

1.

(a)

$$\begin{aligned}
 \vec{E}_{dipole} &= -\nabla \left(\frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} \right) \\
 &= -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\vec{r} \cdot \vec{p}}{r^3} \right) \\
 &= -\frac{\vec{r} \cdot \vec{p}}{4\pi\epsilon_0} \nabla \left(\frac{1}{r^3} \right) - \frac{1}{4\pi\epsilon_0 r^3} \nabla(\vec{r} \cdot \vec{p}) \\
 &= \frac{\vec{r} \cdot \vec{p}}{4\pi\epsilon_0} \frac{3\hat{n}}{r^4} - \frac{\vec{p}}{4\pi\epsilon_0 r^3} \\
 &= \frac{3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}}{4\pi\epsilon_0 r^3}
 \end{aligned}$$

(b)

$$\begin{aligned}
 &\int \vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) d^3x \\
 &= \frac{1}{2} \int \vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) d^3x + \frac{1}{2} \int \left(\hat{n} \times (\vec{J}(\vec{r}') \times \vec{r}') - \vec{r}' (\hat{n} \cdot \vec{J}(\vec{r}')) \right) d^3x \\
 &= \hat{n} \times \frac{1}{2} \int \vec{J}(\vec{r}') \times \vec{r}' d^3x + \frac{1}{2} \int \left(\vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) - \vec{r}' (\hat{n} \cdot \vec{J}(\vec{r}')) \right) d^3x
 \end{aligned}$$

For arbitrary vector \vec{l}

$$\begin{aligned}
 &\nabla' \cdot \left((\vec{l} \cdot \vec{r}')(\hat{n} \cdot \vec{r}')\vec{J}(r') \right) \\
 &= (\hat{n} \cdot \vec{r}')\vec{J}(r') \cdot \nabla'(\vec{l} \cdot \vec{r}') + (\vec{l} \cdot \vec{r}')\vec{J}(r') \cdot \nabla'(\hat{n} \cdot \vec{r}') + (\vec{l} \cdot \vec{r}')(\hat{n} \cdot \vec{r}')\nabla' \cdot \vec{J}(r') \\
 &= (\hat{n} \cdot \vec{r}')\vec{J}(r') \cdot \vec{l} + (\vec{l} \cdot \vec{r}')\vec{J}(r') \cdot \hat{n} \\
 &= \vec{l} \cdot \left((\hat{n} \cdot \vec{r}')\vec{J}(r') + \vec{r}'\vec{J}(r') \cdot \hat{n} \right)
 \end{aligned}$$

Integrate both sides

$$\begin{aligned}
 0 &= \vec{l} \cdot \int d^3x' \left((\hat{n} \cdot \vec{r}')\vec{J}(r') + \vec{r}'\vec{J}(r') \cdot \hat{n} \right) \\
 0 &= \int d^3x' \left((\hat{n} \cdot \vec{r}')\vec{J}(r') + \vec{r}'\vec{J}(r') \cdot \hat{n} \right) \\
 &\quad \int \vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) d^3x \\
 &= \vec{m} \times \vec{n}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \frac{4\pi}{\mu_0} \vec{B}_{dipole} &= \nabla \times \frac{\vec{m} \times \vec{r}}{r^3} \\
 &= \vec{m} \left(\nabla \cdot \frac{\vec{r}}{r^3} \right) - (\vec{m} \cdot \nabla) \frac{\vec{r}}{r^3} \\
 &= -\vec{r} \left(\vec{m} \cdot \nabla \frac{1}{r^3} \right) - \frac{1}{r^3} (\vec{m} \cdot \nabla) \vec{r} \\
 &= \vec{r} \frac{3\vec{m} \cdot \hat{n}}{r^4} - \frac{\vec{m}}{r^3} \\
 &= \frac{3\hat{n}(\vec{m} \cdot \hat{n}) - \vec{m}}{r^3}
 \end{aligned}$$

2.

