Solutions Assignment #7: Due Friday April 10, 2015 at 2:30 pm

Problems

There are only two problems in this problem set. Each one is worth 30 points.

Problem 7-1: E&M Wave Packet: Where is the Angular Momentum?

(a) A circularly polarized plane wave moving in the *z*-direction has a finite extent in the *x*- and *y*-directions. Assuming that the amplitude modulation is slowly varying (the wave is many wavelengths broad), show that the electric and magnetic fields are given approximately by

$$\mathbf{E}(x,y,z,t) \simeq \left[E_o(x,y)(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) + \frac{i}{k}\left(\frac{\partial E_o}{\partial x} \pm i\frac{\partial E_o}{\partial y} \right)\hat{\mathbf{z}} \right] e^{ikz - i\omega t}$$

$$\mathbf{B} \simeq \mp i\sqrt{\mu\varepsilon}\,\mathbf{E}$$

We must have divergence of the electric field zero. That means that if the electric field is of the form

$$\mathbf{E}(x,y,z,t) \simeq \left[E_o(x,y)(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) + \Theta\left(x,y\right)\hat{\mathbf{z}} \right] e^{ikz - i\omega t}$$

then we must have that

$$\left[\frac{\partial}{\partial x} E_o(x,y) e^{ikz - i\omega t} \pm i\frac{\partial}{\partial y} E_o(x,y) e^{ikz - i\omega t} + ik\Theta\left(x,y\right) e^{ikz - i\omega t} \right] = 0$$

Solving this for $\Theta\left(x,y\right)$ gives us the form we were to prove. To find the magnetic field we use Faraday's Law

$$i\omega\mathbf{B} = \nabla \times \mathbf{E} = \nabla \times \left[E_o(x,y)(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) + \frac{i}{k}\left(\frac{\partial E_o}{\partial x} \pm i\frac{\partial E_o}{\partial y} \right)\hat{\mathbf{z}} \right] e^{ikz - i\omega t}$$

$$= ike^{ikz - i\omega t}\hat{\mathbf{z}} \times \left[E_o(x,y)(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) + \frac{i}{k}\left(\frac{\partial E_o}{\partial x} \pm i\frac{\partial E_o}{\partial y} \right)\hat{\mathbf{z}} \right] + e^{ikz - i\omega t} + \hat{\mathbf{x}}\frac{\partial}{\partial y}\frac{i}{k}\left(\frac{\partial E_o}{\partial x} \pm i\frac{\partial E_o}{\partial y} \right)$$

$$- \hat{\mathbf{y}}\frac{\partial}{\partial x}\frac{i}{k}\left(\frac{\partial E_o}{\partial x} \pm i\frac{\partial E_o}{\partial y} \right) + \hat{\mathbf{z}}\left[\pm i\frac{\partial E_o}{\partial x} - \frac{\partial E_o}{\partial y} \right]$$

In the equation above we neglect terms like $\frac{\partial}{\partial x}\frac{i}{k}\left(\frac{\partial E_o}{\partial x} \pm i\frac{\partial E_o}{\partial y} \right)$ compared to $ikE_o(x,y)$ because they are of order λ^2 / L^2 , where L is the length over which $E_o(x,y)$ varies, and we have assumed $\lambda / L << 1$. Thus we have

$$i\omega\mathbf{B} =$$
$$= ike^{ikz-i\omega t}\hat{\mathbf{z}}\times\left[E_o(x,y)(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}})+\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)\hat{\mathbf{z}}\right]+\hat{\mathbf{z}}\left[\pm i\frac{\partial E_o}{\partial x}-\frac{\partial E_o}{\partial y}\right]e^{ikz-i\omega t}$$
$$= ke^{ikz-i\omega t}\left\{E_o(x,y)i\hat{\mathbf{y}}\pm\hat{\mathbf{x}}E_o(x,y)+\frac{\hat{\mathbf{z}}}{k}\left[\pm i\frac{\partial E_o}{\partial x}-\frac{\partial E_o}{\partial y}\right]\right\}$$
$$= e^{ikz-i\omega t}k\left[E_o(x,y)i\hat{\mathbf{y}}\pm\hat{\mathbf{x}}E_o(x,y)+\hat{\mathbf{z}}\frac{i}{k}\left(\pm\frac{\partial E_o}{\partial x}+i\frac{\partial E_o}{\partial y}\right)\right]$$
$$= \pm e^{ikz-i\omega t}k\left[E_o(x,y)\left(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}\right)+\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)\right]$$

$$\mathbf{B}\simeq\mp i\frac{k}{\omega}\mathbf{E}=\mp i\sqrt{\mu\varepsilon}\mathbf{E}$$

