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1.

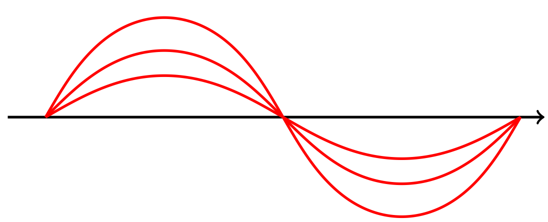
(a)

$$\begin{aligned}
z =& \bar{n}\int_0^x \frac{\mathrm{d}x}{\sqrt{n^2-\bar{n}^2}}\\
=& \bar{n}\int_0^x \frac{\mathrm{d}x}{\sqrt{n_0^2\mathrm{sech}^2(\alpha x)-\bar{n}^2}}\\
=& \frac{\cos\theta_0}{\alpha}\int_0^{\alpha x} \frac{\cosh(\alpha x)\mathrm{d}\alpha x}{\sqrt{1-\cos^2\theta_0\cosh^2(\alpha x)}}\\
=& \frac{\cos\theta_0}{\alpha}\int_0^{\sinh(\alpha x)} \frac{\mathrm{d}\sinh(\alpha x)}{\sqrt{\sin^2\theta_0-\cos^2\theta_0\sinh^2(\alpha x)}}\\
=& \frac{1}{\alpha}\int_0^{\cot\theta_0\sinh(\alpha x)} \frac{\mathrm{d}y}{\sqrt{1-y^2}}\\
=& \frac{1}{\alpha}\arcsin(\cot\theta_0\sinh(\alpha x))\\
\sin(\alpha z) =& \cot\theta_0\sinh(\alpha x)
\end{aligned}$$

Since $\max(\sin(\alpha z)) = 1$

$$\begin{aligned}
\sin(\alpha z) =& \frac{\sinh(\alpha x)}{\sinh(\alpha x_{max})}\\
\alpha x =& \mathrm{arcsinh}(\sinh(\alpha x_{max})\sin(\alpha z))
\end{aligned}$$

Rays for $\theta_0 = \frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3}$



(b)

$$Z = \frac{\pi}{\alpha}$$

Independent from $\bar{n}$

(c)

$$\begin{aligned}
L_{opt} =& \int_{z=0}^{Z} n\mathrm{d}s \\
=& 2\int_0^{x_{max}} n\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
=& 2\int_0^{x_{max}} n\sqrt{1+\left(\frac{\bar{n}}{\sqrt{n^2-\bar{n}^2}}\right)^2}\mathrm{d}x \\
=& 2\int_0^{x_{max}} \frac{n^2}{\sqrt{n^2-\bar{n}^2}}\mathrm{d}x \\
=& 2n_0\int_0^{x_{max}} \frac{\mathrm{sech}^2(\alpha x)}{\sqrt{\mathrm{sech}^2(\alpha x)-\cos^2\theta_0}}\mathrm{d}x \\
=& \frac{2n_0}{\alpha}\int_0^{\sinh(\alpha x_{max})} \frac{\mathrm{d}\sinh(\alpha x)}{\cosh^2(\alpha x)\sqrt{1-\cos^2\theta_0\cosh^2(\alpha x)}} \\
=& \frac{2n_0}{\alpha}\int_0^{\sinh(\alpha x_{max})} \frac{\mathrm{d}y}{(1+y^2)\sqrt{\sin^2\theta_0-\cos^2\theta_0 y^2}} \\
=& \frac{2n_0}{\alpha\cos\theta_0}\frac{\pi}{2\sqrt{1+\tan^2\theta_0}} \\
=& \frac{\pi n_0}{\alpha} \\
=& n_0 Z
\end{aligned}$$

2.

