MIT 8.422 Atomic Physics II — *Spring 2015*
Prof. Martin Zwierlein — March 2, 2015

Assignment #3

Due: Monday, March 9, 2015
TA: Kristin Beck / 26-223 / kbeck@mit.edu
Office Hours: Mar 6th (Friday) & Mar 9nd (Monday), 9am - 11am.

1. **Generation of Squeezed States by Two-Photon Interactions**

   Consider a mode $(\vec{k}, \vec{\varepsilon})$ with wavevector $\vec{k}$ and polarization vector $\vec{\varepsilon}$ of the electromagnetic field with frequency ω whose Hamiltonian H is given by

   $$H = \hbar\omega a^\dagger a + i\hbar\Lambda\left[(a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right] \tag{1}$$

   where $a^\dagger$ and a are the creation and annihilation operators of the mode.

   The first term of Eq. 1 is the energy of the mode for the free field. The second term describes a two-photon interaction process such as parametric amplification (a classical wave of frequency 2ω generating two photons with frequency ω). Λ is a real quantity characterizing the strength of the interaction.

   In this problem, you will show that this Hamiltonian produces squeezed vacuum and explore how it acts on coherent states.

   (a) Write the equation of motion for $a(t)$ using the Heisenberg picture. Take

   $$a(t) = b(t)e^{-i\omega t}. \tag{2}$$

   What are the equations of motion for $b(t)$ and $b^\dagger(t)$?

   (b) The contribution of the mode $(\vec{k}, \vec{\varepsilon})$ to the electric field is

   $$\vec{E}(\vec{r}, t) = i\mathcal{E}_\omega\vec{\varepsilon}\left[a(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger(t)e^{-i\vec{k}\cdot\vec{r}}\right] \tag{3}$$

   where $a(t)$ is the solution of Eq. 2. Show that

   $$b_P(t) = \frac{b(t) + b^\dagger(t)}{2} \qquad \text{and} \qquad b_Q(t) = \frac{b(t) - b^\dagger(t)}{2i} \tag{4}$$

   represent physically two quadrature components of the field. ($b(t)$ was defined Eq. 2.) Find equations of motion for $b_P(t)$ and $b_Q(t)$ and give their solutions, assuming that $b_P(0)$ and $b_Q(0)$ are known. Give the corresponding solutions for $b(t)$ and $b^\dagger(t)$ and express them in terms of $b(0)$ and $b^\dagger(0)$.

   (c) Assume that at $t = 0$, the electromagnetic field is in the vacuum state. Calculate the mean number of photons $\langle N\rangle$ in the mode $(\vec{k}, \vec{\varepsilon})$ at time t and the dispersion $\Delta b_P(t)$ and $\Delta b_Q(t)$ of the two quadrature components of the field. Explain the results.

   (d) In problem set 1, we defined squeezed vacuum with parameter ϵ as

   $$|0_\epsilon\rangle = S(\epsilon)|0\rangle = \exp\left[\frac{1}{2}\epsilon^* a^2 - \frac{1}{2}\epsilon a^{\dagger 2}\right]|0\rangle = \frac{1}{\sqrt{\cosh\epsilon}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}(\tanh\epsilon)^n|2n\rangle \tag{5}$$

   Show that the two-photon interaction in Eq. 1 produces this state (and this operator) when you apply it for a time $t = t_0$. What is ϵ?

(e) Plot the Q function of the states that arise at time t under this two-photon interaction.

(i) Squeezed vacuum: $Q_1(\alpha) = |\langle\alpha|S(\epsilon)|0\rangle|^2$

(ii) A displaced squeezed state: $Q_2(\alpha) = |\langle\alpha|D(\beta)S(\epsilon)|0\rangle|^2$

(iii) A squeezed coherent state: $Q_3(\alpha) = |\langle\alpha|S(\epsilon)D(\beta)|0\rangle|^2$

Here, S is the squeezing operator defined in Eq. 5 and $D = \exp\left[\alpha a^\dagger - \alpha^* a\right]$ is the displacement operator. Compare plots made with $\epsilon = 0.2$, 1.2 and 4. Compare the displaced squeezed state $Q_2(\alpha)$ and the squeezed coherent state $Q_3(\alpha)$. Does the two-photon interaction in Eq. 1 create create amplitude or phase squeezing?

A note and some hints: There will be some ugly math for this question. We want you to understand the states that are produced by this squeezing, so you can just compute $Q(\alpha)$ numerically to plot it.

To compute it numerically, you can make use of the following relations:

$$\begin{aligned} S(\epsilon)|0\rangle &= \frac{1}{\pi}\frac{e^{\epsilon/2}}{\sqrt{e^{2\epsilon}-1}}\int_{-\infty}^{\infty} d\alpha\, e^{-[\alpha^2/(e^{2\epsilon}-1)]}|\alpha\rangle \\ D(\gamma)S(\epsilon) &= S(\epsilon)D(\gamma_+) \\ S(\epsilon)D(\gamma) &= D(\gamma_-)S(\epsilon) \end{aligned}$$

where $\gamma_\pm(\epsilon) = (\cosh\epsilon)\gamma \pm (\sinh\epsilon)\gamma^*$. Also note that ϵ is real for our interaction.

