1.

(a)

Expand the field in derivatives in x and y

$$
\begin{aligned}
\vec{E} =&\left(\vec{E}_0 + \vec{E}_1\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t} + \cdots\\
\vec{B} =&\left(\vec{B}_0 + \vec{B}_1\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t} + \cdots
\end{aligned}
$$

where $\vec{E}_0 = E_0(\hat{x}\pm\mathrm{i}\hat{y})$, Since

$$
\begin{aligned}
\nabla\cdot\vec{E} =&0\\
0 =&\nabla\cdot\vec{E}_0\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t} + \nabla\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\cdot\left(\vec{E}_0 + \vec{E}_1\right)\\
=&\left(\frac{\partial E_0}{\partial x}\pm\mathrm{i}\frac{\partial E_0}{\partial y}\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t} + \mathrm{i}k\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}E_{z_1}\\
E_{z_1} =&\frac{\mathrm{i}}{k}\left(\frac{\partial E_0}{\partial x}\pm\mathrm{i}\frac{\partial E_0}{\partial y}\right)\\
\vec{E} =&\left(E_0(\hat{x}\pm\mathrm{i}\hat{y}) + \frac{\mathrm{i}}{k}\left(\frac{\partial E_0}{\partial x}\pm\mathrm{i}\frac{\partial E_0}{\partial y}\right)\hat{z}\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}
\end{aligned}
$$

B field

$$
\begin{aligned}
\vec{B} =&-\frac{\mathrm{i}}{\omega}\nabla\times\vec{E}\\
=&-\frac{\mathrm{i}}{\omega}\left(\mathrm{i}k\hat{z}\times\left(\vec{E}_0 + \vec{E}_1\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t} + (\nabla E_0)\times(\hat{x}\pm\mathrm{i}\hat{y})\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\right)\\
=&\frac{k}{\omega}\left(E_0(\hat{y}\mp\mathrm{i}\hat{x})\pm\frac{1}{k}\left(\frac{\partial E_0}{\partial x}\pm\mathrm{i}\frac{\partial E_0}{\partial y}\right)\hat{z}\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\\
=&\mp\mathrm{i}\frac{k}{\omega}\left(E_0(\hat{x}\pm\mathrm{i}\hat{y}) + \frac{\mathrm{i}}{k}\left(\frac{\partial E_0}{\partial x}\pm\mathrm{i}\frac{\partial E_0}{\partial y}\right)\hat{z}\right)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\\
=&\mp\mathrm{i}\sqrt{\mu\varepsilon}\vec{E}
\end{aligned}
$$

(b)

Angular momentum density

$$\begin{aligned}
\vec{l} =& \varepsilon_0\vec{r}\times\left(\vec{E}\times\vec{B}\right)\\
\langle l_z\rangle =& \frac{\hat{z}}{2}\cdot\varepsilon_0\vec{r}\times\Re\left(\vec{E}\times\vec{B}^*\right)\\
=& \frac{\hat{z}}{2}\cdot\varepsilon_0\vec{r}\times\Re\left(\vec{E}_0\times\vec{B}_1^*+\vec{E}_1\times\vec{B}_0^*\right)\\
=& \mp\frac{\hat{z}}{2}\cdot\sqrt{\mu_0\varepsilon_0}\varepsilon_0\vec{r}\times\Re\left(\mathrm{i}\left(\vec{E}_0\times\vec{E}_1^*-\vec{E}_0^*\times\vec{E}_1\right)\right)\\
=& \pm\hat{z}\cdot\sqrt{\mu_0\varepsilon_0}\varepsilon_0\vec{r}\times\Im\left(\vec{E}_0\times\vec{E}_1^*\right)\\
=& \pm\hat{z}\cdot\sqrt{\mu_0\varepsilon_0}\varepsilon_0\vec{r}\times\Im\left(E_0(\hat{x}\pm\mathrm{i}\hat{y})\times\frac{-\mathrm{i}}{k}\left(\frac{\partial E_0}{\partial x}\mp\mathrm{i}\frac{\partial E_0}{\partial y}\right)\hat{z}\right)\\
=& \mp\frac{\varepsilon_0E_0}{\omega}\Re\left((-x\mp\mathrm{i}y)\left(\frac{\partial E_0}{\partial x}\mp\mathrm{i}\frac{\partial E_0}{\partial y}\right)\right)\\
=& \pm\frac{\varepsilon_0E_0}{\omega}\left(x\frac{\partial E_0}{\partial x}+y\frac{\partial E_0}{\partial y}\right)\\
=& \pm\frac{\varepsilon_0}{2\omega}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)E_0^2
\end{aligned}$$

1D density

$$\begin{aligned}
\langle L_z\rangle =& \pm\int\mathrm{d}\sigma\frac{1}{2\omega}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u\\
=& \pm\int\mathrm{d}\sigma\frac{u}{\omega}\\
=& \pm\frac{U}{\omega}
\end{aligned}$$

This means a circularly polarized photon of energy $\hbar\omega$ has angular momentum along it's propagation direction of $\pm\hbar$. The transverse component of angular momentum vanish because of symmetry.

2.

(a)

$$\begin{aligned}
\Psi =& \frac{\mathrm{i}k\sqrt{I_0}}{2\pi}\int_{-X}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\\
\approx& \frac{\mathrm{i}k\sqrt{I_0}\mathrm{e}^{\mathrm{i}kZ}}{2\pi Z}\int_{-X}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\exp\left(\frac{\mathrm{i}k\rho^2}{2Z}\right)\\
\approx& \frac{\mathrm{i}k\sqrt{I_0}\mathrm{e}^{\mathrm{i}kZ}}{2\pi Z}\int_{-X}^{\infty}\mathrm{d}x\exp\left(\frac{\mathrm{i}kx^2}{2Z}\right)\int_{-\infty}^{\infty}\mathrm{d}y\exp\left(\frac{\mathrm{i}ky^2}{2Z}\right)\\
=& \frac{\mathrm{i}\sqrt{I_0}\mathrm{e}^{\mathrm{i}kZ}}{\pi}\int_{-\xi}^{\infty}\mathrm{d}x'\mathrm{e}^{\mathrm{i}x'^2}\int_{-\infty}^{\infty}\mathrm{d}y'\mathrm{e}^{\mathrm{i}y'^2}\\
=& \frac{\mathrm{i}\sqrt{I_0}\mathrm{e}^{\mathrm{i}kZ}(i+1)}{2}\sqrt{\frac{2}{\pi}}\int_{-\xi}^{\infty}\mathrm{d}x'\mathrm{e}^{\mathrm{i}x'^2}
\end{aligned}$$

(b)

$$\begin{aligned}
I =&\frac{I_0}{2}\frac{2}{\pi}\left|\int_{-\xi}^{\infty}\mathrm{d}x'\mathrm{e}^{\mathrm{i}x'^2}\right|^2\\
=&\frac{I_0}{2}\frac{2}{\pi}\left(\left(\int_{-\xi}^{\infty}\mathrm{d}x'\cos x'^2\right)^2+\left(\int_{-\xi}^{\infty}\mathrm{d}x'\sin x'^2\right)^2\right)\\
=&\frac{I_0}{2}\left(\left(\mathcal{C}(\xi)+\frac{1}{2}\right)^2+\left(\mathcal{S}(\xi)+\frac{1}{2}\right)^2\right)
\end{aligned}$$