Solutions Assignment #3: Due Friday February 27, 2015 at 2:30 pm

Reading: 8.311 Simple Radiating Systems Course Notes available through Piazza and at http://web.mit.edu/8.311/www/resources/radNotes_8311_Spring2015.pdf

Note: There are only six problems on this problem set, but one of the problems (3.5) carries twice the normal weight (20 points) so the total number of points on this problem set is 70, as in the first two problem sets.

Notation: Below, as compared to Jackson, I use $\mathbf{r} = \mathbf{X}$ and the volume integral $d\tau = d^3x$. I also do this in the course notes referenced above.

Problems

Problem 3-1: The fields of a static electric dipole and of a static magnetic dipole

Why:? Dealing with anything to do with magnetic fields is a pain compared to dealing with electric fields, and this problem shows that once again.

(a) From equation (15) of our 8.311 Simple Radiating Systems Course Notes, we see that the electric field of a static electric dipole of dipole moment vector $\mathbf{p}$ is given by

$$\mathbf{E}_{dipole}(\mathbf{r}) = -\nabla\left[\Phi_{dipole}(\mathbf{r})\right] = -\nabla\left[\frac{1}{4\pi\varepsilon_o}\frac{\hat{\mathbf{n}}\cdot\mathbf{p}}{r^2}\right] \text{ where } \hat{\mathbf{n}} = \mathbf{r}/r$$

Prove that if we avoid the origin

$$\mathbf{E}_{dipole}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}\right]}{r^3}$$

$$-\left\{\nabla\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{p}}{r^2}\right]\right\}_i = -\partial_i\frac{x_m p_m}{r^3} = -\frac{p_i}{r^3} + \frac{3p_m x_m x_i}{r^5} = \left\{\frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) - \mathbf{p}}{r^3}\right\}_i$$

(b) Equation (10) of our 8.311 Simple Radiating Systems Course Notes is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi r}\int d\tau' \,\mathbf{J}(\mathbf{r}',t') + \frac{\mu_o}{4\pi r}\int(\hat{\mathbf{n}}\cdot\mathbf{r}')\left[\frac{\mathbf{J}(\mathbf{r}',t')}{r} + \frac{1}{c}\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right]d\tau' + \ldots$$

We argued in the notes that the first term in the above equation vanishes in statics. For the second term in this equation, again assuming statics, show that

$$\frac{\mu_o}{4\pi}\frac{1}{r^2}\int\mathbf{J}(\mathbf{r}')(\mathbf{r}'\cdot\hat{\mathbf{n}})\,d^3x' = \frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2} \qquad \mathbf{m} = \frac{1}{2}\int\mathbf{r}'\times\mathbf{J}(\mathbf{r}')d^3x'$$

To do this you need to first prove that (see the Appendix of the course notes for *lots* of help)

$$\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}') = \frac{1}{2}\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J}) + \mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')\right] - \frac{1}{2}\left[\hat{\mathbf{n}}\,\mathbf{x}\left(\mathbf{r}'\,\mathbf{x}\,\mathbf{J}\right)\right]$$

Let **J** be any vector. Then

$$\hat{\mathbf{n}}\,\mathbf{x}\left(\mathbf{r}'\,\mathbf{x}\,\mathbf{J}\right)=\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J})-\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')$$
$$=\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J})+\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')-2\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')$$

where in the first step we have used a standard vector identity, and in the second step we have added and subtracted the same term $\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{X}')$ on the right hand side. Solving this equation for $\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{X}')$, we have

$$\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')=\frac{1}{2}\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J})+\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')\right]-\frac{1}{2}\left[\hat{\mathbf{n}}\,\mathbf{x}\left(\mathbf{r}'\,\mathbf{x}\,\mathbf{J}\right)\right]$$

and then that

$$\int\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J})+\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')\right]d^3x'=-\int\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\nabla'\cdot\mathbf{J}(\mathbf{r}')\,d^3x'$$

[Hint: consider the divergence of the vector $x'(\hat{\mathbf{n}}\cdot\mathbf{X}')\mathbf{J}$, that is $\nabla'\cdot\left[x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\mathbf{J}\right]$, and play the usual tricks in converting to a surface integral at infinity and setting that to zero]. Then consider the divergence of $y'(\hat{\mathbf{n}}\cdot\mathbf{X}')\mathbf{J}$ and $z'(\hat{\mathbf{n}}\cdot\mathbf{X}')\mathbf{J}$. Then use the fact that we are in statics with no time dependence.

To show this, consider the divergence of the vector below:

$$\nabla'\cdot\left[x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\mathbf{J}\right]=x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\nabla'\cdot\mathbf{J}+\mathbf{J}\cdot\nabla'(x'(\hat{\mathbf{n}}\cdot\mathbf{r}'))$$
$$=x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\nabla'\cdot\mathbf{J}+\mathbf{J}\cdot\left[\hat{\mathbf{e}}_x(\hat{\mathbf{n}}\cdot\mathbf{r}')+x'\hat{\mathbf{n}}\right]$$
$$\nabla'\cdot\left[x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\mathbf{J}\right]=x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\nabla'\cdot\mathbf{J}+J_x(\hat{\mathbf{n}}\cdot\mathbf{r}')+x'\mathbf{J}\cdot\hat{\mathbf{n}}$$

If we integrate the last equation above over all space, convert the left side to a surface integral and set it to zero, we have

$$\int\left[J_x(\hat{\mathbf{n}}\cdot\mathbf{r}')+x'\mathbf{J}\cdot\hat{\mathbf{n}}\right]d^3x'=-\int x'(\hat{\mathbf{n}}\cdot\mathbf{r}')\nabla'\cdot\mathbf{J}\;d^3x'$$