(a)

$$\begin{aligned}
 E &= \int_R^\infty dr \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \frac{p^2}{32\pi^2 \epsilon_0 r^4} (4 \cos^2 \theta + \sin^2 \theta) \\
 &= \frac{p^2}{16\pi \epsilon_0} \int_R^\infty \frac{dr}{r^4} \int_0^\pi d\theta \sin \theta (4 \cos^2 \theta + \sin^2 \theta) \\
 &= \frac{p^2}{12\pi \epsilon_0 R^3}
 \end{aligned}$$

(b)

Derivatives of the dipole moment

$$\begin{aligned}
 \dot{\vec{p}} &= \dot{p} \hat{z} \\
 \ddot{\vec{p}} &= \ddot{p} \hat{z}
 \end{aligned}$$

Magnetic field

$$\begin{aligned}
 \vec{B} &= \frac{\hat{z} \times \hat{n}}{4\pi \epsilon_0} \left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr} \right) \\
 &= \frac{\sin \theta \hat{\phi}}{4\pi \epsilon_0} \left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr} \right)
 \end{aligned}$$

Electric field

$$\begin{aligned}
 \vec{E} &= \frac{3\hat{n}(\vec{p} \cdot \hat{n}) - \vec{p}}{4\pi \epsilon_0 r^3} + \frac{3\hat{n}(\dot{\vec{p}} \cdot \hat{n}) - \dot{\vec{p}}}{4\pi \epsilon_0 cr^2} + \frac{(\ddot{\vec{p}} \times \hat{n}) \times \hat{n}}{4\pi \epsilon_0 c^2 r} \\
 &= \frac{3p \cos \theta \hat{n} - p \hat{z}}{4\pi \epsilon_0 r^3} + \frac{3\dot{p} \cos \theta \hat{n} - \dot{p} \hat{z}}{4\pi \epsilon_0 cr^2} + \frac{\ddot{p} \sin \theta \hat{\phi} \times \hat{n}}{4\pi \epsilon_0 c^2 r} \\
 &= \frac{3p \cos \theta \hat{n} - p \hat{z}}{4\pi \epsilon_0 r^3} + \frac{3\dot{p} \cos \theta \hat{n} - \dot{p} \hat{z}}{4\pi \epsilon_0 cr^2} + \frac{\ddot{p} \sin \theta \hat{\theta}}{4\pi \epsilon_0 c^2 r}
 \end{aligned}$$

Since $\hat{z} = \cos \theta \hat{n} - \sin \theta \hat{\theta}$

$$\begin{aligned}\vec{E} &= \frac{2p \cos \theta \hat{n} + p \sin \theta \hat{\theta}}{4\pi\epsilon_0 r^3} + \frac{2\dot{p} \cos \theta \hat{n} + \dot{p} \sin \theta \hat{\theta}}{4\pi\epsilon_0 cr^2} + \frac{\ddot{p} \sin \theta \hat{\theta}}{4\pi\epsilon_0 c^2 r} \\ &= \frac{2 \cos \theta \hat{n}}{4\pi\epsilon_0} \left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} \right) + \frac{\sin \theta \hat{\theta}}{4\pi\epsilon_0} \left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r} \right)\end{aligned}$$

Energy flux

$$\begin{aligned}\Phi_E &= \frac{1}{\mu_0 c^2} \int dt \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta r^2 \frac{\sin \theta}{4\pi\epsilon_0} \left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr} \right) \frac{\sin \theta}{4\pi\epsilon_0} \left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r} \right) \\ &= \frac{1}{6\pi\epsilon_0} \int dt \left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c} \right) \left(\frac{p}{r^2} + \frac{\dot{p}}{cr} + \frac{\ddot{p}}{c^2} \right) \\ &= \frac{1}{6\pi\epsilon_0} \left(\frac{1}{2r} \left(\frac{p}{r} + \frac{\dot{p}}{c} \right)^2 \right)_{t_0}^{t_1} + \frac{1}{6\pi\mu_0\epsilon_0^2} \int dt \left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c} \right) \left(\frac{\ddot{p}}{c^2} \right) \\ &= \frac{p_2^2 - p_1^2}{12\pi\epsilon_0 r^3} + \int \frac{\ddot{p}^2}{6\pi\mu_0\epsilon_0^2 c^3} dt\end{aligned}$$

The first term corresponds to change in the energy stored in the field.

3.

