Yichao Yu

1.

(a)

$$
\begin{aligned}
&\left(\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right)_i\\
=&\varepsilon_{ijk}A_j\left(\vec{B}\times\vec{C}\right)_k\\
=&\varepsilon_{kij}\varepsilon_{klm}A_jB_lC_m\\
=&(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})A_jB_lC_m\\
=&B_iA_jC_j-C_iA_jB_j\\
=&\left(\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)\right)_i
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\vec{A}\times\left(\nabla\times\vec{A}\right)\\
=&\left(\nabla\otimes\vec{A}\right)\cdot\vec{A}-\left(\vec{A}\cdot\nabla\right)\vec{A}\\
=&\frac{1}{2}\nabla(A^2)-\left(\vec{A}\cdot\nabla\right)\vec{A}
\end{aligned}
$$

(c)

Use X_c to respresent treating X as constant during the derivative.

$$
\begin{aligned}
&\nabla\times(\vec{A}\times\vec{B})\\
=&\nabla\times(\vec{A}_c\times\vec{B})+\nabla\times(\vec{A}\times\vec{B}_c)\\
=&\vec{A}_c(\nabla\cdot\vec{B})-(\vec{A}_c\cdot\nabla)\vec{B}-\vec{B}_c(\nabla\cdot\vec{A})+(\vec{B}_c\cdot\nabla)\vec{A}\\
=&\vec{A}(\nabla\cdot\vec{B})-(\vec{A}\cdot\nabla)\vec{B}-\vec{B}(\nabla\cdot\vec{A})+(\vec{B}\cdot\nabla)\vec{A}
\end{aligned}
$$

2.

(a)

$$
\begin{aligned}
&\int_{min}^{max}f(x)\Theta'(x-a)\mathrm{d}x\\
=&\int_{min}^{max}f(x)\mathrm{d}\Theta(x-a)\\
=&f(x)\Theta(x-a)|_{min}^{max}-\int_{min}^{max}\Theta(x-a)\mathrm{d}f(x)\\
=&f(max)-\int_{a}^{max}\mathrm{d}f(x)\\
=&f(a)
\end{aligned}
$$

(b)

$$\frac{\mathrm{d}\mathbf{sgn}(t)}{\mathrm{d}t} = \frac{\mathrm{d}2\Theta(t) - 1}{\mathrm{d}t}$$
$$= 2\delta(t)$$

(c)

$$\rho(r, \theta, \phi) = \frac{Q}{4\pi R^2}\delta(r - R)$$

(d)

$$\rho(\rho, \theta, z) = \frac{\lambda}{2\pi b}\delta(\rho - b)$$

(e)

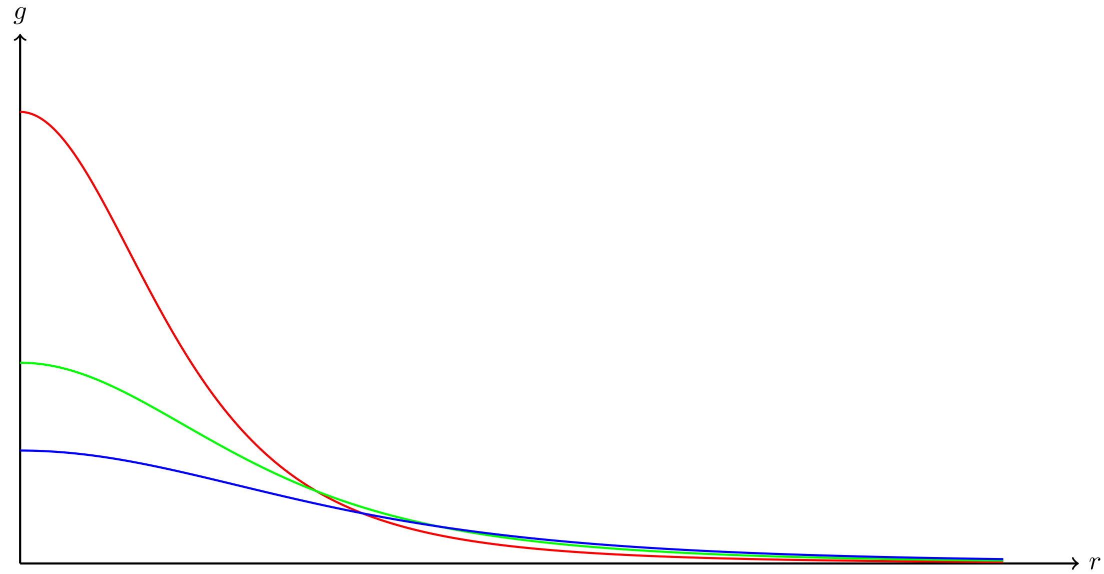
Assuming the disk is parallel to the $x - y$ plane at z_0.

$$\rho(\rho, \theta, z) = \frac{Q}{\pi b^2}\delta(z - z_0)\Theta(b - r)$$

3.

