

1.

(a)

$$\begin{aligned}
 \vec{E}_{dipole} &= -\nabla \left(\frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} \right) \\
 &= -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\vec{r} \cdot \vec{p}}{r^3} \right) \\
 &= -\frac{\vec{r} \cdot \vec{p}}{4\pi\epsilon_0} \nabla \left(\frac{1}{r^3} \right) - \frac{1}{4\pi\epsilon_0 r^3} \nabla(\vec{r} \cdot \vec{p}) \\
 &= \frac{\vec{r} \cdot \vec{p}}{4\pi\epsilon_0} \frac{3\hat{n}}{r^4} - \frac{\vec{p}}{4\pi\epsilon_0 r^3} \\
 &= \frac{3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}}{4\pi\epsilon_0 r^3}
 \end{aligned}$$

(b)

$$\begin{aligned}
 &\int \vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) d^3x \\
 &= \frac{1}{2} \int \vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) d^3x + \frac{1}{2} \int \left(\hat{n} \times (\vec{J}(\vec{r}') \times \vec{r}') - \vec{r}' (\hat{n} \cdot \vec{J}(\vec{r}')) \right) d^3x \\
 &= \hat{n} \times \frac{1}{2} \int \vec{J}(\vec{r}') \times \vec{r}' d^3x + \frac{1}{2} \int \left(\vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) - \vec{r}' (\hat{n} \cdot \vec{J}(\vec{r}')) \right) d^3x
 \end{aligned}$$

For arbitrary vector \vec{l}

$$\begin{aligned}
 &\nabla' \cdot \left((\vec{l} \cdot \vec{r}')(\hat{n} \cdot \vec{r}')\vec{J}(r') \right) \\
 &= (\hat{n} \cdot \vec{r}')\vec{J}(r') \cdot \nabla'(\vec{l} \cdot \vec{r}') + (\vec{l} \cdot \vec{r}')\vec{J}(r') \cdot \nabla'(\hat{n} \cdot \vec{r}') + (\vec{l} \cdot \vec{r}')(\hat{n} \cdot \vec{r}')\nabla' \cdot \vec{J}(r') \\
 &= (\hat{n} \cdot \vec{r}')\vec{J}(r') \cdot \vec{l} + (\vec{l} \cdot \vec{r}')\vec{J}(r') \cdot \hat{n} \\
 &= \vec{l} \cdot \left((\hat{n} \cdot \vec{r}')\vec{J}(r') + \vec{r}'\vec{J}(r') \cdot \hat{n} \right)
 \end{aligned}$$

Integrate both sides

$$\begin{aligned}
 0 &= \vec{l} \cdot \int d^3x' \left((\hat{n} \cdot \vec{r}')\vec{J}(r') + \vec{r}'\vec{J}(r') \cdot \hat{n} \right) \\
 0 &= \int d^3x' \left((\hat{n} \cdot \vec{r}')\vec{J}(r') + \vec{r}'\vec{J}(r') \cdot \hat{n} \right) \\
 &\quad \int \vec{J}(\vec{r}')(\vec{r}' \cdot \hat{n}) d^3x \\
 &= \vec{m} \times \vec{n}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \frac{4\pi}{\mu_0} \vec{B}_{dipole} &= \nabla \times \frac{\vec{m} \times \vec{r}}{r^3} \\
 &= \vec{m} \left(\nabla \cdot \frac{\vec{r}}{r^3} \right) - (\vec{m} \cdot \nabla) \frac{\vec{r}}{r^3} \\
 &= -\vec{r} \left(\vec{m} \cdot \nabla \frac{1}{r^3} \right) - \frac{1}{r^3} (\vec{m} \cdot \nabla) \vec{r} \\
 &= \vec{r} \frac{3\vec{m} \cdot \hat{n}}{r^4} - \frac{\vec{m}}{r^3} \\
 &= \frac{3\hat{n}(\vec{m} \cdot \hat{n}) - \vec{m}}{r^3}
 \end{aligned}$$

2.

