Solutions Assignment #6: Due Friday April 3, 2015 at 2:30 pm

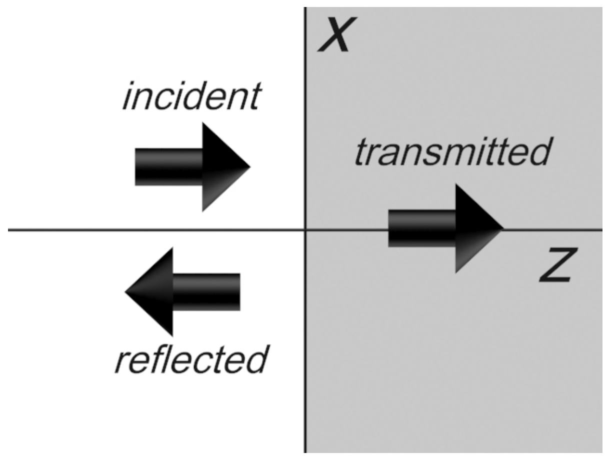
Problems

Problem 2 is worth twice the credit of the other two problems.

Problem 6-1: Transmission and Reflection of Waves Impinging From Vacuum into a Material Medium.

We have an interface between a vacuum and a material media. For $z < 0$ we have vacuum and an incident wave propagating to the right of the form

$$\mathbf{E}_{incident}(z,t) = \hat{\mathbf{x}}\, E_{oi} e^{-i\omega(t-z/c)} \qquad z<0$$



E_{oi} is a real constant. This incident wave in interacting with the material medium for z > 0 will generate a reflected wave propagating to the left of the form

$$\mathbf{E}_{reflected}(z,t) = \hat{\mathbf{x}}\, E_{or} e^{-i\omega(t+z/c)} \qquad z<0$$

E_{or} can be in principle an complex number. We must always take real parts of complex expressions at the end to get the actual solutions for the reflected and transmitted fields, as well as for other quantities. For $z > 0$ we have a non-permeable medium ($\mu = \mu_o$) with index of refraction $n = ck/\omega$ and the incident wave will cause a transmitted wave to exist for $z > 0$ of the form

$$\mathbf{E}_{transmitted}(z,t) = E_{ot}\hat{\mathbf{x}} e^{-i\omega t + ikz} \qquad z>0$$

E_{ot} can be in principle and in fact is in the problems below a complex number.

(a) In lecture on 3/14, we argued that the boundary conditions on the electromagnetic fields across the interface and Faraday's Law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ implies that if **B** and **E** are of the form $\mathbf{B} = \mathbf{B}_o e^{-i\omega t+ikz}$ and $\mathbf{E} = \mathbf{E}_o e^{-i\omega t+ikz}$, then we must have

$$\mathbf{B} = \frac{k}{\omega}\hat{\mathbf{z}} \times \mathbf{E} = \frac{n}{c}\hat{\mathbf{z}} \times \mathbf{E} \tag{1}$$

and that the transmitted, reflected, and incident wave amplitudes are related by

$$E_{or}/E_{oi} = -(n-1)/(n+1) \qquad E_{ot}/E_{oi} = 2/(n+1) \tag{2}$$

In these equations, k and n can in principle be complex. Prove that the statements in (1) and (2) are true, reproducing the arguments we used in lecture.

If $\mathbf{B} = \mathbf{B}_o e^{-i\omega t + ikz}$ then $\partial \mathbf{B} / \partial t = -i\omega \mathbf{B}$ and if $\mathbf{E} = \mathbf{E}_o e^{-i\omega t + ikz}$, then $\partial \mathbf{E} / \partial z = ik\mathbf{E}$ and $\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E} = ik\hat{\mathbf{z}} \times \mathbf{E}$, so we have shown that $\mathbf{B} = \frac{k}{\omega} \hat{\mathbf{z}} \times \mathbf{E} = \frac{n}{c} \hat{\mathbf{z}} \times \mathbf{E}$ where $n = ck / \omega$ and can be complex in principle.

Thus we can find the form of the magnetic fields on either side given the electric fields, and they are

$$\mathbf{B}_{incident}(z,t) = \hat{\mathbf{y}} \frac{E_{oi}}{c} e^{-i\omega(t - z/c)} \quad z < 0 \qquad \mathbf{B}_{reflected}(z,t) = -\hat{\mathbf{y}} \frac{E_{or}}{c} e^{-i\omega(t + z/c)} \quad z < 0$$

$$\mathbf{B}_{transmitted}(z,t) = \hat{\mathbf{y}} \frac{nE_{ot}}{c} e^{-i\omega(t - z/c)} \quad z > 0$$

The requirement that the tangential component of the electric and magnetic fields be continuous at $z = 0$ leads to

$$\hat{\mathbf{x}} E_{oi} e^{-i\omega t} + \hat{\mathbf{x}} E_{or} e^{-i\omega t} = E_{ot} e^{-i\omega t} \quad \text{and} \quad \hat{\mathbf{y}} \frac{E_{oi}}{c} e^{-i\omega t} - \hat{\mathbf{y}} \frac{E_{or}}{c} e^{-i\omega t} = \hat{\mathbf{y}} \frac{n}{c} E_{ot} e^{-i\omega t}$$

Solving these equations leads to the desired result.