(a)

$$\begin{aligned} g_a =& \nabla^2 f_a \\ =& -\frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2 + a^2}}\right)\right) \\ =& \frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{\sqrt{r^2 + a^2}^3}\right) \\ =& \frac{3}{4\pi(r^2 + a^2)}\frac{\partial}{\partial r}\left(\frac{r}{\sqrt{r^2 + a^2}}\right) \\ =& \frac{3a^2}{4\pi(r^2 + a^2)^{5/2}} \end{aligned}$$



(b)

$$\begin{aligned}\int \mathrm{d}r 4\pi r^2 g_a =& \int \mathrm{d}r 4\pi r^2 \frac{3a^2}{4\pi(r^2+a^2)^{5/2}}\\ =& \int \frac{3\rho^2\mathrm{d}\rho}{(1+\rho^2)^{5/2}}\\ =& \int_0^{\pi/2} 3\sin^2\theta \mathrm{d}\sin\theta\\ =&1\end{aligned}$$

(c)

$$\begin{aligned}\lim_{a\to 0, r\neq 0} g_a(r) =& \lim_{a\to 0, r\neq 0} \frac{3a^2}{4\pi(r^2+a^2)^{5/2}}\\ =& \lim_{a\to 0, r\neq 0} \frac{3a^2}{4\pi r^5}\\ =&0\end{aligned}$$

4.

Area of the spherical cap

$$S =2\pi\sqrt{z^2+R^2}\left(\sqrt{z^2+R^2}-|z|\right)$$

Magnitude of solid angle

$$|\Omega| =2\pi\left(1-\frac{|z|}{\sqrt{z^2+R^2}}\right)$$

Solid angle

$$\begin{aligned}\Omega =&2\pi\left(\mathbf{sgn}(z)-\frac{\mathbf{sgn}(z)|z|}{\sqrt{z^2+R^2}}\right)\\ =&2\pi\left(\mathbf{sgn}(z)-\frac{z}{\sqrt{z^2+R^2}}\right)\end{aligned}$$

For $z=\pm 0$, $\Omega=\pm 2\pi$.

5.

$$\begin{aligned}\Phi(z) =&\frac{1}{4\pi\varepsilon_0}D\Omega\\ =&\frac{D}{2\varepsilon_0}\left(\mathbf{sgn}(z)-\frac{z}{\sqrt{z^2+R^2}}\right)\end{aligned}$$

$\Phi(z=\pm 0)=\pm\dfrac{D}{2\varepsilon_0}$, $\Delta\Phi=\dfrac{D}{\varepsilon_0}$

6.

(a)

$$\begin{aligned}\Phi =&\frac{1}{4\pi\varepsilon_0}\frac{p_o\cos\theta}{R^2}\\ \frac{\mathrm{d}\Phi}{\mathrm{d}\vec{n}} =&\frac{1}{2\pi\varepsilon_0}\frac{p\cos\theta}{R^3}\end{aligned}$$

(b)

$$\begin{aligned}\sigma =&-\frac{1}{2\pi}\frac{p\cos\theta}{R^3}\\ D =&-\frac{1}{4\pi}\frac{p_o\cos\theta}{R^2}\end{aligned}$$

7.

$$\begin{aligned}\nabla\times\vec{F} =&\frac{1}{4\pi}\nabla\times\left(\nabla\times\int\frac{\vec{c}(\vec{r}')\mathrm{d}^3x'}{|\vec{r}-\vec{r}'|}\right)\\ =&\frac{1}{4\pi}\int\mathrm{d}^3x'\nabla\times\left(\nabla\times\frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\\ =&\frac{1}{4\pi}\int\mathrm{d}^3x'\left(\nabla\left(\nabla\cdot\frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)-\nabla^2\frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\end{aligned}$$

Since $\vec{c}$ is a curl field, $\nabla\cdot\vec{c}=0$

$$
\begin{aligned}
\nabla\times\vec{F}=&\frac{1}{4\pi}\int\mathrm{d}^3x'\left(\nabla\left(\nabla\frac{1}{|\vec{r}-\vec{r}'|}\cdot\vec{c}(\vec{r}')\right)-\nabla^2\frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\\
=&\frac{1}{4\pi}\left(-\nabla\int\mathrm{d}^3x'\nabla'\cdot\frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}-\int\mathrm{d}^3x'\vec{c}(\vec{r}')\nabla^2\frac{1}{|\vec{r}-\vec{r}'|}\right)\\
=&\frac{1}{4\pi}\left(-\nabla\oiint\mathrm{d}\vec{S}\cdot\frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}+\int\mathrm{d}^3x'\vec{c}(\vec{r}')4\pi\delta(\vec{r}-\vec{r}')\right)
\end{aligned}
$$

If c drops to 0 faster than $\dfrac{1}{r}$

$$
\begin{aligned}
\nabla\times\vec{F}=&\frac{1}{4\pi}(\vec{c}(\vec{r})4\pi)\\
=&\vec{c}(\vec{r})
\end{aligned}
$$