1.

(a)

$$
\begin{aligned}
\vec{E}_{dipole} =& -\nabla\left(\frac{1}{4\pi\varepsilon_0}\frac{\vec{r}\cdot\vec{p}}{r^3}\right)\\
=& -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\vec{r}\cdot\vec{p}}{r^3}\right)\\
=& -\frac{\vec{r}\cdot\vec{p}}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r^3}\right)-\frac{1}{4\pi\varepsilon_0 r^3}\nabla(\vec{r}\cdot\vec{p})\\
=&\frac{\vec{r}\cdot\vec{p}}{4\pi\varepsilon_0}\frac{3\hat{n}}{r^4}-\frac{\vec{p}}{4\pi\varepsilon_0 r^3}\\
=&\frac{3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\int\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})\mathrm{d}^3x\\
=&\frac{1}{2}\int\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})\mathrm{d}^3x+\frac{1}{2}\int\left(\hat{n}\times\left(\vec{J}(\vec{r}')\times\vec{r}'\right)-\vec{r}'\left(\hat{n}\cdot\vec{J}(\vec{r}')\right)\right)\mathrm{d}^3x\\
=&\hat{n}\times\frac{1}{2}\int\vec{J}(\vec{r}')\times\vec{r}'\mathrm{d}^3x+\frac{1}{2}\int\left(\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})-\vec{r}'\left(\hat{n}\cdot\vec{J}(\vec{r}')\right)\right)\mathrm{d}^3x
\end{aligned}
$$

For arbitrary vector $\vec{l}$

$$
\begin{aligned}
&\nabla'\cdot\left(\left(\vec{l}\cdot\vec{r}'\right)(\hat{n}\cdot\vec{r}')\vec{J}(r')\right)\\
=&(\hat{n}\cdot\vec{r}')\vec{J}(r')\cdot\nabla'\left(\vec{l}\cdot\vec{r}'\right)+\left(\vec{l}\cdot\vec{r}'\right)\vec{J}(r')\cdot\nabla'(\hat{n}\cdot\vec{r}')+\left(\vec{l}\cdot\vec{r}'\right)(\hat{n}\cdot\vec{r}')\nabla'\cdot\vec{J}(r')\\
=&(\hat{n}\cdot\vec{r}')\vec{J}(r')\cdot\vec{l}+\left(\vec{l}\cdot\vec{r}'\right)\vec{J}(r')\cdot\hat{n}\\
=&\vec{l}\cdot\left((\hat{n}\cdot\vec{r}')\vec{J}(r')+\vec{r}'\vec{J}(r')\cdot\hat{n}\right)
\end{aligned}
$$

Integrate both sides

$$
\begin{aligned}
0 =&\vec{l}\cdot\int\mathrm{d}^3x'\left((\hat{n}\cdot\vec{r}')\vec{J}(r')+\vec{r}'\vec{J}(r')\cdot\hat{n}\right)\\
0 =&\int\mathrm{d}^3x'\left((\hat{n}\cdot\vec{r}')\vec{J}(r')+\vec{r}'\vec{J}(r')\cdot\hat{n}\right)\\
&\int\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})\mathrm{d}^3x\\
=&\vec{m}\times\vec{n}
\end{aligned}
$$

(c)

$$\begin{aligned}\frac{4\pi}{\mu_0}\vec{B}_{dipole} =&\nabla\times\frac{\vec{m}\times\vec{r}}{r^3}\\ =&\vec{m}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)-(\vec{m}\cdot\nabla)\frac{\vec{r}}{r^3}\\ =&-\vec{r}\left(\vec{m}\cdot\nabla\frac{1}{r^3}\right)-\frac{1}{r^3}(\vec{m}\cdot\nabla)\vec{r}\\ =&\vec{r}\frac{3\vec{m}\cdot\hat{n}}{r^4}-\frac{\vec{m}}{r^3}\\ =&\frac{3\hat{n}(\vec{m}\cdot\hat{n})-\vec{m}}{r^3}\end{aligned}$$

2.