(b) Suppose that we have two functions of space and time of the form $a(z,t) = a_o(z) e^{-i\omega t + i\phi_a(z)}$ and $b(z,t) = b_o(z) e^{-i\omega t + i\phi_b(z)}$. In these expressions, $a_o(z)$, $b_o(z)$, $\phi_a(z)$, and $\phi_a(z)$ are all real functions. Show that at a given *z*, if $\langle \; \rangle$ denotes a time average over one period of the wave, that is,

$$F(z) = \langle f(z,t) \rangle = \frac{1}{T} \int_0^T f(z,t) dt = \frac{2\pi}{\omega} \int_0^T f(z,t) dt$$

then

$$\left\langle \text{Re}\left[a(z,t)\right] \text{Re}\left[b(z,t)\right] \right\rangle = \frac{1}{2} \text{Re}\left[a(z,t) b^*(z,t)\right]$$

where if $b = b_r + ib_i$, then $b^* = b_r - ib_i$,

$$\left\langle \text{Re}\left[a(z,t)\right] \text{Re}\left[b(z,t)\right] \right\rangle = \frac{1}{T} \int_0^T \left[a_o(z) \cos\left(\omega t - \phi_a(z)\right) b_o(z) \cos\left(\omega t - \phi_b(z)\right) \right] dt$$

But $\cos A \cos B = \frac{1}{2} \left[\cos(A+B) + \cos(A-B) \right]$ so

$$\left\langle \mathrm{Re}\left[a(z,t)\right]\mathrm{Re}\left[b(z,t)\right]\right\rangle = \frac{1}{2T}\int_0^T \left[a_o(z)b_o(z)\left\{\cos\left(2\omega t - \phi_a(z) - \phi_b(z)\right) + \cos\left(\phi_b(z) - \phi_a(z)\right)\right\}\right]dt$$

$$\left\langle \mathrm{Re}\left[a(z,t)\right]\mathrm{Re}\left[b(z,t)\right]\right\rangle = \frac{1}{2T}\int_0^T \left[a_o(z)b_o(z)\cos\left(\phi_b(z) - \phi_a(z)\right)\right]dt = \frac{1}{2}\left[a_o(z)b_o(z)\cos\left(\phi_b(z) - \phi_a(z)\right)\right]$$

$$\left\langle \mathrm{Re}\left[a(z,t)\right]\mathrm{Re}\left[b(z,t)\right]\right\rangle = \frac{1}{2}\mathrm{Re}\, a(z,t)b^*(z,t)$$

Problem 6-2: Transmission and Reflection of Waves at a "Good" Conductor

If $\mathbf{J}(z,t) = \sigma_C \mathbf{E}(z,t)$, and we are in a non-permeable material ($\mu = \mu_o$), then $k^2 = i\mu_o\omega\sigma + \omega^2/c^2$, and if furthermore we assume $\omega << \sigma/\varepsilon_o$, then

$$k = (i+i)/\delta \qquad \delta = \sqrt{2/\mu_o\omega\sigma}$$

(a) Show that in this limit $|n|^2 = nn^* >> 1$, and that to first order in $1/n$

$$E_{or}/E_{oi} \cong -(1 - 2/n) \qquad E_{ot}/E_{oi} \cong 2/n$$

(b) Write down vector expressions for the incident, reflected and transmitted electric and magnetic fields, in complex form, keeping only first order terms in the small quantity $1/n$. Also show that the current density **J** for $z > 0$ in complex form is given by

$$\mathbf{J}(z,t) = \hat{\mathbf{x}}\frac{2E_{oi}}{c\mu_o}\left[\frac{\sqrt{2}}{\delta}e^{-i\omega t + i(1+i)z/\delta - i\pi/4}\right]$$

$$\mathbf{E}_{incident}(z,t) = \hat{\mathbf{x}}\, E_{oi}e^{-i\omega(t-z/c)} \qquad \mathbf{B}_{incident}(z,t) = \hat{\mathbf{y}}\,\frac{E_{oi}}{c}e^{-i\omega(t-z/c)}$$

$$E_{or}/E_{oi} = -(1-1/n)/(1+1/n) = -(1-2/n)$$

$$E_{or}/E_{oi} = -(1 - 2\omega\delta/c(1+i))$$

$$E_{or}/E_{oi} = -(1-(1-i)\omega\delta/c) = -(1-\omega\delta/c + i\omega\delta/c)$$

