Solutions Assignment #9: Due Friday April 24, 2015 at 2:30 pm

Problems

There are only three problems in this problem set. Each one is worth 20 points.

Problem 9-1: Jackson 8.14 page 402.

Note: There is no need to do the integral from scratch that you need do to get to the equation in part (a). It is sufficient to show that the equation given satisfies the appropriate differential equation and the boundary conditions.

(a) We are given a graded index of refraction $n(x) = n(0)\operatorname{sech}(\alpha x)$. We first find the differential satisfied by the equation we are given ($\sinh\alpha x = \sinh(\alpha x_{\max})\sin(\alpha z)$), and show that it satisfies eq. (8.117) of Jackson. Taking the derivatives of both sides of the equation given yields

$$\alpha\cosh\alpha x\frac{dx}{dz} = \alpha\sinh(\alpha x_{\max})\cos(\alpha z) \Rightarrow \frac{dz}{dx} = \frac{\cosh\alpha x}{\sinh(\alpha x_{\max})\cos(\alpha z)}$$

But $\cos\alpha z = \sqrt{1-\sin^2(\alpha z)} = \sqrt{1-\frac{\sinh^2\alpha x}{\sinh^2(\alpha x_{\max})}} \Rightarrow \frac{dz}{dx} = \frac{\cosh\alpha x}{\sqrt{\sinh^2(\alpha x_{\max})-\sinh^2\alpha x}}$

But $\cosh^2\alpha x - \sinh^2\alpha x = 1$ so

$$\sinh^2(\alpha x_{\max}) - \sinh^2\alpha x = \cosh^2(\alpha x_{\max}) - \cosh^2\alpha x .$$

Thus we have

$$\frac{dz}{dx} = \frac{\cosh\alpha x}{\sqrt{\cosh^2(\alpha x_{\max})-\cosh^2\alpha x}} = \frac{1}{\sqrt{\frac{\cosh^2(\alpha x_{\max})}{\cosh^2\alpha x}-1}}$$

We also have $\operatorname{sech}(\alpha x_{\max}) = \bar{n}/n(0)$ and $\operatorname{sech}(\alpha x) = n/n(0)$, so $\frac{\operatorname{sech}(\alpha x)}{\operatorname{sech}(\alpha x_{\max})} = n/\bar{n}$,

Thus we find that, as desired

$$\frac{dz}{dx} = \frac{1}{\sqrt{\frac{n^2}{\bar{n}^2}-1}} = \frac{\bar{n}}{\sqrt{n^2-\bar{n}^2}}$$

To plot $\sinh\alpha x = \sinh(\alpha x_{\max})\sin(\alpha z)$ the easiest thing is to plot

$$\alpha z = \arcsin \frac{\sinh \alpha x}{\sinh(\alpha x_{\max})}$$

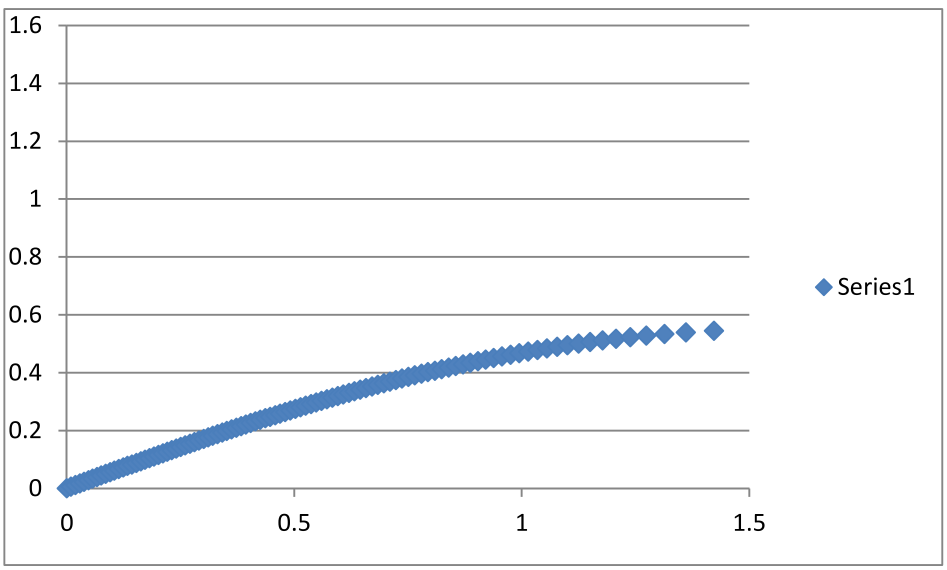
where $\sinh(\alpha x_{\max})$ is determined from

