1.

(a)

From the generic form

$$V = V_0 J_0\left(\frac{x_{01}\rho}{a}\right)\frac{\sinh\left(\frac{x_{01}z}{a}\right)}{\sinh\left(\frac{x_{01}L}{a}\right)}$$

(b)

$$V = V_0 J_0\left(\frac{\rho}{a}\right)\exp\left(-\frac{z}{a}\right)$$

2.

(a)

The flux through a circle defined by r and θ, for $r < R$

$$\Phi = \pi r^2 \sin^2\theta B$$

For $r > R$

$$\begin{aligned}\Phi =& \int_{\cos\theta}^{1} 2\pi r^2 \mathrm{d}z \frac{\mu_0 m}{4\pi r^3} 2z \\ =& \pi\frac{R^3}{r}\sin^2\theta B\end{aligned}$$

E field

$$\begin{aligned}E_\phi =& \frac{1}{2\pi r\sin\theta}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ =& \frac{\mathrm{d}B}{\mathrm{d}t}\sin\theta\begin{cases}\frac{r}{2} & (r < R) \\ \frac{R^3}{2r^2} & (r > R)\end{cases}\end{aligned}$$

(b)

$$\begin{aligned}U_{B_{out}} =& \frac{1}{2\mu_0}\int_R^\infty \mathrm{d}r\int_0^\pi r\mathrm{d}\theta\int_0^{2\pi} r\sin\theta\mathrm{d}\phi\frac{\mu_0^2 m^2}{16\pi^2 r^6}\left(4\cos^2\theta + \sin^2\theta\right) \\ =& \frac{\mu_0 m^2}{16\pi}\int_R^\infty \mathrm{d}r\frac{1}{r^4}\int_{-1}^1 \mathrm{d}\cos\theta\left(4\cos^2\theta + \sin^2\theta\right) \\ =& \frac{\mu_0 m^2}{4\pi}\frac{1}{3R^3} \\ =& \frac{\mu_0 m^2}{12\pi R^3}\end{aligned}$$

Inside

$$\begin{aligned}U_{B_{in}} =&\frac{4\pi}{3}R^3\frac{B^2}{2\mu_0}\\ =&\frac{4\pi}{3}R^3\frac{1}{2\mu_0}\frac{4\mu_0^2}{9}\frac{9}{16\pi^2R^6}m^2\\ =&\frac{\mu_0 m^2}{6\pi R^3}\end{aligned}$$

Total

$$U_B =\frac{\mu_0 m^2}{4\pi R^3}$$

(c)

$$\begin{aligned}W =&\int_0^\pi R\mathrm{d}\theta\int_0^{2\pi}R\sin\theta\mathrm{d}\phi\kappa\sin\theta\frac{\mathrm{d}B}{\mathrm{d}t}\sin\theta\frac{R}{2}\\ =&\pi\frac{\mu_0}{3}R^3\frac{\mathrm{d}\kappa^2}{\mathrm{d}t}\int_{-1}^1\mathrm{d}z\left(1-z^2\right)\\ =&\frac{\mu_0}{4\pi R^3}\frac{\mathrm{d}m^2}{\mathrm{d}t}\\ =&\frac{\mathrm{d}U_B}{\mathrm{d}t}\end{aligned}$$

(d)

For $r = R + 0^+$

$$\begin{aligned}W_+ =&\int_0^\pi R\mathrm{d}\theta\int_0^{2\pi}R\sin\theta\mathrm{d}\phi\frac{\mu_0 m}{4\pi R^3}\sin\theta\frac{1}{\mu_0}\frac{\mathrm{d}B}{\mathrm{d}t}\sin\theta\frac{R}{2}\\ =&\int_0^\pi R\mathrm{d}\theta\int_0^{2\pi}R\sin\theta\mathrm{d}\phi\frac{\kappa}{3}\sin\theta\frac{\mathrm{d}B}{\mathrm{d}t}\sin\theta\frac{R}{2}\\ =&\frac{W}{3}\end{aligned}$$

For $r = R - 0^+$

$$\begin{aligned}W_- =&\int_0^\pi R\mathrm{d}\theta\int_0^{2\pi}R\sin\theta\mathrm{d}\phi B\sin\theta\frac{1}{\mu_0}\frac{\mathrm{d}B}{\mathrm{d}t}\sin\theta\frac{R}{2}\\ =&\int_0^\pi R\mathrm{d}\theta\int_0^{2\pi}R\sin\theta\mathrm{d}\phi\frac{2}{3}\kappa\sin\theta\frac{\mathrm{d}B}{\mathrm{d}t}\sin\theta\frac{R}{2}\\ =&\frac{2W}{3}\end{aligned}$$

This is the same as what one would expect from (b) and (c) with determines the ratio and the sum of W_- and W_+ respectively.

