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1.

(a)

$$\begin{aligned} x'_i x'_i =& a_{ij} a_{ik} x_j x_k \\ x_i x_i =& \delta_{jk} x_j x_k \end{aligned}$$

Since $x'_i x'_i = x_i x_i$ for all x_i

$$\delta_{jk} = a_{ij} a_{ik}$$

(b)

$$\begin{aligned} x_i =& \delta_{ik} x_k \\ =& a_{ji} a_{jk} x_k \\ =& a_{ji} x'_j \end{aligned}$$

(c)

$$\begin{aligned} \frac{\partial f}{\partial x'_i} =& \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x'_i} \\ =& \frac{\partial f}{\partial x_j} \frac{\partial a_{kj} x'_k}{\partial x'_i} \\ =& a_{ij} \frac{\partial f}{\partial x_j} \end{aligned}$$

2.

(a)

$$\begin{aligned} \delta'_{ij} =& a_{im} a_{jn} \delta_{mn} \\ =& a_{ik} a_{jk} \\ =& \delta_{ij} \end{aligned}$$

(b)

$$\begin{aligned} C'_i T'_{ij} =& a_{ik} C_k a_{im} a_{jn} T_{mn} \\ =& a_{ik} a_{im} a_{jn} C_k T_{mn} \\ =& \delta_{km} a_{jn} C_k T_{mn} \\ =& a_{jk} C_i T_{ik} \\ =& (C_i T_{ik})' \end{aligned}$$

(c)

Since both A_iA_j and δ_{ij} are second rank tensors and A^2 is a scalar, T_{ij} is also a second rank tensor.

(d)

$$\begin{aligned}
&\partial_i\left(A_iA_j - \frac{1}{2}\delta_{ij}A_kA_k\right)\\
=&\partial_i(A_iA_j) - \frac{1}{2}\partial_j(A_kA_k)\\
=&A_j\partial_i(A_i) + A_i\partial_i(A_j) - A_k\partial_j(A_k)\\
=&A_j\partial_i(A_i) + A_i\partial_i(A_j) - A_i\partial_j(A_i)\\
=&A_j\partial_i(A_i) + A_i(\delta_{ki}\delta_{jl} - \delta_{jk}\delta il)\partial_k(A_l)\\
=&A_j\partial_i(A_i) + \varepsilon_{mij}A_i\varepsilon_{mkl}\partial_k(A_l)\\
=&A_j\partial_i(A_i) + \varepsilon_{jmi}(\varepsilon_{mkl}\partial_k(A_l))A_i
\end{aligned}$$

(e)

i.

$$\begin{aligned}
T_{ij}n_j =&\left(A_iA_j - \frac{1}{2}\delta_{ij}A^2\right)n_j\\
=&A_iA_jn_j - \frac{1}{2}A^2n_i
\end{aligned}$$

Therefore $T\cdot\vec{n}$ is a linear combination of $\vec{A}$ and $\vec{n}$

ii.

Let $B_i = T_{ij}n_j$

$$\begin{aligned}
B_iB_i =&\left(A_iA_jn_j - \frac{1}{2}A^2n_i\right)\left(A_iA_kn_k - \frac{1}{2}A^2n_i\right)\\
=&(A_jn_j)^2A^2 + \frac{1}{4}A^4 - (A_jn_j)^2A^2 = \qquad \frac{1}{4}A^4\\
B_in_i =&A_iA_jn_jn_i - \frac{1}{2}A^2\\
=&A^2\left(\cos^2\theta - \frac{1}{2}\right)\\
=&\frac{A^2}{2}\cos 2\theta\\
\cos\theta_{Bn} =&\cos 2\theta\\
\theta_{Bn} =&2\theta
\end{aligned}$$

iii.

See above.

3.

