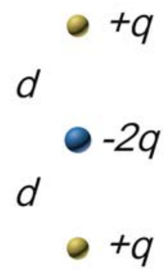
Solutions Assignment #4: Due Friday March 6, 2015 at 2:30 pm

Reading: Jackson 4.1 beginning on p 145. Jackson Section 6.5 beginning on p 246. Jackson Section 9.3 beginning on page 413. Jackson Section 16.through 16.3, beginning on page 745.

Problems

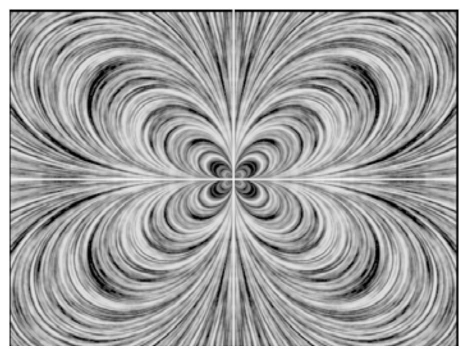
Problem 4-1: Electric Quadrupole radiation
Why:?An example of a more complicated angular dependence of radiation than electric dipole radiation, radiating at twice the frequency you might naively expect.

(a) A static electric quadrupole. A charge $+q$ sits a distance d up on the +z-axis, and a charge $+q$ sits the same distance down on the –z-axis. A charge $+2q$ sits at the origin. Equation 4.10 of Jackson p 142 is



$$\Phi(\mathbf{X}) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{X}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \ldots\right]$$

where $Q_{ij} = \int\left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\mathbf{X}')\,d^3x'$. In spherical polar coordinates, what is the potential due to this distribution of charge? What are the electric field components E_r and E_θ. What is the equation for $r(\theta)$ describing an electrostatic quadrupole field line, assuming $r >> d$? At what angle will a given field line be furthest from the origin?



If we have a set of particles with charges q_m located at $\mathbf{X}^m$ the associated charge density is

$$\rho(\mathbf{X}') = \sum_m q^m \delta^3\left(\mathbf{X}' - \mathbf{X}^m\right)$$

$$Q_{ij} = \int\left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\mathbf{X}')\,d^3x' = \sum_m q^m\left(3X_i^m X_j^m - \left|\mathbf{X}^m\right|^2\delta_{ij}\right)$$

$$Q_{xx} = -2qd^2 \quad Q_{yy} = -2qd^2 \quad Q_{zz} = 4qd^2 \quad \text{otherwise } Q_{ij} \text{ is zero}$$

The potential using the above equation is

$$\Phi(\mathbf{X}) = \frac{1}{8\pi\varepsilon_o}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} = \frac{2qd^2}{8\pi\varepsilon_o}\frac{\left(2z^2 - \left(x^2+y^2\right)\right)}{r^5} = \frac{2qd^2}{8\pi\varepsilon_o}\frac{\left(2\cos^2\theta - \sin^2\theta\right)}{r^3}$$

$$\Phi(\mathbf{X}) = \frac{qd^2}{4\pi\varepsilon_o}\frac{\left(3\cos^2\theta - 1\right)}{r^3}$$

$$E_r = -\frac{\partial\Phi(\mathbf{X})}{\partial r} = \frac{3qd^2}{4\pi\varepsilon_o}\frac{(3\cos^2\theta - 1)}{r^4} \quad E_\theta = -\frac{\partial\Phi(\mathbf{X})}{r\partial\theta} = \frac{qd^2}{4\pi\varepsilon_o}\frac{6\cos\theta\sin\theta}{r^4}$$

Thus the equation for the field line is

$\frac{1}{r}\frac{dr}{d\theta} = \frac{E_r}{E_\theta} = \frac{(3\cos^2\theta - 1)}{2\cos\theta\sin\theta}$ and to solve this equation we have to solve

$$\int\frac{dr}{r} = \int\frac{(3\cos^2\theta - 1)}{2\cos\theta\sin\theta}d\theta$$

$$\int\frac{(3\cos^2\theta - 1)}{2\cos\theta\sin\theta}d\theta = \frac{3}{2}\int\frac{\cos\theta}{\sin\theta}d\theta - \frac{1}{2}\int\frac{1}{\cos\theta\sin\theta}d\theta = \frac{3}{2}\ln\sin\theta - \frac{1}{2}\ln\frac{\sin\theta}{\cos\theta} + \text{constant}$$

$$= \frac{1}{2}\left[\ln\sin^3\theta + \ln\frac{\cos\theta}{\sin\theta}\right] + \text{constant} = \frac{1}{2}\left[\ln\sin^2\theta\cos\theta\right] + \text{constant}$$

So $\ln r = \frac{1}{2}\left[\ln\sin^2\theta\cos\theta\right] + \text{constant}$ or $r = r_o\sin\theta\sqrt{|\cos\theta|}$ where r_o is a positive constant. The field line will be furthest from the origin when its radial component vanishes, or when $(3\cos^2\theta - 1) = 0$ or when $\theta = \arccos 1/\sqrt{3}$ or 54.73 degrees or 125.26 degrees.