(a)

For $r < R$, $B_l = 0$, for $r > R$, $A_l = 0$. Since the charge distribution only have $P_1(\cos\theta)$ component, the only non-zero term in the series is when $l = 1$. Therefore, for $r < R$

$$\phi_- = A_1 r\cos\theta$$

And for $r > R$

$$\phi_+ = \frac{B_1}{r^2}\cos\theta$$

From the boundary condition

$$\begin{aligned}
A_1 R =& \frac{B_1}{R^2} \\
\frac{\sigma}{\varepsilon_0} =& A_1 + \frac{2B_1}{R^3} \\
A_1 =& \frac{\sigma}{3\varepsilon_0} \\
B_1 =& \frac{\sigma R^3}{3\varepsilon_0}
\end{aligned}$$

(b)

For $r < R$, $B_l = 0$, for $r > R$, $A_l = 0$. Therefore, for $r < R$

$$\begin{aligned}
\phi_- =&\sum_l A_l r^l P_l(\cos\theta)\\
B_{r-} =&\frac{\partial\phi_-}{\partial r}\\
=&\sum_l lA_l r^{l-1} P_l(\cos\theta)\\
B_{\theta-} =&\frac{1}{r}\frac{\partial\phi_-}{\partial\theta}\\
=&\sum_l A_l r^{l-1}\frac{\partial P_l(\cos\theta)}{\partial\theta}
\end{aligned}$$

And for $r > R$

$$\begin{aligned}
\phi_+ =&\sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)\\
B_{r+} =&\frac{\partial\phi_+}{\partial r}\\
=&-\sum_l \frac{(l+1)B_l}{r^{l+2}} P_l(\cos\theta)\\
B_{\theta+} =&\frac{1}{r}\frac{\partial\phi_-}{\partial\theta}\\
=&\sum_l \frac{B_l}{r^{l+2}}\frac{\partial P_l(\cos\theta)}{\partial\theta}
\end{aligned}$$

From the boundary condition (integrate the relation for B_θ ignoring a integral/potential constant that doesn't matter)

$$\begin{aligned}
A_l =&-\frac{l+1}{l}\frac{B_l}{R^{2l+1}}\\
\mu_0\kappa_0\sin\theta =&\sum_l \frac{B_l}{R^{l+2}}\frac{\partial P_l(\cos\theta)}{\partial\theta} - \sum_l A_l R^{l-1}\frac{\partial P_l(\cos\theta)}{\partial\theta}\\
\mu_0\kappa_0\cos\theta =&\sum_l\left(\frac{B_l}{R^{l+2}} - A_l R^{l-1}\right) P_l(\cos\theta)
\end{aligned}$$

Therefore only $l = 1$ is not vanishing

$$\begin{aligned}
\mu_0\kappa_0 =&\frac{B_1}{R^3} - A_l\\
A_1 =&-2\frac{B_l}{R^3}\\
B_1 =&\frac{\mu_0\kappa_0 R^3}{3}\\
A_1 =&-\frac{2\mu_0\kappa_0}{3}
\end{aligned}$$

3.

Let the surface charge density be $\dfrac{\sigma_0}{\sqrt{R^2-\rho^2}}$.

On the Z axis

$$\begin{aligned}\phi =&\frac{\sigma_0}{2\varepsilon_0}\int_0^R\frac{\rho\mathrm{d}\rho}{\sqrt{\rho^2+r^2}\sqrt{R^2-\rho^2}}\\ =&\frac{\sigma_0}{4\varepsilon_0}\int_0^{R^2}\frac{\mathrm{d}x}{\sqrt{r^2+x}\sqrt{R^2-x}}\\ =&\frac{\sigma_0}{4\varepsilon_0}\left(\frac{\pi}{2}+\arctan\left(\frac{R^2-r^2}{2rR}\right)\right)\\ =&\frac{\sigma_0}{2\varepsilon_0}\sum_{l=0}^{\infty}(-1)^l\frac{R^{2l+1}}{r^{2l+1}}\end{aligned}$$

For off-axis points

$$\phi =\frac{\sigma_0}{2\varepsilon_0}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos\theta)$$

For the origin (from the expression for on-axis points)

$$V =\frac{\pi\sigma_0}{4\varepsilon_0}$$

Therefore

$$\phi =\frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos\theta)$$