$$\mathbf{E}_{reflected}(z,t) = -(1-\omega\delta/c + i\omega\delta/c)E_{oi}\hat{\mathbf{x}}e^{-i\omega(t-z/c)}$$

$$\mathbf{B}_{reflected}(z,t) \cong \frac{E_{oi}}{c}(1-\omega\delta/c + i\omega\delta/c)\hat{\mathbf{y}}e^{-i\omega(t-z/c)}$$

$$\mathbf{E}_{transmitted}(z,t) \cong E_{oi}\frac{2}{n}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta} = E_{oi}\frac{2\delta\omega}{(1+i)c}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta} = E_{oi}\frac{\sqrt{2}\delta\omega}{c}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta-i\pi/4}$$

$$\mathbf{E}_{transmitted}(z,t) \cong E_{oi}\frac{\sqrt{2}\delta\omega}{c}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta-i\pi/4}$$

$$\mathbf{B}_{transmitted} = \frac{k}{\omega}\hat{\mathbf{z}}\times\mathbf{E}_{transmitted} = \frac{k}{\omega}\hat{\mathbf{y}}E_{oi}\frac{2\delta\omega}{(1+i)c}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta}$$

$$\mathbf{B}_{transmitted} = E_{oi}\frac{2}{c}\hat{\mathbf{y}}e^{-i\omega t+i(1+i)z/\delta}$$

$$\mathbf{J}(z,t) = \sigma_C\mathbf{E}_{transmitted}(z,t) = \sigma_C E_{oi}\frac{\sqrt{2}\delta\omega}{c}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta-i\pi/4} = \frac{2\sqrt{2}}{\delta\mu_o c}E_{oi}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta-i\pi/4}$$

(c) What is the time averaged Poynting flux in the incident wave just at $z=0^-$, that is, what is $\left\langle\left[\mathbf{E}_{incident}\times\mathbf{B}_{incident}\right]_{z=0^-}\right\rangle/\mu_o$? What is $\left\langle\left[\mathbf{E}_{reflected}\times\mathbf{B}_{reflected}\right]_{z=0^-}\right\rangle/\mu_o$? What is $\left\langle\left[\mathbf{E}_{transmitted}\times\mathbf{B}_{transmitted}\right]_{z=0^+}\right\rangle/\mu_o$? Keep only first order terms in small quantities.

$$\frac{\left\langle\left[\mathbf{E}_{incident}\times\mathbf{B}_{incident}\right]_{z=-\varepsilon}\right\rangle}{\mu_o} = \hat{\mathbf{z}}\frac{1}{2}E_{oi}^2/\left(\mu_o c\right) = \hat{\mathbf{z}}\frac{1}{2}c\varepsilon_o E_{oi}^2$$

$$\frac{\left\langle\left[\mathbf{E}_{reflected}\times\mathbf{B}_{reflected}\right]_{z=+\varepsilon}\right\rangle}{\mu_o} = \frac{E_{oi}^2}{2\mu_o c}\left[\left(1-\omega\delta/c\right)^2+\left(\omega\delta/c\right)^2\right]$$

$$\cong c\frac{E_{oi}^2}{2\mu_o c^2}\left(1-2\omega\delta/c\right) = \frac{1}{2}c\varepsilon_o E_{oi}^2\left(1-2\omega\delta/c\right)$$

$$\frac{\left\langle\left[\mathbf{E}_{transmitted}\times\mathbf{B}_{transmitted}\right]_{z=+\varepsilon}\right\rangle}{\mu_o} = \hat{\mathbf{z}}\frac{1}{2\mu_o}\operatorname{Re}E_{oi}\frac{\sqrt{2}\delta\omega}{c}e^{-i\omega t-i\pi/4}\left(E_{oi}\frac{2}{c}e^{+i\omega t}\right) = \hat{\mathbf{z}}\frac{1}{\mu_o}E_{oi}^2\frac{\delta\omega}{c^2} = \hat{\mathbf{z}}\,\delta\,\omega\varepsilon_o E_{oi}^2$$

(d) The reflection coefficient R is the ratio of the reflected time-averaged Poynting flux to the incident time-averaged Poynting flux. The transmission coefficient T is the ratio of the transmitted time-averaged Poynting flux to the incident time-averaged Poynting flux. What are R and T? You should have $R + T = 1$.

