**Solutions Assignment 10: Due Friday May 1, 2015 at 2:30 pm**

**Problems**

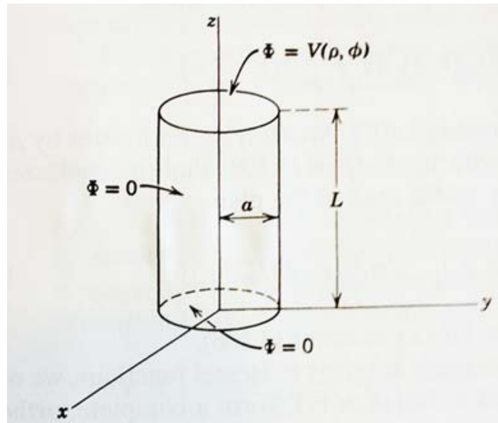
There are two problems in this problem set. The first is worth 20 points, the second is worth 40 points.

**Problem 10.1 : Two electrostatic potential problems in cylindrical coordinates.**

(a) A cylinder has zero potential on its sides and bottom (see figure) and on the top the potential is given by

$$V(\rho,\phi) = V_o J_o(x_{01}\frac{\rho}{a})$$

where $x_{01}$ is the first zero of $J_0$ . What is the potential everywhere inside the cylinder?




Jackson 3.105a page 117 gives the general form of the solution for this problem

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left[A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right]$$

In this case there is no $\phi$ dependence so we only have the $m = 0$ term. From 3.105b we have that for $m = 0$, $A_{mn}$ is zero and

$$B_{0n} = \frac{1}{2}\frac{2\text{cosech}(k_{0n}L)}{\pi a^2 J_1^2(k_{0n}a)}\int_0^{2\pi} d\phi \int_o^a d\rho\rho V(\rho,\phi) J_0(k_{0n}\rho)$$

Using our expression for $V(\rho,\phi)$ from above, we have (using (3.95))

$$B_{0n} = \frac{1}{2}\frac{2\text{cosech}(k_{0n}L)}{\pi a^2 J_1^2(k_{0n}a)}\int_0^{2\pi} d\phi \int_o^a d\rho\rho \left[V_o J_o(x_{01}\frac{\rho}{a})\right] J_0(k_{0n}\rho) = \pi \frac{2\text{cosech}(k_{0n}L)}{\pi a^2 J_1^2(k_{0n}a)} \delta_{0n} \frac{a^2 J_1^2(k_{0n}a)}{2}$$

$$B_{0n} = \frac{1}{2}\frac{2\text{cosech}(k_{0n}L)}{\pi a^2 J_1^2(k_{0n}a)}\int_0^{2\pi} d\phi \int_o^a d\rho\rho \left[V_o J_o(x_{01}\frac{\rho}{a})\right] J_0(k_{0n}\rho) = \text{cosech}(k_{0n}L)\delta_{0n}$$

so

$$\Phi(\rho,\phi,z) = V_o J_0(k_{01}\rho)\frac{\sinh(x_{01}z/a)}{\sinh(x_{01}L/a)}$$

(b) We have a cylindrical potential problem in the region $z \ge 0$ where the potential goes to zero as $z \to \infty$ and at $z = 0$ the potential is $V(\rho,\phi)$ where

$$V(\rho,\phi) = V_o J_o(\frac{\rho}{a})$$

What is the potential everywhere for $z \ge 0$ ?

Jackson equation just after 3.106 p 118 gives the general solution

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\int_0^{\infty} dk e^{-kz} J_m(k\rho)\left[A_m \sin(m\phi) + B_m \cos(m\phi)\right]$$

In this case there is no $\phi$ dependence so we only have the $m = 0$ term. From 3.109 we have that for $m = 0$, $A_m$ is zero and (using 3.108 p 118)

$$B_o = V_o k\int_0^{\infty} \rho d\rho J_o(k\rho) J_o\left(\frac{\rho}{a}\right) = V_o \delta\left(k - \frac{1}{a}\right)$$

