Solutions Assignment #2: Due Friday February 20, 2015 at 2:30 pm

Reading: Jackson Sections 5.15, 6.2, 6.4, 6.5, 6.7, 6.10

Problems

Problem 2-1: Rotation Matrices

Why? More math that we need.

Consider two right handed orthogonal coordinate systems whose basis vectors are $\{\hat{\mathbf{e}}_i\}_{i=1}^3$ and $\{\hat{\mathbf{e}}'_j\}_{j=1}^3$. Let the vector $\vec{\mathbf{X}} = x_i\hat{\mathbf{e}}_i$ (the repeated index implies a summation over *i* from 1 to 3). When we look at the components of this same vector in the second (primed) system, we have $x'_i = a_{ij}x_j$, where $a_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$ and the repeated index implies a summation over *j*.

(a) Show that $a_{ij}a_{ik} = \delta_{jk}$ in order to preserve the length of the vector.

Solution

$$A'^2 = A^2 = A_k A_k = \delta_{jk} A_j A_k$$

but we also have

$$A'^2 = A'_i A'_i = \left(a_{ij}A_j\right)\left(a_{ik}A_k\right) = \left(a_{ij}a_{ik}\right)A_j A_k$$

In order for this to hold for all A_k, we must have $a_{ij}a_{ik} = \delta_{jk}$, as desired.

(b) Using the constraint above, show that $x_i = a_{ji}x'_j$. Note that we can now show that $a_{ji}a_{ki} = \delta_{jk}$ using this relation, in a manner similar to the procedure in (a) (you do not have to show this).

Solution: Take the equation $A'_i = a_{ij}A_j$, multiply by a_{ik}, and sum over *i*, giving

$$a_{ik}A'_i = a_{ik}a_{ij}A_j = \delta_{kj}A_j = A_k$$

Renaming indices, this equation is $A_i = a_{ji}A'_j$, as desired.

(c) Using the chain rule for partial differentiation and the results of (b), show that if *f* is scalar function of **r**, then ∇f transforms as a vector, i.e., show that $\dfrac{\partial f'}{\partial x'_i} = a_{ij}\dfrac{\partial f}{\partial x_j}$.

Solution: The chain rule for partial differentiation is $\dfrac{\partial f'}{\partial x'_i} = \dfrac{\partial f}{\partial x_j}\dfrac{\partial x_j}{\partial x'_i}$, and since

$x_j = a_{kj}x'_k$, $\dfrac{\partial x_j}{\partial x'_i} = \dfrac{\partial}{\partial x'_i}\left(a_{kj}x'_k\right) = a_{kj}\dfrac{\partial}{\partial x'_i}\left(x'_k\right) = a_{kj}\delta_{ik} = a_{ij}$. Thus we have $\dfrac{\partial f'}{\partial x'_i} = a_{ij}\dfrac{\partial f}{\partial x_j}$ as desired.

Problem 2-2: Second Rank Tensors

Why? The flux of a vector quantity (like momentum) is a second rank tensor.

A 2nd rank tensor is any nine components T_{mn} that transform as

$$T'_{ij} = a_{im} a_{jn} T_{mn}$$

(a) Show that $\overleftrightarrow{\mathbf{I}}$ defined in *any* coordinate system by $I_{ij} = \delta_{ij}$ is a second rank tensor.

Solution: The transform of $\overleftrightarrow{\mathbf{I}}$ to the new coordinate system is given by $a_{ij} a_{mn} I_{jn} = a_{ij} a_{mn} \delta_{jn} = a_{ij} a_{mj} = \delta_{im}$, so therefore $\overleftrightarrow{\mathbf{I}}$ transforms into $\overleftrightarrow{\mathbf{I}}$, and thus $\overleftrightarrow{\mathbf{I}}$ is a second rank tensor.

(b) If $\overleftrightarrow{\mathbf{T}}$ is a second rank tensor and **C** is any vector, the dot product of **C** with $\overleftrightarrow{\mathbf{T}}$ *"from the left"* is a vector, denoted by $\mathbf{C} \cdot \overleftrightarrow{\mathbf{T}}$, and is given by

$$(\mathbf{C} \cdot \overleftrightarrow{\mathbf{T}})_j = C_i T_{ij}$$

The dot product of **C** with $\overleftrightarrow{\mathbf{T}}$ *"from the right"* is also a vector, denoted by $\overleftrightarrow{\mathbf{T}} \cdot \mathbf{C}$, and is given by $(\overleftrightarrow{\mathbf{T}} \cdot \mathbf{C})_j = T_{ji} C_i$. For arbitrary $\overleftrightarrow{\mathbf{T}}$, these two different ways of taking the dot product of a vector and a 2nd rank tensor result in different vectors. However, for symmetric 2nd rank tensors ($T_{ij} = T_{ji}$), the two vectors are the same. In practice, in 8.311 we will only deal with symmetric 2nd rank tensors, so the distinction between $\mathbf{C} \cdot \overleftrightarrow{\mathbf{T}}$ and $\overleftrightarrow{\mathbf{T}} \cdot \mathbf{C}$ is unnecessary.

Show that $\mathbf{C} \cdot \overleftrightarrow{\mathbf{T}}$ as defined above is a vector.

Solution:

$$\left(\mathbf{C}' \cdot \overleftrightarrow{\mathbf{T}}'\right)_i = C'_k T'_{ki} = \left(a_{km} C_m\right)\left(a_{kl} a_{in} T_{ln}\right) = \left(a_{km} a_{kl} a_{in}\right)\left(C_m T_{ln}\right) = \delta_{ml} a_{in}\left(C_m T_{ln}\right) = a_{in}\left(C_m T_{mn}\right)$$

thus

$$\left(\mathbf{C}' \cdot \overleftrightarrow{\mathbf{T}}'\right)_i = a_{in} (\mathbf{C} \cdot \overleftrightarrow{\mathbf{T}})_n$$

which proves that $\mathbf{C} \cdot \overleftrightarrow{\mathbf{T}}$ is a vector.

(c) For any vector function **A**(**r**), define the second rank tensor $\overleftrightarrow{\mathbf{T}}(\mathbf{r})$ by

$$\overleftrightarrow{\mathbf{T}} = \mathbf{A}\mathbf{A} - \frac{1}{2} \overleftrightarrow{\mathbf{I}} A^2 \quad \text{(that is, } T_{ij} = A_i A_j - \frac{1}{2} \delta_{ij} A^2 \text{)}$$

Show that $\overleftrightarrow{\mathbf{T}}$ defined in this ways is a 2nd rank tensor if **A** is a vector.

Solution: $\overleftrightarrow{\mathbf{T}}$ defined in this way is a 2nd rank tensor if **A** is a vector, since $\overleftrightarrow{\mathbf{I}}$ is a second rank tensor, as is $\mathbf{AA}$. We can prove that $\mathbf{AA}$ is a second rank tensor easily, since $A'_i A'_j = a_{im} A_m a_{jn} A_n = a_{im} a_{jn} A_m A_n$.

