Solutions Assignment #5: Due Friday March 20, 2015 at 2:30 pm

Problems

Problem 5-1: Synchrotron Radiation

Our general formula for the rate at which a particle radiates energy is

$$\frac{dW_{rad}}{dt'} = \frac{1}{4\pi\,\varepsilon_o}\frac{2\,q^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right|^2\right]$$

We are going to apply this formula to an electron in the Crab Nebula gyrating in a magnetic field B and emitting synchrotron radiation. We will assume that the time scale over which significant energy is radiated is very long compared to the gyration period of the electron in the field. The gyration frequency for the electron is

$$\Omega_e = eB/m_e\gamma_e = \omega_o/\gamma_e \quad \text{where} \quad \omega_o = eB/m_e$$

(a) Show that if the acceleration is perpendicular to the velocity and the electron is gyrating in a magnetic field with the frequency given above, then the rate at which it is radiating energy is

$$\frac{dW_{rad}}{dt'} = \frac{e^2}{4\pi\,\varepsilon_o}\frac{2\omega_o^2}{3c}\gamma^2\beta^2$$

For synchrotron radiation we have that the acceleration a is related to the speed v by $a = \Omega_e \mathrm{v}$

So that we have

$$\frac{dW_{rad}}{dt'} = \frac{1}{4\pi\,\varepsilon_o}\frac{2\,e^2}{3\,c^3}\gamma^4 a^2 = \frac{1}{4\pi\,\varepsilon_o}\frac{2\,e^2}{3\,c}\gamma^4\left(\frac{eB}{m\gamma}\right)^2\beta^2 = \frac{1}{4\pi\,\varepsilon_o}\frac{2e^4B^2}{3cm^2}\gamma^2\beta^2 = \frac{e^2}{4\pi\,\varepsilon_o}\frac{2\omega_o^2}{3c}\gamma^2\beta^2$$

(b) Equate the loss of kinetic energy of the electron to the rate at which energy is being radiated away and show that the following equation holds

$$\frac{d\gamma_e}{dt'} = -\frac{\gamma_e^2\beta^2}{T_o}$$

What is T_o? Write your expression for T_o in terms of the speed of light, ω_o, and the classical electron radius r_e, where

$$r_e = \frac{1}{4\pi\,\varepsilon_o}\frac{e^2}{m_e c^2}$$

$$\frac{d}{dt'}(\gamma_e - 1)mc^2 = -\frac{e^2}{4\pi\varepsilon_o}\frac{2\omega_o^2}{3c}\gamma_e^2\beta^2 \Rightarrow \frac{d\gamma_e}{dt'} = -\frac{\gamma_e^2\beta^2}{T_o} \quad \text{where } T_o = \frac{3}{2}\frac{c}{r_e}\frac{1}{\omega_o^2}$$

(c) We are now going to ***only*** consider radiation for $\gamma_e >> 1$. Suppose that at $t = 0$ an electron has energy $\gamma_e^o >> 1$. At what time $T > 0$ will the electron have a gamma of γ_e, if we assume $\gamma_e^o > \gamma_e >> 1$? Give you answer in terms of T_o, γ_e, and γ_e^0.

$$\frac{d\gamma_e}{dt'} = -\frac{\gamma_e^2\beta^2}{T_o} \Rightarrow \frac{d\gamma_e}{\gamma_e^2} = -\frac{dt'}{T_o} \quad \text{since we can assume that } \beta = 1$$

$$\text{so } -\frac{1}{\gamma_e}\bigg|_o^T = -\frac{T}{T_o} \Rightarrow \frac{1}{\gamma_e} = \frac{1}{\gamma_e^0} + \frac{T}{T_o} \Rightarrow T = T_o\left(\frac{1}{\gamma_e} - \frac{1}{\gamma_e^0}\right)$$

(d) Suppose you make optical observations of the Crab Nebula and you find that there is a sharp break in the optical spectrum at a frequency in radians per second of ω_{break}, such that above this break the spectrum is falling much more steeply with frequency than below this break. Using the fact that an electron of energy $\gamma_e >> 1$ emits synchrotron radiation at frequencies which peak at $3\gamma_e^3\Omega_e$, estimate the $\gamma_e >> 1$ of the particles that produce radiation at this break frequency, assuming that you know $\omega_o = eB/m_e$. Give your answer in terms of ω_{break} and ω_o.

