1. Classical Model of the Light Force

(a) Time averaged force

$$
\begin{aligned}
\left\langle\vec{F}\right\rangle =&\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))\nabla(E_0\cos(\omega t+\theta))\rangle\\
=&\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))(\cos(\omega t+\theta)\nabla E_0-E_0\sin(\omega t+\theta)\nabla\theta)\rangle\\
=&\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})(u\nabla E_0+vE_0\nabla\theta)
\end{aligned}
$$

(b) The potential picture

$$
\begin{aligned}
\langle U\rangle =&-\left\langle\vec{p}\cdot\vec{E}\right\rangle\\
=&-\langle(\hat{p}\cdot\hat{\epsilon})(u\cos(\omega t+\theta)-v\sin(\omega t+\theta))E_0\cos(\omega t+\theta)\rangle\\
=&-\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})uE_0\\
\left\langle\vec{F}\right\rangle =&\frac{1}{2}(\hat{p}\cdot\hat{\epsilon})u\nabla E_0
\end{aligned}
$$

(c) Dipole moment of electron

Let $\vec{r}=\hat{\epsilon}\tilde{r}_0\mathrm{e}^{\mathrm{i}\omega t}$

$$
\begin{aligned}
&-e\hat{\epsilon}E_0\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\theta}\\
=&\left(-m\omega^2+\mathrm{i}\omega\gamma+m\omega_0^2\right)\hat{\epsilon}\tilde{r}_0\mathrm{e}^{\mathrm{i}\omega t}\\
\tilde{r}_0 =&\frac{eE_0\mathrm{e}^{\mathrm{i}\theta}}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\\
\tilde{\vec{p}} =&-\hat{\epsilon}\mathrm{e}^{\mathrm{i}\omega t}\frac{e^2E_0\mathrm{e}^{\mathrm{i}\theta}}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}
\end{aligned}
$$

Real part

$$
\begin{aligned}
\vec{p} =&-\hat{\epsilon}\Re\left(\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\theta}\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)\\
=&-\hat{\epsilon}\left(\cos(\omega t+\theta)\Re\left(\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)-\sin(\omega t+\theta)\Im\left(\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\right)\right)
\end{aligned}
$$

Where

$$
\begin{aligned}
&\frac{e^2E_0}{m\omega^2-\mathrm{i}\omega\gamma-m\omega_0^2}\\
=&e^2E_0\frac{1}{m(\omega^2-\omega_0^2)-\mathrm{i}\omega\gamma}\\
\approx&\frac{e^2E_0}{m\omega_0}\frac{1}{2\delta-\mathrm{i}\Gamma}\\
=&\frac{e^2E_0}{m\omega_0}\frac{2\delta+\mathrm{i}\Gamma}{4\delta^2+\Gamma^2}
\end{aligned}
$$

Compare to the definition of u and v

$$
\begin{aligned}
u =& -\Re\left(\frac{e^2E_0}{m\omega^2 - \mathrm{i}\omega\gamma - m\omega_0^2}\right)\\
=& -\frac{e^2E_0}{m\omega_0}\frac{2\delta}{4\delta^2 + \Gamma^2}\\
v =& -\Im\left(\frac{e^2E_0}{m\omega^2 - \mathrm{i}\omega\gamma - m\omega_0^2}\right)\\
=& -\frac{e^2E_0}{m\omega_0}\frac{\Gamma}{4\delta^2 + \Gamma^2}
\end{aligned}
$$

Force

$$
\begin{aligned}
\left\langle\vec{F}\right\rangle =& -\frac{e^2}{2m\omega_0}\frac{2E_0\nabla E_0\delta + \Gamma E_0^2\nabla\theta}{4\delta^2 + \Gamma^2}\\
=& -\frac{e^2}{2m\omega_0}\frac{\delta\nabla E_0^2 + \Gamma E_0^2\nabla\theta}{4\delta^2 + \Gamma^2}
\end{aligned}
$$

(d) Force on a two-level atom

Since $\omega_R \propto E_0$

$$
\begin{aligned}
\left\langle\vec{F}\right\rangle \propto& -\frac{\delta\nabla\omega_R^2 + \Gamma\omega_R^2\nabla\theta}{4\delta^2 + \Gamma^2}\\
\propto& F_{quantum}
\end{aligned}
$$

2. Master equation for a damped optical cavity

(a)

$$
\begin{aligned}
\langle\tilde{\psi}_1|\tilde{\psi}_1\rangle =& \Gamma\mathrm{d}t\langle\psi|a^\dagger a|\psi\rangle\\
=& \mathrm{d}p
\end{aligned}
$$

Since the imaginary and real part of H commute

$$
\begin{aligned}
\langle\tilde{\psi}_0|\tilde{\psi}_0\rangle =& \langle\psi|\mathrm{e}^{\mathrm{i}H^\dagger\mathrm{d}t/\hbar}\mathrm{e}^{-\mathrm{i}H\mathrm{d}t/\hbar}|\psi\rangle\\
=& \langle\psi|\mathrm{e}^{-\Gamma\mathrm{d}ta^\dagger a}|\psi\rangle\\
=& \exp\left(-\Gamma\mathrm{d}t\langle\psi|a^\dagger a|\psi\rangle\right)\\
=& \mathrm{e}^{-\mathrm{d}p}\\
=& 1 - \mathrm{d}p
\end{aligned}
$$

(b)

$$
\begin{aligned}
\rho+\mathrm{d}\rho=&|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|+|\tilde{\psi}_0\rangle\langle\tilde{\psi}_0|\\
=&\Gamma\mathrm{d}ta|\psi\rangle\langle\psi|a^\dagger+\left(1-\frac{\mathrm{i}H^\dagger\mathrm{d}t}{\hbar}\right)|\psi\rangle\langle\psi|\left(1+\frac{\mathrm{i}H\mathrm{d}t}{\hbar}\right)\\
=&\rho+\Gamma\mathrm{d}ta\rho a^\dagger+\rho\frac{\mathrm{i}H^\dagger\mathrm{d}t}{\hbar}-\frac{\mathrm{i}H\mathrm{d}t}{\hbar}\rho\\
=&\rho+\Gamma\mathrm{d}ta\rho a^\dagger-\frac{\mathrm{i}\mathrm{d}t}{\hbar}[H_0,\rho]-\mathrm{d}t\frac{\Gamma}{2}\rho a^\dagger a-\mathrm{d}t\frac{\Gamma}{2}a^\dagger a\rho
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t}=&\Gamma a\rho a^\dagger-\frac{\mathrm{i}}{\hbar}[H_0,\rho]-\frac{\Gamma}{2}\rho a^\dagger a-\frac{\Gamma}{2}a^\dagger a\rho\\
=&-\frac{\mathrm{i}}{\hbar}[H_0,\rho]-\frac{1}{2}\left(\Gamma\rho a^\dagger a-2\Gamma a\rho a^\dagger+\Gamma a^\dagger a\rho\right)\\
=&-\frac{\mathrm{i}}{\hbar}[H_0,\rho]-\frac{1}{2}\left(\rho C^\dagger C-2C\rho C^\dagger+C^\dagger C\rho\right)
\end{aligned}
$$

where $C=\sqrt{\Gamma}a$