(d) The divergence *"from the left"* of a 2nd rank tensor is a vector, and is defined by

$$\left(\nabla \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \frac{\partial}{\partial x_i} T_{ij}$$

where $x_1 = x, x_2 = y,$ and $x_3 = z$. With this definition, and using the expression in (c) for $\overleftrightarrow{\mathbf{T}}$ in terms of the vector **A**, show that

$$\nabla \cdot \overleftrightarrow{\mathbf{T}} = \mathbf{A}\left(\nabla \cdot \mathbf{A}\right) + \left(\nabla \times \mathbf{A}\right) \times \mathbf{A}$$

Again, we can also define a divergence of $\overleftrightarrow{\mathbf{T}}$ *"from the right"*, in analogy with the dot product above, but if the tensor is symmetric, there is no difference in the two.
Solution:

$$\left(\nabla\cdot\overleftrightarrow{\mathbf{T}}\right)_j = \frac{\partial}{\partial x_i}T_{ij} = \frac{\partial}{\partial x_i}\left[A_iA_j - \frac{1}{2}\delta_{ij}A^2\right] = A_j\frac{\partial}{\partial x_i}A_i + A_i\frac{\partial}{\partial x_i}A_j - \frac{1}{2}\frac{\partial}{\partial x_j}A^2$$

In vector notation, the equation above is

$$\nabla\cdot\overleftrightarrow{\mathbf{T}} = \mathbf{A}\left(\nabla\cdot\mathbf{A}\right) + \left(\mathbf{A}\cdot\nabla\right)\mathbf{A} - \frac{1}{2}\nabla A^2$$

And we can use a vector identity to write the last two terms above as one, viz

$$\nabla\cdot\overleftrightarrow{\mathbf{T}} = \mathbf{A}\left(\nabla\cdot\mathbf{A}\right) + \left(\nabla\times\mathbf{A}\right)\times\mathbf{A}$$

(e) Let $\hat{\mathbf{n}}$ be any unit vector. Prove the following about $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$.

1. $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ lies in the plane defined by **A** and $\hat{\mathbf{n}}$.
2. If I go an angle θ to get to **A** from $\hat{\mathbf{n}}$, I have to go an angle 2θ to get to $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$, in the same sense.
3. The magnitude of $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is always $A^2/2$

Solution: Unless **A** and $\hat{\mathbf{n}}$ are co-linear, they determine a plane. Let the *x* axis in that plane be along the $\hat{\mathbf{n}}$ direction, and the *y* axis be perpendicular to the $\hat{\mathbf{n}}$ direction and in the **A**-$\hat{\mathbf{n}}$ plane. Let **A** make an angle θ with $\hat{\mathbf{n}}$. In this coordinate system, I thus have the components of **A** as $\mathbf{A} = A\cos\theta\,\hat{\mathbf{x}} + A\sin\theta\,\hat{\mathbf{y}}$ and $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. If I look at the definition of $\overleftrightarrow{\mathbf{T}}$, and the definition of $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$, I see that $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is a vector with components

$$\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)_i = T_{ij}n_j = T_{ix}$$

and therefore

$$\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} = T_{xx}\hat{\mathbf{x}} + T_{yx}\hat{\mathbf{y}} + T_{zx}\hat{\mathbf{z}} = (A_xA_x - \frac{1}{2}A^2)\hat{\mathbf{x}} + A_xA_y\hat{\mathbf{y}}$$

If I insert the values of the components of **A**, and use some trig, I have

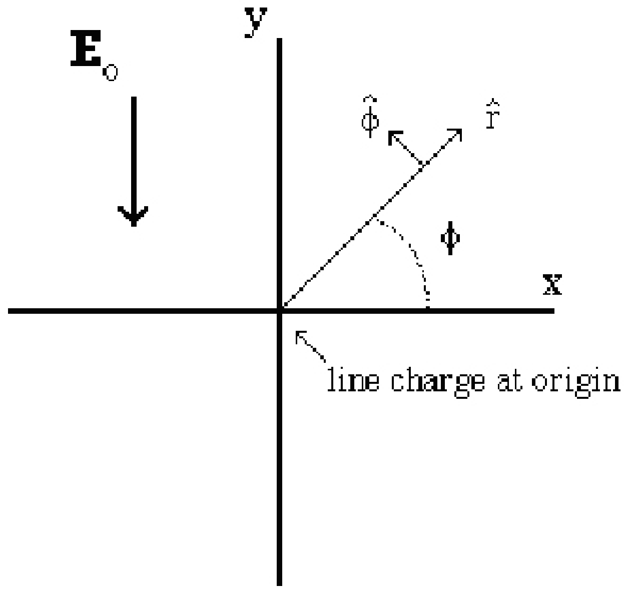
$$\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} = A^2\left(\cos^2\theta - \frac{1}{2}\right)\hat{\mathbf{x}} + A^2\sin\theta\cos\theta\,\hat{\mathbf{y}} = \frac{1}{2}A^2\left(\cos 2\theta\,\hat{\mathbf{x}} + \sin 2\theta\,\hat{\mathbf{y}}\right)$$

Using this equation, we can show all of the properties listed above.

Problem 2-3: Calculating the force on an infinite line charge in a uniform electric field using the stress tensor

Why?: The point of both this problem and the next is that by simply looking at the total field configuration, you can deduce the direction of the magnetic or electric force. That is, the Maxwell stress tensor only contains the field components, and by knowing the shape of the fields you can deduce the direction of the forces they transmit. Fields carry the stresses that result in forces on charges and currents, and those stresses can be deduced from the shape of the fields. This was Faraday's great insight.

An infinite line charge with charge per unit length λ lies along the z-axis (see sketch). It is immersed in a uniform field. We want to determine the force per unit length in the z direction on the line charge by carrying out a surface integral of the stress tensor over the surface of an (imaginary) cylinder of radius *R* whose axis is the z-axis.



(a) What are the E_r and E_ϕ components of the total electric field (line charge plus uniform field). To obtain this you will have to write $\hat{\mathbf{y}}$ in terms of the radial and azimuthal unit vectors in cylindrical coordinates (see sketch)

Solution: $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$, so

$$\mathbf{E} = \hat{\mathbf{r}}\left[\frac{\lambda}{2\pi\varepsilon_o r} - E_o\sin\phi\right] - \hat{\boldsymbol{\phi}}\,E_o\cos\phi$$

(b) Show that the total electric field vanishes at a distance up the positive $\hat{\mathbf{y}}$-axis of. $L = \lambda / 2\pi\varepsilon_o E_o$

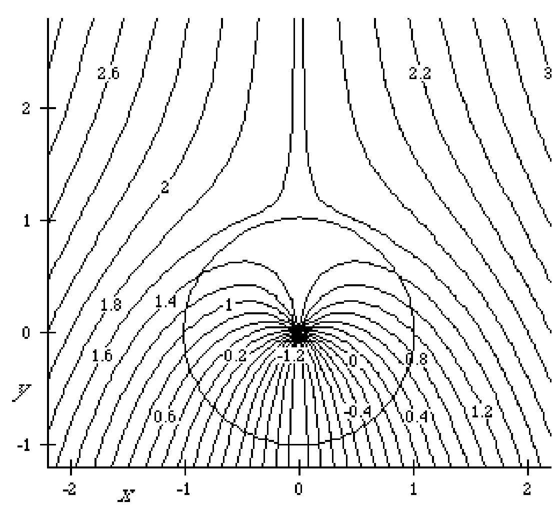
Solution: On the y-axis, $\sin\phi = 1$, $\cos\phi = 0$, and the magnitude of **E** will vanish when

$$\frac{\lambda}{2\pi\varepsilon_o L} - E_o = 0 \text{, or } L = \lambda / 2\pi\varepsilon_o E_o$$

