Solutions Assignment 11: Due Friday May 8, 2015 at 2:30 pm

Problems

Problem 11.1 : The spinning shell of charge in general

A spherical shell of radius R, carries a uniform surface charge σ. Its total charge Q is $4\pi R^2 \sigma$, and its Coulomb electric field is as given on Problem Set 10. The sphere spins at an angular frequency $\Omega(t) = \Omega_o \cos(\omega t)$, where $\Omega_o R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\vec{\mathbf{J}}(\mathbf{X},t) = \mathrm{Re}\left\{\sigma \Omega_o R \delta(r-R)\sin\theta e^{-i\omega t}\,\hat{\boldsymbol{\phi}}\right\}$$

In addition to the radial coulomb electric field, there will be a time varying electric field in the azimuthal direction $E_\phi(\mathbf{X},t)\hat{\boldsymbol{\phi}}$. From arguments in lecture the azimuthal electric field must be of the form

$$E_\phi(r,\theta,t) = \mathrm{Re}\left\{A j_1(kr_<) h_1^1(kr_>)\sin\theta e^{-i\omega t}\right\}$$

where $r_< = \min(r,R)$ and $r_> = \max(r,R)$ and A is a constant to be determined.

(a) Given the definitions on page 426 of Jackson, and applying the proper boundary conditions at $r = R$, show that

$$|A| = \mu_o c\sigma \mathrm{R}\,\Omega_o (kR)^2 = \frac{Q}{4\pi R^2 \varepsilon_o}\frac{\Omega_o \mathrm{R}}{c}(kR)^2$$

$$E_\phi(r,\theta,t) = \mathrm{Re}\left\{A\sin\theta e^{-i\omega t}\begin{Bmatrix} A j_1(kr)h_1^1(kR) & r<R \\ A j_1(kR)h_1^1(kr) & r>R \end{Bmatrix}\right\}$$

$$E_\phi(r,\theta,t) = \mathrm{Re}\left\{A\sin\theta e^{-i\omega t}\begin{Bmatrix} \left[\dfrac{\sin(kr)}{(kr)^2} - \dfrac{\cos(kr)}{(kr)}\right]\left[-\dfrac{e^{ikR}}{kR}\left(1+\dfrac{i}{kR}\right)\right] & r<R \\ \left[\dfrac{\sin(kR)}{(kR)^2} - \dfrac{\cos(kR)}{(kR)}\right]\left[-\dfrac{e^{ikr}}{kr}\left(1+\dfrac{i}{kr}\right)\right] & r>R \end{Bmatrix}\right\} \tag{1}$$

Given the expressions in (1), the magnetic field is $\partial \mathbf{B}/\partial t = -i\omega \mathbf{B} = -\nabla \times \mathbf{E}$.

$$\mathbf{B} = -\frac{i}{\omega}\nabla\times\mathbf{E} = -\frac{i}{\omega}\hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) + \hat{\boldsymbol{\theta}}\frac{i}{\omega}\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)$$

Using our expression in (1), we have

$$B_r(r,\theta,t) = -\frac{2\cos\theta}{r}\mathrm{Re}\left(A\frac{i}{\omega}e^{-i\omega t}\right)\begin{cases}\left[\frac{\sin(kr)}{(kr)^2}-\frac{\cos(kr)}{(kr)}\right]\left[\left(-\frac{e^{ikR}}{kR}\right)\left(1+\frac{i}{kR}\right)\right] & r<R \\ \left[\frac{\sin(kR)}{(kR)^2}-\frac{\cos(kR)}{(kR)}\right]\left[\left(-\frac{e^{ikr}}{kr}\right)\left(1+\frac{i}{kr}\right)\right] & r>R\end{cases} \tag{2}$$