(b) For this circularly polarized wave, calculate the time-averaged component of the angular momentum parallel to the direction of propagation. Show that the ratio of this component of angular momentum to the energy of the wave is

$$\frac{L_z}{U}=\pm\frac{1}{\omega}$$

Interpret this result in terms of quanta of radiation (photons). Show that for a cylindrically symmetric finite plane wave the transverse components of angular momentum vanish.

We will assume that $\mu=\mu_o$ and that $\varepsilon=\varepsilon_o$, so that $\mathbf{B}\simeq\mp i\mathbf{E}/c$ The total angular momentum in the wave packet is (see equation (6.118) of Jackson, p 261)

$$\frac{1}{2}\mathrm{Re}\int\frac{1}{c^2\mu_o}\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}^*\right)d^3x=\frac{1}{2}\mathrm{Re}\pm i\int\frac{1}{c^3\mu_o}\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)d^3x$$
$$\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)=\mathbf{r}\times\left(\left[E_o(x,y)\left(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}\right)+\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)\right]\times\left[E_o(x,y)\left(\hat{\mathbf{x}}\mp i\hat{\mathbf{y}}\right)-\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\mp i\frac{\partial E_o}{\partial y}\right)\right]\right)$$

$$=\mathbf{r}\times\left(\left[E_o^2(x,y)\left(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}\right)\times\left(\hat{\mathbf{x}}\mp i\hat{\mathbf{y}}\right)+\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)\times E_o\left(\hat{\mathbf{x}}\mp i\hat{\mathbf{y}}\right)-E_o(x,y)\left(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}\right)\times\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\mp i\frac{\partial E_o}{\partial y}\right)\right]\right)$$

$$=\mathbf{r}\times\left(\left[E_o^2(x,y)\left(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}\right)\times\left(\hat{\mathbf{x}}\mp i\hat{\mathbf{y}}\right)+\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)\times E_o\left(\hat{\mathbf{x}}\mp i\hat{\mathbf{y}}\right)-E_o(x,y)\left(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}}\right)\times\hat{\mathbf{z}}\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\mp i\frac{\partial E_o}{\partial y}\right)\right]\right)$$

$$=\mathbf{r}\times\left(\left[E_o^2(x,y)\left(\mp 2i\hat{\mathbf{z}}\right)+\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)E_o\left(\hat{\mathbf{y}}\pm i\hat{\mathbf{x}}\right)-E_o\left(-\hat{\mathbf{y}}\pm i\hat{\mathbf{x}}\right)\frac{i}{k}\left(\frac{\partial E_o}{\partial x}\mp i\frac{\partial E_o}{\partial y}\right)\right]\right)$$

$$= \mathbf{r}\times\left(\left[E_o^2(x,y)\left(\mp 2i\hat{\mathbf{z}}\right)+\hat{\mathbf{y}}E_o\frac{i}{k}\left\{\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)+\left(\frac{\partial E_o}{\partial x}\mp i\frac{\partial E_o}{\partial y}\right)\right\}\pm i\hat{\mathbf{x}}E_o\frac{i}{k}\left\{\left(\frac{\partial E_o}{\partial x}\pm i\frac{\partial E_o}{\partial y}\right)-\left(\frac{\partial E_o}{\partial x}\mp i\frac{\partial E_o}{\partial y}\right)\right\}\right]\right)$$

$$= \mathbf{r}\times\left(E_o^2(x,y)\left(\mp 2i\hat{\mathbf{z}}\right)+\hat{\mathbf{y}}E_o\frac{2i}{k}\frac{\partial E_o}{\partial x}-\hat{\mathbf{x}}E_o\frac{2i}{k}\frac{\partial E_o}{\partial y}\right)=\left[x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}\right]\times\left(E_o^2(x,y)\left(\mp 2i\hat{\mathbf{z}}\right)+\hat{\mathbf{y}}E_o\frac{2i}{k}\frac{\partial E_o}{\partial x}-\hat{\mathbf{x}}E_o\frac{2i}{k}\frac{\partial E_o}{\partial y}\right)$$