(a)

$$r = \frac{mV_0}{qB_0}$$

(b)

$$T = \frac{\pi m}{qB}$$

(c)

$$\begin{aligned}\frac{dW}{dt} &= \frac{q^4 V_0^2 B^2}{6\pi\epsilon_0 m^2 c^3} \\ &= \frac{q^2 V_0^4}{6\pi\epsilon_0 c^3 R^2}\end{aligned}$$

(d)

$$\begin{aligned}W &= \frac{q^2 V_0^3}{6\epsilon_0 c^3 R} \\ &= \frac{2\pi m V_0^3}{3cR}\end{aligned}$$

(e)

$$\frac{W}{E_k} = \frac{2\pi V_0 R_{classical}}{3cR}$$

When R is large.

(f)

4.

(a)

$$\begin{aligned} p &= Q_0 d \sin \omega t \\ \left| \frac{dW}{dt} \right| &= \frac{Q_0^2 d^2 \omega^4}{6\pi \epsilon_0 c^3} |\sin^2 \omega t| \\ &= \frac{Q_0^2 d^2 \omega^4}{12\pi \epsilon_0 c^3} \end{aligned}$$

(b)

$$\begin{aligned} E_{rad} &= \frac{Q_0^2 d^2 \omega^4}{12\pi \epsilon_0 c^3} \frac{2\pi}{\omega} \\ &= \frac{Q_0^2 d^2 \omega^3}{6\epsilon_0 c^3} \\ \frac{4CE_{rad}}{Q_0^2} &= \frac{2d^2 C \omega^3}{3\epsilon_0 c^3} \\ &= \frac{2dAk^3}{3} \end{aligned}$$

Therefore if dk and Ak^2 are all small (where k is the wave vector) the radiation is small.

(c)

$$\begin{aligned} R_{rad} &= \frac{Q_0^2 d^2 \omega^4}{12\pi \epsilon_0 c^3} \frac{2}{\omega^2 Q_0^2} \\ &= \frac{d^2 \omega^2}{6\pi \epsilon_0 c^3} \end{aligned}$$

(d)

$$\begin{aligned} R_{rad} &= \frac{d^2}{6\pi\epsilon_0 c^3 LC} \\ &= \frac{hd^3}{6\pi\epsilon_0 c^3 \epsilon_0 A_c \mu_0 N^2 A_L} \\ &= \mu_0 c \frac{hd^3}{6\pi A_c N^2 A_L} \end{aligned}$$

5.

(a)

$$\begin{aligned} \vec{B}_\perp &= \frac{\mu_0}{4\pi} \left(\frac{1}{cr^2} \left(3\hat{n}(\dot{\vec{m}} \cdot \hat{n}) - \dot{\vec{m}} \right) + \frac{1}{rc^2} (\ddot{\vec{m}} \times \hat{n}) \times \hat{n} \right) \\ &= \frac{\mu_0 m_0 \omega_0}{4\pi rc} \left(\frac{1}{r} (3\hat{n}((\cos \omega_0 t \hat{y} - \sin \omega_0 t \hat{x}) \cdot \hat{n}) - (\cos \omega_0 t \hat{y} - \sin \omega_0 t \hat{x})) \right. \\ &\quad \left. - \frac{\omega_0}{c} ((\cos \omega_0 t \hat{x} + \sin \omega_0 t \hat{y}) \times \hat{n}) \times \hat{n} \right) \\ &= \frac{\mu_0 m_0 \omega_0}{4\pi rc} \left(\frac{2\hat{e}_r}{r} (\cos \omega_0 t \sin \theta \sin \phi - \sin \omega_0 t \sin \theta \cos \phi) \right. \\ &\quad \left. - \frac{1}{r} \cos \omega_0 t (\cos \theta \sin \phi \hat{e}_\theta + \cos \phi \hat{e}_\phi) + \frac{1}{r} \sin \omega_0 t (\cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) \right. \\ &\quad \left. - \frac{\omega_0}{c} (\cos \omega_0 t (-\cos \theta \cos \phi \hat{e}_\phi - \sin \phi \hat{e}_\theta) + \sin \omega_0 t (-\cos \theta \sin \phi \hat{e}_\phi + \cos \phi \hat{e}_\theta)) \times \hat{e}_r \right) \\ &= \frac{\mu_0 m_0 \omega_0}{4\pi rc} \left(\frac{2\hat{e}_r \sin \theta \sin(\phi - \omega_0 t) - \cos \theta \sin(\phi - \omega_0 t) \hat{e}_\theta - \cos(\phi - \omega_0 t) \hat{e}_\phi}{r} \right. \\ &\quad \left. + \frac{\omega_0}{c} (\cos \theta \cos(\phi - \omega_0 t) \hat{e}_\theta - \sin(\phi - \omega_0 t) \hat{e}_\phi) \right) \\ \\ \vec{E} &= -\frac{\mu_0}{4\pi} \left(\frac{\ddot{\vec{m}}}{cr} + \frac{\dot{\vec{m}}}{r^2} \right) \times \hat{e}_r \\ &= \frac{\mu_0 m_0}{4\pi r} \left(\frac{\omega_0^2}{c} (\cos \theta \cos(\phi - \omega_0 t) \hat{e}_\theta - \sin(\phi - \omega_0 t) \hat{e}_\phi) - \frac{\omega_0}{r} (\cos \theta \sin(\phi - \omega_0 t) \hat{e}_\theta + \cos(\phi - \omega_0 t) \hat{e}_\phi) \right) \times \hat{e}_r \\ &= \frac{\mu_0 m_0 \omega_0}{4\pi r} \left(-\frac{\omega_0}{c} (\cos \theta \cos(\phi - \omega_0 t) \hat{e}_\phi + \sin(\phi - \omega_0 t) \hat{e}_\theta) + \frac{1}{r} (\cos \theta \sin(\phi - \omega_0 t) \hat{e}_\phi - \cos(\phi - \omega_0 t) \hat{e}_\theta) \right) \end{aligned}$$