(a)

$$\begin{aligned}E =&\int_R^\infty \mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \sin\theta\mathrm{d}\theta\frac{p^2}{32\pi^2\varepsilon_0 r^4}\big(4\cos^2\theta+\sin^2\theta\big)\\ =&\frac{p^2}{16\pi\varepsilon_0}\int_R^\infty\frac{\mathrm{d}r}{r^4}\int_0^\pi \mathrm{d}\theta\sin\theta\big(4\cos^2\theta+\sin^2\theta\big)\\ =&\frac{p^2}{12\pi\varepsilon_0 R^3}\end{aligned}$$

(b)

Direvatives of the dipole moment

$$\begin{aligned}\dot{\vec{p}} =&\dot{p}\hat{z}\\ \ddot{\vec{p}} =&\ddot{p}\hat{z}\end{aligned}$$

Magnetic field

$$\begin{aligned}\vec{B} =&\frac{\hat{z}\times\hat{n}}{4\pi\varepsilon_0}\left(\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{cr}\right)\\ =&\frac{\sin\theta\hat{\phi}}{4\pi\varepsilon_0}\left(\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{cr}\right)\end{aligned}$$

Electric field

$$\begin{aligned}\vec{E} =&\frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{4\pi\varepsilon_0 r^3}+\frac{3\hat{n}\left(\dot{\vec{p}}\cdot\hat{n}\right)-\dot{\vec{p}}}{4\pi\varepsilon_0 cr^2}+\frac{\left(\ddot{\vec{p}}\times\hat{n}\right)\times\hat{n}}{4\pi\varepsilon_0 c^2 r}\\ =&\frac{3p\cos\theta\hat{n}-p\hat{z}}{4\pi\varepsilon_0 r^3}+\frac{3\dot{p}\cos\theta\hat{n}-\dot{p}\hat{z}}{4\pi\varepsilon_0 cr^2}+\frac{\ddot{p}\sin\theta\hat{\phi}\times\hat{n}}{4\pi\varepsilon_0 c^2 r}\\ =&\frac{3p\cos\theta\hat{n}-p\hat{z}}{4\pi\varepsilon_0 r^3}+\frac{3\dot{p}\cos\theta\hat{n}-\dot{p}\hat{z}}{4\pi\varepsilon_0 cr^2}+\frac{\ddot{p}\sin\theta\hat{\theta}}{4\pi\varepsilon_0 c^2 r}\end{aligned}$$

Since $\hat{z} = \cos\theta\hat{n} - \sin\theta\hat{\theta}$

$$\begin{aligned}\vec{E} =& \frac{2p\cos\theta\hat{n} + p\sin\theta\hat{\theta}}{4\pi\varepsilon_0 r^3} + \frac{2\dot{p}\cos\theta\hat{n} + \dot{p}\sin\theta\hat{\theta}}{4\pi\varepsilon_0 c r^2} + \frac{\ddot{p}\sin\theta\hat{\theta}}{4\pi\varepsilon_0 c^2 r}\\ =& \frac{2\cos\theta\hat{n}}{4\pi\varepsilon_0}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2}\right) + \frac{\sin\theta\hat{\theta}}{4\pi\varepsilon_0}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)\end{aligned}$$

Energy flux

$$\begin{aligned}\Phi_E =& \frac{1}{\mu_0 c^2}\int \mathrm{d}t \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta \sin\theta r^2 \frac{\sin\theta}{4\pi\varepsilon_0}\left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr}\right)\frac{\sin\theta}{4\pi\varepsilon_0}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)\\ =& \frac{1}{6\pi\varepsilon_0}\int \mathrm{d}t\left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c}\right)\left(\frac{p}{r^2} + \frac{\dot{p}}{cr} + \frac{\ddot{p}}{c^2}\right)\\ =& \frac{1}{6\pi\varepsilon_0}\left(\frac{1}{2r}\left(\frac{p}{r} + \frac{\dot{p}}{c}\right)^2\right)\Bigg|_{t_0}^{t_1} + \frac{1}{6\pi\mu_0\varepsilon_0^2}\int \mathrm{d}t\left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c}\right)\left(\frac{\ddot{p}}{c^2}\right)\\ =& \frac{p_2^2 - p_1^2}{12\pi\varepsilon_0 r^3} + \int \frac{\ddot{p}^2}{6\pi\mu_0\varepsilon_0^2 c^3}\mathrm{d}t\end{aligned}$$

The first term corresponds to change in the energy stored in the field.

3.

(a)

$$r = \frac{mV_0}{qB_0}$$

(b)

$$T = \frac{\pi m}{qB}$$

(c)

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} =& \frac{q^4 V_0^2 B^2}{6\pi\varepsilon_0 m^2 c^3}\\ =& \frac{q^2 V_0^4}{6\pi\varepsilon_0 c^3 R^2}\end{aligned}$$

(d)

$$\begin{aligned}W =& \frac{q^2 V_0^3}{6\varepsilon_0 c^3 R}\\ =& \frac{2\pi m V_0^3}{3cR}\end{aligned}$$

(e)

$$\frac{W}{E_k} = \frac{2\pi V_0 R_{classical}}{3cR}$$

When R is large.

(f)

4.

