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8.311 Problem Set 3
Recitation Section 1

# 1.

## (a)

$$
\begin{aligned}
\vec{E}_{dipole} =& -\nabla\left(\frac{1}{4\pi\varepsilon_0}\frac{\vec{r}\cdot\vec{p}}{r^3}\right)\\
=& -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\vec{r}\cdot\vec{p}}{r^3}\right)\\
=& -\frac{\vec{r}\cdot\vec{p}}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r^3}\right)-\frac{1}{4\pi\varepsilon_0 r^3}\nabla(\vec{r}\cdot\vec{p})\\
=&\frac{\vec{r}\cdot\vec{p}}{4\pi\varepsilon_0}\frac{3\hat{n}}{r^4}-\frac{\vec{p}}{4\pi\varepsilon_0 r^3}\\
=&\frac{3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

## (b)

$$
\begin{aligned}
&\int\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})\mathrm{d}^3x\\
=&\frac{1}{2}\int\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})\mathrm{d}^3x+\frac{1}{2}\int\left(\hat{n}\times\left(\vec{J}(\vec{r}')\times\vec{r}'\right)-\vec{r}'\left(\hat{n}\cdot\vec{J}(\vec{r}')\right)\right)\mathrm{d}^3x\\
=&\hat{n}\times\frac{1}{2}\int\vec{J}(\vec{r}')\times\vec{r}'\mathrm{d}^3x+\frac{1}{2}\int\left(\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})-\vec{r}'\left(\hat{n}\cdot\vec{J}(\vec{r}')\right)\right)\mathrm{d}^3x
\end{aligned}
$$

For arbitrary vector $\vec{l}$

$$
\begin{aligned}
&\nabla'\cdot\left(\left(\vec{l}\cdot\vec{r}'\right)(\hat{n}\cdot\vec{r}')\vec{J}(r')\right)\\
=&(\hat{n}\cdot\vec{r}')\vec{J}(r')\cdot\nabla'\left(\vec{l}\cdot\vec{r}'\right)+\left(\vec{l}\cdot\vec{r}'\right)\vec{J}(r')\cdot\nabla'(\hat{n}\cdot\vec{r}')+\left(\vec{l}\cdot\vec{r}'\right)(\hat{n}\cdot\vec{r}')\nabla'\cdot\vec{J}(r')\\
=&(\hat{n}\cdot\vec{r}')\vec{J}(r')\cdot\vec{l}+\left(\vec{l}\cdot\vec{r}'\right)\vec{J}(r')\cdot\hat{n}\\
=&\vec{l}\cdot\left((\hat{n}\cdot\vec{r}')\vec{J}(r')+\vec{r}'\vec{J}(r')\cdot\hat{n}\right)
\end{aligned}
$$

Integrate both sides

$$
\begin{aligned}
0=&\vec{l}\cdot\int\mathrm{d}^3x'\left((\hat{n}\cdot\vec{r}')\vec{J}(r')+\vec{r}'\vec{J}(r')\cdot\hat{n}\right)\\
0=&\int\mathrm{d}^3x'\left((\hat{n}\cdot\vec{r}')\vec{J}(r')+\vec{r}'\vec{J}(r')\cdot\hat{n}\right)\\
&\int\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{n})\mathrm{d}^3x\\
=&\vec{m}\times\vec{n}
\end{aligned}
$$

### (c)

$$\begin{aligned}
\frac{4\pi}{\mu_0}\vec{B}_{dipole} =&\nabla\times\frac{\vec{m}\times\vec{r}}{r^3}\\
=&\vec{m}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)-(\vec{m}\cdot\nabla)\frac{\vec{r}}{r^3}\\
=&-\vec{r}\left(\vec{m}\cdot\nabla\frac{1}{r^3}\right)-\frac{1}{r^3}(\vec{m}\cdot\nabla)\vec{r}\\
=&\vec{r}\frac{3\vec{m}\cdot\hat{n}}{r^4}-\frac{\vec{m}}{r^3}\\
=&\frac{3\hat{n}(\vec{m}\cdot\hat{n})-\vec{m}}{r^3}
\end{aligned}$$

## 2.

### (a)

$$\begin{aligned}
E =&\int_R^\infty \mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\sin\theta\mathrm{d}\theta\frac{p^2}{32\pi^2\varepsilon_0 r^4}\left(4\cos^2\theta+\sin^2\theta\right)\\
=&\frac{p^2}{16\pi\varepsilon_0}\int_R^\infty\frac{\mathrm{d}r}{r^4}\int_0^\pi \mathrm{d}\theta\sin\theta\left(4\cos^2\theta+\sin^2\theta\right)\\
=&\frac{p^2}{12\pi\varepsilon_0 R^3}
\end{aligned}$$

### (b)

Direvatives of the dipole moment

$$\begin{aligned}
\dot{\vec{p}} =&\dot{p}\hat{z}\\
\ddot{\vec{p}} =&\ddot{p}\hat{z}
\end{aligned}$$

Magnetic field

$$\begin{aligned}
\vec{B} =&\frac{\hat{z}\times\hat{n}}{4\pi\varepsilon_0}\left(\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{cr}\right)\\
=&\frac{\sin\theta\hat{\phi}}{4\pi\varepsilon_0}\left(\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{cr}\right)
\end{aligned}$$

