Assignment #1

Due: Monday, February 23, 2015
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Office Hours: Feb 20th (Fri) & Feb 23rd (Mon), 9am - 11am.

1. **Properties of the coherent state $|\alpha\rangle$**

   (a) Compute the overlap of two coherent states $\langle\alpha|\beta\rangle$, for arbitrary complex α and β.

   (b) Prove that the coherent states form an over-complete basis, that is

   $$\int |\alpha\rangle\langle\alpha|\,\frac{d^2\alpha}{\pi} = I\,. \tag{1}$$

   (c) The BCH Lemma states that $e^A e^B = e^{A+B+[A,B]/2}$ for operators A and B satisfying $[A,B]=c$, where c is a complex number. Use this lemma to prove that the displacement operator $D(\alpha)$ defined by $D(\alpha)|0\rangle = |\alpha\rangle$ may be written as $D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right]$.

   (d) Let the electric field operator be $E_x = E_0\left[a + a^\dagger\right]\sin(kz)$, where $E_0 = \sqrt{\hbar\omega/V\epsilon_0}$ is the average electric field created by one photon inside the cavity volume V. For a freely evolving coherent state $|\alpha\rangle = |\alpha(t)\rangle$, compute the average electric field $\langle E_x\rangle = \langle\alpha|E_x|\alpha\rangle$ and the root-mean-square deviation of the electric field $\sqrt{\langle\Delta E_x^2\rangle} = \sqrt{\langle\alpha|E_x^2|\alpha\rangle - |\langle E_x\rangle|^2}$. Why is $\sqrt{\langle\Delta E_x^2\rangle}$ independent of time and field strength $|\alpha|$? Why does this result hold even for the vacuum state $\alpha = 0$?

   (e) In quantum mechanics, there is no unique choice for a phase operator. To better understand the phase properties of a state, here we give you one possible definition of a phase probability distribution. For a coherent state $|\alpha\rangle$ with $\alpha = |\alpha|e^{i\theta}$ it can be defined as being

   $$P(\phi) = \frac{1}{2\pi}|\sum_n \langle n|e^{-in\phi}|\alpha\rangle|^2, \tag{2}$$

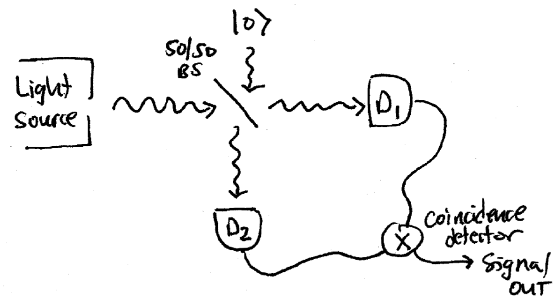
   which can be shown to obey the normalization condition $\int_0^{2\pi} d\phi P(\phi) = 1$. Show that this definition gives the expected results for coherent states (in particular, the value of $\langle\phi\rangle$ and the scaling of $\Delta\phi$ with $|\alpha|$), which for large values of α approach classical states. Show that this definition gives an undefined phase for number states.
   Hint: One (algebraically complex) way to show this is by showing that

   $$P(\phi) \approx \sqrt{\frac{2|\alpha|^2}{\pi}}e^{-2|\alpha|^2(\phi-\theta)^2} \tag{3}$$

   using an approximation of a large-mean Poisson distribution as a Gaussian one.

2. **The Hanbury-Brown Twiss experiment and $g^{(2)}(\tau)$**

The second-order coherence function $g^{(2)}(\tau)$ is often measured in the laboratory using an experiment first developed by Hanbury-Brown and Twiss in the 1950's, for studying the light from distant stars. This experiment involves mixing light from the input source with the vacuum, $|0\rangle$, on a 50/50 beamsplitter, and measuring the intensity-intensity correlation function at the output using two detectors and a coincidence circuit:



This problem examines how this experiment measures $g^{(2)}(\tau)$, and what results are obtained for different input states of light.

a) Let a, $a^\dagger$, and b, $b^\dagger$ be the raising and lowering operators for the two modes of light input to the beamsplitter, and let the unitary transformation performed by the beamsplitter be defined by

$$a_1 = UaU^\dagger \;=\; \frac{a+b}{\sqrt{2}} \tag{4}$$

$$b_1 = UbU^\dagger \;=\; \frac{a-b}{\sqrt{2}}\,. \tag{5}$$

For light input in state $|\psi\rangle$, you are given that the output of the coincidence circuit is a voltage

$$V_\psi = V_0 \langle \psi, 0 |\, a_1^\dagger a_1 b_1^\dagger b_1 \,| \psi, 0\rangle\,, \tag{6}$$

where V_0 is some proportionality constant, and $|\psi, 0\rangle$ denotes a state with $|\psi\rangle$ in mode "a" and $|0\rangle$ in mode "b". In other words, the voltage is the average of the product of the two detected photon signals. Show that V_ψ gives a measure of $g^{(2)}(\tau)$,

$$g^{(2)}(\tau) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} \tag{7}$$

up to an additive offset and normalization.

