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1.

(a)

$$\begin{aligned}
&\left[a^\dagger b - ab^\dagger, a^\dagger a + b^\dagger b\right]\\
=&\left[a^\dagger b, a^\dagger a\right] - \left[ab^\dagger, a^\dagger a\right] + \left[a^\dagger b, b^\dagger b\right] - \left[ab^\dagger, b^\dagger b\right]\\
=&a^\dagger\left[a^\dagger, a\right]b - \left[a, a^\dagger\right]ab^\dagger + a^\dagger\left[b, b^\dagger\right]b - ab^\dagger\left[b^\dagger, b\right]\\
=&-a^\dagger b - ab^\dagger + a^\dagger b + ab^\dagger\\
=&0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[B, n_a + n_b\right] =&\left[\exp\left(\theta\left(a^\dagger b - ab^\dagger\right)\right), n_a + n_b\right]\\
=&0\\
B^\dagger =&\exp\left(\theta\left(a^\dagger b - ab^\dagger\right)^\dagger\right)\\
=&\exp\left(-\theta\left(a^\dagger b - ab^\dagger\right)\right)\\
=&B^{-1}
\end{aligned}$$

(b)

$$\begin{aligned}
\exp\left(\theta A\right)B\exp\left(-\theta A\right) =&\sum_{nm}(-1)^m\frac{\theta^{n+m}A^nBA^m}{n!m!}\\
=&\sum_N\sum_{m=0}^N(-1)^m\frac{\theta^NA^{N-m}BA^m}{(N-m)!m!}\\
=&\sum_N\frac{\theta^N}{N!}\sum_{m=0}^N(-1)^m\frac{N!A^{N-m}BA^m}{(N-m)!m!}\\
=&\sum_N\frac{\theta^N}{N!}[A, B]_N
\end{aligned}$$

where $[A,B]_N$ is defined as $[A,B]_N = \big[A,[A,B]_{N-1}\big]$ and $[A,B]_0 = B$

$$\begin{aligned}
\big[(a^\dagger b - ab^\dagger), a\big]_N =& \begin{cases} (-1)^{N/2}a & (2\mid N)\\ (-1)^{(N+1)/2}b & (2\nmid N)\end{cases}\\
\big[(a^\dagger b - ab^\dagger), b\big]_N =& \begin{cases} (-1)^{N/2}b & (2\mid N)\\ (-1)^{(N-1)/2}a & (2\nmid N)\end{cases}\\
BaB^{-1} =& \sum_N \frac{\theta^N}{N!}\big[(a^\dagger b - ab^\dagger), a\big]_N\\
=& \sum_n \frac{\theta^{2n}}{(2n)!}\big[(a^\dagger b - ab^\dagger), a\big]_{2n} + \sum_n \frac{\theta^{2n+1}}{(2n+1)!}\big[(a^\dagger b - ab^\dagger), a\big]_{2n+1}\\
=& \sum_n \frac{\theta^{2n}}{(2n)!}(-1)^n a + \sum_n \frac{\theta^{2n+1}}{(2n+1)!}(-1)^{n+1} b\\
=& \cos\theta a - \sin\theta b\\
BbB^{-1} =& \sum_N \frac{\theta^N}{N!}\big[(a^\dagger b - ab^\dagger), b\big]_N\\
=& \sum_n \frac{\theta^{2n}}{(2n)!}\big[(a^\dagger b - ab^\dagger), b\big]_{2n} + \sum_n \frac{\theta^{2n+1}}{(2n+1)!}\big[(a^\dagger b - ab^\dagger), b\big]_{2n+1}\\
=& \sum_n \frac{\theta^{2n}}{(2n)!}(-1)^n b + \sum_n \frac{\theta^{2n+1}}{(2n+1)!}(-1)^{n} a\\
=& \cos\theta b + \sin\theta a\\
B|0,\alpha\rangle =& B\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^\dagger}|0,0\rangle\\
=& \mathrm{e}^{-|\alpha|^2/2}B\mathrm{e}^{\alpha a^\dagger}B^{-1}B|0,0\rangle\\
=& \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\left(\cos\theta a^\dagger - \sin\theta b^\dagger\right)}|0,0\rangle\\
=& \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\cos\theta a^\dagger}\mathrm{e}^{-\alpha\sin\theta b^\dagger}|0,0\rangle\\
=& |-\alpha\sin\theta, \alpha\cos\theta\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
s_x =& a^\dagger b + ab^\dagger\\
s_y =& -\mathrm{i}\big(a^\dagger b - ab^\dagger\big)\\
B =& \mathrm{e}^{\mathrm{i}\theta s_y}
\end{aligned}$$

