

1.

(a)

Define

$$Y(x, y) = \frac{x^n y^{n-1} + x^{n-1} y^n}{\sqrt{2n!(n-1)!}}$$

when $[x, y] = 0$

$$Y(x, y) = x^{n-1} y^{n-1} \frac{x + y}{\sqrt{2n!(n-1)!}}$$

we also have, for any number λ

$$Y(\lambda x, \lambda y) = \lambda^{2n-1} Y(x, y)$$

Before the first beam splitter

$$\begin{aligned} |\psi_0\rangle &= Y(a^\dagger, b^\dagger) |0, 0\rangle \\ &= a^{\dagger n-1} b^{\dagger n-1} \frac{a^\dagger + b^\dagger}{\sqrt{2n!(n-1)!}} |0, 0\rangle \end{aligned}$$

After the first beam splitter

$$\begin{aligned} |\psi_1\rangle &= B|\psi_0\rangle \\ &= BY(a^\dagger, b^\dagger) B^\dagger |0, 0\rangle \\ &= Y(Ba^\dagger B^\dagger, Bb^\dagger B^\dagger) |0, 0\rangle \\ &= \frac{\sqrt{2}}{2^n} Y(a^\dagger - ib^\dagger, b^\dagger - ia^\dagger) |0, 0\rangle \\ &= (a^\dagger - ib^\dagger)^{n-1} (b^\dagger - ia^\dagger)^{n-1} \frac{a^\dagger - ib^\dagger + b^\dagger - ia^\dagger}{2^n \sqrt{n!(n-1)!}} |0, 0\rangle \\ &= (-i)^{n-1} (1 - i) \left(a^{\dagger 2} + b^{\dagger 2} \right)^{n-1} \frac{a^\dagger + b^\dagger}{2^n \sqrt{n!(n-1)!}} |0, 0\rangle \end{aligned}$$

After phase shift

$$\begin{aligned} |\psi_2\rangle &= \frac{\sqrt{2}}{2^n} Y(e^{-i\phi} a^\dagger - ib^\dagger, b^\dagger - ie^{-i\phi} a^\dagger) |0, 0\rangle \\ &= (-i)^{n-1} (1 - i) \left(e^{-2i\phi} a^{\dagger 2} + b^{\dagger 2} \right)^{n-1} \frac{e^{-i\phi} a^\dagger + b^\dagger}{2^n \sqrt{n!(n-1)!}} |0, 0\rangle \end{aligned}$$

Output

$$\begin{aligned}
|\psi_3\rangle &= B^\dagger |\psi_2\rangle \\
&= \frac{e^{-i\phi(n-1/2)}}{2^{2n-1}} Y\left(e^{-i\phi/2}(a^\dagger + ib^\dagger) - ie^{i\phi/2}(b^\dagger + ia^\dagger), e^{i\phi/2}(b^\dagger + ia^\dagger) - ie^{-i\phi/2}(a^\dagger + ib^\dagger)\right) |0, 0\rangle \\
&= \frac{e^{-i\phi(n-1/2)}}{2^{2n-1}} Y\left(2\cos\frac{\phi}{2}a^\dagger + 2\sin\frac{\phi}{2}b^\dagger, 2\cos\frac{\phi}{2}b^\dagger - 2\sin\frac{\phi}{2}a^\dagger\right) |0, 0\rangle \\
&= e^{-i\phi(n-1/2)} Y\left(\cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger, \cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger\right) |0, 0\rangle \\
&= \frac{e^{-i\phi(n-1/2)}}{\sqrt{2n!(n-1)!}} \left(\cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger\right)^{n-1} \left(\cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger\right)^{n-1} \\
&\quad \left(\cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger + \cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger\right) |0, 0\rangle
\end{aligned}$$

For $\phi = 0$

$$\begin{aligned}
|\phi_3\rangle_0 &= \frac{1}{\sqrt{n!(n-1)!}} a^{\dagger n-1} b^{\dagger n-1} \frac{a^\dagger + b^\dagger}{\sqrt{2}} |0, 0\rangle \\
&= |\phi_0\rangle
\end{aligned}$$

(b)

Let

$$\begin{aligned}
a' &= \cos\frac{\phi}{2}a + \sin\frac{\phi}{2}b \\
b' &= \cos\frac{\phi}{2}b - \sin\frac{\phi}{2}a
\end{aligned}$$

then

$$\begin{aligned}
a'^\dagger &= \cos\frac{\phi}{2}a^\dagger + \sin\frac{\phi}{2}b^\dagger \\
b'^\dagger &= \cos\frac{\phi}{2}b^\dagger - \sin\frac{\phi}{2}a^\dagger \\
a &= \cos\frac{\phi}{2}a' - \sin\frac{\phi}{2}b' \\
b &= \cos\frac{\phi}{2}b' + \sin\frac{\phi}{2}a' \\
a^\dagger &= \cos\frac{\phi}{2}a'^\dagger - \sin\frac{\phi}{2}b'^\dagger \\
b^\dagger &= \cos\frac{\phi}{2}b'^\dagger + \sin\frac{\phi}{2}a'^\dagger
\end{aligned}$$

$$[a', a'^\dagger] = 1$$

$$[b', b'^\dagger] = 1$$

$$|\phi_3\rangle = \frac{e^{-i\phi(n-1/2)}}{\sqrt{n!(n-1)!}} a'^{\dagger n-1} b'^{\dagger n-1} \frac{a'^\dagger + b'^\dagger}{\sqrt{2}} |0, 0\rangle$$