$$\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)=-xE_o^2(x,y)\left(\mp 2i\right)\hat{\mathbf{y}}+\mathrm{y}E_o^2(x,y)\left(\mp 2i\right)\hat{\mathbf{x}}+\hat{\mathbf{z}}\frac{2i}{k}E_o\left(x\frac{\partial E_o}{\partial x}+y\frac{\partial E_o}{\partial y}\right)-\hat{\mathbf{x}}\frac{2i}{k}zE_o\frac{\partial E_o}{\partial x}-\hat{\mathbf{y}}\frac{2i}{k}xE_o\frac{\partial E_o}{\partial y}$$

$$=-\left[\frac{2i}{k}xE_o\frac{\partial E_o}{\partial y}+xE_o^2(x,y)\left(\mp 2i\right)\right]\hat{\mathbf{y}}+\left[-\frac{2i}{k}zE_o\frac{\partial E_o}{\partial x}+\mathrm{y}E_o^2(x,y)\left(\mp 2i\right)\right]\hat{\mathbf{x}}+\hat{\mathbf{z}}\frac{2i}{k}E_o\left(x\frac{\partial E_o}{\partial x}+y\frac{\partial E_o}{\partial y}\right)$$

$$=\frac{2i}{k}\left\{-\left[xE_o\frac{\partial E_o}{\partial y}\mp kxE_o^2(x,y)\right]\hat{\mathbf{y}}+\left[-zE_o\frac{\partial E_o}{\partial x}\mp kyE_o^2(x,y)\right]\hat{\mathbf{x}}+\hat{\mathbf{z}}E_o\left(x\frac{\partial E_o}{\partial x}+y\frac{\partial E_o}{\partial y}\right)\right\}$$

But $\left[xE_o\dfrac{\partial E_o}{\partial y}\mp kxE_o^2(x,y)\right]\sim kxE_o^2(x,y)\left[1+\dfrac{1}{kE_o(x,y)}\dfrac{\partial E_o}{\partial y}\right]\sim kxE_o^2(x,y)\left[1+\dfrac{\lambda}{2\pi L}\right]$

So to the order we are working,

$$\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)=\frac{2i}{k}\left\{\left[\pm kxE_o^2(x,y)\right]\hat{\mathbf{y}}+\left[\mp kyE_o^2(x,y)\right]\hat{\mathbf{x}}+\hat{\mathbf{z}}E_o\left(x\frac{\partial E_o}{\partial x}+y\frac{\partial E_o}{\partial y}\right)\right\}$$

$$\frac{1}{2}\mathrm{Re}\left[\pm i\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)\right]=\mp\frac{1}{k}\left\{\left[\pm kxE_o^2(x,y)\right]\hat{\mathbf{y}}+\left[\mp kyE_o^2(x,y)\right]\hat{\mathbf{x}}+\hat{\mathbf{z}}E_o\left(x\frac{\partial E_o}{\partial x}+y\frac{\partial E_o}{\partial y}\right)\right\}$$

$$\frac{1}{2}\mathrm{Re}\left[\pm i\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)\right]=\frac{1}{k}\left\{-\left[kxE_o^2(x,y)\right]\hat{\mathbf{y}}+\left[kyE_o^2(x,y)\right]\hat{\mathbf{x}}\mp\hat{\mathbf{z}}E_o\left(x\frac{\partial E_o}{\partial x}+y\frac{\partial E_o}{\partial y}\right)\right\}$$

If we integrate the above over the *x*-*y* plane the first two terms will vanish as long as $E_o(x,y)$ is cylindrically symmetric (which means it is even in both *x* and *y*), and the last term will give

