

Assignment #8

Due: Wednesday, April 22

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Office Hours: April 17th and 20th, 11 am - 1 pm and by arrangement

1. Classical Model of the Light Force

In class, we will discuss light forces using the OBE. In preparation for this, you are asked here to consider a (semi-)classical derivation. Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency ω_0 and damping coefficient γ . The nucleus is fixed at position \mathbf{x}_0 while the electron's position is denoted by \mathbf{x} . Now suppose the atom is illuminated with an electromagnetic wave of the form

$$\mathbf{E}(\mathbf{x}, t) = \hat{\epsilon} E_0(\mathbf{x}) \cos(\theta(\mathbf{x}) + \omega t) \quad (1)$$

where $\theta(\mathbf{x})$ is the phase of the wave as a function of position \mathbf{x} at time $t = 0$. The dipole moment of the electron may be written as

$$\mathbf{p}(\mathbf{x}, t) = \hat{p}(u \cos(\theta(\mathbf{x}) + \omega t) - v \sin(\theta(\mathbf{x}) + \omega t)) \quad (2)$$

Then the force of the light on the atom is

$$\mathbf{F} = (\mathbf{p} \cdot \hat{\epsilon}) \nabla E(\mathbf{x}, t) \quad (3)$$

(a) Time averaged force

Make the dipole approximation that $\mathbf{E}(\mathbf{x}) \approx \mathbf{E}(\mathbf{x}_0)$. Show that the time averaged force is

$$\langle \mathbf{F} \rangle = \frac{1}{2} (\hat{p} \cdot \hat{\epsilon}) (u \nabla E_0(\mathbf{x}_0) + v E_0(\mathbf{x}_0) \nabla \theta(\mathbf{x}_0)) \quad (4)$$

This expression is exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.

(b) The potential picture

Recalculate the time averaged force on the atom from the instantaneous potential energy of a dipole in an electric field. How does this answer differ from that of 1a? Speculate as to why.

(c) Dipole moment of electron

Now we will solve explicitly for the dipole moment of the electron. In complex notation, the equation of motion is

$$m \frac{\partial^2 \mathbf{r}}{\partial t^2} + \gamma \frac{\partial \mathbf{r}}{\partial t} + m \omega_0^2 \mathbf{r} = -e \hat{\epsilon} E_0(\mathbf{x}_0) e^{i(\theta(\mathbf{x}_0) + \omega t)} \quad (5)$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. Solve this equation to find $\mathbf{p} = -e\mathbf{r}$. Substitute the quadrature components of \mathbf{p} into the force equation from part (a) to find that

$$\mathbf{F} = -\frac{e^2}{2m\omega_0} \frac{\delta \nabla E_0^2 + \Gamma E_0^2 \nabla \theta}{4\delta^2 + \Gamma^2} \quad (6)$$

where $\delta = \omega - \omega_0$ and $\Gamma = \gamma/m$. Make the approximation that $\omega \approx \omega_0$.

(d) Force on a two-level atom

The quantum mechanical expression for the light force on a two-level atom in the low intensity limit is

$$\mathbf{F} \approx -\frac{\hbar\delta\nabla\omega_R^2 + \hbar\Gamma\omega_R^2\nabla\theta}{4\delta^2 + \Gamma^2} \quad (7)$$

where ω_R is the Rabi frequency. Show that if we introduce the oscillator strength f_{fi} between the two levels, we may write

$$\mathbf{F}_{\text{quantum}} = f_{fi}\mathbf{F}_{\text{classical}} \quad (8)$$

2. Master equation for a damped optical cavity

A Fabry-Perot cavity can be modeled as being made of a high reflectivity mirror and a perfect mirror with fixed spacing. Clearly, photons stored inside this cavity will gradually leak out the partially reflecting mirror, causing the state inside to change. This process is described by a master equation, much like an atom coupled to fields is described by the optical Bloch equation. In this problem, we explore a simple derivation of such a master equation, for a single mode cavity.

Let a and a^\dagger describe the optical mode of interest within the cavity, with characteristic energy $\hbar\omega$, described by the Hamiltonian $H_0 = \hbar\omega a^\dagger a$. Let $|\psi\rangle$ be the initial cavity state. Let us suppose that photons leak out of the cavity at a rate proportional to the photon number in the cavity and to Γ , which parameterizes the leakiness of the leaky mirror. Thus the probability that a photon leaks out within the infinitesimal timestep between t and $t + dt$ is given by $dp = \Gamma dt \langle \psi | a^\dagger a | \psi \rangle$. We now model the time evolution as follows:

- If a photon leaks out, the cavity state becomes $|\tilde{\psi}_1\rangle = a\sqrt{\Gamma dt}|\psi\rangle$.
- If no photon leaks out, the state becomes $|\tilde{\psi}_0\rangle = e^{-iHt/\hbar}|\psi\rangle$ having evolved under the “Hamiltonian” $H = H_0 - i(\hbar\Gamma/2)a^\dagger a$. Note that H is not Hermitian. It has an imaginary term, so it is “lossy”. Evolving a state forward in time with H yields an un-normalized state.

The two un-normalized states $|\tilde{\psi}_0\rangle$ and $|\tilde{\psi}_1\rangle$ will later be two “components” of the density matrix.

- a) Compute the normalized states $|\psi_0\rangle$ and $|\psi_1\rangle$. Provide some words of justification for the lossy term in H (for instance, why is it proportional to $a^\dagger a$?).
Where you encounter exponentials of H or H^\dagger (which is NOT equal to H), expand them to first order in dt .
- b) At time t , suppose the cavity starts in the pure quantum state $\rho(t) = |\psi\rangle\langle\psi|$. At time $t + dt$ the state of the leaky cavity system is given by

$$\rho(t + dt) = (1 - dp)|\psi_0\rangle\langle\psi_0| + dp|\psi_1\rangle\langle\psi_1|. \quad (9)$$

Rewrite this density matrix, building it out of the the un-normalized states described in part a). The only bras or kets to appear in your final expression should be $|\psi\rangle$ and $\langle\psi|$.

- c) Compute $\rho(t + dt) - \rho(t)$ for small dt , and write the coarse-grained differential equation

$$\frac{d}{dt}\rho(t) \approx \frac{\rho(t + dt) - \rho(t)}{dt}. \quad (10)$$

Show that this has the proper Lindblad form of

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{1}{2}(C^\dagger C\rho - 2C\rho C^\dagger + \rho C^\dagger C), \quad (11)$$

and identify what C and C^\dagger are for this case.