$$\begin{aligned}
M &= a^\dagger a - b^\dagger b \\
&= \left(\cos \frac{\phi}{2} a'^\dagger - \sin \frac{\phi}{2} b'^\dagger \right) \left(\cos \frac{\phi}{2} a' - \sin \frac{\phi}{2} b' \right) - \left(\cos \frac{\phi}{2} b'^\dagger + \sin \frac{\phi}{2} a'^\dagger \right) \left(\cos \frac{\phi}{2} b' + \sin \frac{\phi}{2} a' \right) \\
&= \cos^2 \frac{\phi}{2} a'^\dagger a' + \sin^2 \frac{\phi}{2} b'^\dagger b' - \sin \frac{\phi}{2} \cos \frac{\phi}{2} (a' b'^\dagger + a'^\dagger b') \\
&\quad - \cos^2 \frac{\phi}{2} b'^\dagger b' - \sin \frac{\phi}{2} \cos \frac{\phi}{2} a'^\dagger b' - \sin \frac{\phi}{2} \cos \frac{\phi}{2} a' b'^\dagger - \sin^2 \frac{\phi}{2} a'^\dagger a' \\
&= \cos \phi (a'^\dagger a' - b'^\dagger b') - \sin \phi (a' b'^\dagger + a'^\dagger b')
\end{aligned}$$

Mean

$$\begin{aligned}
\langle M \rangle &= \frac{1}{2n!(n-1)!} \langle 0, 0 | a'^{n-1} b'^{n-1} (a' + b') (\cos \phi (a'^\dagger a' - b'^\dagger b') - \sin \phi (a' b'^\dagger + a'^\dagger b')) \\
&\quad a'^{\dagger n-1} b'^{\dagger n-1} (a'^\dagger + b'^\dagger) | 0, 0 \rangle \\
&= \frac{\cos \phi}{2n!(n-1)!} \langle 0, 0 | a'^{n-1} b'^{n-1} (a' + b') (a'^\dagger a' - b'^\dagger b') a'^{\dagger n-1} b'^{\dagger n-1} (a'^\dagger + b'^\dagger) | 0, 0 \rangle \\
&\quad - \frac{\sin \phi}{2n!(n-1)!} \langle 0, 0 | a'^{n-1} b'^{n-1} (a' + b') (a' b'^\dagger + a'^\dagger b') a'^{\dagger n-1} b'^{\dagger n-1} (a'^\dagger + b'^\dagger) | 0, 0 \rangle \\
&= -n \sin \phi
\end{aligned}$$

Square

$$\begin{aligned}
M^2 &= (\cos \phi (a'^\dagger a' - b'^\dagger b') - \sin \phi (a' b'^\dagger + a'^\dagger b'))^2 \\
&= \cos^2 \phi (a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b') + \sin^2 \phi (a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2) \\
&\quad - \sin \phi \cos \phi (2a'^\dagger a'^2 b'^\dagger + a' b'^\dagger + 2a'^{\dagger 2} a' b' + a'^\dagger b' - 2a' b'^{\dagger 2} b' - a' b'^\dagger - 2a'^\dagger b' b'^2 - a'^\dagger b') \\
&= \cos^2 \phi (a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b') + \sin^2 \phi (a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2) \\
&\quad - 2 \sin \phi \cos \phi (a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2} a' b' - a' b'^{\dagger 2} b' - a'^\dagger b' b'^2) \\
\langle M^2 \rangle &= \cos^2 \phi \langle a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b' \rangle \\
&\quad + \sin^2 \phi \langle a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2 \rangle \\
&\quad - 2 \sin \phi \cos \phi \langle a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2} a' b' - a' b'^{\dagger 2} b' - a'^\dagger b' b'^2 \rangle \\
&= \frac{\cos^2 \phi}{2n} (n^3 + (n-1)^2 n - 2n^2(n-1) - 2n^2(n-1) + (n-1)^2 n + n^3) \\
&\quad + \frac{\sin^2 \phi}{2n} (2(n-1)n(n+1) + 2n^3) \\
&\quad - \frac{\sin \phi \cos \phi}{n} (n^2(n-1) + n^2(n-1) - n^2(n-1) - n^2(n-1)) \\
&= \cos^2 \phi + \sin^2 \phi (2n^2 - 1)
\end{aligned}$$

Fluctuation

$$\begin{aligned}\Delta M^2 &= \cos^2 \phi + \sin^2 \phi (2n^2 - 1) - n^2 \sin^2 \phi \\ &= \cos^2 \phi + \sin^2 \phi (n^2 - 1)\end{aligned}$$

$$\frac{\partial \langle M \rangle}{\partial \phi} = -n \cos \phi$$

$$\begin{aligned}\Delta \phi^2 &= \frac{\cos^2 \phi + \sin^2 \phi (n^2 - 1)}{n^2 \cos^2 \phi} \\ &= \frac{1 + \tan^2 \phi (n^2 - 1)}{n^2}\end{aligned}$$

$$\Delta \phi = \frac{\sqrt{1 + \tan^2 \phi (n^2 - 1)}}{n}$$

for $\phi = 0$

$$\Delta \phi = \frac{1}{n}$$

(c)

(d)

2.

(a)

(b)

(c)

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