(c) The surface element $\hat{\mathbf{n}}da$ on our imaginary cylinder of radius *R*, centered on the *z*-axis, is $\hat{\mathbf{n}}da = \hat{\mathbf{r}}\,Rd\varphi dz$. The stress $d\mathbf{F} = \overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da$ is thus

$$d\mathbf{F} = Rd\phi dz\left[T_{rr}\hat{\mathbf{r}} + T_{r\varphi}\hat{\boldsymbol{\phi}} + T_{rz}\hat{\mathbf{z}}\right]$$

Give expressions for the elements of the stress tensor occurring in the above equation.



Solution:

$$T_{rr} = \frac{\varepsilon_o}{2}\left(E_r^2 - E_\phi^2\right) = \frac{\varepsilon_o}{2}\left(\left[\frac{\lambda}{2\pi\varepsilon_o R} - E_o \sin\phi\right]^2 - \left[E_o \cos\phi\right]^2\right)$$

$$T_{rr} = \frac{\varepsilon_o}{2}\left(\left(\frac{\lambda}{2\pi\varepsilon_o R}\right)^2 - \frac{2\lambda E_o \sin\phi}{2\pi\varepsilon_o R} - E_o^2 \cos 2\phi\right)$$

$$T_{r\phi} = \varepsilon_o E_r E_\phi = \varepsilon_o\left[\frac{\lambda}{2\pi\varepsilon_o R} - E_o \sin\phi\right]\left(-E_o\cos\phi\right) = -\varepsilon_o \frac{\lambda E_o \cos\phi}{2\pi\varepsilon_o R} + \frac{\varepsilon_o}{2} E_o^2 \sin 2\phi$$

$$T_{rz} = \varepsilon_o E_r E_z = 0$$

(d) Integrate your expression for $d\mathbf{F}$ over the surface of the imaginary cylinder to find the total force per unit length on the line charge. Before you do this, you must write the unit vectors in cylindrical coordinates in terms of the (fixed direction) unit vectors in Cartesian coordinates. Also, you can avoid some work in the actual integrations--many of the integrals you will encounter in the integration over $d\phi$ can be argued to vanish simply because they are odd functions of ϕ.

Take a look at this movie: https://www.youtube.com/watch?v=iT7tFqHbn78

Solution: To convert to Cartesian unit vectors we use the equations $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ and $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$. Thus

$$d\mathbf{F} = R\,d\phi\,dz\left[\hat{\mathbf{x}}\left(T_{rr}\cos\phi - T_{r\varphi}\sin\phi\right) + \hat{\mathbf{y}}\left(T_{rr}\sin\phi + T_{r\varphi}\cos\phi\right)\right]$$

$$\int_{-\pi}^{\pi}\left(T_{rr}\cos\phi - T_{r\varphi}\sin\phi\right)d\phi = \int_{-\pi}^{\pi}\left(\begin{array}{l}\frac{\varepsilon_o}{2}\left(\left[\frac{\lambda}{2\pi\varepsilon_o R} - E_o\sin\phi\right]^2 - \left[E_o\cos\phi\right]^2\right)\cos\phi \\ +\left(-\varepsilon_o\frac{\lambda E_o\cos\phi}{2\pi\varepsilon_o R} + \frac{\varepsilon_o}{2}E_o^2\sin 2\phi\right)\sin\phi\end{array}\right)d\phi = 0$$

$$\int_{-\pi}^{\pi}\left(T_{rr}\sin\phi + T_{r\varphi}\cos\phi\right)d\phi = \int_{-\pi}^{\pi}\left(\begin{array}{l}\frac{\varepsilon_o}{2}\left(\left[\frac{\lambda}{2\pi\varepsilon_o R} - E_o\sin\phi\right]^2 - \left[E_o\cos\phi\right]^2\right)\sin\phi \\ +\left(-\varepsilon_o\frac{\lambda E_o\cos\phi}{2\pi\varepsilon_o R} + \frac{\varepsilon_o}{2}E_o^2\sin 2\phi\right)\cos\phi\end{array}\right)d\phi$$

$$= -\int_{-\pi}^{\pi}\left(\varepsilon_o\left[\frac{\lambda E_o\sin^2\phi}{2\pi\varepsilon_o R} + \frac{\lambda E_o\cos^2\phi}{2\pi\varepsilon_o R}\right]\right)d\phi = -\frac{\lambda E_o}{R}$$

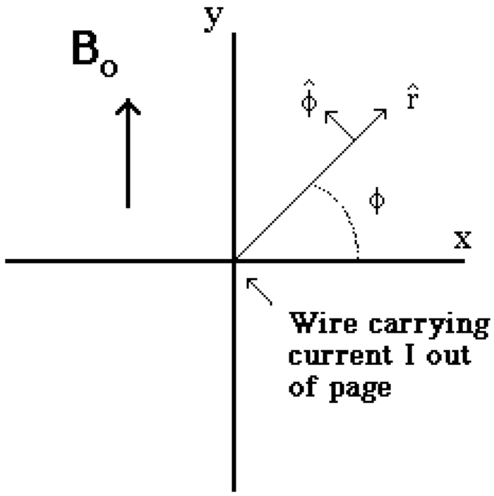
So

$$d\mathbf{F} = -\lambda E_o\,dz\,\hat{\mathbf{y}} \Rightarrow \text{force per unit length} = -\lambda E_o\,\hat{\mathbf{y}} \quad \text{as we expect}$$

Problem 2-4: Calculating the force on an infinite line current in a uniform magnetic field using the stress tensor.

Why?: see above

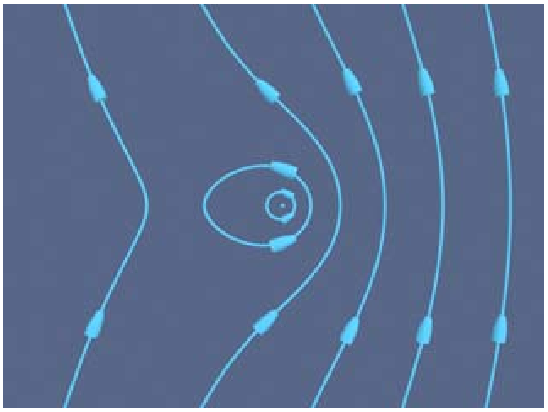
This is similar in many respects to the problem above. Here we have an infinite wire carrying current I lies along the z-axis (see sketch), with the current positive in the +z direction. It is immersed in a uniform field $\mathbf{B_o} = B_\mathbf{o}\,\hat{\mathbf{y}}$. We want to determine the force per unit length on the wire by carrying out a surface integral of the stress tensor over the surface of an (imaginary) cylinder of radius R whose axis is the z-axis.