Since this must be true if we replace x' by y' or z', with the component of **J** changing appropriately, we have

$$\int\left[\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')+\mathbf{r}'(\mathbf{J}\cdot\hat{\mathbf{n}})\right]d^3x'=-\int\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\nabla'\cdot\mathbf{J}\;d^3x'$$

Since we are in statics, and $\nabla'\cdot\mathbf{J}=-\dfrac{\partial}{\partial t'}\rho(\mathbf{r}',t')$, we have the term on the left hand side of is zero. So we have

$$\frac{\mu_o}{4\pi}\frac{1}{r^2}\int \mathbf{J}(\mathbf{r}')(\mathbf{r}'\cdot\hat{\mathbf{n}})\,d^3x' = -\frac{\mu_o}{4\pi}\frac{1}{r^2}\int\frac{1}{2}\left[\hat{\mathbf{n}}\,\mathbf{x}\,(\mathbf{r}'\,\mathbf{x}\,\mathbf{J})\right]d^3x'$$
$$= -\frac{\mu_o}{4\pi}\frac{\hat{\mathbf{n}}\times}{r^2}\int\frac{1}{2}\left[\,(\mathbf{r}'\times\mathbf{J}\,)\right]d^3x' = \frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}$$

(c) Prove that if we avoid the origin, then

$$\mathbf{B}_{dipole}(\mathbf{r}) = \nabla\times\left[\frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}\right] = \frac{\mu_o}{4\pi}\frac{[3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}})-\mathbf{m}]}{r^3}$$

First, note that

$$\partial_j\left(\frac{1}{r^3}\right) = \partial_j\left(\frac{1}{(x_l x_l)^{3/2}}\right) = -\frac{3x_j}{r^5}$$

Then

$$\left\{\nabla\times\left[\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2}\right]\right\}_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\left(\frac{m_l x_m}{r^3}\right) = \varepsilon_{kij}\varepsilon_{klm}\partial_j\left(\frac{m_l x_m}{r^3}\right)$$
$$= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\partial_j\left(\frac{m_l x_m}{r^3}\right) = \partial_j\left(\frac{m_i x_j}{r^3}\right) - \partial_l\left(\frac{m_l x_i}{r^3}\right)$$
$$= \frac{m_i}{r^3}\partial_j x_j - 3\left(\frac{m_i x_j x_j}{r^5}\right) - \left(\frac{m_l}{r^3}\right)\partial_l(x_i) + \frac{3m_l x_i x_l}{r^5}$$
$$= 3\frac{m_i}{r^3} - 3\left(\frac{m_i}{r^3}\right) - \left(\frac{m_l}{r^3}\right)\delta_{il} + \frac{3(\mathbf{m}\cdot\mathbf{n})n_i}{r^3}$$
$$= \left\{\frac{[3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}})-\mathbf{m}]}{r^3}\right\}_i \qquad \text{as desired}$$

Problem 3-2: The Energetics of Electric Dipole Energy

Why:? Emphasizing that there is a lot of energy sloshing around an electric dipole system varying in time, most of which can be recovered, and only a little of which gets radiated away to infinity irreversibly.

(a) For a static electric dipole with $\mathbf{p}(t) = \hat{\mathbf{z}}\,p(t)\,, \mathbf{E}(\mathbf{r}) = \dfrac{2p\cos\theta}{4\pi\,\varepsilon_o r^3}\hat{\mathbf{r}} + \dfrac{p\sin\theta}{4\pi\,\varepsilon_o r^3}\hat{\boldsymbol{\theta}}$.

Calculate the total amount of electric energy outside of a sphere of radius R_o, and show that it is given by

$$\text{Electrostatic energy of dipole outside } \mathrm{R}_o = \frac{p^2}{12\pi\,\varepsilon_o R_o^3}$$

$$\mathbf{E}(\mathbf{r}) = \frac{2p\cos\theta}{4\pi\,\varepsilon_o r^3}\hat{\mathbf{r}} + \frac{p\sin\theta}{4\pi\,\varepsilon_o r^3}\hat{\boldsymbol{\theta}}$$

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \int_{R_o}^{\infty} r^2 dr \frac{\varepsilon_o p^2}{2(4\pi\,\varepsilon_o)^2} \frac{\left(4\cos^2\theta + \sin^2\theta\right)}{r^6} = \frac{p^2}{16\pi\varepsilon_o}\int_{R_o}^{\infty}\frac{1}{r^4}dr\int_{-1}^{1}dx\left(3x^2+1\right) = \frac{p^2}{12\pi\,\varepsilon_o R_o^3}$$

(b) Consider electric dipole radiation for which $\mathbf{p}(t) = \hat{\mathbf{z}}\, p(t)$. Show that Equations (19) and (21) of the class notes in this case become

$$\mathbf{B} = \hat{\boldsymbol{\varphi}}\frac{\sin\theta}{4\pi\,\varepsilon_o}\left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{c\,r}\right) \qquad \mathbf{E} = \hat{\mathbf{r}}\frac{2\cos\theta}{4\pi\,\varepsilon_o}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2}\right) + \hat{\boldsymbol{\theta}}\frac{\sin\theta}{4\pi\,\varepsilon_o}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right)$$

