

1.

(a)

$$\begin{aligned}
 H &= \hbar\omega b^\dagger b + i\hbar\Lambda(b^{\dagger 2} + b^2) \\
 \frac{db}{dt} &= \frac{i}{\hbar}[H, b] + \frac{\partial b}{\partial t} \\
 &= [i\omega b^\dagger b - \Lambda b^{\dagger 2}, b] + i\omega b \\
 &= 2\Lambda b^\dagger \\
 \frac{db^\dagger}{dt} &= 2\Lambda b
 \end{aligned}$$

(b)

$$\begin{aligned}
 \vec{E} &= i\mathcal{E}_\omega \vec{\varepsilon} \left(b e^{i(\vec{k} \cdot \vec{r} - \omega t)} - b^\dagger e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right) \\
 &= i\mathcal{E}_\omega \vec{\varepsilon} \left(b \cos(\vec{k} \cdot \vec{r} - \omega t) + ib \sin(\vec{k} \cdot \vec{r} - \omega t) - b^\dagger \cos(\vec{k} \cdot \vec{r} - \omega t) + ib^\dagger \sin(\vec{k} \cdot \vec{r} - \omega t) \right) \\
 &= -2\mathcal{E}_\omega \vec{\varepsilon} \left(\frac{b - b^\dagger}{2i} \cos(\vec{k} \cdot \vec{r} - \omega t) + \frac{b + b^\dagger}{2} \sin(\vec{k} \cdot \vec{r} - \omega t) \right) \\
 &= -2\mathcal{E}_\omega \vec{\varepsilon} \left(b_Q \cos(\vec{k} \cdot \vec{r} - \omega t) + b_P \sin(\vec{k} \cdot \vec{r} - \omega t) \right) \\
 \frac{db_\pm}{dt} &\equiv \frac{db \pm b^\dagger}{dt} \\
 &= 2\Lambda b^\dagger \pm 2\Lambda b \\
 &= \pm 2\Lambda b_\pm \\
 \frac{db_P}{dt} &= 2\Lambda b_P \\
 \frac{db_Q}{dt} &= -2\Lambda b_Q
 \end{aligned}$$

Therefore

$$\begin{aligned}
 b_P &= e^{2\Lambda t} b_{P0} \\
 b_Q &= e^{-2\Lambda t} b_{Q0} \\
 b &= b_P + ib_Q \\
 &= e^{2\Lambda t} b_{P0} + ie^{-2\Lambda t} b_{Q0} \\
 &= b_0 \cosh 2\Lambda t + b_0^\dagger \sinh 2\Lambda t \\
 b^\dagger &= b_0^\dagger \cosh 2\Lambda t + b_0 \sinh 2\Lambda t
 \end{aligned}$$

(c)

$$\begin{aligned}
 \langle N \rangle &= \langle 0 | b^\dagger b | 0 \rangle \\
 &= \langle 0 | \left(b_0^\dagger \cosh 2\Lambda t + b_0 \sinh 2\Lambda t \right) \left(b_0 \cosh 2\Lambda t + b_0^\dagger \sinh 2\Lambda t \right) | 0 \rangle \\
 &= \sinh^2 2\Lambda t \\
 \Delta b_P &= e^{2\Lambda t} \Delta b_{P0} \\
 &= \frac{1}{2} e^{2\Lambda t} \\
 \Delta b_Q &= e^{-2\Lambda t} \Delta b_{Q0} \\
 &= \frac{1}{2} e^{-2\Lambda t}
 \end{aligned}$$

The state is squeezed in Q direction while the product of the uncertainty in P and Q remains the same.