(a) What are the B_r and B_ϕ components of the total magnetic field (line charge plus uniform field). To obtain this you will have to write $\hat{\mathbf{y}}$ in terms of the radial and azimuthal unit vectors in cylindrical coordinates. .

Solution: $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$, so

$$\mathbf{B} = \hat{\mathbf{r}}\,B_o\sin\phi + \hat{\boldsymbol{\phi}}\left[\frac{\mu_o I}{2\pi r} + B_o\cos\phi\right]$$



(c) Show that the total magnetic field vanishes at a distance along the negative x-axis of $L = \dfrac{\mu_O\, I}{2\pi\, B_o}$. A sketch of the field configuration is given.

(d) Solution: On the negative x-axis, $\sin\phi = 0$, $\cos\phi = -1$ and the magnitude of $\mathbf{B}$ will vanish when $\left[\dfrac{\mu_o I}{2\pi r} - B_o\right]$, or $L = \dfrac{\mu_o I}{2\pi B_o}$

(c) The surface element $\hat{\mathbf{n}}da$ on our imaginary cylinder of radius R, centered on the z-axis, is $\hat{\mathbf{n}}da = \hat{\mathbf{r}}\,Rd\varphi dz$. The stress $d\mathbf{F} = \overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da$ is thus

$$d\mathbf{F} = Rd\phi dz\left[T_{rr}\hat{\mathbf{r}} + T_{r\varphi}\hat{\boldsymbol{\phi}} + T_{rz}\hat{\mathbf{z}}\right]$$

Give expressions for the elements of the stress tensor occurring in the equation above.

Solution:

$$T_{rr} = \frac{1}{2\mu_o}\left(B_r^2 - B_\phi^2\right) = \frac{1}{2\mu_o}\left(B_o^2\sin^2\phi - \left[\frac{\mu_o I}{2\pi R} + B_o\cos\phi\right]^2\right)$$

$$T_{rr} = \frac{1}{2\mu_o}\left(-\left(\frac{\mu_o I}{2\pi R}\right)^2 - \frac{2\mu_o I B_o\cos\phi}{2\pi R} - B_o^2\cos 2\phi\right)$$

$$T_{r\phi} = \frac{1}{\mu_o}B_r B_\phi = \frac{1}{\mu_o}B_o\sin\phi\left[\frac{\mu_o I}{2\pi R} + B_o\cos\phi\right] = \frac{IB_o\sin\phi}{2\pi R} + \frac{1}{2\mu_o}B_o^2\sin 2\phi$$

$$T_{rz} = B_r B_z/\mu_o = 0$$

(d) Integrate your expression for $d\mathbf{F}$ over the surface of the cylinder to find the total force per unit length on the wire. Before you do this, you must write the unit vectors in cylindrical coordinates in terms of the (fixed direction) unit vectors in Cartesian coordinates. Also, you may ignore all terms except the cross-terms (i.e., terms which look like a component of the uniform field times a component of the field of the wire), since we know from experience that other "self-force" terms will integrate to zero.

Take a look at https://www.youtube.com/watch?v=jIbhrRs5Q-Q and https://www.youtube.com/watch?v=zjy9b_PLvbw

Solution: To convert to Cartesian unit vectors we use the equations $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ and $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$. Thus

$$d\mathbf{F} = Rd\phi dz\left[\hat{\mathbf{x}}\left(T_{rr}\cos\phi - T_{r\varphi}\sin\phi\right) + \hat{\mathbf{y}}\left(T_{rr}\sin\phi + T_{r\varphi}\cos\phi\right)\right]$$

$$\int_{-\pi}^{\pi}\left(T_{rr}\cos\phi - T_{r\varphi}\sin\phi\right)d\phi = \int_{-\pi}^{\pi}\left(\begin{array}{l}\left(-\left(\dfrac{\mu_o I}{2\pi R}\right)^2 - \dfrac{2\mu_o IB_o\cos\phi}{2\pi R} - B_o^2\cos 2\phi\right)\dfrac{\cos\phi}{2\mu_o} \\ -\left(\dfrac{IB_o\sin\phi}{2\pi R} + \dfrac{1}{2\mu_o}B_o^2\sin 2\phi\right)\sin\phi\end{array}\right)d\phi$$

$$= \frac{\mu_o IB_o}{2\pi R}\int_{-\pi}^{\pi}\left(\cos^2\phi + \sin^2\phi\right)d\phi = -\frac{IB_o}{R}$$

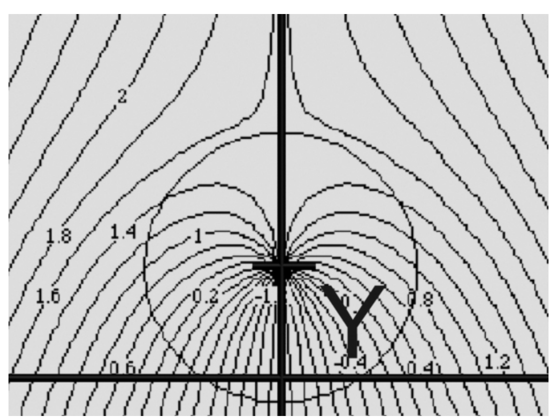
$$\int_{-\pi}^{\pi}\left(T_{rr}\sin\phi + T_{r\varphi}\cos\phi\right)d\phi = \int_{-\pi}^{\pi}\left(\begin{array}{l}\left(-\left(\dfrac{\mu_o I}{2\pi R}\right)^2 - \dfrac{2\mu_o IB_o\cos\phi}{2\pi R} - B_o^2\cos 2\phi\right)\dfrac{\sin\phi}{2\mu_o} \\ +\left(\dfrac{IB_o\sin\phi}{2\pi R} + \dfrac{1}{2\mu_o}B_o^2\sin 2\phi\right)\cos\phi\end{array}\right)d\phi = 0$$

So

$$d\mathbf{F} = -IB_o dz\,\hat{\mathbf{x}} \Rightarrow \text{force per unit length} = -IB_o\,\hat{\mathbf{x}} \text{ as we expect}$$

Problem 2-5: An electrostatic line charge moving in a constant downward field

Why?: In problem 2-3 above, if we allowed the line charge to move in the y-direction, it would lose kinetic energy if it were moving up and gain kinetic energy if it was moving down. Where does that energy go to and come from?



In the figure, the x-axis is horizontal, the y-axis is vertical, and the z-axis is out of the page. An electrostatic line charge with charge per unit length λ lies along the z-axis and is positioned at $x = 0$ and a distance $y = Y(t)$ up the y-axis. The line charge sits in a downward electric field given by $\mathbf{E}_o = -E_o\hat{\mathbf{y}}$. The electrostatic potential ϕ_o associated with this electric field is $\phi_o(y) = +E_o y$ (that is $\mathbf{E}_o = -\nabla\phi_o$). The lines in the plot are electric field lines.