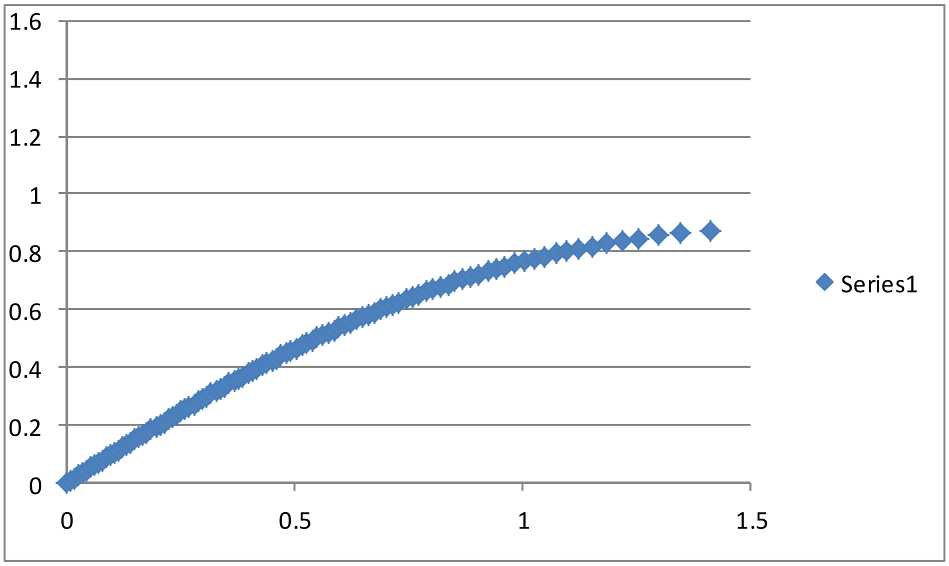
$$\bar{n} = n(x_{\max}) = n(0)\operatorname{sech}(\alpha x_{\max}) = n(0)\cos(0) \Rightarrow \operatorname{sech}(\alpha x_{\max}) = \cos(0)$$

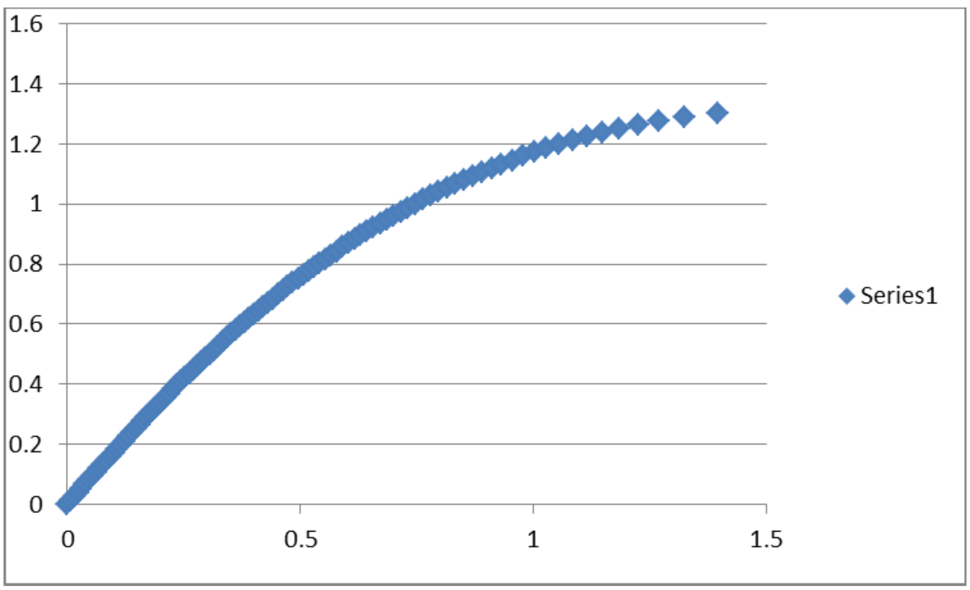
And therefore that $\sinh(\alpha x_{\max}) = \sqrt{\cosh^2(\alpha x_{\max}) - 1} = \sqrt{\frac{1}{\cos^2\theta(0)} - 1} = \tan\theta(0)$. Thus we want to plot

$$\alpha z = \arcsin \frac{\sinh \alpha x}{\tan\theta(0)}$$

In the three plots below we plot a quarter period of the curve with αz horizontal and αx vertical using an excel spreadsheet, for three different values of the initial angle: 30 degrees, 45 degrees, and 60 degrees, respectively. The half period is the obvious extension of the curves below back to the z-axis.







