Assignment 10: Due Friday May 1, 2015 at 2:30 pm

Problems

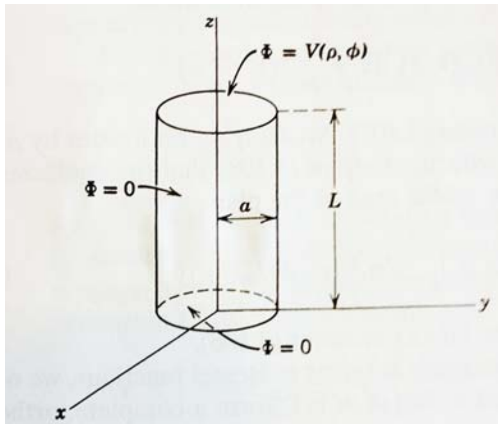
There are only three problems in this problem set. Each one is worth 30 points.

Problem 10.1 : Two electrostatic potential problems in cylindrical coordinates.

(a) A cylinder has zero potential on its sides and bottom (see figure) and on the top the potential is given by

$$V(\rho,\phi) = V_o J_o(x_{01}\frac{\rho}{a})$$

where x_{01} is the first zero of J_0 . What is the potential everywhere inside the cylinder?



(b) We have a cylindrical potential problem in the region $z \geq 0$ where the potential goes to zero as $z \to \infty$ and at $z = 0$ the potential is $V(\rho,\phi)$ where

$$V(\rho,\phi) = V_o J_o(\frac{\rho}{a})$$

What is the potential everywhere for $z \geq 0$?

Problem 10.2 : The spinning shell of charge

The problem below assumes the slow spin up of a shell of charge. We will be able to do this problem with arbitrarily fast spin up in a few more lectures, but I wanted you to work out the "quasi-static" solutions for comparison to that result.

A spherical shell of radius *R*, carries a uniform surface charge σ. Its total charge *Q* is $4\pi R^2 \sigma$, and its Coulomb electric field is

$$\mathbf{E}_{coulomb} = \begin{cases} 0 & r < R \\ \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} & r > R \end{cases}$$

We begin spinning the sphere at a angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

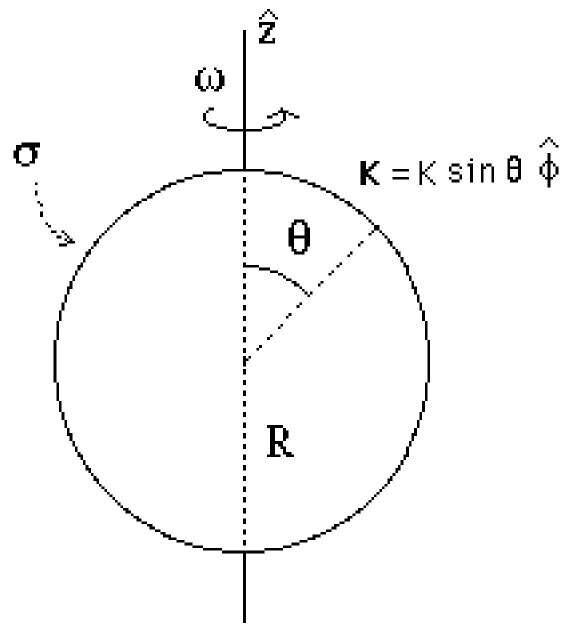
$$\mathbf{K}(t) = \sigma\,\omega(t)\,R\sin\theta\;\hat{\boldsymbol{\phi}} = \kappa(t)\sin\theta\,\hat{\boldsymbol{\phi}}$$

where $\kappa(t) = \sigma\,\omega(t)\,R$. In this problem we assume that we can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (this will be good for variations in $\kappa(t)$ with time scales $T \approx \frac{\kappa}{d\kappa/dt} \gg \frac{R}{c}$) If we define $m(t) = \frac{4\pi R^3}{3}\kappa(t)$ and $B(t) = \frac{2\mu_o}{3}\kappa(t)$, then our quasi-static solution for **B** is

$$\mathbf{B}(\mathbf{r},t) = \begin{cases} \frac{\mu_o m(t)}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) & (r > R) \\ \hat{\mathbf{z}}\ B(t) & (r < R) \end{cases}$$

(a) Given **B** find the induction electric field everywhere in space. You may assume that $\mathbf{E}_{induction} = E_\phi \hat{\boldsymbol{\phi}}$. Then find E_ϕ using $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}$. For $r > R$, calculate the ratio of E_ϕ to the radial Coulomb electric field given above. Is this small or large compared to one?

(b) Show that the magnetic energy outside of $r > R$ is given by $\frac{\mu_o m^2}{12\pi R^3}$. What is the magnetic energy for $r < R$? What is the total magnetic energy?



(c) Show that the total rate at which electromagnetic energy is being created as the sphere is being spun up, $\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\,d^3x$, is equal to the rate at which the total magnetic energy is increasing. If you are spinning up the sphere, it is you who are creating this energy by the additional work you must do to offset the force associated with the induction electric field.

(d) Using the Poynting vector, calculate the flux of electromagnetic energy $\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da$ through a spherical surface of radius r for r a little greater than R and also for r a little smaller than R. Do your results agree with what you expect from (b) and (c)?

(e) Compute the total electromagnetic angular momentum of this spinning charge configuration, ***ignoring*** $\mathbf{E}_{induction}$, that is compute

$$\int_{\text{all space}} \mathbf{r}\times\left[\varepsilon_o \mathbf{E}_{coulomb}\times\mathbf{B}\right] d^3x$$

(f) Show that the total rate at which electromagnetic angular momentum is being created as the sphere is being spun up, $\int\limits_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] d^3x$, is equal to the rate at which the total electromagnetic angular momentum is increasing. Here you cannot ignore the induction electric field. If you are spinning up the sphere, it is you who are creating this angular momentum by the additional torque you must impose to overcome the induction electric field.

(g). Calculate the flux of electromagnetic angular momentum, $\int\limits_{surface} \left[-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right]\cdot\hat{\mathbf{r}}\, da$ through a sphere of radius r for r a little greater than R and for r a little smaller than R. Do your results agree with what you expect from (e) and (f)? As in all stress tensor calculations, figure out what components you are going to need before you calculate anything, and then just calculate those.