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1.

(a)

Define

$$Y(x,y) = \frac{x^n y^{n-1} + x^{n-1} y^n}{\sqrt{2n!(n-1)!}}$$

when $[x,y]=0$

$$Y(x,y) = x^{n-1} y^{n-1} \frac{x+y}{\sqrt{2n!(n-1)!}}$$

we also have, for any number λ

$$Y(\lambda x, \lambda y) = \lambda^{2n-1} Y(x,y)$$

Before the first beam splitter

$$\begin{aligned}|\psi_0\rangle =& Y\left(a^\dagger, b^\dagger\right)|0,0\rangle\\ =& {a^\dagger}^{n-1} {b^\dagger}^{n-1} \frac{a^\dagger + b^\dagger}{\sqrt{2n!(n-1)!}}|0,0\rangle\end{aligned}$$

After the first beam splitter

$$\begin{aligned}|\psi_1\rangle =& B|\psi_0\rangle\\ =& BY\left(a^\dagger, b^\dagger\right)B^\dagger|0,0\rangle\\ =& Y\left(Ba^\dagger B^\dagger, Bb^\dagger B^\dagger\right)|0,0\rangle\\ =& \frac{\sqrt{2}}{2^n} Y\left(a^\dagger - \mathrm{i}b^\dagger, b^\dagger - \mathrm{i}a^\dagger\right)|0,0\rangle\\ =& \left(a^\dagger - \mathrm{i}b^\dagger\right)^{n-1}\left(b^\dagger - \mathrm{i}a^\dagger\right)^{n-1} \frac{a^\dagger - \mathrm{i}b^\dagger + b^\dagger - \mathrm{i}a^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\\ =& (-\mathrm{i})^{n-1}(1-\mathrm{i})\left({a^\dagger}^2 + {b^\dagger}^2\right)^{n-1} \frac{a^\dagger + b^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\end{aligned}$$

After phase shift

$$\begin{aligned}|\psi_2\rangle =& \frac{\sqrt{2}}{2^n} Y\left(\mathrm{e}^{-\mathrm{i}\phi}a^\dagger - \mathrm{i}b^\dagger, b^\dagger - \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}a^\dagger\right)|0,0\rangle\\ =& (-\mathrm{i})^{n-1}(1-\mathrm{i})\left(\mathrm{e}^{-2\mathrm{i}\phi}{a^\dagger}^2 + {b^\dagger}^2\right)^{n-1} \frac{\mathrm{e}^{-\mathrm{i}\phi}a^\dagger + b^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\end{aligned}$$

Output

$$\begin{aligned}
|\psi_3\rangle =& B^\dagger|\psi_2\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{2^{2n-1}}Y\Big(\mathrm{e}^{-\mathrm{i}\phi/2}\big(a^\dagger+\mathrm{i}b^\dagger\big)-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi/2}\big(b^\dagger+\mathrm{i}a^\dagger\big),\mathrm{e}^{\mathrm{i}\phi/2}\big(b^\dagger+\mathrm{i}a^\dagger\big)-\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi/2}\big(a^\dagger+\mathrm{i}b^\dagger\big)\Big)|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{2^{2n-1}}Y\left(2\cos\frac{\phi}{2}a^\dagger+2\sin\frac{\phi}{2}b^\dagger,2\cos\frac{\phi}{2}b^\dagger-2\sin\frac{\phi}{2}a^\dagger\right)|0,0\rangle\\
=&\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}Y\left(\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger,\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\right)|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{\sqrt{2n!(n-1)!}}\left(\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger\right)^{n-1}\left(\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\right)^{n-1}\\
&\left(\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger+\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\right)|0,0\rangle
\end{aligned}$$

For $\phi=0$

$$\begin{aligned}
|\phi_3\rangle_0 =&\frac{1}{\sqrt{n!(n-1)!}}{a^\dagger}^{n-1}{b^\dagger}^{n-1}\frac{a^\dagger+b^\dagger}{\sqrt{2}}|0,0\rangle\\
=&|\phi_0\rangle
\end{aligned}$$

(b)

