

1.

(a)

Define

$$Y(x, y) = \frac{x^n y^{n-1} + x^{n-1} y^n}{\sqrt{2n!(n-1)!}}$$

when $[x, y] = 0$

$$Y(x, y) = x^{n-1} y^{n-1} \frac{x + y}{\sqrt{2n!(n-1)!}}$$

we also have, for any number λ

$$Y(\lambda x, \lambda y) = \lambda^{2n-1} Y(x, y)$$

Before the first beam splitter

$$\begin{aligned} |\psi_0\rangle &= Y(a^\dagger, b^\dagger) |0, 0\rangle \\ &= a^{\dagger n-1} b^{\dagger n-1} \frac{a^\dagger + b^\dagger}{\sqrt{2n!(n-1)!}} |0, 0\rangle \end{aligned}$$

After the first beam splitter

$$\begin{aligned} |\psi_1\rangle &= B|\psi_0\rangle \\ &= BY(a^\dagger, b^\dagger) B^\dagger |0, 0\rangle \\ &= Y(Ba^\dagger B^\dagger, Bb^\dagger B^\dagger) |0, 0\rangle \\ &= \frac{\sqrt{2}}{2^n} Y(a^\dagger - ib^\dagger, b^\dagger - ia^\dagger) |0, 0\rangle \\ &= (a^\dagger - ib^\dagger)^{n-1} (b^\dagger - ia^\dagger)^{n-1} \frac{a^\dagger - ib^\dagger + b^\dagger - ia^\dagger}{2^n \sqrt{n!(n-1)!}} |0, 0\rangle \\ &= (-i)^{n-1} (1 - i) \left(a^{\dagger 2} + b^{\dagger 2} \right)^{n-1} \frac{a^\dagger + b^\dagger}{2^n \sqrt{n!(n-1)!}} |0, 0\rangle \end{aligned}$$

After phase shift

$$\begin{aligned} |\psi_2\rangle &= \frac{\sqrt{2}}{2^n} Y(e^{-i\phi} a^\dagger - ib^\dagger, b^\dagger - ie^{-i\phi} a^\dagger) |0, 0\rangle \\ &= (-i)^{n-1} (1 - i) \left(e^{-2i\phi} a^{\dagger 2} + b^{\dagger 2} \right)^{n-1} \frac{e^{-i\phi} a^\dagger + b^\dagger}{2^n \sqrt{n!(n-1)!}} |0, 0\rangle \end{aligned}$$

Output

$$\begin{aligned}
|\psi_3\rangle &= B^\dagger |\psi_2\rangle \\
&= \frac{e^{-i\phi(n-1/2)}}{2^{2n-1}} Y\left(e^{-i\phi/2}(a^\dagger + ib^\dagger) - ie^{i\phi/2}(b^\dagger + ia^\dagger), e^{i\phi/2}(b^\dagger + ia^\dagger) - ie^{-i\phi/2}(a^\dagger + ib^\dagger)\right) |0, 0\rangle \\
&= \frac{e^{-i\phi(n-1/2)}}{2^{2n-1}} Y\left(2\cos\frac{\phi}{2}a^\dagger + 2\sin\frac{\phi}{2}b^\dagger, 2\cos\frac{\phi}{2}b^\dagger - 2\sin\frac{\phi}{2}a^\dagger\right) |0, 0\rangle \\
&= e^{-i\phi(n-1/2)} Y\left(\cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger, \cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger\right) |0, 0\rangle \\
&= \frac{e^{-i\phi(n-1/2)}}{\sqrt{2n!(n-1)!}} \left(\cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger\right)^{n-1} \left(\cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger\right)^{n-1} \\
&\quad \left(\cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger + \cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger\right) |0, 0\rangle
\end{aligned}$$

For $\phi = 0$

$$\begin{aligned}
|\phi_3\rangle_0 &= \frac{1}{\sqrt{n!(n-1)!}} a^{\dagger n-1} b^{\dagger n-1} \frac{a^\dagger + b^\dagger}{\sqrt{2}} |0, 0\rangle \\
&= |\phi_0\rangle
\end{aligned}$$

(b)

