1.

(a)

$$x_i' x_i' = a_{ij} a_{ik} x_j x_k$$
$$x_i x_i = \delta_{jk} x_j x_k$$

Since $x_i' x_i' = x_i x_i$ for all x_i

$$\delta_{jk} = a_{ij} a_{ik}$$

(b)

$$x_i = \delta_{ik} x_k$$
$$= a_{ji} a_{jk} x_k$$
$$= a_{ji} x_j'$$

(c)

$$\frac{\partial f}{\partial x_i'} = \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x_i'}$$
$$= \frac{\partial f}{\partial x_j} \frac{\partial a_{kj} x_k'}{\partial x_i'}$$
$$= a_{ij} \frac{\partial f}{\partial x_j}$$

2.

(a)

$$\delta_{ij}' = a_{im} a_{jn} \delta_{mn}$$
$$= a_{ik} a_{jk}$$
$$= \delta_{ij}$$

(b)

$$C_i' T_{ij}' = a_{ik} C_k a_{im} a_{jn} T_{mn}$$
$$= a_{ik} a_{im} a_{jn} C_k T_{mn}$$
$$= \delta_{km} a_{jn} C_k T_{mn}$$
$$= a_{jk} C_i T_{ik}$$
$$= (C_i T_{ik})'$$

1

(c)

Since both $A_i A_j$ and δ_{ij} are second rank tensors and A^2 is a scalar, T_{ij} is also a second rank tensor.

(d)

$$\partial_i \left(A_i A_j - \frac{1}{2} \delta_{ij} A_k A_k \right)$$

$$= \partial_i (A_i A_j) - \frac{1}{2} \partial_j (A_k A_k)$$

$$= A_j \partial_i (A_i) + A_i \partial_i (A_j) - A_k \partial_j (A_k)$$

$$= A_j \partial_i (A_i) + A_i \partial_i (A_j) - A_i \partial_j (A_i)$$

$$= A_j \partial_i (A_i) + A_i (\delta_{ki} \delta_{jl} - \delta_{jk} \delta il) \partial_k (A_l)$$

$$= A_j \partial_i (A_i) + \varepsilon_{mij} A_i \varepsilon_{mkl} \partial_k (A_l)$$

$$= A_j \partial_i (A_i) + \varepsilon_{jmi} (\varepsilon_{mkl} \partial_k (A_l)) A_i$$

(e)

i.

$$T_{ij} n_j = \left(A_i A_j - \frac{1}{2} \delta_{ij} A^2 \right) n_j$$

$$= A_i A_j n_j - \frac{1}{2} A^2 n_i$$

Therefore $T \cdot \vec{n}$ is a linear combination of $\vec{A}$ and $\vec{n}$

ii.

Let $B_i = T_{ij} n_j$

$$B_i B_i = \left(A_i A_j n_j - \frac{1}{2} A^2 n_i \right) \left(A_i A_k n_k - \frac{1}{2} A^2 n_i \right)$$

$$= (A_j n_j)^2 A^2 + \frac{1}{4} A^4 - (A_j n_j)^2 A^2 = \qquad\qquad \frac{1}{4} A^4$$

$$B_i n_i = A_i A_j n_j n_i - \frac{1}{2} A^2$$

$$= A^2 \left(\cos^2 \theta - \frac{1}{2} \right)$$

$$= \frac{A^2}{2} \cos 2\theta$$

$$\cos \theta_{Bn} = \cos 2\theta$$

$$\theta_{Bn} = 2\theta$$

iii.

See above.

3.

(a)

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r} - E_0 \sin\phi$$
$$E_\phi = - E_0 \cos\phi$$

(b)

$$E_r = \frac{2\pi\varepsilon_0 E_0 \lambda}{2\pi\varepsilon_0 \lambda} - E_0$$
$$= 0$$
$$E_\phi = 0$$

(c)

$$d\vec{F} = \varepsilon_0 r d\phi dz \left(E_r \vec{E} - \frac{1}{2} E^2 \hat{r} \right)$$
$$= \varepsilon_0 r d\phi dz \left(\frac{E_r^2 - E_\phi^2}{2} \hat{r} + E_r E_\phi \hat{\phi} \right)$$