(a)

$$\begin{aligned}
 E &= \int_R^\infty dr \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \frac{p^2}{32\pi^2 \epsilon_0 r^4} (4 \cos^2 \theta + \sin^2 \theta) \\
 &= \frac{p^2}{16\pi \epsilon_0} \int_R^\infty \frac{dr}{r^4} \int_0^\pi d\theta \sin \theta (4 \cos^2 \theta + \sin^2 \theta) \\
 &= \frac{p^2}{12\pi \epsilon_0 R^3}
 \end{aligned}$$

(b)

Derivatives of the dipole moment

$$\begin{aligned}
 \dot{\vec{p}} &= \dot{p} \hat{z} \\
 \ddot{\vec{p}} &= \ddot{p} \hat{z}
 \end{aligned}$$

Magnetic field

$$\begin{aligned}
 \vec{B} &= \frac{\hat{z} \times \hat{n}}{4\pi \epsilon_0} \left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr} \right) \\
 &= \frac{\sin \theta \hat{\phi}}{4\pi \epsilon_0} \left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr} \right)
 \end{aligned}$$

Electric field

$$\begin{aligned}
 \vec{E} &= \frac{3\hat{n}(\vec{p} \cdot \hat{n}) - \vec{p}}{4\pi \epsilon_0 r^3} + \frac{3\hat{n}(\dot{\vec{p}} \cdot \hat{n}) - \dot{\vec{p}}}{4\pi \epsilon_0 cr^2} + \frac{(\ddot{\vec{p}} \times \hat{n}) \times \hat{n}}{4\pi \epsilon_0 c^2 r} \\
 &= \frac{3p \cos \theta \hat{n} - p \hat{z}}{4\pi \epsilon_0 r^3} + \frac{3\dot{p} \cos \theta \hat{n} - \dot{p} \hat{z}}{4\pi \epsilon_0 cr^2} + \frac{\ddot{p} \sin \theta \hat{\phi} \times \hat{n}}{4\pi \epsilon_0 c^2 r} \\
 &= \frac{3p \cos \theta \hat{n} - p \hat{z}}{4\pi \epsilon_0 r^3} + \frac{3\dot{p} \cos \theta \hat{n} - \dot{p} \hat{z}}{4\pi \epsilon_0 cr^2} + \frac{\ddot{p} \sin \theta \hat{\theta}}{4\pi \epsilon_0 c^2 r}
 \end{aligned}$$

Since $\hat{z} = \cos \theta \hat{n} - \sin \theta \hat{\theta}$

$$\begin{aligned}\vec{E} &= \frac{2p \cos \theta \hat{n} + p \sin \theta \hat{\theta}}{4\pi\epsilon_0 r^3} + \frac{2\dot{p} \cos \theta \hat{n} + \dot{p} \sin \theta \hat{\theta}}{4\pi\epsilon_0 cr^2} + \frac{\ddot{p} \sin \theta \hat{\theta}}{4\pi\epsilon_0 c^2 r} \\ &= \frac{2 \cos \theta \hat{n}}{4\pi\epsilon_0} \left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} \right) + \frac{\sin \theta \hat{\theta}}{4\pi\epsilon_0} \left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r} \right)\end{aligned}$$

Energy flux

$$\begin{aligned}\Phi_E &= \frac{1}{\mu_0 c^2} \int dt \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta r^2 \frac{\sin \theta}{4\pi\epsilon_0} \left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr} \right) \frac{\sin \theta}{4\pi\epsilon_0} \left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r} \right) \\ &= \frac{1}{6\pi\epsilon_0} \int dt \left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c} \right) \left(\frac{p}{r^2} + \frac{\dot{p}}{cr} + \frac{\ddot{p}}{c^2} \right) \\ &= \frac{1}{6\pi\epsilon_0} \left(\frac{1}{2r} \left(\frac{p}{r} + \frac{\dot{p}}{c} \right)^2 \right)_{t_0}^{t_1} + \frac{1}{6\pi\mu_0\epsilon_0^2} \int dt \left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c} \right) \left(\frac{\ddot{p}}{c^2} \right) \\ &= \frac{p_2^2 - p_1^2}{12\pi\epsilon_0 r^3} + \int \frac{\ddot{p}^2}{6\pi\mu_0\epsilon_0^2 c^3} dt\end{aligned}$$

The first term corresponds to change in the energy stored in the field.

3.

- (a)
- (b)
- (c)
- (d)
- (e)
- (f)

4.

- (a)
- (b)
- (c)
- (d)

5.

- (a)
- (b)
- (c)
- (d)
- (e)
- (f)

6.

- (a)
- (b)
- (c)