1.

(a)

$$\begin{aligned}\frac{\mathrm{d}W_{rad}}{\mathrm{d}t'}=&\frac{e^2}{4\pi\varepsilon_0}\frac{2}{3c}\gamma^6\left(\Omega^2\beta^2-\Omega^2\beta^4\right)\\=&\frac{e^2}{4\pi\varepsilon_0}\frac{2}{3c}\gamma^4\Omega^2\beta^2\\=&\frac{e^2}{4\pi\varepsilon_0}\frac{2\omega_0^2}{3c}\gamma^2\beta^2\end{aligned}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t}mc^2=&-\frac{e^2}{4\pi\varepsilon_0}\frac{2\omega_0^2}{3c}\gamma^2\beta^2\\\frac{\mathrm{d}\gamma}{\mathrm{d}t}=&-\frac{e^2}{4\pi\varepsilon_0}\frac{2\omega_0^2}{3mc^3}\gamma^2\beta^2\\=&-\frac{2\omega_0^2r_e}{3c}\gamma^2\beta^2\\T_0=&\frac{3c}{2\omega_0^2r_e}\end{aligned}$$

(c)

For $\gamma\gg1$, $\beta\approx1$

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t}=&-\frac{\gamma^2}{T_0}\\\frac{1}{\gamma}=&\frac{1}{\gamma_0}+\frac{t}{T_0}\\T=&\frac{\gamma_0-\gamma}{\gamma\gamma_0}T_0\end{aligned}$$

(d)

$$\begin{aligned}\omega_{break}=&3\gamma_e^2\omega_0\\\gamma_e=&\sqrt{\frac{\omega_{break}}{3\omega_0}}\end{aligned}$$

(e)

$$\begin{aligned}T=&\frac{T_0}{\gamma}\\=&T_0\sqrt{\frac{3\omega_0}{\omega_{break}}}\end{aligned}$$

(f)

$$\begin{aligned}\omega_0 =& \frac{eB}{m}\\ =& 1.76\cdot 10^3\\ T =& \frac{3c}{2\omega_0^2 r_e}\sqrt{\frac{3\omega_0}{\omega_{break}}}\\ =& 5.16\cdot 10^{10}s\\ =& 1635\text{a}\end{aligned}$$

2.

(a)

$$\begin{aligned}E_r =& \begin{cases}\dfrac{2p_0\cos\theta}{4\pi\varepsilon_0 r^3} & (r>R)\\ E_0\cos\theta & (r<R)\end{cases}\\ E_\theta =& \begin{cases}\dfrac{p_0\sin\theta}{4\pi\varepsilon_0 r^3} & (r>R)\\ -E_0\sin\theta & (r<R)\end{cases}\\ E_0 =& -\frac{p_0}{4\pi\varepsilon_0 R^3}\\ \frac{\sigma_0}{\varepsilon_0} =& \frac{2p_0}{4\pi\varepsilon_0 R^3} - E_0\\ p_0 =& \frac{4\pi R^3\sigma_0}{3}\\ E_0 =& \frac{\sigma_0}{3\varepsilon_0}\end{aligned}$$

(b)

$$\begin{aligned}B_r =& \begin{cases}\dfrac{2\mu_0 m_0\cos\theta}{4\pi r^3} & (r>R)\\ B_0\cos\theta & (r<R)\end{cases}\\ B_\theta =& \begin{cases}\dfrac{\mu_0 m_0\sin\theta}{4\pi r^3} & (r>R)\\ -B_0\sin\theta & (r<R)\end{cases}\\ B_0 =& \frac{\mu_0 m_0}{2\pi R^3}\\ \mu_0\kappa_0 =& \frac{\mu_0 m_0}{4\pi R^3} + B_0\\ m_0 =& \frac{4\pi R^3\kappa_0}{3}\\ B_0 =& \frac{2\mu_0\kappa_0}{3}\end{aligned}$$

3.