(b) The half period of the wave is given by

$$\alpha z_{half-period} = 2\arcsin\frac{\sinh(\alpha x_{\max})}{\sinh(\alpha x_{\max})} = \pi \Rightarrow z_{half-period} = \frac{2}{\alpha}$$

This expression is independent of $\bar{n}$.

(c) From Jackson (8.119) page 383, we have

$$L_{opt} = 2\int_0^{x\max}\frac{n^2 dx}{\sqrt{n^2-\bar{n}^2}} = 2\int_0^{x\max}\frac{n(0)\text{sech}^2(\alpha x)dx}{\sqrt{\text{sech}^2(\alpha x)\text{-sech}^2(\alpha x_{\max})}}$$

Following the hint, we let $\sinh(\alpha x) = \sinh(\alpha x_{\max})\sin t$, so that

$$\alpha \cosh(\alpha x)dx = \sinh(\alpha x_{\max})\cos t \; dt$$

And

$$L_{opt} = 2\int_0^{\pi/2} \frac{n(0)\mathrm{sech}^2(\alpha x)\sinh(\alpha x_{\max})\cos t \; dt}{\alpha\sqrt{1\text{-}\cosh^2(\alpha x)\mathrm{sech}^2(\alpha x_{\max})}} = 2\int_0^{\pi/2} \frac{n(0)\mathrm{sech}^2(\alpha x)\sinh(\alpha x_{\max})\cosh(\alpha x_{\max})\cos t \; dt}{\alpha\sqrt{\cosh^2(\alpha x_{\max})\text{-}\cosh^2(\alpha x)}}$$

$$L_{opt} = 2n(0)\int_0^{\pi/2} \frac{\mathrm{sech}^2(\alpha x)\sinh(\alpha x_{\max})\cosh(\alpha x_{\max})\cos t \; dt}{\alpha\sqrt{\sinh^2(\alpha x_{\max})\text{-}\sinh^2(\alpha x)}} = 2n(0)\int_0^{\pi/2} \frac{\mathrm{sech}^2(\alpha x)\cosh(\alpha x_{\max})\cos t \; dt}{\alpha\sqrt{1\text{-}\sin^2 t}} = \frac{2n(0)}{\alpha}\int_0^{\pi/2} \mathrm{sech}^2(\alpha x)\cosh(\alpha x_{\max})dt$$

$$\mathrm{sech}^2(\alpha x)\cosh(\alpha x_{\max}) = \frac{\cosh(\alpha x_{\max})}{\cosh^2(\alpha x)} = \frac{\cosh(\alpha x_{\max})}{1+\sinh^2(\alpha x)} = \frac{\cosh(\alpha x_{\max})}{1+\sinh^2(\alpha x_{\max})\sin^2 t}$$

so

$$L_{opt} = \frac{2n(0)\cosh(\alpha x_{\max})}{\alpha}\int_0^{\pi/2} \frac{dt}{1+\sinh^2(\alpha x_{\max})\sin^2 t} = \frac{2n(0)\pi\cosh(\alpha x_{\max})}{\alpha 2\sqrt{1+\sinh^2(\alpha x_{\max})}} = \frac{n(0)\pi}{\alpha} = n(0)z_{half-period}$$

where we have used the definite integral given in the problem statement and the results from part (b) above. This is the result we were looking for.