So we have

$$\Phi(\rho,\phi,z) = V_o e^{-z/a} J_0(\frac{\rho}{a})$$

**Problem 10.2 : The spinning shell of charge**

*The problem below assumes the slow spin up of a shell of charge. We will be able to do this problem with arbitrarily fast spin up in a few more lectures, but I wanted you to work out the "quasi-static" solutions for comparison to that result.*

A spherical shell of radius $R$, carries a uniform surface charge $\sigma$. Its total charge $Q$ is $4\pi R^2 \sigma$, and its Coulomb electric field is

$$\mathbf{E}_{coulomb} = \begin{cases} 0 & r < R \\ \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} & r > R \end{cases}$$

We begin spinning the sphere at a angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\mathbf{K}(t) = \sigma\,\omega(t)\,R\sin\theta\,\hat{\boldsymbol{\phi}} = \kappa(t)\sin\theta\,\hat{\boldsymbol{\phi}}$$

where $\kappa(t) = \sigma\,\omega(t)\,R$. In this problem we assume that we can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (this will be good for variations in $\kappa(t)$ with time scales $T \approx \dfrac{\kappa}{d\kappa / dt} \gg \dfrac{R}{c}$) If we define $m(t) = \dfrac{4\pi\, R^3}{3}\kappa(t)$ and $B(t) = \dfrac{2\mu_o}{3}\kappa(t)$, then our quasi-static solution for **B** is

$$\mathbf{B}(\mathbf{r},t) = \begin{cases} \dfrac{\mu_o m(t)}{4\pi\, r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) & (r > R) \\ \hat{\mathbf{z}}\; B(t) & (r < R) \end{cases}$$

(a) Given **B** find the induction electric field everywhere in space. You may assume that $\mathbf{E}_{induction} = E_\phi \hat{\boldsymbol{\phi}}$. Then find $E_\phi$ using $\nabla\times\mathbf{E} = -\dfrac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}$. For $r > R$, calculate the ratio of $E_\phi$ to the radial Coulomb electric field given above. Is this small or large compared to one?

For $r < R$, take a circle of radius $r$ at an angle $\theta$ and apply Faraday's Law in integral form to that circle. This will give us $2\pi r\sin\theta E_\phi = -\pi\left(r\sin\theta\right)^2 \dfrac{dB}{dt}$ or $\mathbf{E} = -\hat{\boldsymbol{\phi}}\dfrac{r\sin\theta}{2}\dfrac{dB}{dt} = -\hat{\boldsymbol{\phi}}\dfrac{r\mu_o}{3}\dfrac{d\kappa}{dt}\sin\theta$ where $B(t) = \dfrac{2\mu_o}{3}\kappa(t)$. For r > R, if we assume $\mathbf{E}_{induction} = E_\phi\hat{\boldsymbol{\phi}}$, then $\nabla\times\mathbf{E} = \hat{\mathbf{r}}\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \hat{\boldsymbol{\theta}}\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rE_\phi\right)$. If we compare this expression to the expression for $\dfrac{\partial \mathbf{B}}{\partial t}$ that we get from taking the time derivative, we see that $\mathbf{E}_\phi = -\dfrac{\mu_o \sin\theta}{4\pi r^2}\dfrac{dm}{dt}$.

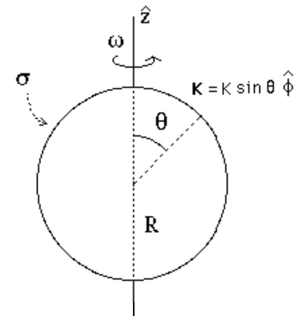
$$\mathbf{E} = \mathbf{E}_{coulomb} + \mathbf{E}_{induction} = \begin{cases} -\hat{\boldsymbol{\phi}}\dfrac{r\mu_o\sin\theta}{4\pi R^3}\dfrac{dm}{dt} & r < R \\ \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} - \hat{\boldsymbol{\phi}}\dfrac{\mu_o\sin\theta}{4\pi r^2}\dfrac{dm}{dt} & r > R \end{cases}$$

