Yichao Yu

1.

(a)

For atoms that are closed enough, the interaction between them can be treated as dipole-dipole interaction (effectively integrate out the photon field). To second order of this effective Hamiltonian, the (non-degenerate) second order perturbation theory gives

$$\Delta E \propto \frac{d_a^2 d_b^2}{R^6\left(E_i^{(a)} + E_g^{(b)} - E_i^{(b)} - E_g^{(a)}\right)}$$

which scale with R^{-6}. The same is true for $|g_a i_b\rangle$. The perturbation theory breaks down when $E_i^{(a)} + E_g^{(b)} - E_i^{(b)} - E_g^{(a)} = 0$.

(b)

When $|g_a i_b\rangle$ and $|i_a g_b\rangle$ are degenerate, the energy shift become first order

$$\Delta E \propto \frac{d^2}{R^3}$$

(c)

$$\begin{aligned}\Gamma \propto& \omega^3 d^2\\ \frac{\Gamma}{\Delta E} =& \omega^3\end{aligned}$$

2.

$$\begin{aligned}F_c =& -\frac{\hbar c \pi^2}{240 a^2}\\ F_e =& \frac{e^2}{16\pi\varepsilon_0 a^2}\\ \frac{\hbar c\pi^2}{240} =& \frac{e^2}{16\pi\varepsilon_0}\\ \alpha =& \frac{e^2}{4\pi\varepsilon_0\hbar c}\\ =& \frac{\pi^2}{60}\end{aligned}$$

3.

(a)

$$\begin{aligned}
\rho(\theta) =& \begin{pmatrix} a & b\mathrm{e}^{\mathrm{i}\theta} \\ c\mathrm{e}^{-\mathrm{i}\theta} & d \end{pmatrix} \\
\langle\rho\rangle =& \int_{-\infty}^{\infty}\mathrm{d}\theta \frac{1}{\sqrt{4\pi\lambda t}} \begin{pmatrix} a & b\mathrm{e}^{\mathrm{i}\theta} \\ c\mathrm{e}^{-\mathrm{i}\theta} & d \end{pmatrix} \exp\left(-\frac{\theta^2}{2\lambda t}\right) \\
=& \begin{pmatrix} a & \frac{b}{\sqrt{4\pi\lambda t}}\int_{-\infty}^{\infty}\mathrm{d}\theta \exp\left(\mathrm{i}\theta - \frac{\theta^2}{2\lambda t}\right) \\ \frac{c}{\sqrt{4\pi\lambda t}}\int_{-\infty}^{\infty}\mathrm{d}\theta \exp\left(-\mathrm{i}\theta - \frac{\theta^2}{2\lambda t}\right) & d \end{pmatrix} \\
=& \begin{pmatrix} a & \frac{b}{\sqrt{4\pi\lambda t}}\int_{-\infty}^{\infty}\mathrm{d}\theta \exp\left(-\frac{\left(\theta - \mathrm{i}\lambda t\right)^2 + \left(\lambda t\right)^2}{2\lambda t}\right) \\ \frac{c}{\sqrt{4\pi\lambda t}}\int_{-\infty}^{\infty}\mathrm{d}\theta \exp\left(-\frac{\left(\theta + \mathrm{i}\lambda t\right)^2 + \left(\lambda t\right)^2}{2\lambda t}\right) & d \end{pmatrix} \\
=& \begin{pmatrix} a & b\exp\left(-\frac{\lambda t}{2}\right) \\ c\exp\left(-\frac{\lambda t}{2}\right) & d \end{pmatrix}
\end{aligned}$$

(b)

Since the Hamiltonian does nothing to $|g\rangle$, $|g0\rangle$ will remain the same. For $|e*\rangle$, the environment will undergo Rabi flopping. Also assume γ is real since the phase of it can be absorbed in $|1\rangle$. The evolution of state,

$$|\psi(t)\rangle = a|g0\rangle + b\left(\cos\frac{\gamma t}{\hbar}|e0\rangle + \mathrm{i}\sin\frac{\gamma t}{\hbar}|e1\rangle\right)$$

Atomic density matrix

$$\rho = \begin{pmatrix} |a|^2 & ab^*\cos\frac{\gamma t}{\hbar} \\ a^*b\cos\frac{\gamma t}{\hbar} & |b|^2 \end{pmatrix}$$

(c)

At time t the average density matrix is

$$\begin{aligned}
\langle\rho\rangle =& \frac{1 + \mathrm{e}^{-\lambda t}}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \frac{1 - \mathrm{e}^{-\lambda t}}{2}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \\
=& \begin{pmatrix} a & b\mathrm{e}^{-\lambda t} \\ c\mathrm{e}^{-\lambda t} & d \end{pmatrix}
\end{aligned}$$