

**1.**

(a)

$$\begin{aligned}x'_i x'_i &= a_{ij} a_{ik} x_j x_k \\x_i x_i &= \delta_{jk} x_j x_k\end{aligned}$$

Since  $x'_i x'_i = x_i x_i$  for all  $x_i$

$$\delta_{jk} = a_{ij} a_{ik}$$

(b)

$$\begin{aligned}x_i &= \delta_{ik} x_k \\&= a_{ji} a_{jk} x_k \\&= a_{ji} x'_j\end{aligned}$$

(c)

$$\begin{aligned}\frac{\partial f}{\partial x'_i} &= \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x'_i} \\&= \frac{\partial f}{\partial x_j} \frac{\partial a_{kj} x'_k}{\partial x'_i} \\&= a_{ij} \frac{\partial f}{\partial x_j}\end{aligned}$$

**2.**

(a)

$$\begin{aligned}\delta'_{ij} &= a_{im} a_{jn} \delta_{mn} \\&= a_{ik} a_{jk} \\&= \delta_{ij}\end{aligned}$$

(b)

$$\begin{aligned}C'_i T'_{ij} &= a_{ik} C_k a_{im} a_{jn} T_{mn} \\&= a_{ik} a_{im} a_{jn} C_k T_{mn} \\&= \delta_{km} a_{jn} C_k T_{mn} \\&= a_{jk} C_i T_{ik} \\&= (C_i T_{ik})'\end{aligned}$$

(c)

Since both  $A_i A_j$  and  $\delta_{ij}$  are second rank tensors and  $A^2$  is a scalar,  $T_{ij}$  is also a second rank tensor.

(d)

$$\begin{aligned}
 & \partial_i \left( A_i A_j - \frac{1}{2} \delta_{ij} A_k A_k \right) \\
 &= \partial_i (A_i A_j) - \frac{1}{2} \partial_j (A_k A_k) \\
 &= A_j \partial_i (A_i) + A_i \partial_i (A_j) - A_k \partial_j (A_k) \\
 &= A_j \partial_i (A_i) + A_i \partial_i (A_j) - A_i \partial_j (A_i) \\
 &= A_j \partial_i (A_i) + A_i (\delta_{ki} \delta_{jl} - \delta_{jk} \delta_{il}) \partial_k (A_l) \\
 &= A_j \partial_i (A_i) + \varepsilon_{mij} A_i \varepsilon_{mkl} \partial_k (A_l) \\
 &= A_j \partial_i (A_i) + \varepsilon_{jmi} (\varepsilon_{mkl} \partial_k (A_l)) A_i
 \end{aligned}$$

(e)

i.

$$\begin{aligned}
 T_{ij} n_j &= \left( A_i A_j - \frac{1}{2} \delta_{ij} A^2 \right) n_j \\
 &= A_i A_j n_j - \frac{1}{2} A^2 n_i
 \end{aligned}$$

Therefore  $T \cdot \vec{n}$  is a linear combination of  $\vec{A}$  and  $\vec{n}$

ii.

Let  $B_i = T_{ij} n_j$

$$\begin{aligned}
 B_i B_i &= \left( A_i A_j n_j - \frac{1}{2} A^2 n_i \right) \left( A_i A_k n_k - \frac{1}{2} A^2 n_i \right) \\
 &= (A_j n_j)^2 A^2 + \frac{1}{4} A^4 - (A_j n_j)^2 A^2 = \frac{1}{4} A^4 \\
 B_i n_i &= A_i A_j n_j n_i - \frac{1}{2} A^2 \\
 &= A^2 \left( \cos^2 \theta - \frac{1}{2} \right) \\
 &= \frac{A^2}{2} \cos 2\theta \\
 \cos \theta_{Bn} &= \cos 2\theta \\
 \theta_{Bn} &= 2\theta
 \end{aligned}$$

iii.

See above.

**3.**

(a)

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r} - E_0 \sin \phi$$

$$E_\phi = -E_0 \cos \phi$$

(b)

$$E_r = \frac{2\pi\epsilon_0 E_0 \lambda}{2\pi\epsilon_0 \lambda} - E_0$$

$$= 0$$

$$E_\phi = 0$$

(c)

$$d\vec{F} = \epsilon_0 r d\phi dz \left( E_r \vec{E} - \frac{1}{2} E^2 \hat{r} \right)$$

$$= \epsilon_0 r d\phi dz \left( \frac{E_r^2 - E_\phi^2}{2} \hat{r} + E_r E_\phi \hat{\phi} \right)$$

