Solutions Assignment #8: Due Friday April 17, 2015 at 2:30 pm

Problems

There are only three problems in this problem set. Each one is worth 20 points.

Problem 8-1: Attenuation in a Wave Guide

In class we have been focusing on the $TE_{1,0}$ mode of a rectangular wave-guide of dimension $a > b$ (see equation (8.46) of Jackson page 362). For this mode, evaluate the Poynting flux $P_{1,0}$, the value of $dP_{1,0} / dz$, and thus the attenuation constant $\beta_{1,0} = -\dfrac{1}{2P_{1,0}} \dfrac{dP_{1,0}}{dz}$ (see equations (8.59) and (8.63) Jackson p. 865). Plot $\beta_{1,0}$ versus $\omega / \omega_{1,0}$ for this mode. At what value of the frequency ω is $\beta_{1,0}$ a minimum?

Our mode is given by

$$B_z(x,y) = B_o \cos\left(\frac{\pi x}{a}\right) \quad \text{with} \quad \gamma_{10} = \frac{\pi}{a}$$

$$\omega_{10} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{\pi}{a} \qquad k = \omega\sqrt{\mu\varepsilon}\left(1 - \omega^2 / \omega_{10}^2\right)^{1/2}$$

Using Jackson (8.26) page 358, we have

$$\mathbf{E}_t = \hat{\mathbf{y}} \frac{ia\omega}{\pi} B_o \sin\left(\frac{\pi x}{a}\right) \quad \text{and} \quad \mathbf{B} = \hat{\mathbf{z}} B_o \cos\left(\frac{\pi x}{a}\right) - \hat{\mathbf{x}} \frac{iak}{\pi} B_o \sin\left(\frac{\pi x}{a}\right)$$

We are going to take the real part of the above expressions, assuming an over all $e^{ikz - i\omega t}$ time dependence. This gives

$$\mathbf{E}_t = -\hat{\mathbf{y}} \frac{a\omega}{\pi} B_o \sin\left(\frac{\pi x}{a}\right) \sin(kz - \omega t)$$

$$\mathbf{B} = \hat{\mathbf{z}} B_o \cos\left(\frac{\pi x}{a}\right) \cos(kz - \omega t) + \hat{\mathbf{x}} B_o \frac{ak}{\pi} \sin\left(\frac{\pi x}{a}\right) \sin(kz - \omega t)$$

With these expressions we can calculate the Poynting vector as

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu} = \frac{1}{\mu}\left[\frac{a\omega}{\pi} B_o \sin\left(\frac{\pi x}{a}\right) \sin(kz - \omega t)\right] \begin{bmatrix} -\hat{\mathbf{x}} B_o \cos\left(\dfrac{\pi x}{a}\right) \cos(kz - \omega t) \\ +\hat{\mathbf{z}} B_o \dfrac{ak}{\pi} \sin\left(\dfrac{\pi x}{a}\right) \sin(kz - \omega t) \end{bmatrix}$$

Averaging over one period of the wave gives

$$\langle \mathbf{S} \rangle = \hat{\mathbf{z}} \frac{a^2 \omega k}{\pi^2} \sin^2\left(\frac{\pi x}{a}\right) \frac{B_o^2}{2\mu}$$

Thus we have

$$P = \int_0^b dy \int_0^a dx\, \hat{\mathbf{z}} \cdot \langle \mathbf{S} \rangle = \int_0^b dy \int_0^a dx\ \frac{a^2 \omega k}{\pi^2} \sin^2\left(\frac{\pi x}{a}\right) \frac{B_o^2}{2\mu} = \frac{a^2 \omega k}{\pi^2} \frac{B_o^2}{2\mu} \frac{ba}{2}$$

which we can rewrite as

$$P = \frac{1}{\sqrt{\mu\varepsilon}} \left(\frac{\omega^2}{\omega_{10}^2}\right) \left(1 - \frac{\omega_{10}^2}{\omega^2}\right)^{1/2} \frac{B_o^2}{2\mu} \frac{ba}{2}$$