Let

$$\begin{aligned}
a' =&\cos\frac{\phi}{2}a+\sin\frac{\phi}{2}b\\
b' =&\cos\frac{\phi}{2}b-\sin\frac{\phi}{2}a
\end{aligned}$$

then

$$\begin{aligned}
a'^\dagger =&\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger\\
b'^\dagger =&\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\\
a =&\cos\frac{\phi}{2}a'-\sin\frac{\phi}{2}b'\\
b =&\cos\frac{\phi}{2}b'+\sin\frac{\phi}{2}a'\\
a^\dagger =&\cos\frac{\phi}{2}a'^\dagger-\sin\frac{\phi}{2}b'^\dagger\\
b^\dagger =&\cos\frac{\phi}{2}b'^\dagger+\sin\frac{\phi}{2}a'^\dagger\\
\left[a',a'^\dagger\right] =&1\\
\left[b',b'^\dagger\right] =&1\\
|\phi_3\rangle =&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{\sqrt{n!(n-1)!}}{a'^\dagger}^{n-1}{b'^\dagger}^{n-1}\frac{a'^\dagger+b'^\dagger}{\sqrt{2}}|0,0\rangle
\end{aligned}$$

$$\begin{aligned}
M =& a^\dagger a - b^\dagger b\\
=& \left(\cos\frac{\phi}{2}a'^\dagger - \sin\frac{\phi}{2}b'^\dagger\right)\left(\cos\frac{\phi}{2}a' - \sin\frac{\phi}{2}b'\right) - \left(\cos\frac{\phi}{2}b'^\dagger + \sin\frac{\phi}{2}a'^\dagger\right)\left(\cos\frac{\phi}{2}b' + \sin\frac{\phi}{2}a'\right)\\
=& \cos^2\frac{\phi}{2}a'^\dagger a' + \sin^2\frac{\phi}{2}b'^\dagger b' - \sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(a'b'^\dagger + a'^\dagger b'\right)\\
& - \cos^2\frac{\phi}{2}b'^\dagger b' - \sin\frac{\phi}{2}\cos\frac{\phi}{2}a'^\dagger b' - \sin\frac{\phi}{2}\cos\frac{\phi}{2}a'b'^\dagger - \sin^2\frac{\phi}{2}a'^\dagger a'\\
=& \cos\phi\left(a'^\dagger a' - b'^\dagger b'\right) - \sin\phi\left(a'b'^\dagger + a'^\dagger b'\right)
\end{aligned}$$

Mean

$$\begin{aligned}
\langle M\rangle =& \frac{1}{2n!(n-1)!}\langle 0,0|a'^{n-1}b'^{n-1}(a'+b')\left(\cos\phi\left(a'^\dagger a' - b'^\dagger b'\right) - \sin\phi\left(a'b'^\dagger + a'^\dagger b'\right)\right)\\
& a'^{\dagger n-1}b'^{\dagger n-1}\left(a'^\dagger + b'^\dagger\right)|0,0\rangle\\
=& \frac{\cos\phi}{2n!(n-1)!}\langle 0,0|a'^{n-1}b'^{n-1}(a'+b')\left(a'^\dagger a' - b'^\dagger b'\right)a'^{\dagger n-1}b'^{\dagger n-1}\left(a'^\dagger + b'^\dagger\right)|0,0\rangle\\
& - \frac{\sin\phi}{2n!(n-1)!}\langle 0,0|a'^{n-1}b'^{n-1}(a'+b')\left(a'b'^\dagger + a'^\dagger b'\right)a'^{\dagger n-1}b'^{\dagger n-1}\left(a'^\dagger + b'^\dagger\right)|0,0\rangle\\
=& -n\sin\phi
\end{aligned}$$