Problem 9-2: Two problems with azimuthal symmetry

(a) Consider a charge distribution given by

$$\rho(\mathbf{X}) = \sigma_o \delta(r-R)\cos\theta$$

Since this charge distribution is azimuthal symmetric, the electrostatic potential can be written in the form given in equation (3.33) of Jackson p 101. Determine the coefficients A_l and B_l for this distribution of charge, using the appropriate boundary conditions on **E.**

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l / r^{l+1}\right]P_l(\cos\theta)$$

To prevent the potential from diverging at 0 or at infinity, we must choose

$$\Phi(r,\theta)=\begin{cases}\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) & r<R\\ \sum_{l=0}^{\infty}B_l / r^{l+1}P_l(\cos\theta) & r>R\end{cases}$$

So we have

$$E_r(r,\theta)=\begin{cases}-\sum_{l=0}^{\infty}A_l l r^{l-1}P_l(\cos\theta) & r<R\\ +\sum_{l=0}^{\infty}B_l(l+1)/r^{l+2}P_l(\cos\theta) & r>R\end{cases}$$

$$E_\theta(r,\theta)=\begin{cases}-\sum_{l=0}^{\infty}A_l r^{l-1}\frac{d}{d\theta}P_l(\cos\theta) & r<R\\ -\sum_{l=0}^{\infty}B_l / r^{l+2}\frac{d}{d\theta}P_l(\cos\theta) & r>R\end{cases}$$

The radial component of the electric field must have a jump given by the charge density,

$$E_{r=R+}(R,\theta)-E_{r=R-}(R,\theta)=\sum_{l=0}^{\infty}\left[B_l(l+1)/R^{l+2}+A_l l R^{l-1}\right]P_l(\cos\theta)=\sigma_o\cos\theta/\varepsilon_o$$

Since $\cos\theta\propto P_1$, we must have all the A_l and B_l equal to zero except for $l=1$ and also

$$2B_1/R^3+A_1=\sigma_o/\varepsilon_o$$

The requirement that the tangential component of the electric field be continuous across $r=R$ requires $A_1=B_1/R^3$, so that we have $B_1=R^3\sigma_o/3\varepsilon_o$ and $A_1=\sigma_o/3\varepsilon_o$, so we have

$$\Phi(r,\theta)=\begin{cases}\sigma_o r\cos\theta/3\varepsilon_o & r<R\\ \left(\sigma_o R^3/3\varepsilon_o r^2\right)\cos\theta & r>R\end{cases}$$

This potential leads to a constant electric field $-\sigma_o/3\varepsilon_o\hat{\mathbf{z}}$ inside the sphere and an electric dipole field with dipole moment $\mathbf{p}=\dfrac{4\pi R^3\sigma_o}{3}\hat{\mathbf{z}}$ outside the sphere.

(b) Consider a current distribution given by

$$\mathbf{J}(\mathbf{X})=\kappa_o\delta(r-R)\sin\theta\,\hat{\phi}$$

Since $\nabla\times\mathbf{B}=0$ almost everywhere (except at $r=R$), we can in this situation write the magnetic field as the negative gradient of a scalar function $\Phi_{magnetic}(r,\theta)$, where equation (3.33) holds for this scalar function as well. Determine the coefficients A_l and

B_l for $\Phi_{magnetic}\left(r,\theta\right)$ for this distribution of current using the appropriate boundary conditions on **B**.

To prevent the potential from diverging at 0 or at infinity, we must choose

$$\Phi_{magnetic}\left(r,\theta\right)=\begin{cases}\sum_{l=0}^{\infty}A_l r^l P_l\left(\cos\theta\right) & r<R\\ \sum_{l=0}^{\infty}B_l / r^{l+1}P_l\left(\cos\theta\right) & r>R\end{cases}$$

So we have

$$B_r\left(r,\theta\right)=\begin{cases}-\sum_{l=0}^{\infty}A_l l r^{l-1}P_l\left(\cos\theta\right) & r<R\\ +\sum_{l=0}^{\infty}B_l\left(l+1\right)/ r^{l+2}P_l\left(\cos\theta\right) & r>R\end{cases}$$

$$B_\theta\left(r,\theta\right)=\begin{cases}-\sum_{l=0}^{\infty}A_l r^{l-1}\frac{d}{d\theta}P_l\left(\cos\theta\right) & r<R\\ -\sum_{l=0}^{\infty}B_l / r^{l+2}\frac{d}{d\theta}P_l\left(\cos\theta\right) & r>R\end{cases} \quad n$$

The tangential component of the magnetic field must have a jump given by the surface current,

$$B_{\theta,r=R+}\left(R,\theta\right)-B_{\theta,r=R-}\left(R,\theta\right)=\sum_{l=0}^{\infty}\left[-B_l / R^{l+2}+A_l R^{l-1}\right]\frac{d}{d\theta}P_l\left(\cos\theta\right)=\mu_o\kappa_o\sin\theta$$