Now to calculate $-dP/dz$ (cf. Jackson eq. (8.58) p 364), we need

$$\oint_C \left|\hat{\mathbf{n}} \times \mathbf{B}\right|^2 dl = \int_0^a dx \left|\hat{\mathbf{n}} \times \mathbf{B}\right|_{y=0}^2 + \int_0^a dx \left|\hat{\mathbf{n}} \times \mathbf{B}\right|_{y=b}^2 + \int_0^b dy \left|\hat{\mathbf{n}} \times \mathbf{B}\right|_{x=0}^2 + \int_0^b dy \left|\hat{\mathbf{n}} \times \mathbf{B}\right|_{x=a}^2$$

$$\left\langle \int_0^a dx \left|\hat{\mathbf{n}} \times \mathbf{B}\right|_{y=0}^2 \right\rangle = \frac{a}{4} B_o^2 \left(1 + \frac{a^2 k^2}{\pi^2}\right) = \frac{a}{4} B_o^2 \left(1 + \frac{a^2 \mu\varepsilon}{\pi^2}\left(\omega^2 - \omega_{1o}^2\right)\right) = \frac{a}{4} B_o^2 \left(\frac{a^2 \mu\varepsilon\omega^2}{\pi^2}\right) = \frac{a}{4} B_o^2 \left(\frac{\omega^2}{\omega_{1o}^2}\right)$$

$$\left\langle \int_0^b dy \left|\hat{\mathbf{n}} \times \mathbf{B}\right|_{x=0}^2 \right\rangle = \left\langle \int_0^b dy B_o^2 \cos^2\left(kz - \omega t\right) \right\rangle = B_o^2 \frac{b}{2}$$

$$\oint_C \left|\hat{\mathbf{n}} \times \mathbf{B}\right|^2 dl = \frac{a}{2} B_o^2 \left(\frac{\omega^2}{\omega_{1o}^2}\right) + b B_o^2$$

$$-dP/dz = \frac{1}{2\sigma\delta} \oint_C \left|\hat{\mathbf{n}} \times \mathbf{H}\right|^2 dl = \frac{1}{2\sigma\delta\mu^2} \oint_C \left|\hat{\mathbf{n}} \times \mathbf{B}\right|^2 dl = \frac{B_o^2}{2\sigma\delta\mu^2} \left(\frac{a}{2}\left(\frac{\omega^2}{\omega_{1o}^2}\right) + b\right)$$

Thus

$$-\frac{1}{P}\frac{dP}{dz} = \frac{\dfrac{B_o^2}{2\sigma\delta\mu^2}\left(\dfrac{a}{2}\left(\dfrac{\omega^2}{\omega_{1o}^2}\right) + b\right)}{\dfrac{a^2 \omega k}{\pi^2} \dfrac{B_o^2}{2\mu} \dfrac{ba}{2}} = \frac{2}{\sigma\delta\mu\omega kba} \left(\omega_{10}^2 \mu\varepsilon\right) \left(\frac{a}{2}\left(\frac{\omega^2}{\omega_{1o}^2}\right) + b\right)$$

$$-\frac{1}{P}\frac{dP}{dz}=\frac{2}{\sigma\delta\mu\omega^2\sqrt{1-\frac{\omega_{10}^2}{\omega^2}}\sqrt{\mu\varepsilon}ba}\left(\omega_{10}^2\mu\varepsilon\right)\left(\frac{a}{2}\left(\frac{\omega^2}{\omega_{1o}^2}\right)+b\right)$$

$$-\frac{1}{P}\frac{dP}{dz}=\frac{4}{\sigma\delta\mu\sqrt{1-\frac{\omega_{10}^2}{\omega^2}}\sqrt{\mu\varepsilon}ba}\left(\mu\varepsilon\right)\left(a+2b\left(\frac{\omega_{1o}^2}{\omega^2}\right)\right)$$