Take any *p(t)* which goes smoothly from one constant value p_1 to another constant value p_2 over a time *T*. Consider a spherical surface of radius R_o, centered at the origin, and show that integrating the Poynting flux associated with the fields above over the surface of the sphere and over time gives the following expression for the energy in joules flowing through the surface of the sphere during this process of changing the dipole moment

$$\text{Energy through } \mathrm{R}_o = \frac{\left(p_2^2 - p_1^2\right)}{12\pi\,\varepsilon_o R_o^3} + \int_{-\infty}^{\infty}\frac{\ddot{p}^2}{6\pi\,\varepsilon_o c^3}dt$$

Note that the second term on the right side of this equation is independent of R_o and can never be negative. This term represents the energy radiated away to infinity, and this is an irreversible process. What does the first term in the above equation represent? It can be either positive or negative. What does that mean?

$$\int dt \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,R_0^2 = \int dt\int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\phi \frac{\sin^2\theta}{(4\pi)^2\,\varepsilon_o}\left(\frac{\dot{p}}{R_0^2} + \frac{\ddot{p}}{c\,R_0}\right)\left(\frac{p}{R_0^3} + \frac{\dot{p}}{cR_0^2} + \frac{\ddot{p}}{c^2 R_0}\right)R_0^2$$

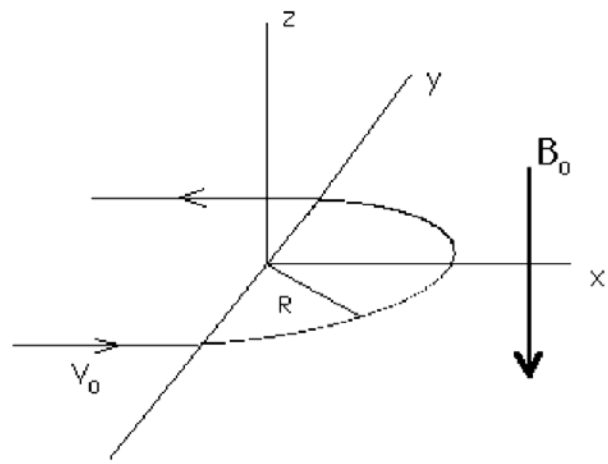
$$= \int dt \frac{1}{8\pi\,\varepsilon_o}\left[\frac{d}{dt}\left(\frac{1}{2}\frac{p^2}{R_0^3} + \frac{(\dot{p})^2}{c^2\,R_0} + \frac{\dot{p}p}{c\,R_0^2}\right) + \frac{(\ddot{p})^2}{c^3}\right]\int_{-1}^{1}dx\left(1-x^2\right)$$

$$= \int dt \frac{1}{6\pi\,\varepsilon_o}\left[\frac{d}{dt}\left(\frac{1}{2}\frac{p^2}{R_0^3} + \frac{(\dot{p})^2}{c^2\,R_0} + \frac{\dot{p}p}{c\,R_0^2}\right) + \frac{(\ddot{p})^2}{c^3}\right] =$$

$$\frac{1}{6\pi\,\varepsilon_o}\left(\frac{1}{2}\frac{p^2}{R_0^3} + \frac{(\dot{p})^2}{c^2\,R_0} + \frac{\dot{p}p}{c\,R_0^2}\right)\Bigg|_{-\infty}^{\infty} + \int dt\frac{(\ddot{p})^2}{6\pi\,\varepsilon_o c^3} = \frac{\left(p_2^2 - p_1^2\right)}{12\pi\,\varepsilon_o R_o^3} + \int_{-\infty}^{\infty}\frac{\ddot{p}^2}{6\pi\,\varepsilon_o c^3}dt$$

The fact that the first term in the equation above can be positive or negative means that energy flows out into the region beyond R_0 or flows back from the region beyond R_0 depending on whether we are increasing p$\left(p_2 > p_1\right)$ or decreasing p $\left(p_2 < p_1\right)$. This is reversible energy flow that we can create and then get back if we desire.

Problem 3-3: A Burst of Electric Dipole Radiation *Why:? Again to point out that in systems satisfying the electric dipole approximation, the energy radiated away to infinity is small compared to other energies in the problem.*



We have a static magnetic field **B** which is given by $\mathbf{B} = \begin{cases} 0 & x < 0 \\ -B_o\hat{\mathbf{z}} & x > 0 \end{cases}$

A particle of mass m and charge q is located at $x < 0$ at $t = 0$, with velocity $\mathbf{V} = \hat{\mathbf{x}}\,V_o$, where $V_o << c$. The particle crosses into the region $x > 0$, gyrates in the magnetic field and then exits the region $x > 0$ and returns back down the x-axis at speed $\mathbf{V} = -\hat{\mathbf{x}}\,V_o$.

(a) What is the radius R of the circle of gyration, in terms of q, m, B_o and V_o?

$$R = \frac{mV_o}{qB_o}$$

(b) What is the time T that the particle spends in the region $x > 0$?

$$\text{Time in the region } x > 0 \text{ is } \frac{\pi R}{V_o} = \frac{\pi m}{qB_o}$$

(c) What is the rate that the particle radiates energy into all solid angles in electric dipole radiation when it is in the region $x > 0$--that is, what is $\dfrac{dW_{\text{rad}}}{dt}$?

$$\text{The acceleration in the region } x > 0 \text{ is } \frac{qV_oB_o}{m}$$

$$\text{So the rate at which power is radiated is } \frac{\mu_o q^2}{6\pi c}\left[\frac{qV_oB_o}{m}\right]^2 = \frac{\mu_o q^4 V_o^2 B_o^2}{6\pi c m^2}$$

(d) What is the total energy radiation in electric dipole radiation in this process?

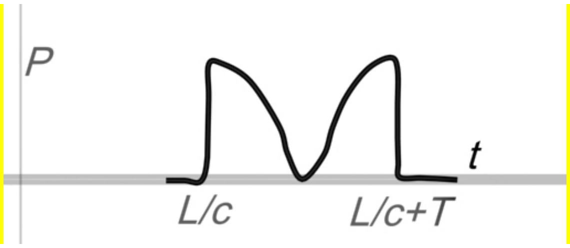
$$\text{So the total power radiated is } \frac{\mu_o q^4 V_o^2 B_o^2}{6\pi c m^2}\frac{\pi m}{qB_o} = \frac{\mu_o q^3 V_o^2 B_o}{6mc}$$

(e) Let $R_{classical} = \dfrac{1}{4\pi\,\varepsilon_o}\dfrac{q^2}{m\,c^2}$. What is the ratio of the total radiated energy to the kinetic energy of the particle? Write this ratio in terms of the quantities $R_{classical}$, R, V_O and c and π ***ONLY***. Under what conditions will the radiated energy be small compared to the kinetic energy of the charge?

$$\frac{\mu_o q^3 V_o^2 B_o}{6mc} \Big/ \left[\tfrac{1}{2} m V_o^2\right] = \frac{2\mu_o q^3 B_o}{6m^2 c} = \frac{8\pi}{6}\left[\frac{qB_o}{mV_o}\right]\left[\frac{q^2}{4\pi\varepsilon_o mc^2}\right]\left[\frac{V_o}{c}\right] = \frac{4\pi}{3}\left[\frac{R_{classical}}{R}\right]\left[\frac{V_o}{c}\right]$$

So as long as the last expression is small compared to one, the energy radiated will be small compared to the kinetic energy.

(f) Suppose you are an observer sitting on the x-axis at a distance $L \gg cT \gg d$. Make a sketch of the power radiated into unit solid angle as a function of time at your position. Remember, the particle is just entering $x > 0$ at $t = 0$. [Hint: the instantaneous radiation pattern always has a $\sin^2\theta$ dependence, where θ is the angle between the direction to the observer and the vector acceleration, properly retarded of course.]



Problem 3-4: Electric dipole radiation from an LC oscillator--"radiation resistance" *Why:? All circuits radiate, and that power loss can look like "resistance".*

A capacitor with capacitance C is in a circuit with an inductor with inductance L. There is no resistance as we normally think of it in this circuit. The capacitor has maximum energy and charge Q_o at $t = 0$, and the energy then sloshes back and forth between the inductor and capacitor at a frequency $\omega = 1/\sqrt{LC}$. We assume that the dimensions of this circuit are such that the speed of light transit time across the circuit is much shorter than $2\pi/\omega$. The distance between the plates of the capacitor is d, and the area of the plates is A_C, so that $C = \dfrac{\varepsilon_o A_C}{d}$. The inductance is a solenoid, with N turns, cross-sectional area A_L, and length h, so that $L = \dfrac{\mu_o N^2 A_L}{h}$.

(a) What is the time-averaged rate at which this system radiates electric dipole radiation, in terms of d, ω, c, ε_O, and Q_O? The electric dipole moment of the capacitor is just Qd.

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle = \frac{1}{4\pi\varepsilon_o}\frac{2\left\langle (\ddot{p})^2 \right\rangle}{c^3} = \frac{d^2 Q_o^2 \omega^4}{4\pi\varepsilon_o c^3}$$

(b) Take the total energy radiated in one period of the oscillation (your answer in (a) times $\frac{2\pi}{\omega}$ and divide it by the average energy in the capacitor, $\frac{Q_o{}^2}{4C}$. Show this ratio is small if the speed of light transit time across the capacitor is small.

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle \frac{2\pi}{\omega} = \frac{2\pi}{\omega} \frac{d^2 Q_o^2 \omega^4}{4\pi\varepsilon_o c^3} = \frac{d^2 Q_o^2 \omega^3}{2\varepsilon_o c^3}$$

$$\frac{4C}{Q_o^2} \left\langle \frac{dW_{rad}}{dt} \right\rangle \frac{2\pi}{\omega} = \frac{4C}{Q_o^2} \frac{d^2 Q_o^2 \omega^3}{2\varepsilon_o c^3} = \frac{2d^2\omega^3}{\varepsilon_o c^3} C = \frac{2d^2\omega^3}{\varepsilon_o c^3} \frac{\varepsilon_o A_C}{d} = \frac{2d\omega^3}{c^3} A_C$$

$$\frac{4C}{Q_o^2} \left\langle \frac{dW_{rad}}{dt} \right\rangle \frac{2\pi}{\omega} = 2(2\pi)^3 \frac{A_C d}{c^3 T^3} << 1$$ as long as the transit time across any dimension of the capacitor is small compared to T

(c) The current $I(t)$ in this circuit is given by $I(t) = \frac{d}{dt} Q(t)$, so that it is clear that the time-averaged value of I^2 is $\left\langle I^2 \right\rangle = \frac{\omega^2 Q_o{}^2}{2}$. Use this relation to write your answer in (a) for the energy radiated as $\left\langle I^2 \right\rangle R_{radiation}$, where $R_{radiation}$ is the "radiation resistance", and has units of ohms. Give an expression for $R_{radiation}$ in terms of d, ω, c, and ε_o.

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle = \frac{d^2 Q_o^2 \omega^4}{4\pi\varepsilon_o c^3} = \left\langle I^2 \right\rangle \frac{d^2\omega^2}{2\pi\varepsilon_o c^3} = \left\langle I^2 \right\rangle R_{radiation} \qquad R_{radiation} = \frac{d^2\omega^2}{2\pi\varepsilon_o c^3}$$

(d) Using $\omega = \frac{1}{\sqrt{LC}}$, and the equations for L and C given above, to show that the radiation resistance you have from (b) can be written in the form $c\mu_o$ times a dimensionless expression which involves the geometry of the capacitor and inductor, and N. The constant $c\mu_o$ has dimensions of ohms ($c\mu_o$ = 377 ohms), and is sometimes called the radiation resistance of free space. This "radiation resistance" has the same effect as a true resistance--the energy in the circuit slowly decreases as it is irreversibly lost to the system through radiation.

$$R_{radiation} = \frac{d^2\omega^2}{2\pi\varepsilon_o c^3} = \frac{d^2}{2\pi\varepsilon_o c^3} \frac{1}{LC} = \frac{d^2}{2\pi\varepsilon_o c^3} \frac{hd}{(\mu_o N^2 A_L)(\varepsilon_o A_C)}$$
$$= \frac{1}{\varepsilon_o c^3} \frac{1}{\varepsilon_o \mu_o} \left[\frac{1}{2\pi} \frac{hd^3}{N^2 A_L A_C} \right] = \frac{1}{\varepsilon_o c} \left[\frac{1}{2\pi} \frac{hd^3}{N^2 A_L A_C} \right] = \mu_o c \left[\frac{1}{2\pi} \frac{hd^3}{N^2 A_L A_C} \right]$$

Problem 3-5: Energy and angular momentum radiated by a spinning magnet (carries twice the normal weight).

A spinning magnet has a dipole moment that is given by

$$\mathbf{m} = m_o\left[\cos\omega_o t\,\hat{\mathbf{x}} + \sin\omega_o t\,\hat{\mathbf{y}}\right]$$

(a) Equation (48) of the class notes is

$$\boxed{\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left\{\underbrace{\frac{1}{r^3}\left[3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}}) - \mathbf{m}\right]}_{\text{quasi-static}} \quad \underbrace{+\frac{1}{c\,r^2}\left[3\hat{\mathbf{n}}(\dot{\mathbf{m}}\cdot\hat{\mathbf{n}}) - \dot{\mathbf{m}}\right]}_{\text{induction}} \underbrace{+\frac{1}{rc^2}(\ddot{\mathbf{m}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}}_{\text{radiation}}\right\}}$$

Keeping only the induction and radiation terms, write out the expression for $\mathbf{B}(\mathbf{r},t)$ for this spinning magnet. **Do this in terms of spherical polar unit vectors.** Do the same for the electric field, given by

$$\mathbf{E}(\mathbf{r},t) = -\frac{\mu_o}{4\pi}\left[\frac{\ddot{\mathbf{m}}}{c\,r} + \frac{\dot{\mathbf{m}}}{r^2}\right]\mathbf{x}\,\hat{\mathbf{n}}$$

[Useful formula:

$$\begin{aligned}
\hat{\mathbf{e}}_x &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{e}}_y &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{e}}_z &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

]

$$\sin(\phi-\omega_o t) = \sin\phi\cos\omega_o t - \sin\omega_o t\cos\phi$$

$$\dot{\mathbf{m}} = \omega_o m_o\left[-\sin\omega_o t\,\hat{\mathbf{x}} + \cos\omega_o t\,\hat{\mathbf{y}}\right] = \omega_o m_o\left[\sin(\phi-\omega_o t)(\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta) + \hat{\boldsymbol{\phi}}\cos(\phi-\omega_o t)\right]$$

$$3\hat{\mathbf{n}}(\dot{\mathbf{m}}\cdot\hat{\mathbf{n}}) - \dot{\mathbf{m}} = \omega_o m_o\left[2\hat{\mathbf{r}}\sin\theta\sin(\phi-\omega_o t) - \hat{\boldsymbol{\theta}}\cos\theta\sin(\phi-\omega_o t) - \hat{\boldsymbol{\phi}}\cos(\phi-\omega_o t)\right]$$

$$\dot{\mathbf{m}}\times\hat{\mathbf{n}} = \omega_o m_o\left[-\sin\omega_o t\,\hat{\mathbf{x}}\times\hat{\mathbf{r}} + \cos\omega_o t\,\hat{\mathbf{y}}\times\hat{\mathbf{r}}\right] = \omega_o m_o\left[\hat{\boldsymbol{\theta}}\cos(\phi-\omega_o t) - \hat{\boldsymbol{\phi}}\cos\theta\sin(\phi-\omega_o t)\right]$$

$$\ddot{\mathbf{m}}\times\hat{\mathbf{n}} = \omega_o^2 m_o\left[+\hat{\boldsymbol{\theta}}\sin(\phi-\omega_o t) + \hat{\boldsymbol{\phi}}\cos\theta\cos(\phi-\omega_o t)\right]$$

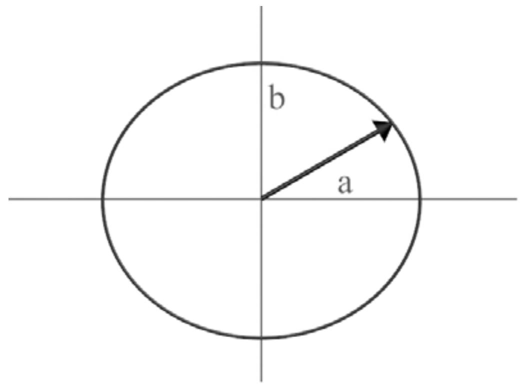
$$(\ddot{\mathbf{m}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} = \omega_o^2 m_o\left[-\hat{\boldsymbol{\phi}}\sin(\phi-\omega_o t) + \hat{\boldsymbol{\theta}}\cos\theta\cos(\phi-\omega_o t)\right]$$

So keeping only the radiation and induction terms we have

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o\omega_o m_o}{4\pi}\left\{\begin{array}{l}\dfrac{1}{c\,r^2}\left[2\hat{\mathbf{r}}\sin\theta\sin(\phi-\omega_o t') - \hat{\boldsymbol{\theta}}\cos\theta\sin(\phi-\omega_o t') - \hat{\boldsymbol{\phi}}\cos(\phi-\omega_o t')\right] \\ +\dfrac{\omega_o}{rc^2}\left[-\hat{\boldsymbol{\phi}}\sin(\phi-\omega_o t') + \hat{\boldsymbol{\theta}}\cos\theta\cos(\phi-\omega_o t')\right] \qquad t' = t - r/c\end{array}\right\}$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{\mu_o \omega_o m_o}{4\pi}\left[\frac{\left[\hat{\boldsymbol{\theta}}\cos(\phi-\omega_o t') - \hat{\boldsymbol{\phi}}\cos\theta\sin(\phi-\omega_o t')\right]}{r^2} + \frac{\omega_o\left[\hat{\boldsymbol{\theta}}\sin(\phi-\omega_o t') + \hat{\boldsymbol{\phi}}\cos\theta\cos(\phi-\omega_o t')\right]}{c\,r}\right]$$

(b) Now consider only the radiation terms and consider any direction in space. We characterize the polarization ellipse and the helicity of the emitted radiation as follows. Consider the figure. In this figure we are sitting at a fixed point in space and the Poynting vector is out of the plane of the paper. If the **B** field rotates counter clockwise as time progresses at this fixed point, we call this positive helicity; if the field rotates clockwise, we call this negative helicity. The ellipticity is specified by the direction of the semi-major axis (the horizontal axis here) and the ratio of b/a. If $b = 0$ this is linear polarization, and if $b = a$ this is circular polarization. Characterize the polarization ellipse and the helicity of the emitted radiation at polar angles of $\theta = 0$, $\theta = \pi/4$, $\theta = \pi/2$, $\theta = 3\pi/4$, and $\theta = \pi$.



If we look only at the radiation terms for the magnetic field, we have

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o \omega_o m_o}{4\pi}\left\{\frac{\omega_o}{rc^2}\left[-\hat{\boldsymbol{\phi}}\sin(\phi-\omega_o t') + \hat{\boldsymbol{\theta}}\cos\theta\cos(\phi-\omega_o t')\right] \qquad t' = t - r/c\right\}$$

In the equatorial plane, $\theta = \pi/2$, this is linearly polarized in the $\hat{\boldsymbol{\phi}}$ direction. At $\theta = \pi/4$, this is elliptically polarized with positive helicity and a ratio of *b/a* of $\cos\pi/4 = 1/\sqrt{2}$; at $\theta = 3\pi/4$, this is elliptically polarized with negative helicity and the same ratio of b/a. At $\theta = 0$ we have circular polarization with positive helicity, and at $\theta = \pi$ we have circular polarization with negative helicity.

(c) Now again consider only the radiation terms and consider any direction in space. What is the *time averaged* angular distribution of radiation radiated into the solid angle about the direction to the observer, that is $\dfrac{dW_{rad}}{d\Omega\, dt}$ (compare equation (29) of the class notes. Sketch the angular distribution of this time-averaged radiation.

$$\frac{dW_{rad}}{d\Omega\, dt} = \frac{c\,r^2 B_{rad}^2}{\mu_o} = \frac{c\,r^2}{\mu_o}\left[\frac{\omega_o}{rc^2}\frac{\mu_o\omega_o m_o}{4\pi}\right]^2\left[\sin^2(\phi-\omega_o t') + \cos^2\theta\cos^2(\phi-\omega_o t')\right]$$

Time averaging gives

$$\left\langle\frac{dW_{rad}}{d\Omega\, dt}\right\rangle = \frac{c}{2\mu_o}\left[\frac{1}{c^2}\frac{\mu_o\omega_0^2 m_o}{4\pi}\right]^2\left[1+\cos^2\theta\right]$$

(d) Integrate over all solid angle to find $\left\langle \frac{dW_{rad}}{dt} \right\rangle$, the total rate at energy is being emitted.

$$\int \left[1+\cos^2\theta\right] d\left(\cos\theta\right) d\phi = 2\pi\int_{-1}^{1} dx(1+x^2) = \frac{16\pi}{3} \qquad \left\langle \frac{dW_{rad}}{dt} \right\rangle = \frac{2}{3}\left[\frac{\mu_o\omega_0^4 m_o^2}{4\pi c^3}\right]$$

(e) Equation (31) for the rate of radiation of electromagnetic angular momentum per unit solid angle $\frac{d\mathbf{L}_{rad}}{dt\, d\Omega} = \; -r^3\,\hat{\mathbf{n}}\,\mathbf{x}(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}})$. Calculate this quantity. In doing this calculation, as in all stress tensor calculations, make sure you only calculate the components of $\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)$ that you actually need. That is, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ so we only need T_{ir}, and so on, and the cross product means that you will only end up needing $T_{\theta r}$ and $T_{\phi r}$. When you get to a form that looks like $r\,T_{\theta r}\hat{\boldsymbol{\phi}}$, ***you must express*** $\hat{\boldsymbol{\phi}}$ in terms of Cartesian unit vectors, whose direction does not vary in space, before you do any angular integrations.

Sketch the angular distribution of the z-component of the time-averaged radiated angular momentum. Note how different this is from your distribution in (c)

[Useful formula:

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \hat{\mathbf{e}}_x \sin\theta\cos\phi + \hat{\mathbf{e}}_y \sin\theta\sin\phi + \hat{\mathbf{e}}_z \cos\theta \\
\hat{\mathbf{e}}_\theta &= \hat{\mathbf{e}}_x \cos\theta\cos\phi + \hat{\mathbf{e}}_y \cos\theta\sin\phi - \hat{\mathbf{e}}_z \sin\theta \\
\hat{\mathbf{e}}_\phi &= -\hat{\mathbf{e}}_x \sin\phi + \hat{\mathbf{e}}_y \cos\phi
\end{aligned}$$

]

We first calculate the relevant terms of $\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)$

$$\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right) = r\,\hat{\mathbf{r}}\times\left[T_{rr}\hat{\mathbf{r}} + T_{\theta r}\hat{\boldsymbol{\theta}} + T_{\phi r}\hat{\boldsymbol{\phi}}\right] = rT_{\theta r}\hat{\boldsymbol{\phi}} - rT_{\phi r}\hat{\boldsymbol{\theta}}$$

If we look at our expression for **E**, there are no radial terms, and therefore the electric field will not contribute to $T_{\theta r}$ or $T_{\phi r}$. For the magnetic field, we have

$$T_{\theta r} = \frac{1}{\mu_o}B_\theta B_r = \left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2 \left(\frac{\omega_o}{rc^2}\cos\theta\cos\left(\phi-\omega_o t'\right)\right)\frac{1}{\mu_o}\left(\frac{2}{c\,r^2}\sin\theta\sin(\phi-\omega_o t')\right)$$

Where I have dropped terms in B_θ that fall off faster than inverse *r*, since these will give me no contribution at infinity. Simlarly

$$T_{\phi r} = \frac{1}{\mu_o}B_\phi B_r = -\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2 \left(\frac{\omega_o}{rc^2}\sin\left(\phi-\omega_o t'\right)\right)\frac{1}{\mu_o}\left(\frac{2}{c\,r^2}\sin\theta\sin(\phi-\omega_o t')\right)$$

If I time average, $\left\langle T_{\theta r}\right\rangle = 0$ and

$$\left\langle T_{\phi r}\right\rangle = -\frac{1}{r^3}\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^2\mu_o c}\right)\sin\theta$$

So that $$\left\langle \mathbf{r}\times\left(\ddot{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)\right\rangle = -\left\langle rT_{\phi r}\hat{\boldsymbol{\theta}}\right\rangle = \frac{1}{r^2}\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^2\mu_o c}\right)\sin\theta\,\hat{\boldsymbol{\theta}}$$

$$\left\langle \int_S\left[-\mathbf{r}\times\left(\ddot{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)\right]da\right\rangle = -\int\left[\frac{1}{r^2}\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^2\mu_o c}\right)\sin\theta\right]\hat{\boldsymbol{\theta}}\left[r^2 d\Omega\right] = \left\langle\frac{d\mathbf{L}_{rad}}{dt\,d\Omega}\right\rangle d\Omega$$

$$\left\langle\frac{d\mathbf{L}_{rad}}{dt\,d\Omega}\right\rangle = -\int\left[\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^2\mu_o c}\right)\sin\theta\right]\left[\hat{\mathbf{e}}_x\cos\theta\cos\phi+\hat{\mathbf{e}}_y\cos\theta\sin\phi-\hat{\mathbf{e}}_z\sin\theta\right]$$

The x and y components of this will vanish when we integrate over azimuth because of the $\cos\phi d\phi$ and $\sin\phi d\phi$ factors, so that

$$\left\langle\frac{d\mathbf{L}_{rad}}{dt\,d\Omega}\right\rangle = \left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^2\mu_o c}\right)\sin^2\theta\,\hat{\mathbf{e}}_z$$

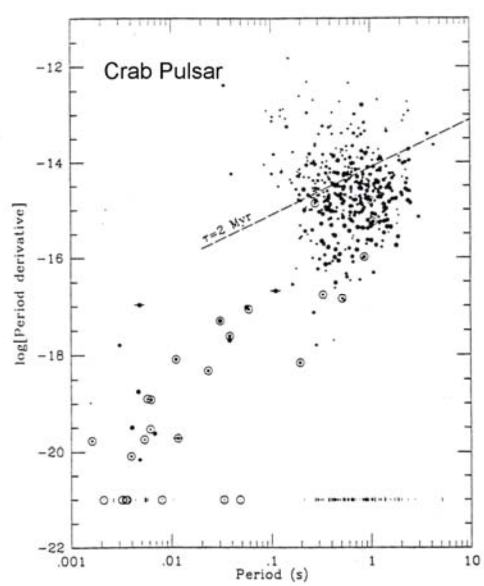
(f) Integrate over all solid angle to find $\left\langle\frac{d\mathbf{L}_{rad}}{dt}\right\rangle$, the total rate at which angular momentum is being emitted. How does this quantity compare to $\left\langle\frac{dW_{rad}}{dt}\right\rangle$ from (e). Is this what you expect?