$$R = \frac{\left\langle\left[\mathbf{E}_{reflected}\times\mathbf{B}_{reflected}\right]_{z=+\varepsilon}\right\rangle}{\left\langle\left[\mathbf{E}_{incident}\times\mathbf{B}_{incident}\right]_{z=-\varepsilon}\right\rangle} = \frac{\frac{1}{2}c\varepsilon_o E_{oi}^2\left(1-2\omega\delta/c\right)}{\frac{1}{2}c\varepsilon_o E_{oi}^2} = (1-2\omega\delta/c)$$

$$T = \frac{\left\langle\left[\mathbf{E}_{transmitted}\times\mathbf{B}_{transmitted}\right]_{z=+\varepsilon}\right\rangle}{\left\langle\left[\mathbf{E}_{incident}\times\mathbf{B}_{incident}\right]_{z=-\varepsilon}\right\rangle} = \frac{\delta\omega\varepsilon_o E_{oi}^2}{\frac{1}{2}c\varepsilon_o E_{oi}^2} = 2\delta\omega/c$$

(e) Calculate the time-averaged rate at which energy is being dissipated by ohmic heating at a location $z > 0$. Calculate the total rate at which energy is being dissipated by ohmic heating from $z = 0$ to $z = \infty$. Show explicitly that this rate is equal to the energy being transmitted into the conductor at $z = 0^+$

$$\frac{1}{2}\mathrm{Re}\,\mathbf{J}\cdot\mathbf{E}^*_{transmitted} = \frac{1}{2}\sigma_C\mathbf{E}_{transmitted}\cdot\mathbf{E}^*_{transmitted} = \sigma_C E_{oi}^2\frac{\delta^2\omega^2}{c^2}e^{-2z/\delta}$$

$$\int_0^\infty \frac{1}{2}\mathrm{Re}\,\mathbf{J}\cdot\mathbf{E}^*_{transmitted}dz = \sigma_C E_{oi}^2\frac{\delta^3\omega^2}{2c^2} = \sigma_C\delta E_{oi}^2\frac{\omega^2}{2c^2}\frac{2}{\mu_o\sigma_C\omega} = \varepsilon_o\omega\delta E_{oi}^2$$

From above we have $\frac{\left\langle\left[\mathbf{E}_{transmitted}\times\mathbf{B}_{transmitted}\right]_{z=+\varepsilon}\right\rangle}{\mu_o} = \hat{\mathbf{z}}\frac{1}{\mu_o}E_{oi}^2\frac{\delta\omega}{c^2} = \hat{\mathbf{z}}\,\delta\,\omega\varepsilon_o E_{oi}^2$ so that we see that we have the equality we desire.

(f) Calculate the time-averaged $\mathbf{J}\times\mathbf{B}$ forced per unit volume at a location $z > 0$. Calculate the total force exerted by the $\mathbf{J}\times\mathbf{B}$ force from $z = 0$ to $z = \infty$.

$$\frac{1}{2}\mathrm{Re}\,\mathbf{J}\times\mathbf{B}^* = \frac{1}{2}\mathrm{Re}\frac{4}{\delta\mu_o}B_{oi}^2\sqrt{2}e^{-2z/\delta - i\pi/4}\hat{\mathbf{z}} = \frac{2}{\delta\mu_o}B_{oi}^2 e^{-2z/\delta}\hat{\mathbf{z}}$$

$$\int_0^\infty \frac{1}{2}\mathrm{Re}\,\mathbf{J}\times\mathbf{B}^* dz = \frac{2}{\delta\mu_o}\frac{\delta}{2}B_{oi}^2 e^{-2z/\delta}\hat{\mathbf{z}} = \frac{1}{\mu_o}B_{oi}^2\hat{\mathbf{z}}$$

(g) If we take the real part of the complex current we gave above, we find that

$$\mathbf{J}(z,t) = \hat{\mathbf{x}}\frac{2E_{oi}}{\mu_o c}\left[\frac{\sqrt{2}}{\delta}e^{-z/\delta}\cos(\omega t - \frac{z}{\delta} + \frac{\pi}{4})\right]$$

Show that in the limit that $\delta \to 0$, this expression becomes

$$\mathbf{J}(z,t) = \hat{\mathbf{x}}\frac{2B_{oi}}{\mu_o}\cos\left(\omega t\right)\delta(z)$$

where $\delta(z)$ is a delta function in z.

We need to show that $\frac{\sqrt{2}}{\delta}e^{-z/\delta}\cos(\omega t - \frac{z}{\delta} + \frac{\pi}{4})$ goes to zero for any $z > 0$. This is obvious. We also need to show that

$$\int_0^\infty \frac{\sqrt{2}}{\delta}e^{-z/\delta}\cos(\omega t - \frac{z}{\delta} + \frac{\pi}{4})\,dz = \cos\left(\omega t\right)$$

$$\int_0^\infty \frac{\sqrt{2}}{\delta} e^{-z/\delta} \cos(\omega t - \frac{z}{\delta} + \frac{\pi}{4})\, dz = \int_0^\infty \frac{\sqrt{2}}{\delta} e^{-z/\delta} \frac{1}{2}\left(e^{i(\omega t - \frac{z}{\delta} + \frac{\pi}{4})} + e^{-i(\omega t - \frac{z}{\delta} + \frac{\pi}{4})} \right) dz$$

