Assignment #4

Due: Monday, March 16, 2015
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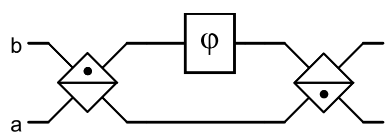
Office Hours: Mar 13th (Fri) 6pm - 8pm.& Mar 16th (Mon), 9am - 10:30am.

1. **Heisenberg-limited interferometry with the Yurke state**

   The Yurke state $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$ allows one to obtain a measurement of an unknown phase ϕ with uncertainty $\langle\Delta\phi\rangle = \frac{1}{n}$, using a Mach-Zehnder interferometer. Methods for experimentally realizing these states have been proposed, for example, using Bose-Einstein condenstates [Castin & Dalibard, Phys. Rev. A vol. 55, p. 4330, 1997]. For this problem, use the following definition for the beamsplitter:

$$\begin{aligned} BaB^\dagger &= \frac{1}{\sqrt{2}}(a+ib) \\ BbB^\dagger &= \frac{1}{\sqrt{2}}(b+ia) \end{aligned} \tag{1}$$

   a) Let us now analyze the Mach-Zehnder interferometer, fed with a Yurke state as input. Use this setup:



   and work in the Schrodinger picture, by doing the following. Let the input be the Yurke state, $|\phi_0\rangle = |\psi\rangle$, let the state after the first 50/50 beamsplitter be $|\phi_1\rangle = B|\phi_0\rangle$, the state after the phase shifter be $|\phi_2\rangle = P|\phi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\phi_3\rangle = B^\dagger|\phi_2\rangle$. Give expressions for $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$. Note that the transform of the phase shifter P is $PaP^\dagger = ae^{i\phi}$. Double-check that when $\phi = 0$, the output is the same as the input $|\phi_3\rangle = |\phi_0\rangle$. Hint: write these states in terms of operators acting on the vacuum.

   b) What is the uncertainty with which you can determine ϕ using the Yurke state input? This is

$$\langle\Delta\phi^2\rangle = \frac{\langle\Delta M^2\rangle}{\left|\frac{\partial\langle M\rangle}{\partial\phi}\right|^2}\,, \tag{2}$$

   where $M = a^\dagger a - b^\dagger b$ is the difference in the photon numbers measured at the outputs of the interferometer. Compute $\langle\Delta\phi^2\rangle$, evaluated at $\phi = 0$ (the point at which the interferometer is balanced), using the $|\phi_3\rangle$ you obtained above. You should find $\langle\Delta\phi\rangle = \sqrt{\langle\Delta\phi^2\rangle} = \frac{1}{n}$.

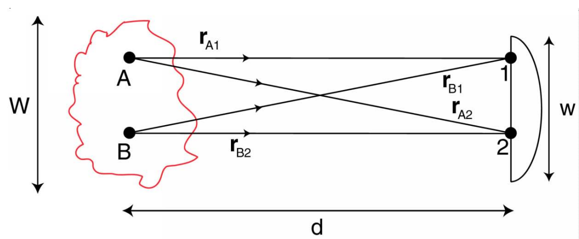
   c) Using the same diagram as above, let the input now be a coherent state and a vacuum state, $|\psi_0\rangle = |\alpha\rangle|0\rangle$. Just as above, let the state after the first 50/50 beamsplitter be $|\psi_1\rangle = B|\psi_0\rangle$, the state after the phase shifter be $|\psi_2\rangle = P|\psi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\psi_3\rangle = B^\dagger|\psi_2\rangle$. Give expressions for $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$.

d) Given the quantum fluctuations of the coherent state, use uncertainty propagation to determne the uncertainty with which you can determine ϕ using the coherent state input, as a function of $\bar{n} = |\alpha|^2$ and the phase shift angle ϕ.

2. **Hanbury Brown and Twiss Experiment with Atoms**

This problem illustrates the coherence and collimation requirements for performing a Hanbury Brown and Twiss (HBT) experiment with atoms. In fact the HBT experiment was done for both bosons (^{4}He) and fermions (^{3}He) by Jeltes and company in 2007 (T. Jeltes et al., Nature **445**, 402 (2007)). (Note: Ignore gravity in this problem.)

If a free particle starts at point A at time $t = 0$ with an amplitude (wavefunction) ψ_A, then the amplitude at another point 1 and time $t = \tau$ is proportional to $\psi_A e^{i(\mathbf{k}\cdot\mathbf{r}_{A1} - \omega\tau)}$, where $\mathbf{r}_{A1}$ is the vector from A to 1, $\mathbf{k}$ is the particle's wavevector, and $\hbar\omega$ is its total energy. This can be regarded as Huygen's principle for matter waves, and is a special case of the Feynman path integral formulation of quantum mechanics.