(b)

For radiation part

$$\vec{E}_{rad} = -\frac{\mu_0 m_0 \omega_0^2}{4\pi rc} (\cos \theta \cos(\phi - \omega_0 t) \hat{e}_\phi + \sin(\phi - \omega_0 t) \hat{e}_\theta)$$

Helicity is positive for $\theta > 0$ and negative for $\theta < 0$

Ellipticity is $|\cos \theta|$

(c)

$$\begin{aligned}\frac{dW_{rad}}{d\Omega dt} &= \frac{1}{\mu_0 c} \frac{\mu_0^2 m_0^2 \omega_0^4}{16\pi^2 r^2 c^2} \frac{1}{2} (\cos^2 \theta + 1) \\ &= \frac{\mu_0 m_0^2 \omega_0^4 (\cos^2 \theta + 1)}{32\pi^2 r^2 c^3}\end{aligned}$$

(d)

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{3\mu_0 m_0^2 \omega_0^4}{16\pi r^2 c^3}$$

(e)

$$\begin{aligned}\frac{dW}{dt d\Omega} &= -r^3 \hat{n} \times \left(\left(\epsilon_0 \vec{E} \vec{E} + \frac{1}{\mu_0} \vec{B} \vec{B} \right) \cdot \hat{n} \right) \\ &= -r^3 \left(\epsilon_0 \hat{n} \times \vec{E} \vec{E} \cdot \hat{n} + \frac{1}{\mu_0} \hat{n} \times \vec{B} \vec{B} \cdot \hat{n} \right) \\ &= -\frac{m_0 \omega_0 r}{2\pi c} \sin \theta \sin(\phi - \omega_0 t) \hat{n} \times \vec{B}\end{aligned}$$

Ignoring the terms that vanishes for large r

$$\frac{dW}{dt d\Omega} = -\frac{\mu_0 m_0^2 \omega_0^3}{8\pi^2 c^3} \sin \theta \sin(\phi - \omega_0 t) (\cos \theta \cos(\phi - \omega_0 t) \hat{e}_\phi + \sin(\phi - \omega_0 t) \hat{e}_\theta)$$

Time averaging

$$\left\langle \frac{dW}{dt d\Omega} \right\rangle = -\frac{\mu_0 m_0^2 \omega_0^3}{16\pi^2 c^3} \sin \theta \hat{e}_\theta$$

(f)

After the angular integral, only the z component can be not zero

$$\begin{aligned}\left\langle \frac{dW}{d\Omega} \right\rangle &= \hat{z} \frac{\mu_0 m_0^2 \omega_0^3}{16\pi^2 c^3} \int d\Omega \sin^2 \theta \\ &= \hat{z} \frac{\mu_0 m_0^2 \omega_0^3}{8\pi c^3}\end{aligned}$$

6.

(a)

(b)

(c)