(d)

Under the transformation $U = e^{i\omega t a^\dagger a}$

$$\begin{aligned}
 U a U^\dagger &= e^{i\omega t a^\dagger a} a e^{-i\omega t a^\dagger a} \\
 &= \sum_N \frac{(i\omega t)^N}{N!} [a^\dagger a, a]_N \\
 &= e^{-i\omega t} a \\
 U a^\dagger U^\dagger &= e^{i\omega t} a^\dagger \\
 \frac{d}{dt} |\psi'\rangle &= \frac{d}{dt} U |\psi\rangle \\
 &= \frac{dU}{dt} |\psi\rangle + U \frac{d}{dt} |\psi\rangle \\
 &= \frac{d e^{i\omega t a^\dagger a}}{dt} |\psi\rangle + \frac{U}{i\hbar} H |\psi\rangle \\
 &= i\omega a^\dagger a |\psi'\rangle + \frac{1}{i\hbar} U H U^\dagger |\psi'\rangle \\
 &= \Lambda U \left(a^{\dagger 2} e^{-2i\omega t} - a^2 e^{2i\omega t} \right) U^\dagger |\psi'\rangle \\
 &= \Lambda \left(a^{\dagger 2} - a^2 \right) |\psi'\rangle
 \end{aligned}$$

Therefore, the state $|\psi'\rangle$ is transforming as

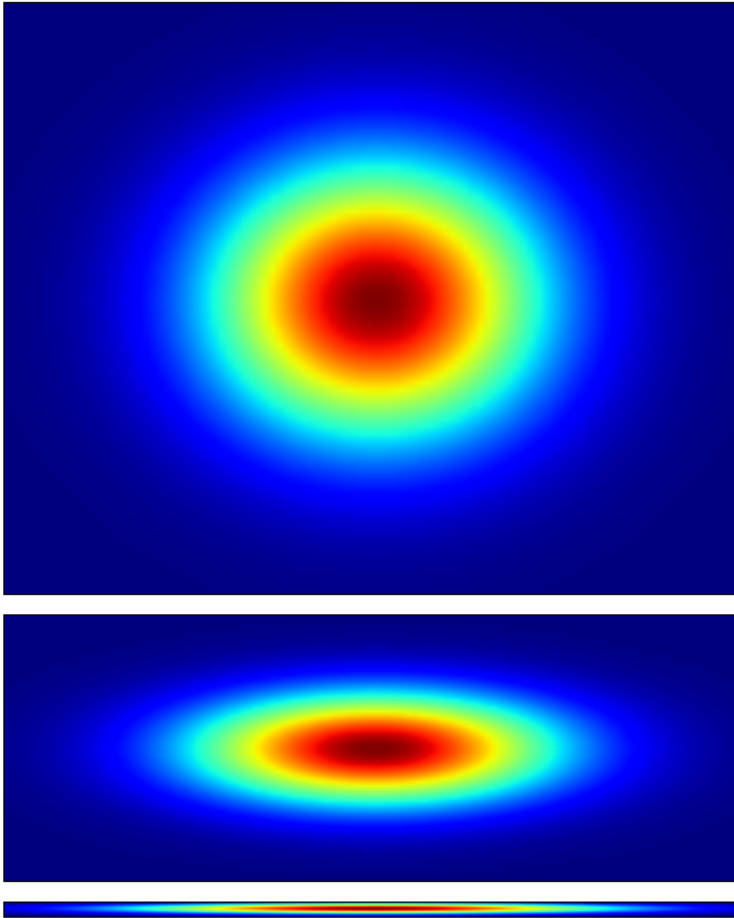
$$e^{\Lambda t (a^{\dagger 2} - a^2)}$$

where

$$\varepsilon = -2\Lambda t$$

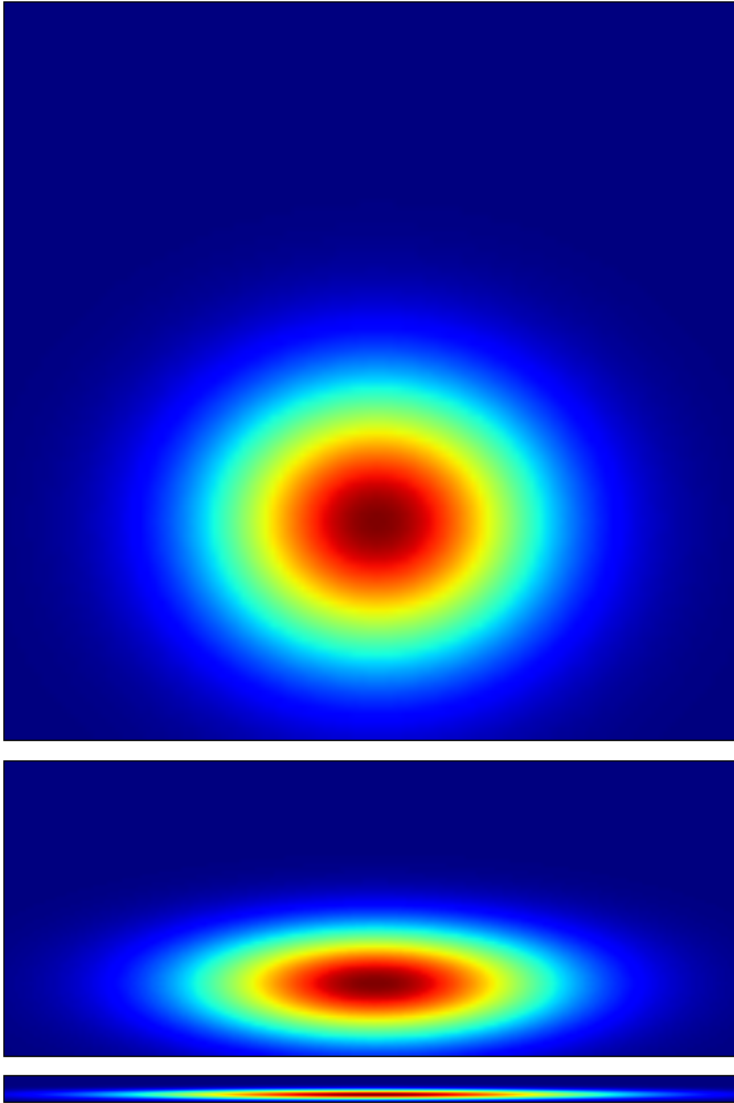
(e)

i.



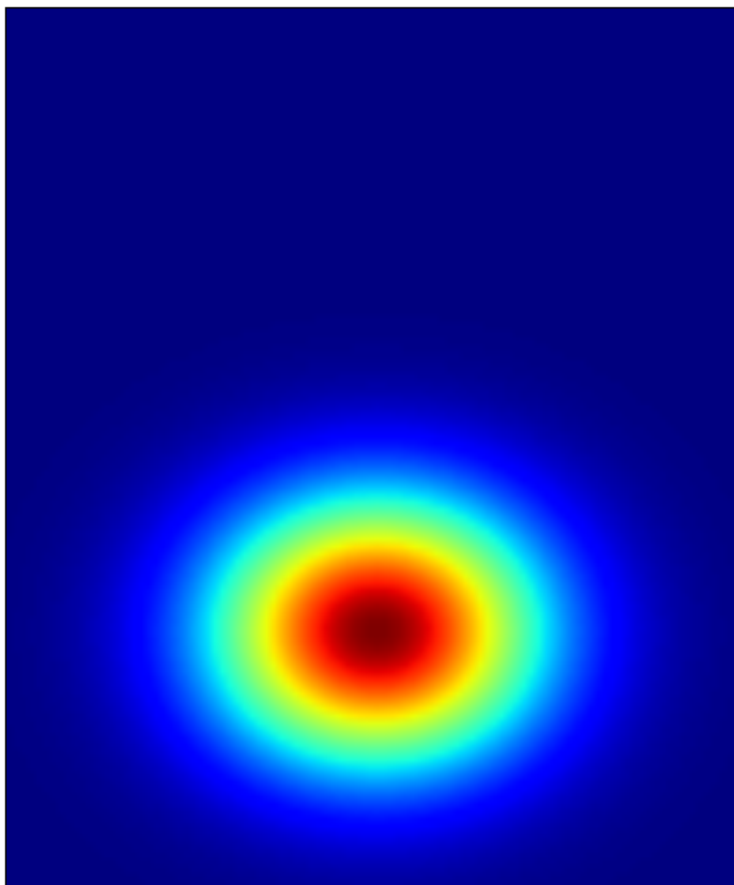
ii.

