1.

(a)

$$\begin{aligned}
&\left[a^\dagger b - a b^\dagger, a^\dagger a + b^\dagger b\right]\\
=&\left[a^\dagger b, a^\dagger a\right] - \left[a b^\dagger, a^\dagger a\right] + \left[a^\dagger b, b^\dagger b\right] - \left[a b^\dagger, b^\dagger b\right]\\
=&a^\dagger\left[a^\dagger, a\right]b - \left[a, a^\dagger\right]a b^\dagger + a^\dagger\left[b, b^\dagger\right]b - a b^\dagger\left[b^\dagger, b\right]\\
=&-a^\dagger b - a b^\dagger + a^\dagger b + a b^\dagger\\
=&0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[B, n_a + n_b\right] =&\left[\exp\left(\theta\left(a^\dagger b - a b^\dagger\right)\right), n_a + n_b\right]\\
=&0\\
B^\dagger =&\exp\left(\theta\left(a^\dagger b - a b^\dagger\right)^\dagger\right)\\
=&\exp\left(-\theta\left(a^\dagger b - a b^\dagger\right)\right)\\
=&B^{-1}
\end{aligned}$$

(b)

$$\begin{aligned}
\exp(\theta A) B \exp(-\theta A) =&\sum_{nm}(-1)^m\frac{\theta^{n+m}A^n B A^m}{n!m!}\\
=&\sum_N\sum_{m=0}^N(-1)^m\frac{\theta^N A^{N-m} B A^m}{(N-m)!m!}\\
=&\sum_N\frac{\theta^N}{N!}\sum_{m=0}^N(-1)^m\frac{N!A^{N-m} B A^m}{(N-m)!m!}\\
=&\sum_N\frac{\theta^N}{N!}[A, B]_N
\end{aligned}$$

where $[A,B]_N$ is defined as $[A,B]_N=\big[A,[A,B]_{N-1}\big]$ and $[A,B]_0=B$

$$
\begin{aligned}
\big[\big(a^\dagger b-ab^\dagger\big),a\big]_N=&\left\{\begin{array}{ll}(-1)^{N/2}a & (2\mid N)\\ (-1)^{(N+1)/2}b & (2\nmid N)\end{array}\right.\\
\big[\big(a^\dagger b-ab^\dagger\big),b\big]_N=&\left\{\begin{array}{ll}(-1)^{N/2}b & (2\mid N)\\ (-1)^{(N-1)/2}a & (2\nmid N)\end{array}\right.\\
BaB^{-1}=&\sum_N\frac{\theta^N}{N!}\big[\big(a^\dagger b-ab^\dagger\big),a\big]_N\\
=&\sum_n\frac{\theta^{2n}}{(2n)!}\big[\big(a^\dagger b-ab^\dagger\big),a\big]_{2n}+\sum_n\frac{\theta^{2n+1}}{(2n+1)!}\big[\big(a^\dagger b-ab^\dagger\big),a\big]_{2n+1}\\
=&\sum_n\frac{\theta^{2n}}{(2n)!}(-1)^na+\sum_n\frac{\theta^{2n+1}}{(2n+1)!}(-1)^{n+1}b\\
=&\cos\theta a-\sin\theta b\\
BbB^{-1}=&\sum_N\frac{\theta^N}{N!}\big[\big(a^\dagger b-ab^\dagger\big),b\big]_N\\
=&\sum_n\frac{\theta^{2n}}{(2n)!}\big[\big(a^\dagger b-ab^\dagger\big),b\big]_{2n}+\sum_n\frac{\theta^{2n+1}}{(2n+1)!}\big[\big(a^\dagger b-ab^\dagger\big),b\big]_{2n+1}\\
=&\sum_n\frac{\theta^{2n}}{(2n)!}(-1)^nb+\sum_n\frac{\theta^{2n+1}}{(2n+1)!}(-1)^na\\
=&\cos\theta b+\sin\theta a\\
B|0,\alpha\rangle=&B\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^\dagger}|0,0\rangle\\
=&\mathrm{e}^{-|\alpha|^2/2}B\mathrm{e}^{\alpha a^\dagger}B^{-1}B|0,0\rangle\\
=&\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\left(\cos\theta a^\dagger-\sin\theta b^\dagger\right)}|0,0\rangle\\
=&\mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\cos\theta a^\dagger}\mathrm{e}^{-\alpha\sin\theta b^\dagger}|0,0\rangle\\
=&|-\alpha\sin\theta,\alpha\cos\theta\rangle
\end{aligned}
$$