(a)

$$\begin{aligned}
E_r =&\frac{\lambda}{2\pi\varepsilon_0 r} - E_0\sin\phi\\
E_\phi =& - E_0\cos\phi
\end{aligned}$$

(b)

$$\begin{aligned}
E_r =&\frac{2\pi\varepsilon_0 E_0\lambda}{2\pi\varepsilon_0\lambda} - E_0\\
=&0\\
E_\phi =&0
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{d}\vec{F} =&\varepsilon_0 r\mathrm{d}\phi\mathrm{d}z\left(E_r\vec{E} - \frac{1}{2}E^2\hat{r}\right)\\
=&\varepsilon_0 r\mathrm{d}\phi\mathrm{d}z\left(\frac{E_r^2 - E_\phi^2}{2}\hat{r} + E_rE_\phi\hat{\phi}\right)
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{\mathrm{d}\vec{F}}{\mathrm{d}z} =&\int\varepsilon_0 r\mathrm{d}\phi\left(\frac{E_r^2 - E_\phi^2}{2}\hat{r} + E_rE_\phi\hat{\phi}\right)\\
=&\frac{\varepsilon_0 r}{2}\hat{y}\int\mathrm{d}\phi\left(\left(\left(\frac{\lambda}{2\pi\varepsilon_0 r} - E_0\sin\phi\right)^2 - E_0\cos^2\phi\right)\sin\phi - 2\left(\frac{\lambda}{2\pi\varepsilon_0 r} - E_0\sin\phi\right)E_0\cos^2\phi\right)\\
=&\frac{\varepsilon_0 r}{2}\hat{y}\int\mathrm{d}\phi\left(-2\frac{\lambda}{2\pi\varepsilon_0 r}E_0\sin^2\phi - \frac{\lambda}{\pi\varepsilon_0 r}E_0\cos^2\phi\right)\\
=&\frac{1}{2}\hat{y}\int\mathrm{d}\phi\left(-\frac{\lambda}{\pi}E_0\sin^2\phi - \frac{\lambda}{\pi}E_0\cos^2\phi\right)\\
=&-\frac{1}{2}\hat{y}2\pi\frac{\lambda}{\pi}E_0\\
=&-E_0\lambda\hat{y}
\end{aligned}$$

4.

(a)

$$\begin{aligned}
B_r =& B_0 \sin\phi \\
B_\phi =& \frac{\mu_0 I}{2\pi r} + B_0 \cos\phi
\end{aligned}$$

(b)

$$\begin{aligned}
B_r =& 0 \\
B_\phi =& \frac{\mu_0 I 2\pi B_0}{2\pi \mu_0 I} + B_0 \\
=& 0
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{d}\vec{F} =& \frac{1}{\mu_0} r\mathrm{d}\phi\mathrm{d}z\left(B_r\vec{B} - \frac{1}{2}B^2\hat{r}\right) \\
=& \frac{1}{\mu_0} r\mathrm{d}\phi\mathrm{d}z\left(\frac{B_r^2 - B_\phi^2}{2}\hat{r} + B_r B_\phi\hat{\phi}\right)
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{\mathrm{d}\vec{F}}{\mathrm{d}z} =& \frac{1}{2}\int \frac{r}{\mu_0}\mathrm{d}\phi\Big(\big(B_r^2 - B_\phi^2\big)\hat{r} + 2B_r B_\phi\hat{\phi}\Big) \\
=& \frac{r\hat{x}}{2\mu_0}\int \mathrm{d}\phi\left(\left(B_0^2\sin^2\phi - \left(\frac{\mu_0 I}{2\pi r} + B_0\cos\phi\right)^2\right)\cos\phi - 2B_0\sin\phi\left(\frac{\mu_0 I}{2\pi r} + B_0\cos\phi\right)\sin\phi\right) \\
=& \frac{r\hat{x}}{2\mu_0}\int \mathrm{d}\phi\left(-\left(\frac{\mu_0 I}{2\pi r} + B_0\cos\phi\right)^2\cos\phi - B_0\sin^2\phi\frac{\mu_0 I}{\pi r}\right) \\
=& \frac{\hat{x}}{2}\int \mathrm{d}\phi\left(-\frac{I}{\pi}B_0\cos^2\phi - B_0\sin^2\phi\frac{I}{\pi}\right) \\
=& -B_0 I\hat{x}
\end{aligned}$$

5.