(d)

$$\frac{d\vec{F}}{dz} = \int \varepsilon_0 r d\phi \left(\frac{E_r^2 - E_\phi^2}{2} \hat{r} + E_r E_\phi \hat{\phi} \right)$$
$$= \frac{\varepsilon_0 r}{2} \hat{y} \int d\phi \left(\left(\left(\frac{\lambda}{2\pi\varepsilon_0 r} - E_0 \sin\phi \right)^2 - E_0 \cos^2\phi \right) \sin\phi - 2 \left(\frac{\lambda}{2\pi\varepsilon_0 r} - E_0 \sin\phi \right) E_0 \cos^2\phi \right)$$
$$= \frac{\varepsilon_0 r}{2} \hat{y} \int d\phi \left(-2 \frac{\lambda}{2\pi\varepsilon_0 r} E_0 \sin^2\phi - \frac{\lambda}{\pi\varepsilon_0 r} E_0 \cos^2\phi \right)$$
$$= \frac{1}{2} \hat{y} \int d\phi \left(-\frac{\lambda}{\pi} E_0 \sin^2\phi - \frac{\lambda}{\pi} E_0 \cos^2\phi \right)$$
$$= -\frac{1}{2} \hat{y} 2\pi \frac{\lambda}{\pi} E_0$$
$$= - E_0 \lambda \hat{y}$$

4.

(a)

$$B_r = B_0 \sin \phi$$
$$B_\phi = \frac{\mu_0 I}{2\pi r} + B_0 \cos \phi$$

(b)

$$B_r = 0$$
$$B_\phi = \frac{\mu_0 I 2\pi B_0}{2\pi \mu_0 I} + B_0$$
$$= 0$$

(c)

$$\mathrm{d}\vec{F} = \frac{1}{\mu_0} r\mathrm{d}\phi \mathrm{d}z \left(B_r \vec{B} - \frac{1}{2} B^2 \hat{r} \right)$$
$$= \frac{1}{\mu_0} r\mathrm{d}\phi \mathrm{d}z \left(\frac{B_r^2 - B_\phi^2}{2} \hat{r} + B_r B_\phi \hat{\phi} \right)$$

(d)

$$\frac{\mathrm{d}\vec{F}}{\mathrm{d}z} = \frac{1}{2} \int \frac{r}{\mu_0} \mathrm{d}\phi \left((B_r^2 - B_\phi^2)\hat{r} + 2B_r B_\phi \hat{\phi} \right)$$
$$= \frac{r\hat{x}}{2\mu_0} \int \mathrm{d}\phi \left(\left(B_0^2 \sin^2 \phi - \left(\frac{\mu_0 I}{2\pi r} + B_0 \cos \phi \right)^2 \right) \cos \phi - 2B_0 \sin \phi \left(\frac{\mu_0 I}{2\pi r} + B_0 \cos \phi \right) \sin \phi \right)$$
$$= \frac{r\hat{x}}{2\mu_0} \int \mathrm{d}\phi \left(- \left(\frac{\mu_0 I}{2\pi r} + B_0 \cos \phi \right)^2 \cos \phi - B_0 \sin^2 \phi \frac{\mu_0 I}{\pi r} \right)$$
$$= \frac{\hat{x}}{2} \int \mathrm{d}\phi \left(-\frac{I}{\pi} B_0 \cos^2 \phi - B_0 \sin^2 \phi \frac{I}{\pi} \right)$$
$$= -B_0 I \hat{x}$$

5.

(a)

$$\vec{E}_{line} = \frac{\lambda\left(\vec{r} - \vec{Y}\right)}{2\pi\varepsilon_0\left(\vec{r} - \vec{Y}\right)^2}$$

$$= \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\varepsilon_0\left(x^2 + (y - Y)^2\right)}$$

$$E = \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\varepsilon_0\left(x^2 + (y - Y)^2\right)} - E_0\hat{y}$$

(b)

