1.

(a)

Define

$$Y(x,y) = \frac{x^n y^{n-1} + x^{n-1} y^n}{\sqrt{2n!(n-1)!}}$$

when $[x,y]=0$

$$Y(x,y) = x^{n-1}y^{n-1}\frac{x+y}{\sqrt{2n!(n-1)!}}$$

we also have, for any number λ

$$Y(\lambda x, \lambda y) = \lambda^{2n-1} Y(x,y)$$

Before the first beam splitter

$$\begin{aligned}|\psi_0\rangle =& Y\left(a^\dagger, b^\dagger\right)|0,0\rangle\\ =& {a^\dagger}^{n-1}{b^\dagger}^{n-1}\frac{a^\dagger + b^\dagger}{\sqrt{2n!(n-1)!}}|0,0\rangle\end{aligned}$$

After the first beam splitter

$$\begin{aligned}|\psi_1\rangle =& B|\psi_0\rangle\\ =& BY\left(a^\dagger, b^\dagger\right)B^\dagger|0,0\rangle\\ =& Y\left(Ba^\dagger B^\dagger, Bb^\dagger B^\dagger\right)|0,0\rangle\\ =& \frac{\sqrt{2}}{2^n}Y\left(a^\dagger - \mathrm{i}b^\dagger, b^\dagger - \mathrm{i}a^\dagger\right)|0,0\rangle\\ =& \left(a^\dagger - \mathrm{i}b^\dagger\right)^{n-1}\left(b^\dagger - \mathrm{i}a^\dagger\right)^{n-1}\frac{a^\dagger - \mathrm{i}b^\dagger + b^\dagger - \mathrm{i}a^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\\ =& (-\mathrm{i})^{n-1}(1-\mathrm{i})\left({a^\dagger}^2 + {b^\dagger}^2\right)^{n-1}\frac{a^\dagger + b^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\end{aligned}$$

After phase shift

$$\begin{aligned}|\psi_2\rangle =& \frac{\sqrt{2}}{2^n}Y\left(\mathrm{e}^{-\mathrm{i}\phi}a^\dagger - \mathrm{i}b^\dagger, b^\dagger - \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}a^\dagger\right)|0,0\rangle\\ =& (-\mathrm{i})^{n-1}(1-\mathrm{i})\left(\mathrm{e}^{-2\mathrm{i}\phi}{a^\dagger}^2 + {b^\dagger}^2\right)^{n-1}\frac{\mathrm{e}^{-\mathrm{i}\phi}a^\dagger + b^\dagger}{2^n\sqrt{n!(n-1)!}}|0,0\rangle\end{aligned}$$

Output

$$\begin{aligned}
|\psi_3\rangle =&B^\dagger|\psi_2\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{2^{2n-1}}Y\Big(\mathrm{e}^{-\mathrm{i}\phi/2}\big(a^\dagger+\mathrm{i}b^\dagger\big)-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi/2}\big(b^\dagger+\mathrm{i}a^\dagger\big),\mathrm{e}^{\mathrm{i}\phi/2}\big(b^\dagger+\mathrm{i}a^\dagger\big)-\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi/2}\big(a^\dagger+\mathrm{i}b^\dagger\big)\Big)|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{2^{2n-1}}Y\left(2\cos\frac{\phi}{2}a^\dagger+2\sin\frac{\phi}{2}b^\dagger,2\cos\frac{\phi}{2}b^\dagger-2\sin\frac{\phi}{2}a^\dagger\right)|0,0\rangle\\
=&\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}Y\left(\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger,\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\right)|0,0\rangle\\
=&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{\sqrt{2n!(n-1)!}}\left(\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger\right)^{n-1}\left(\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\right)^{n-1}\\
&\left(\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger+\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\right)|0,0\rangle
\end{aligned}$$

For $\phi=0$

$$\begin{aligned}
|\phi_3\rangle_0 =&\frac{1}{\sqrt{n!(n-1)!}}{a^\dagger}^{n-1}{b^\dagger}^{n-1}\frac{a^\dagger+b^\dagger}{\sqrt{2}}|0,0\rangle\\
=&|\phi_0\rangle
\end{aligned}$$

(b)

Let

$$\begin{aligned}
a' =&\cos\frac{\phi}{2}a+\sin\frac{\phi}{2}b\\
b' =&\cos\frac{\phi}{2}b-\sin\frac{\phi}{2}a
\end{aligned}$$

then

$$\begin{aligned}
a'^\dagger =&\cos\frac{\phi}{2}a^\dagger+\sin\frac{\phi}{2}b^\dagger\\
b'^\dagger =&\cos\frac{\phi}{2}b^\dagger-\sin\frac{\phi}{2}a^\dagger\\
a =&\cos\frac{\phi}{2}a'-\sin\frac{\phi}{2}b'\\
b =&\cos\frac{\phi}{2}b'+\sin\frac{\phi}{2}a'\\
a^\dagger =&\cos\frac{\phi}{2}a'^\dagger-\sin\frac{\phi}{2}b'^\dagger\\
b^\dagger =&\cos\frac{\phi}{2}b'^\dagger+\sin\frac{\phi}{2}a'^\dagger\\
\left[a',a'^\dagger\right] =&1\\
\left[b',b'^\dagger\right] =&1\\
|\phi_3\rangle =&\frac{\mathrm{e}^{-\mathrm{i}\phi(n-1/2)}}{\sqrt{n!(n-1)!}}{a'^\dagger}^{n-1}{b'^\dagger}^{n-1}\frac{a'^\dagger+b'^\dagger}{\sqrt{2}}|0,0\rangle
\end{aligned}$$

$$\begin{aligned}
M =& a^\dagger a - b^\dagger b\\
=& \left(\cos\frac{\phi}{2}a'^\dagger - \sin\frac{\phi}{2}b'^\dagger\right)\left(\cos\frac{\phi}{2}a' - \sin\frac{\phi}{2}b'\right) - \left(\cos\frac{\phi}{2}b'^\dagger + \sin\frac{\phi}{2}a'^\dagger\right)\left(\cos\frac{\phi}{2}b' + \sin\frac{\phi}{2}a'\right)\\
=& \cos^2\frac{\phi}{2}a'^\dagger a' + \sin^2\frac{\phi}{2}b'^\dagger b' - \sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(a'b'^\dagger + a'^\dagger b'\right)\\
& - \cos^2\frac{\phi}{2}b'^\dagger b' - \sin\frac{\phi}{2}\cos\frac{\phi}{2}a'^\dagger b' - \sin\frac{\phi}{2}\cos\frac{\phi}{2}a'b'^\dagger - \sin^2\frac{\phi}{2}a'^\dagger a'\\
=& \cos\phi\left(a'^\dagger a' - b'^\dagger b'\right) - \sin\phi\left(a'b'^\dagger + a'^\dagger b'\right)
\end{aligned}$$

Mean

$$\begin{aligned}
\langle M\rangle =& \frac{1}{2n!(n-1)!}\langle 0,0|a'^{n-1}b'^{n-1}(a'+b')\left(\cos\phi\left(a'^\dagger a' - b'^\dagger b'\right) - \sin\phi\left(a'b'^\dagger + a'^\dagger b'\right)\right)\\
& a'^{\dagger n-1}b'^{\dagger n-1}\left(a'^\dagger + b'^\dagger\right)|0,0\rangle\\
=& \frac{\cos\phi}{2n!(n-1)!}\langle 0,0|a'^{n-1}b'^{n-1}(a'+b')\left(a'^\dagger a' - b'^\dagger b'\right)a'^{\dagger n-1}b'^{\dagger n-1}\left(a'^\dagger + b'^\dagger\right)|0,0\rangle\\
& - \frac{\sin\phi}{2n!(n-1)!}\langle 0,0|a'^{n-1}b'^{n-1}(a'+b')\left(a'b'^\dagger + a'^\dagger b'\right)a'^{\dagger n-1}b'^{\dagger n-1}\left(a'^\dagger + b'^\dagger\right)|0,0\rangle\\
=& -n\sin\phi
\end{aligned}$$