(a)

$$\begin{aligned} p =& Q_0 d \sin \omega t \\ \left|\frac{\mathrm{d}W}{\mathrm{d}t}\right| =& \frac{Q_0^2 d^2 \omega^4}{6\pi\varepsilon_0 c^3} \left|\sin^2 \omega t\right| \\ =& \frac{Q_0^2 d^2 \omega^4}{12\pi\varepsilon_0 c^3} \end{aligned}$$

(b)

$$\begin{aligned} E_{rad} =& \frac{Q_0^2 d^2 \omega^4}{12\pi\varepsilon_0 c^3} \frac{2\pi}{\omega} \\ =& \frac{Q_0^2 d^2 \omega^3}{6\varepsilon_0 c^3} \\ \frac{4CE_{rad}}{Q_0^2} =& \frac{2d^2 C\omega^3}{3\varepsilon_0 c^3} \\ =& \frac{2dAk^3}{3} \end{aligned}$$

Therefore if dk and Ak^2 are all small (where k is the wave vector) the radiation is small.

(c)

$$\begin{aligned} R_{rad} =& \frac{Q_0^2 d^2 \omega^4}{12\pi\varepsilon_0 c^3} \frac{2}{\omega^2 Q_0^2} \\ =& \frac{d^2 \omega^2}{6\pi\varepsilon_0 c^3} \end{aligned}$$

(d)

$$\begin{aligned}
R_{rad} =&\frac{d^2}{6\pi\varepsilon_0 c^3 LC}\\
=&\frac{hd^3}{6\pi\varepsilon_0 c^3\varepsilon_0 A_c\mu_0 N^2 A_L}\\
=&\mu_0 c\frac{hd^3}{6\pi A_c N^2 A_L}
\end{aligned}$$

5.

(a)

$$\begin{aligned}
\vec{B}_\perp =&\frac{\mu_0}{4\pi}\left(\frac{1}{cr^2}\left(3\hat{n}\left(\dot{\vec{m}}\cdot\hat{n}\right)-\dot{\vec{m}}\right)+\frac{1}{rc^2}\left(\ddot{\vec{m}}\times\hat{n}\right)\times\hat{n}\right)\\
=&\frac{\mu_0 m_0\omega_0}{4\pi rc}\left(\frac{1}{r}(3\hat{n}((\cos\omega_0 t\hat{y}-\sin\omega_0 t\hat{x})\cdot\hat{n})-(\cos\omega_0 t\hat{y}-\sin\omega_0 t\hat{x}))\right.\\
&\left.-\frac{\omega_0}{c}((\cos\omega_0 t\hat{x}+\sin\omega_0 t\hat{y})\times\hat{n})\times\hat{n}\right)\\
=&\frac{\mu_0 m_0\omega_0}{4\pi rc}\left(\frac{2\hat{e}_r}{r}(\cos\omega_0 t\sin\theta\sin\phi-\sin\omega_0 t\sin\theta\cos\phi)\right.\\
&-\frac{1}{r}\cos\omega_0 t(\cos\theta\sin\phi\hat{e}_\theta+\cos\phi\hat{e}_\phi)+\frac{1}{r}\sin\omega_0 t(\cos\theta\cos\phi\hat{e}_\theta-\sin\phi\hat{e}_\phi)\\
&\left.-\frac{\omega_0}{c}(\cos\omega_0 t(-\cos\theta\cos\phi\hat{e}_\phi-\sin\phi\hat{e}_\theta)+\sin\omega_0 t(-\cos\theta\sin\phi\hat{e}_\phi+\cos\phi\hat{e}_\theta))\times\hat{e}_r\right)\\
=&\frac{\mu_0 m_0\omega_0}{4\pi rc}\left(\frac{2\hat{e}_r\sin\theta\sin(\phi-\omega_0 t)-\cos\theta\sin(\phi-\omega_0 t)\hat{e}_\theta-\cos(\phi-\omega_0 t)\hat{e}_\phi}{r}\right.\\
&\left.+\frac{\omega_0}{c}(\cos\theta\cos(\phi-\omega_0 t)\hat{e}_\theta-\sin(\phi-\omega_0 t)\hat{e}_\phi)\right)
\end{aligned}$$

$$\begin{aligned}
\vec{E} =&-\frac{\mu_0}{4\pi}\left(\frac{\ddot{\vec{m}}}{cr}+\frac{\dot{\vec{m}}}{r^2}\right)\times\hat{e}_r\\
=&\frac{\mu_0 m_0}{4\pi r}\left(\frac{\omega_0^2}{c}(\cos\theta\cos(\phi-\omega_0 t)\hat{e}_\theta-\sin(\phi-\omega_0 t)\hat{e}_\phi)-\frac{\omega_0}{r}(\cos\theta\sin(\phi-\omega_0 t)\hat{e}_\theta+\cos(\phi-\omega_0 t)\hat{e}_\phi)\right)\times\hat{e}_r\\
=&\frac{\mu_0 m_0\omega_0}{4\pi r}\left(-\frac{\omega_0}{c}(\cos\theta\cos(\phi-\omega_0 t)\hat{e}_\phi+\sin(\phi-\omega_0 t)\hat{e}_\theta)+\frac{1}{r}(\cos\theta\sin(\phi-\omega_0 t)\hat{e}_\phi-\cos(\phi-\omega_0 t)\hat{e}_\theta)\right)
\end{aligned}$$