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{-L}^{L} \mathrm{d}y \varepsilon_0 \vec{E}_0 \cdot \vec{E}_{line}$$

$$= -\int_{-\infty}^{\infty} \mathrm{d}x \int_{-L}^{L} \mathrm{d}y \varepsilon_0 \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\varepsilon_0\left(x^2 + (y - Y)^2\right)} E_0 \cdot \hat{y}$$

$$= -\frac{E_0\lambda}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-L}^{L} \mathrm{d}y \frac{y - Y}{x^2 + (y - Y)^2}$$

$$= -\frac{E_0\lambda}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}x \ln\left(\frac{x^2 + (L - Y)^2}{x^2 + (-L - Y)^2}\right)$$

Let $\alpha = \dfrac{x}{L}$, $\beta = \dfrac{Y}{L}$

$$\frac{\mathrm{d}E}{\mathrm{d}z} = -\frac{E_0\lambda}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\alpha \ln\left(\frac{\alpha^2 + (1 - \beta)^2}{\alpha^2 + (1 + \beta)^2}\right)$$

$$= -\frac{E_0\lambda}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\alpha \ln\left(\frac{\alpha^2 + 1 - 2\beta}{\alpha^2 + 1 + 2\beta}\right)$$

$$\approx -\frac{E_0\lambda L}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\alpha \frac{4\beta}{1 + \alpha^2}$$

$$= -E_0\lambda Y$$

6.

(a)

Let $\mathrm{d}S$ be the area between the boundary at t and $t + \mathrm{d}t$ (pointing outward), S and S' are the surface at t and $t + \mathrm{d}t$ respectively.

$$\oiint_{-S+S'+\mathrm{d}S} \vec{G} \cdot \mathrm{d}\vec{a} = \iiint_{\mathrm{d}V} \nabla \cdot \vec{G}\mathrm{d}v$$

$$= \mathrm{d}t \iint_S \left(\nabla \cdot \vec{G}\right)\vec{V} \cdot \mathrm{d}\vec{a}$$

$$\mathrm{d}\left(\iint_S \vec{G}(t) \cdot \mathrm{d}\vec{a}\right) = \iint_S \vec{\mathrm{d}G} \cdot \mathrm{d}\vec{a} + \left(\iint_{S'} \vec{G} \cdot \mathrm{d}\vec{a} - \iint_S \vec{G} \cdot \mathrm{d}\vec{a}\right)$$

$$= \mathrm{d}t \iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} - \iint_{\mathrm{d}S} \vec{G} \cdot \mathrm{d}\vec{a} + \mathrm{d}t \iint_S \left(\nabla \cdot \vec{G}\right)\vec{V} \cdot \mathrm{d}\vec{a}$$

$$= \mathrm{d}t\left(\iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{G}\right)\vec{V} \cdot \mathrm{d}\vec{a}\right) - \oint_C \vec{G} \cdot \left(\mathrm{d}\vec{l} \times \vec{V}\mathrm{d}t\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{G}(t) \cdot \mathrm{d}\vec{a} = \iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{G}\right)\vec{V} \cdot \mathrm{d}\vec{a} - \oint_C \vec{G} \cdot \left(\mathrm{d}\vec{l} \times \vec{V}\right)$$

$$= \iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{G}\right)\vec{V} \cdot \mathrm{d}\vec{a} - \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{V} \times \vec{G}\right)$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{B}(t) \cdot \mathrm{d}\vec{a} = \iint_S \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{B}\right)\vec{V} \cdot \mathrm{d}\vec{a} - \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{V} \times \vec{B}\right)$$

$$= \iint_S \left(\nabla \times \vec{E}\right) \cdot \mathrm{d}\vec{a} - \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{V} \times \vec{B}\right)$$

$$= \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{E} - \vec{V} \times \vec{B}\right)$$

7.

(a)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G$$

$$= -\frac{c}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Theta\left(t - t' - \frac{|x - x'|}{c}\right)$$

$$= -\frac{c}{2}\left(\frac{\partial^2}{\partial x^2}\Theta(ct - ct' - |x - x'|) - \frac{\partial^2}{\partial(ct)^2}\Theta(ct - ct' - |x - x'|)\right)$$

$$= -\frac{c}{2}\left(-\frac{\partial}{\partial x}\delta(ct - ct' - |x - x'|)\textbf{sign}(x - x') - \frac{\partial}{\partial ct}\delta(ct - ct' - |x - x'|)\right)$$