Since $\sin\theta=-dP_1/d\theta$, we must have all the A_l and B_l equal to zero except for $l=1$ and also

$$+B_1/R^3-A_1=\mu_o\kappa_o$$

The requirement that the normal component of the magnetic field be continuous across $r=R$ requires $A_1=-2B_1/R^3$, so that we have $B_1=+R^3\kappa_o/3\varepsilon_o$ and $A_1=-2\mu_o\kappa_o/3$, so we have

$$\Phi\left(r,\theta\right)=\begin{cases}-2\mu_o\kappa_o r\cos\theta/3\varepsilon_o & r<R\\ \frac{\mu_o\kappa_o R^3}{3r^2}\cos\theta & r>R\end{cases}$$

This potential leads to a constant electric field $+\left(2\mu_o\kappa_o/3\right)\hat{\mathbf{z}}$ inside the sphere and a magnetic dipole field with dipole moment $\mathbf{m}=\frac{4\pi R^3\kappa_o}{3}\hat{\mathbf{z}}$ outside the sphere.

Problem 9-3: Jackson 3.3 page 136 part (a) ONLY

We first find an expression for the potential on the axis of a ring of charge of radius ρ whose center is at the origin. The potential at $\mathbf{X} = r\,\hat{\mathbf{z}}$ on the axis of this ring of charge with charge per unit length λ is given by

$$\Phi_{\text{on axis}} = \frac{1}{4\pi\varepsilon_o\sqrt{r^2+\rho^2}}\oint \lambda dl = \frac{1}{2\varepsilon_o}\frac{\lambda\rho}{\sqrt{r^2+\rho^2}}$$

We now use this expression for a ring to get the potential on the axis of our conducting disk. We divide up our disk into a number of ring segments of width $d\rho$ and charge per unit length $\lambda = \sigma d\rho$. We are told that the surface charge density σ on the disk goes as $\sigma = \sigma_o / \sqrt{1-\left(\frac{\rho}{R}\right)^2}$ where σ_o is a quantity we will determine below. So we have

$$\lambda = \sigma d\rho = \sigma_o d\rho / \sqrt{1-\left(\frac{\rho}{R}\right)^2} = \frac{\sigma_o R d\rho}{\sqrt{R^2-\rho^2}}$$

$$d\Phi_{\text{on axis conducting disk}} = \frac{\rho}{2\varepsilon_o\sqrt{r^2+\rho^2}}\frac{\sigma_o R d\rho}{\sqrt{R^2-\rho^2}}$$

Summing up the potential due to all those rings gives us

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\int_0^R \frac{\rho d\rho}{\sqrt{r^2+\rho^2}}\frac{1}{\sqrt{R^2-\rho^2}}$$

From Jackson equation (3.38) page 102, we have that $\frac{1}{|\mathbf{X}-\mathbf{X}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$. For $\mathbf{X} = r\,\hat{\mathbf{z}}$ and $\mathbf{X}' = \rho\,\hat{\boldsymbol{\rho}}$ we have $\cos\gamma = 0$ and if $|\mathbf{X}| > |\mathbf{X}'|$, this equation comes

$$\frac{1}{|\mathbf{X}-\mathbf{X}'|} = \sum_{l=0}^{\infty}\frac{\rho^l}{r^{l+1}}P_l(0) = \frac{1}{\sqrt{r^2+\rho^2}}$$

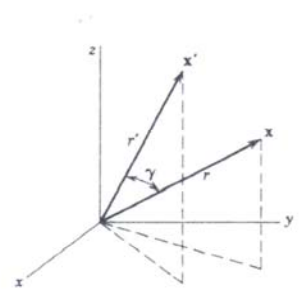


Figure 3.3

Using this expansion for $\frac{1}{\sqrt{r^2+\rho^2}}$ we have

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\int_0^R \frac{\rho d\rho}{\sqrt{R^2-\rho^2}}\sum_{l=0}^{\infty}\frac{\rho^l}{r^{l+1}}P_l(0)$$

Since $P_l(0) = 0$ for odd l , we can rewrite out sum as

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\int_0^R \frac{\rho d\rho}{\sqrt{R^2-\rho^2}}\sum_{l=0}^{\infty}\frac{\rho^{2l}}{r^{2l+1}}P_{2l}\left(0\right)$$