Square

$$\begin{aligned}
M^2 =& \left(\cos\phi\left(a'^\dagger a' - b'^\dagger b'\right) - \sin\phi\left(a'b'^\dagger + a'^\dagger b'\right)\right)^2\\
=& \cos^2\phi\left(a'^\dagger a'a'^\dagger a' - 2a'^\dagger a'b'^\dagger b' + b'^\dagger b'b'^\dagger b'\right) + \sin^2\phi\left(a'^2 b'^{\dagger 2} + a'a'^\dagger b'^\dagger b' + a'^\dagger a'b'b'^\dagger + a'^{\dagger 2}b'^2\right)\\
& - \sin\phi\cos\phi\left(2a'^\dagger a'^2 b'^\dagger + a'b'^\dagger + 2a'^{\dagger 2}a'b' + a'^\dagger b' - 2a'b'^{\dagger 2}b' - a'b'^\dagger - 2a'^\dagger b'^\dagger b'^2 - a'^\dagger b'\right)\\
=& \cos^2\phi\left(a'^\dagger a'a'^\dagger a' - 2a'^\dagger a'b'^\dagger b' + b'^\dagger b'b'^\dagger b'\right) + \sin^2\phi\left(a'^2 b'^{\dagger 2} + a'a'^\dagger b'^\dagger b' + a'^\dagger a'b'b'^\dagger + a'^{\dagger 2}b'^2\right)\\
& - 2\sin\phi\cos\phi\left(a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2}a'b' - a'b'^{\dagger 2}b' - a'^\dagger b'^\dagger b'^2\right)\\
\left\langle M^2\right\rangle =& \cos^2\phi\left\langle a'^\dagger a'a'^\dagger a' - 2a'^\dagger a'b'^\dagger b' + b'^\dagger b'b'^\dagger b'\right\rangle\\
& + \sin^2\phi\left\langle a'^2 b'^{\dagger 2} + a'a'^\dagger b'^\dagger b' + a'^\dagger a'b'b'^\dagger + a'^{\dagger 2}b'^2\right\rangle\\
& - 2\sin\phi\cos\phi\left\langle a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2}a'b' - a'b'^{\dagger 2}b' - a'^\dagger b'^\dagger b'^2\right\rangle\\
=& \frac{\cos^2\phi}{2n}\left(n^3 + (n-1)^2 n - 2n^2(n-1) - 2n^2(n-1) + (n-1)^2 n + n^3\right)\\
& + \frac{\sin^2\phi}{2n}\left(2(n-1)n(n+1) + 2n^3\right)\\
& - \frac{\sin\phi\cos\phi}{n}\left(n^2(n-1) + n^2(n-1) - n^2(n-1) - n^2(n-1)\right)\\
=& \cos^2\phi + \sin^2\phi\left(2n^2 - 1\right)
\end{aligned}$$

Fluctuation

$$\begin{aligned}\Delta M^2 =&\cos^2\phi + \sin^2\phi\big(2n^2 - 1\big) - n^2\sin^2\phi\\=&\cos^2\phi + \sin^2\phi\big(n^2 - 1\big)\\\frac{\partial\langle M\rangle}{\partial\phi} =&-n\cos\phi\\\Delta\phi^2 =&\frac{\cos^2\phi + \sin^2\phi\big(n^2 - 1\big)}{n^2\cos^2\phi}\\=&\frac{1 + \tan^2\phi\big(n^2 - 1\big)}{n^2}\\\Delta\phi =&\frac{\sqrt{1 + \tan^2\phi(n^2 - 1)}}{n}\end{aligned}$$

for $\phi = 0$

$$\Delta\phi = \frac{1}{n}$$

(c)

$$\begin{aligned}|\psi_0\rangle =&|\alpha, 0\rangle\\|\psi_1\rangle =&\left|\frac{\alpha}{\sqrt{2}}, -\frac{\mathrm{i}\alpha}{\sqrt{2}}\right\rangle\\|\psi_2\rangle =&\left|\frac{\alpha\mathrm{e}^{-\mathrm{i}\phi}}{\sqrt{2}}, -\frac{\mathrm{i}\alpha}{\sqrt{2}}\right\rangle\\|\psi_3\rangle =&\left|\frac{\alpha\mathrm{e}^{-\mathrm{i}\phi}}{2} + \frac{\alpha}{2}, \mathrm{i}\left(\frac{\alpha\mathrm{e}^{-\mathrm{i}\phi}}{2} - \frac{\alpha}{2}\right)\right\rangle\\=&\left|\alpha\mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{\phi}{2}, \alpha\mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{\phi}{2}\right\rangle\end{aligned}$$

(d)

Mean

$$\begin{aligned}\langle M\rangle =&\big\langle a^\dagger a - b^\dagger b\big\rangle\\=&\alpha^*\mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{\phi}{2}\alpha\mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{\phi}{2} - \alpha^*\mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{\phi}{2}\alpha\mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{\phi}{2}\\=&|\alpha|^2\cos^2\frac{\phi}{2} - \alpha^2\sin^2\frac{\phi}{2}\\=&|\alpha|^2\cos\phi\end{aligned}$$

Square

$$\begin{aligned}\big\langle M^2\big\rangle =&\big\langle\big(a^\dagger a - b^\dagger b\big)\big(a^\dagger a - b^\dagger b\big)\big\rangle\\=&\big\langle a^\dagger a a^\dagger a - a^\dagger a b^\dagger b - b^\dagger b a^\dagger a + b^\dagger b b^\dagger b\big\rangle\\=&\langle M\rangle^2 + \big\langle a^\dagger a + b^\dagger b\big\rangle\\=&\langle M\rangle^2 + |\alpha|^2\end{aligned}$$

Uncertainty

$$\begin{aligned}\left\langle\Delta\phi^2\right\rangle =&\frac{|\alpha|^2}{|\alpha|^4\sin^2\phi}\\ =&\frac{1}{|\alpha|^2\sin^2\phi}\end{aligned}$$

2.