Let

$$\begin{aligned}
a' &= \cos\frac{\phi}{2}a + \sin\frac{\phi}{2}b \\
b' &= \cos\frac{\phi}{2}b - \sin\frac{\phi}{2}a
\end{aligned}$$

then

$$\begin{aligned}
a'^\dagger &= \cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger \\
b'^\dagger &= \cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger \\
a &= \cos\frac{\phi}{2}a' - \sin\frac{\phi}{2}b' \\
b &= \cos\frac{\phi}{2}b' + \sin\frac{\phi}{2}a' \\
a^\dagger &= \cos\frac{\phi}{2}a'^\dagger - \sin\frac{\phi}{2}b'^\dagger \\
b^\dagger &= \cos\frac{\phi}{2}b'^\dagger + \sin\frac{\phi}{2}a'^\dagger \\
[a', a'^\dagger] &= 1 \\
[b', b'^\dagger] &= 1 \\
|\phi_3\rangle &= \frac{e^{-i\phi(n-1/2)}}{\sqrt{n!(n-1)!}} a'^{\dagger n-1} b'^{\dagger n-1} \frac{a'^\dagger + b'^\dagger}{\sqrt{2}} |0, 0\rangle
\end{aligned}$$

$$\begin{aligned}
M &= a^\dagger a - b^\dagger b \\
&= \left(\cos \frac{\phi}{2} a'^\dagger - \sin \frac{\phi}{2} b'^\dagger \right) \left(\cos \frac{\phi}{2} a' - \sin \frac{\phi}{2} b' \right) - \left(\cos \frac{\phi}{2} b'^\dagger + \sin \frac{\phi}{2} a'^\dagger \right) \left(\cos \frac{\phi}{2} b' + \sin \frac{\phi}{2} a' \right) \\
&= \cos^2 \frac{\phi}{2} a'^\dagger a' + \sin^2 \frac{\phi}{2} b'^\dagger b' - \sin \frac{\phi}{2} \cos \frac{\phi}{2} (a' b'^\dagger + a'^\dagger b') \\
&\quad - \cos^2 \frac{\phi}{2} b'^\dagger b' - \sin \frac{\phi}{2} \cos \frac{\phi}{2} a'^\dagger b' - \sin \frac{\phi}{2} \cos \frac{\phi}{2} a' b'^\dagger - \sin^2 \frac{\phi}{2} a'^\dagger a' \\
&= \cos \phi (a'^\dagger a' - b'^\dagger b') - \sin \phi (a' b'^\dagger + a'^\dagger b')
\end{aligned}$$

Mean

$$\begin{aligned}
\langle M \rangle &= \frac{1}{2n!(n-1)!} \langle 0, 0 | a'^{n-1} b'^{n-1} (a' + b') (\cos \phi (a'^\dagger a' - b'^\dagger b') - \sin \phi (a' b'^\dagger + a'^\dagger b')) \\
&\quad a'^{\dagger n-1} b'^{\dagger n-1} (a'^\dagger + b'^\dagger) | 0, 0 \rangle \\
&= \frac{\cos \phi}{2n!(n-1)!} \langle 0, 0 | a'^{n-1} b'^{n-1} (a' + b') (a'^\dagger a' - b'^\dagger b') a'^{\dagger n-1} b'^{\dagger n-1} (a'^\dagger + b'^\dagger) | 0, 0 \rangle \\
&\quad - \frac{\sin \phi}{2n!(n-1)!} \langle 0, 0 | a'^{n-1} b'^{n-1} (a' + b') (a' b'^\dagger + a'^\dagger b') a'^{\dagger n-1} b'^{\dagger n-1} (a'^\dagger + b'^\dagger) | 0, 0 \rangle \\
&= -n \sin \phi
\end{aligned}$$

