1. The Dressed Atom

(a)

AC Stark shift

$$
\begin{aligned}
E_{Stark} =& \frac{1}{2}(\Omega - |\delta|) \\
=& \frac{1}{2}\left(\sqrt{\Omega_1^2 + \delta^2} - |\delta|\right)
\end{aligned}
$$

In the far detune limit

$$
E_{Stark} \approx \frac{\Omega_1^2}{4\delta}
$$

The sign depends on the state and the detuning (positive for excited state red detune and ground state blue detune and negetive otherwise)

(b)

(c)

(d)

(e)

(f)

2. Sideband Cooling

(a)

Interaction term

$$
\begin{aligned}
H_I =& \hbar\Omega(\sigma_+ + \sigma_-)\cos\left(k\hat{x} - \omega t\right) \\
=& \hbar\Omega(\sigma_+ + \sigma_-)\cos\left(\eta\left(a + a^\dagger\right) - \omega t\right) \\
=& \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{\mathrm{i}\left(\eta\left(a+a^\dagger\right) - \omega t\right)} + \mathrm{e}^{-\mathrm{i}\left(\eta\left(a+a^\dagger\right) - \omega t\right)}\right)
\end{aligned}
$$

In the Lamb-Dicke limit

$$
\begin{aligned}
H_I =& \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{\mathrm{i}\eta\left(a+a^\dagger\right)}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\eta\left(a+a^\dagger\right)}\mathrm{e}^{\mathrm{i}\omega t}\right) \\
=& \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\left(1 + \mathrm{i}\eta\left(a + a^\dagger\right)\right)\mathrm{e}^{-\mathrm{i}\omega t} + \left(1 - \mathrm{i}\eta\left(a + a^\dagger\right)\right)\mathrm{e}^{\mathrm{i}\omega t}\right) \\
=& \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{\mathrm{i}\omega t}\right) + \mathrm{i}\eta\frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)\left(a + a^\dagger\right)\left(\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{e}^{\mathrm{i}\omega t}\right)
\end{aligned}
$$

In the interaction picture

$$\begin{aligned}
\sigma'_{\pm} =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}\sigma_{\pm}\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&\sigma_{\pm}\mathrm{e}^{\mathrm{i}\omega_0t}\\
a' =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}a\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&a\mathrm{e}^{-\mathrm{i}\nu t}\\
a'^{\dagger} =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}a^{\dagger}\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&a^{\dagger}\mathrm{e}^{\mathrm{i}\nu t}
\end{aligned}$$

In the rotating wave approximation

$$\begin{aligned}
H'_I =&\mathrm{e}^{\mathrm{i}H_0t/\hbar}H_I\mathrm{e}^{-\mathrm{i}H_0t/\hbar}\\
=&\frac{\hbar\Omega}{2}\left(\sigma_+\mathrm{e}^{-\mathrm{i}\delta t}+\sigma_-\mathrm{e}^{\mathrm{i}\delta t}\right)+\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a\mathrm{e}^{-\mathrm{i}(\delta+\nu)t}-\sigma_-a^{\dagger}\mathrm{e}^{\mathrm{i}(\delta+\nu)t}\right)\\
&+\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a^{\dagger}\mathrm{e}^{-\mathrm{i}(\delta-\nu)t}-\sigma_-a\mathrm{e}^{\mathrm{i}(\delta-\nu)t}\right)
\end{aligned}$$

When the light is in resonance with the sideband, the Rabi frequencies

$$\begin{aligned}
\Omega_{n,n-1} =&\left|\langle n-1,e|\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a-\sigma_-a^{\dagger}\right)|n,g\rangle\right|\\
=&\frac{\eta\hbar\Omega}{2}\langle n-1|a|n\rangle\\
=&\frac{\eta\hbar\Omega}{2}\sqrt{n}\\
\Omega_{n,n+1} =&\left|\langle n+1,e|\mathrm{i}\eta\frac{\hbar\Omega}{2}\left(\sigma_+a^{\dagger}-\sigma_-a\right)|n,g\rangle\right|\\
=&\frac{\eta\hbar\Omega}{2}\langle n+1|a^{\dagger}|n\rangle\\
=&\frac{\eta\hbar\Omega}{2}\sqrt{n+1}
\end{aligned}$$

(b)

For spontanious decay, when the emitted photon is θ from the axis of the trap, the probability to go from n to $n+1$ is $\eta^2\cos^2\theta(n+1)$ Assuming isotropic emission patter (i.e. non-polarized) the averate probability is

$$\begin{aligned}
p_{n,n+1} =&\frac{1}{4\pi}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\sin\theta\mathrm{d}\phi\eta^2\cos^2\theta(n+1)\\
=&\frac{\eta^2(n+1)}{2}\int_{-1}^{1}\mathrm{d}zz^2\\
=&\frac{\eta^2(n+1)}{3}
\end{aligned}$$