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdy\frac{1}{2}\mathrm{Re}\left[\pm i\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)\right]=\mp\frac{4}{k}\hat{\mathbf{z}}\left\{\int_0^{\infty}\int_0^{\infty}\left(x\frac{1}{2}\frac{\partial E_o^2}{\partial x}+y\frac{1}{2}\frac{\partial E_o^2}{\partial y}\right)dxdy\right\}=\pm\frac{4}{k}\hat{\mathbf{z}}\left\{\int_0^{\infty}\int_0^{\infty}E_o^2dxdy\right\}$$

where we have done the last integral by parts, which flips the sign. So we have

$$\frac{1}{2}\mathrm{Re}\int\frac{1}{c^2\mu_o}\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}^*\right)d^3x=\frac{1}{2}\mathrm{Re}\pm i\int\frac{1}{c^3\mu_o}\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)d^3x=\pm\frac{1}{ck}\hat{\mathbf{z}}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varepsilon_oE_o^2dxdy\right\}$$

$$\frac{1}{2}L_z\hat{\mathbf{z}}=\mathrm{Re}\int\frac{1}{c^2\mu_o}\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}^*\right)d^3x=\frac{1}{2}\mathrm{Re}\pm i\int\frac{1}{c^3\mu_o}\mathbf{r}\times\left(\mathbf{E}\times\mathbf{E}^*\right)d^3x=\pm\frac{1}{\omega}\hat{\mathbf{z}}U$$

as we wanted to show.

Problem 7-2: Jackson 10.11 page 510 parts (a) and (b). Do not do part (c).

These solutions are due to Patrick John Fitzpatrick.

(a) In the Kirchhoff approximation (eq. 10.85 of Jackson, p 481), we have

$$\psi(\mathbf{x}) = \frac{k}{2\pi i}\int_{S_1}\frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\hat{\mathbf{n}}'\cdot\mathbf{R}}{R}\psi(\mathbf{x}')\,da'$$

$$\hat{\mathbf{n}}' = \hat{\mathbf{z}} \qquad \mathbf{R} = \mathbf{X}-\mathbf{X}'$$

The coordinates of the observation point are $(X,0,Z)$ and we assume $Z >> X$ and $\sqrt{kZ} >> 1$. $R = \sqrt{(x'-X)^2+(y')^2+(z'-Z)^2}$.

$$\psi(\mathbf{x}) \simeq \frac{k}{2\pi i}\sqrt{I_o}\int_0^\infty dx'\int_{-\infty}^{\infty} dy'\,\frac{e^{ikR}}{R} = \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_0^\infty dy'\,\frac{e^{ik\sqrt{(x'-X)^2+(y')^2+(z'-Z)^2}}}{\sqrt{(x'-X)^2+(y')^2+Z^2}}$$

Let $r^2 = (x'-X)^2+Z^2$. Then

$$\psi(\mathbf{x}) \simeq \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_0^\infty dy'\,\frac{e^{ik\sqrt{r^2+(y')^2}}}{\sqrt{r^2+(y')^2}}$$

Let $\alpha^2 = r^2+(y')^2$. Then $y' = \sqrt{\alpha^2-r^2}$ and $dy' = \dfrac{\alpha\,d\alpha}{\sqrt{\alpha^2-r^2}}$

$$\psi(\mathbf{x}) \simeq \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_r^\infty \frac{\alpha\,d\alpha}{\sqrt{\alpha^2-r^2}}\frac{e^{ik\alpha}}{\alpha} = \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_r^\infty \frac{e^{ik\alpha}\,d\alpha}{\sqrt{\alpha^2-r^2}}$$

$$\psi(\mathbf{x}) \simeq \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_r^\infty \frac{\alpha\,d\alpha}{\sqrt{\alpha^2-r^2}}\frac{e^{ik\alpha}}{\alpha} = \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_r^\infty \frac{\cos(k\alpha)+i\sin(k\alpha)}{r\sqrt{\alpha^2/r^2-1}}d\alpha$$

Let $\xi = \dfrac{\alpha}{r}$. Then

$$\psi(\mathbf{x}) \simeq \frac{k}{\pi i}\sqrt{I_o}\int_0^\infty dx'\int_1^\infty \frac{\cos(k\xi r)+i\sin(k\xi r)}{\sqrt{\xi^2-1}}d\xi$$