(e)

From the symmetry of the problem, only z component of angular momentum can be non-zero. Since the "static" E field is 0 inside the sphere, we only need to consider the space outside the sphere.

$$\begin{aligned}
L_z =& \int_R^\infty \mathrm{d}r \int_0^\pi r\mathrm{d}\theta \int_0^{2\pi} r\sin\theta \mathrm{d}\phi \hat{z}\cdot\left(\vec{r}\times\left(\varepsilon_0\vec{E}\times\vec{B}\right)\right)\\
=&2\pi\varepsilon_0\int_R^\infty \mathrm{d}r\int_0^\pi \mathrm{d}\theta r^2\sin\theta\left(\hat{r}\times\hat{\theta}\right)\cdot(\hat{z}\times\vec{r})EB_\theta\\
=&2\pi\varepsilon_0\int_R^\infty \mathrm{d}r\int_0^\pi \mathrm{d}\theta r^2\sin\theta\hat{\phi}\cdot\hat{\phi}r\sin\theta\frac{Q}{4\pi\varepsilon_0 r^2}\frac{\mu_0 m}{4\pi r^3}\sin\theta\\
=&\frac{\mu_0 mQ}{8\pi}\int_R^\infty\frac{\mathrm{d}r}{r^2}\int_{-1}^1\mathrm{d}z\left(1-z^2\right)\\
=&\frac{\mu_0 mQ}{6\pi R}
\end{aligned}$$

(f)

Torque,

$$\begin{aligned}
\tau_z =& \int_0^\pi R\mathrm{d}\theta\int_0^{2\pi} R\sin\theta\mathrm{d}\phi\hat{z}\cdot\left(\vec{R}\times\left(\sigma\vec{E}+\vec{\kappa}\times\vec{B}\right)\right)\\
=&2\pi\int_0^\pi \mathrm{d}\theta R^2\sin\theta\left(\sigma\vec{E}+\vec{\kappa}\times\vec{B}\right)\cdot\left(\hat{z}\times\vec{R}\right)\\
=&2\pi\int_0^\pi \mathrm{d}\theta R^2\sin\theta\left(\sigma\vec{E}\cdot\hat{\phi}+\left(\vec{\kappa}\times\vec{B}\right)\cdot\hat{\phi}\right)R\sin\theta\\
=&2\pi\int_0^\pi \mathrm{d}\theta R^3\sin^2\theta\left(\sigma E_\phi+\left(\hat{\phi}\times\vec{\kappa}\right)\cdot\vec{B}\right)\\
=&\pi R^4\frac{\mathrm{d}B}{\mathrm{d}t}\sigma\int_0^\pi\mathrm{d}\theta\sin^3\theta\\
=&\pi R^4\frac{\mathrm{d}m}{\mathrm{d}t}\frac{3}{4\pi R^3}\frac{2\mu_0}{3}\frac{Q}{4\pi R^2}\int_{-1}^1\mathrm{d}z\left(1-z^2\right)\\
=&\frac{\mu_0 Q}{6\pi R}\frac{\mathrm{d}m}{\mathrm{d}t}
\end{aligned}$$

(g)

$$\begin{aligned}
\tau_z =& \int_0^\pi R\mathrm{d}\theta\int_0^{2\pi} R\sin\theta\mathrm{d}\phi\hat{z}\cdot\left(\vec{R}\times\left(\varepsilon_0\vec{E}\vec{E}+\frac{1}{\mu_0}\vec{B}\vec{B}-\frac{\varepsilon}{2}E^2+\frac{1}{2\mu_0}B^2\right)\right)\cdot\hat{R}\\
=& \int_0^\pi R\mathrm{d}\theta\int_0^{2\pi} R\sin\theta\mathrm{d}\phi\hat{z}\cdot\left(\vec{R}\times\left(\varepsilon_0\vec{E}\left(\vec{E}\cdot\hat{R}\right)+\frac{1}{\mu_0}\vec{B}\left(\vec{B}\cdot\hat{R}\right)\right)\right)\\
=& \int_0^\pi R\mathrm{d}\theta\int_0^{2\pi} R\sin\theta\mathrm{d}\phi\left(\varepsilon_0\vec{E}E_r+\frac{1}{\mu_0}\vec{B}B_r\right)\cdot\left(\hat{z}\times\vec{R}\right)\\
=&2\pi R^3\int_0^\pi\mathrm{d}\theta\sin^2\theta\left(\varepsilon_0\vec{E}E_r+\frac{1}{\mu_0}\vec{B}B_r\right)\cdot\hat{\phi}\\
=&2\pi\varepsilon_0 R^3\int_0^\pi\mathrm{d}\theta\sin^2\theta E_\phi E_r
\end{aligned}$$

Since $E_r = 0$ for $r < R$, the flux is only non-zero outside the shell. For $r = R + 0^+$

$$\begin{aligned}
\tau_z =& 2\pi\varepsilon_0 R^3 \frac{Q}{4\pi\varepsilon_0 R^2} \int_0^\pi \mathrm{d}\theta \sin^2\theta \frac{\mathrm{d}B}{\mathrm{d}t} \sin\theta \frac{R}{2} \\
=& \frac{QR^2}{4} \frac{\mathrm{d}B}{\mathrm{d}t} \int_0^\pi \mathrm{d}\theta \sin^3\theta \\
=& \frac{QR^2}{4} \frac{\mathrm{d}m}{\mathrm{d}t} \frac{3}{4\pi R^3} \frac{2\mu_0}{3} \frac{4}{3} \\
=& \frac{\mu_0 Q}{6\pi R} \frac{\mathrm{d}m}{\mathrm{d}t}
\end{aligned}$$