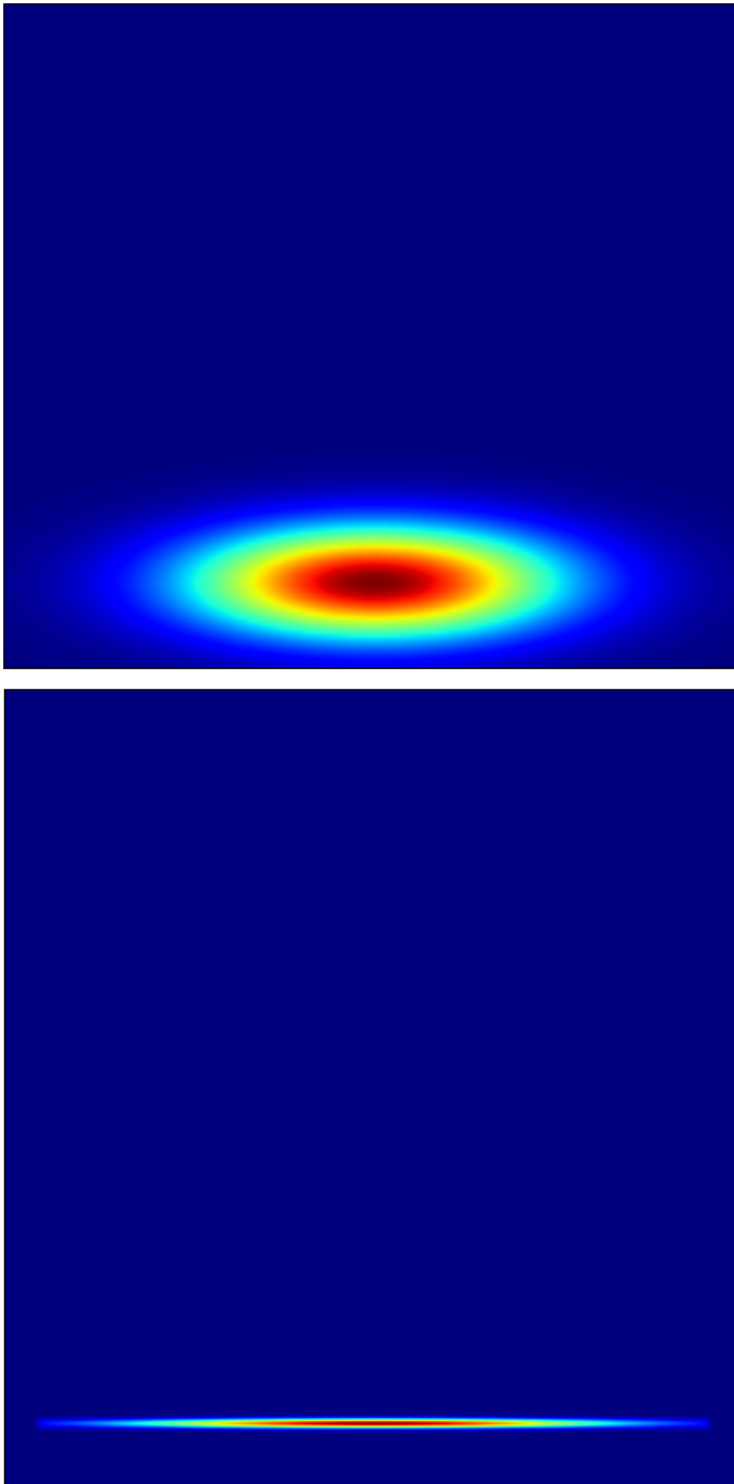
$$\beta = i$$



iii.

$$\beta = i$$





Whether a phase or number squeezed state is created depends on the initial state.

2.

