1.

(a)

Expand the field in derivatives in x and y

$$\vec{E} = \left(\vec{E}_0 + \vec{E}_1\right)e^{ikz-i\omega t} + \cdots$$

$$\vec{B} = \left(\vec{B}_0 + \vec{B}_1\right)e^{ikz-i\omega t} + \cdots$$

where $\vec{E}_0 = E_0(\hat{x} \pm i\hat{y})$, Since

$$\nabla \cdot \vec{E} = 0$$

$$0 = \nabla \cdot \vec{E}_0 e^{ikz-i\omega t} + \nabla e^{ikz-i\omega t} \cdot \left(\vec{E}_0 + \vec{E}_1\right)$$

$$= \left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)e^{ikz-i\omega t} + ike^{ikz-i\omega t}E_{z_1}$$

$$E_{z_1} = \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)$$

$$\vec{E} = \left(E_0(\hat{x} \pm i\hat{y}) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{z}\right)e^{ikz-i\omega t}$$

B field

$$\vec{B} = -\frac{i}{\omega}\nabla \times \vec{E}$$

$$= -\frac{i}{\omega}\left(ik\hat{z} \times \left(\vec{E}_0 + \vec{E}_1\right)e^{ikz-i\omega t} + (\nabla E_0) \times (\hat{x} \pm i\hat{y})e^{ikz-i\omega t}\right)$$

$$= \frac{k}{\omega}\left(E_0(\hat{y} \mp i\hat{x}) \pm \frac{1}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{z}\right)e^{ikz-i\omega t}$$

$$= \mp i\frac{k}{\omega}\left(E_0(\hat{x} \pm i\hat{y}) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{z}\right)e^{ikz-i\omega t}$$

$$= \mp i\sqrt{\mu\varepsilon}\vec{E}$$

(b)

Angular momentum density

$$\vec{l} = \varepsilon_0 \vec{r} \times \left(\vec{E} \times \vec{B} \right)$$

$$\langle l_z \rangle = \frac{\hat{z}}{2} \cdot \varepsilon_0 \vec{r} \times \Re \left(\vec{E} \times \vec{B}^* \right)$$

$$= \frac{\hat{z}}{2} \cdot \varepsilon_0 \vec{r} \times \Re \left(\vec{E}_0 \times \vec{B}_1^* + \vec{E}_1 \times \vec{B}_0^* \right)$$

$$= \mp \frac{\hat{z}}{2} \cdot \sqrt{\mu_0 \varepsilon_0} \varepsilon_0 \vec{r} \times \Re \left(i \left(\vec{E}_0 \times \vec{E}_1^* - \vec{E}_0^* \times \vec{E}_1 \right) \right)$$

$$= \pm \hat{z} \cdot \sqrt{\mu_0 \varepsilon_0} \varepsilon_0 \vec{r} \times \Im \left(\vec{E}_0 \times \vec{E}_1^* \right)$$

$$= \pm \hat{z} \cdot \sqrt{\mu_0 \varepsilon_0} \varepsilon_0 \vec{r} \times \Im \left(E_0 (\hat{x} \pm i\hat{y}) \times \frac{-i}{k} \left(\frac{\partial E_0}{\partial x} \mp i \frac{\partial E_0}{\partial y} \right) \hat{z} \right)$$

$$= \mp \frac{\varepsilon_0 E_0}{\omega} \Re \left((-x \mp iy) \left(\frac{\partial E_0}{\partial x} \mp i \frac{\partial E_0}{\partial y} \right) \right)$$

$$= \pm \frac{\varepsilon_0 E_0}{\omega} \left(x \frac{\partial E_0}{\partial x} + y \frac{\partial E_0}{\partial y} \right)$$

$$= \pm \frac{\varepsilon_0}{2\omega} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) E_0^2$$

1D density

$$\langle L_z \rangle = \pm \int d\sigma \frac{1}{2\omega} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) u$$

$$= \pm \int d\sigma \frac{u}{\omega}$$

$$= \pm \frac{U}{\omega}$$

This means a circularly polarized photon of energy $\hbar\omega$ has angular momentum along it's propagation direction of $\pm\hbar$. The transverse component of angular momentum vanish because of symmetry.

2.

(a)

(b)