Yichao Yu
8.311 Problem Set 4
Recitation Section 1

1.

(a)

In x, y, z basis

$$\begin{aligned}
Q_{ij} =& \begin{pmatrix} -2d^2q & & \\ & -2d^2q & \\ & & 4d^2q \end{pmatrix}\\
=& 2d^2q \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}
\end{aligned}$$

Potential

$$\begin{aligned}
\Phi =& \frac{1}{8\pi\varepsilon_0}\frac{Q_{ij}x_ix_j}{r^5}\\
=& \frac{d^2q}{4\pi\varepsilon_0}\frac{-x^2-y^2+2z^2}{r^5}\\
=& \frac{d^2q}{4\pi\varepsilon_0}\frac{3\cos^2\theta-1}{r^3}
\end{aligned}$$

Electric field

$$\begin{aligned}
E_r =& \frac{3d^2q}{4\pi\varepsilon_0}\frac{3\cos^2\theta-1}{r^4}\\
E_\theta =& \frac{d^2q}{4\pi\varepsilon_0}\frac{3\sin 2\theta}{r^4}
\end{aligned}$$

Field line

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} =& r\frac{3\cos^2\theta-1}{\sin 2\theta}\\
\frac{\mathrm{d}r}{r} =& \frac{3\cos^2\theta-1}{\sin 2\theta}\mathrm{d}\theta\\
r =& R_0\sin\theta\sqrt{\cos\theta}
\end{aligned}$$

The field line is the furthest when $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 0$

$$\theta = \arccos\frac{1}{\sqrt{3}}$$

(b)

Magnetic field

$$\begin{aligned}\vec{B} =& -\frac{\mu_0}{24\pi c^2 r}\hat{r}\times\left(\hat{r}\cdot\dddot{Q}\right)\\
=& -\frac{\mu_0}{24\pi c^2 r}\hat{r}\times\left(\hat{r}\cdot\frac{\mathrm{d}^3 2d^2 q(3\hat{z}\hat{z}-1)}{\mathrm{d}t^3}\right)\\
=& -\frac{q\mu_0 d_0^2}{12\pi c^2 r}\hat{r}\times(3(\hat{r}\cdot\hat{z})\hat{z}-\hat{r})\frac{\mathrm{d}^3\cos^2\omega t}{\mathrm{d}t^3}\\
=&\frac{q\mu_0 d_0^2\omega^3}{\pi c^2 r}(\hat{r}\cdot\hat{z})(\hat{r}\times\hat{z})\sin 2\omega t\\
=& -\frac{q\mu_0 d_0^2\omega^3}{2\pi c^2 r}\sin 2\theta\sin 2\omega t\hat{e}_\phi\end{aligned}$$

Electric field

$$\begin{aligned}\vec{E} =& c\vec{B}\times\vec{r}\\
=& -\frac{q\mu_0 d_0^2\omega^3}{2\pi c r}\sin 2\theta\sin 2\omega t\hat{e}_\theta\end{aligned}$$

Power

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2 d_0^4\omega^6}{8\pi^2\varepsilon_0 c^5}\sin^2 2\theta$$

The radiation is the strongest for $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

$$\begin{aligned}P =& \int_0^\pi \mathrm{d}\theta\sin\theta\int_0^{2\pi}\mathrm{d}\phi\frac{q^2 d_0^4\omega^6}{8\pi^2\varepsilon_0 c^5}\sin^2 2\theta\\
=&\frac{4q^2 d_0^4\omega^6}{15\pi\varepsilon_0 c^5}\end{aligned}$$

The radiation appears at 2ω

(c)

The radiation will be a dipole radiation appears at ω with total power

$$P = \frac{q^2 d_0^2\omega^4}{12\pi\varepsilon_0 c^3}$$

Ratio of power

$$\begin{aligned}\frac{P_{quad}}{P} =& \frac{16 d_0^2\omega^2}{5c^2}\\
\ll& 1\end{aligned}$$

2.

(a)

$$
\begin{aligned}
\vec{B} =&\nabla \times A\\
=&\frac{\mu_0}{4\pi}\int \mathrm{d}^3\vec{r}'\nabla\times\left(\frac{1}{|\vec{r}-\vec{r}'|}\vec{j}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right)\\
=&\frac{\mu_0}{4\pi}\int \mathrm{d}^3\vec{r}'\frac{1}{|\vec{r}-\vec{r}'|}\nabla\times\vec{j}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)+\left(\nabla\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{j}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\\
=&\frac{\mu_0}{4\pi}\int \mathrm{d}^3\vec{r}'\vec{j}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}+\frac{\partial}{\partial t}\vec{j}\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\frac{\vec{r}-\vec{r}'}{c|\vec{r}-\vec{r}'|^2}
\end{aligned}
$$

(b)

The ratio of the two term is on the order of (replacing time derivative with T and replace $|\vec{r}-\vec{r}'|$ with d)

$$\frac{d}{cT}\ll 1$$

3.