And

$$B_\theta(r,\theta,t) = \frac{\sin\theta}{r}\mathrm{Re}\left(A\frac{i}{\omega}e^{-i\omega t}\right)\begin{cases}\left[\frac{\cos(kr)}{kr}+\sin(kr)\left(1-\frac{1}{k^2r^2}\right)\right]\left[\left(-\frac{e^{ikR}}{kR}\right)\left(1+\frac{i}{kR}\right)\right] & r<R \\ \left[\frac{\sin(kR)}{(kR)^2}-\frac{\cos(kR)}{(kR)}\right]\left(e^{ikr}\right)\left(-i\left(1-\frac{1}{k^2r^2}\right)+\frac{1}{kr}\right) & r>R\end{cases} \tag{3}$$

Our boundary condition on tangential **B** across the surface of the sphere is

$$B_\theta\left(R^+,\theta,t\right)-B_\theta\left(R^-,\theta,t\right)=\mu_o\sigma\,\Omega_o\sin\theta\,\mathrm{R}\cos\omega t \tag{4}$$

Using our expression for $B_\theta(r,\theta,t)$ in 3), we have

$$B_\theta\left(R^+,\theta,t\right)-B_\theta\left(R^-,\theta,t\right)=$$

$$=\frac{\sin\theta}{R}\mathrm{Re}\left(A\frac{i}{\omega}e^{-i\omega t+ikR}\right)\left\{\begin{array}{l}\left[\frac{\sin(kR)}{(kR)^2}-\frac{\cos(kR)}{(kR)}\right]\left(-i\left(1-\frac{1}{k^2R^2}\right)+\frac{1}{kR}\right) \\ -\left[\frac{\cos(kR)}{kR}+\sin(kR)\left(1-\frac{1}{k^2R^2}\right)\right]\left[-\left(\frac{1}{kR}+\frac{i}{k^2R^2}\right)\right]\end{array}\right\}$$

Which simplifies to

$$B_\theta\left(R^+,\theta,t\right)-B_\theta\left(R^-,\theta,t\right)=$$

$$=\frac{\sin\theta}{R}\mathrm{Re}\left(\frac{iA}{\omega}e^{-i\omega t+ikR}\right)\left\{\left[+\sin(kR)(1)\right]\left(\frac{1}{kR}\right)+i\left\{\left[\frac{\cos(kR)}{(kR)}\right]\right\}\right\} \tag{5}$$

$$=-\frac{\sin\theta}{R}\mathrm{Re}\left(\frac{A}{\omega}e^{-i\omega t+ikR}\right)\frac{e^{-ikR}}{kR}=-\frac{1}{k^2R^2}\frac{A}{c}\sin\theta\cos\omega t$$

Comparing (4) and (5), we see that we must have, as desired

$$A=-ck^2R^2\mu_o\sigma R\Omega_o=-\frac{Q}{4\pi\varepsilon_o R^2}\frac{R\Omega_o}{c}k^2R^2 \tag{6}$$

(b) Calculate the time average over one period of $\int\limits_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, and multiply this quantity by the period T. Divide this energy by the magnetostatic energy stored in a sphere spinning at a constant rate Ω_o . Plot this quantity versus $kR=2\pi R/\lambda$ from $kR=0$ to $kR=10$. This is the energy radiated away in one period normalized to the magnetostatic energy.

Using our expression in (1) and evaluating it at $r=R$, we have

$$E_\phi\left(R,\theta,t\right)=\mathrm{Re}\left\{-\frac{A\sin\theta}{kR}e^{-i\omega t+ikR}\left[\frac{\sin\left(kR\right)}{\left(kR\right)^2}-\frac{\cos\left(kR\right)}{\left(kR\right)}\right]\left[\left(1+\frac{i}{kR}\right)\right]\right\}$$

$$=-\frac{A\sin\theta}{kR}\cos\left(\omega t-kR\right)\left[\frac{\sin\left(kR\right)}{\left(kR\right)^2}-\frac{\cos\left(kR\right)}{\left(kR\right)}\right]-\frac{A\sin\theta}{k^2R^2}\sin\left(\omega t-kR\right)\left[\frac{\sin\left(kR\right)}{\left(kR\right)^2}-\frac{\cos\left(kR\right)}{\left(kR\right)}\right]$$

$$=-A\sin\theta\left[\frac{\sin\left(kR\right)}{\left(kR\right)^2}-\frac{\cos\left(kR\right)}{\left(kR\right)}\right]\left\{+\frac{1}{kR}\cos\left(\omega t-kR\right)+\frac{1}{k^2R^2}\sin\left(\omega t-kR\right)\right\}$$

