MIT 8.422 Atomic Physics II — *Spring, 2015*
Prof. Martin Zweierlein — April 07, 2015

Assignment #7

Due: Wednesday, April 15, 2015
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Office Hours (in 26-214): April 13th (Mon) 9am - 11am & April 14th (Tues) 9am - 11am & by arrangement

1. **Optical Bloch Equations: weak and short-time limits**.

   The time-independent form of the optical Bloch equations (eg, see API p. 359), including spontaneous emission and the rotating wave approximation, is:

$$\dot{\rho}_{ee} = i\frac{\Omega}{2}(\rho_{eg} - \rho_{ge}) - \Gamma\rho_{ee} \tag{1}$$

$$\dot{\rho}_{ge} = i(\omega_0 - \omega_L)\rho_{ge} - i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) - \frac{\Gamma}{2}\rho_{ge}\,, \tag{2}$$

   where the remaining two components of the density matrix are given by $\rho_{gg} = 1 - \rho_{ee}$, and $\rho_{eg} = \rho_{ge}^*$. It is insightful to study these equations in the limit of weak excitation, and for short evolution times.

   a) Show that the solution of these equations to lowest order in $|\Omega|$, and in the limit $|\Omega| \ll \Gamma$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{\frac{1}{4}|\Omega|^2}{(\omega_0 - \omega_L)^2 + \left(\frac{\Gamma}{2}\right)^2}\left[1 + e^{-\Gamma t} - 2\cos[(\omega_0 - \omega_L)t]e^{-\Gamma t/2}\right]. \tag{3}$$

   What does this solution reduce to in the limit of an infinitely narrow linewidth ($\Gamma \to 0$)?

   b) Show that the solution of these equations to lowest order in $|\Omega|$ in the limit $|\Omega|t \ll 1$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{1}{4}|\Omega|^2 t^2\,. \tag{4}$$

   This result is independent of the detuning $\omega_0 - \omega_L$ and the decay rate Γ. Why?

2. **One atom and one photon: spontaneous emission**

   A single atom coupled to a single mode of electromagnetic radiation undergoes spontaneous emission. What is the state of the atom during such spontaneous emission?

   Let us model the interaction of one atom with a single optical mode using the Jaynes-Cummings interaction

$$H = \hbar\omega a^\dagger a + \frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\Omega_1}{2}(a^\dagger\sigma_- + a\sigma_+) \tag{5}$$

   where $\delta = \omega - \omega_0$ is the detuning of the cavity frequency ω from the atomic frequency ω_0, Ω_1 is the coupling of the atom to the field and we defined $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_+ = |e\rangle\langle g|$,

$\sigma_- = |g\rangle\langle e|$.
In the subspace where at most one quantum of energy is exchanged with the optical mode, we may write this Hamiltonian as a matrix

$$H = \frac{\hbar}{2} \begin{bmatrix} -\delta - 2\omega_0 & 0 & 0 \\ 0 & -\delta & \Omega_1 \\ 0 & \Omega_1 & \delta \end{bmatrix}, \tag{6}$$

where the basis states are $|0g\rangle$, $|0e\rangle$ and $|1g\rangle$. Here $|g\rangle$ and $|e\rangle$ are the ground and excited states of the atom. $|0\rangle$ and $|1\rangle$ denote the number of photons in the optical mode.

a) Compute the full unitary evolution $U(t) = e^{-iHt/\hbar}$ under this Hamiltonian and obtain

$$\begin{aligned} U(t) &= e^{i(\delta/2+\omega_0)t}|0g\rangle\langle 0g| + \left(\cos(\Omega t/2) + i\frac{\delta}{\Omega}\sin(\Omega t/2)\right)|0e\rangle\langle 0e| \\ &+ \left(\cos(\Omega t/2) - i\frac{\delta}{\Omega}\sin(\Omega t/2)\right)|1g\rangle\langle 1g| - i\frac{\Omega_1}{\Omega}\sin(\Omega t/2)\left(|0e\rangle\langle 1g| + |1g\rangle\langle 0e|\right) \end{aligned} \tag{7}$$

Here $\Omega = \sqrt{\Omega_1^2 + \delta^2}$ is the generalized Rabi frequency.

b) Suppose the atom starts out in the excited state $|e\rangle$ and the cavity with no photon $|0\rangle$. What is the state of the atom after time t, *if the cavity is measured and found to have no photon*? What if one photon is found to be in the cavity?

c) A reduced density matrix describes the state of part of a system, averaged over the possible states of the remainder. Suppose that at $t = 0$ the atom starts out in a general mixed state with no photons in the cavity:

$$\begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix} \otimes |0\rangle\langle 0| \tag{8}$$

Give a reduced density matrix describing the state of the atom at time $t > 0$.

d) Suppose that the atom interacts with the cavity for a short time Δt, after which the cavity is measured. Determine how its density matrix evolves under repeated short evolutions with the cavity. The cavity state is reset to $|0\rangle$ after each interaction. What is the fixed point of this process? This shows how spontaneous emission emerges as the limiting case of a continuous readout of the photon field.

e) Let $|e\rangle$ and $|g\rangle$ be depicted as the south and north poles of a Bloch sphere representation of the atom. Plot the points $(|e\rangle + |g\rangle)/\sqrt{2}$, $(|e\rangle - |g\rangle)/\sqrt{2}$ and $|e\rangle$. Recall that a two-dimensional density matrix ρ can be represented by a point $\vec{r}$ *inside* the Bloch sphere, using

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}. \tag{9}$$

Plot how these three initial states evolve under repeated short evolutions with the cavity according to d).

f) If the cavity is tuned far off resonance from the atom, emission into the cavity is strongly suppressed. It seems surprising that the results you should have obtained in d) and e) are independent of the detuning δ. How can this be explained physically?
Hint: What assumptions have you made in d)?

3. **Driven two-level atom: dressed states**

A two-level atom driven by a classical laser field is often conveniently studied in a stationary basis, that is, the basis defined by the eigenstates of the Hamiltonian. These basis states are known as *dressed states*, and they are useful for interpreting many phenomena and solving problems in atomic physics.

Let the Hamiltonian for the classically driven atom be

$$H = \frac{\hbar\omega_0}{2} Z + \frac{\hbar\Omega_1}{2}\left[X\cos(\omega_L t) + Y\sin(\omega_L t)\right], \tag{10}$$

where X, Y, and Z are the usual Pauli matrices, $\hbar\omega_0$ is the energy difference between the atomic levels $|e\rangle$ and $|g\rangle$, ω_L is the laser frequency, and $\Omega_1 = eE_0\langle g|z|e\rangle/\hbar$ is the Rabi frequency.

a) Write the coupled time-dependent Schrodinger equations using a solution anzatz of the form $|\psi(t)\rangle = ae^{i\omega_1 t}|g\rangle + be^{i\omega_2 t}|e\rangle$. Choose the frequencies such that the equations are steady-state, containing no oscillating terms.

b) The Schrodinger equations you have just written are identical to those for a system with a Hamiltonian that is a function of the Rabi frequency and the detuning $\delta_L = \omega_L - \omega_0$ only. Give this Hamiltonian; denote it as H'.

c) Write H' in terms of trigonometric functions, where $\sin 2\theta = \Omega_1/\Omega$, where $\Omega = \sqrt{\Omega_1^2 + \delta_L^2}$ is the "effective" Rabi frequency.

d) Diagonalize H' to find the eigenvalues and associated eigenvectors.

e) Use these results to find the time-dependent solutions to the Schrodinger equations for H, in the original frame of reference.