To compare the induction term for r > R to the Coulomb term, we take the ratio.

$$\frac{E_{induction}}{E_{coulomb}} = \frac{4\pi\varepsilon_o r^2}{Q}\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt} = \frac{\sin\theta}{Qc^2}\frac{d}{dt}\left[\frac{4\pi R^3}{3}\omega R\frac{Q}{4\pi R^2}\right]$$

$$\frac{E_{induction}}{E_{coulomb}} = \frac{\sin\theta R^2}{3c^2}\frac{d\omega}{dt} = \frac{\omega\sin\theta R^2}{3c^2}\frac{1}{\omega}\frac{d\omega}{dt} \approx \frac{\omega R^2}{3c^2 T} \approx \frac{1}{3}\left(\frac{\omega R}{c}\right)\left(\frac{R}{cT}\right)$$

Where to get the final form above we have used the time scale $T$ for changes in the angular velocity defined by $\frac{1}{T}=\frac{1}{\omega}\frac{d\omega}{dt}$. Our final result above shows that the ratio is the product of two terms, both of which we are assuming to be small, so the ratio is second order small in small quantities.




(b) Show that the magnetic energy outside of $r > R$ is given by $\frac{\mu_o m^2}{12\pi R^3}$. What is the magnetic energy for $r < R$? What is the total magnetic energy?

$$\int_{r>R}\left[\frac{B^2}{2\mu_o}\right]d^3x = 2\pi\int_{-1}^{1} d\left(\cos\theta\right)\int_R^\infty\left[\frac{B^2}{2\mu_o}\right]r^2dr$$

$$\int_{r>R}\left[\frac{B^2}{2\mu_o}\right]d^3x = 2\pi\int_{-1}^{1} d\left(\cos\theta\right)\int_R^\infty\left[\frac{1}{2\mu_o}\left(\frac{\mu_o m}{4\pi\, r^3}\right)^2\left(4\cos^2\theta+\sin^2\theta\right)\right]r^2dr$$

$$\int_{r>R}\left[\frac{B^2}{2\mu_o}\right]d^3x = 2\pi\int_{-1}^{1}\left(3\cos^2\theta+1\right)d\left(\cos\theta\right)\int_R^\infty\left[\frac{\mu_o m^2}{32\pi^2}\frac{1}{r^4}\right]dr$$

$$\int_{r>R}\left[\frac{B^2}{2\mu_o}\right]d^3x = \frac{\mu_o m^2}{12\pi R^3}$$

$$\int_{r<R}\left[\frac{B^2}{2\mu_o}\right]d^3x = \frac{B^2}{2\mu_o}\frac{4\pi R^3}{3} = \frac{2\pi R^3}{3\mu_o}\left[\frac{2\mu_o}{3}\kappa(t)\right]^2 = \frac{\mu_o m^2}{6\pi R^3}$$

So the magnetic energy inside the sphere is twice that outside the sphere.

(b) Show that the total rate at which electromagnetic energy is being created as the sphere is being spun up, $\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, is equal to the rate at which the total magnetic energy is increasing. If you are spinning up the sphere, it is you who are creating this energy by the additional work you must do to offset the force associated with the induction electric field.

$$\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x = \int_{\text{all space}} -\left[\hat{\boldsymbol{\phi}}\kappa(t)\sin\theta\,\delta(r-R)\right]\cdot\left[-\hat{\boldsymbol{\phi}}\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt}\right]d^3x$$

$$\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x = \frac{\mu_o\kappa(t)}{2}\frac{dm}{dt}\int_{-1}^{1}\sin^2\theta\, d\left(\cos\theta\right) = \frac{2\mu_o\kappa(t)}{3}\frac{dm}{dt} = \frac{\mu_o m}{2\pi\, R^3}\frac{dm}{dt} = \frac{d}{dt}\left[\frac{\mu_o m^2}{4\pi\, R^3}\right]$$

The total energy we have at time $t$ in the magnetic field is given by the sum of the two terms we calculated above, or $\frac{\mu_o m^2}{6\pi R^3}+\frac{\mu_o m^2}{12\pi R^3}$, which is $\frac{\mu_o m^2}{4\pi R^3}$, so we are doing work at just the rate needed to increase the total magnetic energy density we have at a given time.

