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1. The Dressed Atom

(a)

AC Stark shift

$$\begin{aligned}
E_{Stark} =& \frac{1}{2}(\Omega - |\delta|) \\
=& \frac{1}{2}\left(\sqrt{\Omega_1^2 + \delta^2} - |\delta|\right)
\end{aligned}$$

In the far detune limit

$$E_{Stark} \approx \frac{\Omega_1^2}{4\delta}$$

The sign depends on the state and the detuning (positive for excited state red detune and ground state blue detune and negetive otherwise)

(b)

Without loss of generality, consider the ground state and red detuning

$$\begin{aligned}
\langle P \rangle =& \langle -|P|- \rangle \\
=& e(\sin\theta\langle e'| + \cos\theta\langle g'|) z (\sin\theta|e'\rangle + \cos\theta|g'\rangle) \\
=& e\sin\theta\cos\theta(\langle e'|z|g'\rangle + \langle g'|z|e'\rangle) \\
=& e\frac{\Omega_1}{\Omega}\langle e|z|g\rangle \cos\omega_L t \\
=& e\frac{e\langle e|z|g\rangle E_0}{\hbar\Omega}\langle e|z|g\rangle \cos\omega_L t \\
\alpha =& \frac{e^2}{\hbar\Omega}|\langle e|z|g\rangle|^2 \\
=& \frac{e^2}{\hbar\Omega}\frac{\hbar f_{ab}}{2m\omega_0} \\
=& \frac{e^2 f_{ab}}{2m\omega_0\Omega}
\end{aligned}$$

The two agrees in the far detuned high frequency limit (where $\Omega = \delta$ and $\omega_0 + \omega_L \approx 2\omega_0$). When $\omega_L \to \omega_0$ the perturbation treatment breaks down and the effect becomes none linear (or non-quadratic). The linewidth / decay can also be important in real system.

(c)

(d)

(e)

(f)

2. Sideband Cooling

(a)

Interaction term

$$\begin{aligned}H_I =&\hbar\Omega(\sigma_+ + \sigma_-)\cos\left(k\hat{x} - \omega t\right)\\ =&\hbar\Omega(\sigma_+ + \sigma_-)\cos\left(\eta\left(a + a^\dagger\right) - \omega t\right)\\ =&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{\mathrm{i}\left(\eta\left(a+a^\dagger\right)-\omega t\right)} + \mathrm{e}^{-\mathrm{i}\left(\eta\left(a+a^\dagger\right)-\omega t\right)}\right)\end{aligned}$$

In the Lamb-Dicke limit

$$\begin{aligned}H_I =&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{\mathrm{i}\eta\left(a+a^\dagger\right)}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\eta\left(a+a^\dagger\right)}\mathrm{e}^{\mathrm{i}\omega t}\right)\\ =&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\left(1 + \mathrm{i}\eta\left(a + a^\dagger\right)\right)\mathrm{e}^{-\mathrm{i}\omega t} + \left(1 - \mathrm{i}\eta\left(a + a^\dagger\right)\right)\mathrm{e}^{\mathrm{i}\omega t}\right)\\ =&\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{\mathrm{i}\omega t}\right) + \mathrm{i}\eta\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(a + a^\dagger\right)\left(\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{e}^{\mathrm{i}\omega t}\right)\end{aligned}$$

In the interaction picture

$$\begin{aligned}\sigma'_\pm =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}\sigma_\pm\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\ =&\sigma_\pm\mathrm{e}^{\mathrm{i}\omega_0 t}\\ a' =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}a\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\ =&a\mathrm{e}^{-\mathrm{i}\nu t}\\ a'^\dagger =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}a^\dagger\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\ =&a^\dagger\mathrm{e}^{\mathrm{i}\nu t}\end{aligned}$$

In the rotating wave approximation

$$\begin{aligned}H'_I =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}H_I\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\ =&\frac{\hbar\Omega}{2}\left(\sigma_+\mathrm{e}^{-\mathrm{i}\delta t} + \sigma_-\mathrm{e}^{\mathrm{i}\delta t}\right) + \mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+ a\mathrm{e}^{-\mathrm{i}(\delta+\nu)t} - \sigma_- a^\dagger\mathrm{e}^{\mathrm{i}(\delta+\nu)t}\right)\\ &+ \mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+ a^\dagger\mathrm{e}^{-\mathrm{i}(\delta-\nu)t} - \sigma_- a\mathrm{e}^{\mathrm{i}(\delta-\nu)t}\right)\end{aligned}$$

When the light is in resonance with the sideband, the Rabi frequencies

$$\begin{aligned}
\Omega_{n,n-1} =&\left|\langle n-1,e|\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a-\sigma_-a^\dagger\right)|n,g\rangle\right|\\
=&\frac{\eta\hbar\Omega}{2}\langle n-1|a|n\rangle\\
=&\frac{\eta\hbar\Omega}{2}\sqrt{n}\\
\Omega_{n,n+1} =&\left|\langle n+1,e|\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a^\dagger-\sigma_-a\right)|n,g\rangle\right|\\
=&\frac{\eta\hbar\Omega}{2}\langle n+1|a^\dagger|n\rangle\\
=&\frac{\eta\hbar\Omega}{2}\sqrt{n+1}
\end{aligned}$$

(b)

For spontanious decay, when the emitted photon is θ from the axis of the trap, the probability to go from n to $n+1$ is $\eta^2\cos^2\theta(n+1)$ Assuming isotropic emission patter (i.e. non-polarized) the averate probability is

$$\begin{aligned}
p_{n,n+1} =&\frac{1}{4\pi}\int_0^\pi\mathrm{d}\theta\int_0^{2\pi}\sin\theta\mathrm{d}\phi\eta^2\cos^2\theta(n+1)\\
=&\frac{\eta^2(n+1)}{2}\int_{-1}^1\mathrm{d}zz^2\\
=&\frac{\eta^2(n+1)}{3}
\end{aligned}$$

Similarly, the probability of going to $n-1$ is

$$p_{n,n-1} =\frac{\eta^2n}{3}$$

To the lowest order in η, the rate at which $|n,g\rangle\to|n,e\rangle\to|n-1,g\rangle$ happens is

$$\frac{n\Omega^2\eta^2}{3}\frac{\Gamma}{\Gamma^2+4\delta^2}$$

For $|n,g\rangle\to|n-1,e\rangle\to|n-1,g\rangle$

$$\begin{aligned}
&n\Omega^2\eta^2\frac{\Gamma}{\Gamma^2+4(\delta+\nu)^2}\\
A_- =&\Omega^2\Gamma\eta^2\left(\frac{1}{\Gamma^2+4(\delta+\nu)^2}+\frac{1}{3}\frac{1}{\Gamma^2+4\delta^2}\right)
\end{aligned}$$

Similarly

$$A_+ =\Omega^2\Gamma\eta^2\left(\frac{1}{\Gamma^2+4(\delta-\nu)^2}+\frac{1}{3}\frac{1}{\Gamma^2+4\delta^2}\right)$$

(c)

$$
\begin{aligned}
\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} =& \sum_{n=0}^{\infty} n\frac{\mathrm{d}p_n}{\mathrm{d}t}\\
=& \sum_{n=0}^{\infty} n(nA_+p_{n-1} + (n+1)A_-p_{n+1} - (n+1)A_+p_n - nA_-p_n)\\
=& \sum_{n=0}^{\infty} \left(n^2A_+p_{n-1} + n(n+1)A_-p_{n+1} - n(n+1)A_+p_n - n^2A_-p_n\right)\\
=& \sum_{n=0}^{\infty} \left((n+1)^2A_+p_n + (n-1)nA_-p_n - n(n+1)A_+p_n - n^2A_-p_n\right)\\
=& \sum_{n=0}^{\infty} ((n+1)A_+p_n - nA_-p_n)\\
=& A_+ - (A_- - A_+)\langle n\rangle
\end{aligned}
$$

The solution is a exponential decay to $\langle n\rangle = \dfrac{A_+}{A_- - A_+}$ with decay rate $A_- - A_+$. For driving cooling sideband in the resolved limit

$$
\begin{aligned}
A_- \approx& \frac{\Omega^2\eta^2}{\Gamma}\\
A_+ \approx& \frac{7\Omega^2\Gamma\eta^2}{48\nu^2}
\end{aligned}
$$

Final temperature

$$
\begin{aligned}
\langle n\rangle_\infty \approx& \frac{A_+}{A_-}\\
=& \frac{7\Gamma^2}{48\nu^2}\\
T_\infty =& \langle n\rangle_\infty\frac{\hbar\nu}{k_B}\\
\approx& \frac{7\hbar\Gamma^2}{48k_B\nu}
\end{aligned}
$$

Decay time

$$
\begin{aligned}
\tau \approx& \frac{1}{A_-}\\
\approx& \frac{\Gamma}{\Omega^2\eta^2}
\end{aligned}
$$

(d)

For the narrow line

$$\begin{aligned}\eta =& k\sqrt{\frac{\hbar}{2m\nu}}\\ =&0.022\\ T_\infty \approx& \frac{7\hbar\Gamma^2}{48k_B\nu}\\ =&0.11\text{aK}\\ \tau \approx& \frac{\Gamma}{\Omega^2\eta^2}\\ =&22\text{hr}\end{aligned}$$

For broadened line

$$\begin{aligned}T_\infty \approx& \frac{7\hbar\Gamma^2}{48k_B\nu}\\ =&7.0\text{nK}\\ \tau \approx& \frac{\Gamma}{\Omega^2\eta^2}\\ =&0.32\text{s}\end{aligned}$$

3. Optical dipole trap

(a)

Phase shift

$$\begin{aligned}\Delta\phi_D =& \Delta n d k\\ =& \frac{n}{2}\alpha d k\\ =& \frac{\dot{n}}{2}\alpha d k t\end{aligned}$$

Frequency shift

$$\Delta\omega = \frac{\dot{n}}{2}\alpha d k$$

(b)

Power loss

$$\begin{aligned}\Delta P =& P\frac{\Delta\omega}{\omega}\\ =& \frac{\dot{n}\alpha V k}{2\omega}\varepsilon_0 c\left\langle E^2\right\rangle\\ =& \frac{\dot{n}\alpha V}{2}\varepsilon_0\left\langle E^2\right\rangle\end{aligned}$$

Total energy

$$\Delta E_{EM} = \frac{N\alpha}{2}\varepsilon_0\left\langle E^2\right\rangle$$

(c)

Start shift

$$U_{Stark} = \frac{\varepsilon_0 \langle pE \rangle}{2}$$
$$= \frac{\varepsilon_0 \alpha \left\langle E^2 \right\rangle}{2}$$

Total energy

$$\Delta E_{Stark} = \frac{N \varepsilon_0 \alpha \left\langle E^2 \right\rangle}{2}$$
$$= \Delta E_{EM}$$