$$= \frac{\sqrt{2}}{2\delta} \int_0^\infty \left(e^{i(\omega t - \frac{z}{\delta} + \frac{\pi}{4}) - z/\delta} + e^{-i(\omega t - \frac{z}{\delta} + \frac{\pi}{4}) - z/\delta} \right) dz = \frac{\sqrt{2}}{2\delta} \int_0^\infty \left(e^{i\omega t - \frac{z}{\delta}(1+i) + \frac{i\pi}{4}} + e^{-i\omega t - \frac{z}{\delta}(1-i) - \frac{i\pi}{4}} \right) dz$$

$$\int_0^\infty \frac{\sqrt{2}}{\delta} e^{-z/\delta} \cos(\omega t - \frac{z}{\delta} + \frac{\pi}{4})\, dz = \frac{\sqrt{2}}{2\delta}\left[e^{i\omega t + \frac{i\pi}{4}} \int_0^\infty \left(e^{-\frac{z}{\delta}(1+i)} \right) dz + e^{-i\omega t - \frac{i\pi}{4}} \int_0^\infty \left(e^{-\frac{z}{\delta}(1-i)} \right) dz \right]$$

$$= \frac{\sqrt{2}}{2\delta}\left[-e^{i\omega t + \frac{i\pi}{4}} \frac{\delta}{1+i} - e^{-i\omega t - \frac{i\pi}{4}} \frac{\delta}{1-i} \right] = \frac{\sqrt{2}}{\delta} \mathrm{Re}\left[-e^{i\omega t + \frac{i\pi}{4}} \frac{\delta}{1+i} \right] = \frac{\sqrt{2}}{\delta} \mathrm{Re}\left[-e^{i\omega t + \frac{i\pi}{4}} \frac{\delta e^{-\frac{i\pi}{4}}}{\sqrt{2}} \right] = \cos(\omega t)$$

(h) Show that the current given above is what we expect to see in the limit of a perfect conductor which totally excludes electric and magnetic fields from its interior.

In the perfect conductor limit, our electric field for $z = 0^-$ is zero and out magnetic field is $\hat{\mathbf{y}} 2\cos(\omega t) E_{oi} / c = \hat{\mathbf{y}} 2\cos(\omega t) B_{oi}$. Since the magnetic field is zero at $z = 0^+$, that means we must have a current sheet at $z = 0$ of $\hat{\mathbf{x}} \frac{2B_{oi}}{\mu_o} \cos(\omega t)$. This will cause a jump in the tangential component of the y-component of the magnetic field of $2B_{oi} \cos(\omega t)$, which is just what we need to take the magnetic field from $2\cos(\omega t) B_{oi}$ in vacuum at $z = 0^-$ to zero in the conductor.

Problem 6-3: Transmission and Reflection of Waves at a Plasma Interface

Consider transverse waves in a simple plasma with n_e electrons per cubic meter and $n_p = n_e$ protons per cubic meter, with no collisions, and taking the protons to be infinitely massive.

(a) If we assume a time dependence for all quantities of $e^{-i\omega t + ikz}$ show that the dispersion relation for waves in this plasma is $c^2 k^2 = \omega^2 - \omega_p^2$, where $\omega_p^2 = \frac{e^2 n_e}{m_e \varepsilon_o}$. In consider the force on the electrons oscillating in the transverse electromagnetic wave, ignore the $\mathbf{v} \times \mathbf{B}$ force.

Our equation for the motion of the electron is $m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E}$ and we also have $\mathbf{J}_e = -e n_e \mathbf{v}_e$. Combining these two equations gives us a relation between **J** and **E** known as a "generalized" ohm's law, that is $\frac{d}{dt}\mathbf{J}_e = -en_e \frac{d}{dt}\mathbf{v}_e = \frac{n_e e^2}{m_e}\mathbf{E}$. If we then assume a time dependence of the form $e^{-i\omega t + ikz}$, we have $\mathbf{J}_e = i\frac{ne^2}{\omega m_e}\mathbf{E}$, and inserting this form into

$$-\nabla^2\mathbf{B} = \mu_o i\frac{n_e e^2}{\omega m_e}\nabla\times\mathbf{E} + \mu_o\varepsilon_o\frac{\partial}{\partial t}\nabla\times\mathbf{E} \Rightarrow k^2 = -\mu_o\frac{n_e e^2}{m_e} + \frac{\omega^2}{c^2} \quad \text{or } c^2k^2 = \omega^2 - \omega_p^2$$

From now on we will consider only the situation where $\omega < \omega_p$ and that $k = \frac{i}{\delta}$ is purely imaginary with $\delta = c/\sqrt{\omega_p^2 - \omega^2}$.