$$-\frac{1}{P}\frac{dP}{dz}=\frac{4\mu\varepsilon}{\sigma\delta\mu\sqrt{1-\frac{\omega_{10}^2}{\omega^2}}\sqrt{\mu\varepsilon}ba}\left(a+2b\left(\frac{\omega_{1o}^2}{\omega^2}\right)\right)$$

$$\delta=\sqrt{\frac{2}{\mu\sigma\omega}}=\sqrt{\frac{2}{\mu\sigma\omega_{10}}}\left(\frac{\omega_{10}}{\omega}\right)^{1/2}=\delta_{10}\left(\frac{\omega_{10}}{\omega}\right)^{1/2}$$

$$-\frac{1}{P}\frac{dP}{dz}=\frac{4}{\sigma\delta_{10}ba}\sqrt{\frac{\varepsilon}{\mu}}\frac{\left(\frac{\omega}{\omega_{10}}\right)^{1/2}}{\sqrt{1-\frac{\omega_{10}^2}{\omega^2}}}\left(a+2b\left(\frac{\omega_{10}^2}{\omega^2}\right)\right)$$

$$\beta_{10}=-\frac{1}{2P}\frac{dP}{dz}=\frac{2}{\sigma\delta_{10}ba}\sqrt{\frac{\varepsilon}{\mu}}\frac{\left(\frac{\omega}{\omega_{10}}\right)^{1/2}}{\sqrt{1-\frac{\omega_{10}^2}{\omega^2}}}\left(a+2b\left(\frac{\omega_{10}^2}{\omega^2}\right)\right)$$

Problem 8-2: Q of a mode in a cavity

Suppose we have a cavity with dimension $0<x<a$, $0<y<b$ and $0<z<d$ containing a resonant mode in the cavity given by

$$B_z=B_o\cos\left(\pi x/a\right)\sin\left(\pi z/d\right)$$

(this is just our $TE_{1,0}$ mode from above converted to a standing wave in the z-direction satisfying the appropriate boundary conditions). Calculate the Q of this mode in this cavity, following the development leading up to equation (8.96) of Jackson p 373). Assume that the skin depth in the conductor $\delta=\sqrt{2/\mu_c\sigma\omega}$ is small compared to any of *a*, *b*, or *d*.

We need the magnetic field everywhere in space, and since $\nabla\cdot\mathbf{B}=0$ we must have

$\frac{\partial B_x}{\partial x} = -\frac{\partial B_z}{\partial z} = -B_o \frac{\pi}{d}\cos(\pi x/a)\cos(\pi z/d) \Rightarrow B_x = -B_o \frac{a}{d}\sin(\pi x/a)\cos(\pi z/d)$. We will also need the electric field, which we can find from $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ which gives us

$\mathbf{E} = i\hat{\mathbf{y}}\frac{c^2}{\omega}B_o \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right)\frac{\pi}{a}\left(1+\frac{a^2}{d^2}\right)$. We also have for the resonant frequency $\omega = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}$.

We now begin with Eq. (8.93) of Jackson, page 373.

$$P_{loss} = \frac{1}{2\sigma\delta}\left[\oint_C dl \int_0^d dz \left|\hat{\mathbf{n}}\times\mathbf{H}\right|^2_{sides} + \int_A da\left|\hat{\mathbf{n}}\times\mathbf{H}\right|^2_{top} + \int_A da\left|\hat{\mathbf{n}}\times\mathbf{H}\right|^2_{bottom}\right]$$

$$P_{loss} = \frac{1}{2\sigma\delta\mu_o^2}\left[\oint_C dl \int_0^d dz \left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{sides} + \int_A da\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{top} + \int_A da\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{bottom}\right]$$

$$\int_A da\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{top} = \int_0^a\int_0^b dxdy\left[B_x^2(x,y,d)+B_y^2(x,y,d)\right] = \int_0^a\int_0^b dxdy\left[B_o\frac{a}{d}\sin(\pi x/a)\right]^2$$

$$= \left(B_o\frac{a}{d}\right)^2 b\left(\frac{a}{2}\right) = \frac{B_o^2}{2}\frac{a^3 b}{d} = \int_A da\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{bottom}$$