(c)

$$
\begin{aligned}
s_x=&a^\dagger b+ab^\dagger\\
s_y=&-\mathrm{i}\big(a^\dagger b-ab^\dagger\big)\\
B=&\mathrm{e}^{\mathrm{i}\theta s_y}
\end{aligned}
$$

which is a rotation around y

$$
\begin{aligned}
&\left(n_a+n_b\right)^2\\
=&s_z^2+4a^\dagger ab^\dagger b\\
=&s_z^2+4s^+s^-\\
=&s_x^2+s_y^2+s_z^2\\
=&S^2
\end{aligned}
$$

which is the total spin

$$
\begin{aligned}
[s_x, s_y] =& \left[a^\dagger b + a b^\dagger, -\mathrm{i}\left(a^\dagger b - a b^\dagger\right)\right]\\
=& -\mathrm{i}\left[a^\dagger b + a b^\dagger, a^\dagger b - a b^\dagger\right]\\
=& 2\mathrm{i}\left[a^\dagger b, a b^\dagger\right]\\
=& 2\mathrm{i} a^\dagger a\left[b, b^\dagger\right] + 2\mathrm{i}\left[a^\dagger, a\right] b^\dagger b\\
=& 2\mathrm{i} a^\dagger a - 2\mathrm{i} b^\dagger b\\
=& 2\mathrm{i} s_z\\
[s_y, s_z] =& \left[-\mathrm{i}\left(a^\dagger b - a b^\dagger\right), a^\dagger a - b^\dagger b\right]\\
=& -\mathrm{i} a^\dagger\left[a^\dagger, a\right] b + \mathrm{i} a^\dagger\left[b, b^\dagger\right] b + \mathrm{i}\left[a, a^\dagger\right] a b^\dagger - \mathrm{i} a b^\dagger\left[b^\dagger, b\right]\\
=& 2\mathrm{i} s_x\\
[s_z, s_x] =& \left[a^\dagger a - b^\dagger b, a^\dagger b + a b^\dagger\right]\\
=& a^\dagger\left[a, a^\dagger\right] b + \left[a^\dagger, a\right] a b^\dagger - a^\dagger\left[b^\dagger, b\right] b - a b^\dagger\left[b, b^\dagger\right]\\
=& 2 a^\dagger b - 2 a b^\dagger\\
=& 2\mathrm{i} s_y
\end{aligned}
$$

(d)

$$
\begin{aligned}
B|0, n\rangle =& B\frac{{a^\dagger}^n}{n!}|0, 0\rangle\\
=& \frac{\left(a^\dagger - b^\dagger\right)^n}{2^{n/2} n!}|0, 0\rangle\\
=& \sum_i \frac{{a^\dagger}^i {b^\dagger}^{n-i}}{2^{n/2} i!(n-i)!}|0, 0\rangle\\
=& \sum_i \frac{|n-i, i\rangle}{\sqrt{2^n i!(n-i)!}}
\end{aligned}
$$

The state(s) with the largest amplitude is $|n/2, n/2\rangle$ (when n is even) or $|(n+1)/2, (n-1)/2\rangle$ and $|(n-1)/2, (n+1)/2\rangle$ when n is odd

The variance of the distribution is $\dfrac{n}{4}$ so the relative width is getting narrower for larger n although the absolute width is getting wider.

2.

(a)

$$
\begin{aligned}
(\Delta n')^2 =& \left\langle n'^2 \right\rangle - \langle n' \rangle^2 \\
=& |t|^4 (\Delta n)^2 \\
=& |t|^4 |\alpha|^2 \\
\langle n' \rangle =& |t|^2 |\alpha|^2 \\
>& |t|^4 |\alpha|^2 \qquad (\text{when } 0 < |t| < 1 \text{ and } \alpha \neq 0)
\end{aligned}
$$

(b)

As shown in problem one.

$$
\begin{aligned}
\rho' =& \mathrm{Tr}_b \big(B |0, \alpha\rangle\langle 0, \alpha| B^\dagger \big) \\
=& \mathrm{Tr}_b (|r\alpha, t\alpha\rangle\langle r\alpha, t\alpha|) \\
=& |t\alpha\rangle\langle t\alpha|
\end{aligned}
$$

3.

(a)

(b)

(c)

(d)

4.

(a)

(b)

(c)

(d)

(e)