(b) Now let the positive charges move up and down the z-axis, with the top charge having a time-dependent position $d\cos(\omega_o t)$ and the bottom charge having a time-dependent position $-d\cos(\omega_o t)$. The subsequent electric quadrupolke radiation magnetic field in the radiation zone can be shwn to be contained in equation (42) page 12 of our 8.311 Simple Radiating System Notes, and is given by

$$\mathbf{B}_{el\ quad}(\mathbf{r},t) = -\frac{\mu_o}{24\pi c^2}\frac{1}{r}\hat{\mathbf{n}}\ \mathbf{x}\left[\hat{\mathbf{n}}\cdot\dddot{\mathbf{Q}}\right]$$

What are is the magnetic and electric fields in spherical polar coordinates for this particular time varying electric quadrapole moment. What is the angular distribution of the emitted radiation? At what angle is the maximum energy radiated? What is the total energy radiated into all solid angles? At what frequency does that radiation appear?

$$\ddot{\mathbf{Q}} = 2qd^2\cos^2(\omega_o t)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = qd^2\left(1+\cos 2\omega_o t\right)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\hat{\mathbf{n}}\cdot\frac{d^3}{dt^3}\ddot{\mathbf{Q}} =$$

$$\left(2\omega_o\right)^3 qd^2\sin 2\omega_o t\,\hat{\mathbf{n}}\cdot\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{8\omega_o^3 qd^2\sin 2\omega_o t}{r}\left[-x\hat{\mathbf{x}} - y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}}\right]$$

$$= \frac{8\omega_o^3 qd^2\sin 2\omega_o t}{r}\left[-x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - z\hat{\mathbf{z}} + 3z\hat{\mathbf{z}}\right] = \frac{8\omega_o^3 qd^2\sin 2\omega_o t}{r}\left[-r\hat{\mathbf{n}} + 3z\hat{\mathbf{z}}\right]$$

$$\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\cdot\frac{d^3}{dt^3}\ddot{\mathbf{Q}}\right] = \frac{8\omega_o^3 qd^2\sin 2\omega_o t}{r}\hat{\mathbf{n}}\times\left[-r\hat{\mathbf{n}} + 3z\hat{\mathbf{z}}\right] = -\left(24\omega_o^3 qd^2\sin 2\omega_o t\right)\cos\theta\sin\theta\,\hat{\boldsymbol{\varphi}}$$

$$\mathbf{B}_{el\ quad}(\mathbf{r},t) = \frac{\mu_o}{24\pi c^2}\frac{1}{r}\left(24\omega_o^3 qd^2\sin 2\omega_o t\right)\cos\theta\sin\theta\,\hat{\boldsymbol{\varphi}}$$

$$\mathbf{E}_{el\ quad}(\mathbf{r},t) = \frac{c\mu_o}{24\pi c^2}\frac{1}{r}\left(24\omega_o^3 qd^2\sin 2\omega_o t\right)\cos\theta\sin\theta\,\hat{\boldsymbol{\theta}}$$

$$\frac{r^2}{\mu_o}\left(\mathbf{E}_{el\ quad}(\mathbf{r},t)\times\mathbf{B}_{el\ quad}(\mathbf{r},t)\right)\cdot\hat{\mathbf{r}} = \frac{c}{\mu_o}\left(\frac{\mu_o}{2\pi c^2}\left(\omega_o^3 qd^2\sin 2\omega_o t\sin 2\theta\right)\right)^2$$

$$= \frac{\mu_o\omega_o^6 q^2 d^4}{4\pi^2 c^3}\sin^2 2\omega_o t\sin^2 2\theta = \frac{dP}{d\Omega}$$ so maximum energy at $\theta = 45$ and 135 degrees

If we integrate over solid angle we get

$$P = \frac{\mu_o\omega_o^6 q^2 d^4}{2\pi c^3}\sin^2 2\omega_o t\int d\cos\theta\left(1-2\cos^2\theta\right)^2$$

$$= \frac{\mu_o\omega_o^6 q^2 d^4}{2\pi c^3}\sin^2 2\omega_o t\left[2 - \frac{8}{3} + \frac{8}{5}\right] = \frac{14\mu_o\omega_o^6 q^2 d^4}{30\pi c^3}\sin^2 2\omega_o t$$

$$\left\langle P\right\rangle = \frac{7\mu_o\omega_o^6 q^2 d^4}{30\pi c^3}$$

(c) Suspect we only had the top charge moving up and down with the given motion, and no other charges. What would be the total rate at which energy is radiated into electric dipole radiation into all solid angles? At what frequency does this appear? What is the ratio of your answer for the total rate at which energy is radiated into electric quadrupole from above to the rate energy is radiated into electric dipole here?

