

1. The Dressed Atom

- (a)
- (b)
- (c)
- (d)
- (e)
- (f)

2. Sideband Cooling

- (a)

Interaction term

$$\begin{aligned} H_I &= \hbar\Omega(\sigma_+ + \sigma_-) \cos(k\hat{x} - \omega t) \\ &= \hbar\Omega(\sigma_+ + \sigma_-) \cos(\eta(a + a^\dagger) - \omega t) \\ &= \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) \left(e^{i(\eta(a+a^\dagger) - \omega t)} + e^{-i(\eta(a+a^\dagger) - \omega t)} \right) \end{aligned}$$

In the Lamb-Dicke limit

$$\begin{aligned} H_I &= \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) \left(e^{i\eta(a+a^\dagger)} e^{-i\omega t} + e^{-i\eta(a+a^\dagger)} e^{i\omega t} \right) \\ &= \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) \left((1 + i\eta(a + a^\dagger)) e^{-i\omega t} + (1 - i\eta(a + a^\dagger)) e^{i\omega t} \right) \\ &= \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) (e^{-i\omega t} + e^{i\omega t}) + i\eta \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)(a + a^\dagger) (e^{-i\omega t} - e^{i\omega t}) \end{aligned}$$

In the interaction picture

$$\begin{aligned} \sigma'_\pm &= e^{iH_0 t/\hbar} \sigma_\pm e^{-iH_0 t/\hbar} \\ &= \sigma_\pm e^{i\omega_0 t} \\ a' &= e^{iH_0 t/\hbar} a e^{-iH_0 t/\hbar} \\ &= a e^{-i\nu t} \\ a'^\dagger &= e^{iH_0 t/\hbar} a^\dagger e^{-iH_0 t/\hbar} \\ &= a^\dagger e^{i\nu t} \end{aligned}$$

In the rotating wave approximation

$$\begin{aligned} H'_I &= e^{iH_0 t/\hbar} H_I e^{-iH_0 t/\hbar} \\ &= \frac{\hbar\Omega}{2}(\sigma_+ e^{-i\delta t} + \sigma_- e^{i\delta t}) + i\eta \frac{\hbar\Omega}{2} (\sigma_+ a e^{-i(\delta+\nu)t} - \sigma_- a^\dagger e^{i(\delta+\nu)t}) \\ &\quad + i\eta \frac{\hbar\Omega}{2} (\sigma_+ a^\dagger e^{-i(\delta-\nu)t} - \sigma_- a e^{i(\delta-\nu)t}) \end{aligned}$$

When the light is in resonance with the sideband, the Rabi frequencies

$$\begin{aligned}
 \Omega_{n,n-1} &= \left| \langle n-1, e | i\eta \frac{\hbar\Omega}{2} (\sigma_+ a - \sigma_- a^\dagger) | n, g \rangle \right| \\
 &= \frac{\eta\hbar\Omega}{2} \langle n-1 | a | n \rangle \\
 &= \frac{\eta\hbar\Omega}{2} \sqrt{n} \\
 \Omega_{n,n+1} &= \left| \langle n+1, e | i\eta \frac{\hbar\Omega}{2} (\sigma_+ a^\dagger - \sigma_- a) | n, g \rangle \right| \\
 &= \frac{\eta\hbar\Omega}{2} \langle n+1 | a^\dagger | n \rangle \\
 &= \frac{\eta\hbar\Omega}{2} \sqrt{n+1}
 \end{aligned}$$

(b)

For spontaneous decay, when the emitted photon is θ from the axis of the trap, the probability to go from n to $n+1$ is $\eta^2 \cos^2 \theta (n+1)$. Assuming isotropic emission pattern (i.e. non-polarized) the average probability is

$$\begin{aligned}
 p_{n,n+1} &= \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \eta^2 \cos^2 \theta (n+1) \\
 &= \frac{\eta^2 (n+1)}{2} \int_{-1}^1 dz z^2 \\
 &= \frac{\eta^2 (n+1)}{3}
 \end{aligned}$$