Integrating above over solid angle, we have

$$\left\langle\frac{d\mathbf{L}_{rad}}{dt}\right\rangle = 2\pi\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^2\mu_o c}\right)\int_{-\pi}^{\pi} d\left(\cos\theta\right)\sin^2\theta\,\hat{\mathbf{e}}_z$$

$$= \hat{\mathbf{e}}_z\frac{8\pi}{3}\left(\frac{\mu_o\omega_o m_o}{4\pi}\right)^2\left(\frac{\omega_o}{c^3\mu_o}\right) = \frac{2\omega_o}{3}\left[\frac{\mu_o\omega_0^4 m_o^2}{4\pi c^3}\right] = \omega_o\left\langle\frac{dW_{rad}}{dt}\right\rangle$$

Problem 3-6: Magnetic Dipole Radiation from a Rotating Neutron Star (10 points)

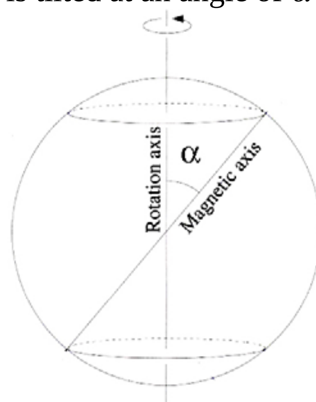


Pulsars are rotating neutron stars with a solar mass and a radius on the order of 10 km, and strong magnetic fields. Observationally one has both the period of the pulsar and the rate at which it is slowing down (all isolated pulsars slow down; some pulsars in binary systems are speeding up because of accretion of material from a companion). The period is usually denoted as *P* (seconds) and the slowdown rate as *dP/dt* (dimensionless). An estimate of the characteristic spin down time, is *P/[dP/dt]*, which is typically a few million years. The figure gives the observed slowdown rate *dP/dt* versus the period *P* for many pulsars.

Frequently the above graph data is re-plotted as in the figure on the last page, which shows the surface polar magnetic field strength of the pulsar versus its period. This plot is derived from the plot to the above, assuming that the observed slowdown is due to magnetic dipole radiation, and then estimating what field at the surface of the neutron star you need to get the observed slow down rate. In this problem you reproduce the calculation needed to go from the first plot to the second plot.

Assume that the magnetic dipole axis of the neutron star is tilted at an angle of α to its rotation axis, which is along the vertical axis, so that

$$\mathbf{m} = m_o \cos\alpha\,\hat{\mathbf{z}} + m_o \sin\alpha\left[\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right]$$



(a) What is the rate at which this spinning neutron star is emitting magnetic dipole radiation? Express this rate in terms of the period P of the rotation and the magnetic field strength B_p at the north magnetic pole of the neutron star, assuming that the field of the neutron star is perfectly dipolar outside of its radius *R*.

$$\frac{dW_{rad}}{dt} = \frac{\mu_o}{4\pi}\frac{2\left|\ddot{\mathbf{m}}\right|^2}{3c^3} \qquad \ddot{\mathbf{m}} = -\omega^2 m_o \sin\alpha\left[\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right] \qquad \left|\ddot{\mathbf{m}}\right|^2 = \omega^4 m_o^2 \sin^2\alpha$$

$$\frac{dW_{rad}}{dt} = \frac{\mu_o}{4\pi}\frac{2\omega^4 m_o^2 \sin^2\alpha}{3c^3} \qquad \mathbf{B}_{\theta=0} = \hat{\mathbf{e}}_r \frac{2m_o}{R^3} \Rightarrow m_o = \frac{1}{2}B_p R^3$$

$$\frac{dW_{rad}}{dt} = \frac{\mu_o}{4\pi}\frac{2\omega^4 \sin^2\alpha}{3c^3}\left[\frac{1}{2}B_p R^3\right]^2 = \frac{\mu_o}{8\pi}\frac{\omega^4 B_p^2 R^6 \sin^2\alpha}{3c^3}$$

(b) Assume that the mass of the neutron star is uniformly distributed through its volume, so that its moment of inertia is $\left[\frac{2}{5}MR^2\right]$ and its rotational energy is $\frac{1}{2}\left[\frac{2}{5}MR^2\right]\omega^2 = \frac{1}{2}\left[\frac{2}{5}MR^2\right]\left(\frac{2\pi}{P}\right)^2$. Equate the rate at which the neutron is losing rotational energy to the rate at which it is radiating magnetic dipole radiation. Find an equation that gives B_p in terms of the mass of the neutron star, its radius, the angle α, the observed period P and slowdown rate *dP/dt*.

$$\frac{d}{dt}\frac{1}{2}\left[\frac{2}{5}MR^2\right]\left(\frac{2\pi}{P}\right)^2 = -\left[\frac{2}{5}MR^2\right]\frac{(2\pi)^2}{P^3}\frac{dP}{dt} = -\frac{\mu_o}{8\pi}\frac{(2\pi)^4}{P^4}\frac{B_p^2 R^6 \sin^2\alpha}{3c^3}$$

$$B_p = \sqrt{\frac{12c^3 M}{5\pi\mu_o R^4 \sin^2\alpha} P \frac{dP}{dt}}$$

(c) Calculate B_p for the Crab Pulsar, with P = 33 ms and dP/dt = 4.17×10^{-13}, assuming it has a solar mass and the angle α is 90 degrees.

Putting in the appropriate units, we have

$$B_p = 5.7\times10^{19}\,gauss\sqrt{\frac{M}{M_\odot}}\left(\frac{10km}{R}\right)^2\frac{1}{\sin\alpha}\sqrt{P\frac{dP}{dt}} \quad \approx 7\times10^{12}\,gauss \text{ for the Crab}$$