$$\left\langle P_{\text{electric dipole}}\right\rangle = \frac{q^2 d^2 \omega_o^4}{12\pi\varepsilon_o c^3} \qquad \frac{\left\langle P_{quadrupole}\right\rangle}{\left\langle P_{\text{electric dipole}}\right\rangle} = \frac{7\,\mu_o \omega_o^6 q^2 d^4}{30\pi c^3} \Big/ \frac{q d^2 \omega_o^4}{12\pi\varepsilon_o c^3} = \frac{42}{15}\frac{\omega_o^2 d^2}{c^2} << 1$$

Problem 4-2: Whatever happened to Biot-Savart?
Why:? Now that we have introduced $\mathbf{A}$ *and* Φ *it seems we never talk about laws from statics, like the Biot-Savart Law. Is there a time-dependent version of Biot-Savart that is still valid in some regions in some cases? Yes there is!*

(a) Show that we can write the correct time-dependent solution for $\mathbf{B}(\mathbf{X},t)$ without ever introducing the vector potential **A**, and that the form below is the correct expression (compare this to Jackson equation 6.56 p 247 which is slightly different but equivalent)

$$\mathbf{B}(\mathbf{X},t) = \frac{\mu_o}{4\pi}\int_{\text{all space}} d^3x' \left\{ \mathbf{J}(\mathbf{X}',t-\frac{|\mathbf{X}-\mathbf{X}'|}{c}) \times \frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3} + \frac{1}{c}\left[\frac{1}{|\mathbf{X}-\mathbf{X}'|}\frac{\partial \mathbf{J}(\mathbf{X}',t-\frac{|\mathbf{X}-\mathbf{X}'|}{c})}{\partial t'}\right] \times \frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|}\right\}$$

In doing this you may assume that by parts we can write

$$\int d^3x' \frac{1}{|\mathbf{X}-\mathbf{X}'|}\frac{\partial J_k}{\partial x'_j} = -\int d^3x' J_k \left(\frac{\partial}{\partial x'_j}\frac{1}{|\mathbf{X}-\mathbf{X}'|}\right)$$

Solution: Since $\nabla\times\mathbf{B} = \mu_o \mathbf{J} + \mu_o\varepsilon_o \frac{\partial}{\partial t}\mathbf{E}$, $\nabla\times(\nabla\times\mathbf{B}) = \mu_o\nabla\times\mathbf{J} + \mu_o\varepsilon_o\frac{\partial}{\partial t}\nabla\times\mathbf{E}$ or

$$-\nabla^2\mathbf{B} = \mu_o\nabla\times\mathbf{J} + \mu_o\varepsilon_o\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{and thus} \quad \left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{B} = \mu_o\nabla\times\mathbf{J}$$

The last equation above is of the form equation 6.32 of Jackson p 243, and thus we can write down the solution as

$$\mathbf{B}(\mathbf{X},t) = \frac{\mu_o}{4\pi}\int_{\text{all space}} d^3x' \frac{1}{|\mathbf{X}-\mathbf{X}'|}\{\nabla'\times\mathbf{J}\}_{t-|\mathbf{X}-\mathbf{X}'|/c}$$

From the form of the equation we are trying to prove, we see that what we want is $\mathbf{J}$ evaluated at the retarded time, not $\nabla'\times\mathbf{J}$ evaluated at the retarded time. According to Jackson bottom of page 246, “the meaning of ∇' [in the above equation] ….is a spatial gradient with respect in x' with t' fixed [then evaluated at the retarded time]; the meaning outside the retarded bracket is a spatial gradient with respect to x' with x and t

fixed…it is necessary to correct for the x' dependence introduced … when the gradient operator is taken outside. “. In mathematics,

$$\left\{\nabla'\times\mathbf{J}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}=\nabla'\times\left\{\mathbf{J}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}+\left\{\frac{\partial\mathbf{J}}{\partial t'}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}\times\nabla'\left(t-|\mathbf{X}-\mathbf{X}'|/c\right)$$
$$=\nabla'\times\left\{\mathbf{J}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}+\frac{1}{c}\left\{\frac{\partial\mathbf{J}}{\partial t'}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}\times\left(\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|}\right)$$

Inserting this into the first equation for $\mathbf{B}$ above, we have

$$\mathbf{B}(\mathbf{X},t)=\frac{\mu_o}{4\pi}\int_{\text{all space}}d^3x'\frac{1}{|\mathbf{X}-\mathbf{X}'|}\left[\nabla'\times\left\{\mathbf{J}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}+\frac{1}{c}\left\{\frac{\partial\mathbf{J}}{\partial t'}\right\}_{t-|\mathbf{X}-\mathbf{X}'|/c}\times\left(\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|}\right)\right]$$

But if we integrate the first term here by parts and throw away the values at infinity [that is,

