

1.

(a)

$$\begin{aligned}
 & [a^\dagger b - ab^\dagger, a^\dagger a + b^\dagger b] \\
 &= [a^\dagger b, a^\dagger a] - [ab^\dagger, a^\dagger a] + [a^\dagger b, b^\dagger b] - [ab^\dagger, b^\dagger b] \\
 &= a^\dagger [a^\dagger, a] b - [a, a^\dagger] ab^\dagger + a^\dagger [b, b^\dagger] b - ab^\dagger [b^\dagger, b] \\
 &= -a^\dagger b - ab^\dagger + a^\dagger b + ab^\dagger \\
 &= 0
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 [B, n_a + n_b] &= [\exp(\theta(a^\dagger b - ab^\dagger)), n_a + n_b] \\
 &= 0 \\
 B^\dagger &= \exp(\theta(a^\dagger b - ab^\dagger)^\dagger) \\
 &= \exp(-\theta(a^\dagger b - ab^\dagger)) \\
 &= B^{-1}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \exp(\theta A) B \exp(-\theta A) &= \sum_{nm} (-1)^m \frac{\theta^{n+m} A^n B A^m}{n! m!} \\
 &= \sum_N \sum_{m=0}^N (-1)^m \frac{\theta^N A^{N-m} B A^m}{(N-m)! m!} \\
 &= \sum_N \frac{\theta^N}{N!} \sum_{m=0}^N (-1)^m \frac{N! A^{N-m} B A^m}{(N-m)! m!} \\
 &= \sum_N \frac{\theta^N}{N!} [A, B]_N
 \end{aligned}$$

where $[A, B]_N$ is defined as $[A, B]_N = [A, [A, B]_{N-1}]$ and $[A, B]_0 = B$

$$\begin{aligned}
 [(a^\dagger b - ab^\dagger), a]_N &= \begin{cases} (-1)^{N/2} a & (2 \mid N) \\ (-1)^{(N+1)/2} b & (2 \nmid N) \end{cases} \\
 [(a^\dagger b - ab^\dagger), b]_N &= \begin{cases} (-1)^{N/2} b & (2 \mid N) \\ (-1)^{(N-1)/2} a & (2 \nmid N) \end{cases} \\
 BaB^{-1} &= \sum_N \frac{\theta^N}{N!} [(a^\dagger b - ab^\dagger), a]_N \\
 &= \sum_n \frac{\theta^{2n}}{(2n)!} [(a^\dagger b - ab^\dagger), a]_{2n} + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} [(a^\dagger b - ab^\dagger), a]_{2n+1} \\
 &= \sum_n \frac{\theta^{2n}}{(2n)!} (-1)^n a + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} (-1)^{n+1} b \\
 &= \cos \theta a - \sin \theta b \\
 BbB^{-1} &= \sum_N \frac{\theta^N}{N!} [(a^\dagger b - ab^\dagger), b]_N \\
 &= \sum_n \frac{\theta^{2n}}{(2n)!} [(a^\dagger b - ab^\dagger), b]_{2n} + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} [(a^\dagger b - ab^\dagger), b]_{2n+1} \\
 &= \sum_n \frac{\theta^{2n}}{(2n)!} (-1)^n b + \sum_n \frac{\theta^{2n+1}}{(2n+1)!} (-1)^{n+1} a \\
 &= \cos \theta b + \sin \theta a \\
 B|0, \alpha\rangle &= B e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0, 0\rangle \\
 &= e^{-|\alpha|^2/2} B e^{\alpha a^\dagger} B^{-1} B|0, 0\rangle \\
 &= e^{-|\alpha|^2/2} e^{\alpha (\cos \theta a^\dagger - \sin \theta b^\dagger)} |0, 0\rangle \\
 &= e^{-|\alpha|^2/2} e^{\alpha \cos \theta a^\dagger} e^{-\alpha \sin \theta b^\dagger} |0, 0\rangle \\
 &= |-\alpha \sin \theta, \alpha \cos \theta\rangle
 \end{aligned}$$

(c)

$$\begin{aligned}
 s_x &= a^\dagger b + ab^\dagger \\
 s_y &= -i(a^\dagger b - ab^\dagger) \\
 B &= e^{i\theta s_y}
 \end{aligned}$$

which is a rotation around y

$$\begin{aligned}
 &(n_a + n_b)^2 \\
 &= s_z^2 + 4a^\dagger ab^\dagger b \\
 &= s_z^2 + 4s^+ s^- \\
 &= s_x^2 + s_y^2 + s_z^2 \\
 &= S^2
 \end{aligned}$$

which is the total spin

$$\begin{aligned}
 [s_x, s_y] &= [a^\dagger b + ab^\dagger, -i(a^\dagger b - ab^\dagger)] \\
 &= -i[a^\dagger b + ab^\dagger, a^\dagger b - ab^\dagger] \\
 &= 2i[a^\dagger b, ab^\dagger] \\
 &= 2ia^\dagger a [b, b^\dagger] + 2i[a^\dagger, a] b^\dagger b \\
 &= 2ia^\dagger a - 2ib^\dagger b \\
 &= 2is_z \\
 [s_y, s_z] &= [-i(a^\dagger b - ab^\dagger), a^\dagger a - b^\dagger b] \\
 &= -ia^\dagger [a^\dagger, a] b + ia^\dagger [b, b^\dagger] b + i[a, a^\dagger] ab^\dagger - iab^\dagger [b^\dagger, b] \\
 &= 2is_x \\
 [s_z, s_x] &= [a^\dagger a - b^\dagger b, a^\dagger b + ab^\dagger] \\
 &= a^\dagger [a, a^\dagger] b + [a^\dagger, a] ab^\dagger - a^\dagger [b^\dagger, b] b - ab^\dagger [b, b^\dagger] \\
 &= 2a^\dagger b - 2ab^\dagger \\
 &= 2is_y
 \end{aligned}$$

(d)

$$\begin{aligned}
 B|0, n\rangle &= B \frac{a^{\dagger n}}{n!} |0, 0\rangle \\
 &= \frac{(a^\dagger - b^\dagger)^n}{2^{n/2} n!} |0, 0\rangle \\
 &= \sum_i \frac{a^{\dagger i} b^{\dagger n-i}}{2^{n/2} i! (n-i)!} |0, 0\rangle \\
 &= \sum_i \frac{|n-i, i\rangle}{\sqrt{2^n i! (n-i)!}}
 \end{aligned}$$

The state(s) with the largest amplitude is $|n/2, n/2\rangle$ (when n is even) or $|(n+1)/2, (n-1)/2\rangle$ and $|(n-1)/2, (n+1)/2\rangle$ when n is odd

The variance of the distribution is $\frac{n}{4}$ so the relative width is getting narrower for larger n although the absolute width is getting wider.

2.

- (a)
- (b)

3.

- (a)
- (b)
- (c)
- (d)

4.

- (a)
- (b)
- (c)
- (d)
- (e)