$$3\gamma_e^3\Omega_e = \omega_{break} = 3\gamma_e^2\omega_o \Rightarrow \gamma_e = \sqrt{\frac{\omega_{break}}{3\omega_o}}$$

(e) Assume that the particles now radiating at the break frequency had an infinite energy at the time the Crab was born, and have lost energy for a time T where T is the age of the Crab, so that now they have an energy corresponding to your estimate of $\gamma_e >> 1$ above for the electrons radiating at the break frequency. What is the lifetime T of the Crab using your answers above. Your answer should be in terms of ω_{break}, ω_o, and T_o.

$$\text{for } \gamma_e^0 = \infty,\ T = \frac{T_o}{\gamma_e} = T_o\sqrt{\frac{3\omega_o}{\omega_{break}}}$$

(f) Assume that the magnetic field strength of the Crab is $10^{-4}\,\text{gauss} = 10^{-8}\,\text{Tesla}$, and that the break in the spectrum is at $\omega_{break} = 3\times10^{12}\omega_o$, estimate the lifetime of the Crab in years. Historically we know that the Crab exploded in 1054 AD. Is your estimate roughly consistent with this historical date?

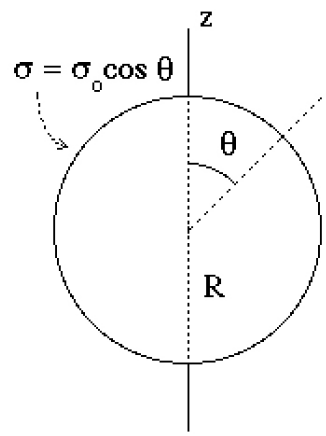
Putting in the numbers, we find that $T_o \approx 5\times10^{16}\,s \approx 1.6\times10^{9}\,\text{years}$, so that

$T = T_o\sqrt{\dfrac{3\omega_o}{\omega_{break}}} = \dfrac{3\times 10^9}{10^6} = 1600$ years, roughly consistent with 2011-1054=957 years

Problem 5-2: Interesting Statics Problems

Why:? These statics problems are interesting in themselves and useful in the next problem.

(a) A spherical shell of radius R has a surface charge $\sigma = \sigma_o \cos\theta$ placed on its surface (see sketch). You may *assume* that the resultant electric field in this case is exactly dipolar outside and constant inside. That is, you may assume that the field outside is due to an electric dipole with dipole moment $\mathbf{p}_o = p_o\hat{\mathbf{z}}$ that inside the electric field is given by $\mathbf{E}_o = E_o\hat{\mathbf{z}}$



$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{p_o}{4\pi\varepsilon_o r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) & (r > R) \\ \mathbf{E}_o & (r < R) \end{cases}$$

Using appropriate boundary condition on the electric field (the normal component of **E** jumps by σ/ε_o and the tangential component of **E** is continuous) applied at the poles and the equator of this sphere and only at those points, derive an equation for p_o and E_o in terms of R and σ_o for the this electrostatic problem. What is the relation between $\mathbf{p}_o$ and $\mathbf{E}_o$?

The tangential component of E must be continuous at the equator, and therefore $E_o = -\dfrac{p_o}{4\pi\varepsilon_o R^3}$. The normal component of E must jump by σ/ε_o at the poles, so we must have $\dfrac{2p_o}{4\pi\varepsilon_o R^3} - E_o = \dfrac{\sigma}{\varepsilon_o}$. Taken together this two equations give us $E_o = \dfrac{\sigma}{3\varepsilon_o}$ and $E_o = -\dfrac{p_o}{4\pi\varepsilon_o R^3}$ and $\mathbf{E}_o = -\dfrac{1}{3\varepsilon_o}\dfrac{\mathbf{p}_o}{\frac{4}{3}\pi R^3}$.