Square

$$\begin{aligned}
M^2 &= (\cos \phi (a'^\dagger a' - b'^\dagger b') - \sin \phi (a' b'^\dagger + a'^\dagger b'))^2 \\
&= \cos^2 \phi (a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b') + \sin^2 \phi (a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2) \\
&\quad - \sin \phi \cos \phi (2a'^\dagger a'^2 b'^\dagger + a' b'^\dagger + 2a'^{\dagger 2} a' b' + a'^\dagger b' - 2a' b'^{\dagger 2} b' - a' b'^\dagger - 2a'^\dagger b' b'^2 - a'^\dagger b') \\
&= \cos^2 \phi (a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b') + \sin^2 \phi (a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2) \\
&\quad - 2 \sin \phi \cos \phi (a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2} a' b' - a' b'^{\dagger 2} b' - a'^\dagger b' b'^2) \\
\langle M^2 \rangle &= \cos^2 \phi \langle a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b' \rangle \\
&\quad + \sin^2 \phi \langle a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2 \rangle \\
&\quad - 2 \sin \phi \cos \phi \langle a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2} a' b' - a' b'^{\dagger 2} b' - a'^\dagger b' b'^2 \rangle \\
&= \frac{\cos^2 \phi}{2n} (n^3 + (n-1)^2 n - 2n^2(n-1) - 2n^2(n-1) + (n-1)^2 n + n^3) \\
&\quad + \frac{\sin^2 \phi}{2n} (2(n-1)n(n+1) + 2n^3) \\
&\quad - \frac{\sin \phi \cos \phi}{n} (n^2(n-1) + n^2(n-1) - n^2(n-1) - n^2(n-1)) \\
&= \cos^2 \phi + \sin^2 \phi (2n^2 - 1)
\end{aligned}$$

Fluctuation

$$\begin{aligned}\Delta M^2 &= \cos^2 \phi + \sin^2 \phi (2n^2 - 1) - n^2 \sin^2 \phi \\ &= \cos^2 \phi + \sin^2 \phi (n^2 - 1) \\ \frac{\partial \langle M \rangle}{\partial \phi} &= -n \cos \phi \\ \Delta \phi^2 &= \frac{\cos^2 \phi + \sin^2 \phi (n^2 - 1)}{n^2 \cos^2 \phi} \\ &= \frac{1 + \tan^2 \phi (n^2 - 1)}{n^2} \\ \Delta \phi &= \frac{\sqrt{1 + \tan^2 \phi (n^2 - 1)}}{n}\end{aligned}$$

for $\phi = 0$

$$\Delta \phi = \frac{1}{n}$$

(c)

$$\begin{aligned}|\psi_0\rangle &= |\alpha, 0\rangle \\ |\psi_1\rangle &= \left| \frac{\alpha}{\sqrt{2}}, -\frac{i\alpha}{\sqrt{2}} \right\rangle \\ |\psi_2\rangle &= \left| \frac{\alpha e^{-i\phi}}{\sqrt{2}}, -\frac{i\alpha}{\sqrt{2}} \right\rangle \\ |\psi_3\rangle &= \left| \frac{\alpha e^{-i\phi}}{2} + \frac{\alpha}{2}, i \left(\frac{\alpha e^{-i\phi}}{2} - \frac{\alpha}{2} \right) \right\rangle \\ &= \left| \alpha e^{-i\phi/2} \cos \frac{\phi}{2}, \alpha e^{-i\phi/2} \sin \frac{\phi}{2} \right\rangle\end{aligned}$$

(d)

Mean

$$\begin{aligned}\langle M \rangle &= \langle a^\dagger a - b^\dagger b \rangle \\ &= \alpha^* e^{i\phi/2} \cos \frac{\phi}{2} \alpha e^{-i\phi/2} \cos \frac{\phi}{2} - \alpha^* e^{i\phi/2} \sin \frac{\phi}{2} \alpha e^{-i\phi/2} \sin \frac{\phi}{2} \\ &= |\alpha|^2 \cos^2 \frac{\phi}{2} - \alpha^2 \sin^2 \frac{\phi}{2} \\ &= |\alpha|^2 \cos \phi\end{aligned}$$

Square

$$\begin{aligned}\langle M^2 \rangle &= \langle (a^\dagger a - b^\dagger b)(a^\dagger a - b^\dagger b) \rangle \\ &= \langle a^\dagger a a^\dagger a - a^\dagger a b^\dagger b - b^\dagger b a^\dagger a + b^\dagger b b^\dagger b \rangle \\ &= \langle M \rangle^2 + \langle a^\dagger a + b^\dagger b \rangle \\ &= \langle M \rangle^2 + |\alpha|^2\end{aligned}$$

Uncertainty

$$\begin{aligned}\langle \Delta\phi^2 \rangle &= \frac{|\alpha|^2}{|\alpha|^4 \sin^2 \phi} \\ &= \frac{1}{|\alpha|^2 \sin^2 \phi}\end{aligned}$$

2.

