1. The Dressed Atom

(a)

(b)

(c)

(d)

(e)

(f)

2. Sideband Cooling

(a)

Interaction term

$$\begin{aligned}
H_I =&\hbar\Omega(\sigma_+ + \sigma_-)\cos\left(k\hat{x} - \omega t\right)\\
=&\hbar\Omega(\sigma_+ + \sigma_-)\cos\left(\eta\left(a + a^\dagger\right) - \omega t\right)\\
=&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{\mathrm{i}\left(\eta\left(a+a^\dagger\right)-\omega t\right)} + \mathrm{e}^{-\mathrm{i}\left(\eta\left(a+a^\dagger\right)-\omega t\right)}\right)
\end{aligned}$$

In the Lamb-Dicke limit

$$\begin{aligned}
H_I =&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{\mathrm{i}\eta\left(a+a^\dagger\right)}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\eta\left(a+a^\dagger\right)}\mathrm{e}^{\mathrm{i}\omega t}\right)\\
=&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\left(1 + \mathrm{i}\eta\left(a + a^\dagger\right)\right)\mathrm{e}^{-\mathrm{i}\omega t} + \left(1 - \mathrm{i}\eta\left(a + a^\dagger\right)\right)\mathrm{e}^{\mathrm{i}\omega t}\right)\\
=&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{\mathrm{i}\omega t}\right) + \mathrm{i}\eta\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(a + a^\dagger\right)\left(\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{e}^{\mathrm{i}\omega t}\right)
\end{aligned}$$

In the interaction picture

$$\begin{aligned}
\sigma'_\pm =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}\sigma_\pm\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&\sigma_\pm\mathrm{e}^{\mathrm{i}\omega_0t}\\
a' =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}a\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&a\mathrm{e}^{-\mathrm{i}\nu t}\\
a'^\dagger =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}a^\dagger\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&a^\dagger\mathrm{e}^{\mathrm{i}\nu t}
\end{aligned}$$

In the rotating wave approximation

$$\begin{aligned}
H'_I =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}H_I\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&\frac{\hbar\Omega}{2}\left(\sigma_+\mathrm{e}^{-\mathrm{i}\delta t} + \sigma_-\mathrm{e}^{\mathrm{i}\delta t}\right) + \mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a\mathrm{e}^{-\mathrm{i}(\delta+\nu)t} - \sigma_-a^\dagger\mathrm{e}^{\mathrm{i}(\delta+\nu)t}\right)\\
&+ \mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a^\dagger\mathrm{e}^{-\mathrm{i}(\delta-\nu)t} - \sigma_-a\mathrm{e}^{\mathrm{i}(\delta-\nu)t}\right)
\end{aligned}$$

When the light is in resonance with the sideband, the Rabi frequencies

$$\begin{aligned}
\Omega_{n,n-1} =&\left|\langle n-1,e|\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a-\sigma_-a^\dagger\right)|n,g\rangle\right|\\
=&\frac{\eta\hbar\Omega}{2}\langle n-1|a|n\rangle\\
=&\frac{\eta\hbar\Omega}{2}\sqrt{n}\\
\Omega_{n,n+1} =&\left|\langle n+1,e|\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a^\dagger-\sigma_-a\right)|n,g\rangle\right|\\
=&\frac{\eta\hbar\Omega}{2}\langle n+1|a^\dagger|n\rangle\\
=&\frac{\eta\hbar\Omega}{2}\sqrt{n+1}
\end{aligned}$$

(b)

For spontanious decay, when the emitted photon is θ from the axis of the trap, the probability to go from n to $n+1$ is $\eta^2\cos^2\theta(n+1)$ Assuming isotropic emission patter (i.e. non-polarized) the averate probability is

$$\begin{aligned}
p_{n,n+1} =&\frac{1}{4\pi}\int_0^\pi\mathrm{d}\theta\int_0^{2\pi}\sin\theta\mathrm{d}\phi\eta^2\cos^2\theta(n+1)\\
=&\frac{\eta^2(n+1)}{2}\int_{-1}^1\mathrm{d}zz^2\\
=&\frac{\eta^2(n+1)}{3}
\end{aligned}$$

Similarly, the probability of going to $n-1$ is

$$p_{n,n-1} =\frac{\eta^2n}{3}$$

To the lowest order in η, the rate at which $|n,g\rangle\to|n,e\rangle\to|n-1,g\rangle$ happens is

$$\frac{n\Omega^2\eta^2}{3}\frac{\Gamma}{\Gamma^2+4\delta^2}$$

For $|n,g\rangle\to|n-1,e\rangle\to|n-1,g\rangle$

$$\begin{aligned}
&n\Omega^2\eta^2\frac{\Gamma}{\Gamma^2+4(\delta+\nu)^2}\\
A_- =&\Omega^2\Gamma\eta^2\left(\frac{1}{\Gamma^2+4(\delta+\nu)^2}+\frac{1}{3}\frac{1}{\Gamma^2+4\delta^2}\right)
\end{aligned}$$