2. **Noise Properties of Squeezed Vacuum**

Ths problem explores the noise properties of squeezed vacuum, in particular the squeezed vacuum we generated in question 1 by applying the two-photon interaction Hamiltonian Eq. 1 for a time $t = t_0$. You will need the results of parts (a)-(c) of that problem, in particular the transformations under squeezing of b and $b^\dagger$, of the quadrature components b_P and b_Q and of their dispersions Δb_P and Δb_Q. You can express your answers in terms of the ϵ you found in 1(d), or leave it in terms of the parameters in Eq. 1.

(a) Calculate the photon number variance $\langle\Delta N^2\rangle$ of this squeezed vacuum state. How is it related to the variance of the two quadrature components of the field $\langle\Delta b_P^2\rangle$ and $\langle\Delta b_Q^2\rangle$? Take the limit of large squeezing $\epsilon \gg 1$. How does the photon number variance scale with the mean photon number of the squeezed vacuum state? Compare this scaling to a strong coherent state (classical laser beam).

(b) One way to quantify the quality of the squeezed state is the degree of squeezing, which we calculate as the logarithm of variance of the quadrature with increased noise

$$10\log_{10}(4\langle\Delta a_P^2\rangle) \tag{6}$$

for minimum uncertainty states. (This has units of decibels). Calculate the degree of squeezing for the squeezed vacuum state. What is the degree of squeezing when the ratio of the variances differs by a factor of 2? Now, calculate the degree of squeezing for an attenuated squeezed vacuum state by modeling the loss as unitary beam splitter with a vacuum state input at the second port. Let $T = t^2$ be the transmission coefficient of the beam splitter (t is the amplitude transmission coefficient) and $R = 1-T$ be the reflection coefficient. At what attenuation (value of T) is the degree of squeezing reduced by 3dB?

(c) Now let us see what happens to the squeezed vacuum state when we replace the vacuum state input at the second port of the beam splitter with a coherent state $|\beta\rangle$ which serves as the local oscillator (LO), a setup called 'homodyne detection'. Assume that the coherent state displaces the squeezed vacuum along the lower-variance quadrature (this corresponds to a choice of the phase of β).

i. Sketch the Q function of the output state.

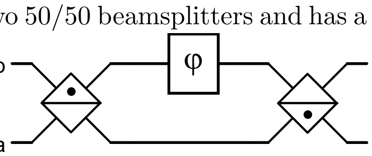
ii. What is the mean photon number at the output? Express the result in terms of the mean photon numbers of the input states.

iii. What is the photon number variance at the output? Identify the different contributions. Take the limit of large squeezing and small attenuation (so that it does not degrade the quality of squeezing).

iv. Can sub-Poissonian photon number fluctuations be achieved at the output (that is, fluctuations less than $\sqrt{\langle N\rangle}$)? If so, what are the constraints on $|\beta|^2$, the strength of the LO? Give intuitive reasoning for why mixing with the LO changes the nature of photon number fluctuations of the squeezed state.

3. **Better Phase Measurements with Squeezed Vacuum**

One application of squeezed light is to measure phase shifts with better precision than can be achieved with the same number of photons in a coherent state. In this problem, we revisit the Mach-Zehnder interferometer from last week's homework with coherent and squeezed input states. This interferometer uses two 50/50 beamsplitters and has a phase shift in one path of ϕ:



and has output ports b_{out} and a_{out}.

(a) Calculate the output signal $\langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\rangle$ as a function of ϕ and its the variance $\langle \Delta(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out})^2\rangle$ for the Mach-Zehnder interferometer with a coherent state and vacuum at its inputs ($|a\rangle = |\alpha\rangle$ and $|b\rangle = |0\rangle$). Calculate the Signal-to-Noise Ratio (SNR) for this measurement, $\frac{\langle N\rangle}{\sqrt{\langle \Delta N^2\rangle}}$.

(b) Find the minimal detectible phase $\phi_{\min}$ for the Mach-Zehnder interferometer with a coherent state and vacuum at its inputs. (Use a small-angle approximation for ϕ.) The minimal detectible phase is the phase for which the SNR is 1.

(c) Repeat (a-b) using squeezed vacuum in port b ($|a\rangle = |\alpha\rangle$ and $|b\rangle = S(\epsilon)|0\rangle$). How what degree of squeezing do you need to get factor of 2 increase in the phase resolution? Does squeezing change the average photon number in the device?

(d) The LIGO experiment is a long-arm Michaelson interferometer that measures differences in the length of one arm compared to the other as a signature of gravitational waves. Model LIGO as the Mach-Zehnder interferometer studied in (a)-(c). If the interferometer uses 5W of 1064nm light, what is the minimal differential length change that can be detected with coherent state inputs? If each path is 4 km long, what is the minimal detectable strain? What if a 6dB squeezed vacuum state is used instead of a coherent vacuum state?

The LIGO team measured their sentativity with squeezed vacuum. To learn about what they saw, see *Nature Photonics* **7**, 613619 (2013).