1. Classical Model of the Light Force

(a) Time averaged force

$$\begin{aligned}\left\langle\vec{F}\right\rangle =&\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))\nabla(E_0\cos(\omega t+\theta))\rangle\\ =&\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))(\cos(\omega t+\theta)\nabla E_0-E_0\sin(\omega t+\theta)\nabla\theta)\rangle\\ =&\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})(u\nabla E_0+vE_0\nabla\theta)\end{aligned}$$

(b) The potential picture

$$\begin{aligned}\langle U\rangle =&-\left\langle\vec{p}\cdot\vec{E}\right\rangle\\ =&-\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))E_0\cos(\omega t+\theta)\rangle\\ =&-\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})uE_0\\ \left\langle\vec{F}\right\rangle =&\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})u\nabla E_0\end{aligned}$$

(c) Dipole moment of electron

Let $\vec{r}=\hat{\epsilon}\tilde{r}_0\mathrm{e}^{\mathrm{i}\omega t}$

$$\begin{aligned}&-e\hat{\epsilon}E_0\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\theta}\\ =&\left(-m\omega^2+\mathrm{i}\omega\gamma+m\omega_0^2\right)\hat{\epsilon}\tilde{r}_0\mathrm{e}^{\mathrm{i}\omega t}\\ \tilde{r}_0 =&\frac{eE_0\mathrm{e}^{\mathrm{i}\theta}}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\\ \tilde{\vec{p}} =&-\hat{\epsilon}\mathrm{e}^{\mathrm{i}\omega t}\frac{e^2E_0\mathrm{e}^{\mathrm{i}\theta}}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\end{aligned}$$

Real part

$$\begin{aligned}\vec{p} =&-\hat{\epsilon}\Re\left(\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\theta}\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)\\ =&-\hat{\epsilon}\left(\cos(\omega t+\theta)\Re\left(\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)-\sin(\omega t+\theta)\Im\left(\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)\right)\end{aligned}$$

Where

$$\begin{aligned}&\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\\ =&e^2E_0\frac{1}{m(\omega^2-\omega_0^2)-\mathrm{i}\omega\gamma}\\ \approx&\frac{e^2E_0}{m\omega_0}\frac{1}{2\delta-\mathrm{i}\Gamma}\\ =&\frac{e^2E_0}{m\omega_0}\frac{2\delta+\mathrm{i}\Gamma}{4\delta^2+\Gamma^2}\end{aligned}$$

Compare to the definition of u and v

$$\begin{aligned}
u =& -\Re\left(\frac{e^2E_0}{m\omega^2 - \mathrm{i}\omega\gamma - m\omega_0^2}\right)\\
=& -\frac{e^2E_0}{m\omega_0}\frac{2\delta}{4\delta^2+\Gamma^2}\\
v =& -\Im\left(\frac{e^2E_0}{m\omega^2 - \mathrm{i}\omega\gamma - m\omega_0^2}\right)\\
=& -\frac{e^2E_0}{m\omega_0}\frac{\Gamma}{4\delta^2+\Gamma^2}
\end{aligned}$$

Force

$$\begin{aligned}
\left\langle\vec{F}\right\rangle =& -\frac{e^2}{2m\omega_0}\frac{2E_0\nabla E_0\delta + \Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2}\\
=& -\frac{e^2}{2m\omega_0}\frac{\delta\nabla E_0^2 + \Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2}
\end{aligned}$$

(d) Force on a two-level atom

Since $\omega_R \propto E_0$

$$\begin{aligned}
\left\langle\vec{F}\right\rangle \propto& -\frac{\delta\nabla\omega_R^2 + \Gamma\omega_R^2\nabla\theta}{4\delta^2+\Gamma^2}\\
\propto& F_{quantum}
\end{aligned}$$

2. Master equation for a damped optical cavity

(a)

(b)

(c)