Similarly

$$A_+ =\Omega^2\Gamma\eta^2\left(\frac{1}{\Gamma^2+4(\delta-\nu)^2}+\frac{1}{3}\frac{1}{\Gamma^2+4\delta^2}\right)$$

(c)

$$\begin{aligned}
\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} =& \sum_{n=0}^{\infty} n\frac{\mathrm{d}p_n}{\mathrm{d}t}\\
=& \sum_{n=0}^{\infty} n(nA_+p_{n-1} + (n+1)A_-p_{n+1} - (n+1)A_+p_n - nA_-p_n)\\
=& \sum_{n=0}^{\infty} \left(n^2A_+p_{n-1} + n(n+1)A_-p_{n+1} - n(n+1)A_+p_n - n^2A_-p_n\right)\\
=& \sum_{n=0}^{\infty} \left((n+1)^2A_+p_n + (n-1)nA_-p_n - n(n+1)A_+p_n - n^2A_-p_n\right)\\
=& \sum_{n=0}^{\infty} ((n+1)A_+p_n - nA_-p_n)\\
=& A_+ - (A_- - A_+)\langle n\rangle
\end{aligned}$$

The solution is a exponential decay to $\langle n\rangle = \dfrac{A_+}{A_- - A_+}$ with decay rate $A_- - A_+$. For driving cooling sideband in the resolved limit

$$\begin{aligned}
A_- \approx& \frac{\Omega^2\eta^2}{\Gamma}\\
A_+ \approx& \frac{7\Omega^2\Gamma\eta^2}{48\nu^2}
\end{aligned}$$

Final temperature

$$\begin{aligned}
\langle n\rangle_\infty \approx& \frac{A_+}{A_-}\\
=& \frac{7\Gamma^2}{48\nu^2}\\
T_\infty =& \langle n\rangle_\infty \frac{\hbar\nu}{k_B}\\
\approx& \frac{7\hbar\Gamma^2}{48k_B\nu}
\end{aligned}$$

Decay time

$$\begin{aligned}
\tau \approx& \frac{1}{A_-}\\
\approx& \frac{\Gamma}{\Omega^2\eta^2}
\end{aligned}$$

(d)

For the narrow line

$$\begin{aligned}\eta =&k\sqrt{\frac{\hbar}{2m\nu}}\\ =&0.022\\ T_\infty \approx&\frac{7\hbar\Gamma^2}{48k_B\nu}\\ =&0.11\mathrm{aK}\\ \tau \approx&\frac{\Gamma}{\Omega^2\eta^2}\\ =&22\mathrm{hr}\end{aligned}$$

For broadened line

$$\begin{aligned}T_\infty \approx&\frac{7\hbar\Gamma^2}{48k_B\nu}\\ =&7.0\mathrm{nK}\\ \tau \approx&\frac{\Gamma}{\Omega^2\eta^2}\\ =&0.32\mathrm{s}\end{aligned}$$

3. Optical dipole trap

(a)

Phase shift

$$\begin{aligned}\Delta\phi_D =&\Delta ndk\\ =&\frac{n}{2}\alpha dk\\ =&\frac{\dot{n}}{2}\alpha dkt\end{aligned}$$

Frequency shift

$$\Delta\omega =\frac{\dot{n}}{2}\alpha dk$$

(b)

Power loss

$$\begin{aligned}\Delta P =&P\frac{\Delta\omega}{\omega}\\ =&\frac{\dot{n}\alpha Vk}{2\omega}\varepsilon_0 c\left\langle E^2\right\rangle\\ =&\frac{\dot{n}\alpha V}{2}\varepsilon_0\left\langle E^2\right\rangle\end{aligned}$$

Total energy

$$\Delta E_{EM} =\frac{N\alpha}{2}\varepsilon_0\left\langle E^2\right\rangle$$

(c)

Start shift

$$\begin{aligned}U_{Stark} =&\frac{\varepsilon_0\langle pE\rangle}{2}\\=&\frac{\varepsilon_0\alpha\left\langle E^2\right\rangle}{2}\end{aligned}$$

Total energy

$$\begin{aligned}\Delta E_{Stark} =&\frac{N\varepsilon_0\alpha\left\langle E^2\right\rangle}{2}\\=&\Delta E_{EM}\end{aligned}$$