(b) Again consider the situation we had in Problem 6-1, where now the medium for *z > 0* is a plasma. Write down vector expressions for the incident, reflected and transmitted electric and magnetic fields, in complex form. What is the current density **J** for *z > 0* in complex form?

$$\mathbf{E}_{incident}(z,t) = \hat{\mathbf{x}}\,E_{oi}e^{-i\omega(t-z/c)} \qquad \mathbf{B}_{incident}(z,t) = \hat{\mathbf{y}}\,\frac{E_{oi}}{c}e^{-i\omega(t-z/c)}$$

$$\mathbf{E}_{reflected}(z,t) \cong -E_{oi}\left(\frac{n-1}{n+1}\right)\hat{\mathbf{x}}e^{-i\omega(t-z/c)} = -E_{oi}\left(\frac{\frac{ci}{\delta\omega}-1}{\frac{ci}{\delta\omega}+1}\right)\hat{\mathbf{x}}e^{-i\omega(t-z/c)} = \mathbf{E}_{reflected}(z,t) = -\frac{ci/\delta-\omega}{ci/\delta+\omega}E_{oi}\hat{\mathbf{x}}e^{-i\omega(t-z/c)}$$

$$\mathbf{B}_{reflected}(z,t) \cong -\frac{E_{oi}}{c}\frac{ci/\delta-\omega}{ci/\delta+\omega}\hat{\mathbf{y}}e^{-i\omega(t-z/c)}$$

$$\mathbf{E}_{transmitted}(z,t) \cong E_{oi}\frac{2}{1+n}\hat{\mathbf{x}}e^{-i\omega t-cz/\delta} = E_{oi}\frac{2}{1+ci/(\delta\omega)}\hat{\mathbf{x}}e^{-i\omega t+i(1+i)z/\delta} = E_{oi}\frac{2}{1+ci/(\delta\omega)}\hat{\mathbf{x}}e^{-i\omega t-cz/\delta}$$

$$\mathbf{B}_{transmitted} = \frac{k}{\omega}\hat{\mathbf{z}}\times\mathbf{E}_{transmitted} = \frac{i}{\omega\delta}\hat{\mathbf{y}}E_{oi}\frac{2}{1+ci/(\delta\omega)}e^{-i\omega t-cz/\delta}$$

$$\mathbf{J}_e = i\frac{ne^2}{\omega m_e}\mathbf{E}_{transmitted} = i\frac{\varepsilon_o\omega_p^2}{\omega}\mathbf{E}_{transmitted}$$

(c) What is the time averaged Poynting flux in the incident wave just at $z = 0^-$, that is, what is $\left\langle \left[\mathbf{E}_{incident} \times \mathbf{B}_{incident} \right]_{z=0^-} \right\rangle / \mu_o$? What is $\left\langle \left[\mathbf{E}_{reflected} \times \mathbf{B}_{reflected} \right]_{z=0^-} \right\rangle / \mu_o$? What is $\left\langle \left[\mathbf{E}_{transmitted} \times \mathbf{B}_{transmitted} \right]_{z=0^+} \right\rangle / \mu_o$?

$$\frac{\left\langle \left[\mathbf{E}_{incident} \times \mathbf{B}_{incident} \right]_{z=-\varepsilon} \right\rangle}{\mu_o} = \hat{\mathbf{z}} \frac{1}{2} E_{oi}^2 / \left(\mu_o c \right) = \hat{\mathbf{z}} \frac{1}{2} c \varepsilon_o E_{oi}^2$$

$$\frac{\left\langle \left[\mathbf{E}_{reflected} \times \mathbf{B}_{reflected} \right]_{z=+\varepsilon} \right\rangle}{\mu_o} = \frac{E_{oi}^2}{2\mu_o c} \left[\left(\frac{ci/\delta - \omega}{ci/\delta + \omega} \right) \left(\frac{-ci/\delta - \omega}{-ci/\delta + \omega} \right) \right] = \frac{E_{oi}^2}{2\mu_o c}$$

$$\frac{\left\langle \left[\mathbf{E}_{transmitted} \times \mathbf{B}_{transmitted} \right]_{z=+\varepsilon} \right\rangle}{\mu_o} = \hat{\mathbf{z}} \frac{1}{2\mu_o} \operatorname{Re} E_{oi}^2 \frac{2}{1 + ci/\left(\delta\omega \right)} \frac{-i}{\omega\delta} \left(\frac{2}{1 - ci/\left(\delta\omega \right)} \right) = 0$$

(d) The reflection coefficient R is the ratio of the reflected time-averaged Poynting flux to the incident time-averaged Poynting flux. The transmission coefficient T is the ratio of the transmitted time-averaged Poynting flux to the incident time-averaged Poynting flux. What are R and T? You should have $R + T = 1$.

