1.

(a)

$$
\begin{aligned}
&\left[a^\dagger b - ab^\dagger, a^\dagger a + b^\dagger b\right] \\
=&\left[a^\dagger b, a^\dagger a\right] - \left[ab^\dagger, a^\dagger a\right] + \left[a^\dagger b, b^\dagger b\right] - \left[ab^\dagger, b^\dagger b\right] \\
=&a^\dagger\left[a^\dagger, a\right]b - \left[a, a^\dagger\right]ab^\dagger + a^\dagger\left[b, b^\dagger\right]b - ab^\dagger\left[b^\dagger, b\right] \\
=&-a^\dagger b - ab^\dagger + a^\dagger b + ab^\dagger \\
=&0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left[B, n_a + n_b\right] =&\left[\exp\left(\theta\left(a^\dagger b - ab^\dagger\right)\right), n_a + n_b\right] \\
=&0 \\
B^\dagger =&\exp\left(\theta\left(a^\dagger b - ab^\dagger\right)^\dagger\right) \\
=&\exp\left(-\theta\left(a^\dagger b - ab^\dagger\right)\right) \\
=&B^{-1}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\exp\left(\theta A\right)B\exp\left(-\theta A\right) =&\sum_{nm}(-1)^m\frac{\theta^{n+m}A^n BA^m}{n!m!} \\
=&\sum_N\sum_{m=0}^N(-1)^m\frac{\theta^N A^{N-m}BA^m}{(N-m)!m!} \\
=&\sum_N\frac{\theta^N}{N!}\sum_{m=0}^N(-1)^m\frac{N!A^{N-m}BA^m}{(N-m)!m!} \\
=&\sum_N\frac{\theta^N}{N!}[A, B]_N
\end{aligned}
$$

where $[A, B]_N$ is defined as $[A, B]_N = \left[A, [A, B]_{N-1}\right]$ and $[A, B]_0 = B$

$$\left[\left(a^\dagger b - ab^\dagger\right), a\right]_N = \begin{cases} (-1)^{N/2} a & (2 \mid N) \\ (-1)^{(N+1)/2} b & (2 \nmid N) \end{cases}$$

$$\left[\left(a^\dagger b - ab^\dagger\right), b\right]_N = \begin{cases} (-1)^{N/2} b & (2 \mid N) \\ (-1)^{(N-1)/2} a & (2 \nmid N) \end{cases}$$

$$BaB^{-1} = \sum_N \frac{\theta^N}{N!} \left[\left(a^\dagger b - ab^\dagger\right), a\right]_N$$

$$= \sum_n \frac{\theta^{2n}}{(2n)!} \left[\left(a^\dagger b - ab^\dagger\right), a\right]_{2n} + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} \left[\left(a^\dagger b - ab^\dagger\right), a\right]_{2n+1}$$

$$= \sum_n \frac{\theta^{2n}}{(2n)!} (-1)^n a + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} (-1)^{n+1} b$$

$$= \cos\theta\, a - \sin\theta\, b$$

$$BbB^{-1} = \sum_N \frac{\theta^N}{N!} \left[\left(a^\dagger b - ab^\dagger\right), b\right]_N$$

$$= \sum_n \frac{\theta^{2n}}{(2n)!} \left[\left(a^\dagger b - ab^\dagger\right), b\right]_{2n} + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} \left[\left(a^\dagger b - ab^\dagger\right), b\right]_{2n+1}$$

$$= \sum_n \frac{\theta^{2n}}{(2n)!} (-1)^n b + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} (-1)^n a$$

$$= \cos\theta\, b + \sin\theta\, a$$

$$B|0, \alpha\rangle = B e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0, 0\rangle$$

$$= e^{-|\alpha|^2/2} B e^{\alpha a^\dagger} B^{-1} B |0, 0\rangle$$

$$= e^{-|\alpha|^2/2} e^{\alpha\left(\cos\theta a^\dagger - \sin\theta b^\dagger\right)} |0, 0\rangle$$

$$= e^{-|\alpha|^2/2} e^{\alpha \cos\theta a^\dagger} e^{-\alpha \sin\theta b^\dagger} |0, 0\rangle$$

$$= |-\alpha \sin\theta, \alpha \cos\theta\rangle$$

(c)