Electric field

$$\begin{aligned}
\vec{E} =&\frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{4\pi\varepsilon_0 r^3}+\frac{3\hat{n}\left(\dot{\vec{p}}\cdot\hat{n}\right)-\dot{\vec{p}}}{4\pi\varepsilon_0 cr^2}+\frac{\left(\ddot{\vec{p}}\times\hat{n}\right)\times\hat{n}}{4\pi\varepsilon_0 c^2 r}\\
=&\frac{3p\cos\theta\hat{n}-p\hat{z}}{4\pi\varepsilon_0 r^3}+\frac{3\dot{p}\cos\theta\hat{n}-\dot{p}\hat{z}}{4\pi\varepsilon_0 cr^2}+\frac{\ddot{p}\sin\theta\hat{\phi}\times\hat{n}}{4\pi\varepsilon_0 c^2 r}\\
=&\frac{3p\cos\theta\hat{n}-p\hat{z}}{4\pi\varepsilon_0 r^3}+\frac{3\dot{p}\cos\theta\hat{n}-\dot{p}\hat{z}}{4\pi\varepsilon_0 cr^2}+\frac{\ddot{p}\sin\theta\hat{\theta}}{4\pi\varepsilon_0 c^2 r}
\end{aligned}$$

Since $\hat{z} = \cos\theta\hat{n} - \sin\theta\hat{\theta}$

$$\begin{aligned}\vec{E} =& \frac{2p\cos\theta\hat{n} + p\sin\theta\hat{\theta}}{4\pi\varepsilon_0 r^3} + \frac{2\dot{p}\cos\theta\hat{n} + \dot{p}\sin\theta\hat{\theta}}{4\pi\varepsilon_0 c r^2} + \frac{\ddot{p}\sin\theta\hat{\theta}}{4\pi\varepsilon_0 c^2 r}\\ =& \frac{2\cos\theta\hat{n}}{4\pi\varepsilon_0}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2}\right) + \frac{\sin\theta\hat{\theta}}{4\pi\varepsilon_0}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)\end{aligned}$$

Energy flux

$$\begin{aligned}\Phi_E =& \frac{1}{\mu_0 c^2}\int \mathrm{d}t \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta \sin\theta r^2 \frac{\sin\theta}{4\pi\varepsilon_0}\left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{cr}\right)\frac{\sin\theta}{4\pi\varepsilon_0}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)\\ =& \frac{1}{6\pi\varepsilon_0}\int \mathrm{d}t\left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c}\right)\left(\frac{p}{r^2} + \frac{\dot{p}}{cr} + \frac{\ddot{p}}{c^2}\right)\\ =& \frac{1}{6\pi\varepsilon_0}\left(\frac{1}{2r}\left(\frac{p}{r} + \frac{\dot{p}}{c}\right)^2\right)\Bigg|_{t_0}^{t_1} + \frac{1}{6\pi\mu_0\varepsilon_0^2}\int \mathrm{d}t\left(\frac{\dot{p}}{r} + \frac{\ddot{p}}{c}\right)\left(\frac{\ddot{p}}{c^2}\right)\\ =& \frac{p_2^2 - p_1^2}{12\pi\varepsilon_0 r^3} + \int \frac{\ddot{p}^2}{6\pi\mu_0\varepsilon_0^2 c^3}\mathrm{d}t\end{aligned}$$

The first term corresponds to change in the energy stored in the field.

# 3.

## (a)

$$r = \frac{mV_0}{qB_0}$$

## (b)

$$T = \frac{\pi m}{qB}$$

## (c)

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} =& \frac{q^4 V_0^2 B^2}{6\pi\varepsilon_0 m^2 c^3}\\ =& \frac{q^2 V_0^4}{6\pi\varepsilon_0 c^3 R^2}\end{aligned}$$

## (d)

$$\begin{aligned}W =& \frac{q^2 V_0^3}{6\varepsilon_0 c^3 R}\\ =& \frac{2\pi m V_0^3}{3cR}\end{aligned}$$

### (e)

$$\frac{W}{E_k} = \frac{2\pi V_0 R_{classical}}{3cR}$$

When $R$ is large.

### (f)

## 4.

### (a)

$$\begin{aligned} p =& Q_0 d \sin\omega t \\ \left|\frac{\mathrm{d}W}{\mathrm{d}t}\right| =& \frac{Q_0^2 d^2 \omega^4}{6\pi\varepsilon_0 c^3} \left|\sin^2\omega t\right| \\ =& \frac{Q_0^2 d^2 \omega^4}{12\pi\varepsilon_0 c^3} \end{aligned}$$

### (b)

$$\begin{aligned} E_{rad} =& \frac{Q_0^2 d^2 \omega^4}{12\pi\varepsilon_0 c^3} \frac{2\pi}{\omega} \\ =& \frac{Q_0^2 d^2 \omega^3}{6\varepsilon_0 c^3} \\ \frac{4CE_{rad}}{Q_0^2} =& \frac{2d^2 C\omega^3}{3\varepsilon_0 c^3} \\ =& \frac{2dAk^3}{3} \end{aligned}$$

Therefore if $dk$ and $Ak^2$ are all small (where $k$ is the wave vector) the radiation is small.

### (c)

$$\begin{aligned} R_{rad} =& \frac{Q_0^2 d^2 \omega^4}{12\pi\varepsilon_0 c^3} \frac{2}{\omega^2 Q_0^2} \\ =& \frac{d^2\omega^2}{6\pi\varepsilon_0 c^3} \end{aligned}$$

### (d)

$$
\begin{aligned}
R_{rad} =& \frac{d^2}{6\pi\varepsilon_0 c^3 LC} \\
=& \frac{hd^3}{6\pi\varepsilon_0 c^3 \varepsilon_0 A_c \mu_0 N^2 A_L} \\
=& \mu_0 c \frac{hd^3}{6\pi A_c N^2 A_L}
\end{aligned}
$$

## 5.

### (a)

### (b)

### (c)

### (d)

### (e)

### (f)

## 6.

### (a)

### (b)

### (c)