Yichao Yu

1.

(a)

$$
\begin{aligned}
x_i'x_i' =&a_{ij}a_{ik}x_jx_k\\
x_ix_i =&\delta_{jk}x_jx_k
\end{aligned}
$$

Since $x_i'x_i' = x_ix_i$ for all x_i

$$
\delta_{jk} =a_{ij}a_{ik}
$$

(b)

$$
\begin{aligned}
x_i =&\delta_{ik}x_k\\
=&a_{ji}a_{jk}x_k\\
=&a_{ji}x_j'
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\partial f}{\partial x_i'} =&\frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_i'}\\
=&\frac{\partial f}{\partial x_j}\frac{\partial a_{kj}x_k'}{\partial x_i'}\\
=&a_{ij}\frac{\partial f}{\partial x_j}
\end{aligned}
$$

2.

(a)

$$
\begin{aligned}
\delta_{ij}' =&a_{im}a_{jn}\delta_{mn}\\
=&a_{ik}a_{jk}\\
=&\delta_{ij}
\end{aligned}
$$

(b)

$$
\begin{aligned}
C_i'T_{ij}' =&a_{ik}C_ka_{im}a_{jn}T_{mn}\\
=&a_{ik}a_{im}a_{jn}C_kT_{mn}\\
=&\delta_{km}a_{jn}C_kT_{mn}\\
=&a_{jk}C_iT_{ik}\\
=&(C_iT_{ik})'
\end{aligned}
$$

(c)

Since both A_iA_j and δ_{ij} are second rank tensors and A^2 is a scalar, T_{ij} is also a second rank tensor.

(d)

$$\begin{aligned}
&\partial_i\left(A_iA_j - \frac{1}{2}\delta_{ij}A_kA_k\right)\\
=&\partial_i(A_iA_j) - \frac{1}{2}\partial_j(A_kA_k)\\
=&A_j\partial_i(A_i) + A_i\partial_i(A_j) - A_k\partial_j(A_k)\\
=&A_j\partial_i(A_i) + A_i\partial_i(A_j) - A_i\partial_j(A_i)\\
=&A_j\partial_i(A_i) + A_i(\delta_{ki}\delta_{jl} - \delta_{jk}\delta il)\partial_k(A_l)\\
=&A_j\partial_i(A_i) + \varepsilon_{mij}A_i\varepsilon_{mkl}\partial_k(A_l)\\
=&A_j\partial_i(A_i) + \varepsilon_{jmi}(\varepsilon_{mkl}\partial_k(A_l))A_i
\end{aligned}$$

(e)

i.

$$\begin{aligned}
T_{ij}n_j =&\left(A_iA_j - \frac{1}{2}\delta_{ij}A^2\right)n_j\\
=&A_iA_jn_j - \frac{1}{2}A^2n_i
\end{aligned}$$

Therefore $T\cdot\vec{n}$ is a linear combination of $\vec{A}$ and $\vec{n}$

ii.

Let $B_i = T_{ij}n_j$

$$\begin{aligned}
B_iB_i =&\left(A_iA_jn_j - \frac{1}{2}A^2n_i\right)\left(A_iA_kn_k - \frac{1}{2}A^2n_i\right)\\
=&(A_jn_j)^2A^2 + \frac{1}{4}A^4 - (A_jn_j)^2A^2 = \qquad \frac{1}{4}A^4\\
B_in_i =&A_iA_jn_jn_i - \frac{1}{2}A^2\\
=&A^2\left(\cos^2\theta - \frac{1}{2}\right)\\
=&\frac{A^2}{2}\cos 2\theta\\
\cos\theta_{Bn} =&\cos 2\theta\\
\theta_{Bn} =&2\theta
\end{aligned}$$

iii.

See above.

3.

