

1.

(a)

In x, y, z basis

$$\begin{aligned} Q_{ij} &= \begin{pmatrix} -2d^2q & & \\ & -2d^2q & \\ & & 4d^2q \end{pmatrix} \\ &= 2d^2q \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix} \end{aligned}$$

Potential

$$\begin{aligned} \Phi &= \frac{1}{8\pi\epsilon_0} \frac{Q_{ij}x_ix_j}{r^5} \\ &= \frac{d^2q}{4\pi\epsilon_0} \frac{-x^2 - y^2 + 2z^2}{r^5} \\ &= \frac{d^2q}{4\pi\epsilon_0} \frac{3\cos^2\theta - 1}{r^3} \end{aligned}$$

Electric field

$$\begin{aligned} E_r &= \frac{3d^2q}{4\pi\epsilon_0} \frac{3\cos^2\theta - 1}{r^4} \\ E_\theta &= \frac{d^2q}{4\pi\epsilon_0} \frac{3\sin 2\theta}{r^4} \end{aligned}$$

Field line

$$\begin{aligned} \frac{dr}{d\theta} &= r \frac{3\cos^2\theta - 1}{\sin 2\theta} \\ \frac{dr}{r} &= \frac{3\cos^2\theta - 1}{\sin 2\theta} d\theta \\ r &= R_0 \sin \theta \sqrt{\cos \theta} \end{aligned}$$

The field line is the furthest when $\frac{dr}{d\theta} = 0$

$$\theta = \arccos \frac{1}{\sqrt{3}}$$

(b)

Magnetic field

$$\begin{aligned}
 \vec{B} &= -\frac{\mu_0}{24\pi c^2 r} \hat{r} \times (\hat{r} \cdot \ddot{\vec{Q}}) \\
 &= -\frac{\mu_0}{24\pi c^2 r} \hat{r} \times \left(\hat{r} \cdot \frac{d^3 2d^2 q (3\hat{z}\hat{z} - 1)}{dt^3} \right) \\
 &= -\frac{q\mu_0 d_0^2}{12\pi c^2 r} \hat{r} \times (3(\hat{r} \cdot \hat{z})\hat{z} - \hat{r}) \frac{d^3 \cos^2 \omega t}{dt^3} \\
 &= \frac{q\mu_0 d_0^2 \omega^3}{\pi c^2 r} (\hat{r} \cdot \hat{z})(\hat{r} \times \hat{z}) \sin 2\omega t \\
 &= -\frac{q\mu_0 d_0^2 \omega^3}{2\pi c^2 r} \sin 2\theta \sin 2\omega t \hat{e}_\phi
 \end{aligned}$$

Electric field

$$\begin{aligned}
 \vec{E} &= c\vec{B} \times \vec{r} \\
 &= -\frac{q\mu_0 d_0^2 \omega^3}{2\pi c r} \sin 2\theta \sin 2\omega t \hat{e}_\theta
 \end{aligned}$$

Power

$$\frac{dP}{d\Omega} = \frac{q^2 d_0^4 \omega^6}{8\pi^2 \varepsilon_0 c^5} \sin^2 2\theta$$

The radiation is the strongest for $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

$$\begin{aligned}
 P &= \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \frac{q^2 d_0^4 \omega^6}{8\pi^2 \varepsilon_0 c^5} \sin^2 2\theta \\
 &= \frac{4q^2 d_0^4 \omega^6}{15\pi \varepsilon_0 c^5}
 \end{aligned}$$

The radiation appears at 2ω

(c)

The radiation will be a dipole radiation appears at ω with total power

$$P = \frac{q^2 d_0^2 \omega^4}{12\pi \varepsilon_0 c^3}$$

Ratio of power

$$\begin{aligned}
 \frac{P_{quad}}{P} &= \frac{16d_0^2 \omega^2}{5c^2} \\
 &\ll 1
 \end{aligned}$$

2.

