Assignment #2

Due: Monday, March 2, 2015
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Office Hours: Feb 27th (Friday) & Mar 2nd (Monday), 9:30am - 11am.

1. **The quantum beamsplitter**.

   Let the beamsplitter operator B, acting with angle θ on modes a and b, be defined by

   $$B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right] \, . \tag{1}$$

   a) We can show that B conserves the total photon number and leaves coherent states as coherent states. Prove that B leaves $n_a + n_b = a^\dagger a + b^\dagger b$ unchanged. Also prove that $B^\dagger B = I$.

   b) Let $|\alpha\rangle$ be a coherent state. Compute $B|0\rangle_b|\alpha\rangle_a$, and show that the output is a tensor product of coherent states for all θ. Your result should be consistent with the intuition that the beamsplitter has well defined transmission and reflection coefficients; give these as a function of θ.

   c) There is close connection between the Lie group $SU(2)$ and the algebra of two coupled harmonic oscillators, which is useful for understanding B. Let's define

   $$s_z = a^\dagger a - b^\dagger b \qquad s_+ = a^\dagger b \qquad s_- = ab^\dagger \, , \tag{2}$$

   and let $s_\pm = (s_x \pm i s_y)/2$. What is $B(\theta)$ in spin space? What is $a^\dagger a + b^\dagger b$ in spin space? Show that s_x, s_y, and s_z have the same commutation relations as the Pauli matrices. This relationship also explains why $a^\dagger a + b^\dagger b$ is invariant; it is the Casimir operator of the algebra.

   d) How does a beamsplitter transform an input photon-number eigenstate? Let

   $$B(\theta) = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right] \, , \tag{3}$$

   and $B = B(\pi/4)$ be a 50/50 beamsplitter, such that

   $$BaB^\dagger = \frac{a-b}{\sqrt{2}} \qquad \text{and} \qquad BbB^\dagger = \frac{a+b}{\sqrt{2}} \, . \tag{4}$$

   Compute $B|0\rangle|n\rangle$, where the first label is mode b, and the second label is mode a. Note that the result is *not* $|n/2\rangle|n/2\rangle$, because $|n\rangle$ is a photon number eigenstate, and not a coherent state. What photon number states have the largest amplitude? How sharp is the distribution for $n = 10$, and $n = 100$, or as a function of n, if a general solution exists? Hint: use the binomial expansion on $(a^\dagger + b^\dagger)^n$.

2. **Understanding a beam attenuator**

   In this problem, we study the attenuation of coherent state using a beamsplitter model.

   First, let's devise an intuitive definition for a quantum optical attenuator. In the Heisenberg picture, might think to define an attenuator as the annihilation operator transformed by some attenuating operator $\mathcal{T}$, such that $a' = \mathcal{T} a \mathcal{T}^\dagger = ta$, where t is a complex transmission coefficient with $|t| < 1$.

a) In the last homework we found that a coherent state has $\Delta n^2 = \bar{n} = |\alpha|^2$ Show that a coherent state acted on by this attenuator would have $\Delta n'^2 < \bar{n'}$, i.e. photon number fluctuations below shot noise. This definition for an attenuator gives states that violate minimum uncertainty.

b) We see that our intuitive definition gives an unphysical result for an attenuated coherent state. In fact, it is not a unitary transformation and it does not correspond to quantum time evolution. One correct model, instead, is to formulate attenuation as a beamsplitter operation, where the attenuation comes from mixing our initial state with the vacuum state. The output ports follow

$$a' = ta + rb, \tag{5}$$

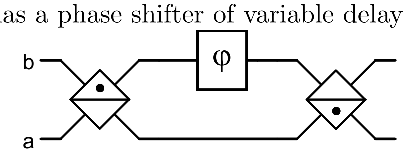
$$b' = ra + tb, \tag{6}$$

where r is a reflection scaling factor with $|r|^2 + |t|^2 = 1$. Show that in this model, coherent states are indeed transmitted as attenuated coherent states, and the photon number fluctuations of an attenuated coherent state do follow shot noise.

This problem shows that any kind of attenuation (e.g. dissipation) couples the quantum system to an open port which allows vacuum fluctuations to enter. Any kind of dissipation adds fluctuations to the attenuated quantum state.

3. **Lossy interferometer**. A real interferometer usually suffers from loss of photons in its arms, or due to imperfect beamsplitters. The effect of such loss can be modeled by a beamsplitter, as we explore in this problem.

a) Our starting point is the standard Mach-Zehnder interferometer, which uses two 50/50 beamsplitters, and has a phase shifter of variable delay ϕ:

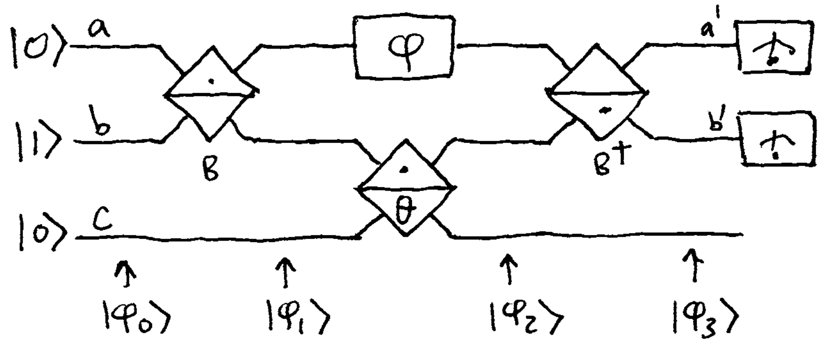


In the absence of loss, and given an input $|10\rangle$ (notation $|ba\rangle$), a photon is measured to be output in mode b with probability $P_b = \cos^2(\phi/2)$. The *fringe visibility* of this output signal is defined as

$$V = \frac{\max(P_b) - \min(P_b)}{\max(P_b) + \min(P_b)}, \tag{7}$$

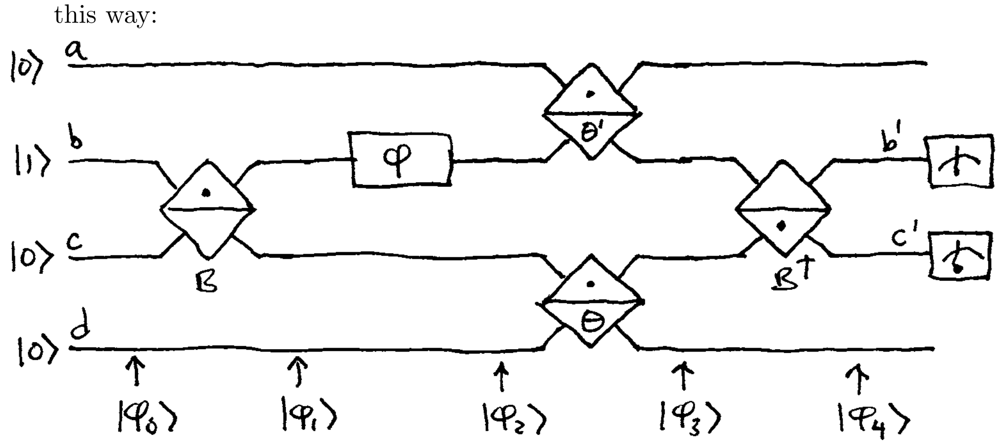
where the min and max are taken over ϕ. Show that $V = 1$ for this ideal situation.

b) Suppose now that one arm of the interferometer is lossy, and we model this loss with a beamsplitter, in this way:



Let the input state be $|\phi_0\rangle = |010\rangle$ (using notation $|cba\rangle$ for the three modes). The two outer beamsplitters are the usual 50/50 beamsplitters of the Mach-Zehnder interferometer, while the new middle beamsplitter has angle θ. When $\theta = 0$, no photons are lost from b into c, and when $\theta = \pi/2$, any photon in b is moved into mode c, modeling complete loss. Give the intermediate states $|\phi_1\rangle$ through $|\phi_3\rangle$, and compute the probability that a photon is found at the output in mode b, as a function of ϕ and θ. What is the fringe visibility as a function of θ?

c) Now let both arms of the interferometer be lossy, modeled with two beamsplitters, in this way:



The top beamsplitter has angle θ', and the bottom one θ. Give the intermediate states $|\phi_1\rangle$ through $|\phi_4\rangle$, and compute the probability that a photon is found at the output in mode b, as a function of ϕ, $\Delta\theta = \theta - \theta'$ and $\theta_{tot} = \theta + \theta'$. What is the fringe visibility as a function of $\Delta\theta$? Suppose you include only cases when at least one photon is measured to be output in b or c (in other words, throw away the case when no photons are measured at the output). How does the visibility change, when using this conditional probability? The case when $\Delta\theta = 0$ is known as the *balanced loss* Mach-Zehnder interferometer, and your results should show why balancing the loss in the two arms is helpful. Restricting cases to when at least one photon is detected implements a simple kind of quantum error detection code!

d) In the scenarios above, a single photon input was used. Now look at the lossy interferometer in the case of a coherent state input. In this case several meaningful definitions of visibility are possible. One way is to carry over the definition from part a), but instead of using the probability of detecting a single photon in one of the output arms, it is more meaningful to use the probability of detecting no photons (this kind of detector would discriminate whether any photons are detected or not during a trial). Another natural definition of visibility would simply be that of a classical interferometer, based on the intensity measured by a photodiode. How do the visibilities of the interferometer differ for a coherent state input and a single photon input? Describe a situation in which the effects of loss would allow the interferometer to distinguish between these two types of input.

4. **The Schmidt measure of pure state entanglement**

We will talk soon in class about entanglements, but this problem can already be addressed without special knowledge.

Entanglement is a property of a composite quantum system that cannot be changed by local operations and classical communications. How do we mathematically determine if a given state is entangled or not? And if a state is entangled, how entangled is it?

In this problem, we explore a measure of bi-partite entanglement known as the *Schmidt number*, which is particularly easy to compute. This is based on the Schmidt decomposition, which, for a pure state $|\psi\rangle$ in the Hilbert space of systems A and B, is the expression of $|\psi\rangle$ in the form

$$|\psi\rangle = \sum_k \lambda_k |k_A\rangle |k_B\rangle \,, \tag{8}$$

where $|k_A\rangle$ and $|k_B\rangle$ are orthonormal states of systems A and B, respectively, and $\sum_k \lambda_k^2 = 1$. Note that this is essentially just a singular value decomposition. The Schmidt number is defined as the number of nonzero λ_k. You may also find the following definition of the Schmidt number helpful, as given in Nielsen and Chuang (2005): $Sch(|\psi\rangle) = \mathrm{Rank}(\mathrm{Tr}_B(\rho))$, where $\rho = |\psi\rangle\langle\psi|$ is the density matrix and Tr_B means the trace over the eigenstates of B, $|k_B\rangle$.

a) Prove that $|\psi\rangle$ is a product state, that is $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$, for some states $|\psi_A\rangle$ and $|\psi_B\rangle$ of systems A and B, if and only if the Schmidt number of $|\psi\rangle$ is 1.

b) Prove that the Schmidt number cannot be changed by local unitary transforms. It turns out that even with any amount of additional classical communication, the Schmidt number still cannot be changed. Because this number is invariant under "local operations and classical communication," it is a measure of entanglement.

c) Give the Schmidt numbers for each of the following states:

$$|\phi_1\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}} \tag{9}$$

$$|\phi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \tag{10}$$

$$|\phi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \tag{11}$$

$$|\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} \,. \tag{12}$$

d) It would seem to follow that the more entangled a state you have (eg the higher Schmidt number a state has), the better you can do with interferometric measurements.

The Yurke state, $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$, is an input state of interferometers that gives a precision higher than the standard shot noise limit. What is the Schmidt number of the Yurke state? What is the Schmidt number of the state after the Yurke state is transformed by a 50/50 beamsplitter B, $|\phi_1\rangle = B|\psi\rangle$? Explain why this is a better measure for "how entangled" the interferometer is.

What is the Schmidt number of the state given by feeding a coherent state $|\alpha\rangle$ and vacuum $|0\rangle$ into B, i.e. $|\psi_1\rangle = B|\alpha\rangle|0\rangle$?

e) Often, the Schmidt number can lack meaning as a quantitative measure of entanglement, because it includes even the smallest non-zero coefficients in its count. One improvement on this is to count only coefficients above a certain threshold, as illustrated by the following example.

A two-mode squeezed state can be generated in the laboratory by a certain kind of optical parameteric oscillator; suppose this is the state

$$|\Psi\rangle = \exp\left[-\frac{r}{2}(a_1 a_2 - a_1^\dagger a_2^\dagger)\right]|0\rangle_1|0\rangle_2 \propto \frac{1}{\cosh r}\sum_n \tanh^n r|n\rangle_1|n\rangle_2\,. \tag{13}$$

Plot the number of Schmidt coefficients of $|\Psi\rangle$ which are above a fixed threshold, say 0.01, as a function of the squeezing parameter r. Show that this measure of entanglement increases with increasing r, as intuitively desired.