$$\oint_C dl\int_0^d dz\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{sides} = \int_0^a dx\int_0^d dz B_x^2(x,0,z) + \int_0^a dx\int_0^d dz B_x^2(x,b,z) + \int_0^b dy\int_0^d dz B_y^2(0,y,z) + \int_0^b dy\int_0^d dz B_y^2(a,y,z)$$

$$\oint_C dl\int_0^d dz\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{sides} = \int_0^a dx\int_0^d dz\left(B_x^2(x,0,z)+B_z^2(x,0,z)\right) + \int_0^a dx\int_0^d dz\left(B_x^2(x,b,z)+B_z^2(x,b,z)\right)$$
$$+\int_0^b dy\int_0^d dz\left(B_y^2(0,y,z)+B_z^2(0,y,z)\right) + \int_0^b dy\int_0^d dz\left(B_y^2(a,y,z)+B_z^2(a,y,z)\right)$$

$$\oint_C dl\int_0^d dz\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{sides} = 2\int_0^a dx\int_0^d dz\left\{\left(B_o\cos(\pi x/a)\sin(\pi z/d)\right)^2 + \left(B_o\frac{a}{d}\sin(\pi x/a)\cos(\pi z/d)\right)^2\right\}$$
$$+2\int_0^b dy\int_0^d dz\left(B_o\sin(\pi z/d)\right)^2$$

$$\oint_C dl\int_0^d dz\left|\hat{\mathbf{n}}\times\mathbf{B}\right|^2_{sides} = 2B_o^2\frac{ad}{4}\left(1+\frac{a^2}{d^2}\right) + 2B_o^2\frac{bd}{2} = B_o^2\left(\frac{ad}{2}\left(1+\frac{a^2}{d^2}\right)+bd\right)$$

So we have $P_{loss} = \frac{1}{2\sigma\delta\mu_o^2} B_o^2 \left(\frac{ad}{2}\left(1+\frac{a^2}{d^2}\right) + bd + \frac{a^3 b}{d} \right)$

To get Q we must compute the energy of this mode. The average energy in the electric field is

$$\left\langle \int \frac{1}{2}\varepsilon_o E_y^2 dxdydx \right\rangle = \frac{1}{4}\left\langle \int \varepsilon_o \left\{ \frac{c^2}{c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}} B_o \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{d}\right) \frac{\pi}{a}\left(1+\frac{a^2}{d^2}\right) \right\}^2 dxdydz \right\rangle$$

$$= \frac{B_o^2}{4\mu_o} \frac{\left(\frac{\pi}{a}\right)^2 \left(1+\frac{a^2}{d^2}\right)^2}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2} \frac{abd}{4} = \frac{B_o^2}{16\mu_o} abd\left(1+\frac{a^2}{d^2}\right)$$

Similarly,

$$\left\langle \int \frac{B_x^2 + B_z^2}{2\mu_o} dxdydx \right\rangle = \frac{B_o^2}{4\mu_o} \left\langle \int \left\{ \begin{matrix} \left[\cos(\pi x/a)\sin(\pi z/d)\right]^2 \\ +\left[\frac{a}{d}\sin(\pi x/a)\cos(\pi z/d)\right]^2 \end{matrix} \right\} dxdydz \right\rangle$$

$$\left\langle \int \frac{B_x^2 + B_z^2}{2\mu_o} dxdydx \right\rangle = \frac{B_o^2}{4\mu_o} \left\langle \frac{a}{2}\frac{d}{2}b + \left(\frac{a}{d}\right)^2 \frac{a}{2}\frac{d}{2}b \right\rangle = \frac{B_o^2}{16\mu_o}(abd)\left(1+\left(\frac{a}{d}\right)^2\right)$$