(a)

$$\begin{aligned}
 \langle N^2 \rangle &= \langle 0 | b^\dagger b b^\dagger b | 0 \rangle \\
 &= \langle 0 | \left(b_0^\dagger \cosh \varepsilon + b_0 \sinh \varepsilon \right) \left(b_0 \cosh \varepsilon + b_0^\dagger \sinh \varepsilon \right) \\
 &\quad \left(b_0^\dagger \cosh \varepsilon + b_0 \sinh \varepsilon \right) \left(b_0 \cosh \varepsilon + b_0^\dagger \sinh \varepsilon \right) | 0 \rangle \\
 &= \sinh^2 \varepsilon \langle 0 | \left(b_0^2 \cosh \varepsilon + b_0 b_0^\dagger \sinh \varepsilon \right) \left(b_0^{\dagger 2} \cosh \varepsilon + b_0 b_0^\dagger \sinh \varepsilon \right) | 0 \rangle \\
 &= \sinh^2 \varepsilon (\sinh^2 \varepsilon + 2 \cosh^2 \varepsilon) \\
 \Delta N^2 &= 2 \sinh^2 \varepsilon \cosh^2 \varepsilon \\
 &= 2 (\Delta b_P^2 - \Delta b_Q^2)^2
 \end{aligned}$$

For large ε

$$\begin{aligned}
 N &\approx e^{2\varepsilon} \\
 \Delta N &\approx \sqrt{2} e^{2\varepsilon} \\
 &= \sqrt{2} N
 \end{aligned}$$

The fluctuation is larger than classical state which has $\Delta N \propto \sqrt{N}$

(b)

$$\begin{aligned}
 &10 \log_{10} (4 \Delta a_P^2) \\
 &= 20 \varepsilon \log_{10} (e)
 \end{aligned}$$

which scales linearly with ε

$$\begin{aligned}
 \Delta a_P'^2 &= \langle a_P'^2 \rangle - \langle a_P' \rangle^2 \\
 &= \langle (t a_P + r a_{P0})^2 \rangle - \langle (t a_P + r a_{P0}) \rangle^2 \\
 &= \langle t^2 a_P^2 + r^2 a_{P0}^2 \rangle - \langle t a_P \rangle^2 \\
 &= t^2 \Delta a_P^2 + r^2 \Delta a_{P0}^2
 \end{aligned}$$

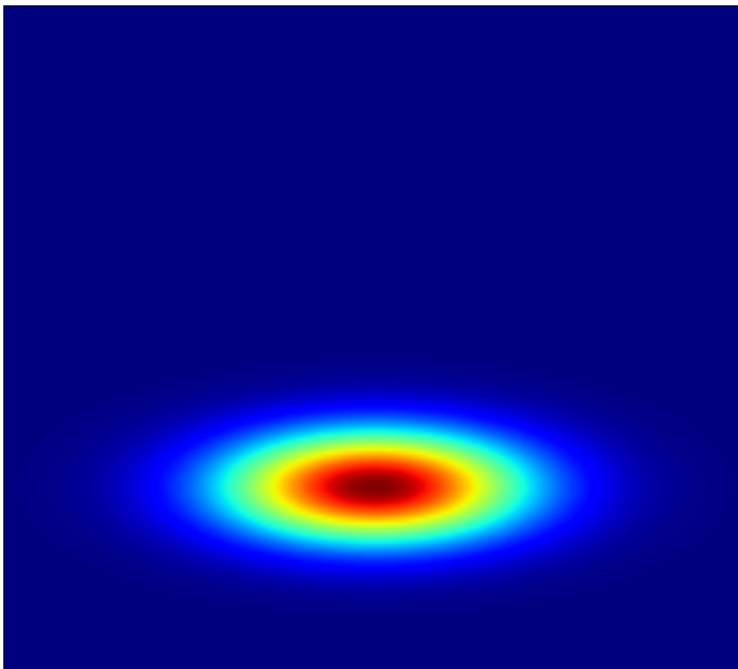
where a_P is the operator for the squeezed input state and a_{P0} is the operator for the vacuum input. In order to decrease it by 3dB

$$\begin{aligned}
 \Delta a_P'^2 &= \frac{1}{2} \Delta a_P^2 \\
 \Delta a_P^2 &= 2 \left(T \Delta a_P^2 + \frac{1}{4} (1 - T) \right) \\
 T &= \frac{2 \Delta a_P^2 - 1}{4 \Delta a_P^2 - 1}
 \end{aligned}$$

At the limit of strong squeezing, $T \rightarrow \frac{1}{2}$

(c)

i.



ii.