$$R = \frac{\left\langle \left[\mathbf{E}_{reflected} \times \mathbf{B}_{reflected} \right]_{z=+\varepsilon} \right\rangle}{\left\langle \left[\mathbf{E}_{incident} \times \mathbf{B}_{incident} \right]_{z=-\varepsilon} \right\rangle} = 1$$

$$T = \frac{\left\langle \left[\mathbf{E}_{transmitted} \times \mathbf{B}_{transmitted} \right]_{z=+\varepsilon} \right\rangle}{\left\langle \left[\mathbf{E}_{incident} \times \mathbf{B}_{incident} \right]_{z=-\varepsilon} \right\rangle} = 0$$

(e) Calculate the time-averaged rate at which energy is being dissipated by ohmic heating at a location $z > 0$. Calculate the total rate at which energy is being dissipated by ohmic heating from $z = 0$ to $z = \infty$. Show explicitly that this rate is equal to the energy being transmitted into the conductor at $z = 0^+$

$$\frac{1}{2} \operatorname{Re} \mathbf{J} \cdot \mathbf{E}_{transmitted}^* = \frac{1}{2} \operatorname{Re} i \frac{\varepsilon_o \omega_p^2}{\omega} \mathbf{E}_{transmitted} \cdot \mathbf{E}_{transmitted}^* = 0$$

$$\int_0^\infty \frac{1}{2} \operatorname{Re} \mathbf{J} \cdot \mathbf{E}_{transmitted}^* dz = 0$$

From above we have $\dfrac{\left\langle \left[\mathbf{E}_{transmitted} \times \mathbf{B}_{transmitted} \right]_{z=+\varepsilon} \right\rangle}{\mu_o} = \hat{\mathbf{z}} 0$ so that we see that we have the equality we desire.

(f) Calculate the time-averaged $\mathbf{J} \times \mathbf{B}$ forced per unit volume at a location $z > 0$. Calculate the total force exerted by the $\mathbf{J} \times \mathbf{B}$ force from $z = 0$ to $z = \infty$.

$$\frac{1}{2}\mathrm{Re}\,\mathbf{J}\times\mathbf{B}^* = \frac{1}{2}\mathrm{Re}\,i\frac{\varepsilon_o\omega_p^2}{\omega}E_{oi}\left(\frac{2}{1+ci/(\delta\omega)}\right)\frac{i}{\omega\delta}E\left(\frac{2}{1-ci/(\delta\omega)}\right)e^{-i\omega t-cz/\delta}e^{-2z/\delta}\hat{\mathbf{z}}$$

$$=\frac{\varepsilon_o\omega_p^2}{\omega}\frac{2}{\omega\delta}\frac{c^2\omega^2/\left(\omega_p^2-\omega^2\right)}{c^2\omega^2/\left(\omega_p^2-\omega^2\right)+c^2}E_{oi}^2e^{-2z/\delta}\hat{\mathbf{z}} = -\frac{\omega_p^2}{\omega}\frac{2}{\omega\delta}\frac{\omega^2}{\omega_p^2}E_{oi}^2e^{-2z/\delta}\hat{\mathbf{z}} = \frac{2\varepsilon_o}{\delta}E_{oi}^2e^{-2z/\delta}\hat{\mathbf{z}}$$

$$\int_0^\infty \frac{1}{2}\mathrm{Re}\,\mathbf{J}\times\mathbf{B}^* dz = \varepsilon_o E_{oi}^2\hat{\mathbf{z}} = \frac{B_{oi}^2}{\mu_o}\hat{\mathbf{z}}$$

Problem 6-4: Radiation from a Sheet of Charge

A sheet of charge has charge per unit area σ_o and is located in the *xy* plane at $z = 0$. Each segment of the sheet of charge at time *t* has exactly the same acceleration, given by $\mathbf{a} = \hat{\mathbf{x}}a(t)$ where $a(t) = \frac{d}{dt}V(t)$. We have previously shown that the electric field generated by this moving sheet is given by

$$\mathbf{E}(z,t) = -\hat{\mathbf{x}}\frac{\sigma_o}{2\varepsilon_o c}V\left(t-|z|/c\right) \tag{4.1}$$

And I have repeatedly said in class that you could start from the radiation field of a single accelerated charge, which we know to be

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_o}\frac{\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\mathbf{a}\left(t-r/c\right)\right]}{r} \tag{4.2}$$

then break the infinite sheet above into infinitesimal areas radiating according to (4.2), add then all up, and recover equation (4.1).