(a)

$$\begin{aligned}P =&|\psi_A|^2|\psi_B|^2\left|\mathrm{e}^{\mathrm{i}\left(\vec{k}_A\cdot\vec{r}_{A1}+\vec{k}_B\cdot\vec{r}_{B2}\right)}\pm\mathrm{e}^{\mathrm{i}\left(\vec{k}_A\cdot\vec{r}_{A2}+\vec{k}_B\cdot\vec{r}_{B1}\right)}\right|^2\\ =&|\psi_A|^2|\psi_B|^2\left|\mathrm{e}^{\mathrm{i}\vec{k}_A\cdot\vec{r}_{21}}\pm\mathrm{e}^{\mathrm{i}\vec{k}_B\cdot\vec{r}_{21}}\right|^2\\ =&|\psi_A|^2|\psi_B|^2\left|\mathrm{e}^{\mathrm{i}\left(\vec{k}_A-\vec{k}_B\right)\cdot\vec{r}_{21}/2}\pm\mathrm{e}^{-\mathrm{i}\left(\vec{k}_A-\vec{k}_B\right)\cdot\vec{r}_{21}/2}\right|^2\\ =&|\psi_A|^2|\psi_B|^2\left|2\pm\mathrm{e}^{\mathrm{i}\left(\vec{k}_A-\vec{k}_B\right)\cdot\vec{r}_{21}}\pm\mathrm{e}^{-\mathrm{i}\left(\vec{k}_A-\vec{k}_B\right)\cdot\vec{r}_{21}}\right|\\ =&2|\psi_A|^2|\psi_B|^2\left(1\pm\cos\left(\left(\vec{k}_A-\vec{k}_B\right)\cdot\vec{r}_{21}\right)\right)\end{aligned}$$

(b)

In order to see the correlation $\Delta\phi_t\ll 1$,

$$\begin{aligned}1\gg&\Delta\vec{k}_t\cdot\vec{r}_{21t}\\ =&k_0\frac{W}{d}w\\ =&\frac{2\pi Ww}{\lambda d}\\ \lambda d\gg&Ww\end{aligned}$$

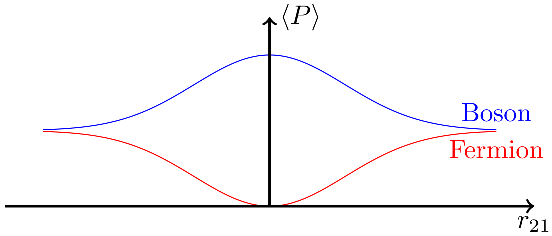
For ^{6}Li at 500μK

$$\begin{aligned}\lambda =&\frac{\hbar}{p}\\ =&\frac{\hbar}{\sqrt{mk_BT}}\\ \approx&12\mathrm{nm}\\ w_{max}\approx&\sqrt{\lambda d}\\ \approx&36\mu\mathrm{m}\end{aligned}$$

(c)

i.

$$
\begin{aligned}
\langle P\rangle =&2|\psi_A|^2|\psi_B|^2\sqrt{\frac{\gamma}{\pi}}\int(1\pm\cos\Delta k_r r_{21})\mathrm{e}^{-\Delta k_r^2\gamma^2}\mathrm{d}\Delta k_r\\
=&2|\psi_A|^2|\psi_B|^2\left(1\pm\exp\left(-\frac{r_{21}^2}{4\gamma^2}\right)\right)\\
=&2|\psi_A|^2|\psi_B|^2\left(1\pm\exp\left(-\frac{r_{21l}^2+r_{21t}^2}{4\gamma^2}\right)\right)
\end{aligned}
$$



ii.

$$
\begin{aligned}
\Delta k =&\frac{\Delta p}{\hbar}\\
=&\frac{m}{\hbar}\Delta v\\
=&\frac{m}{\hbar}\Delta\left(\frac{d}{\tau}\right)\\
=&\frac{mL}{\hbar\tau}\\
=&\frac{mLv}{\hbar d}
\end{aligned}
$$

$$
\begin{aligned}
vt_{min} <&\frac{\hbar d}{mvL}\\
t_{min} <&\frac{\hbar\tau^2}{mdL}\\
=&\frac{\hbar\tau^2}{mdL}\\
\approx&35\mu\mathrm{s}
\end{aligned}
$$

(d)