1. Classical Model of the Light Force

(a) Time averaged force

$$\begin{aligned}
\left\langle\vec{F}\right\rangle =&\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))\nabla(E_0\cos(\omega t+\theta))\rangle\\
=&\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))(\cos(\omega t+\theta)\nabla E_0-E_0\sin(\omega t+\theta)\nabla\theta)\rangle\\
=&\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})(u\nabla E_0+vE_0\nabla\theta)
\end{aligned}$$

(b) The potential picture

$$\begin{aligned}
\langle U\rangle =&-\left\langle\vec{p}\cdot\vec{E}\right\rangle\\
=&-\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))E_0\cos(\omega t+\theta)\rangle\\
=&-\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})uE_0\\
\left\langle\vec{F}\right\rangle =&\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})u\nabla E_0
\end{aligned}$$

The potential picture includes an extra term from the change of dipole moment when the dipole moves.

(c) Dipole moment of electron

Let $\vec{r}=\hat{\epsilon}\tilde{r}_0\mathrm{e}^{\mathrm{i}\omega t}$

$$\begin{aligned}
&-e\hat{\epsilon}E_0\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\theta}\\
=&\left(-m\omega^2+\mathrm{i}\omega\gamma+m\omega_0^2\right)\hat{\epsilon}\tilde{r}_0\mathrm{e}^{\mathrm{i}\omega t}\\
\tilde{r}_0=&\frac{eE_0\mathrm{e}^{\mathrm{i}\theta}}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\\
\tilde{\vec{p}}=&-\hat{\epsilon}\mathrm{e}^{\mathrm{i}\omega t}\frac{e^2E_0\mathrm{e}^{\mathrm{i}\theta}}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}
\end{aligned}$$

Real part

$$\begin{aligned}
\vec{p}=&-\hat{\epsilon}\Re\left(\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\theta}\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)\\
=&-\hat{\epsilon}\left(\cos(\omega t+\theta)\Re\left(\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)-\sin(\omega t+\theta)\Im\left(\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)\right)
\end{aligned}$$

Where

$$\begin{aligned}
&\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\\
=&e^2E_0\frac{1}{m(\omega^2-\omega_0^2)-\mathrm{i}\omega\gamma}\\
\approx&\frac{e^2E_0}{m\omega_0}\frac{1}{2\delta-\mathrm{i}\Gamma}\\
=&\frac{e^2E_0}{m\omega_0}\frac{2\delta+\mathrm{i}\Gamma}{4\delta^2+\Gamma^2}
\end{aligned}$$

Compare to the definition of u and v

$$\begin{aligned}
u =& -\Re\left(\frac{e^2E_0}{m\omega^2 - \mathrm{i}\omega\gamma - m\omega_0^2}\right)\\
=& -\frac{e^2E_0}{m\omega_0}\frac{2\delta}{4\delta^2+\Gamma^2}\\
v =& -\Im\left(\frac{e^2E_0}{m\omega^2 - \mathrm{i}\omega\gamma - m\omega_0^2}\right)\\
=& -\frac{e^2E_0}{m\omega_0}\frac{\Gamma}{4\delta^2+\Gamma^2}
\end{aligned}$$

Force

$$\begin{aligned}
\left\langle\vec{F}\right\rangle =& -\frac{e^2}{2m\omega_0}\frac{2E_0\nabla E_0\delta + \Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2}\\
=& -\frac{e^2}{2m\omega_0}\frac{\delta\nabla E_0^2 + \Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2}
\end{aligned}$$

(d) Force on a two-level atom

Since $\omega_R \propto E_0$

$$\begin{aligned}
\left\langle\vec{F}\right\rangle \propto& -\frac{\delta\nabla\omega_R^2 + \Gamma\omega_R^2\nabla\theta}{4\delta^2+\Gamma^2}\\
\propto& F_{quantum}
\end{aligned}$$

2. Master equation for a damped optical cavity

(a)

$$\begin{aligned}
\langle\tilde{\psi}_1|\tilde{\psi}_1\rangle =& \Gamma\mathrm{d}t\langle\psi|a^\dagger a|\psi\rangle\\
=& \mathrm{d}p
\end{aligned}$$

Since the imaginary and real part of H commute

$$\begin{aligned}
\langle\tilde{\psi}_0|\tilde{\psi}_0\rangle =& \langle\psi|\mathrm{e}^{\mathrm{i}H^\dagger\mathrm{d}t/\hbar}\mathrm{e}^{-\mathrm{i}H\mathrm{d}t/\hbar}|\psi\rangle\\
=& \langle\psi|\mathrm{e}^{-\Gamma\mathrm{d}ta^\dagger a}|\psi\rangle\\
=& \exp\left(-\Gamma\mathrm{d}t\langle\psi|a^\dagger a|\psi\rangle\right)\\
=& \mathrm{e}^{-\mathrm{d}p}\\
=& 1-\mathrm{d}p
\end{aligned}$$

Therefore

$$\begin{aligned}
|\tilde{\psi}_1\rangle =& \frac{|\tilde{\psi}_1\rangle}{\mathrm{d}p}\\
|\tilde{\psi}_0\rangle =& \frac{|\tilde{\psi}_0\rangle}{1-\mathrm{d}p}
\end{aligned}$$

The lossy term in H is proportional to $a^\dagger a$ because the loss is proportional to $a^\dagger a$. The prefactor gives the correct overall normalization.

(b)

$$\begin{aligned}
\rho + \mathrm{d}\rho =& |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| + |\tilde{\psi}_0\rangle\langle\tilde{\psi}_0| \\
=& \Gamma \mathrm{d}t a|\psi\rangle\langle\psi|a^\dagger + \left(1 - \frac{\mathrm{i}H^\dagger \mathrm{d}t}{\hbar}\right)|\psi\rangle\langle\psi|\left(1 + \frac{\mathrm{i}H \mathrm{d}t}{\hbar}\right) \\
=& \rho + \Gamma \mathrm{d}t a\rho a^\dagger + \rho\frac{\mathrm{i}H^\dagger \mathrm{d}t}{\hbar} - \frac{\mathrm{i}H \mathrm{d}t}{\hbar}\rho \\
=& \rho + \Gamma \mathrm{d}t a\rho a^\dagger - \frac{\mathrm{i}\mathrm{d}t}{\hbar}[H_0, \rho] - \mathrm{d}t\frac{\Gamma}{2}\rho a^\dagger a - \mathrm{d}t\frac{\Gamma}{2}a^\dagger a\rho
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} =& \Gamma a\rho a^\dagger - \frac{\mathrm{i}}{\hbar}[H_0, \rho] - \frac{\Gamma}{2}\rho a^\dagger a - \frac{\Gamma}{2}a^\dagger a\rho \\
=& -\frac{\mathrm{i}}{\hbar}[H_0, \rho] - \frac{1}{2}\left(\Gamma\rho a^\dagger a - 2\Gamma a\rho a^\dagger + \Gamma a^\dagger a\rho\right) \\
=& -\frac{\mathrm{i}}{\hbar}[H_0, \rho] - \frac{1}{2}\left(\rho C^\dagger C - 2C\rho C^\dagger + C^\dagger C\rho\right)
\end{aligned}$$

where $C = \sqrt{\Gamma}a$