(a)

$$\begin{aligned}
 \vec{B} &= \nabla \times A \\
 &= \frac{\mu_0}{4\pi} \int d^3\vec{r}' \nabla \times \left(\frac{1}{|\vec{r} - \vec{r}'|} \vec{j} \left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c} \right) \right) \\
 &= \frac{\mu_0}{4\pi} \int d^3\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|} \nabla \times \vec{j} \left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c} \right) + \left(\nabla \frac{1}{|\vec{r} - \vec{r}'|} \right) \times \vec{j} \left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c} \right) \\
 &= \frac{\mu_0}{4\pi} \int d^3\vec{r}' \vec{j} \left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c} \right) \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} + \frac{\partial}{\partial t} \vec{j} \left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c} \right) \times \frac{\vec{r} - \vec{r}'}{c|\vec{r} - \vec{r}'|^2}
 \end{aligned}$$

(b)

The ratio of the two term is on the order of (replacing time derivative with T and replace $|\vec{r} - \vec{r}'|$ with d)

$$\frac{d}{cT} \ll 1$$

3.

(a)

$$\begin{aligned}
 P &= \frac{e^4 E_0^2}{12\pi\epsilon_0 m_e^2 c^3} \\
 &= \langle S \rangle \frac{e^4}{6\pi\epsilon_0^2 m_e^2 c^4} \\
 &= \langle S \rangle \frac{8\pi}{3} r_e^2
 \end{aligned}$$

(b)

The radiation (and therefore the stress tensor) has mirror symmetry along each axis.

(c)

$$\begin{aligned}
 F &= \frac{P}{c} \\
 &= \langle S \rangle \frac{8\pi}{3c} r_e^2
 \end{aligned}$$

4.

$$\begin{aligned}
 a_y &= -\frac{eE_0}{m_e} \cos \omega t - \frac{e^2}{6\pi\epsilon_0 m_e c^3} \dot{a}_y \\
 v_y &= -\frac{eE_0}{m_e \omega} \sin \omega t - \frac{e^2}{6\pi\epsilon_0 m_e c^3} \dot{v}_y \\
 \tilde{v}_y &= i \frac{eE_0}{m_e \omega} e^{i\omega t} - i\omega \frac{e^2}{6\pi\epsilon_0 m_e c^3} \tilde{v}_y \\
 \tilde{v}_y &= i \frac{eE_0}{m_e \omega} e^{i\omega t} - i \frac{2}{3} \omega \tau_e \tilde{v}_y \\
 \tilde{v}_y &= i \frac{eE_0}{m_e \omega} \frac{1}{1 + i \frac{2}{3} \omega \tau_e} e^{i\omega t}
 \end{aligned}$$

Force

$$\begin{aligned}
 \langle F \rangle &= \frac{1}{2} \Re \left(i e \frac{eE_0}{m_e c \omega} \frac{1}{1 + i \frac{2}{3} \omega \tau_e} e^{i\omega t} E_0 e^{-i\omega t} \right) \\
 &= -\frac{1}{2} \Im \left(\frac{e^2 E_0^2}{m_e c \omega} \frac{1}{1 + i \frac{2\omega \tau_e}{3c}} \right) \\
 &= \frac{1}{2} \frac{e^2 E_0^2}{m_e c \omega} \frac{\frac{2\omega \tau_e}{3c}}{1 + \frac{4\omega^2 \tau_e^2}{9c^2}} \\
 &\approx \frac{e^4 E_0^2}{12\pi\epsilon_0 m_e^2 c^4}
 \end{aligned}$$

5.

$$\begin{aligned}
 \frac{\sigma_T L c}{4\pi} &\leq GM m_p \\
 L &\leq \frac{4\pi GM m_p c}{\sigma_T} \\
 &= 1.2 \cdot 10^{31} J \cdot s^{-1} \quad (\text{for the sun})
 \end{aligned}$$

6.

- (a)
- (b)
- (c)
- (d)
- (e)