$\int d^3x'\frac{1}{|\mathbf{X}-\mathbf{X}'|}\frac{\partial J_k}{\partial x'_j}=-\int d^3x' J_k\left(\frac{\partial}{\partial x'_j}\frac{1}{|\mathbf{X}-\mathbf{X}'|}\right)$], we find that the above becomes

$$\mathbf{B}(\mathbf{X},t)=\frac{\mu_o}{4\pi}\int_{\text{all space}}d^3x'\left\{\mathbf{J}(\mathbf{X}',t-\frac{|\mathbf{X}-\mathbf{X}'|}{c})\times\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}+\frac{1}{c}\left[\frac{1}{|\mathbf{X}-\mathbf{X}'|}\frac{\partial\mathbf{J}(\mathbf{X}',t-\frac{|\mathbf{X}-\mathbf{X}'|}{c})}{\partial t'}\right]\times\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|}\right\}$$

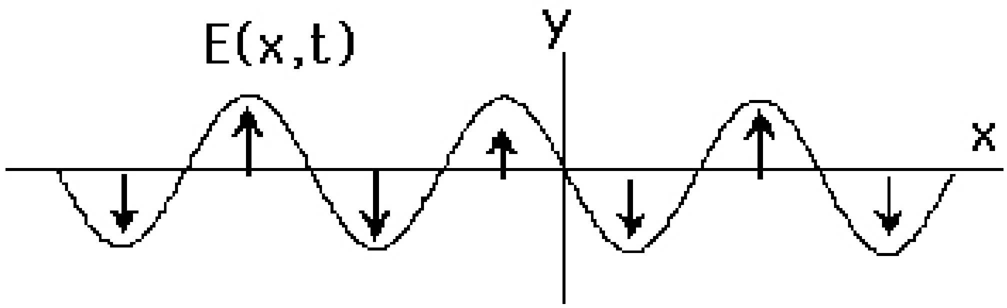
as desired.

(b) The first term in the above equation for $\mathbf{B}$ is just something that looks like the Biot-Savat Law except we have retarded the evaluation of the time dependence of the current. Take a spatial limited distribution of currents (that is, the currents vanish outside of $r > d$) and assume that the currents vary slowly in time ($T >> d/c$). Make a simple dimensional argument that if we are calculating $\mathbf{B}$ in the quasi-static region $r << cT$ (which may be quite large if T is quite large), that we can ignore the second term in the above equation compared to the first. What we have left is a “time-dependent” form of Biot-Savart. Note also that in the quasi-static region region $t-|\mathbf{X}-\mathbf{X}'|/c$ is not that different from t, so that again we do not make much error if we simply ignore the retardation in this region. Thus, Biot-Savart is still a reasonable formula to use if we are in the quasi-static region, which can be very large indeed. There is a similar statement for the “time-dependent” form of Coulomb’s Law (see equation 6.55 of Jackson, page 247).

Solution:

$$\frac{1}{c}\left[\frac{1}{|\mathbf{X}-\mathbf{X}'|}\frac{\partial \mathbf{J}(\mathbf{X}',t-\frac{|\mathbf{X}-\mathbf{X}'|}{c})}{\partial t'}\times\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|}\right] / \left[\mathbf{J}(\mathbf{X}',t-\frac{|\mathbf{X}-\mathbf{X}'|}{c})\times\frac{\mathbf{X}-\mathbf{X}'}{|\mathbf{X}-\mathbf{X}'|^3}\right]\approx\left(\frac{r}{cT}\text{ or }\frac{d}{cT}\right)<<1$$

Problem 4.3: The Thompson cross-section



Consider an electromagnetic wave propagating in the x direction, and polarized in the y direction; that is, $\mathbf{E}(x,t)=\hat{\mathbf{y}}\,E_o\cos(\omega_o\,t-k\,x)$. The time-average Poynting flux in the x-direction is given by $\langle S\rangle=\frac{c}{2}\varepsilon_o E_o{}^2$. An electron sits at the origin and oscillates in the electric field of this wave. The equation of motion of this electron is

$$m_e\,\mathbf{a}=-e(\mathbf{E}+\mathbf{V}\mathbf{x}\mathbf{B})\cong -e\,E_o\cos(\omega_o\,t)\;\hat{\mathbf{y}}=\text{Re}(-eE_oe^{-i\omega_o t})\hat{\mathbf{y}} \tag{1}$$

where **v** is the velocity of the electron, and **a** its acceleration. In the last term on the right in equation (1), we have dropped the $\mathbf{v}\times\mathbf{B}$ term because the ratio $\frac{|\mathbf{v}|B}{E}=\frac{|\mathbf{v}|}{c}$, since $\frac{B}{E}=\frac{1}{c}$ for an electromagnetic wave, and we assume that ω and E_O are such that the electron motion is non-relativistic. To a good approximation, the acceleration of the electron is therefore given by

$$\mathbf{a}(t)=\text{Re}\left[-\frac{e\,E_o}{m_e}e^{-i\omega_o t}\right]\hat{\mathbf{y}}=-\frac{e\,E_o}{m_e}\cos(\omega_o t)\hat{\mathbf{y}} \tag{2}$$

with the velocity **v** of the electron given by

$$\mathbf{v}(t)=\text{Re}\left[-\frac{e\,E_o}{(-i\omega_o)m_e}e^{-i\omega_o t}\right]\hat{\mathbf{y}}=-\frac{e\,E_o}{\omega_o m_e}\sin(\omega_o t)\hat{\mathbf{y}} \tag{3}$$

