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1.

(a)

(Shouldn't the condition be $\Omega \ll \delta$ instead of $\Omega \ll \Gamma$?)
Let $\delta = (\omega_0 - \omega_L)$

$$\begin{aligned}
\rho_{ee} =&\rho'_{ee}\mathrm{e}^{-\Gamma t}\\
\rho_{ge} =&\rho'_{ge}\mathrm{e}^{-\Gamma t/2+\mathrm{i}\delta t}\\
\dot{\rho}'_{ee} =&\mathrm{i}\frac{\Omega\mathrm{e}^{\Gamma t/2}}{2}\left({\rho'_{ge}}^*\mathrm{e}^{-\mathrm{i}\delta t} - \rho'_{ge}\mathrm{e}^{\mathrm{i}\delta t}\right)\\
\dot{\rho}'_{ge} =&-\mathrm{i}\frac{\Omega\mathrm{e}^{\Gamma t/2}}{2}\left(2\rho'_{ee}\mathrm{e}^{-\Gamma t} - 1\right)\mathrm{e}^{-\mathrm{i}\delta t}
\end{aligned}$$

0th order in Ω

$$\begin{aligned}
\rho'_{ee0} =&0\\
\rho'_{ge0} =&0
\end{aligned}$$

1st order in Ω

$$\begin{aligned}
\rho'_{ee1} =&0\\
\dot{\rho}'_{ge1} =&\mathrm{i}\frac{\Omega}{2}\mathrm{e}^{\Gamma t/2-\mathrm{i}\delta t}\\
\rho'_{ge1} =&\mathrm{i}\frac{\Omega}{2(\Gamma/2-\mathrm{i}\delta)}\left(\mathrm{e}^{\Gamma t/2-\mathrm{i}\delta t} - 1\right)\\
{\rho'_{ge1}}^* =&-\mathrm{i}\frac{\Omega}{2(\Gamma/2+\mathrm{i}\delta)}\left(\mathrm{e}^{\Gamma t/2+\mathrm{i}\delta t} - 1\right)
\end{aligned}$$

2nd order in Ω

$$\begin{aligned}
\dot{\rho}'_{ee2} =&\mathrm{i}\frac{\Omega\mathrm{e}^{\Gamma t/2}}{2}\left({\rho'_{ge1}}^*\mathrm{e}^{-\mathrm{i}\delta t} - \rho'_{ge1}\mathrm{e}^{\mathrm{i}\delta t}\right)\\
=&\mathrm{i}\frac{\Omega\mathrm{e}^{\Gamma t/2}}{2}\left(-\mathrm{i}\frac{\Omega}{2(\Gamma/2+\mathrm{i}\delta)}\left(\mathrm{e}^{\Gamma t/2+\mathrm{i}\delta t} - 1\right)\mathrm{e}^{-\mathrm{i}\delta t} - \mathrm{i}\frac{\Omega}{2(\Gamma/2-\mathrm{i}\delta)}\left(\mathrm{e}^{\Gamma t/2-\mathrm{i}\delta t} - 1\right)\mathrm{e}^{\mathrm{i}\delta t}\right)\\
=&\frac{\Omega^2}{4}\left(\frac{1}{\Gamma/2+\mathrm{i}\delta}\left(\mathrm{e}^{\Gamma t} - \mathrm{e}^{\Gamma t/2-\mathrm{i}\delta t}\right) + \frac{1}{\Gamma/2-\mathrm{i}\delta}\left(\mathrm{e}^{\Gamma t} - \mathrm{e}^{\Gamma t/2+\mathrm{i}\delta t}\right)\right)\\
\rho'_{ee2} =&C_0 + \frac{\Omega^2}{4}\left(\frac{1}{\Gamma/2+\mathrm{i}\delta}\left(\frac{1}{\Gamma}\mathrm{e}^{\Gamma t} - \frac{1}{\Gamma/2-\mathrm{i}\delta}\mathrm{e}^{\Gamma t/2-\mathrm{i}\delta t}\right) + \frac{1}{\Gamma/2-\mathrm{i}\delta}\left(\frac{1}{\Gamma}\mathrm{e}^{\Gamma t} - \frac{1}{\Gamma/2+\mathrm{i}\delta}\mathrm{e}^{\Gamma t/2+\mathrm{i}\delta t}\right)\right)\\
=&C_0 + \frac{\Omega^2}{\Gamma^2+4\delta^2}\left(\frac{\Gamma/2-\mathrm{i}\delta}{\Gamma}\mathrm{e}^{\Gamma t} - \mathrm{e}^{\Gamma t/2-\mathrm{i}\delta t} + \frac{\Gamma/2+\mathrm{i}\delta}{\Gamma}\mathrm{e}^{\Gamma t} - \mathrm{e}^{\Gamma t/2+\mathrm{i}\delta t}\right)\\
=&C_0 + \frac{\Omega^2}{\Gamma^2+4\delta^2}\left(\mathrm{e}^{\Gamma t} - 2\cos(\delta t)\mathrm{e}^{\Gamma t/2}\right)
\end{aligned}$$

Since $\rho'(0) = 0$

$$\rho'_{ee2} =\frac{\Omega^2}{\Gamma^2+4\delta^2}\left(1 + \mathrm{e}^{\Gamma t} - 2\cos(\delta t)\mathrm{e}^{\Gamma t/2}\right)$$

Therefore, to the second order in Ω

$$\rho_{ee} =\frac{\Omega^2}{\Gamma^2 + 4\delta^2}\left(1 + \mathrm{e}^{-\Gamma t} - 2\cos{(\delta t)}\mathrm{e}^{-\Gamma t/2}\right)$$

In the limit of $\Gamma \to 0$, this turns into a Rabi flopping at the frequency of the detuning.

(b)

Expand the solution to second order in t

$$\begin{aligned}
\rho_{ee} =&\frac{\Omega^2}{\Gamma^2 + 4\delta^2}\left(1 + 1 - \Gamma t + \frac{\Gamma^2 t^2}{2} - 2\left(1 - \frac{\delta^2 t^2}{2}\right)\left(1 - \frac{\Gamma t}{2} + \frac{\Gamma^2 t^2}{8}\right)\right)\\
=&\frac{\Omega^2}{\Gamma^2 + 4\delta^2}\left(1 + 1 - \Gamma t + \frac{\Gamma^2 t^2}{2} - 2 + \delta^2 t^2 + \Gamma t - \frac{\Gamma^2 t^2}{4}\right)\\
=&\frac{\Omega^2}{\Gamma^2 + 4\delta^2}\left(\frac{\Gamma^2 t^2}{4} + \delta^2 t^2\right)\\
=&\frac{\Omega^2 t^2}{4}
\end{aligned}$$

When the pulse is very short, the decay haven't started yet and the atom also doesn't have enough time to figure out that the frequency is wrong.

2.

(a)

(b)

(c)

(d)

(e)

(f)

3.

(a)

(b)

(c)

(d)

(e)