H}$, and note that the $-\nabla^2 \mathbf{H}$ will give us $\mathbf{c}(\mathbf{r},t)$, since

$$-\nabla^2 \frac{1}{4\pi} \int_{all\,space} \frac{\mathbf{c}(\mathbf{r}',t)\, d^3x'}{|\mathbf{r}-\mathbf{r}'|} = \int_{all\,space} \mathbf{c}(\mathbf{r}',t)\, d^3x' \left[-\nabla^2 \frac{1}{4\pi} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right]$$
$$= \int_{all\,space} \mathbf{c}(\mathbf{r}',t)\, d^3x' \delta^3(\mathbf{r}-\mathbf{r}') = \mathbf{c}(\mathbf{r},t)$$

Thus we need only prove that

$$\nabla\left[\nabla\cdot\mathbf{H}\right] = \nabla\left[\nabla\cdot\left(\frac{1}{4\pi}\int_{all\,space}\frac{\mathbf{c}(\mathbf{r}',t)\,d^3x'}{|\mathbf{r}-\mathbf{r}'|}\right)\right] = 0$$

To do this, first write

$$\nabla\cdot\left(\frac{1}{4\pi}\int_{all\,space}\frac{\mathbf{c}(\mathbf{r}',t)\,d^3x'}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{1}{4\pi}\int_{all\,space}\left(\mathbf{c}(\mathbf{r}',t)\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3x'$$

Now we use the fact that $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ to write the above as

$$\nabla\cdot\left(\frac{1}{4\pi}\int_{all\,space}\frac{\mathbf{c}(\mathbf{r}',t)\,d^3x'}{|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{1}{4\pi}\int_{all\,space}\left(\mathbf{c}(\mathbf{r}',t)\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3x'$$

But we have the vector identity that

$$\nabla'\cdot\left[\frac{\mathbf{c}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{c}(\mathbf{r}',t) + \mathbf{c}(\mathbf{r}',t)\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

But $\mathbf{c}(\mathbf{r}',t)$ is the curl of a vector and therefore $\nabla'\cdot\mathbf{c}(\mathbf{r}',t) = 0$. Thus

$$\nabla\cdot\left(\frac{1}{4\pi}\int_{all\,space}\frac{\mathbf{c}(\mathbf{r}',t)\,d^3x'}{|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{1}{4\pi}\int_{all\,space}\nabla'\cdot\left(\frac{\mathbf{c}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)d^3x' = -\frac{1}{4\pi}\int_{surface}\left(\frac{\mathbf{c}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\hat{\mathbf{n}}\,da'$$

where we have used the divergence theorem to get the form on the right hand side of the above equation. . As long as $\mathbf{c}(\mathbf{r}',t)$ goes to zero faster than $1/r'$ as r' goes to ∞, the surface integral will vanish, since da' only goes to infinity as the square of r' as r' goes to infinity. QED.