which is a rotation around y

$$\begin{aligned}
&(n_a + n_b)^2\\
=& s_z^2 + 4a^\dagger a b^\dagger b\\
=& s_z^2 + 4s^+s^-\\
=& s_x^2 + s_y^2 + s_z^2\\
=& S^2
\end{aligned}$$

which is the total spin

$$\begin{aligned}
[s_x, s_y] =& \left[a^\dagger b + ab^\dagger, -\mathrm{i}\left(a^\dagger b - ab^\dagger\right)\right]\\
=& -\mathrm{i}\left[a^\dagger b + ab^\dagger, a^\dagger b - ab^\dagger\right]\\
=& 2\mathrm{i}\left[a^\dagger b, ab^\dagger\right]\\
=& 2\mathrm{i}a^\dagger a\left[b, b^\dagger\right] + 2\mathrm{i}\left[a^\dagger, a\right]b^\dagger b\\
=& 2\mathrm{i}a^\dagger a - 2\mathrm{i}b^\dagger b\\
=& 2\mathrm{i}s_z\\
[s_y, s_z] =& \left[-\mathrm{i}\left(a^\dagger b - ab^\dagger\right), a^\dagger a - b^\dagger b\right]\\
=& -\mathrm{i}a^\dagger\left[a^\dagger, a\right]b + \mathrm{i}a^\dagger\left[b, b^\dagger\right]b + \mathrm{i}\left[a, a^\dagger\right]ab^\dagger - \mathrm{i}ab^\dagger\left[b^\dagger, b\right]\\
=& 2\mathrm{i}s_x\\
[s_z, s_x] =& \left[a^\dagger a - b^\dagger b, a^\dagger b + ab^\dagger\right]\\
=& a^\dagger\left[a, a^\dagger\right]b + \left[a^\dagger, a\right]ab^\dagger - a^\dagger\left[b^\dagger, b\right]b - ab^\dagger\left[b, b^\dagger\right]\\
=& 2a^\dagger b - 2ab^\dagger\\
=& 2\mathrm{i}s_y
\end{aligned}$$

(d)

$$\begin{aligned}
B|0, n\rangle =& B\frac{{a^\dagger}^n}{n!}|0, 0\rangle\\
=& \frac{\left(a^\dagger - b^\dagger\right)^n}{2^{n/2}n!}|0, 0\rangle\\
=& \sum_i \frac{{a^\dagger}^i {b^\dagger}^{n-i}}{2^{n/2}i!(n-i)!}|0, 0\rangle\\
=& \sum_i \frac{|n-i, i\rangle}{\sqrt{2^n i!(n-i)!}}
\end{aligned}$$

The state(s) with the largest amplitude is $|n/2, n/2\rangle$ (when n is even) or $|(n+1)/2, (n-1)/2\rangle$ and $|(n-1)/2, (n+1)/2\rangle$ when n is odd

The variance of the distribution is $\dfrac{n}{4}$ so the relative width is getting narrower for larger n although the absolute width is getting wider.

2.

(a)

$$\begin{aligned}(\Delta n')^2 =&\left\langle n'^2\right\rangle - \langle n'\rangle^2\\ =&|t|^4(\Delta n)^2\\ =&|t|^4|\alpha|^2\\ \langle n'\rangle =&|t|^2|\alpha|^2\\ >&|t|^4|\alpha|^2 \qquad (\text{when } 0<|t|<1 \text{ and } \alpha\neq 0)\end{aligned}$$

(b)

As shown in problem one.

$$\begin{aligned}\rho' =&\mathrm{Tr}_b\left(B|0,\alpha\rangle\langle 0,\alpha|B^\dagger\right)\\ =&\mathrm{Tr}_b(|r\alpha,t\alpha\rangle\langle r\alpha,t\alpha|)\\ =&|t\alpha\rangle\langle t\alpha|\end{aligned}$$

3.