However I find that I cannot actually show that. What I can do starting from (4.2) is show that the following is true

$$\mathbf{E}(z,t) = -\hat{\mathbf{x}}\int_{t-|z|/c}^{t-\infty/c}\frac{\sigma}{4\varepsilon_o c}a\left(\tau\right)\left\{\frac{\rho'^2}{\left(\rho'^2+z^2\right)}-2\right\}d\tau \qquad \tau = t-\frac{\sqrt{\left(\rho'^2+z^2\right)}}{c} \tag{4.3}$$

If I did not have the first term in brackets above, and if I didn't bother my mind about what $e^{t-\infty/c}$ means, I could recover what I want, e.g. (4.1). But I can't. Here is the problem I want you to do and how you will get credit. If you can reproduce (4.3) and it is correct as I wrote it down, you get 10 points. If I made a mistake in getting to (4.3) and you find my mistake and get the answer I think should be right, (4.1), you get 15 points. If (4.3) is correct and you figure out how to get rid of the first term in brackets in your final answer, you get 15 points.

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_o}\frac{(\mathbf{X}-\mathbf{X}')\times\left[(\mathbf{X}-\mathbf{X}')\times\mathbf{a}\left(t-|\mathbf{X}-\mathbf{X}'|/c\right)\right]}{|\mathbf{X}-\mathbf{X}'|^3}$$

$$\mathbf{X}' = \hat{\mathbf{x}}\rho'\cos\phi' + \hat{\mathbf{y}}\rho'\sin\phi' \qquad \mathbf{X} = z\,\hat{\mathbf{z}} \quad \mathbf{X}-\mathbf{X}' = -\hat{\mathbf{x}}\rho'\cos\phi' - \hat{\mathbf{y}}\rho'\sin\phi' + z\,\hat{\mathbf{z}}$$

$$|\mathbf{X}-\mathbf{X}'| = \sqrt{\rho'^2 + z^2}$$

$$d\mathbf{E} = \frac{\sigma\rho' d\rho' d\phi'}{4\pi\varepsilon_o}\frac{(\mathbf{X}-\mathbf{X}')\times\left[(\mathbf{X}-\mathbf{X}')\times\mathbf{a}\left(t-|\mathbf{X}-\mathbf{X}'|/c\right)\right]}{|\mathbf{X}-\mathbf{X}'|^3}$$

$$d\mathbf{E} = \frac{\sigma\rho' d\rho' d\phi' a\left(t-|\mathbf{X}-\mathbf{X}'|/c\right)}{4\pi\varepsilon_o|\mathbf{X}-\mathbf{X}'|^3}\left\{(\mathbf{X}-\mathbf{X}')\left[\hat{\mathbf{x}}\cdot(\mathbf{X}-\mathbf{X}')\right] - \hat{\mathbf{x}}|\mathbf{X}-\mathbf{X}'|^2\right\}$$

$$d\mathbf{E} = \frac{\sigma\rho' d\rho' d\phi' a\left(t-\sqrt{\rho'^2+z^2}\,/\,c\right)}{4\pi\varepsilon_o\left(\rho'^2+z^2\right)^{3/2}}\left\{\left(-\hat{\mathbf{x}}\rho'\cos\phi' - \hat{\mathbf{y}}\rho'\sin\phi' + z\,\hat{\mathbf{z}}\right)\left(-\rho'\cos\phi'\hat{\mathbf{x}}\right) - \hat{\mathbf{x}}\left(\rho'^2+z^2\right)\right\}$$

$$d\mathbf{E} = \hat{\mathbf{x}}\frac{\pi\sigma\rho' d\rho' a\left(t-\sqrt{\rho'^2+z^2}\,/\,c\right)}{4\pi\varepsilon_o\left(\rho'^2+z^2\right)^{3/2}}\left\{\rho'^2 - 2\left(\rho'^2+z^2\right)\right\}$$

$$d\mathbf{E} = \hat{\mathbf{x}}\frac{\pi\sigma}{4\pi\varepsilon_o}\rho' d\rho' a\left(t-\sqrt{\rho'^2+z^2}\,/\,c\right)\left\{\frac{\rho'^2}{\left(\rho'^2+z^2\right)^{3/2}} - \frac{2}{\left(\rho'^2+z^2\right)^{1/2}}\right\}$$

$$\tau = t - \sqrt{z^2+\rho'^2}\,/\,c \quad d\tau = -\frac{\rho' d\rho'}{c\sqrt{z^2+\rho'^2}} \qquad \frac{\rho' d\rho'}{\sqrt{z^2+\rho'^2}} = -c\,d\tau$$

$$d\mathbf{E} = -\hat{\mathbf{x}}\frac{\pi\sigma c\,d\tau}{4\pi\varepsilon_o}\left(\rho'^2+z^2\right)^{1/2} a(\tau)\left\{\frac{\rho'^2}{\left(\rho'^2+z^2\right)^{3/2}} - \frac{2}{\left(\rho'^2+z^2\right)^{1/2}}\right\}$$

$$d\mathbf{E} = -\hat{\mathbf{x}}\frac{\pi\sigma c\,d\tau}{4\pi\varepsilon_o}a(\tau)\left\{\frac{\rho'^2}{\left(\rho'^2+z^2\right)} - 2\right\}$$