(a) What is the average rate at which energy is radiated by this electron into all solid angle? Write this rate (joules per second) in terms of an area σ_T times the incident time-average Poynting flux. This area σ_T is the "Thompson cross-section" of the electron.

Your answer should involve the distance $r_e = \dfrac{e^2}{4\pi\varepsilon_o m_e c^2}$, the classical electron radius. This is the radius at which the electrostatic energy required to assemble the electron using its observed charge is equal to its observed mass time the speed of light squared. The classical electron radius occurs over and over in radiation problems in classical electromagnetism.

The total rate at which energy is radiated into all solid angles for an acceleration *a* is given by $\dfrac{2}{3}\dfrac{e^2a^2(t)}{4\pi\varepsilon_o c^3}$. Since the acceleration a is given $\mathbf{a} = \dfrac{d\mathrm{v}}{dt} = -\dfrac{eE_o}{m_e}\cos(\omega_o t)\hat{\mathbf{y}}$, we have $\dfrac{2}{3}\dfrac{e^2a^2(t)}{4\pi\varepsilon_o c^3} = \dfrac{2}{3}\dfrac{e^2}{4\pi\varepsilon_o c^3}\left[\dfrac{eE_o}{m_e}\cos(\omega_o t)\right]^2 = \dfrac{2}{3}\dfrac{e^4E_o^2}{4\pi\varepsilon_o c^3 m_e^2}\cos^2(\omega_o t)$. When we time average we find that the average rate at which energy is radiated is $\dfrac{1}{3}\dfrac{e^4E_o^2}{4\pi\varepsilon_o c^3 m_e^2} = \dfrac{1}{3}c\varepsilon_o 4\pi r_e^2 E_o^2$, where in the last form we have used the definition of r .

The incident time average Poynting flux is given by the time average of $\dfrac{EB}{\mu_o}\hat{\mathbf{x}} = \dfrac{E_o^2\cos^2\omega_o t}{c\mu_o}\hat{\mathbf{x}}$, or $\dfrac{E_o^2}{2c\mu_o}\hat{\mathbf{x}}$. If we take the ratio $\dfrac{1}{3}c\varepsilon_o 4\pi r_e^2 E_o^2$ to $\dfrac{E_o^2}{2c\mu_o}$, we obtain the Thompson cross-section,

$$\sigma_T = \frac{1}{3}c\varepsilon_o 4\pi r_e^2 E_o^2 \Big/ \frac{E_o^2}{2c\mu_o} = \frac{8\pi}{3}r_e^2$$

(b) The energy scattered out of the electromagnetic wave into electric dipole radiation carries away no net momentum. Justify this statement using the Maxwell stress tensor for the radiation fields (only) of this radiating charge. Don't do too much work here, use symmetry arguments and (as always) calculate only the components of $-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\,da$ that you need.

Since we are only considering radiation fields, there is no radial component of **E** or **B**, and therefore the momentum flux associated with the electric field is given by

$$-\overleftrightarrow{\mathbf{T}}_E\cdot\hat{\mathbf{r}}da = -\varepsilon_o\left[\mathbf{E}\mathbf{E} - \frac{1}{2}\mathbf{I}E^2\right]\cdot\hat{\mathbf{r}}da = -\varepsilon_o\left[\mathbf{E}E_r - \hat{\mathbf{r}}\frac{1}{2}E^2\right]da = +\varepsilon_o\hat{\mathbf{r}}\frac{1}{2}E^2$$

with a similar expression for that carried by the magnetic field. Since both of these are in the radial direction, they will average to zero. That is, we are radiating momentum into a given solid angle, as we must, since we are radiating energy into that solid angle, but if we average over all directions we have no radiated momentum.

(c) The incident electromagnetic wave loses momentum because some of its energy is carried away by the electric dipole radiation. Since that momentum is not carried away by the scattered wave, it must be absorbed by the electron. From this conservation of momentum argument, deduce the radiation pressure force felt by the electron. Demonstrate explicitly that your answer has units of force.

The incident wave is carrying momentum in the *x*-direction, and the momentum it has lost is the energy it has lost divided by c. Therefore the time averaged momentum that the electron must be absorbing per unit time is given by

$$\frac{1}{c}\sigma_T \frac{E_o^2}{2c\mu_o}\hat{\mathbf{x}} = \sigma_T \frac{\varepsilon_o E_o^2}{2}\hat{\mathbf{x}}$$

This clearly has units of force because we have an area times an energy density, and an energy density is a force per unit area.