(a)

$$\begin{aligned}\vec{E}_{line} =& \frac{\lambda\left(\vec{r} - \vec{Y}\right)}{2\pi\varepsilon_0\left(\vec{r} - \vec{Y}\right)^2}\\ =& \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\varepsilon_0\left(x^2 + (y - Y)^2\right)}\\ E =& \frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\varepsilon_0\left(x^2 + (y - Y)^2\right)} - E_0\hat{y}\end{aligned}$$

(b)

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\int_{-L}^{L}\mathrm{d}y\varepsilon_0\vec{E}_0\cdot\vec{E}_{line}\\ =& -\int_{-\infty}^{\infty}\mathrm{d}x\int_{-L}^{L}\mathrm{d}y\varepsilon_0\frac{\lambda(x\hat{x} + (y - Y)\hat{y})}{2\pi\varepsilon_0\left(x^2 + (y - Y)^2\right)}E_0\cdot\hat{y}\\ =& -\frac{E_0\lambda}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-L}^{L}\mathrm{d}y\frac{y - Y}{x^2 + (y - Y)^2}\\ =& -\frac{E_0\lambda}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}x\ln\left(\frac{x^2 + (L - Y)^2}{x^2 + (-L - Y)^2}\right)\end{aligned}$$

Let $\alpha = \dfrac{x}{L}$, $\beta = \dfrac{Y}{L}$

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}z} =& -\frac{E_0\lambda}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\ln\left(\frac{\alpha^2 + (1 - \beta)^2}{\alpha^2 + (1 + \beta)^2}\right)\\ =& -\frac{E_0\lambda}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\ln\left(\frac{\alpha^2 + 1 - 2\beta}{\alpha^2 + 1 + 2\beta}\right)\\ \approx& -\frac{E_0\lambda L}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{4\beta}{1 + \alpha^2}\\ =& -E_0\lambda Y\end{aligned}$$

6.

(a)

Let $\mathrm{d}S$ be the area between the boundary at t and $t + \mathrm{d}t$ (pointing outward), S and S' are the surface at t and $t + \mathrm{d}t$ respectively.

$$\begin{aligned}
\oiint_{-S+S'+\mathrm{d}S} \vec{G} \cdot \mathrm{d}\vec{a} =& \iiint_{\mathrm{d}V} \nabla \cdot \vec{G} \mathrm{d}v \\
=& \mathrm{d}t \iint_S \left(\nabla \cdot \vec{G}\right) \vec{V} \cdot \mathrm{d}\vec{a} \\
\mathrm{d}\left(\iint_S \vec{G}(t) \cdot \mathrm{d}\vec{a}\right) =& \iint_S \vec{\mathrm{d}}G \cdot \mathrm{d}\vec{a} + \left(\iint_{S'} \vec{G} \cdot \mathrm{d}\vec{a} - \iint_S \vec{G} \cdot \mathrm{d}\vec{a}\right) \\
=& \mathrm{d}t \iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} - \iint_{\mathrm{d}S} \vec{G} \cdot \mathrm{d}\vec{a} + \mathrm{d}t \iint_S \left(\nabla \cdot \vec{G}\right) \vec{V} \cdot \mathrm{d}\vec{a} \\
=& \mathrm{d}t \left(\iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{G}\right) \vec{V} \cdot \mathrm{d}\vec{a}\right) - \oint_C \vec{G} \cdot \left(\mathrm{d}\vec{l} \times \vec{V} \mathrm{d}t\right) \\
\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{G}(t) \cdot \mathrm{d}\vec{a} =& \iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{G}\right) \vec{V} \cdot \mathrm{d}\vec{a} - \oint_C \vec{G} \cdot \left(\mathrm{d}\vec{l} \times \vec{V}\right) \\
=& \iint_S \frac{\partial \vec{G}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{G}\right) \vec{V} \cdot \mathrm{d}\vec{a} - \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{V} \times \vec{G}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{B}(t) \cdot \mathrm{d}\vec{a} =& \iint_S \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{a} + \iint_S \left(\nabla \cdot \vec{B}\right) \vec{V} \cdot \mathrm{d}\vec{a} - \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{V} \times \vec{B}\right) \\
=& \iint_S \left(\nabla \times \vec{E}\right) \cdot \mathrm{d}\vec{a} - \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{V} \times \vec{B}\right) \\
=& \oint_C \mathrm{d}\vec{l} \cdot \left(\vec{E} - \vec{V} \times \vec{B}\right)
\end{aligned}$$