$$\begin{aligned}\langle N' \rangle &= \langle (ta^\dagger + rb^\dagger)(ta + rb) \rangle \\ &= \langle t^2 n_a + r^2 n_b \rangle \\ &= TN_a + (1 - T)N_b\end{aligned}$$

iii.

$$\begin{aligned}\Delta n'^2 &= \langle n'^2 \rangle - \langle n' \rangle^2 \\ &= \langle ((ta^\dagger + rb^\dagger)(ta + rb))^2 \rangle - \langle (ta^\dagger + rb^\dagger)(ta + rb) \rangle^2 \\ &= \langle (t^2 n_a + r^2 n_b + tr(a^\dagger b + b^\dagger a))^2 \rangle - (TN_a + RN_b)^2 \\ &= T^2 \Delta N_a^2 + R^2 \Delta N_b^2 + TR \langle (a^\dagger b + b^\dagger a)^2 \rangle\end{aligned}$$

For large N_a and if the displacement is along the lower variant direction

$$\Delta n'^2 \approx T^2 \Delta N_a^2 + R^2 \Delta N_b^2$$

iv.

$$\Delta n'^2 \approx 2T^2 N_a^2 + R^2 N_b$$

In order to be smaller than $\langle N' \rangle$

$$N_b > \frac{(2TN_a - 1)N_a}{R}$$

And the squeezing should be strong enough that $2TN_a > 1$

3.

(a)

$$\begin{aligned} a_{out} &= \frac{a_2 - b_2}{\sqrt{2}} \\ &= \frac{a_1 e^{i\varphi} - b_1}{\sqrt{2}} \\ &= \frac{1}{2} ((a - b)e^{i\varphi} - a - b) \\ &= e^{i\varphi/2} \left(ia \sin \frac{\varphi}{2} - b \cos \frac{\varphi}{2} \right) \\ a_{out}^\dagger &= e^{-i\varphi/2} \left(-ia^\dagger \sin \frac{\varphi}{2} - b^\dagger \cos \frac{\varphi}{2} \right) \end{aligned}$$

$$\begin{aligned} b_{out} &= \frac{a_2 + b_2}{\sqrt{2}} \\ &= \frac{a_1 e^{i\varphi} + b_1}{\sqrt{2}} \\ &= \frac{1}{2} ((a - b)e^{i\varphi} + a + b) \\ &= e^{i\varphi/2} \left(a \cos \frac{\varphi}{2} - ib \sin \frac{\varphi}{2} \right) \\ b_{out}^\dagger &= e^{-i\varphi/2} \left(a^\dagger \cos \frac{\varphi}{2} + ib^\dagger \sin \frac{\varphi}{2} \right) \end{aligned}$$

$$\begin{aligned} n_{out_a} &= \left(-ia^\dagger \sin \frac{\varphi}{2} - b^\dagger \cos \frac{\varphi}{2} \right) \left(ia \sin \frac{\varphi}{2} - b \cos \frac{\varphi}{2} \right) \\ &= n_a \sin^2 \frac{\varphi}{2} + n_b \cos^2 \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} (a^\dagger b - ab^\dagger) \end{aligned}$$

$$\begin{aligned} n_{out_b} &= \left(a^\dagger \cos \frac{\varphi}{2} + ib^\dagger \sin \frac{\varphi}{2} \right) \left(a \cos \frac{\varphi}{2} - ib \sin \frac{\varphi}{2} \right) \\ &= n_a \cos^2 \frac{\varphi}{2} + n_b \sin^2 \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} (ab^\dagger - a^\dagger b) \end{aligned}$$

$$\begin{aligned} & b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \\ &= n_a \cos \varphi - n_b \cos \varphi + i \sin \varphi (ab^\dagger - a^\dagger b) \end{aligned}$$

Average

$$\begin{aligned} & \langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \rangle \\ &= \langle n_a \cos \varphi \rangle \\ &= |\alpha|^2 \cos \varphi \end{aligned}$$