Problem 4-4: Radiation pressure on an electron using radiation reaction

Just as in the case with electromagnetic radiation reflected from a conductor, the radiation force on the electron is a $\mathbf{v}\times\mathbf{B}$ force. This is not obvious when we look at the solution given above, since if we compute the time average of $\mathbf{v}\times\mathbf{B}$, we get zero over one cycle of the wave. This happens because we have not included the radiation reaction force in the equation of motion of the electron, which introduces a small phase shift between the motion of the electron and phase of the incident wave, such that the time average of $-e\,\mathbf{v}\times\mathbf{B}$ gives the same answer as in 10-4(c). I ask you to show this in this problem. That is, I want you to derive the equation the radiation pressure on an electron by looking at time-averaged forces, rather than relying on conservation of momentum arguments (conservation arguments are fine, they just don't give you a feel for the actual mechanisms--this is a perfect example of that).

(a) Consider again the solution for the velocity of our electron in the electric field of the plane wave, given above. Show that if you include the radiation reaction force in the equation of motion for the electron (still neglecting the $\mathbf{v}\times\mathbf{B}$ force), you obtain the following form for $\mathbf{v}$(t), (assuming that the product $\omega_o\tau_e$ is much less than one, where τ_e is the speed of light travel time across the classical electron radius):

$$\mathbf{v}(t) = \mathrm{Re}\left[-\frac{e\,E_o}{(-i\omega_o)m_e}\frac{1}{1+i\frac{2}{3}\omega_o\tau_e}e^{-i\omega_o t}\right]\hat{\mathbf{y}} = -\frac{e\,E_o}{\omega_o m_e}\sin(\omega_o t - \tfrac{2}{3}\omega_o\tau_e)\hat{\mathbf{y}}$$

Show that with this form for $\mathbf{v}$(t), which differs from the original form above only by a *very small* phase shift, we find that the time-average of $-e\,\mathbf{v}\times\mathbf{B}$ is no longer zero, but exactly our radiation pressure force given above.

If we include the radiation reaction force, our equation of motion becomes

$$m_e\,\mathbf{a} = -e(\mathbf{E}+\mathbf{VxB}) + \frac{2}{3}m_e\tau_e\frac{d\mathbf{a}}{dt} \Rightarrow m_e\,\mathbf{a}\left(1+\frac{2}{3}i\tau_e\omega_o\right) = -e\,E_o\cos(\omega_o\,t)\;\hat{\mathbf{y}}$$

And thus we recover the second term in **Error! Reference source not found.**.

Furthermore, if $\tau_e\omega_o << 1$,

$$\frac{1}{1+\frac{2}{3}i\tau_e\omega_o} \approx 1-\frac{2}{3}i\tau_e\omega_o \approx e^{-2i\tau_e\omega_o/3}$$. If we now look at the **v** x **B** force, it is

$$-e\mathbf{v}\times\mathbf{B} = +\frac{e^2 E_o}{\omega_o m_e}\sin(\omega_o t - \tfrac{2}{3}\omega_o\tau_e)\hat{\mathbf{y}}\times\frac{E_o}{c}\cos(\omega_o t)\hat{\mathbf{z}} = \hat{\mathbf{x}}\frac{e^2 E_o^2}{\omega_o m_e c}\sin(\omega_o t - \tfrac{2}{3}\omega_o\tau_e)\cos(\omega_o t)$$

Using the trig identity $\sin(A-B) = \sin A\cos B - \sin B\cos A$, we see that

$$\sin(\omega_o t - \tfrac{2}{3}\omega_o\tau_e)\cos(\omega_o t) = \sin\omega_o t\cos\tfrac{2}{3}\omega_o\tau_e\cos(\omega_o t) - \sin\tfrac{2}{3}\omega_o\tau_e\cos^2(\omega_o t)$$

and since $\sin\tfrac{2}{3}\omega_o\tau_e \approx \tfrac{2}{3}\omega_o\tau_e$, we have that the time average of the **v** x **B** is

$$\hat{\mathbf{x}}\frac{e^2 E_o^2}{\omega_o m_e c}\tfrac{1}{2}\tfrac{2}{3}\omega_o\tau_e = \hat{\mathbf{x}}\frac{e^2 E_o^2}{m_e c}\frac{1}{3}\frac{r_e}{c} = \hat{\mathbf{x}}4\pi\varepsilon_o E_o^2\frac{1}{3}r_e^2 = \sigma_T\frac{1}{2}\varepsilon_o E_o^2\hat{\mathbf{x}}$$ as desired