Square

$$\begin{aligned}
M^2 =& \left(\cos\phi\left(a'^\dagger a' - b'^\dagger b'\right) - \sin\phi\left(a'b'^\dagger + a'^\dagger b'\right)\right)^2\\
=& \cos^2\phi\left(a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b'\right) + \sin^2\phi\left(a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2\right)\\
& - \sin\phi\cos\phi\left(2a'^\dagger a'^2 b'^\dagger + a'b'^\dagger + 2a'^{\dagger 2} a' b' + a'^\dagger b' - 2a' b'^{\dagger 2} b' - a' b'^\dagger - 2a'^\dagger b'^\dagger b'^2 - a'^\dagger b'\right)\\
=& \cos^2\phi\left(a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b'\right) + \sin^2\phi\left(a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2\right)\\
& - 2\sin\phi\cos\phi\left(a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2} a' b' - a' b'^{\dagger 2} b' - a'^\dagger b'^\dagger b'^2\right)\\
\left\langle M^2\right\rangle =& \cos^2\phi\left\langle a'^\dagger a' a'^\dagger a' - 2a'^\dagger a' b'^\dagger b' + b'^\dagger b' b'^\dagger b'\right\rangle\\
& + \sin^2\phi\left\langle a'^2 b'^{\dagger 2} + a' a'^\dagger b'^\dagger b' + a'^\dagger a' b' b'^\dagger + a'^{\dagger 2} b'^2\right\rangle\\
& - 2\sin\phi\cos\phi\left\langle a'^\dagger a'^2 b'^\dagger + a'^{\dagger 2} a' b' - a' b'^{\dagger 2} b' - a'^\dagger b'^\dagger b'^2\right\rangle\\
=& \frac{\cos^2\phi}{2n}\left(n^3 + (n-1)^2 n - 2n^2(n-1) - 2n^2(n-1) + (n-1)^2 n + n^3\right)\\
& + \frac{\sin^2\phi}{2n}\left(2(n-1)n(n+1) + 2n^3\right)\\
& - \frac{\sin\phi\cos\phi}{n}\left(n^2(n-1) + n^2(n-1) - n^2(n-1) - n^2(n-1)\right)\\
=& \cos^2\phi + \sin^2\phi\left(2n^2 - 1\right)
\end{aligned}$$

Fluctuation

$$\begin{aligned}\Delta M^2 =&\cos^2\phi + \sin^2\phi\left(2n^2 - 1\right) - n^2\sin^2\phi\\ =&\cos^2\phi + \sin^2\phi\left(n^2 - 1\right)\\ \frac{\partial\langle M\rangle}{\partial\phi} =& -n\cos\phi\\ \Delta\phi^2 =&\frac{\cos^2\phi + \sin^2\phi\left(n^2 - 1\right)}{n^2\cos^2\phi}\\ =&\frac{1 + \tan^2\phi\left(n^2 - 1\right)}{n^2}\\ \Delta\phi =&\frac{\sqrt{1 + \tan^2\phi(n^2 - 1)}}{n}\end{aligned}$$

for $\phi = 0$

$$\Delta\phi = \frac{1}{n}$$

(c)

(d)

2.

(a)

$$\begin{aligned}P =&|\psi_A|^2|\psi_B|^2\left|\mathrm{e}^{\mathrm{i}\left(\vec{k}_A\cdot\vec{r}_{A1} + \vec{k}_B\cdot\vec{r}_{B2}\right)} \pm \mathrm{e}^{\mathrm{i}\left(\vec{k}_A\cdot\vec{r}_{A2} + \vec{k}_B\cdot\vec{r}_{B1}\right)}\right|^2\\ =&|\psi_A|^2|\psi_B|^2\left|\mathrm{e}^{\mathrm{i}\vec{k}_A\cdot\vec{r}_{21}} \pm \mathrm{e}^{\mathrm{i}\vec{k}_B\cdot\vec{r}_{21}}\right|^2\\ =&|\psi_A|^2|\psi_B|^2\left|\mathrm{e}^{\mathrm{i}\left(\vec{k}_A - \vec{k}_B\right)\cdot\vec{r}_{21}/2} \pm \mathrm{e}^{-\mathrm{i}\left(\vec{k}_A - \vec{k}_B\right)\cdot\vec{r}_{21}/2}\right|^2\\ =&|\psi_A|^2|\psi_B|^2\left|2 \pm \mathrm{e}^{\mathrm{i}\left(\vec{k}_A - \vec{k}_B\right)\cdot\vec{r}_{21}} \pm \mathrm{e}^{-\mathrm{i}\left(\vec{k}_A - \vec{k}_B\right)\cdot\vec{r}_{21}}\right|\\ =&2|\psi_A|^2|\psi_B|^2\left(1 \pm \cos\left(\left(\vec{k}_A - \vec{k}_B\right)\cdot\vec{r}_{21}\right)\right)\end{aligned}$$

(b)

In order to see the correlation $\Delta\phi_t \ll 1$,

$$\begin{aligned}1 \gg&\Delta\vec{k}_t\cdot\vec{r}_{21t}\\ =&k_0\frac{W}{d}w\\ =&\frac{2\pi Ww}{\lambda d}\\ \lambda d \gg&Ww\end{aligned}$$

$$\begin{aligned}
\lambda =&\frac{\hbar}{p}\\
=&\frac{\hbar}{\sqrt{mk_BT}}\\
\approx&12\text{nm}\\
w_{max} \approx&\sqrt{\lambda d}\\
\approx&36\mu\text{m}
\end{aligned}$$

(c)

(d)