(d)

$$\frac{d\vec{F}}{dz} = \int \epsilon_0 r d\phi \left( \frac{E_r^2 - E_\phi^2}{2} \hat{r} + E_r E_\phi \hat{\phi} \right)$$

$$= \frac{\epsilon_0 r}{2} \hat{y} \int d\phi \left( \left( \left( \frac{\lambda}{2\pi\epsilon_0 r} - E_0 \sin \phi \right)^2 - E_0 \cos^2 \phi \right) \sin \phi - 2 \left( \frac{\lambda}{2\pi\epsilon_0 r} - E_0 \sin \phi \right) E_0 \cos^2 \phi \right)$$

$$= \frac{\epsilon_0 r}{2} \hat{y} \int d\phi \left( -2 \frac{\lambda}{2\pi\epsilon_0 r} E_0 \sin^2 \phi - \frac{\lambda}{\pi\epsilon_0 r} E_0 \cos^2 \phi \right)$$

$$= \frac{1}{2} \hat{y} \int d\phi \left( -\frac{\lambda}{\pi} E_0 \sin^2 \phi - \frac{\lambda}{\pi} E_0 \cos^2 \phi \right)$$

$$= -\frac{1}{2} \hat{y} 2\pi \frac{\lambda}{\pi} E_0$$

$$= -E_0 \lambda \hat{y}$$

4.

(a)

$$B_r = B_0 \sin \phi$$

$$B_\phi = \frac{\mu_0 I}{2\pi r} + B_0 \cos \phi$$

(b)

$$B_r = 0$$

$$B_\phi = \frac{\mu_0 I 2\pi B_0}{2\pi \mu_0 I} + B_0$$

$$= 0$$

(c)

$$d\vec{F} = \frac{1}{\mu_0} r d\phi dz \left( B_r \vec{B} - \frac{1}{2} B^2 \hat{r} \right)$$

$$= \frac{1}{\mu_0} r d\phi dz \left( \frac{B_r^2 - B_\phi^2}{2} \hat{r} + B_r B_\phi \hat{\phi} \right)$$

(d)

$$\frac{d\vec{F}}{dz} = \frac{1}{2} \int \frac{r}{\mu_0} d\phi \left( (B_r^2 - B_\phi^2) \hat{r} + 2B_r B_\phi \hat{\phi} \right)$$

$$= \frac{r \hat{x}}{2\mu_0} \int d\phi \left( \left( B_0^2 \sin^2 \phi - \left( \frac{\mu_0 I}{2\pi r} + B_0 \cos \phi \right)^2 \right) \cos \phi - 2B_0 \sin \phi \left( \frac{\mu_0 I}{2\pi r} + B_0 \cos \phi \right) \sin \phi \right)$$

$$= \frac{r \hat{x}}{2\mu_0} \int d\phi \left( - \left( \frac{\mu_0 I}{2\pi r} + B_0 \cos \phi \right)^2 \cos \phi - B_0 \sin^2 \phi \frac{\mu_0 I}{\pi r} \right)$$

$$= \frac{\hat{x}}{2} \int d\phi \left( - \frac{I}{\pi} B_0 \cos^2 \phi - B_0 \sin^2 \phi \frac{I}{\pi} \right)$$

$$= -B_0 I \hat{x}$$

5.

(a)

$$\begin{aligned}\vec{E}_{line} &= \frac{\lambda(\vec{r} - \vec{Y})}{2\pi\epsilon_0(\vec{r} - \vec{Y})^2} \\ &= \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\epsilon_0(x^2 + (y - Y)^2)} \\ E &= \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\epsilon_0(x^2 + (y - Y)^2)} - E_0\hat{y}\end{aligned}$$

(b)

$$\begin{aligned}& \int_{-\infty}^{\infty} dx \int_{-L}^L dy \epsilon_0 \vec{E}_0 \cdot \vec{E}_{line} \\ &= - \int_{-\infty}^{\infty} dx \int_{-L}^L dy \epsilon_0 \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\epsilon_0(x^2 + (y - Y)^2)} E_0 \cdot \hat{y} \\ &= - \frac{E_0\lambda}{2\pi} \int_{-\infty}^{\infty} dx \int_{-L}^L dy \frac{y - Y}{x^2 + (y - Y)^2} \\ &= - \frac{E_0\lambda}{4\pi} \int_{-\infty}^{\infty} dx \ln \left( \frac{x^2 + (L - Y)^2}{x^2 + (-L - Y)^2} \right)\end{aligned}$$

Let  $\alpha = \frac{x}{L}$ ,  $\beta = \frac{Y}{L}$

$$\begin{aligned}\frac{dE}{dz} &= - \frac{E_0\lambda}{4\pi} \int_{-\infty}^{\infty} d\alpha \ln \left( \frac{\alpha^2 + (1 - \beta)^2}{\alpha^2 + (1 + \beta)^2} \right) \\ &= - \frac{E_0\lambda}{4\pi} \int_{-\infty}^{\infty} d\alpha \ln \left( \frac{\alpha^2 + 1 - 2\beta}{\alpha^2 + 1 + 2\beta} \right) \\ &\approx - \frac{E_0\lambda L}{4\pi} \int_{-\infty}^{\infty} d\alpha \frac{4\beta}{1 + \alpha^2} \\ &= - E_0\lambda Y\end{aligned}$$