$$s_x = a^\dagger b + ab^\dagger$$
$$s_y = -\mathrm{i}\left(a^\dagger b - ab^\dagger\right)$$
$$B = e^{\mathrm{i}\theta s_y}$$

which is a rotation around y

$$(n_a + n_b)^2$$
$$= s_z^2 + 4a^\dagger a b^\dagger b$$
$$= s_z^2 + 4s^+ s^-$$
$$= s_x^2 + s_y^2 + s_z^2$$
$$= S^2$$

which is the total spin

$$
\begin{aligned}
[s_x, s_y] &= \left[a^\dagger b + ab^\dagger, -i\left(a^\dagger b - ab^\dagger\right)\right] \\
&= -i\left[a^\dagger b + ab^\dagger, a^\dagger b - ab^\dagger\right] \\
&= 2i\left[a^\dagger b, ab^\dagger\right] \\
&= 2ia^\dagger a\left[b, b^\dagger\right] + 2i\left[a^\dagger, a\right]b^\dagger b \\
&= 2ia^\dagger a - 2ib^\dagger b \\
&= 2is_z \\
[s_y, s_z] &= \left[-i\left(a^\dagger b - ab^\dagger\right), a^\dagger a - b^\dagger b\right] \\
&= -ia^\dagger\left[a^\dagger, a\right]b + ia^\dagger\left[b, b^\dagger\right]b + i\left[a, a^\dagger\right]ab^\dagger - iab^\dagger\left[b^\dagger, b\right] \\
&= 2is_x \\
[s_z, s_x] &= \left[a^\dagger a - b^\dagger b, a^\dagger b + ab^\dagger\right] \\
&= a^\dagger\left[a, a^\dagger\right]b + \left[a^\dagger, a\right]ab^\dagger - a^\dagger\left[b^\dagger, b\right]b - ab^\dagger\left[b, b^\dagger\right] \\
&= 2a^\dagger b - 2ab^\dagger \\
&= 2is_y
\end{aligned}
$$

(d)

$$
\begin{aligned}
B|0, n\rangle &= B\frac{a^{\dagger n}}{n!}|0, 0\rangle \\
&= \frac{\left(a^\dagger - b^\dagger\right)^n}{2^{n/2}n!}|0, 0\rangle \\
&= \sum_i \frac{a^{\dagger i}b^{\dagger n-i}}{2^{n/2}i!(n-i)!}|0, 0\rangle \\
&= \sum_i \frac{|n-i, i\rangle}{\sqrt{2^n i!(n-i)!}}
\end{aligned}
$$

The state(s) with the largest amplitude is $|n/2, n/2\rangle$ (when n is even) or $|(n+1)/2, (n-1)/2\rangle$ and $|(n-1)/2, (n+1)/2\rangle$ when n is odd

The variance of the distribution is $\dfrac{n}{4}$ so the relative width is getting narrower for larger n although the absolute width is getting wider.

2.

(a)

$$
\begin{aligned}
(\Delta n')^2 =& \left\langle n'^2 \right\rangle - \langle n' \rangle^2 \\
=& |t|^4 (\Delta n)^2 \\
=& |t|^4 |\alpha|^2 \\
\langle n' \rangle =& |t|^2 |\alpha|^2 \\
>& |t|^4 |\alpha|^2 \qquad (\text{when } 0 < |t| < 1 \text{ and } \alpha \neq 0)
\end{aligned}
$$

(b)

As shown in problem one.

$$
\begin{aligned}
\rho' =& \mathrm{Tr}_b\left(B|0,\alpha\rangle\langle 0,\alpha|B^\dagger\right) \\
=& \mathrm{Tr}_b(|r\alpha, t\alpha\rangle\langle r\alpha, t\alpha|) \\
=& |t\alpha\rangle\langle t\alpha|
\end{aligned}
$$

3.

(a)

After the first beam splitter

$$
|\psi_1\rangle = \frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}}
$$

After the phase shift

$$
|\psi_2\rangle = \frac{\mathrm{e}^{-\mathrm{i}\phi/2}|0,1\rangle + \mathrm{e}^{\mathrm{i}\phi/2}|1,0\rangle}{\sqrt{2}}
$$

After the second beam splitter

$$
\begin{aligned}
|\psi_3\rangle =& \frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2}}\left(\frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}}\right) + \frac{\mathrm{e}^{\mathrm{i}\phi/2}}{\sqrt{2}}\left(\frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}}\right) \\
=& \frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{2}(|0,1\rangle - |1,0\rangle) + \frac{\mathrm{e}^{\mathrm{i}\phi/2}}{2}(|0,1\rangle + |1,0\rangle) \\
=& \cos\frac{\phi}{2}|0,1\rangle + \mathrm{i}\sin\frac{\phi}{2}|1,0\rangle \\
P_b =& \cos^2\frac{\phi}{2} \\
V =& \frac{\max(P_b) - \min(P_b)}{\max(P_b) + \min(P_b)} \\
=& \frac{1-0}{1+0} \\
=& 1
\end{aligned}
$$

4

(b)

$$|\psi_0\rangle = |0,1,0\rangle$$

$$|\psi_1\rangle = \frac{|0,0,1\rangle + |0,1,0\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\phi/2}|0,0,1\rangle + \mathrm{e}^{-\mathrm{i}\phi/2}(\cos\theta|0,1,0\rangle + \sin\theta|1,0,0\rangle)\right)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\phi/2}\frac{|0,0,1\rangle + |0,1,0\rangle}{\sqrt{2}} + \cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}\frac{|0,1,0\rangle - |0,0,1\rangle}{\sqrt{2}} + \sin\theta\mathrm{e}^{-\mathrm{i}\phi/2}|1,0,0\rangle\right)$$

$$= \frac{\left(\mathrm{e}^{\mathrm{i}\phi/2} - \cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}\right)|0,0,1\rangle - \left(\mathrm{e}^{\mathrm{i}\phi/2} + \cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}\right)|0,1,0\rangle}{2} + \frac{\sin\theta\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2}}|1,0,0\rangle$$

$$P_b = \left|\frac{\mathrm{e}^{\mathrm{i}\phi/2} + \cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}}{2}\right|^2$$

$$= \left(\frac{1+\cos\theta}{2}\cos\frac{\phi}{2}\right)^2 + \left(\frac{1-\cos\theta}{2}\sin\frac{\phi}{2}\right)^2$$

$$= \frac{1+\cos^2\theta}{4} + \frac{\cos\theta}{2}\cos\phi$$

$$V = \frac{2\cos\theta}{1+\cos^2\theta}$$

(c)

$$|\psi_0\rangle = |0,0,1,0\rangle$$

$$|\psi_1\rangle = \frac{|0,1,0,0\rangle + |0,0,1,0\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\phi/2}|0,1,0,0\rangle + \mathrm{e}^{-\mathrm{i}\phi/2}|0,0,1,0\rangle\right)$$

$$|\psi_3\rangle = \frac{\mathrm{e}^{\mathrm{i}\phi/2}}{\sqrt{2}}(\cos\theta|0,1,0,0\rangle + \sin\theta|1,0,0,0\rangle) + \frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2}}(\cos\theta'|0,0,1,0\rangle + \sin\theta'|0,0,0,1\rangle)$$

$$= \frac{\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta}{\sqrt{2}}|1,0,0,0\rangle + \frac{\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'}{\sqrt{2}}|0,0,0,1\rangle + \frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta}{\sqrt{2}}|0,1,0,0\rangle + \frac{\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{\sqrt{2}}|0,0,1,0\rangle$$

$$|\psi_4\rangle = \frac{\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta}{\sqrt{2}}|1,0,0,0\rangle + \frac{\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'}{\sqrt{2}}|0,0,0,1\rangle$$

$$+ \frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta}{2}(|0,0,1,0\rangle + |0,1,0,0\rangle) + \frac{\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{2}(|0,0,1,0\rangle - |0,1,0,0\rangle)$$

$$= \frac{\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta}{\sqrt{2}}|1,0,0,0\rangle + \frac{\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'}{\sqrt{2}}|0,0,0,1\rangle$$

$$+ \frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta + \mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{2}|0,0,1,0\rangle + \frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta - \mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{2}|0,1,0,0\rangle$$

Maximum and minimum probabilities

$$\max P_b = \frac{1}{4}(\cos\theta + \cos\theta')^2$$
$$\min P_b = \frac{1}{4}(\cos\theta - \cos\theta')^2$$
$$V = \frac{2\cos\theta\cos\theta'}{\cos^2\theta + \cos^2\theta'}$$

Changing to condition probability does not change the visibility.