(a)

After the first beam splitter

$$|\psi_1\rangle =\frac{|0,1\rangle+|1,0\rangle}{\sqrt{2}}$$

After the phase shift

$$|\psi_2\rangle =\frac{\mathrm{e}^{-\mathrm{i}\phi/2}|0,1\rangle+\mathrm{e}^{\mathrm{i}\phi/2}|1,0\rangle}{\sqrt{2}}$$

After the second beam splitter

$$\begin{aligned}|\psi_3\rangle =&\frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2}}\left(\frac{|0,1\rangle-|1,0\rangle}{\sqrt{2}}\right)+\frac{\mathrm{e}^{\mathrm{i}\phi/2}}{\sqrt{2}}\left(\frac{|0,1\rangle+|1,0\rangle}{\sqrt{2}}\right)\\ =&\frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{2}(|0,1\rangle-|1,0\rangle)+\frac{\mathrm{e}^{\mathrm{i}\phi/2}}{2}(|0,1\rangle+|1,0\rangle)\\ =&\cos\frac{\phi}{2}|0,1\rangle+\mathrm{i}\sin\frac{\phi}{2}|1,0\rangle\\ P_b =&\cos^2\frac{\phi}{2}\\ V =&\frac{\max(P_b)-\min(P_b)}{\max(P_b)+\min(P_b)}\\ =&\frac{1-0}{1+0}\\ =&1\end{aligned}$$

(b)

$$
\begin{aligned}
|\psi_0\rangle =& |0,1,0\rangle\\
|\psi_1\rangle =& \frac{|0,0,1\rangle+|0,1,0\rangle}{\sqrt{2}}\\
|\psi_2\rangle =& \frac{1}{\sqrt{2}}\Big(\mathrm{e}^{\mathrm{i}\phi/2}|0,0,1\rangle+\mathrm{e}^{-\mathrm{i}\phi/2}(\cos\theta|0,1,0\rangle+\sin\theta|1,0,0\rangle)\Big)\\
|\psi_3\rangle =& \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\phi/2}\frac{|0,0,1\rangle+|0,1,0\rangle}{\sqrt{2}}+\cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}\frac{|0,1,0\rangle-|0,0,1\rangle}{\sqrt{2}}+\sin\theta\mathrm{e}^{-\mathrm{i}\phi/2}|1,0,0\rangle\right)\\
=& \frac{\big(\mathrm{e}^{\mathrm{i}\phi/2}-\cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}\big)|0,0,1\rangle-\big(\mathrm{e}^{\mathrm{i}\phi/2}+\cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}\big)|0,1,0\rangle}{2}+\frac{\sin\theta\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2}}|1,0,0\rangle\\
P_b =& \left|\frac{\mathrm{e}^{\mathrm{i}\phi/2}+\cos\theta\mathrm{e}^{-\mathrm{i}\phi/2}}{2}\right|^2\\
=& \left(\frac{1+\cos\theta}{2}\cos\frac{\phi}{2}\right)^2+\left(\frac{1-\cos\theta}{2}\sin\frac{\phi}{2}\right)^2\\
=& \frac{1+\cos^2\theta}{4}+\frac{\cos\theta}{2}\cos\phi\\
V =& \frac{2\cos\theta}{1+\cos^2\theta}
\end{aligned}
$$

(c)

$$
\begin{aligned}
|\psi_0\rangle =& |0,0,1,0\rangle\\
|\psi_1\rangle =& \frac{|0,1,0,0\rangle+|0,0,1,0\rangle}{\sqrt{2}}\\
|\psi_2\rangle =& \frac{1}{\sqrt{2}}\Big(\mathrm{e}^{\mathrm{i}\phi/2}|0,1,0,0\rangle+\mathrm{e}^{-\mathrm{i}\phi/2}|0,0,1,0\rangle\Big)\\
|\psi_3\rangle =& \frac{\mathrm{e}^{\mathrm{i}\phi/2}}{\sqrt{2}}(\cos\theta|0,1,0,0\rangle+\sin\theta|1,0,0,0\rangle)+\frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2}}(\cos\theta'|0,0,1,0\rangle+\sin\theta'|0,0,0,1\rangle)\\
=& \frac{\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta}{\sqrt{2}}|1,0,0,0\rangle+\frac{\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'}{\sqrt{2}}|0,0,0,1\rangle+\frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta}{\sqrt{2}}|0,1,0,0\rangle+\frac{\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{\sqrt{2}}|0,0,1,0\rangle\\
|\psi_4\rangle =& \frac{\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta}{\sqrt{2}}|1,0,0,0\rangle+\frac{\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'}{\sqrt{2}}|0,0,0,1\rangle\\
&+\frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta}{2}(|0,0,1,0\rangle+|0,1,0,0\rangle)+\frac{\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{2}(|0,0,1,0\rangle-|0,1,0,0\rangle)\\
=& \frac{\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta}{\sqrt{2}}|1,0,0,0\rangle+\frac{\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'}{\sqrt{2}}|0,0,0,1\rangle\\
&+\frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta+\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{2}|0,0,1,0\rangle+\frac{\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta-\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'}{2}|0,1,0,0\rangle
\end{aligned}
$$

Maximum and minimum probabilities

$$\begin{aligned}\max P_b =&\frac{1}{4}(\cos\theta + \cos\theta')^2\\ \min P_b =&\frac{1}{4}(\cos\theta - \cos\theta')^2\\ V =&\frac{2\cos\theta\cos\theta'}{\cos^2\theta + \cos^2\theta'}\end{aligned}$$

Changing to condition probability does not change the visibility.