(7)

We have

$$\int -J_\phi E_\phi\left(r,\theta,t\right)d^3x=\int-\left[\sigma\Omega_o R\sin\theta\cos\left(\omega t\right)\delta\left(r-R\right)\right]E_\phi\left(R,\theta,t\right)r^2drd\phi d\left(\cos\theta\right)$$

$$=-2\pi\sigma\Omega_o R^3\cos\left(\omega t\right)\int\sin\theta E_\phi\left(R,\theta,t\right)d\left(\cos\theta\right)$$

Using (7) this becomes

$$\int -J_\phi E_\phi d^3x=2\pi\sigma\Omega_o R^3 A\cos\left(\omega t\right)\left[\frac{\sin\left(kR\right)}{\left(kR\right)^2}-\frac{\cos\left(kR\right)}{\left(kR\right)}\right]\left\{\begin{matrix}+\frac{1}{kR}\cos\left(\omega t-kR\right)\\+\frac{1}{k^2R^2}\sin\left(\omega t-kR\right)\end{matrix}\right\}\int_{-1}^{1}\left(1-x^2\right)dx$$

And using (6)

$$\int -J_\phi E_\phi d^3x=-\frac{8\pi c\mu_o}{3}\left(k^2R^2\right)\sigma^2R^4\Omega_o^2\cos\left(\omega t\right)\left[\frac{\sin\left(kR\right)}{\left(kR\right)^2}-\frac{\cos\left(kR\right)}{\left(kR\right)}\right]\left\{\begin{matrix}+\frac{1}{kR}\cos\left(\omega t-kR\right)\\+\frac{1}{k^2R^2}\sin\left(\omega t-kR\right)\end{matrix}\right\}$$

(8)

We need to time average (8), and in doing so we need to know that

$$\left\langle\cos\left(\omega t-kR\right)\cos\omega t\right\rangle=\left\langle(\cos\omega t\cos(kR)+\sin(\omega t)\sin\left(kR\right))\cos\omega t\right\rangle=\frac{1}{2}\cos(kR)$$

$$\langle \sin(\omega t - kR)\cos\omega t\rangle = \langle(\sin\omega t\cos(kR) - \cos(\omega t)\sin(kR))\cos\omega t\rangle = -\frac{1}{2}\sin(kR)$$

