1.

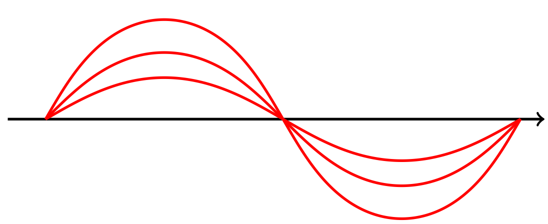
(a)

$$
\begin{aligned}
z =& \bar{n} \int_0^x \frac{\mathrm{d}x}{\sqrt{n^2 - \bar{n}^2}} \\
=& \bar{n} \int_0^x \frac{\mathrm{d}x}{\sqrt{n_0^2 \mathrm{sech}^2(\alpha x) - \bar{n}^2}} \\
=& \frac{\cos\theta_0}{\alpha} \int_0^{\alpha x} \frac{\cosh(\alpha x) \mathrm{d}\alpha x}{\sqrt{1 - \cos^2\theta_0 \cosh^2(\alpha x)}} \\
=& \frac{\cos\theta_0}{\alpha} \int_0^{\sinh(\alpha x)} \frac{\mathrm{d}\sinh(\alpha x)}{\sqrt{\sin^2\theta_0 - \cos^2\theta_0 \sinh^2(\alpha x)}} \\
=& \frac{1}{\alpha} \int_0^{\cot\theta_0 \sinh(\alpha x)} \frac{\mathrm{d}y}{\sqrt{1 - y^2}} \\
=& \frac{1}{\alpha} \arcsin(\cot\theta_0 \sinh(\alpha x)) \\
\sin(\alpha z) =& \cot\theta_0 \sinh(\alpha x)
\end{aligned}
$$

Since $\max(\sin(\alpha z)) = 1$

$$
\begin{aligned}
\sin(\alpha z) =& \frac{\sinh(\alpha x)}{\sinh(\alpha x_{max})} \\
\alpha x =& \mathrm{arcsinh}(\sinh(\alpha x_{max}) \sin(\alpha z))
\end{aligned}
$$

Rays for $\theta_0 = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$



(b)

$$
Z = \frac{\pi}{\alpha}
$$

Independent from $\bar{n}$

(c)

$$
\begin{aligned}
L_{opt} =& \int_{z=0}^{Z} n\mathrm{d}s \\
=& 2\int_0^{x_{max}} n\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
=& 2\int_0^{x_{max}} n\sqrt{1+\left(\frac{\bar{n}}{\sqrt{n^2-\bar{n}^2}}\right)^2}\mathrm{d}x \\
=& 2\int_0^{x_{max}} \frac{n^2}{\sqrt{n^2-\bar{n}^2}}\mathrm{d}x \\
=& 2n_0\int_0^{x_{max}} \frac{\mathrm{sech}^2(\alpha x)}{\sqrt{\mathrm{sech}^2(\alpha x)-\cos^2\theta_0}}\mathrm{d}x \\
=& \frac{2n_0}{\alpha}\int_0^{\sinh(\alpha x_{max})} \frac{\mathrm{d}\sinh(\alpha x)}{\cosh^2(\alpha x)\sqrt{1-\cos^2\theta_0\cosh^2(\alpha x)}} \\
=& \frac{2n_0}{\alpha}\int_0^{\sinh(\alpha x_{max})} \frac{\mathrm{d}y}{(1+y^2)\sqrt{\sin^2\theta_0-\cos^2\theta_0 y^2}} \\
=& \frac{2n_0}{\alpha\cos\theta_0}\frac{\pi}{2\sqrt{1+\tan^2\theta_0}} \\
=& \frac{\pi n_0}{\alpha} \\
=& n_0 Z
\end{aligned}
$$

2.

(a)

For $r < R$, $B_l = 0$, for $r > R$, $A_l = 0$. Since the charge distribution only have $P_1(\cos\theta)$ component, the only non-zero term in the series is when $l = 1$. Therefore, for $r < R$

$$\phi_- = A_1 r\cos\theta$$

And for $r > R$

$$\phi_+ = \frac{B_1}{r^2}\cos\theta$$

From the boundary condition

$$
\begin{aligned}
A_1 R =& \frac{B_1}{R^2} \\
\frac{\sigma}{\varepsilon_0} =& A_1 + \frac{2B_1}{R^3} \\
A_1 =& \frac{\sigma}{3\varepsilon_0} \\
B_1 =& \frac{\sigma R^3}{3\varepsilon_0}
\end{aligned}
$$

(b)

3.