If this line charge is allowed to move only in the y-direction, and has mass per unit length *m*, the equation of motion for the line charge in this field is

$$m\frac{d^2Y}{dt^2} = -\lambda E_o = \lambda\nabla\phi_o$$

If we multiply both sides of this equation by dY/dt, we have

$$m\frac{d^2Y}{dt^2}\frac{dY}{dt} = -\lambda E_o\frac{dY}{dt} \Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dY}{dt}\right)^2 + \lambda E_o Y\right] = 0$$

The equation above is usually written as a conservation of energy equation,

$$\frac{1}{2}m\left(\frac{dY}{dt}\right)^2 + \lambda E_o Y = \frac{1}{2}m\left(\frac{dY}{dt}\right)^2 + \lambda\phi_o = \text{Constant}$$

where we have the sum of the kinetic energy per unit length of the line charge and the "electrostatic potential energy" per unit length of the line charge in the back ground field. You show here that the term $\lambda\phi_o = \lambda E_o Y$ represents the excess energy stored in the total electric field (the sum of the line charge field and the background electric field).

(a) The total electric field in this problem for the configuration in the figure is given by

$$\mathbf{E} = \mathbf{E}_o + \mathbf{E}_{\text{line charge}}$$

Given an expression for the total electric field in this case, remembering that the line charge is located not at the origin but a distance *Y(t)* up the y-axis. You of course are writing down the quasi electrostatic solution, or the near zone solution, which is fine as long as the speed of light transit time across the distance *Y* is short compared to any characteristic time involving the motion of the line charge, or as long as $dY/dt << c$

Solution: $\mathbf{E} = -E_o\hat{\mathbf{y}} + \dfrac{\lambda}{2\pi\varepsilon_o\left[x^2 + (y-Y)^2\right]}\left[\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-Y)\right]$

(b) We are only interested in the "excess" energy stored in the electric field over and above that stored in the background field and the line charge field separately. Thus we are only interested in the cross term in the expression $\frac{1}{2}\varepsilon_0\left(\mathbf{E}_o + \mathbf{E}_{\text{line charge}}\right)^2$. Evaluate the integral

$$\int_{-\infty}^{\infty} dx \int_{-L}^{L} dy \left[\varepsilon_o \mathbf{E}_o \cdot \mathbf{E}_{\text{line charge}}\right]$$

This quantity gives the excess electromagnetic energy per unit length in the z-direction in a box running from $-\infty$ to ∞ in the x-direction and – *L* to *L* in the y-direction. Do the y-integration first, which is easy, and then take the limit that *L* >> *Y*. Keeping only first order terms in *Y/L*, then do the x-integration. The end result you get should just be the "electrostatic potential energy". [Hint: $\int_{-\infty}^{\infty}\frac{dx}{x^2+L^2} = \frac{\pi}{L}$]

So we see that as the line charge moves upward (downward), it loses (gains) kinetic energy per unit length in the z-direction, and that that lost (gained) kinetic energy is appearing as an excess (deficit) of electrostatic energy in the electric field.
Solution:

$$\int_{-\infty}^{\infty} dx \int_{-L}^{L} dy \left[\varepsilon_o \mathbf{E}_o \cdot \mathbf{E}_{\text{line charge}} \right] = -\int_{-\infty}^{\infty} dx \int_{-L}^{L} dy \frac{\lambda E_o (y-Y)}{2\pi \left[x^2 + (y-Y)^2 \right]} = -\frac{\lambda E_o}{4\pi} \int_{-\infty}^{\infty} dx \ln \left[x^2 + (y-Y)^2 \right] \Big|_{-L}^{L}$$

$$= -\frac{\lambda E_o}{4\pi} \int_{-\infty}^{\infty} dx \ln \frac{\left[x^2 + (L-Y)^2 \right]}{\left[x^2 + (-L-Y)^2 \right]} \approx -\frac{\lambda E_o}{4\pi} \int_{-\infty}^{\infty} dx \ln \frac{\left[x^2 + L^2 - 2LY \right]}{\left[x^2 + L^2 + 2LY \right]}$$

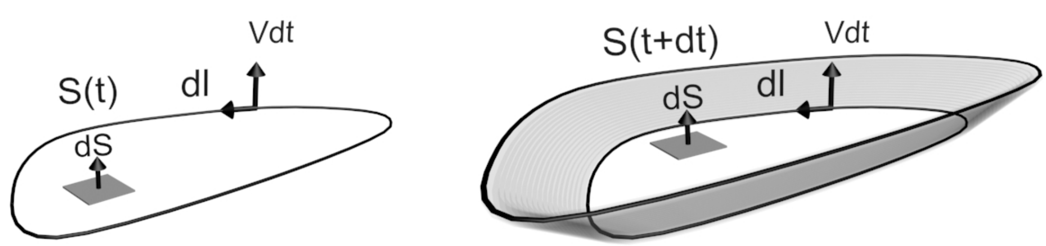
$$\approx -\frac{\lambda E_o}{4\pi} \int_{-\infty}^{\infty} dx \ln \frac{\left[1 - \frac{2LY}{x^2 + L^2} \right]}{\left[1 + \frac{2LY}{x^2 + L^2} \right]} \quad \text{but for small } \varepsilon,\ \ln \frac{(1-\varepsilon)}{(1+\varepsilon)} \approx \ln(1-2\varepsilon) \approx -2\varepsilon$$

$$\int_{-\infty}^{\infty} dx \int_{-L}^{L} dy \left[\varepsilon_o \mathbf{E}_o \cdot \mathbf{E}_{\text{line charge}} \right] = \frac{\lambda E_o}{4\pi} \int_{-\infty}^{\infty} dx \frac{4YL}{x^2 + L^2} \approx \lambda E_o Y$$

Problem 2-6: A General Vector Theorem and Faraday's Law
Why?: There are a lot of subtleties in written Faraday's Law in integral form
Theorem: If $\mathbf{G}(\mathbf{r},t)$ is *any* vector field, the time rate of change of the flux of that field through an open surface *S* bounded by a contour *C* that moves with velocity $\mathbf{V}(\mathbf{r},t)$ is given by

$$\frac{d}{dt}\int_S \mathbf{G}\cdot\hat{\mathbf{n}}\,da = \int_S \frac{\partial \mathbf{G}}{\partial t}\cdot\hat{\mathbf{n}}\,da + \int_S (\nabla\cdot\mathbf{G})\mathbf{V}\cdot\hat{\mathbf{n}}\,da - \oint_C (\mathbf{V}\times\mathbf{G})\cdot d\mathbf{l} \quad (\text{note } da = dS)$$

(a) Prove this theorem. See figure on next page (note $dS = da$ in this figure). (Hint: consider the closed surface at time *t* that consists of the surface *S* at time *t* plus the curve *S* at time $t + \Delta t$, plus the surface swept out by the contour *C* as it moves from its position at *t* to its position at *t to t*. Apply the divergence theorem for **G** to the volume thus defined ***at time t***. At one point you will have to use the vector identity $\mathrm{A}\cdot(\mathrm{B}\times\mathrm{C}) = \mathrm{B}\cdot(\mathrm{C}\times\mathrm{A})$. You will also have to use a Taylor series to expand
G(**r**, *t*+*dt*)).



