1.

(a)

Before the first beam splitter

$$
\begin{aligned}
|\psi_0\rangle =&\frac{{a^\dagger}^n {b^\dagger}^{n-1} + {a^\dagger}^{n-1} {b^\dagger}^n}{\sqrt{2n!(n-1)!}}|0,0\rangle\\
=&{a^\dagger}^{n-1} {b^\dagger}^{n-1}\frac{a^\dagger + b^\dagger}{\sqrt{2n!(n-1)!}}|0,0\rangle
\end{aligned}
$$

After the first beam splitter

$$
\begin{aligned}
|\psi_1\rangle =&B|\psi_0\rangle\\
=&B{a^\dagger}^{n-1} {b^\dagger}^{n-1}\frac{a^\dagger + b^\dagger}{\sqrt{2n!(n-1)!}}B^\dagger B|0,0\rangle\\
=&\left(a^\dagger - \mathrm{i}b^\dagger\right)^{n-1}\left(b^\dagger - \mathrm{i}a^\dagger\right)^{n-1}\frac{a^\dagger - \mathrm{i}b^\dagger + b^\dagger - \mathrm{i}a^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\\
=&(-\mathrm{i})^{n-1}(1-\mathrm{i})\left({a^\dagger}^2 + {b^\dagger}^2\right)^{n-1}\frac{a^\dagger + b^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle
\end{aligned}
$$

After phase shift

$$
|\psi_2\rangle =(-\mathrm{i})^{n-1}(1-\mathrm{i})\left(\mathrm{e}^{-2\mathrm{i}\phi}{a^\dagger}^2 + {b^\dagger}^2\right)^{n-1}\frac{\mathrm{e}^{-\mathrm{i}\phi}a^\dagger + b^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle
$$

Output

$$
\begin{aligned}
|\psi_3\rangle =&B^\dagger|\psi_2\rangle\\
=&(-\mathrm{i})^{n-1}(1-\mathrm{i})B^\dagger\left(\mathrm{e}^{-2\mathrm{i}\phi}{a^\dagger}^2 + {b^\dagger}^2\right)^{n-1}\frac{\mathrm{e}^{-\mathrm{i}\phi}a^\dagger + b^\dagger}{2^{2n-1/2}\sqrt{n!(n-1)!}}B|0,0\rangle\\
=&(-\mathrm{i})^{n-1}(1-\mathrm{i})\left(\mathrm{e}^{-2\mathrm{i}\phi}\left(a^\dagger + \mathrm{i}b^\dagger\right)^2 + \left(b^\dagger + \mathrm{i}a^\dagger\right)^2\right)^{n-1}\frac{\mathrm{e}^{-\mathrm{i}\phi}\left(a^\dagger + \mathrm{i}b^\dagger\right) + \left(b^\dagger + \mathrm{i}a^\dagger\right)}{2^{2n-1/2}\sqrt{n!(n-1)!}}|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}(-\mathrm{i})^{n-1}(1-\mathrm{i})}{2^{2n-1/2}\sqrt{n!(n-1)!}}\left(\mathrm{e}^{-\mathrm{i}\phi}\left(a^\dagger + \mathrm{i}b^\dagger\right)^2 + \mathrm{e}^{\mathrm{i}\phi}\left(b^\dagger + \mathrm{i}a^\dagger\right)^2\right)^{n-1}\\
&\left(\mathrm{e}^{-\mathrm{i}\phi/2}\left(a^\dagger + \mathrm{i}b^\dagger\right) + \mathrm{e}^{\mathrm{i}\phi/2}\left(b^\dagger + \mathrm{i}a^\dagger\right)\right)|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}(-\mathrm{i})^{n-1}(1-\mathrm{i})}{2^{2n-1/2}\sqrt{n!(n-1)!}}\left(4\mathrm{i}\cos\phi a^\dagger b^\dagger + 2\mathrm{i}\sin\phi\left({b^\dagger}^2 - {a^\dagger}^2\right)\right)^{n-1}\\
&\left((1+\mathrm{i})\cos\frac{\phi}{2}\left(a^\dagger + b^\dagger\right) - \mathrm{i}(1-\mathrm{i})\sin\frac{\phi}{2}\left(a^\dagger - b^\dagger\right)\right)|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{2^{n-1/2}\sqrt{n!(n-1)!}}\left(2\cos\phi a^\dagger b^\dagger + \sin\phi\left({b^\dagger}^2 - {a^\dagger}^2\right)\right)^{n-1}\\
&\left(\cos\frac{\phi}{2}\left(a^\dagger + b^\dagger\right) - \sin\frac{\phi}{2}\left(a^\dagger - b^\dagger\right)\right)|0,0\rangle
\end{aligned}
$$

For $\phi = 0$

$$\begin{aligned}
|\phi_3\rangle_0 =& \frac{1}{2^{n-1/2}\sqrt{n!(n-1)!}}\left(2a^\dagger b^\dagger\right)^{n-1}\left(a^\dagger + b^\dagger\right)|0,0\rangle \\
=& \frac{1}{\sqrt{n!(n-1)!}}{a^\dagger}^{n-1}{b^\dagger}^{n-1}\frac{a^\dagger + b^\dagger}{\sqrt{2}}|0,0\rangle \\
=& |\phi_0\rangle
\end{aligned}$$

(b)

(c)

(d)

2.

(a)

(b)

(c)

(d)