(b) A spherical shell of radius R has a surface current $\kappa_o \sin\theta\,\hat{\boldsymbol{\phi}}$ placed on its surface. You may *assume* that the resultant magnetic field in this case is exactly dipolar outside and constant inside. That is, you may assume outside the magnetic field is that due do a magnetic dipole with dipole moment $\mathbf{m}_o = m_o\hat{\mathbf{z}}$ that inside the electric field is given by $\mathbf{B}_o = B_o\hat{\mathbf{z}}$

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_o m_o}{4\pi\, r^3}\left(2\cos\theta\;\hat{\mathbf{r}} + \sin\theta\;\hat{\boldsymbol{\theta}}\right) & (r > R) \\ \mathbf{B}_o & (r < R) \end{cases}$$

Using appropriate boundary condition on the magnetic field (the normal component of **B** is continuous and the tangential component of **B** jumps by $\mu_o\kappa$) applied at the poles and the equator of this sphere and only at those points, derive an equation for m_o and B_o in terms of R and κ_o for the this magnetostatic problem. What is the relation between $\mathbf{m}_o$ and $\mathbf{B}_o$?

We have $\mu_o\kappa = \dfrac{\mu_o m_o}{4\pi R^3} - B_o$ and $\dfrac{2\mu_o m_o}{4\pi R^3} = B_o$ which implies that $\mathbf{B}_o = \dfrac{2\mu_o}{3}\dfrac{\mathbf{m}_o}{\frac{4}{3}\pi R^3}$

Problem 5-3: The average fields of a collection of perfect point dipoles

Why:? One of the most important things you should know when considering the effects of matter on electromagnetic fields is contained in this problem.

(a) Show that if **p** is a constant vector, and **r** is the spherical polar **r**, with $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, then

$$-\nabla\left[\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{4\pi\varepsilon_o r^2}\right] = \frac{1}{4\pi\varepsilon_o}\left[\frac{3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}}{r^3}\right] - \frac{\mathbf{p}}{3\varepsilon_o}\delta^3(\mathbf{r}) \qquad (1)$$

Just worry about getting the delta function here, since the ordinary dipole terms are straightforward and you have already done them in a previous assignment (where we ignored the origin, which we do not do here). You may assume the dipole is along the z-axis for simplicity.

For the delta function I am only interested in the following terms, assuming the dipole moment is along the z axis.

$$-\nabla\left[\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{4\pi\varepsilon_o r^2}\right] = -\nabla\left[\frac{\mathbf{p}\cdot}{4\pi\varepsilon_o}\left(-\nabla\frac{1}{r}\right)\right] = \nabla\left[\frac{p}{4\pi\varepsilon_o}\left(\frac{\partial}{\partial z}\frac{1}{r}\right)\right] = \frac{p}{4\pi\varepsilon_o}\left[\frac{\partial^2}{\partial x\partial z}\frac{1}{r}\hat{\mathbf{x}} + \frac{\partial^2}{\partial y\partial z}\frac{1}{r}\hat{\mathbf{y}} + \frac{\partial^2}{\partial z^2}\frac{1}{r}\hat{\mathbf{z}}\right]$$

If we integrate the above over the volume of a sphere of radius *r*, terms like $\dfrac{\partial^2}{\partial x\partial z}\dfrac{1}{r}\hat{\mathbf{x}}$ will integrate to zero, where as $\int\dfrac{\partial^2}{\partial z^2}\dfrac{1}{r}d^3x = \int\dfrac{\partial^2}{\partial x^2}\dfrac{1}{r}d^3x = \int\dfrac{\partial^2}{\partial y^2}\dfrac{1}{r}d^3x$. Thus we have that the delta function part is

$$-\nabla\left[\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{4\pi\varepsilon_o r^2}\right]=\frac{p}{4\pi\varepsilon_o}\left[\frac{\partial^2}{\partial z^2}\frac{1}{r}\hat{\mathbf{z}}\right]=\hat{\mathbf{z}}\frac{p}{4\pi\varepsilon_o}\frac{1}{3}\left[\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{1}{r}\right]$$

$$=-\hat{\mathbf{z}}\frac{p}{3\varepsilon_o}\delta^3(\mathbf{r})=-\frac{\mathbf{p}}{3\varepsilon_o}\delta^3(\mathbf{r})$$

(b) Show that if $\mathbf{m}$ is a constant vector, and $\mathbf{r}$ is the spherical polar $\mathbf{r}$, with $\hat{\mathbf{n}}=\hat{\mathbf{r}}$, then

$$\nabla\times\left[\frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}\right]=\frac{\mu_o}{4\pi}\left[\frac{3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}})-\mathbf{m}}{r^3}\right]+\frac{2\mu_o}{3}\mathbf{m}\delta^3(\mathbf{r}) \qquad (2)$$

Just worry about getting the delta function here, since the ordinary dipole terms are straightforward and you have already done them in a previous assignment (where we ignored the origin, which we do not do here). You may assume the dipole is along the z-axis for simplicity.