Square

$$\begin{aligned} & \langle (b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out})^2 \rangle \\ &= \langle (n_a \cos \varphi - n_b \cos \varphi + i \sin \varphi (ab^\dagger - a^\dagger b)) (n_a \cos \varphi - n_b \cos \varphi + i \sin \varphi (ab^\dagger - a^\dagger b)) \rangle \\ &= \langle (n_a \cos \varphi - i \sin \varphi \alpha^* b) (n_a \cos \varphi + i \sin \varphi \alpha b^\dagger) \rangle \\ &= \cos^2 \varphi \langle n_a^2 \rangle + |\alpha|^2 \sin^2 \varphi \end{aligned}$$

Fluctuation

$$\begin{aligned} & \Delta(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out})^2 \\ &= \cos^2 \varphi \Delta n_a^2 + |\alpha|^2 \sin^2 \varphi \\ &= |\alpha|^2 \\ SNR &= \frac{|\alpha|^2 \cos \varphi}{|\alpha|} \\ &= |\alpha| \cos \varphi \end{aligned}$$

(b)

$$\varphi' = \frac{\pi}{2} - \varphi$$

$$SNR \approx |\alpha| \varphi'$$

$$\varphi'_{min} = \frac{1}{|\alpha|}$$

(c)

Average

$$\begin{aligned}
& b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \\
&= n_a \cos \varphi - (t^\dagger \cosh \varepsilon - t \sinh \varepsilon)(t \cosh \varepsilon - t^\dagger \sinh \varepsilon) \cos \varphi \\
&\quad + i \sin \varphi (a(t^\dagger \cosh \varepsilon - t \sinh \varepsilon) - a^\dagger(t \cosh \varepsilon - t^\dagger \sinh \varepsilon)) \\
&= n_a \cos \varphi - \cos \varphi (\cosh 2\varepsilon n_t + \sinh^2 \varepsilon - \sinh \varepsilon \cosh \varepsilon (t^{\dagger 2} + t^2)) \\
&\quad + i \sin \varphi (a(t^\dagger \cosh \varepsilon - t \sinh \varepsilon) - a^\dagger(t \cosh \varepsilon - t^\dagger \sinh \varepsilon)) \\
&\quad \langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \rangle \\
&= \langle n_a \cos \varphi - \cos \varphi (\cosh 2\varepsilon n_t + \sinh^2 \varepsilon - \sinh \varepsilon \cosh \varepsilon (t^{\dagger 2} + t^2)) \\
&\quad + i \sin \varphi (a(t^\dagger \cosh \varepsilon - t \sinh \varepsilon) - a^\dagger(t \cosh \varepsilon - t^\dagger \sinh \varepsilon)) \rangle \\
&= |\alpha|^2 \cos \varphi - \cos \varphi \sinh^2 \varepsilon
\end{aligned}$$