(d)

$$\begin{aligned}|\psi_0\rangle =&|0, 0, \alpha, 0\rangle\\ |\psi_1\rangle =&|0, \frac{\alpha}{2}, \frac{\alpha}{2}, 0\rangle\\ |\psi_2\rangle =&|0, \mathrm{e}^{\mathrm{i}\phi/2}\frac{\alpha}{2}, \mathrm{e}^{-\mathrm{i}\phi/2}\frac{\alpha}{2}, 0\rangle\\ |\psi_3\rangle =&|\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta\frac{\alpha}{2}, \mathrm{e}^{\mathrm{i}\phi/2}\cos\theta\frac{\alpha}{2}, \mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'\frac{\alpha}{2}, \mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'\frac{\alpha}{2}\rangle\\ |\psi_4\rangle =&|\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta\frac{\alpha}{2}, \mathrm{e}^{\mathrm{i}\phi/2}\cos\theta\frac{\alpha}{2} - \mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'\frac{\alpha}{2}, \mathrm{e}^{\mathrm{i}\phi/2}\cos\theta\frac{\alpha}{2} + \mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta'\frac{\alpha}{2}, \mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta'\frac{\alpha}{2}\rangle\end{aligned}$$

Maximum and minimum intensity

$$\begin{aligned}\max P_b =&\frac{\alpha^2}{4}(\cos\theta + \cos\theta')^2\\ \min P_b =&\frac{\alpha^2}{4}(\cos\theta - \cos\theta')^2\\ V =&\frac{2\cos\theta\cos\theta'}{\cos^2\theta + \cos^2\theta'}\end{aligned}$$

4.

(a)

From the definition of entangled state, the state is a product state if Schmidt number is 1. If a product state has a Schmidt number greater than 1

$$\begin{aligned}|\psi_A\rangle|\psi_B\rangle =&\sum_k \lambda_k|k_A\rangle|k_B\rangle\\ |\psi_A\rangle =&\sum_k \lambda_k\langle\psi_B|k_B\rangle|k_A\rangle\\ |\psi_B\rangle =&\sum_k \lambda_k\langle\psi_A|k_A\rangle|k_B\rangle\\ |\psi_A\rangle|\psi_B\rangle =&\sum_{k_1,k_2} \lambda_{k_1}\langle\psi_B|k_{1B}\rangle|k_{1A}\rangle\lambda_{k_2}\langle\psi_A|k_{2A}\rangle|k_{2B}\rangle\\ \neq&\sum_k \lambda_k|k_A\rangle|k_B\rangle\end{aligned}$$

(b)

With the same local unitary transformation on the right hand side, it is still a valid expansion of the new state and therefore the Schmidt number does not change.

(c)

State	Schmidt number
ϕ_1	3
ϕ_2	1
ϕ_3	2
ϕ_4	2

(d)

Schmidt number is 2.

$$\begin{aligned}|\phi_1\rangle =&B|\psi\rangle\\ =&\frac{1}{\sqrt{n!(n-1)!2^n}}\left(\left(a^\dagger+b^\dagger\right)^n\left(a^\dagger-b^\dagger\right)^{n-1}+\left(a^\dagger+b^\dagger\right)^{n-1}\left(a^\dagger-b^\dagger\right)^n\right)|0,0\rangle\\ =&\frac{1}{\sqrt{n!(n-1)!2^{n-1}}}\left(a^\dagger+b^\dagger\right)^{n-1}a^\dagger\left(a^\dagger-b^\dagger\right)^{n-1}|0,0\rangle\\ =&\frac{1}{\sqrt{n!(n-1)!2^{n-1}}}\left({a^\dagger}^2-{b^\dagger}^2\right)^{n-1}a^\dagger|0,0\rangle\end{aligned}$$

Therefore the Schmidt number is n. This is a better measure since these are the states that interfere later.
The Schmidt number for a coherent state after a beam splitter is 1 since it is still a product state.

(e)