(d) Using the Poynting vector, calculate the flux of electromagnetic energy $\int\limits_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da$ through a spherical surface of radius $r$ for $r$ a little greater than $R$ and also for $r$ a little smaller than $R$. Do your results agree with what you expect from (b) and (c)?

First let's do this for $r$ a little greater than $R$. There we have

$$\mathbf{E}\times\mathbf{B}=\left[\frac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}}-\hat{\boldsymbol{\phi}}\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt}\right]\times\left[\frac{\mu_o m(t)}{4\pi\, r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)\right]$$

$$\mathbf{E}\times\mathbf{B}=\frac{\mu_o m(t)}{4\pi\, r^3}\left[\frac{Q}{4\pi\varepsilon_o r^2}\sin\theta\left(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\right)-\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt}2\cos\theta\left(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{r}}\right)-\frac{\mu_o\sin^2\theta}{4\pi r^2}\frac{dm}{dt}\left(\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\theta}}\right)\right]$$

$$\mathbf{E}\times\mathbf{B}=\frac{\mu_o m(t)}{4\pi\, r^3}\left[\frac{Q}{4\pi\varepsilon_o r^2}\sin\theta\,\hat{\boldsymbol{\phi}}-\frac{\mu_o 2\sin\theta\cos\theta}{4\pi r^2}\frac{dm}{dt}\hat{\boldsymbol{\theta}}+\frac{\mu_o\sin^2\theta}{4\pi r^2}\frac{dm}{dt}\hat{\mathbf{r}}\right]$$

So

$$\int\limits_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da=\int\limits_{surface}\left[\frac{1}{\mu_o}\frac{\mu_o m(t)}{4\pi\, R^3}\frac{\mu_o\sin^2\theta}{4\pi R^2}\frac{dm}{dt}\right]da=\frac{\mu_o m(t)}{6\pi\, R^3}\frac{dm}{dt}=\frac{d}{dt}\left[\frac{\mu_o m^2(t)}{12\pi\, R^3}\right]$$

And this is exactly the energy flow outward we need to increase the magnetic energy outside the sphere at a given time.

Now let's consider the energy flow across a sphere of radius $r < R$. There we have

$$\mathbf{E}\times\mathbf{B}=\left[-\hat{\boldsymbol{\phi}}\frac{r\mu_o\sin\theta}{3}\frac{d\kappa}{dt}\right]\times\left[\hat{\mathbf{z}}\;B(t)\right]=-\frac{r\mu_o}{3}\frac{d\kappa}{dt}\sin\theta B(t)\left(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}\right)$$

$$\mathbf{E}\times\mathbf{B}=-\frac{r\left(\mu_o\right)^2}{9}\frac{d\kappa^2}{dt}\sin\theta\left(\hat{\boldsymbol{\phi}}\times\left(\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right)\right)=-\frac{d}{dt}\frac{r\left(\mu_o\right)^2 m^2}{\left(4\pi R^3\right)^2}\sin\theta\left(\hat{\boldsymbol{\theta}}\cos\theta+\hat{\mathbf{r}}\sin\theta\right)$$

$$\text{So}\int\limits_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da=-2\pi\frac{\mu_o}{\left(4\pi\right)^2R^3}\frac{dm^2}{dt}\int_{-1}^{1}d(\cos\theta)\sin^2\theta=-\frac{d}{dt}\frac{\mu_o m^2}{6\pi R^3}$$

The minus sign in this equation means the energy flow is inward, and it is exactly the amount we need to account for the rate at which magnetic energy is building up in the interior.

So all of this makes sense, we are creating energy at the shell where we are doing work, and it is flowing out from where we create it at exactly the rate that we need for the build up of magnetic energy inside and outside of the sphere.