(a) Time t *(b) Time (t + dt)*

Solution:

To prove this theorem, consider the closed surface at time *t* that consists of the surface *S* at time *t* plus the surface *S* at time *t+dt*, plus the surface swept out by the contour *C* as it moves from its position at *t* to its position at *t* + *dt*. To understand what this means, consider Figures (a) and (b) above. The gray area in Figure (b) is the surface swept out by the contour *C* as the open surface moves from its position at *t* to its position at *t* + *dt*. The infinitesimal area element for that surface is $d\mathbf{S}_{sides} = d\mathbf{l}\times\mathbf{V}dt$ and the volume it sweeps out in time *dt* is obtained by the integral of $d^3x = \mathbf{V}dt\cdot d\mathbf{S}$. If we apply Gauss's theorem to the vector $\mathbf{G}(\mathbf{r},t)$ at time *t* using the surface defined in Figure (b), we have

$$\int_{vol} \nabla\cdot\mathbf{G}(\mathbf{r},t)\ d^3x = \int_{\text{S(t+dt)+S(t)+sides}} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da =$$

$$= \int_{S(t+dt)} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da - \int_{S(t)} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da + \oint_{Contour} \mathbf{G}(\mathbf{r},t)\cdot d\mathbf{l}\times\mathbf{V}dt$$

where the sign of the second term on the right side of the equation above is negative because for Gauss's Law the surface element vector always points away from the volume of interest, and on the bottom this is the opposite of the vector $d\mathbf{S}$ defined in Figure (a). Since $\mathbf{G}\cdot d\mathbf{l}\times\mathbf{V} = d\mathbf{l}\cdot\mathbf{V}\times\mathbf{G}$, we can write the equation above as

$$\int_{S(t+dt)} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da - \int_{S(t)} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da = \int_{vol} \nabla\cdot\mathbf{G}(\mathbf{r},t)\ (\mathbf{V}dt\cdot d\mathbf{S}) - \oint_{Contour} d\mathbf{l}\cdot\big(\mathbf{V}\times\mathbf{G}(\mathbf{r},t)\big)\,dt$$

Now we are only one step from the theorem. The time derivative of the change in flux of **G** through d**S** is

$$\frac{d}{dt}\int_S \mathbf{G}\cdot\hat{\mathbf{n}}\,da = \lim_{dt\to 0}\frac{\displaystyle\int_{S(t+dt)} \mathbf{G}(\mathbf{X},t+dt)\cdot\hat{\mathbf{n}}\,da - \int_{S(t)} \mathbf{G}(\mathbf{X},t)\cdot\hat{\mathbf{n}}\,da}{dt}$$

But we can expand **G**(**r**, *t*+*dt*) as $\mathbf{G}(\mathbf{r},t) + dt\frac{\partial}{\partial t}\mathbf{G}(\mathbf{r},t) + \ldots$, so that the equation above becomes

$$\frac{d}{dt}\int_S \mathbf{G}\cdot\hat{\mathbf{n}}\,da = \lim_{dt\to 0}\frac{\displaystyle\int_{S(t+dt)} \left[\mathbf{G}(\mathbf{r},t) + dt\frac{\partial}{\partial t}\mathbf{G}(\mathbf{r},t)\right]\cdot\hat{\mathbf{n}}\,da - \int_{S(t)} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da}{dt}$$

$$\frac{d}{dt}\int_S \mathbf{G}\cdot\hat{\mathbf{n}}\,da = \int_S \frac{\partial}{\partial t}\mathbf{G}(\mathbf{r},t)\cdot d\mathbf{S} + \lim_{dt\to 0}\frac{\displaystyle\int_{S(t+dt)} [\mathbf{G}(\mathbf{r},t)]\cdot\hat{\mathbf{n}}\,da - \int_{S(t)} \mathbf{G}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da}{dt}$$

If we use our previous result above in this equation, we recover what was to be proved.

(b) Use this general theorem, and the fact that $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ and $\nabla\cdot\mathbf{B} = 0$, to write an expression for $\frac{d}{dt}\int_S \mathbf{B}\cdot\hat{\mathbf{n}}\,da$ in terms of a line integral involving $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{V}$. This is Faraday's Law for an open surface whose bounding contour is changing in time.

You may want to ponder this form while you play with this visualization, where you can move either the magnet or the coil, and interpret the results.
http://public.mitx.mit.edu/gwt-teal/FaradaysLaw2.html

Solution: $$\oint_{C(t)} \left[\mathbf{E}+\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\cdot d\mathbf{l} = -\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot\hat{\mathbf{n}}\, da$$

Problem 2-7: The Time Dependent Free-Space Green's Function in 1D

(a) The time dependent free-space Green's function in one dimension by definition satisfies the differential equation

$$\left[\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G\left(x,t,x',t'\right)=\delta(x-x')\,\delta(t-t')$$

Show by substitution that the solution for G is given by

$$G\left(x,t,x',t'\right)=-\frac{c}{2}\Theta(t-t'-\frac{|x-x'|}{c}) \qquad \text{where} \qquad \Theta(\eta)=\begin{cases}0 & \eta<0\\ 1 & \eta>0\end{cases}$$

Hint: first show that $\left[\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=\delta(x)\,\delta(t)$

Thus if we consider one-dimensional electromagnetism, that is, problems where there is only a dependence on one spatial dimension x and the time t, then our solutions for the vector and scalar potential become

$$\mathbf{A}(x,t)=-\mu_o\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx'\mathbf{J}(x',t')\,G\left(x,t,x',t'\right)$$

$$\phi(x,t)=-\frac{1}{\varepsilon_o}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx'\rho(x',t')\,G\left(x,t,x',t'\right)$$

Solution: First, we note that

$$\frac{d\Theta}{d\eta}=\delta\left(\eta\right) \text{ since this is zero if } \eta\neq 0 \text{ and } \int_{-\varepsilon}^{\varepsilon}\frac{d\Theta}{d\eta}d\eta=\theta(+\varepsilon)-\theta(-\varepsilon)=1$$

and $$\frac{d\,|x|}{dx}=sign(x)=\begin{cases}-1 & x<0\\ 1 & x>0\end{cases}$$ and by the same reasong as above, $$\frac{d^2\,|x|}{dx^2}=2\delta(x)$$

Using the chain rule, we have

$$\left[\frac{\partial}{\partial x}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=-\frac{c}{2}\left[-\frac{1}{c}\Theta'\frac{d\,|x|}{dx}\right]$$

$$\left[\frac{\partial^2}{\partial x^2}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=\frac{\partial}{\partial x}\left\{\left[\frac{1}{2}\Theta'\frac{d|x|}{dx}\right]\right\}=\frac{1}{2}\left(-\frac{1}{c}\right)\Theta''\left[\frac{d|x|}{dx}\right]^2+\frac{1}{2}\Theta'\frac{d^2|x|}{dx^2}$$

We therefore have

$$\left[\frac{\partial^2}{\partial x^2}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=\left(-\frac{1}{2c}\right)\Theta''+\Theta'\delta(x)$$

Similarly

$$\left[\frac{1}{c^2}\frac{\partial}{\partial t}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=-\frac{1}{2c}\Theta'(t-\frac{|x|}{c})$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=-\frac{1}{2c}\Theta''(t-\frac{|x|}{c})$$

We thus have

$$\left[\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left[-\frac{c}{2}\Theta(t-\frac{|x|}{c})\right]=\delta(x)\,\delta(t-\frac{|x|}{c})$$

But since the left hand side of the above equation is non-zero only when $x = 0$, from the behavior of the first delta function in x, we can replace the argument in second delta function with t, and this is what we wanted to show. Having shown that, we can simply shift our time origin and spatial origin to obtain the general solution.