(d)

$$|\psi_0\rangle = |0, 0, \alpha, 0\rangle$$
$$|\psi_1\rangle = |0, \frac{\alpha}{2}, \frac{\alpha}{2}, 0\rangle$$
$$|\psi_2\rangle = |0, e^{i\phi/2}\frac{\alpha}{2}, e^{-i\phi/2}\frac{\alpha}{2}, 0\rangle$$
$$|\psi_3\rangle = |e^{i\phi/2}\sin\theta\frac{\alpha}{2}, e^{i\phi/2}\cos\theta\frac{\alpha}{2}, e^{-i\phi/2}\cos\theta'\frac{\alpha}{2}, e^{-i\phi/2}\sin\theta'\frac{\alpha}{2}\rangle$$
$$|\psi_4\rangle = |e^{i\phi/2}\sin\theta\frac{\alpha}{2}, e^{i\phi/2}\cos\theta\frac{\alpha}{2} - e^{-i\phi/2}\cos\theta'\frac{\alpha}{2}, e^{i\phi/2}\cos\theta\frac{\alpha}{2} + e^{-i\phi/2}\cos\theta'\frac{\alpha}{2}, e^{-i\phi/2}\sin\theta'\frac{\alpha}{2}\rangle$$

Maximum and minimum intensity

$$\max P_b = \frac{\alpha^2}{4}(\cos\theta + \cos\theta')^2$$
$$\min P_b = \frac{\alpha^2}{4}(\cos\theta - \cos\theta')^2$$
$$V = \frac{2\cos\theta\cos\theta'}{\cos^2\theta + \cos^2\theta'}$$

4.

(a)

From the definition of entangled state, the state is a product state if Schmidt number is 1. If a product state has a Schmidt number greater than 1

$$|\psi_A\rangle|\psi_B\rangle = \sum_k \lambda_k |k_A\rangle|k_B\rangle$$
$$|\psi_A\rangle = \sum_k \lambda_k \langle\psi_B|k_B\rangle|k_A\rangle$$
$$|\psi_B\rangle = \sum_k \lambda_k \langle\psi_A|k_A\rangle|k_B\rangle$$
$$|\psi_A\rangle|\psi_B\rangle = \sum_{k_1,k_2} \lambda_{k_1}\langle\psi_B|k_{1B}\rangle|k_{1A}\rangle\lambda_{k_2}\langle\psi_A|k_{2A}\rangle|k_{2B}\rangle$$
$$\neq \sum_k \lambda_k |k_A\rangle|k_B\rangle$$

(b)

With the same local unitary transformation on the right hand side, it is still a valid expansion of the new state and therefore the Schmidt number does not change.

(c)

State	Schmidt number
ϕ_1	3
ϕ_2	1
ϕ_3	2
ϕ_4	2

(d)

Schmidt number is 2.

$$\begin{aligned}
|\phi_1\rangle &= B|\psi\rangle \\
&= \frac{1}{\sqrt{n!(n-1)!2^n}}\left(\left(a^\dagger+b^\dagger\right)^n\left(a^\dagger-b^\dagger\right)^{n-1}+\left(a^\dagger+b^\dagger\right)^{n-1}\left(a^\dagger-b^\dagger\right)^n\right)|0,0\rangle \\
&= \frac{1}{\sqrt{n!(n-1)!2^{n-1}}}\left(a^\dagger+b^\dagger\right)^{n-1}a^\dagger\left(a^\dagger-b^\dagger\right)^{n-1}|0,0\rangle \\
&= \frac{1}{\sqrt{n!(n-1)!2^{n-1}}}\left(a^{\dagger 2}-b^{\dagger 2}\right)^{n-1}a^\dagger|0,0\rangle
\end{aligned}$$

Therefore the Schmidt number is $n-1$. This is a better measure since these are the states that interfere later.

The Schmidt number for a coherent state after a beam splitter is 1 since it is still a product state.

(e)