Square

$$\begin{aligned}
& \langle (b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out})^2 \rangle \\
&= \langle (n_a \cos \varphi - \cos \varphi (\cosh 2\varepsilon n_t + \sinh^2 \varepsilon - \sinh \varepsilon \cosh \varepsilon (t^{\dagger 2} + t^2)) \\
&\quad + i \sin \varphi (a(t^\dagger \cosh \varepsilon - t \sinh \varepsilon) - a^\dagger(t \cosh \varepsilon - t^\dagger \sinh \varepsilon))) \\
&\quad (n_a \cos \varphi - \cos \varphi (\cosh 2\varepsilon n_t + \sinh^2 \varepsilon - \sinh \varepsilon \cosh \varepsilon (t^{\dagger 2} + t^2)) \\
&\quad + i \sin \varphi (a(t^\dagger \cosh \varepsilon - t \sinh \varepsilon) - a^\dagger(t \cosh \varepsilon - t^\dagger \sinh \varepsilon))) \rangle \\
&= \langle (n_a \cos \varphi - \cos \varphi (\sinh^2 \varepsilon - \sinh \varepsilon \cosh \varepsilon t^2) - i t \sin \varphi (a \sinh \varepsilon + a^\dagger \cosh \varepsilon)) \\
&\quad (n_a \cos \varphi - \cos \varphi (\sinh^2 \varepsilon - \sinh \varepsilon \cosh \varepsilon t^{\dagger 2}) + i \sin \varphi t^\dagger (a \cosh \varepsilon + a^\dagger \sinh \varepsilon)) \rangle \\
&= \langle (n_a \cos \varphi - \cos \varphi \sinh^2 \varepsilon + \cos \varphi \sinh \varepsilon \cosh \varepsilon t^2 - i t \sin \varphi (a \sinh \varepsilon + \alpha^* \cosh \varepsilon)) \\
&\quad (n_a \cos \varphi - \cos \varphi \sinh^2 \varepsilon + \cos \varphi \sinh \varepsilon \cosh \varepsilon t^{\dagger 2} + i t^\dagger \sin \varphi (\alpha \cosh \varepsilon + a^\dagger \sinh \varepsilon)) \rangle \\
&= \langle (n_a \cos \varphi - \cos \varphi \sinh^2 \varepsilon)^2 \rangle + \langle \cos^2 \varphi \sinh^2 \varepsilon \cosh^2 \varepsilon t t^\dagger t t^\dagger \rangle \\
&\quad + \langle (\cos \varphi \sinh \varepsilon \cosh \varepsilon t - i \sin \varphi (a \sinh \varepsilon + \alpha^* \cosh \varepsilon)) \\
&\quad (\cos \varphi \sinh \varepsilon \cosh \varepsilon t^\dagger + i \sin \varphi (\alpha \cosh \varepsilon + a^\dagger \sinh \varepsilon)) \rangle \\
&= \langle (n_a \cos \varphi - \cos \varphi \sinh^2 \varepsilon)^2 \rangle + 2 \cos^2 \varphi \sinh^2 \varepsilon \cosh^2 \varepsilon \\
&\quad + \sin^2 \varphi |\alpha \sinh \varepsilon + \alpha^* \cosh \varepsilon|^2 + \sin^2 \varphi \sinh^2 \varepsilon
\end{aligned}$$

Fluctuation

$$\begin{aligned}
& \Delta (b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out})^2 \\
&= |\alpha|^2 \cos^2 \varphi + 2 \cos^2 \varphi \sinh^2 \varepsilon \cosh^2 \varepsilon + \sin^2 \varphi |\alpha \sinh \varepsilon + \alpha^* \cosh \varepsilon|^2 + \sin^2 \varphi \sinh^2 \varepsilon \\
&\approx |\alpha|^2 \varphi'^2 + 2 \varphi'^2 \sinh^2 \varepsilon \cosh^2 \varepsilon + (1 - \varphi'^2) |\alpha \sinh \varepsilon + \alpha^* \cosh \varepsilon|^2 + (1 - \varphi'^2) \sinh^2 \varepsilon
\end{aligned}$$

Minimum angle

$$\varphi'^2 (|\alpha|^2 - \sinh^2 \varepsilon)^2 = |\alpha|^2 \varphi'^2 + 2 \varphi'^2 \sinh^2 \varepsilon \cosh^2 \varepsilon + (1 - \varphi'^2) |\alpha \sinh \varepsilon + \alpha^* \cosh \varepsilon|^2 + (1 - \varphi'^2) \sinh^2 \varepsilon$$

In high power limit

$$\varphi'^2 |\alpha|^4 = |\alpha \sinh \varepsilon + \alpha^* \cosh \varepsilon|^2$$

To get a factor of β in phase resolution

$$\frac{\alpha^2}{\beta^2} = |\alpha \sinh \varepsilon + \alpha^* \cosh \varepsilon|^2$$

With correct phase of α

$$\begin{aligned}\beta &= e^\varepsilon \\ \varepsilon &= \ln \beta\end{aligned}$$

Squeezing changes the photon number because squeezed vacuum has non-zero photon number.

(d)

For coherent state

$$\begin{aligned}\alpha &= \sqrt{\frac{PL\lambda}{\pi\hbar c^2}} \\ &= 2.7 \cdot 10^7 \\ \varphi_{min} &= 3.7 \cdot 10^{-8} \\ l_{min} &= 6.3 \cdot 10^{-15} m \\ \varepsilon_{min} &= 1.6 \cdot 10^{-18}\end{aligned}$$

With a 6dB squeezed vacuum, the sensitivity can go up by a factor of 4