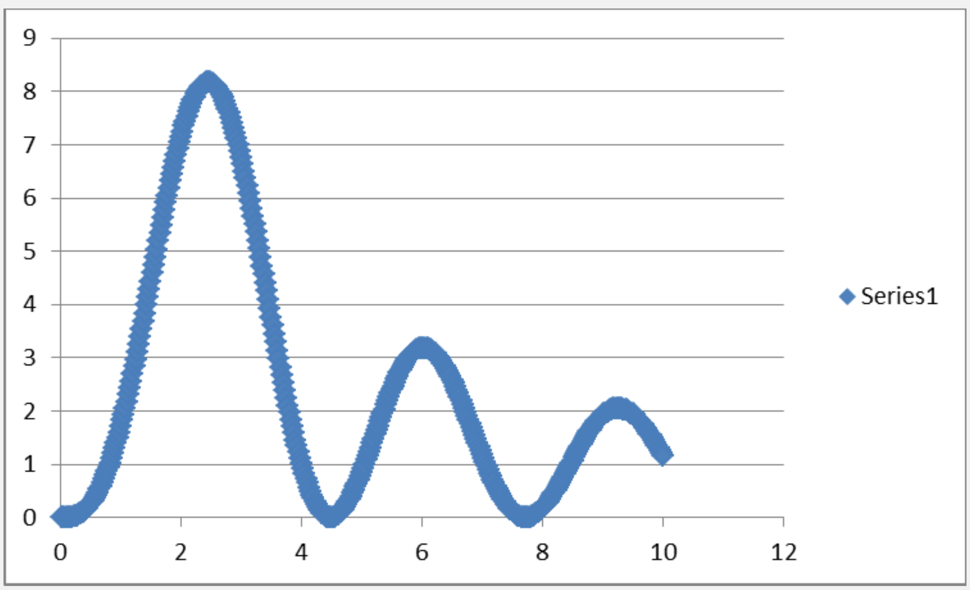
We find that

$$\int -J_\phi E_\phi d^3x = \frac{4\pi c\mu_o}{3}\left(k^2R^2\right)\sigma^2 R^4\Omega_o^2\left[\frac{\sin(kR)}{(kR)^2} - \frac{\cos(kR)}{(kR)}\right]^2 \qquad (9)$$

We are told to multiply this by $T = 2\pi/\omega$ and divide by the energy in a static magnetic dipole. From the last problem set, we know that the energy in a static magnetic dipole is

$$U_{static} = \frac{\mu_o m_o^2}{4\pi R^3} \qquad m_o = \frac{4\pi\,\sigma\Omega_o R^4}{3} \qquad U_{static} = \frac{\mu_o}{4\pi R^3}\left(\frac{4\pi\,\sigma\Omega_o R^4}{3}\right)^2 = \frac{4\pi}{9}\mu_o\sigma^2\Omega_o^2 R^5 \qquad (10)$$

$$\frac{2\pi}{\omega}\frac{1}{U_{static}}\int -J_\phi E_\phi d^3x = 12\pi kR\left[\frac{\sin(kR)}{(kR)^2} - \frac{\cos(kR)}{(kR)}\right]^2 = 6\pi(kR)\,j_1^2(kR) \qquad (11)$$



(c) Does the small argument behavior of your expression in (b) make sense to you? Explain *quantitatively*.

From p 427 of Jackson eq (9.88) we see that the small argument limit is $j_1(kR) = kR/3$, so for small argument we have

$$\frac{2\pi}{\omega}\frac{1}{U_{static}}\int -J_\phi E_\phi d^3x \simeq \frac{6\pi}{9}(kR)^3 = \frac{2\pi}{3}(kR)^3 \quad \text{for } kR << 1$$

The small argument limit is the electric dipole approximation, for which $T >> R/c$ or $kR << 2\pi$. In the electric dipole limit we know that the magnetic dipole radiates at a rate

$$\frac{dW_{mag\ dip}}{dt} = \frac{\mu_o}{4\pi}\frac{2|\ddot{\mathbf{m}}|^2}{3c^3} = \frac{\mu_o}{4\pi}\frac{2}{3c^3}\omega^4 m_o^2 \cos^2(\omega t)$$

$$\frac{2\pi}{\omega}\frac{1}{U_{static}}\left\langle \frac{dW_{mag\ dip}}{dt}\right\rangle = \frac{2\pi}{\omega}\frac{4\pi\ R^3}{\mu_o m_o^2}\frac{\mu_o}{4\pi}\frac{\omega^4 m_o^2}{3c^3} = \frac{2\pi}{3}\frac{\omega^3 R^3}{c^3} = \frac{2\pi}{3}(kR)^3$$

So our small argument limit of (11) is exactly what we expect in the electric dipole limit.

(d) Does the behavior of this energy radiated over one period between $kR = 1$ and $kR = 10$ make sense to you? Explain *qualitatively*. Is there a frequency at which there is a maximum in the energy radiated during one period?

The surprising thing is that there are zeros in the radiated energy at the zeroes of $j_1(kR)$. The reason this happens is that at the zeros of $j_1(kR)$ the electric field at the surface of the sphere goes to zero, because radiation from one side of the sphere arrives at the other side 180 degrees out of phase with the electric field there, producing no net work done and no radiation. Acceleration without radiation. Surprising.

There is a peak at a frequency, around $kR = 2.46$ or $\lambda = 2.55R$ or $T = 2.55R/c$.