And thus we have

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\sum_{l=0}^{\infty}\frac{P_{2l}\left(0\right)}{r^{2l+1}}\int_0^R \frac{\rho^{2l+1}d\rho}{\sqrt{R^2-\rho^2}}$$

Below we use the three expressions given in the hints, which are

$$\int_0^{\pi/2}\sin^{2l+1}x\ dx = \frac{2*4*6*....*2l}{1*3*5*...*(2l+1)} \quad l=1,2,3... \qquad \int_0^{\pi/2}\sin x\ dx = 1$$

$$P_{2l}(0) = \left(-1\right)^l \frac{1*3*5*...*(2l-1)}{2*4*6*....*2l} \quad l=1,2,3... \qquad P_{2l}(0)=1 \quad l=0$$

$$1+\sum_{l=1}^{\infty}\frac{1*3*5*...*(2l-1)}{2*4*6*....*2l*\left(2l+1\right)} = \frac{\pi}{2}$$

To evaluate the integral, let $\rho = R\sin\theta$ so that $d\rho = R\cos\theta d\theta$ and $\sqrt{R^2-\rho^2} = R\cos\theta$, so that the integral becomes

$$\int_0^R \frac{\rho^{2l+1}d\rho}{\sqrt{R^2-\rho^2}} = R^{2l+1}\int_0^{\pi/2}\sin^{2l+1}\theta d\theta = \begin{cases} R \quad l=0 \\ R^{2l+1}\dfrac{2*4*6*....*2l}{1*3*5*...*(2l+1)} \quad l=1,2,3.. \end{cases}$$

so

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\sum_{l=0}^{\infty}\frac{P_{2l}\left(0\right)}{r^{2l+1}}\int_0^R \frac{\rho^{2l+1}d\rho}{\sqrt{R^2-\rho^2}} = \frac{\sigma_o R}{2\varepsilon_o}\left\{\frac{R}{r}+\sum_{l=1}^{\infty}P_{2l}\left(0\right)\frac{R^{2l+1}}{r^{2l+1}}\frac{2*4*6*....*2l}{1*3*5*...*(2l+1)}\right\}$$

But using the identity above involving $P_{2l}\left(0\right)$, we have

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\left\{\frac{R}{r}+\sum_{l=1}^{\infty}\left(-1\right)^l\frac{1*3*5*...*(2l-1)}{2*4*6*....*2l}\frac{R^{2l+1}}{r^{2l+1}}\frac{2*4*6*....*2l}{1*3*5*...*(2l+1)}\right\}$$

and this reduces to

$$\Phi_{\text{on axis conducting disk}} = \frac{\sigma_o R}{2\varepsilon_o}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{2l+1}\frac{R^{2l+1}}{r^{2l+1}}$$

This is the potential on the axis of the disk. Taking advantage of the symmetry and the consideration of Jackson Section 3.3 starting on page 101, we see that in general we have

$$\Phi\left(r,\theta\right) = \frac{\sigma_o R}{2\varepsilon_o}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{2l+1}\frac{R^{2l+1}}{r^{2l+1}}P_{2l}\left(\cos\theta\right)$$

We now want to evaluate σ_o in terms of the potential of the conducting disk *V*. If we evaluate the above potential at $r = R$ and $\theta = \pi/2$, we have

$$\Phi\left(r = R, \theta = \pi/2\right) = \frac{\sigma_o R}{2\varepsilon_o} \sum_{l=0}^{\infty} \frac{\left(-1\right)^l}{2l+1} P_{2l}\left(0\right) = \frac{\sigma_o R}{2\varepsilon_o} \left\{ 1 + \sum_{l=1}^{\infty} \frac{\left(-1\right)^l}{2l+1} P_{2l}\left(0\right) \right\}$$

$$= \frac{\sigma_o R}{2\varepsilon_o} \left\{ 1 + \sum_{l=1}^{\infty} \frac{1*3*5*...*(2l-1)}{2*4*6*....*2l*(2l+1)} \right\} = \frac{\sigma_o R}{2\varepsilon_o} \frac{\pi}{2} = V$$

So our final expression for the potential is, as we desired,

$$\Phi\left(r, \theta\right) = \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{\left(-1\right)^l}{2l+1} \frac{R^{2l}}{r^{2l}} P_{2l}\left(\cos\theta\right)$$