Similarly, the probability of going to $n-1$ is

$$p_{n,n-1} =\frac{\eta^2n}{3}$$

To the lowest order in η, the rate at which $|n,g\rangle \to |n,e\rangle \to |n-1,g\rangle$ happens is

$$\frac{n\Omega^2\eta^2}{3}\frac{\Gamma}{\Gamma^2+4\delta^2}$$

For $|n,g\rangle \to |n-1,e\rangle \to |n-1,g\rangle$

$$\begin{aligned}
&n\Omega^2\eta^2\frac{\Gamma}{\Gamma^2+4(\delta+\nu)^2}\\
A_- =&\Omega^2\Gamma\eta^2\left(\frac{1}{\Gamma^2+4(\delta+\nu)^2}+\frac{1}{3}\frac{1}{\Gamma^2+4\delta^2}\right)
\end{aligned}$$

Similarly

$$A_+ =\Omega^2\Gamma\eta^2\left(\frac{1}{\Gamma^2+4(\delta-\nu)^2}+\frac{1}{3}\frac{1}{\Gamma^2+4\delta^2}\right)$$

(c)

$$\begin{aligned}
\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} =&\sum_{n=0}^{\infty} n\frac{\mathrm{d}p_n}{\mathrm{d}t}\\
=&\sum_{n=0}^{\infty} n(nA_+p_{n-1}+(n+1)A_-p_{n+1}-(n+1)A_+p_n-nA_-p_n)\\
=&\sum_{n=0}^{\infty}\left(n^2A_+p_{n-1}+n(n+1)A_-p_{n+1}-n(n+1)A_+p_n-n^2A_-p_n\right)\\
=&\sum_{n=0}^{\infty}\left((n+1)^2A_+p_n+(n-1)nA_-p_n-n(n+1)A_+p_n-n^2A_-p_n\right)\\
=&\sum_{n=0}^{\infty}((n+1)A_+p_n-nA_-p_n)\\
=&A_+-(A_--A_+)\langle n\rangle
\end{aligned}$$

The solution is a exponential decay to $\langle n\rangle = \dfrac{A_+}{A_- - A_+}$ with decay rate $A_- - A_+$. For driving cooling sideband in the resolved limit

$$\begin{aligned}
A_- \approx&\frac{\Omega^2\eta^2}{\Gamma}\\
A_+ \approx&\frac{7\Omega^2\Gamma\eta^2}{48\nu^2}
\end{aligned}$$

Final temperature

$$\begin{aligned}\langle n\rangle_\infty \approx&\frac{A_+}{A_-}\\=&\frac{7\Gamma^2}{48\nu^2}\\T_\infty =&\langle n\rangle_\infty\frac{\hbar\nu}{k_B}\\\approx&\frac{7\hbar\Gamma^2}{48k_B\nu}\end{aligned}$$

Decay time

$$\begin{aligned}\tau \approx&\frac{1}{A_-}\\\approx&\frac{\Gamma}{\Omega^2\eta^2}\end{aligned}$$

(d)

For the narrow line

$$\begin{aligned}\eta =&k\sqrt{\frac{\hbar}{2m\nu}}\\=&0.022\\T_\infty \approx&\frac{7\hbar\Gamma^2}{48k_B\nu}\\=&0.11\text{aK}\\\tau \approx&\frac{\Gamma}{\Omega^2\eta^2}\\=&22\text{hr}\end{aligned}$$

For broadened line

$$\begin{aligned}T_\infty \approx&\frac{7\hbar\Gamma^2}{48k_B\nu}\\=&7.0\text{nK}\\\tau \approx&\frac{\Gamma}{\Omega^2\eta^2}\\=&0.32\text{s}\end{aligned}$$

3. Optical dipole trap

(a)

Phase shift

$$\begin{aligned}\Delta\phi_D =&\Delta ndk\\=&\frac{n}{2}\alpha dk\\=&\frac{\dot{n}}{2}\alpha dkt\end{aligned}$$

Frequency shift

$$\Delta\omega = \frac{\dot{n}}{2}\alpha dk$$

(b)

Power loss

$$\begin{aligned}\Delta P =& P\frac{\Delta\omega}{\omega}\\ =& \frac{\dot{n}\alpha Vk}{2\omega}\varepsilon_0 c\left\langle E^2\right\rangle\\ =& \frac{\dot{n}\alpha V}{2}\varepsilon_0\left\langle E^2\right\rangle\end{aligned}$$

Total energy

$$\Delta E_{EM} = \frac{N\alpha}{2}\varepsilon_0\left\langle E^2\right\rangle$$

(c)

Start shift

$$\begin{aligned}U_{Stark} =& \frac{\varepsilon_0\langle pE\rangle}{2}\\ =& \frac{\varepsilon_0\alpha\left\langle E^2\right\rangle}{2}\end{aligned}$$

Total energy

$$\begin{aligned}\Delta E_{Stark} =& \frac{N\varepsilon_0\alpha\left\langle E^2\right\rangle}{2}\\ =& \Delta E_{EM}\end{aligned}$$