(a)

$$\begin{aligned}
E_r =& \frac{\lambda}{2\pi\varepsilon_0 r} - E_0\sin\phi\\
E_\phi =& -E_0\cos\phi
\end{aligned}$$

(b)

$$\begin{aligned}
E_r =& \frac{2\pi\varepsilon_0 E_0\lambda}{2\pi\varepsilon_0\lambda} - E_0\\
=& 0\\
E_\phi =& 0
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{d}\vec{F} =& \varepsilon_0 r\mathrm{d}\phi\mathrm{d}z\left(E_r\vec{E} - \frac{1}{2}E^2\hat{r}\right)\\
=& \varepsilon_0 r\mathrm{d}\phi\mathrm{d}z\left(\frac{E_r^2 - E_\phi^2}{2}\hat{r} + E_rE_\phi\hat{\phi}\right)
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{\mathrm{d}\vec{F}}{\mathrm{d}z} =& \int\varepsilon_0 r\mathrm{d}\phi\left(\frac{E_r^2 - E_\phi^2}{2}\hat{r} + E_rE_\phi\hat{\phi}\right)\\
=& \frac{\varepsilon_0 r}{2}\hat{y}\int\mathrm{d}\phi\left(\left(\left(\frac{\lambda}{2\pi\varepsilon_0 r} - E_0\sin\phi\right)^2 - E_0\cos^2\phi\right)\sin\phi - 2\left(\frac{\lambda}{2\pi\varepsilon_0 r} - E_0\sin\phi\right)E_0\cos^2\phi\right)\\
=& \frac{\varepsilon_0 r}{2}\hat{y}\int\mathrm{d}\phi\left(-2\frac{\lambda}{2\pi\varepsilon_0 r}E_0\sin^2\phi - \frac{\lambda}{\pi\varepsilon_0 r}E_0\cos^2\phi\right)\\
=& \frac{1}{2}\hat{y}\int\mathrm{d}\phi\left(-\frac{\lambda}{\pi}E_0\sin^2\phi - \frac{\lambda}{\pi}E_0\cos^2\phi\right)\\
=& -\frac{1}{2}\hat{y}2\pi\frac{\lambda}{\pi}E_0\\
=& -E_0\lambda\hat{y}
\end{aligned}$$

4.

(a)

$$
\begin{aligned}
B_r =& B_0 \sin\phi \\
B_\phi =& \frac{\mu_0 I}{2\pi r} + B_0 \cos\phi
\end{aligned}
$$

(b)

$$
\begin{aligned}
B_r =& 0 \\
B_\phi =& \frac{\mu_0 I 2\pi B_0}{2\pi\mu_0 I} + B_0 \\
=& 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathrm{d}\vec{F} =& \frac{1}{\mu_0} r\mathrm{d}\phi\mathrm{d}z\left(B_r\vec{B} - \frac{1}{2}B^2\hat{r}\right) \\
=& \frac{1}{\mu_0} r\mathrm{d}\phi\mathrm{d}z\left(\frac{B_r^2 - B_\phi^2}{2}\hat{r} + B_r B_\phi\hat{\phi}\right)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\frac{\mathrm{d}\vec{F}}{\mathrm{d}z} =& \frac{1}{2}\int\frac{r}{\mu_0}\mathrm{d}\phi\Big(\big(B_r^2 - B_\phi^2\big)\hat{r} + 2B_r B_\phi\hat{\phi}\Big) \\
=& \frac{r\hat{x}}{2\mu_0}\int\mathrm{d}\phi\left(\left(B_0^2\sin^2\phi - \left(\frac{\mu_0 I}{2\pi r} + B_0\cos\phi\right)^2\right)\cos\phi - 2B_0\sin\phi\left(\frac{\mu_0 I}{2\pi r} + B_0\cos\phi\right)\sin\phi\right) \\
=& \frac{r\hat{x}}{2\mu_0}\int\mathrm{d}\phi\left(-\left(\frac{\mu_0 I}{2\pi r} + B_0\cos\phi\right)^2\cos\phi - B_0\sin^2\phi\frac{\mu_0 I}{\pi r}\right) \\
=& \frac{\hat{x}}{2}\int\mathrm{d}\phi\left(-\frac{I}{\pi}B_0\cos^2\phi - B_0\sin^2\phi\frac{I}{\pi}\right) \\
=& -B_0 I\hat{x}
\end{aligned}
$$