(b) Maxwell showed that plane electromagnetic waves with **E** and **B** perpendicular to each other and both perpendicular to the direction of propagation were a solution to his vacuum equations. However, he had no idea of how to generate such waves. You will do Maxwell one better in this part of the problem. Assume that you have an infinite sheet of charge with uniform charge per unit area σ_o in the y-z plane and located at $x = 0$, and that it is moving in the y-direction with some given velocity v(t). The corresponds to a current density **J** and a charge density ρ given by

$$\mathbf{J}(x,t)=\hat{\mathbf{y}}\,\sigma_o \mathrm{v}(t)\,\delta(x) \qquad \rho(x,t)=\sigma_o\,\delta(x)$$

Use the source functions above to show that the expressions for $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$ in terms of σ_o and $\mathrm{v}(t)$ are given by

$$\mathbf{B}(x,t)=-\hat{\mathbf{z}}\frac{1}{2}\mu_o\,\sigma_o \mathrm{v}(t-\frac{|x|}{c})\,sign(x)$$

$$\mathbf{E}(x,t)=-\hat{\mathbf{y}}\frac{c}{2}\mu_o\,\sigma_o \mathrm{v}(t-\frac{|x|}{c})+\hat{\mathbf{x}}\frac{\sigma_o}{2\varepsilon_o}sign(x)$$

Solution: $\mathbf{A}(x,t)=\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx'\mu_o\hat{\mathbf{y}}\,\sigma_o \mathrm{v}(t')\,\delta(x')\frac{c}{2}\Theta(t-t'-\frac{|x-x'|}{c})$

$$\mathbf{A}(x,t)=\int_{-\infty}^{\infty}dt'\mu_o\hat{\mathbf{y}}\,\sigma_o \mathrm{v}(t')\frac{c}{2}\Theta(t-t'-\frac{|x|}{c})$$

Similarly,

$$\phi(x,t)=\frac{1}{\varepsilon_o}\int_{-\infty}^{\infty}dt'\,\sigma_o\,\frac{c}{2}\Theta(t-t'-\frac{|x|}{c})$$

Given the potentials, we first calculate **B**.

$$\mathbf{B}(x,t)=\nabla\times\mathbf{A}=\frac{\partial A_y}{\partial x}\hat{\mathbf{z}}=\hat{\mathbf{z}}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}dt'\mu_o\hat{\mathbf{y}}\,\sigma_o \mathrm{v}(t')\frac{c}{2}\Theta(t-t'-\frac{|x|}{c})$$

$$=\ -\hat{\mathbf{z}}\int_{-\infty}^{\infty}dt'\mu_o\,\sigma_o \mathrm{v}(t')\frac{1}{2}\delta(t-t'-\frac{|x|}{c})\frac{\partial|x|}{\partial x}$$

$$\mathbf{B}(x,t)=-\hat{\mathbf{z}}\frac{1}{2}\mu_o\,\sigma_o \mathrm{v}(t-\frac{|x|}{c})\,sign(x)$$

Now, $\mathbf{E}(x,t)=-\dfrac{\partial\mathbf{A}}{\partial t}-\nabla\phi$, so

$$\mathbf{E}(x,t)=-\hat{\mathbf{y}}\frac{c}{2}\mu_o\,\sigma_o \mathrm{v}(t-\frac{|x|}{c})+\hat{\mathbf{x}}\frac{\sigma_o}{2\varepsilon_o}sign(x)$$

These solutions have the following properties. First, there is the expected static electric field in the *x*-direction, of magnitude $\sigma_o/2\varepsilon_o$, which reverses sign across the sheet of charge. Then there is a component of **E** in the *y*-direction, such that right at the sheet, whichever direction the sheet is moving, it is pointed in the opposite direction. This results in a creation rate for electromagnetic energy, $-\mathbf{J}\cdot\mathbf{E}$, which is always positive and non-zero at *x = 0*. Finally there is a magnetic field component which reverses across *x = 0,* is perpendicular to the electric field in the *y*-direction, and such that $\mathbf{E}\times\mathbf{B}$ is always away from the sheet, representing the flow of the energy being created at x = 0 out to $\pm\infty$ infinity. The magnitude of this *B* field is just the magnitude of the *y* component of the electric field divided by the speed of light, as we expect for an electromagnetic plane wave.

(c) Suppose the velocity of the sheet is given by

$$\mathrm{v}(t)=-V_o\Theta(t)\quad\text{where}\quad\Theta(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$$

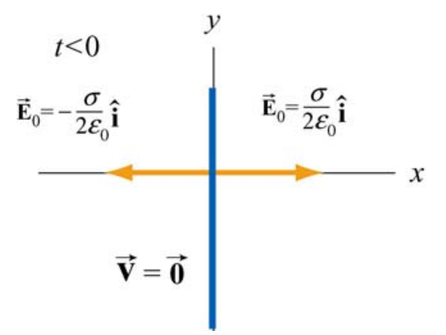
Show for this source function that the fields above are

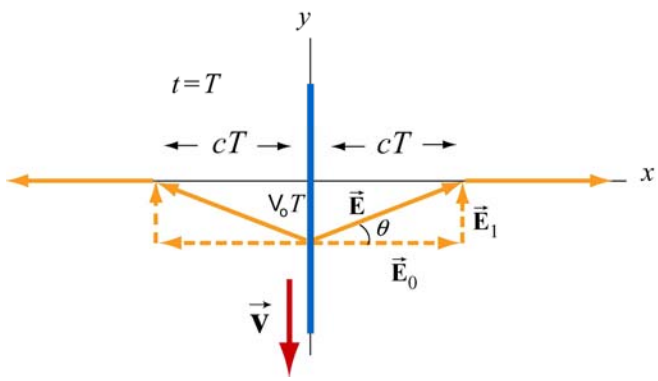
$$\mathbf{B}(x,t)=+\hat{\mathbf{z}}\frac{1}{2}\mu_o\,\sigma_o \mathrm{V}_o\Theta(t-\frac{|x|}{c})\,sign(x)$$