(e) Compute the total electromagnetic angular momentum of this spinning charge configuration, ***ignoring*** $\mathbf{E}_{induction}$, that is compute

$$\int\limits_{\text{all space}} \mathbf{r}\times\left[\varepsilon_o \mathbf{E}_{coulomb}\times\mathbf{B}\right] d^3x$$

We all ready have the expressions for $\mathbf{E}\times\mathbf{B}$ above, so

$$\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \begin{cases} \mathbf{r}\times\left[-\dfrac{d}{dt}\dfrac{r(\mu_o)^2 m^2}{(4\pi R^3)^2}\sin\theta\left(\hat{\boldsymbol{\theta}}\cos\theta+\hat{\mathbf{r}}\sin\theta\right)\right] & r<R \\ \dfrac{\mu_o m(t)}{4\pi\, r^3}\mathbf{r}\times\left[\dfrac{Q}{4\pi\varepsilon_o r^2}\sin\theta\,\hat{\boldsymbol{\phi}} - \dfrac{\mu_o 2\sin\theta\cos\theta}{4\pi r^2}\dfrac{dm}{dt}\hat{\boldsymbol{\theta}} + \dfrac{\mu_o\sin^2\theta}{4\pi r^2}\dfrac{dm}{dt}\hat{\mathbf{r}}\right] & r>R \end{cases}$$

$$\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \begin{cases} \left[-\dfrac{d}{dt}\dfrac{r^2(\mu_o)^2 m^2}{(4\pi R^3)^2}\sin\theta\left(\hat{\boldsymbol{\phi}}\cos\theta\right)\right] & r<R \\ -\dfrac{\mu_o m(t)}{4\pi\, r^3}\left[\dfrac{Q}{4\pi\varepsilon_o r}\sin\theta\,\hat{\boldsymbol{\theta}} - \dfrac{\mu_o 2\sin\theta\cos\theta}{4\pi r}\dfrac{dm}{dt}\hat{\boldsymbol{\phi}}\right] & r>R \end{cases}$$

We were instructed to ignore the *dm/dt* terms, and these terms (a) integrate to zero because of the $\hat{\boldsymbol{\phi}}$ dependence and (b) are small compared to the terms we keep in any case. So we have

$$\mathbf{r}\times(\varepsilon_o\mathbf{E}\times\mathbf{B}) = \begin{cases} 0 & r<R \\ -\dfrac{m(t)}{4\pi\, r^3 c^2}\dfrac{Q}{4\pi\varepsilon_o r}\sin\theta\,\hat{\boldsymbol{\theta}} = & r>R \end{cases}$$

Using $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}\sin\theta + \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi$, and realizing in advance that the x and y components will average to zero because of the $\cos\phi$ and $\sin\phi$ terms, we have

$$\mathbf{r}\times(\varepsilon_o\mathbf{E}\times\mathbf{B}) = \begin{cases} 0 & r<R \\ \dfrac{m(t)}{4\pi\, r^3 c^2}\dfrac{Q\sin^2\theta}{4\pi\varepsilon_o r}\hat{\mathbf{z}} & r>R \end{cases}$$

And the total electromagnetic angular momentum is

$$\int \mathbf{r}\times\left(\varepsilon_o\mathbf{E}\times\mathbf{B}\right)d^3x=\hat{\mathbf{z}}\frac{m(t)Q}{\left(8\pi\varepsilon_o\right)c^2}\int_R^\infty\frac{dr}{r^2}\int_{-1}^{1}d(\cos\theta)\sin^2\theta=\hat{\mathbf{z}}\frac{mQ}{\left(6\pi\varepsilon_o\right)R\,c^2}=\hat{\mathbf{z}}\frac{4\pi R^3\sigma\omega RQ}{3\left(6\pi\varepsilon_o\right)R\,c^2}$$

$$\int \mathbf{r}\times\left(\varepsilon_o\mathbf{E}\times\mathbf{B}\right)d^3x=\hat{\mathbf{z}}\frac{\omega RQ^2}{18\pi\varepsilon_o\,c^2}$$