(b)

For radiation part

$$\vec{E}_{rad} = -\frac{\mu_0 m_0\omega_0^2}{4\pi rc}(\cos\theta\cos(\phi-\omega_0 t)\hat{e}_\phi+\sin(\phi-\omega_0 t)\hat{e}_\theta)$$

Helicity is positive for $\theta > 0$ and negative for $\theta < 0$
Ellipticity is $|\cos\theta|$

(c)

$$
\begin{aligned}
\frac{\mathrm{d}W_{rad}}{\mathrm{d}\Omega\mathrm{d}t} =& \frac{1}{\mu_0 c}\frac{\mu_0^2 m_0^2\omega_0^4}{16\pi^2c^2}\frac{1}{2}\left(\cos^2\theta+1\right)\\
=&\frac{\mu_0 m_0^2\omega_0^4\left(\cos^2\theta+1\right)}{32\pi^2c^3}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\left\langle\frac{\mathrm{d}W}{\mathrm{d}t}\right\rangle =& \frac{\mu_0 m_0^2\omega_0^4}{32\pi^2c^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\left(\cos^2\theta+1\right)\\
=&\frac{\mu_0 m_0^2\omega_0^4}{6\pi c^3}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\Omega} =& -r^3\hat{n}\times\left(\left(\varepsilon_0\vec{E}\vec{E}+\frac{1}{\mu_0}\vec{B}\vec{B}\right)\cdot\hat{n}\right)\\
=& -r^3\left(\varepsilon_0\hat{n}\times\vec{E}\vec{E}\cdot\hat{n}+\frac{1}{\mu_0}\hat{n}\times\vec{B}\vec{B}\cdot\hat{n}\right)\\
=& -\frac{m_0\omega_0 r}{2\pi c}\sin\theta\sin\left(\phi-\omega_0t\right)\hat{n}\times\vec{B}
\end{aligned}
$$

Ignoring the terms that vanishes for large r

$$
\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\Omega} = -\frac{\mu_0 m_0^2\omega_0^3}{8\pi^2c^3}\sin\theta\sin\left(\phi-\omega_0t\right)(\cos\theta\cos\left(\phi-\omega_0t\right)\hat{e}_\phi+\sin\left(\phi-\omega_0t\right)\hat{e}_\theta)
$$

Time averaging

$$
\left\langle\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\Omega}\right\rangle = -\frac{\mu_0 m_0^2\omega_0^3}{16\pi^2c^3}\sin\theta\hat{e}_\theta
$$

(f)

After the angular integral, only the z component can be not zero

$$
\begin{aligned}
\left\langle\frac{\mathrm{d}W}{\mathrm{d}\Omega}\right\rangle =& \hat{z}\frac{\mu_0 m_0^2\omega_0^3}{16\pi^2c^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\sin^2\theta\\
=&\hat{z}\frac{\mu_0 m_0^2\omega_0^3}{6\pi c^3}
\end{aligned}
$$

6.

(a)

$$\begin{aligned}\left\langle\frac{\mathrm{d}W}{\mathrm{d}t}\right\rangle =&\frac{\mu_0 m_0^2\sin^2\alpha\omega_0^4}{6\pi c^3}\\ B_p =&\frac{\mu_0 m_0}{2\pi R^3}\\ \left\langle\frac{\mathrm{d}W}{\mathrm{d}t}\right\rangle =&\frac{2\pi\sin^2\alpha\omega_0^4 B_p^2 R^6}{3\mu_0 c^3}\\ =&\frac{32\pi^5\sin^2\alpha B_p^2 R^6}{3\mu_0 c^3 P^4}\end{aligned}$$

(b)

$$\begin{aligned}\frac{32\pi^5\sin^2\alpha B_p^2 R^6}{3\mu_0 c^3 P^4} =&\frac{8\pi^2}{5}MR^2\frac{\dot{P}}{P^3}\\ B_p^2 =&\frac{3\mu_0 c^3 P\dot{P}M}{20\pi^3\sin^2\alpha R^4}\end{aligned}$$

(c)

$$B_p \approx 7\cdot 10^8 T$$