1.

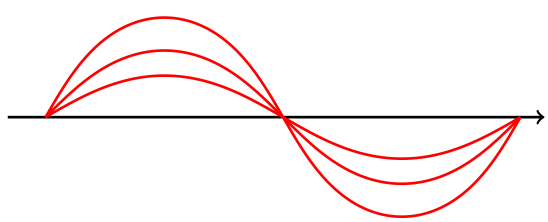
(a)

$$
\begin{aligned}
z =&\bar{n}\int_0^x\frac{\mathrm{d}x}{\sqrt{n^2-\bar{n}^2}}\\
=&\bar{n}\int_0^x\frac{\mathrm{d}x}{\sqrt{n_0^2\mathrm{sech}^2(\alpha x)-\bar{n}^2}}\\
=&\frac{\cos\theta_0}{\alpha}\int_0^{\alpha x}\frac{\cosh(\alpha x)\mathrm{d}\alpha x}{\sqrt{1-\cos^2\theta_0\cosh^2(\alpha x)}}\\
=&\frac{\cos\theta_0}{\alpha}\int_0^{\sinh(\alpha x)}\frac{\mathrm{d}\sinh(\alpha x)}{\sqrt{\sin^2\theta_0-\cos^2\theta_0\sinh^2(\alpha x)}}\\
=&\frac{1}{\alpha}\int_0^{\cot\theta_0\sinh(\alpha x)}\frac{\mathrm{d}y}{\sqrt{1-y^2}}\\
=&\frac{1}{\alpha}\arcsin(\cot\theta_0\sinh(\alpha x))\\
\sin(\alpha z) =&\cot\theta_0\sinh(\alpha x)
\end{aligned}
$$

Since $\max(\sin(\alpha z))=1$

$$
\begin{aligned}
\sin(\alpha z) =&\frac{\sinh(\alpha x)}{\sinh(\alpha x_{max})}\\
\alpha x =&\mathrm{arcsinh}(\sinh(\alpha x_{max})\sin(\alpha z))
\end{aligned}
$$

Rays for $\theta_0=\frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3}$



(b)

$$
Z=\frac{\pi}{\alpha}
$$

Independent from $\bar{n}$

(c)

$$
\begin{aligned}
L_{opt}=&\int_{z=0}^{Z}n\mathrm{d}s\\
=&2\int_0^{x_{max}}n\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\mathrm{d}x\\
=&2\int_0^{x_{max}}n\sqrt{1+\left(\frac{\bar{n}}{\sqrt{n^2-\bar{n}^2}}\right)^2}\mathrm{d}x\\
=&2\int_0^{x_{max}}\frac{n^2}{\sqrt{n^2-\bar{n}^2}}\mathrm{d}x\\
=&2n_0\int_0^{x_{max}}\frac{\mathrm{sech}^2(\alpha x)}{\sqrt{\mathrm{sech}^2(\alpha x)-\cos^2\theta_0}}\mathrm{d}x\\
=&\frac{2n_0}{\alpha}\int_0^{\sinh(\alpha x_{max})}\frac{\mathrm{d}\sinh(\alpha x)}{\cosh^2(\alpha x)\sqrt{1-\cos^2\theta_0\cosh^2(\alpha x)}}\\
=&\frac{2n_0}{\alpha}\int_0^{\sinh(\alpha x_{max})}\frac{\mathrm{d}y}{(1+y^2)\sqrt{\sin^2\theta_0-\cos^2\theta_0y^2}}\\
=&\frac{2n_0}{\alpha\cos\theta_0}\frac{\pi}{2\sqrt{1+\tan^2\theta_0}}\\
=&\frac{\pi n_0}{\alpha}\\
=&n_0Z
\end{aligned}
$$

2.

(a)

For $r<R$, $B_l=0$, for $r>R$, $A_l=0$. Since the charge distribution only have $P_1(\cos\theta)$ component, the only non-zero term in the series is when $l=1$. Therefore, for $r<R$

$$\phi_-=A_1r\cos\theta$$

And for $r>R$

$$\phi_+=\frac{B_1}{r^2}\cos\theta$$

From the boundary condition

$$
\begin{aligned}
A_1R=&\frac{B_1}{R^2}\\
\frac{\sigma}{\varepsilon_0}=&A_1+\frac{2B_1}{R^3}\\
A_1=&\frac{\sigma}{3\varepsilon_0}\\
B_1=&\frac{\sigma R^3}{3\varepsilon_0}
\end{aligned}
$$

(b)

For $r < R$, $B_l = 0$, for $r > R$, $A_l = 0$. Therefore, for $r < R$

$$\begin{aligned}
\phi_- =& \sum_l A_l r^l P_l(\cos\theta)\\
B_{r-} =& \frac{\partial\phi_-}{\partial r}\\
=& \sum_l l A_l r^{l-1} P_l(\cos\theta)\\
B_{\theta-} =& \frac{1}{r}\frac{\partial\phi_-}{\partial\theta}\\
=& \sum_l A_l r^{l-1} \frac{\partial P_l(\cos\theta)}{\partial\theta}
\end{aligned}$$

And for $r > R$

$$\begin{aligned}
\phi_+ =& \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)\\
B_{r+} =& \frac{\partial\phi_+}{\partial r}\\
=& -\sum_l \frac{(l+1)B_l}{r^{l+2}} P_l(\cos\theta)\\
B_{\theta+} =& \frac{1}{r}\frac{\partial\phi_-}{\partial\theta}\\
=& \sum_l \frac{B_l}{r^{l+2}} \frac{\partial P_l(\cos\theta)}{\partial\theta}
\end{aligned}$$

From the boundary condition (integrate the relation for B_θ ignoring a integral/potential constant that doesn't matter)

$$\begin{aligned}
A_l =& -\frac{l+1}{l}\frac{B_l}{R^{2l+1}}\\
\mu_0\kappa_0\sin\theta =& \sum_l \frac{B_l}{R^{l+2}}\frac{\partial P_l(\cos\theta)}{\partial\theta} - \sum_l A_l R^{l-1}\frac{\partial P_l(\cos\theta)}{\partial\theta}\\
\mu_0\kappa_0\cos\theta =& \sum_l \left(\frac{B_l}{R^{l+2}} - A_l R^{l-1}\right) P_l(\cos\theta)
\end{aligned}$$

Therefore only $l = 1$ is not vanishing

$$\begin{aligned}
\mu_0\kappa_0 =& \frac{B_1}{R^3} - A_l\\
A_1 =& -2\frac{B_l}{R^3}\\
B_1 =& \frac{\mu_0\kappa_0 R^3}{3}\\
A_1 =& -\frac{2\mu_0\kappa_0}{3}
\end{aligned}$$

3.