So the total time averaged electromagnetic energy in the cavity is

$$\langle U \rangle = \frac{B_o^2}{16\mu_o}(abd)\left(1+\left(\frac{a}{d}\right)^2\right) + \frac{B_o^2}{16\mu_o} abd\left(1+\frac{a^2}{d^2}\right) = \frac{B_o^2}{8\mu_o} abd\left(1+\frac{a^2}{d^2}\right)$$

$$\frac{P_{loss}}{\langle U \rangle} = \frac{1}{2\sigma\delta\mu_o^2} B_o^2 \left(\frac{ad}{2}\left(1+\frac{a^2}{d^2}\right) + bd + \frac{a^3 b}{d} \right) / \frac{B_o^2}{8\mu_o} abd\left(1+\frac{a^2}{d^2}\right)$$

$$= \frac{4}{\sigma\delta\mu_o} \frac{\left(ad\left(1+\frac{a^2}{d^2}\right) + 2bd + \frac{2a^3 b}{d} \right)}{abd\left(1+\frac{a^2}{d^2}\right)}$$

$$Q = \frac{\omega\sigma\delta\mu_o}{4}\frac{abd\left(1+\frac{a^2}{d^2}\right)}{\left(\frac{ad}{2}\left(1+\frac{a^2}{d^2}\right)+bd+\frac{a^3b}{d}\right)} = \frac{1}{2\delta}\frac{abd\left(1+\frac{a^2}{d^2}\right)}{\left(\frac{ad}{2}\left(1+\frac{a^2}{d^2}\right)+bd+\frac{a^3b}{d}\right)}$$

Problem 8-3: Jackson 8.5 part (a) only, page 398.

We want to explore whether we can use the solutions to a square waveguide properly combined to form symmetric or anti-symmetric functions to find the solution to this triangular wave guide. For the square wave guide, the TE modes are given by

$$B_z^{square} = B_o \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a}$$

where we have chosen the spatial dependence so that we have $\left.\frac{\partial B_z}{\partial y}\right|_{y=0,a} = 0$ and $\left.\frac{\partial B_z}{\partial x}\right|_{x=0,a} = 0$. So we are ok on those surfaces. For the diagonal, we have $\hat{\mathbf{n}} = -\hat{\mathbf{x}}/\sqrt{2} + \hat{\mathbf{y}}/\sqrt{2}$ so $\frac{\partial B_z}{\partial n} = \frac{1}{\sqrt{2}}\left(-\frac{\partial B_z}{\partial x}+\frac{\partial B_z}{\partial y}\right)$. Let's try

$$B_z^{triangle} = B_o\left[\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a}+\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a}\right] \tag{1}$$

For this combination, we have in general that

$$\begin{aligned}\frac{\partial B_z^{triangle}}{\partial n} &= B_o\left[+\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a}+\frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{a}\right] \\ &-B_o\left[+\frac{n\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{a}+\frac{m\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\right]\end{aligned}$$

and on the diagonal where $x = y$ we have

$$\frac{\partial B_z^{triangle}}{\partial n} = B_o\frac{\pi}{a}\left\{\begin{array}{l}\left[m\sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}+n\sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a}\right] \\ -\left[n\cos\frac{m\pi x}{a}\sin\frac{n\pi x}{a}+m\cos\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\right]\end{array}\right\} = 0$$

Thus the form of $B_z^{triangle}$ given in equation (1) satisfies the appropriate boundary conditions on all the surfaces of our triangular wave guide. The cutoff frequencies of this mode are given by $\omega_{cutoff} = \frac{c\pi}{a}\sqrt{\left(m^2+n^2\right)}$.

For the square wave guide, the TM modes are given by

$$E_z^{square} = E_o \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

where we have chosen the spatial dependence so that we have $E_z\big|_{y=0,a} = 0$ and $E_z\big|_{x=0,a} = 0$. So we are ok on those surfaces. For the diagonal, we want $E_z\big|_{x=y} = 0$ so let's try

$$E_z^{triangle} = E_o\left[\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\right] \qquad (2)$$

This vanishes at $x = y$ as we desire and again has $\omega_{cutoff} = \frac{c\pi}{a}\sqrt{\left(m^2+n^2\right)}$

To get the corresponding electric or magnetic fields for either mode we use Jackson eq (8.26) page 358.