(Based on figure 19-5, in G. Baym, *Lectures on Quantum Mechanics*)

(a) Correlation function

Assume we have a particle at A with amplitude ψ_A and one at B with amplitude ψ_B. The joint probability P of finding one particle at 1 and one at 2 is

$$P = \left|\psi_A e^{i\phi_{A1}} \psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}} \psi_B e^{i\phi_{B1}}\right|^2 \tag{3}$$

and is proportional to the second-order coherence function $g^{(2)}(1,2)$. The $\pm$ is for bosons/fermions and makes the two-particle wavefunction symmetric/antisymmetric under the exchange of particles. Here, $\phi_{A1} = \mathbf{k}_A \cdot \mathbf{r}_{A1} - \omega\tau$ is the phase factor for the path from point A to detector 1, etc. Calculate P as a function of $\mathbf{r}_{21}$, the vector from point 2 to point 1 on the detector.

(b) Transverse Collimation

Assume you are given a source (e.g. a ball of trapped atoms) with transverse dimension W and detector with transverse dimension w where $|\mathbf{r}_{21}| \leq w$. The distance between source and detector d is much greater than all other distances. The transverse component of the phase factor in part (a) can be written: $\phi_t = (\mathbf{k}_A - \mathbf{k}_B)_t \cdot (\mathbf{r}_{21})_t$. Assume that the signal at the detector is mainly due to atoms with wavevectors distributed around $\mathbf{k}_0$. Argue that the transverse collimation required to see second order correlation effects can be expressed as $Ww \ll d\lambda_{dB}$, where λ_{dB} is the deBroglie wavelength corresponding to $\mathbf{k}_0$. (Hint: How does ϕ_t vary for atoms originating at different points in the source and being detected at different points on the detector?)

Consider a ^{6}Li MOT at 500 μK. Calculate the deBroglie wavelength. Assuming a MOT and detector of approximately equal size ($W \approx w$), estimate an upper bound on the MOT and detector size using $d = 10$ cm.

(c) Longitudinal Collimation

(i) The longitudinal component of the phase factor in part (a) can be written: $\phi_l = (\mathbf{k}_A - \mathbf{k}_B)_l \cdot (\mathbf{r}_{21})_l$. Assume a Gaussian distribution of wavevector differences $p(\mathbf{k}_A - \mathbf{k}_B) = e^{-|\mathbf{k}_A - \mathbf{k}_B|^2 \gamma^2}$ where the width γ is related to the temperature of the atoms. Calculate $\langle P \rangle$ using this distribution and your result from part (a). Sketch $\langle P \rangle$ for both fermions and bosons, indicating the extent of $(\mathbf{r}_{21})_l$ over which the second order correlation effect can be seen. (Hint: Use the fact that $\phi_t \ll 2\pi$ from part (b) to simplify the integral.)

(ii) Now assume you have a pulsed source of atoms with longitudinal dimension L. Atoms are released at time $t = 0$ and detected at some later time $t = \tau$. Give geometric arguments to show that the wavevectors of detected atoms must obey $|(\mathbf{k}_A - \mathbf{k}_B)_l| \leq \frac{mvL}{\hbar d}$, where the velocity $v = \frac{d}{\tau}$. This implies that the different velocity groups separate during the expansion, narrowing (by a factor $\frac{L}{d}$) the velocity distribution of atoms detected at any particular time.

Consider again the ^{6}Li MOT from part (b). Assuming $\tau = 0.1$s and $L \approx W$, estimate the necessary timing resolution of the detector in order to see second order correlation effects?

(d) Phase-Space Volume Enhancement

We now pull all the pieces together. The peak in $g^{(2)}(1,2)$ is visible for $(\mathbf{k}_A - \mathbf{k}_B) \cdot \mathbf{r}_{21} \leq 2\pi$. This is equivalent to saying that we must detect atoms from within a single phase space cell, defined by $\delta p_x \delta x \leq h$ (and likewise for y and z). In our trapped atom sample, the 3D volume of a phase space cell is $\delta x \delta y \delta z = (\lambda_{dB})^3$. Liouville's theorem says that as our ball of atoms expands, the number of phase space cells remains constant. Verify that, by using this pulsed source, the volume of a coherent phase space cell is increased by a factor $d^3/W^2 L$ by the time atoms reach the detector. What is the order of magnitude of this increase (assuming $L \approx W$)?

Estimate the average occupation of a cell of phase space for the ^{6}Li MOT from parts (b) and (c). Use the following numbers for the ^{6}Li MOT: 10^{10} atoms in 1 cm^3. How does this compare with the average occupation of a BEC or a degenerate Fermi cloud?