$$= \frac{c}{2}\left(\textbf{sign}(x - x')\frac{\partial}{\partial x}\delta(ct - ct' - |x - x'|) + \delta(ct - ct' - |x - x'|)\frac{\partial}{\partial x}\textbf{sign}(x - x') + \frac{\partial}{\partial ct}\delta(ct - ct' - |x - x'|)\right)$$

$$= \frac{c}{2}\left(-\textbf{sign}^2(x - x')\delta'(ct - ct' - |x - x'|) + 2\delta(ct - ct' - |x - x'|)\delta(x - x') + \delta'(ct - ct' - |x - x'|)\right)$$

$$= c\delta(ct - ct' - |x - x'|)\delta(x - x')$$

$$= \delta(t - t')\delta(x - x')$$

(Last equation is using scaling law of δ function and the fact that the second δ limits $x - x' = 0$)

(b)

$$\vec{A} = \frac{c}{2}\mu_0\int dt'\int dx'\hat{y}\sigma_0 v\delta(x')\Theta\left(t - t' - \frac{|x - x'|}{c}\right)$$

$$= \frac{\sigma_0\mu_0 c}{2}\hat{y}\int dt'v\Theta\left(t - t' - \frac{|x|}{c}\right)$$

$$\phi = \frac{c}{2}\frac{1}{\varepsilon_0}\int dt'\int dx'\sigma_0\delta(x')\Theta\left(t - t' - \frac{|x - x'|}{c}\right)$$

$$= \frac{\sigma_0 c}{2\varepsilon_0}\int dt'\Theta\left(t - t' - \frac{|x|}{c}\right)$$

$$\vec{B} = \nabla \times \vec{A}$$

$$= \frac{c}{2}\frac{\partial}{\partial x}\hat{x} \times \sigma_0\mu_0\hat{y}\int dt'v\Theta\left(t - t' - \frac{|x|}{c}\right)$$

$$= \frac{c\sigma_0\mu_0\hat{z}}{2}\int dt'v\frac{\partial}{\partial x}\Theta\left(t - t' - \frac{|x|}{c}\right)$$

$$= -\frac{\sigma_0\mu_0}{2}\hat{z}\int dt'v\delta\left(t - t' - \frac{|x|}{c}\right)\textbf{sign}(x)$$

$$= -\frac{\sigma_0\mu_0}{2}\hat{z}v\left(t - \frac{|x|}{c}\right)\textbf{sign}(x)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

$$= \frac{\sigma_0 \mu_0 c}{2} \hat{y} \int \mathrm{d}t' v \frac{\partial}{\partial t} \Theta\left(t - t' - \frac{|x|}{c}\right) + \frac{\sigma_0 c}{2\varepsilon_0} \hat{x} \int \mathrm{d}t' \frac{\mathrm{d}}{\mathrm{d}x} \Theta\left(t - t' - \frac{|x|}{c}\right)$$

$$= \frac{\sigma_0 \mu_0 c}{2} \hat{y} \int \mathrm{d}t' v \delta\left(t - t' - \frac{|x|}{c}\right) + \frac{\sigma_0}{2\varepsilon_0} \hat{x} \int \mathrm{d}t' \delta\left(t - t' - \frac{|x|}{c}\right) \mathbf{sign}(x)$$

$$= \frac{\sigma_0 \mu_0 c}{2} \hat{y} v\left(t - \frac{|x|}{c}\right) + \frac{\sigma_0}{2\varepsilon_0} \hat{x} \mathbf{sign}(x)$$

(c)

$$\vec{B} = \hat{z} \frac{\sigma_0 \mu_0 V_0}{2} \theta\left(t - \frac{|x|}{c}\right) \mathbf{sign}(x)$$

$$\vec{E} = \frac{\sigma_0 \mu_0 c V_0}{2} \hat{y} \theta\left(t - \frac{|x|}{c}\right) + \frac{\sigma_0}{2\varepsilon_0} \hat{x} \mathbf{sign}(x)$$

$$= \frac{\sigma_0}{2\varepsilon_0} \left(\hat{y} \frac{V_0}{c} \theta\left(t - \frac{|x|}{c}\right) + \hat{x} \mathbf{sign}(x) \right)$$