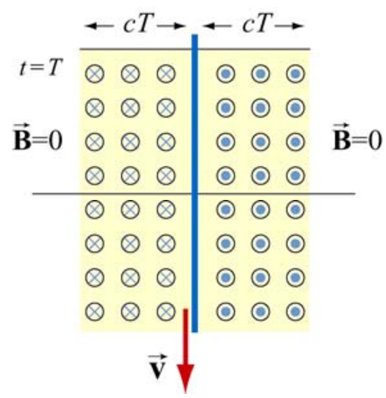
$$\mathbf{E}(x,t) = +\hat{\mathbf{y}}\frac{c}{2}\mu_o\sigma_o \mathrm{V}_o\Theta(t-\frac{|x|}{c}) + \hat{\mathbf{x}}\frac{\sigma_o}{2\varepsilon_o}sign(x) = \frac{\sigma_o}{2\varepsilon_o}\left[\hat{\mathbf{y}}c\mu_o\varepsilon_o \mathrm{V}_o\Theta(t-\frac{|x|}{c}) + \hat{\mathbf{x}}sign(x)\right]$$

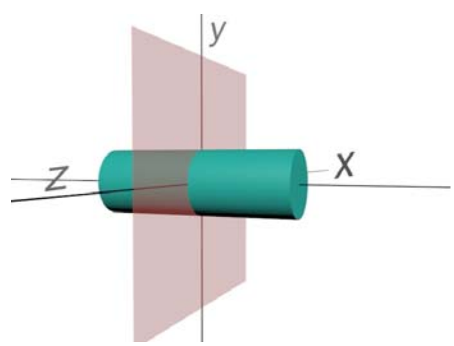
$$\mathbf{E}(x,t) = \frac{\sigma_o}{2\varepsilon_o}\left[\hat{\mathbf{y}}\frac{\mathrm{V}_o}{c}\Theta(t-\frac{|x|}{c}) + \hat{\mathbf{x}}\,sign(x)\right] \quad \text{using } \mu_o\varepsilon_o = \frac{1}{c^2}$$

These solutions are shown the figures below. The first figure below shows the solution for $t < 0$, before the sheet has started to move downward, when **B** is zero and **E** is only in the $\pm\hat{\mathbf{x}}$ direction with magnitude $\sigma_o / 2\varepsilon_o$ The second two figures below show the configuration for **E** and **B**, respectively, at a time $t = T$.









Take a cylinder of length L as shown in the figure to the right such that at time $t = T$, $L=5cT$ (that is, at a time when the fields generated by the motion of the sheet have ***not yet propagated*** to the ends of the cylinder at time T). Show that energy is conserved for this cylinder in integral form, i.,e. show that

$$\frac{d}{dt}\int_{volume}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right]d^3x + \int_{surface}\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)\cdot\hat{\mathbf{n}}\,da = \int_{volume}\left[-\mathbf{E}\cdot\mathbf{J}\right]d^3x$$

Give a physical interpretation of this equation; in particular explain the physical meaning of the zero and the non-zero terms.

Note: We see the field configuration looks like we are pulling down on the electric field line which is "rooted" in the sheet. That is, the "base" of the electric field is attached to the charges that generate it, and is moving downward with those charges at velocity $-V_o\hat{\mathbf{y}}$. The information that the base of the field line is being pulled down is propagating outward in the $\pm\hat{\mathbf{x}}$ direction at the speed of light. Therefore we see a field line with is tilted for $|x| < ct$ so that $\cos\theta = \frac{V_o}{c}$. Note how similar this is to putting a wave on a string. Here the electric field is the string and the tension in the string corresponds to a similar *tension* in the electric field. When you try to displace an electric field line transverse to itself, you set up a restoring tension, just as you do on a string.

To generate your own plane E&M waves of whatever shape you desire, go here:

http://public.mitx.mit.edu/gwt-teal/PlaneWave2.html

(Uncheck the "motion on" box and left click and drag up and down on the sphere. Note you can change the perspective by left clicking and dragging away from the sphere.)

Solution: The form of **E** and **B** for this source function follows directly from our solutions in (b) above. Using these fields, lets calculate the total energy in the cylinder a time *t*, realizing that our fields transverse to the *x*-direction are zero outside of $|x| = ct$ and the length of the cylinder and the time are such that $L > ct$

$$\int_{volume}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right]d^3x = \frac{\sigma_o^2}{8\varepsilon_o}\left[2L\pi R_o^2\right] + \left[\frac{\sigma_o^2}{4\varepsilon_o}\frac{V_o^2}{c^2}\right]\left[2ct\pi R_o^2\right]$$

So

$$\frac{d}{dt}\int_{volume}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right]d^3x = \pi R_o^2\left[\frac{\sigma_o^2 V_o^2}{2\varepsilon_o c}\right]$$

Now let's calculate the rate at which energy is being created at the sheet for *t > 0*

$$\int_{volume}\left[-\mathbf{E}\cdot\mathbf{J}\right]d^3x = \pi R_o^2\left[\frac{V_o}{c}\frac{\sigma_o}{2\varepsilon_o}\right]\left[\sigma_o V_o\right] = \pi R_o^2\frac{\sigma_o^2 V_o^2}{2\varepsilon_o c}$$

Finally let's do the energy flux out of the surface of the cylinder. For *t > 0*

$$\frac{\mathbf{E}\times\mathbf{B}}{\mu_o} = \frac{1}{\mu_o}\left[-\hat{\mathbf{y}}\frac{c}{2}\mu_o\sigma_o \mathrm{v}(t-\frac{|x|}{c}) + \hat{\mathbf{x}}\frac{\sigma_o}{2\varepsilon_o}sign(x)\right]\times\left[-\hat{\mathbf{z}}\frac{1}{2}\mu_o\sigma_o\mathrm{v}(t-\frac{|x|}{c})\,sign(x)\right]$$

$$= \hat{\mathbf{x}}\frac{c}{2}\mu_o\mathrm{v}^2(t-\frac{|x|}{c})\sigma_0^2 sign(x) + \hat{\mathbf{y}}\frac{\sigma_o^2}{4\varepsilon_o}\mathrm{v}(t-\frac{|x|}{c})$$

So if we have that $\mathbf{E}\times\mathbf{B} = 0$ if $|x| > ct$ and for $|x| < ct < L$

$$\frac{\mathbf{E}\times\mathbf{B}}{\mu_o} = \hat{\mathbf{x}}\frac{c}{2}\mu_o V_o^2\sigma_0^2 sign(x) + \hat{\mathbf{y}}\frac{\sigma_o^2 V_o}{4\varepsilon_o}$$

We now want to evaluate this integral over the surface of our cylinder which has *L > ct*. We will get no contribution from the ends of the cylinder because on the ends, the

Poynting flux is zero. If we look at the sides of the cylinder, where the surface normal $\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{x}}$, we will get no contribution from the sides, because the only surviving component of $\hat{\mathbf{n}}\cdot\left[\mathbf{E}\times\mathbf{B}\right]$ will be in the $\hat{\mathbf{n}}\cdot\hat{\mathbf{y}}$ direction, and when we integrate over the sides this will yield zero.

Therefore what we see is that the cylinder of length $L > ct$ is filling up with electromagnetic energy from the transverse components, and that energy is being provided by the agent creating energy at the origin. No energy is flowing out of the surface of the cylinder.