b) The classical expression for $g^{(2)}$ is

$$g^{(2)}_{cl}(\tau) = \frac{\langle \bar{I}(t)\bar{I}(t+\tau)\rangle}{\langle \bar{I}\rangle^2}\,. \tag{8}$$

Prove that $g^{(2)}_{cl}(0) \geq 1$. It is helpful to use the fact that $\langle (\bar{I}(t) - \langle \bar{I}(t)\rangle)^2\rangle \geq 0$.

c) Compute $g^{(2)}(\tau)$ for the following input light states:

$$|\psi_1\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_k \frac{\alpha^k}{\sqrt{k!}} |k\rangle \quad \text{(a coherent state)} \tag{9}$$

$$|\psi_2\rangle = |2\rangle \quad \text{(the number state } n = 2) \tag{10}$$

d) Compute $g^{(2)}(\tau)$ for the following input light states, as a function of α:

$$|\psi_3\rangle = \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2}} \tag{11}$$

$$|\psi_4\rangle = \frac{|\alpha\rangle - |-\alpha\rangle}{\sqrt{2}} \tag{12}$$

Do either of these two states show non-classical second-order coherence? Why (or why not)?
Hint: Are these states normalized?

3. **States of light: pseudo-probability distribution plots**

Pseudo-probability distributions such as the $Q(\alpha)$ function provide useful and insightful ways to depict quantum states of light. In this problem, we explore some important states and their depictions. Due to the technical challenge of analytically calculating $Q(\alpha)$, it is satisfactory to give numerical answers for some parts of this problem. In particular, the $Q(\alpha)$ function is defined as

$$Q_\rho(\alpha) \equiv \langle\alpha|\rho|\alpha\rangle\,, \tag{13}$$

and it is readily computed numerically by using the fact that any pure state $\rho = |\psi\rangle\langle\psi|$ can be represented using

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle\,, \tag{14}$$

and

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}\,. \tag{15}$$

Compute and plot the following:

(i)

$$Q_1(\alpha) = |\langle\alpha|\psi_1\rangle|^2\,, \tag{16}$$

for

$$|\psi_1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|12\rangle\,, \tag{17}$$

where $|12\rangle$ is the twelve photon number eigenstate. Plot for various values of ϕ and θ. Is this a minimum uncertainty state? Is it a squeezed state?

(ii)

$$Q_2(\alpha) = |\langle\alpha|\psi_2\rangle|^2 \,, \tag{18}$$

for

$$|\psi_2\rangle = \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2}} \,, \tag{19}$$

say, with $\alpha = 3$. Try plotting this on a logarithmic scale, and you should see interference fringes around the origin. What is that due to?

(iii)

$$Q_3(\alpha) = |\langle\alpha|\psi_3\rangle|^2 \,, \tag{20}$$

for

$$|\psi_3\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ik\phi}|k\rangle \,, \tag{21}$$

where $|k\rangle$ is a Fock state of k photons, say with $N = 10$ and $\phi = \pi/4$. How would you interpret the physical meaning of this state? What happens as $N \to \infty$? Is this a squeezed state, and how so?

(iv)

$$Q_4(\alpha) = |\langle\alpha|0_\epsilon\rangle|^2 \,, \tag{22}$$

where $|0_\epsilon\rangle = S(\epsilon)|0\rangle$ is the squeezed vacuum with parameter ϵ. You may use

$$S(\epsilon)|0\rangle = \frac{1}{\sqrt{\cosh\epsilon}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh\epsilon)^n |2n\rangle \,. \tag{23}$$

Compare plots made with $\epsilon = 0.2$, 1.2, and 4, for example. Look for the signatures of squeezing.

(v)

$$Q_5(\alpha) = |\langle\alpha|e^{iH_{\mathrm{kerr}}t}|\beta\rangle|^2 \,, \tag{24}$$

where the Kerr effect Hamiltonian is

$$H_{\mathrm{kerr}} = \xi a^\dagger a(a^\dagger a - 1) = \xi n(n-1) \,. \tag{25}$$

This is easily numerically computed by using the number basis representation of the coherent state $|\beta\rangle$. Take $\beta = 4$, and $\xi = \pi/128$, for example, and generate a sequence of plots as a function of time. At what time does the initial coherent state evolve to become two superposed coherent states? When does it return to its original state? The Kerr-type nonlinearity is important both in nonlinear optics, and in interacting cold atomic gasses. It produces interesting and useful squeezed states, such as that depicted in this sample $Q(\alpha)$ plot: