1.

(a)

$$
\begin{aligned}
H =&\hbar\omega b^\dagger b + \mathrm{i}\hbar\Lambda\left({b^\dagger}^2 + b^2\right)\\
\frac{\mathrm{d}b}{\mathrm{d}t} =&\frac{\mathrm{i}}{\hbar}[H, b] + \frac{\partial b}{\partial t}\\
=&\left[\mathrm{i}\omega b^\dagger b - \Lambda {b^\dagger}^2, b\right] + \mathrm{i}\omega b\\
=&2\Lambda b^\dagger\\
\frac{\mathrm{d}b^\dagger}{\mathrm{d}t} =&2\Lambda b
\end{aligned}
$$

(b)

$$
\begin{aligned}
\vec{E} =&\mathrm{i}\mathcal{E}_\omega\vec{\varepsilon}\left(b\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{r}-\omega t\right)} - b^\dagger\mathrm{e}^{-\mathrm{i}\left(\vec{k}\cdot\vec{r}-\omega t\right)}\right)\\
=&\mathrm{i}\mathcal{E}_\omega\vec{\varepsilon}\left(b\cos\left(\vec{k}\cdot\vec{r}-\omega t\right) + \mathrm{i}b\sin\left(\vec{k}\cdot\vec{r}-\omega t\right) - b^\dagger\cos\left(\vec{k}\cdot\vec{r}-\omega t\right) + \mathrm{i}b^\dagger\sin\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)\\
=&-2\mathcal{E}_\omega\vec{\varepsilon}\left(\frac{b-b^\dagger}{2\mathrm{i}}\cos\left(\vec{k}\cdot\vec{r}-\omega t\right) + \frac{b+b^\dagger}{2}\sin\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)\\
=&-2\mathcal{E}_\omega\vec{\varepsilon}\left(b_Q\cos\left(\vec{k}\cdot\vec{r}-\omega t\right) + b_P\sin\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)\\
\frac{\mathrm{d}b_\pm}{\mathrm{d}t} \equiv&\frac{\mathrm{d}b\pm b^\dagger}{\mathrm{d}t}\\
=&2\Lambda b^\dagger \pm 2\Lambda b\\
=&\pm 2\Lambda b_\pm\\
\frac{\mathrm{d}b_P}{\mathrm{d}t} =&2\Lambda b_P\\
\frac{\mathrm{d}b_Q}{\mathrm{d}t} =&-2\Lambda b_Q
\end{aligned}
$$

Therefore

$$
\begin{aligned}
b_P =&\mathrm{e}^{2\Lambda t}b_{P0}\\
b_Q =&\mathrm{e}^{-2\Lambda t}b_{Q0}\\
b =&b_P + \mathrm{i}b_Q\\
=&\mathrm{e}^{2\Lambda t}b_{P0} + \mathrm{i}\mathrm{e}^{-2\Lambda t}b_{Q0}\\
=&b_0\cosh 2\Lambda t + b_0^\dagger\sinh 2\Lambda t\\
b^\dagger =&b_0^\dagger\cosh 2\Lambda t + b_0\sinh 2\Lambda t
\end{aligned}
$$

(c)

$$\begin{aligned}
\langle N\rangle =&\langle 0|b^\dagger b|0\rangle\\
=&\langle 0|\left(b_0^\dagger\cosh 2\Lambda t+b_0\sinh 2\Lambda t\right)\left(b_0\cosh 2\Lambda t+b_0^\dagger\sinh 2\Lambda t\right)|0\rangle\\
=&\sinh^2 2\Lambda t\\
\Delta b_P =&\mathrm{e}^{2\Lambda t}\Delta b_{P0}\\
=&\frac{1}{2}\mathrm{e}^{2\Lambda t}\\
\Delta b_Q =&\mathrm{e}^{-2\Lambda t}\Delta b_{Q0}\\
=&\frac{1}{2}\mathrm{e}^{-2\Lambda t}
\end{aligned}$$

The state is squeezed in Q direction while the product of the uncertainty in P and Q remains the same.

(d)

Under the transformation $U=\mathrm{e}^{\mathrm{i}\omega t a^\dagger a}$

$$\begin{aligned}
UaU^\dagger =&\mathrm{e}^{\mathrm{i}\omega t a^\dagger a}a\mathrm{e}^{-\mathrm{i}\omega t a^\dagger a}\\
=&\sum_N\frac{(\mathrm{i}\omega t)^N}{N!}\left[a^\dagger a,a\right]_N\\
=&\mathrm{e}^{-\mathrm{i}\omega t}a\\
Ua^\dagger U^\dagger =&\mathrm{e}^{\mathrm{i}\omega t}a^\dagger\\
\frac{\mathrm{d}}{\mathrm{d}t}|\psi'\rangle =&\frac{\mathrm{d}}{\mathrm{d}t}U|\psi\rangle\\
=&\frac{\mathrm{d}U}{\mathrm{d}t}|\psi\rangle+U\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle\\
=&\frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\omega t a^\dagger a}}{\mathrm{d}t}|\psi\rangle+\frac{U}{\mathrm{i}\hbar}H|\psi\rangle\\
=&\mathrm{i}\omega a^\dagger a|\psi'\rangle+\frac{1}{\mathrm{i}\hbar}UHU^\dagger|\psi'\rangle\\
=&\Lambda U\left({a^\dagger}^2\mathrm{e}^{-2\mathrm{i}\omega t}-a^2\mathrm{e}^{2\mathrm{i}\omega t}\right)U^\dagger|\psi'\rangle\\
=&\Lambda\left({a^\dagger}^2-a^2\right)|\psi'\rangle
\end{aligned}$$

Therefore, the state $|\psi'\rangle$ is transforming as

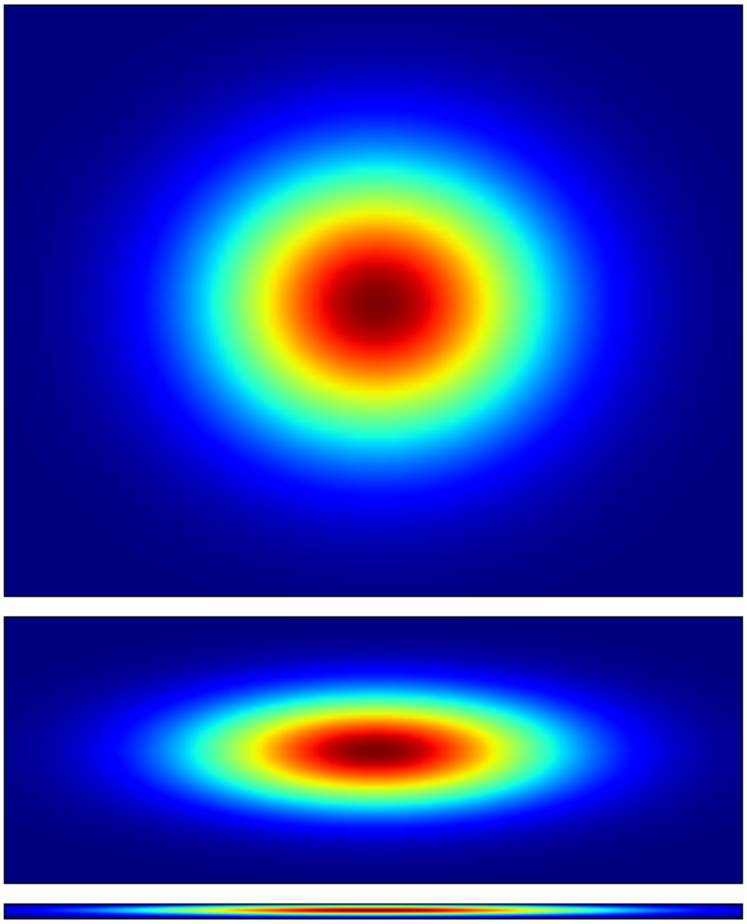
$$\mathrm{e}^{\Lambda t\left({a^\dagger}^2-a^2\right)}$$

where

$$\varepsilon=-2\Lambda t$$

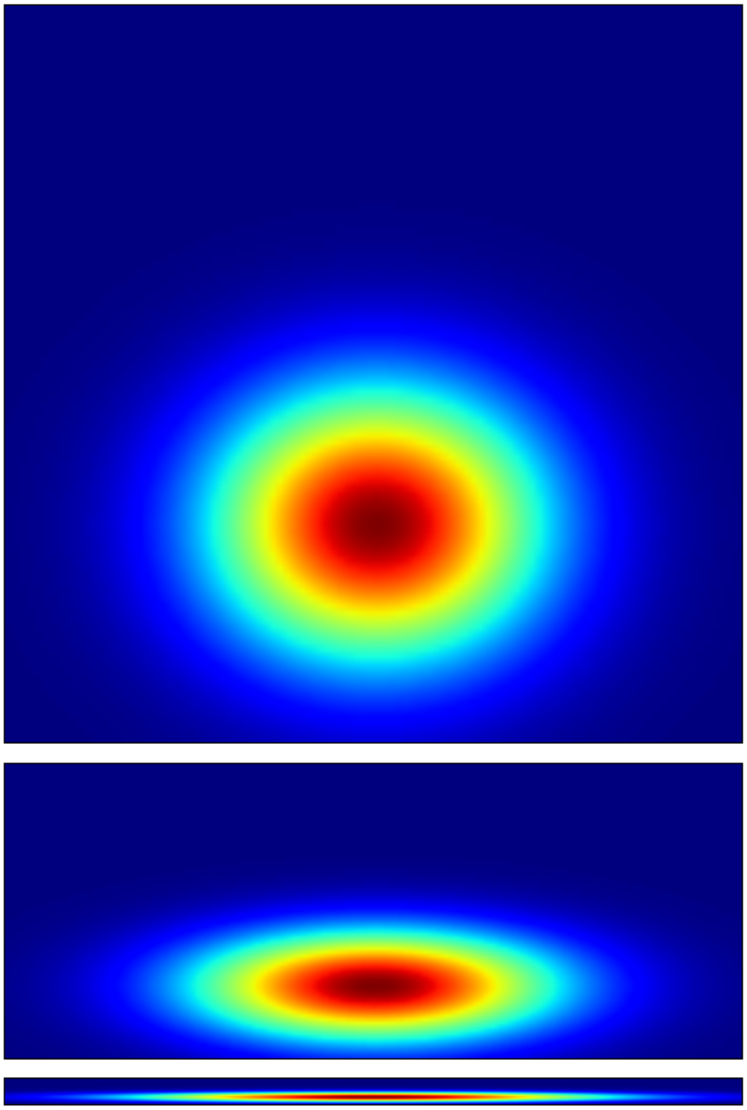
(e)

i.



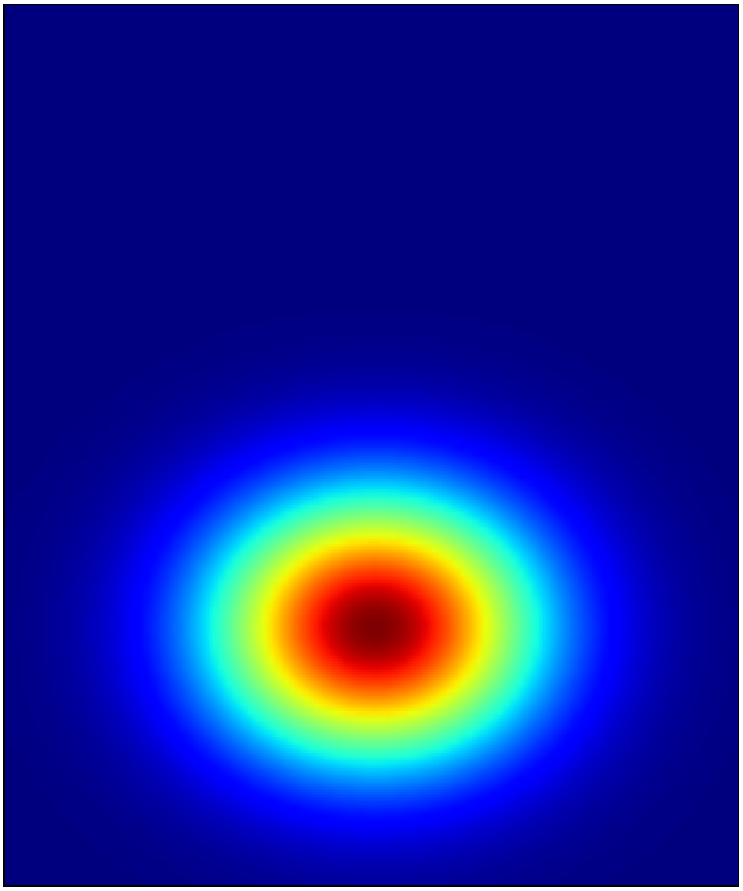
ii.

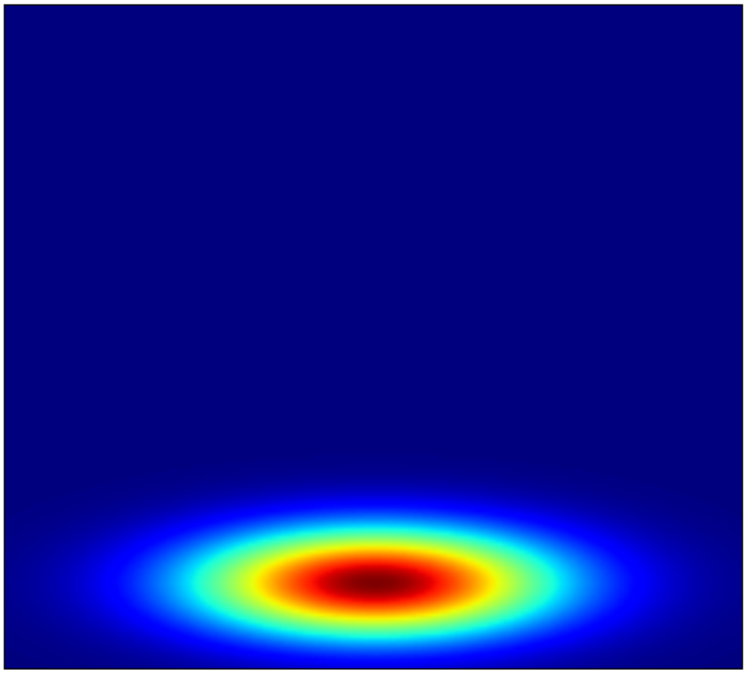
$\beta = \mathrm{i}$

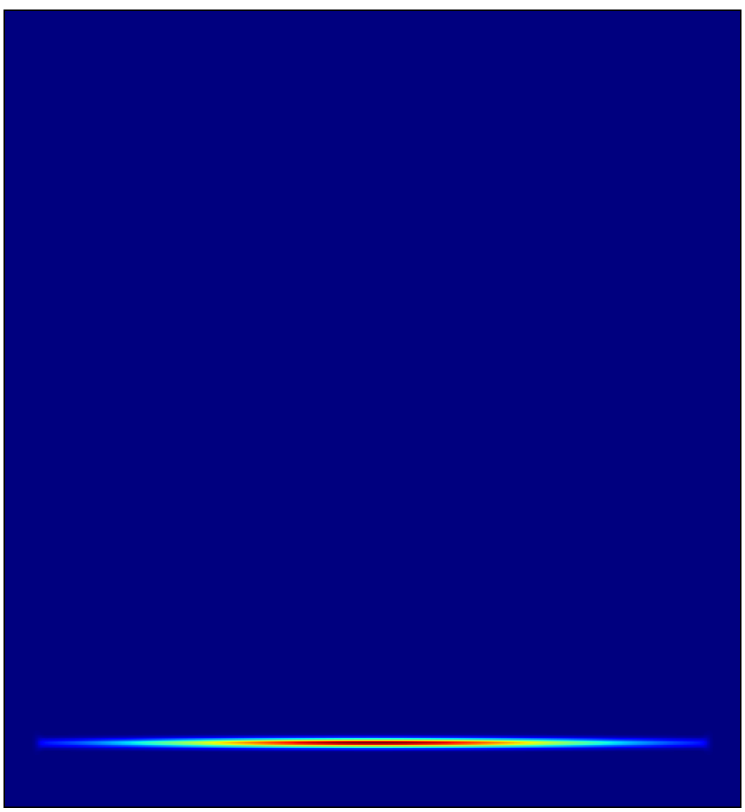


iii.

$\beta = \mathrm{i}$







Weither a phase or number squeezed state is created depends on the initial state.

2.

(a)

$$
\begin{aligned}
\left\langle N^2\right\rangle =&\langle 0|b^\dagger b b^\dagger b|0\rangle\\
=&\langle 0|\left(b_0^\dagger\cosh\varepsilon+b_0\sinh\varepsilon\right)\left(b_0\cosh\varepsilon+b_0^\dagger\sinh\varepsilon\right)\\
&\left(b_0^\dagger\cosh\varepsilon+b_0\sinh\varepsilon\right)\left(b_0\cosh\varepsilon+b_0^\dagger\sinh\varepsilon\right)|0\rangle\\
=&\sinh^2\varepsilon\langle 0|\left(b_0^2\cosh\varepsilon+b_0b_0^\dagger\sinh\varepsilon\right)\left({b_0^\dagger}^2\cosh\varepsilon+b_0b_0^\dagger\sinh\varepsilon\right)|0\rangle\\
=&\sinh^2\varepsilon\big(\sinh^2\varepsilon+2\cosh^2\varepsilon\big)\\
\Delta N^2 =&2\sinh^2\varepsilon\cosh^2\varepsilon\\
=&2\big(\Delta b_P^2-\Delta b_Q^2\big)^2
\end{aligned}
$$

For large ε

$$
\begin{aligned}
N\approx&\mathrm{e}^{2\varepsilon}\\
\Delta N\approx&\sqrt{2}\mathrm{e}^{2\varepsilon}\\
=&\sqrt{2}N
\end{aligned}
$$

The fluctuation is larger than classical state which has $\Delta N\propto\sqrt{N}$

(b)

$$
\begin{aligned}
&10\log_{10}\left(4\Delta a_P^2\right)\\
=&20\varepsilon\log_{10}\left(\mathrm{e}\right)
\end{aligned}
$$

which scales linearly with ε

$$
\begin{aligned}
\Delta a_P'^2 =&\left\langle a_P'^2\right\rangle-\left\langle a_P'\right\rangle^2\\
=&\left\langle\left(ta_P+ra_{P0}\right)^2\right\rangle-\left\langle\left(ta_P+ra_{P0}\right)\right\rangle^2\\
=&\left\langle t^2a_P^2+r^2a_{P0}^2\right\rangle-\left\langle ta_P\right\rangle^2\\
=&t^2\Delta a_P^2+r^2\Delta a_{P0}^2
\end{aligned}
$$

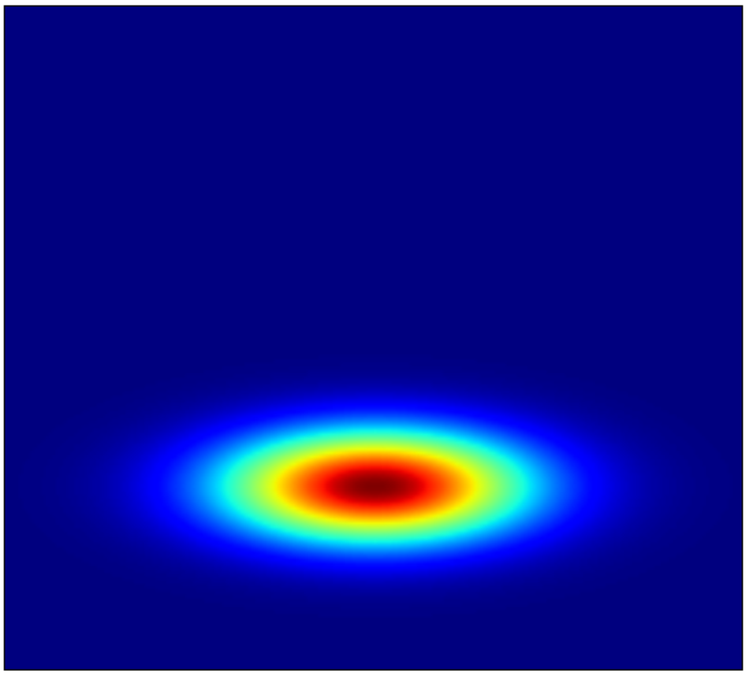
where a_P is the operator for the squeezed input state and a_{P0} is the operator for the vacuum input. In order to decrease it by 3dB

$$
\begin{aligned}
\Delta a_P'^2 =&\frac{1}{2}\Delta a_P^2\\
\Delta a_P^2 =&2\left(T\Delta a_P^2+\frac{1}{4}(1-T)\right)\\
T =&\frac{2\Delta a_P^2-1}{4\Delta a_P^2-1}
\end{aligned}
$$

At the limit of strong squeezing, $T \to \dfrac{1}{2}$

(c)

i.



ii.

$$\begin{aligned}\langle N' \rangle =& \left\langle \left(ta^\dagger + rb^\dagger\right)(ta + rb)\right\rangle \\ =& \left\langle t^2 n_a + r^2 n_b \right\rangle \\ =& TN_a + (1 - T)N_b\end{aligned}$$

iii.

$$\begin{aligned}\Delta n'^2 =& \left\langle n'^2 \right\rangle - \langle n' \rangle^2 \\ =& \left\langle \left(\left(ta^\dagger + rb^\dagger\right)(ta + rb)\right)^2 \right\rangle - \left\langle \left(ta^\dagger + rb^\dagger\right)(ta + rb)\right\rangle^2 \\ =& \left\langle \left(t^2 n_a + r^2 n_b + tr\left(a^\dagger b + b^\dagger a\right)\right)^2 \right\rangle - (TN_a + RN_b)^2 \\ =& T^2 \Delta N_a^2 + R^2 \Delta N_b^2 + TR\left\langle \left(a^\dagger \beta + \beta^* a\right)^2 \right\rangle\end{aligned}$$

For large N_a and if the displacement is along the lower variant direction

$$\Delta n'^2 \approx T^2 \Delta N_a^2 + R^2 \Delta N_b^2$$

iv.

$$\Delta n'^2 \approx 2T^2 N_a^2 + R^2 N_b$$

In order to be smaller than $\langle N' \rangle$

$$N_b > \frac{(2TN_a - 1)N_a}{R}$$

And the squeezing should be strong enough that $2TN_a > 1$

3.

(a)

$$\begin{aligned}
a_{out} =& \frac{a_2 - b_2}{\sqrt{2}} \\
=& \frac{a_1 e^{i\varphi} - b_1}{\sqrt{2}} \\
=& \frac{1}{2}\left((a - b)e^{i\varphi} - a - b\right) \\
=& e^{i\varphi/2}\left(ia \sin\frac{\varphi}{2} - b\cos\frac{\varphi}{2}\right) \\
a_{out}^\dagger =& e^{-i\varphi/2}\left(-ia^\dagger \sin\frac{\varphi}{2} - b^\dagger \cos\frac{\varphi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
b_{out} =& \frac{a_2 + b_2}{\sqrt{2}} \\
=& \frac{a_1 e^{i\varphi} + b_1}{\sqrt{2}} \\
=& \frac{1}{2}\left((a - b)e^{i\varphi} + a + b\right) \\
=& e^{i\varphi/2}\left(a \cos\frac{\varphi}{2} - ib\sin\frac{\varphi}{2}\right) \\
b_{out}^\dagger =& e^{-i\varphi/2}\left(a^\dagger \cos\frac{\varphi}{2} + ib^\dagger \sin\frac{\varphi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
n_{out_a} =& \left(-ia^\dagger \sin\frac{\varphi}{2} - b^\dagger \cos\frac{\varphi}{2}\right)\left(ia\sin\frac{\varphi}{2} - b\cos\frac{\varphi}{2}\right) \\
=& n_a \sin^2\frac{\varphi}{2} + n_b\cos^2\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\left(a^\dagger b - ab^\dagger\right)
\end{aligned}$$

$$\begin{aligned}
n_{out_b} =& \left(a^\dagger \cos\frac{\varphi}{2} + ib^\dagger \sin\frac{\varphi}{2}\right)\left(a\cos\frac{\varphi}{2} - ib\sin\frac{\varphi}{2}\right) \\
=& n_a \cos^2\frac{\varphi}{2} + n_b\sin^2\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\left(ab^\dagger - a^\dagger b\right)
\end{aligned}$$

$$\begin{aligned}
& b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \\
=& n_a \cos\varphi - n_b \cos\varphi + \mathrm{i}\sin\varphi\left(ab^\dagger - a^\dagger b\right)
\end{aligned}$$

Average

$$\begin{aligned}
& \left\langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \right\rangle \\
=& \langle n_a \cos\varphi \rangle \\
=& |\alpha|^2 \cos\varphi
\end{aligned}$$

Square

$$\begin{aligned}
& \left\langle \left(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\right)^2 \right\rangle \\
=& \left\langle \left(n_a \cos\varphi - n_b \cos\varphi + \mathrm{i}\sin\varphi\left(ab^\dagger - a^\dagger b\right)\right)\left(n_a \cos\varphi - n_b \cos\varphi + \mathrm{i}\sin\varphi\left(ab^\dagger - a^\dagger b\right)\right)\right\rangle \\
=& \left\langle (n_a \cos\varphi - \mathrm{i}\sin\varphi\alpha^* b)\left(n_a \cos\varphi + \mathrm{i}\sin\varphi\alpha b^\dagger\right)\right\rangle \\
=& \cos^2\varphi\langle n_a\rangle^2 + |\alpha|^2 \sin^2\varphi
\end{aligned}$$

Fluctuation

$$\begin{aligned}
& \Delta\left(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\right)^2 \\
=& \cos^2\varphi\Delta n_a^2 + |\alpha|^2 \sin^2\varphi \\
=& |\alpha|^2 \\
SNR =& \frac{|\alpha|^2 \cos\varphi}{|\alpha|} \\
=& |\alpha| \cos\varphi
\end{aligned}$$

(b)

$\varphi' = \dfrac{\pi}{2} - \varphi$

$$\begin{aligned}
SNR \approx& |\alpha|\varphi' \\
\varphi'_{min} =& \frac{1}{|\alpha|}
\end{aligned}$$

(c)

Average

$$\begin{aligned}
&b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\\
=&n_a\cos\varphi - \left(t^\dagger\cosh\varepsilon - t\sinh\varepsilon\right)\left(t\cosh\varepsilon - t^\dagger\sinh\varepsilon\right)\cos\varphi\\
&+\mathrm{i}\sin\varphi\left(a\left(t^\dagger\cosh\varepsilon - t\sinh\varepsilon\right) - a^\dagger\left(t\cosh\varepsilon - t^\dagger\sinh\varepsilon\right)\right)\\
=&n_a\cos\varphi - \cos\varphi\left(\cosh 2\varepsilon n_t + \sinh^2\varepsilon - \sinh\varepsilon\cosh\varepsilon\left({t^\dagger}^2 + t^2\right)\right)\\
&+\mathrm{i}\sin\varphi\left(a\left(t^\dagger\cosh\varepsilon - t\sinh\varepsilon\right) - a^\dagger\left(t\cosh\varepsilon - t^\dagger\sinh\varepsilon\right)\right)\\
&\left\langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\right\rangle\\
=&\langle n_a\cos\varphi - \cos\varphi\left(\cosh 2\varepsilon n_t + \sinh^2\varepsilon - \sinh\varepsilon\cosh\varepsilon\left({t^\dagger}^2 + t^2\right)\right)\\
&+\mathrm{i}\sin\varphi\left(a\left(t^\dagger\cosh\varepsilon - t\sinh\varepsilon\right) - a^\dagger\left(t\cosh\varepsilon - t^\dagger\sinh\varepsilon\right)\right)\rangle\\
=&|\alpha|^2\cos\varphi - \cos\varphi\sinh^2\varepsilon
\end{aligned}$$

Square

$$\begin{aligned}
&\left\langle\left(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\right)^2\right\rangle\\
=&\langle\left(n_a\cos\varphi - \cos\varphi\left(\cosh 2\varepsilon n_t + \sinh^2\varepsilon - \sinh\varepsilon\cosh\varepsilon\left({t^\dagger}^2 + t^2\right)\right)\right.\\
&\left.+\mathrm{i}\sin\varphi\left(a\left(t^\dagger\cosh\varepsilon - t\sinh\varepsilon\right) - a^\dagger\left(t\cosh\varepsilon - t^\dagger\sinh\varepsilon\right)\right)\right)\\
&\left(n_a\cos\varphi - \cos\varphi\left(\cosh 2\varepsilon n_t + \sinh^2\varepsilon - \sinh\varepsilon\cosh\varepsilon\left({t^\dagger}^2 + t^2\right)\right)\right.\\
&\left.+\mathrm{i}\sin\varphi\left(a\left(t^\dagger\cosh\varepsilon - t\sinh\varepsilon\right) - a^\dagger\left(t\cosh\varepsilon - t^\dagger\sinh\varepsilon\right)\right)\right)\rangle\\
=&\langle\left(n_a\cos\varphi - \cos\varphi\left(\sinh^2\varepsilon - \sinh\varepsilon\cosh\varepsilon t^2\right) - \mathrm{i}t\sin\varphi\left(a\sinh\varepsilon + a^\dagger\cosh\varepsilon\right)\right)\\
&\left(n_a\cos\varphi - \cos\varphi\left(\sinh^2\varepsilon - \sinh\varepsilon\cosh\varepsilon {t^\dagger}^2\right) + \mathrm{i}\sin\varphi t^\dagger\left(a\cosh\varepsilon + a^\dagger\sinh\varepsilon\right)\right)\rangle\\
=&\langle\left(n_a\cos\varphi - \cos\varphi\sinh^2\varepsilon + \cos\varphi\sinh\varepsilon\cosh\varepsilon t^2 - \mathrm{i}t\sin\varphi\left(a\sinh\varepsilon + \alpha^*\cosh\varepsilon\right)\right)\\
&\left(n_a\cos\varphi - \cos\varphi\sinh^2\varepsilon + \cos\varphi\sinh\varepsilon\cosh\varepsilon {t^\dagger}^2 + \mathrm{i}t^\dagger\sin\varphi\left(\alpha\cosh\varepsilon + a^\dagger\sinh\varepsilon\right)\right)\rangle\\
=&\left\langle\left(n_a\cos\varphi - \cos\varphi\sinh^2\varepsilon\right)^2\right\rangle + \langle\cos^2\varphi\sinh^2\varepsilon\cosh^2\varepsilon tt^\dagger tt^\dagger\rangle\\
&+\langle(\cos\varphi\sinh\varepsilon\cosh\varepsilon t - \mathrm{i}\sin\varphi(a\sinh\varepsilon + \alpha^*\cosh\varepsilon))\\
&(\cos\varphi\sinh\varepsilon\cosh\varepsilon t^\dagger + \mathrm{i}\sin\varphi\left(\alpha\cosh\varepsilon + a^\dagger\sinh\varepsilon\right))\rangle\\
=&\left\langle\left(n_a\cos\varphi - \cos\varphi\sinh^2\varepsilon\right)^2\right\rangle + 2\cos^2\varphi\sinh^2\varepsilon\cosh^2\varepsilon\\
&+\sin^2\varphi|\alpha\sinh\varepsilon + \alpha^*\cosh\varepsilon|^2 + \sin^2\varphi\sinh^2\varepsilon
\end{aligned}$$

Fluctuation

$$\begin{aligned}
&\Delta\left(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\right)^2\\
=&|\alpha|^2\cos^2\varphi + 2\cos^2\varphi\sinh^2\varepsilon\cosh^2\varepsilon + \sin^2\varphi|\alpha\sinh\varepsilon + \alpha^*\cosh\varepsilon|^2 + \sin^2\varphi\sinh^2\varepsilon\\
\approx&|\alpha|^2\varphi'^2 + 2\varphi^2\sinh^2\varepsilon\cosh^2\varepsilon + \left(1-\varphi^2\right)|\alpha\sinh\varepsilon + \alpha^*\cosh\varepsilon|^2 + \left(1-\varphi^2\right)\sinh^2\varepsilon
\end{aligned}$$

Minimum angle

$$\varphi'^2\left(|\alpha|^2 - \sinh^2\varepsilon\right)^2 = |\alpha|^2\varphi'^2 + 2\varphi'^2\sinh^2\varepsilon\cosh^2\varepsilon + \left(1-\varphi'^2\right)|\alpha\sinh\varepsilon + \alpha^*\cosh\varepsilon|^2 + \left(1-\varphi'^2\right)\sinh^2\varepsilon$$

In high power limit

$$\varphi'^2|\alpha|^4 = |\alpha\sinh\varepsilon + \alpha^*\cosh\varepsilon|^2$$

To get a factor of β in phase resolution

$$\frac{\alpha^2}{\beta^2} = |\alpha\sinh\varepsilon + \alpha^*\cosh\varepsilon|^2$$

With correct phase of α

$$\begin{aligned}\beta =& \mathrm{e}^{\varepsilon}\\ \varepsilon =& \ln\beta\end{aligned}$$

Squeezing changes the photon number because squeezed vacuum has non-zero photon number.

(d)

For coherent state

$$\begin{aligned}\alpha =& \sqrt{\frac{PL\lambda}{\pi\hbar c^2}}\\ =& 2.7\cdot 10^7\\ \varphi_{min} =& 3.7\cdot 10^{-8}\\ l_{min} =& 6.3\cdot 10^{-15}m\\ \varepsilon_{min} =& 1.6\cdot 10^{-18}\end{aligned}$$

With a 6dB squeezed vaccum, the sensitivity can go up by a factor of 4