(a)

$$\begin{aligned}P &= |\psi_A|^2 |\psi_B|^2 \left| e^{i(\vec{k}_A \cdot \vec{r}_{A1} + \vec{k}_B \cdot \vec{r}_{B2})} \pm e^{i(\vec{k}_A \cdot \vec{r}_{A2} + \vec{k}_B \cdot \vec{r}_{B1})} \right|^2 \\ &= |\psi_A|^2 |\psi_B|^2 \left| e^{i\vec{k}_A \cdot \vec{r}_{21}} \pm e^{i\vec{k}_B \cdot \vec{r}_{21}} \right|^2 \\ &= |\psi_A|^2 |\psi_B|^2 \left| e^{i(\vec{k}_A - \vec{k}_B) \cdot \vec{r}_{21}/2} \pm e^{-i(\vec{k}_A - \vec{k}_B) \cdot \vec{r}_{21}/2} \right|^2 \\ &= |\psi_A|^2 |\psi_B|^2 \left| 2 \pm e^{i(\vec{k}_A - \vec{k}_B) \cdot \vec{r}_{21}} \pm e^{-i(\vec{k}_A - \vec{k}_B) \cdot \vec{r}_{21}} \right| \\ &= 2|\psi_A|^2 |\psi_B|^2 \left(1 \pm \cos \left((\vec{k}_A - \vec{k}_B) \cdot \vec{r}_{21} \right) \right)\end{aligned}$$

(b)

In order to see the correlation $\Delta\phi_t \ll 1$,

$$\begin{aligned}1 &\gg \Delta\vec{k}_t \cdot \vec{r}_{21t} \\ &= k_0 \frac{W}{d} w \\ &= \frac{2\pi W w}{\lambda d} \\ \lambda d &\gg W w\end{aligned}$$

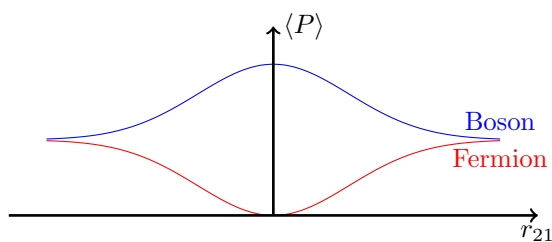
For ${}^6\text{Li}$ at $500\mu\text{K}$

$$\begin{aligned}\lambda &= \frac{\hbar}{p} \\ &= \frac{\hbar}{\sqrt{mk_B T}} \\ &\approx 12\text{nm} \\ w_{max} &\approx \sqrt{\lambda d} \\ &\approx 36\mu\text{m}\end{aligned}$$

(c)

i.

$$\begin{aligned}
 \langle P \rangle &= 2|\psi_A|^2|\psi_B|^2 \sqrt{\frac{\gamma}{\pi}} \int (1 \pm \cos \Delta k_r r_{21}) e^{-\Delta k_r^2 \gamma^2} d\Delta k_r \\
 &= 2|\psi_A|^2|\psi_B|^2 \left(1 \pm \exp\left(-\frac{r_{21}^2}{4\gamma^2}\right) \right) \\
 &= 2|\psi_A|^2|\psi_B|^2 \left(1 \pm \exp\left(-\frac{r_{21}^2 l + r_{21}^2 t}{4\gamma^2}\right) \right)
 \end{aligned}$$



ii.

$$\begin{aligned}
 \Delta k &= \frac{\Delta p}{\hbar} \\
 &= \frac{m}{\hbar} \Delta v \\
 &= \frac{m}{\hbar} \Delta \left(\frac{d}{\tau} \right) \\
 &= \frac{mL}{\hbar \tau} \\
 &= \frac{mLv}{\hbar d}
 \end{aligned}$$

$$\begin{aligned}
 vt_{min} &< \frac{\hbar d}{mvL} \\
 t_{min} &< \frac{\hbar \tau^2}{mdL} \\
 &= \frac{\hbar \tau^2}{mdL} \\
 &\approx 35 \mu s
 \end{aligned}$$

(d)

Size of a coherent cell

$$\begin{aligned} & \left(\frac{\lambda d}{W} \right)^2 \frac{\hbar d}{mvL} \\ & \approx \left(\frac{\lambda d}{W} \right)^2 \frac{\lambda d}{L} \\ & = \lambda^3 \frac{d^3}{W^2 L} \\ & \approx 2 \cdot 10^{10} \lambda^3 \end{aligned}$$

Phase space density of the MOT

$$\begin{aligned} n\lambda^3 &= 1.7 \cdot 10^{-8} \\ &\ll 1 \end{aligned}$$