(f) Show that the total rate at which electromagnetic angular momentum is being created as the sphere is being spun up, $\int\limits_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x$, is equal to the rate at which the total electromagnetic angular momentum is increasing. Here you cannot ignore the induction electric field. If you are spinning up the sphere, it is you who are creating this angular momentum by the additional torque you must impose to overcome the induction electric field.

$$\int\limits_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x=\int\limits_{surface}-\mathbf{r}\times\sigma\mathbf{E}\,da=\int\limits_{surface}\left(\mathbf{r}\times\hat{\boldsymbol{\phi}}\right)\sigma\left(\frac{\mu_o\sin\theta}{4\pi R^2}\frac{dm}{dt}\right)da$$

$$=-\int\limits_{surface}\hat{\boldsymbol{\theta}}\left(\frac{\sigma\mu_o\sin\theta}{4\pi R}\frac{dm}{dt}\right)da=\hat{\mathbf{z}}\left(\frac{\sigma\mu_o}{4\pi R}\frac{dm}{dt}\right)\int\limits_{surface}\sin^2\theta\,da=\hat{\mathbf{z}}\left(\frac{\sigma\mu_o}{4\pi R}\frac{dm}{dt}\right)2\pi R^2\frac{4}{3}$$

$$\hat{\mathbf{z}}\frac{d}{dt}\left(\frac{2\sigma\mu_o R\left(4\pi R^3\kappa\right)}{9}\right)=\hat{\mathbf{z}}\frac{d}{dt}\left(\frac{2\sigma\mu_o 4\pi R^4\omega R\sigma}{9}\right)=\hat{\mathbf{z}}\frac{d}{dt}\left(\left(\frac{Q}{4\pi R^2}\right)^2\frac{8\mu_o\pi R^5\omega}{9}\right)=\hat{\mathbf{z}}\frac{d}{dt}\left(\frac{Q^2\omega R}{18\pi\varepsilon_o c^2}\right)$$

In the second we have ignored a term that looks like $\hat{\mathbf{r}}\sin\theta\cos\theta$ because it will integrate to zero when we do the theta integration, and in the last line we have used the definitions of $m$ and $\kappa$. This is the rate we want to see, in that it agrees with our answer in (e) when integrated over time.

(g). Calculate the flux of electromagnetic angular momentum, $\int\limits_{surface}\left[-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right]\cdot\hat{\mathbf{r}}\,da$ through a sphere of radius $r$ for $r$ a little greater than $R$ and for $r$ a little smaller than $R$. Do your results agree with what you expect from (e) and (f)? As in all stress tensor calculations, figure out what components you are going to need before you calculate anything, and then just calculate those.

$$\int\limits_{surface}\left[-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right]\cdot\hat{\mathbf{r}}\,da=\int\limits_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da=\int\limits_{surface}\left[-\mathbf{r}\times\left(T_{rr}\hat{\mathbf{r}}+T_{r\theta}\hat{\boldsymbol{\theta}}+T_{r\phi}\hat{\boldsymbol{\phi}}\right)\right]da$$

$$\int_{surface} \left[ -\mathbf{r} \times T_{r\phi} \hat{\boldsymbol{\phi}} \right] da = \int_{surface} R T_{r\phi} \hat{\boldsymbol{\theta}} \, da = -\varepsilon_o \int_{surface} \hat{\boldsymbol{\theta}} R \frac{Q}{4\pi\varepsilon_o R^2} \frac{\mu_o \sin\theta}{4\pi R^2} \frac{dm}{dt} da =$$

$$-\varepsilon_o \frac{d}{dt} \int_{surface} \hat{\boldsymbol{\theta}} R \frac{Q}{4\pi\varepsilon_o R^2} \frac{\mu_o \sin\theta}{4\pi R^2} \frac{4\pi R^3 \sigma\omega R}{3} da$$

$$= +\hat{\mathbf{z}} \frac{d}{dt} \frac{Q^2 \omega R}{18\pi c^2 \varepsilon_o}$$