Problem 4-5: The Eddington Luminosity

An electron and a proton pair in the outer layers of a star can be considered as a unit dynamically, even though the outer layers are ionized, because any attempt to separate the electrons and protons gives rise to large electrostatic restoring forces. The gravitational force on the proton is much larger than that on the electron, and conversely, the radiation pressure force on the electron is much greater than that on the proton, since that force goes inversely as the square of the mass. Since the electron/proton pair can be considered as a unit, the total force (gravitational plus radiation pressure) on the pair can be written as

$$\mathbf{F} = \hat{\mathbf{r}}\left[\sigma_T\left\langle\varepsilon_o E^2\right\rangle - \frac{G\,M\,m_p}{r^2}\right]$$

If this force is positive, the outer layers will be blown off by the radiation pressure of the star. The luminosity L of a star is defined to be the average rate at which energy is being carried away radially outward from the star. At radius r in the outer layers of the star, find a relation between L and the quantity $\left\langle\varepsilon_o E^2\right\rangle$ that appears in the equation above Show that the requirement that the force in this equation be negative places an upper limit on the luminosity of a star that is determined by its mass and fundamental constants, independent of r. This is known as the Eddington limit, and stars are observed to exceed this limit only on short time scales. What is this limit for a one solar mass star? [The mass of the sun is 2 x 10^{30} kilograms, and its luminosity is 3.9 x 10^{26} joules/sec,].

The luminosity of a star is the average Poynting flux times the area of a sphere of radius r, that is $L = 4\pi r^2 \left[\frac{1}{2}\varepsilon_o E_o^2\right] \Rightarrow \left[\frac{1}{2}\varepsilon_o E_o^2\right] = \frac{L}{4\pi r^2}$. Thus

$$\mathbf{F} = \hat{\mathbf{r}}\left[\sigma_T \left\langle \varepsilon_o E^2 \right\rangle - \frac{G\,M\,m_p}{r^2}\right] = \frac{\hat{\mathbf{r}}}{r^2}\left[\sigma_T \frac{L}{4\pi} - G\,M\,m_p\right]$$

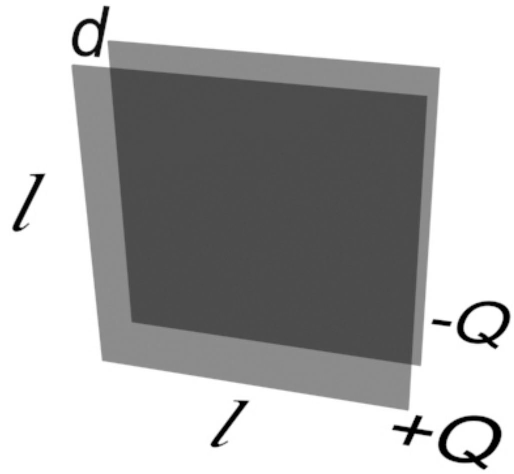
Thus if we want radiation pressure to be less than gravity, we must have

$$L < \frac{4\pi G\,M\,m_p}{\sigma_T} = \text{Eddington Luminosity}$$

For a solar mass star the Eddington Luminosity is 1.25×10^{31} joules per second, much greater than the solar luminosity, so the sun is nowhere near the Eddingtion limit.

Problem 4-6: More on the electromagnetic self-force

Why:? In addition to the electromagnetic self-force associated with irreversible energy loss to infinity due to radiation, there is also an electromagnetic self-force associated with reversible momentum going into the field.



(a) Read carefully the development reading up to equation 16.28 of Jackson, especially noting equation (16.20) on page 751. Note that this chapter of Jackson uses Gaussian units! What are the first two terms of equation (16.28) in SI units and the total electrostatic energy required to put together the static charge distribution. The negative of these terms are the two self-force terms left in the series when we shrink the charge away to nothing (letting its radius go to zero). What is the problem we take this limit?

The first term and second terms of 16.20 in SI units are

$$\frac{4U_E}{3c^2}\dot{\mathbf{v}} - \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3c^3}\ddot{\mathbf{v}}$$

where $2U_E = \int d^3x\rho(\mathbf{X})\Phi(\mathbf{X}') = \frac{1}{4\pi\varepsilon_o}\int d^3x\int d^3x' \frac{\rho(\mathbf{X})\rho(\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|}$ and thus the self-force terms are $-\frac{4U_E}{3c^2}\dot{\mathbf{v}} + \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3c^3}\ddot{\mathbf{v}}$. If we shrink the charge away to zero maintain q constant, the expression for U_E diverges as the inverse of the radius of the charge distribution.