If we refer to http://en.wikipedia.org/wiki/Bessel_function we find that

$$J_o(z) = \frac{2}{\pi}\int_1^\infty \frac{\sin(zu)}{\sqrt{u^2-1}}du \quad \text{and} \quad N_o(z) = -\frac{2}{\pi}\int_1^\infty \frac{\cos(zu)}{\sqrt{u^2-1}}du$$

so we have

$$\psi(\mathbf{x}) \simeq \frac{k}{2i}\sqrt{I_o}\int_0^\infty dx'\left[-N_o(kr)+iJ_o(kr)\right]$$

But we have $\sqrt{kZ} >> 1$ and therefore $kr >> 1$, and for large argument

$$J_o(kr) \simeq \frac{2}{\sqrt{\pi kr}}\cos(kr-\pi/4) \text{ and } N_o(kr) \simeq \frac{2}{\sqrt{\pi kr}}\sin(kr-\pi/4) \ ,$$

and therefore

$$\psi(\mathbf{x}) \simeq k\sqrt{\frac{I_o}{2\pi}}\int_0^\infty dx'\frac{e^{i(kr-\pi/4)}}{\sqrt{kr}}$$

$$r = \sqrt{(x'-X)^2+Z^2} = Z\sqrt{1+(x'-X)^2/Z^2} \approx Z\left(1+\frac{1}{2}(x'-X)^2/Z^2\right) \approx Z+\frac{1}{2}(x'-X)^2/Z$$

$$\psi(\mathbf{x}) \simeq ke^{-i\pi/4+ikZ}\sqrt{\frac{I_o}{2\pi}}\int_0^\infty dx'\frac{e^{ik(x'-X)^2/2Z}}{\sqrt{kZ+k\frac{1}{2}(x'-X)^2/Z}} \approx ke^{-i\pi/4+ikZ}\sqrt{\frac{I_o}{2\pi kZ}}\int_0^\infty dx' e^{ik(x'-X)^2/2Z}$$

Let $\varsigma = \sqrt{\frac{k}{2Z}}(x'-X)$. Then we have $\psi(\mathbf{x}) \simeq e^{-i\pi/4+ikZ}\sqrt{\frac{I_o}{\pi}}\int_{-X\sqrt{\frac{k}{2Z}}}^{\infty} d\varsigma e^{i\varsigma^2}$

If we use the fact that $e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i) = \sqrt{2}\frac{(1+i)}{2i}$ and we put back in the time dependence, we have, as desired,

$$\psi(\mathbf{x},t) \simeq \sqrt{I_o}e^{ikZ-i\omega t}\frac{1+i}{2i}\sqrt{\frac{2}{\pi}}\int_{-X\sqrt{\frac{k}{2Z}}}^{\infty} d\varsigma e^{i\varsigma^2}$$

(b)

$$I = |\psi|^2 = \frac{I_o}{4}\left|\sqrt{\frac{2}{\pi}}\int_{-X\sqrt{\frac{k}{2Z}}}^{\infty} d\varsigma e^{i\varsigma^2}\right|^2 = \frac{I_o}{4}\left|\sqrt{\frac{2}{\pi}}\left(\int_{-X\sqrt{\frac{k}{2Z}}}^{0} d\varsigma e^{i\varsigma^2}+\int_0^\infty d\varsigma e^{i\varsigma^2}\right)\right|^2$$

$$= \frac{I_o}{4}\left|\sqrt{\frac{2}{\pi}}\left(-\int_0^{X\sqrt{\frac{k}{2Z}}} dte^{it^2}+\int_0^\infty d\varsigma e^{i\varsigma^2}\right)\right|$$

The Fresnel Integrals are defined by $C(\lambda)=\int_0^{\lambda}\cos\left(\pi x^2/2\right)dx$ and $S(\lambda)=\int_0^{\lambda}\sin\left(\pi x^2/2\right)dx$. Using these in the expression above we have after a lot of algebra, and using the fact that $C(\infty)=S(\infty)=1/2$, we find that

$$I=\frac{I_o}{2}\left[\left(C\left(\sqrt{\frac{k}{2Z}}X\right)+\frac{1}{2}\right)^2+\left(S\left(\sqrt{\frac{k}{2Z}}X\right)+\frac{1}{2}\right)^2\right]$$