Similarly, the probability of going to $n-1$ is

$$p_{n,n-1} = \frac{\eta^2 n}{3}$$

To the lowest order in η , the rate at which $|n, g\rangle \rightarrow |n, e\rangle \rightarrow |n-1, g\rangle$ happens is

$$\frac{n\Omega^2\eta^2}{3} \frac{\Gamma}{\Gamma^2 + 4\delta^2}$$

For $|n, g\rangle \rightarrow |n-1, e\rangle \rightarrow |n-1, g\rangle$

$$\begin{aligned}
 A_- &= \Omega^2 \Gamma \eta^2 \left(\frac{1}{\Gamma^2 + 4(\delta + \nu)^2} + \frac{1}{3} \frac{1}{\Gamma^2 + 4\delta^2} \right)
 \end{aligned}$$

Similarly

$$A_+ = \Omega^2 \Gamma \eta^2 \left(\frac{1}{\Gamma^2 + 4(\delta - \nu)^2} + \frac{1}{3} \frac{1}{\Gamma^2 + 4\delta^2} \right)$$

(c)

$$\begin{aligned}
 \frac{d\langle n \rangle}{dt} &= \sum_{n=0}^{\infty} n \frac{dp_n}{dt} \\
 &= \sum_{n=0}^{\infty} n(nA_{+p_{n-1}} + (n+1)A_{-p_{n+1}} - (n+1)A_{+p_n} - nA_{-p_n}) \\
 &= \sum_{n=0}^{\infty} (n^2A_{+p_{n-1}} + n(n+1)A_{-p_{n+1}} - n(n+1)A_{+p_n} - n^2A_{-p_n}) \\
 &= \sum_{n=0}^{\infty} ((n+1)^2A_{+p_n} + (n-1)nA_{-p_n} - n(n+1)A_{+p_n} - n^2A_{-p_n}) \\
 &= \sum_{n=0}^{\infty} ((n+1)A_{+p_n} - nA_{-p_n}) \\
 &= A_{+} - (A_{-} - A_{+})\langle n \rangle
 \end{aligned}$$

The solution is an exponential decay to $\langle n \rangle = \frac{A_{+}}{A_{-} - A_{+}}$ with decay rate $A_{-} - A_{+}$. For driving cooling sideband in the resolved limit

$$\begin{aligned}
 A_{-} &\approx \frac{\Omega^2 \eta^2}{\Gamma} \\
 A_{+} &\approx \frac{7\Omega^2 \Gamma \eta^2}{48\nu^2}
 \end{aligned}$$

Final temperature

$$\begin{aligned}
 \langle n \rangle_{\infty} &\approx \frac{A_{+}}{A_{-}} \\
 &= \frac{7\Gamma^2}{48\nu^2} \\
 T_{\infty} &= \langle n \rangle_{\infty} \frac{\hbar\nu}{k_B} \\
 &\approx \frac{7\hbar\Gamma^2}{48k_B\nu}
 \end{aligned}$$

Decay time

$$\begin{aligned}
 \tau &\approx \frac{1}{A_{-}} \\
 &\approx \frac{\Gamma}{\Omega^2 \eta^2}
 \end{aligned}$$

(d)

For the narrow line

$$\begin{aligned}\eta &= k \sqrt{\frac{\hbar}{2m\nu}} \\ &= 0.022 \\ T_\infty &\approx \frac{7\hbar\Gamma^2}{48k_B\nu} \\ &= 0.11 \text{aK} \\ \tau &\approx \frac{\Gamma}{\Omega^2\eta^2} \\ &= 22 \text{hr}\end{aligned}$$

For broadened line

$$\begin{aligned}T_\infty &\approx \frac{7\hbar\Gamma^2}{48k_B\nu} \\ &= 7.0 \text{nK} \\ \tau &\approx \frac{\Gamma}{\Omega^2\eta^2} \\ &= 0.32 \text{s}\end{aligned}$$

3. Optical dipole trap

(a)

(b)

(c)