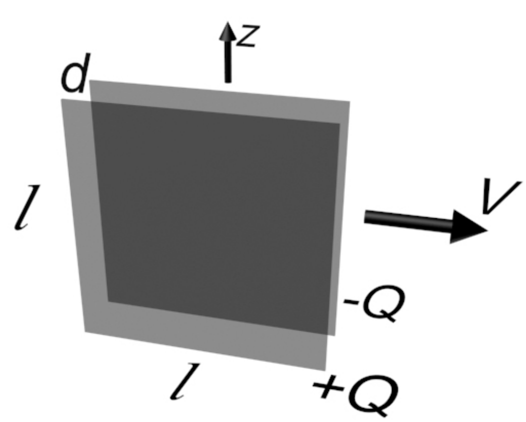
(b) To get some idea physically of why the first term in your above equations exists, consider a capacitor consisting of two metal plates of length *l* separated by a distance *d*, with charge *Q* on one plate and –*Q* on the opposite plate (see sketch in (a)). What is total energy U_E in the electrostatic field required to charge this capacitor (neglect fringing effects), in terms of the given parameters.

The field inside the capacitor is $E = \sigma/\varepsilon_o = Q/A\varepsilon_o$ so the total energy is

$$U_E = l^2 d\frac{1}{2}\varepsilon_o E^2 = l^2 d\frac{1}{2}\varepsilon_o\left(\frac{Q}{l^2\varepsilon_o}\right)^2 = \frac{dQ^2}{2\varepsilon_o l^2}$$

(c) Now suppose we move the capacitor to the right with speed $\mathbf{V} = V\,\hat{\mathbf{x}}$. There will now be a magnetic field in the z-direction because we have current sheets in $\pm\sigma V\hat{\mathbf{x}}$ because of the moving sheets of charge. What is that magnetic field? Neglect fringing effects.

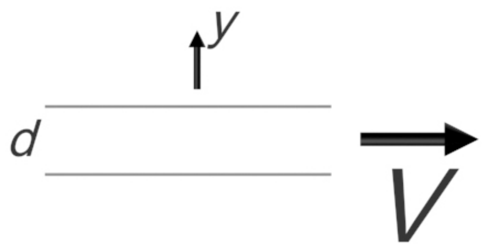
$$\mathbf{B} = \mu_o\sigma V\,\hat{\mathbf{z}}$$



(d) What is the total electromagnetic momentum $\int \varepsilon_o \mathbf{E}\times\mathbf{B}\, d^3x$ in this moving capacitor, neglecting fringing fields. Write your answer in terms of U_E / c^2 and the velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$.

$$\varepsilon_o \mathbf{E}\times\mathbf{B} A d = \varepsilon_o \frac{Q}{l^2\varepsilon_o}\mu_o \frac{Q}{l^2} V l^2 d\,\hat{\mathbf{x}} = \frac{1}{c^2}\frac{Q^2 d}{\varepsilon_o l^2} = \frac{2U_E}{c^2} V\,\hat{\mathbf{x}}$$

(e) Now suppose you have the capacitor moving at velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$ and you want to increase its speed, e.g. $\mathbf{V}(t) = V(t)\,\hat{\mathbf{x}}$ and $dV(t)/dt > 0$. When you try to increase the speed that will increase the magnetic field strength, and Faraday's Law tells you that you will get an induced electric field in the sheets. What is that induced electric field in plates? Below is a top view. Indicate the direction if the induced electric field in the top and bottom plates.



Apply Faraday's Law to a rectangle of length l in the $\hat{\mathbf{x}}$ direction and width d in the $\hat{\mathbf{y}}$ direction with unit normal out of the page.

$$\oint \mathbf{E}\cdot d\mathbf{l} = -E_x^{top} l + E_x^{bottom} l = -\frac{d}{dt} Bld = -\mu_o l d \frac{Q}{l^2}\frac{d}{dt}V$$

$$-E_x^{top} + E_x^{bottom} = -\mu_o d \frac{Q}{l^2}\frac{d}{dt}V \quad \text{by symmetry } E_x^{bottom} = -E_x^{top}$$

$$\text{So } \mathbf{E}_x^{top} = \hat{\mathbf{x}}\mu_o d \frac{Q}{2l^2}\frac{d}{dt}V \qquad \mathbf{E}_x^{bottom} = -\hat{\mathbf{x}}\mu_o d \frac{Q}{2l^2}\frac{d}{dt}V$$

What is the magnitude and direction of the associated total electric force on the top and bottom plates, neglecting fringing fields, in terms of U_E / c^2 and $dV(t)/dt > 0$.

The total electric force on top and bottom plates is

$$-Q\mathbf{E}_x^{top} + Q\mathbf{E}_x^{bottom} = -\hat{\mathbf{x}}\mu_o d \frac{Q^2}{l^2}\frac{d}{dt}V = -\hat{\mathbf{x}}\mu_o\varepsilon_o \frac{dQ^2}{l^2\varepsilon_o}\frac{d}{dt}V = -\frac{2U_E}{c^2}\frac{d}{dt}V$$

This force is always opposite the acceleration, and the "agent" accelerating the capacitor has to provide an additional momentum which ends up stored in the electromagnetic field of the moving capacitor. This momentum can be recovered by the external agent when she stops the capacitor.