(a)

$$
\begin{aligned}
E_i =& -\frac{p_j}{4\pi\varepsilon_0}\partial_i\frac{r_j}{r^3}\\
=&\frac{p_j}{4\pi\varepsilon_0}\partial_i\partial_j\frac{1}{r}
\end{aligned}
$$

Since $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$

$$
\begin{aligned}
\partial_i\frac{r_j}{r^3} =&\left(\frac{\delta_{ij}}{r^3} - 3\frac{x_ix_j}{r^5}\right) + \frac{4\pi}{3}\delta_{ij}\delta(\vec{r})\\
E_i =&\frac{1}{4\pi\varepsilon_0}\Big(3\frac{x_ix_jp_j}{r^5} - \frac{p_i}{r^3}\Big) - \frac{1}{3\varepsilon_0}p_i\delta(\vec{r})
\end{aligned}
$$

(b)

$$
\begin{aligned}
\nabla\times\left(\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}\right) =&\frac{\mu_0}{4\pi}\left(\vec{m}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right) - \vec{m}\cdot\left(\nabla\frac{\vec{r}}{r^3}\right)\right)\\
=&\frac{\mu_0}{4\pi}\left(\vec{m}4\pi\delta(\vec{r}) + 3\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3} - \frac{4\pi\vec{m}}{3}\delta(\vec{r})\right)\\
=&\frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right) + \frac{2\mu_0\vec{m}}{3}\delta(\vec{r})
\end{aligned}
$$

(c)

For electric field the integral of the field inside the sphere is

$$\int E = -\frac{\vec{p}_0}{3\varepsilon_0}$$

Therefore as $R\to 0$ the field becomes

$$E = -\delta(\vec{r})\frac{\vec{p}_0}{3\varepsilon_0}$$

For magnetic field the integral of the field inside the sphere is

$$\int B = \frac{2\mu_0\vec{m}_0}{3}$$

Therefore as $R\to 0$ the field becomes

$$B = \delta(\vec{r})\frac{2\mu_0\vec{m}_0}{3}$$

(d)

By symmetry, the field is along $\vec{p}$ for electric field

$$\begin{aligned}\overline{\vec{E}} =&\frac{3}{4\pi a^3}\frac{p^2}{4\pi\varepsilon_0}\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\sin\theta\mathrm{d}\theta\int_0^a\mathrm{d}r\frac{3\cos^2\theta-1}{r^3}\\ =&\frac{3}{4\pi a^3}\frac{p^2}{2\varepsilon_0}\int_{-1}^1\mathrm{d}\cos\theta\int_0^a\mathrm{d}r\frac{3\cos^2\theta-1}{r^3}\\ =&0\end{aligned}$$

Similarly

$$\overline{\vec{B}} =0$$

(e)

Since the average field generated by the normal part is 0, the sign of the field is determined by the δ part.

4.

(a)

$$\vec{E}(x,t) = -\hat{y}\frac{c\mu_0}{2}\int_{-\infty}^\infty\mathrm{d}x' J_0\exp\left(\mathrm{i}\left(kx'-\omega\left(t-\frac{|x-x'|}{c}\right)\right)\right)$$

(b)

$$\begin{aligned}\vec{E}(x,t) =& -\hat{y}\frac{c\mu_0J_0\mathrm{e}^{\mathrm{i}(kx-\omega t)}}{2}\left(\int_{-\infty}^0\mathrm{d}x''\exp\left(\mathrm{i}\left(k-\frac{\omega}{c}\right)x''\right)+\int_0^\infty\mathrm{d}x''\exp\left(\mathrm{i}\left(k+\frac{\omega}{c}\right)x''\right)\right)\\ =& -\hat{y}\frac{c\mu_0J_0\mathrm{e}^{\mathrm{i}(kx-\omega t)}}{2\mathrm{i}}\left(\frac{1}{k-\omega/c}-\frac{1}{k+\omega/c}\right)\\ =&\mathrm{i}\hat{y}c\mu_0J_0\mathrm{e}^{\mathrm{i}(kx-\omega t)}\frac{\omega/c}{k^2-\omega^2/c^2}\end{aligned}$$

(c)

$$\begin{aligned}J_{pol} =&\varepsilon_0(k_e-1)\omega\hat{y}c\mu_0J_0\mathrm{e}^{\mathrm{i}(kx-\omega t)}\frac{\omega/c}{k^2-\omega^2/c^2}\\ =&J_0\hat{y}\mathrm{e}^{\mathrm{i}(kx-\omega t)}\\ 1 =&\varepsilon_0(k_e-1)\omega c\mu_0\frac{\omega/c}{k^2-\omega^2/c^2}\end{aligned}$$

Let $v \equiv \frac{\omega}{k}$

$$\begin{aligned}
1 =& (k_e - 1)\frac{v^2}{c^2 - v^2} \\
\frac{c^2}{v^2} =& k_e \\
v =& \frac{c}{\sqrt{k_e}}
\end{aligned}$$