For the delta function I am only interested in the following terms, assuming the dipole moment is along the z axis.

$$\nabla\times\left[\frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}\right]=-\frac{\mu_o}{4\pi}\nabla\times\left(\mathbf{m}\times\nabla\frac{1}{r}\right)=-\frac{\mu_o}{4\pi}\left[-(\mathbf{m}\cdot\nabla)\nabla\frac{1}{r}+\mathbf{m}\nabla^2\frac{1}{r}\right]$$

where to get the final form we have used the vector identity

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})$$

So

$$\nabla\times\left[\frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}\right]=\left[+\frac{\mu_o}{4\pi}(\mathbf{m}\cdot\nabla)\nabla\frac{1}{r}-\frac{\mu_o}{4\pi}\mathbf{m}\nabla^2\frac{1}{r}\right]=\left[+\frac{\mu_o}{4\pi}(\mathbf{m}\cdot\nabla)\nabla\frac{1}{r}+\mu_o\mathbf{m}\delta^3(\mathbf{r})\right]$$

As in (a), we can argue the first term in the above equation gives us $-\frac{1}{3}\mu_o\mathbf{m}\delta^3(\mathbf{r})$, so our final form is (including only the delta function terms)

$$\nabla\times\left[\frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}\right]=\frac{2}{3}\mu_o\mathbf{m}\delta^3(\mathbf{r})$$

(c) To get a feel for where these delta functions come from in the expressions above consider the results we obtained in the problem above. Argue that in the limit *R* goes to zero, the solutions Problem 2a and 2b reduces to those given in equation (1) and (2), respectively.

The limit is true in 2a because we found that $\mathbf{E}_o = -\frac{1}{3\varepsilon_o}\frac{\mathbf{p}_o}{\frac{4}{3}\pi R^3}$. The electric field is inversely proportional to the volume, so in the limit the volume goes to zero we will have a delta function behavior. That is,

$$\lim_{R\to 0}\iiint_{\text{sphere of radius R}} \mathbf{E}_o\, d^3x = -\frac{\mathbf{p}_o}{3\varepsilon_o}\iiint_{\text{sphere of radius R}} \frac{1}{\frac{4}{3}\pi R^3}d^3x = -\frac{\mathbf{p}_o}{3\varepsilon_o}$$

The limit is obviously true. In 2b we found that $\mathbf{B}_o = \frac{2\mu_o}{3}\frac{\mathbf{m}_o}{\frac{4}{3}\pi R^3}$. The magnetic field is inversely proportional to the volume, so in the limit the volume goes to zero we will have a delta function behavior. That is,

$$\lim_{R\to 0}\iiint_{\text{sphere of radius R}} \mathbf{B}_o\, d^3x = \frac{2\mu_o\mathbf{m}_o}{3}\iiint_{\text{sphere of radius R}} \frac{1}{\frac{4}{3}\pi R^3}d^3x = \frac{2\mu_o\mathbf{m}_o}{3}$$

(e) If we look at the expression in (1) and (2), ignoring the delta functions, we have terms that look like

$$\frac{1}{4\pi\varepsilon_o}\left[\frac{3\hat{\mathbf{n}}\left(\mathbf{p}\cdot\hat{\mathbf{n}}\right)-\mathbf{p}}{r^3}\right] \quad \text{and} \quad \frac{\mu_o}{4\pi}\left[\frac{3\hat{\mathbf{n}}\left(\mathbf{m}\cdot\hat{\mathbf{n}}\right)-\mathbf{m}}{r^3}\right]$$

These expressions are mathematically identical in form (although very different in physical meaning. Find the *average* field of a field of the forms just above (without the delta functions!) above over the volume of a sphere of radius *a* centered at the origin. Do the angular integrals first, and remember to express the components of the dipole in terms of Cartesian unit vectors before integrating.