(a)

$$
\begin{aligned}
P =&\frac{e^4E_0^2}{12\pi\varepsilon_0m_e^2c^3}\\
=&\langle S\rangle\frac{e^4}{6\pi\varepsilon_0^2m_e^2c^4}\\
=&\langle S\rangle\frac{8\pi}{3}r_e^2
\end{aligned}
$$

(b)

The radiation (and therefore the stress tensor) has mirror symmetry along each axis.

(c)

$$
\begin{aligned}
F =&\frac{P}{c}\\
=&\langle S\rangle\frac{8\pi}{3c}r_e^2
\end{aligned}
$$

4.

$$
\begin{aligned}
a_y =& -\frac{eE_0}{m_e}\cos\omega t - \frac{e^2}{6\pi\varepsilon_0 m_e c^3}\dot{a}_y \\
v_y =& -\frac{eE_0}{m_e\omega}\sin\omega t - \frac{e^2}{6\pi\varepsilon_0 m_e c^3}\dot{v}_y \\
\tilde{v}_y =& \mathrm{i}\frac{eE_0}{m_e\omega}\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{i}\omega\frac{e^2}{6\pi\varepsilon_0 m_e c^3}\tilde{v}_y \\
\tilde{v}_y =& \mathrm{i}\frac{eE_0}{m_e\omega}\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{i}\frac{2}{3}\omega\tau_e\tilde{v}_y \\
\tilde{v}_y =& \mathrm{i}\frac{eE_0}{m_e\omega}\frac{1}{1+\mathrm{i}\frac{2}{3}\omega\tau_e}\mathrm{e}^{\mathrm{i}\omega t}
\end{aligned}
$$

Force

$$
\begin{aligned}
\langle F\rangle =& \frac{1}{2}\Re\left(\mathrm{i}e\frac{eE_0}{m_e c\omega}\frac{1}{1+\mathrm{i}\frac{2}{3}\omega\tau_e}\mathrm{e}^{\mathrm{i}\omega t}E_0\mathrm{e}^{-\mathrm{i}\omega t}\right) \\
=& -\frac{1}{2}\Im\left(\frac{e^2E_0^2}{m_e c\omega}\frac{1}{1+\mathrm{i}\frac{2\omega r_e}{3c}}\right) \\
=& \frac{1}{2}\frac{e^2E_0^2}{m_e c\omega}\frac{\frac{2\omega r_e}{3c}}{1+\frac{4\omega^2 r_e^2}{9c^2}} \\
\approx& \frac{e^4E_0^2}{12\pi\varepsilon_0 m_e^2 c^4}
\end{aligned}
$$

5.

$$
\begin{aligned}
\frac{\sigma_T Lc}{4\pi} \leqslant& GMm_p \\
L \leqslant& \frac{4\pi GMm_p c}{\sigma_T} \\
=& 1.2\cdot 10^{31}J\cdot s^{-1} \qquad \text{(for the sun)}
\end{aligned}
$$

6.

(a)

First term

$$
\frac{1}{6\pi\varepsilon_0 c^2}\vec{a}\int \mathrm{d}^3x'\int \mathrm{d}^3x\frac{\rho\rho'}{R}
$$

Second term

$$
-\frac{1}{6\pi\varepsilon_0 c^3}\dot{\vec{a}}\int \mathrm{d}^3x'\int \mathrm{d}^3x\rho\rho'
$$

The problem is that the first term (EM momentum) diverge.

(b)

$$
\begin{aligned}
E =& \frac{Q}{\varepsilon_0 l^2} \\
U_E =& \frac{Q^2 d}{2\varepsilon_0 l^2}
\end{aligned}
$$

(c)

$$
\vec{B} = \frac{\mu_0 Q V}{l^2} \hat{z}
$$

(d)

$$
\begin{aligned}
\vec{P} =& \varepsilon_0 \frac{Q}{\varepsilon_0 l^2} \frac{\mu_0 Q V}{l^2} \hat{x} l^2 d \\
=& \frac{\mu_0 V Q^2 d}{l^2} \hat{x} \\
=& 2V \frac{U_e}{c^2} \hat{x}
\end{aligned}
$$

(e)

Field is pointing to $-x$ in the plate on $-y$ and $+x$ in the plate on $+y$ side.

$$
\begin{aligned}
E =& \frac{\mu_0 Q a d}{2 l^2} \\
F =& -\frac{\mu_0 Q^2 a d}{l^2} \hat{x} \\
=& -2a \frac{U_e}{c^2} \hat{x}
\end{aligned}
$$