Let us do this assuming the dipole moment is along the z-axis, which is sufficiently general. In cylindrical coordinates, we have (*r* is the cylindrical *r*)

$$\mathbf{E}_{dipole} = -\nabla\frac{pz}{4\pi\varepsilon_o\left(z^2+r^2\right)^{3/2}} = \frac{p}{4\pi\varepsilon_o}\frac{\left[3rz\hat{\mathbf{r}}+\hat{\mathbf{z}}(2z^2-r^2)\right]}{\left(z^2+r^2\right)^{5/2}}$$

It is clear that the $\hat{\mathbf{r}}$ component here will average to zero over a sphere, and the z-component is proportional to $2\cos^2\theta-\sin^2\theta=\cos^2\theta-\sin^2\theta=\cos 2\theta$, where θ is the spherical polar theta, and therefore this will also average to zero when we do the θ integration. Therefore we get an average field of zero.

(f) When considering electrostatic fields in matter, we will assert that the average electric field *due to the electric dipoles alone*, of a bunch of electric dipoles with their dipole moments all lined up in the +z direction, is in the -z direction. In considering magnetostatic fields in matter, we will assert that the magnetic field *due to the magnetic dipoles alone* of a bunch of magnetic dipoles with their dipole moments all lined up in the +z direction, is in the +z-direction. Justify these statements in light of your results in (a), (b) and (e).

It is clear that these two forms agree, and will give us average fields in agreement with the statements above.

The reason magnetic effects in matter are so different from electric effects in matter is because the "guts" of electric dipoles are so different from the "guts" of magnetic dipoles, as evidenced by the delta function behaviors given above for the electric and magnetic fields..

Problem 5-4: Electromagnetic waves in a dielectric

Suppose we have a charges and currents defined for all space and time such that

$$\rho(\mathbf{r},t) = 0 \qquad \mathbf{J}(\mathbf{r},t) = \hat{\mathbf{y}}\, J_o\, e^{i(kx-\omega t)}$$

For the moment we do not ask why these currents are present, we are just given this as a fact. Note that we are NOT assuming that ω/k is c in this expression--for now this parameter can have any value, and of course the expression above represents a traveling wave moving in the + x direction at speed ω/k, whatever that value is. It could be a centimeter per second, or any other value.

This current will generate electromagnetic waves and thus electric fields. How do we calculate the electric fields that it produces? Well we have seen that if we have a current sheet at the origin varying in time as given by $\hat{\mathbf{y}}\kappa(t)$, it will generate traveling electromagnetic waves with the electric field given by

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\frac{c}{2}\mu_o\,\kappa(t-\frac{|x|}{c})$$

And if the current sheet were sitting at x' instead of the origin it would clearly generate electromagnetic waves with the electric field given by

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\frac{c}{2}\mu_o\,\kappa(t-\frac{|x-x'|}{c})$$

With the current density in $J_o\, e^{i(kx-\omega t)}$ we also know that a given interval dx' around x' will have a local current sheet given by $d\kappa = dx'\, J(x',t)$.

(a) Write down an integral equation for the electric field $\mathbf{E}(x,t)$ generated by $\mathbf{J}(\mathbf{r},t)$. In the integral you should be integrating with respect to dx' from $-\infty$ to ∞ (don't do the integral yet, just write it down).

$$\mathbf{E}(x,t) = -\frac{c\,\mu_o}{2}\int_{-\infty}^{\infty} dx'\,\mathbf{J}(x', t - \frac{|x-x'|}{c})$$

(b) Now break this integral with respect to x' into two parts, one from $x' = -\infty$ to $x' = x$ and one from $x' = x$ to $x' = +\infty$. The reason we do this is so we can handle the meaning of $|x - x'|$ properly in these two intervals. When you break it up in this way, you will find that the electric field at the observer is the sum of plane waves generated to the left of the observer, and propagating *at the speed of light in the positive x* direction to the observer, *and* the sum of plane waves generated to the right of the observer, and propagating *at the speed of light in the negative x direction* to the observer. This is what we would expect. Our extended set of currents generate electromagnetic waves propagating both to the left and the right from any given source region, at the speed of light. Now do the integrals, and throw away the evaluation of the integrands at $\pm\infty$. This is a little bogus, but suppose the integrand amplitude decreases extremely slowly to zero as we go to $x' = \pm\infty$, rather than oscillating all the way to $\pm\infty$. Show that the result you get for the electric field generated by this set of currents is given by

$$\mathbf{E}\left(x,t\right) = -ic\,\mu_o\hat{\mathbf{y}}\,J_o\,e^{i(kx-\omega t)}\frac{\omega / c}{\left(\omega^2/c^2 - k^2\right)}$$

Amazingly enough, we have ended up with an electric field which is propagating in the positive x direction, with speed ω/k, *not necessarily the speed of light*, ***even though we constructed this wave out of many waves going in both directions, all moving at the speed of light.*** That is the wonder of adding things up with a definite phase--you get all sorts of things you wouldn't expect.

$$\mathbf{E}(x,t) = -\frac{c\,\mu_o}{2}\left[\int_{-\infty}^{x} dx'\,\mathbf{J}(x', t - \frac{x-x'}{c}) + \int_{x}^{\infty} dx'\,\mathbf{J}(x', t - \frac{x'-x}{c})\right]$$

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\,J_o\,\frac{c\,\mu_o}{2}\left[\int_{-\infty}^{x} dx'\,e^{i\left(kx' - \omega\left(t - \frac{x-x'}{c}\right)\right)} + \int_{x}^{\infty} dx'\,e^{i\left(kx' - \omega\left(t - \frac{x'-x}{c}\right)\right)}\right]$$

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\,J_o\,e^{-i\omega t}\frac{c\,\mu_o}{2}\left[e^{+i\omega x/c}\int_{-\infty}^{x} dx'\,e^{ix'(k-\omega/c)} + e^{-i\omega x/c}\int_{x}^{\infty} dx'\,e^{ix'(k+\omega/c)}\right]$$

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\,J_o\,e^{-i\omega t}\frac{c\,\mu_o}{2}\left[e^{+i\omega x/c}\frac{1}{i\left(k-\omega/c\right)}e^{ix'(k-\omega/c)}\Big|_{-\infty}^{x} + e^{-i\omega x/c}\frac{1}{i\left(k+\omega/c\right)}e^{ix'(k+\omega/c)}\Big|_{x}^{\infty}\right]$$

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\,J_o\,e^{-i\omega t}\frac{c\,\mu_o}{2}\left[e^{+i\omega x/c}\frac{1}{i\left(k-\omega/c\right)}e^{ix(k-\omega/c)} - e^{-i\omega x/c}\frac{1}{i\left(k+\omega/c\right)}e^{ix(k+\omega/c)}\right]$$

$$\mathbf{E}(x,t) = -\hat{\mathbf{y}}\, J_o\, e^{-i\omega t + ikx} \frac{c\,\mu_o}{2i} \left[\frac{1}{(k-\omega/c)} - \frac{1}{(k+\omega/c)} \right]$$

Which leads to what we want.

(c) Up to this point we have said nothing about being in a dielectric, we have just talked about the electric field that is generated by a given set of currents. Now here is where we let the electric fields generated by the currents affect the currents that generate them. We assume that we are in a dielectric, and that there is a polarization current $\mathbf{J}_{polarization}(\mathbf{r},t)$ which is

$$\mathbf{J}_{polarization}(\mathbf{r},t) = \varepsilon_o \chi_e \frac{\partial}{\partial t}\mathbf{E} = \varepsilon_o (K_e - 1)\frac{\partial}{\partial t}\mathbf{E}$$

We derived this equation on lecture on Friday March 6. You will see this relationship derived in the Course Notes, Section 26-3, see equation (26.3.4). We get a natural mode of this system when the electric field producing the polarization current according to the above equation is the same electric field found in part (b) above that these polarization currents generate. That is, when the electric field that *generates* the currents is the same as the electric field *generated by* the currents. Show that this will happen only when

$$\frac{\omega}{k} = \frac{c}{\sqrt{K_e}}$$

and this of course is the speed of light in a dielectric. Which is made up of lots of waves propagating individually at the speed of light but adding up in phase so that the net propagation speed is $c/\sqrt{K_e}$.

We have $\mathbf{E}(x,t) = -ic\,\mu_o \hat{\mathbf{y}}\, J_o\, e^{i(kx-\omega t)} \dfrac{\omega/c}{(\omega^2/c^2 - k^2)}$, and if we plug this into the equation for the polarization current, we have

$$\mathbf{J}_{polarization}(\mathbf{r},t) = \frac{K_e - 1}{k^2c^2/\omega^2 - 1}\hat{\mathbf{y}}\, J_o\, e^{i(kx-\omega t)}$$

But for consistency this must also be our original current

$$\mathbf{J}(\mathbf{r},t) = \hat{\mathbf{y}}\, J_o\, e^{i(kx-\omega t)}$$

so we must have

$$k^2c^2/\omega^2 = K_e$$