AM205: Assignment 2 (due 5 PM, October 9)

Program files: A number of program and data files for this homework can be downloaded as a single ZIP file from the course website.

1. **Norms and Newton root-finding.** Define a matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, \tag{1}$$

which represents a linear transformation in $\mathbb{R}^2$.

   (a) Find four points $b \in \mathbb{R}^2$ such that $||b||_2 = 1$ and $||Ab||_2 = 1$. Plot the two curves $||x||_2 = 1$ and $||Ax||_2 = 1$ and mark the points b on this plot.

   (b) Find four points $c \in \mathbb{R}^2$ such that $||c||_\infty = 1$ and $||Ac||_\infty = 1$. Plot the two curves $||x||_\infty = 1$ and $||Ax||_\infty = 1$ and mark the points c on this plot.

   (c) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(d) = (||d||_4 - 1, ||Ad||_4 - 1), \tag{2}$$

   where $d \in \mathbb{R}^2$. Write a program to find four solutions to the equation $f(d) = (0, 0)$ using the vector generalization of the Newton root finding method.[1] Plot the two curves $||x||_4 = 1$ and $||Ax||_4 = 1$ and mark the solutions d on this plot.

   (d) Show that the families of points b, c, and d (12 points in total) lie on two straight lines, and explain why this is true.

2. **LU factorization for binary numbers.** In this course we usually calculate using the set of real numbers $\mathbb{R}$. This question takes a different approach of calculating using the binary set $\mathbb{B} = \{0, 1\}$ with just two elements. Within $\mathbb{B}$, addition is defined as

$$0 + 0 = 0, \qquad 0 + 1 = 1, \qquad 1 + 1 = 0,$$

corresponding to regular addition, and then taking the remainder after division by two. Multiplication is defined as

$$0 \times 0 = 0, \qquad 0 \times 1 = 0, \qquad 1 \times 1 = 1.$$

Subtraction is the same as addition, so that $x - y = x + y$ for all $x, y \in \mathbb{B}$. For division, $x/1 = x$ for all $x \in \mathbb{B}$, and $x/0$ is undefined, giving a "division by zero" error. With these rules $\mathbb{B}$ becomes a *field*,[2] in that addition and muliplication have all the usual properties. In most cases, linear algebra calculations that work for numbers in $\mathbb{R}$ will work just as well for numbers in $\mathbb{B}$, as long as the arithmetic operations are interpreted as defined above.

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[1] You can make use of linear algebra routines in your code, but the actual Newton iteration should be coded yourself, without using a library function.

[2] More specifically, $\mathbb{B}$ is the field of integers modulo p for $p = 2$, which is frequently studied in number theory courses.

There are various ways to write programs that calculate in $\mathbb{B}$. In the homework files, there are programs `bin_mul.py` and `bin_mul.m` that demonstrate this in Python and MATLAB, respectively. They both implement matrix multiplication in $\mathbb{B}$, using the test matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

The two programs calculate LU and show that it is equal to A. We now consider adapting the algorithms presented in class to carry out the LU factorization to solve linear systems in $\mathbb{B}$.

(a) Write a function `fsolve` that takes a lower triangular matrix L and vector b and returns the solution x to the linear system $Lx = b$ using forward substitution described in the lectures. If any diagonal element of L is zero, the function should give an error and report that the matrix is singular.

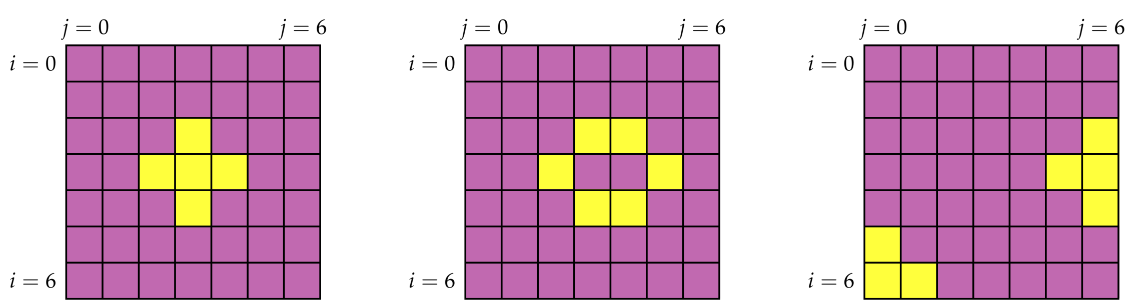
(b) Write a function `rsolve` that takes an upper triangular matrix U and vector b and returns the solution x the the linear system $Ux = b$ using reverse substitution described in the lectures. If any diagonal element of U is zero, the function should give an error and report that the matrix is singular.

(c) Write a program to calculate the LU factorization with partial pivoting as described in the lecture 7 slides. The program should return P, L, and U so that $PA = LU$, and it should also work on singular matrices.

(d) In the homework files, there are two directories called `q2_small` and `q2_large`. Each has a text file containing a binary matrix A and a text file containing source data b. Find the solutions x to both linear systems $Ax = b$.

(e) **Optional.** What is the probability that a random $n \times n$ binary matrix will be singular?

3. **The light game.** An electronic children's toy consists of a 7×7 grid of lights, which are initially all switched off. Pressing on a light toggles it on or off, and toggles its orthogonally adjacent neighbors on or off. A single press in the interior of the grid therefore creates set of lights in the shape of a plus sign, while several presses may lead to more complicated patterns. Three examples of the lights after different presses (i, j) are shown below.



Press $(3, 3)$ Press $(3, 3)$ and $(3, 4)$ Press $(6, 0)$ and $(3, 6)$

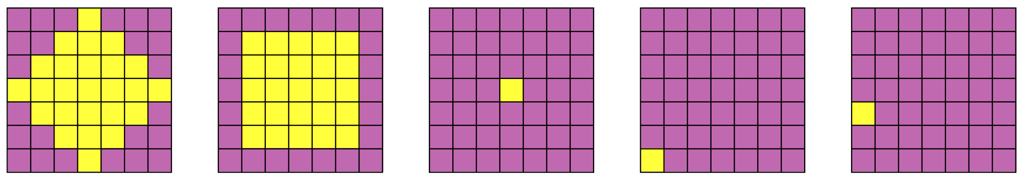
(a) Let x be a vector in $\mathbb{B}^{49}$ that represents which lights have been pressed, and let b be a vector in $\mathbb{B}^{49}$ that represents which lights are lit. Write a program that creates a 49×49

binary matrix A such that

$$Ax = b \tag{3}$$

in the binary arithmetic scheme introduced in Question 2.

(b) The toy presents different patterns of lights and the aim is to determine the correct presses to switch off all of the lights. For each of the patterns given below, use your binary LU solver from Question 2 to determine the correct presses.



For each case, present your results as in a 7×7 grid, such as by using the `spy` command in NumPy and MATLAB. In addition, create a light pattern of your own[3] and solve it.

(c) For the 7×7 grid the matrix A in Eq. 3 is non-singular, so that every combination of lights can be created with presses. However, this is not always the case for a general $m \times n$ grid. For each $m \times n$ grid with $m, n \in \{1, 2, \ldots, 9\}$ determine the dimension of the null space, $f(m, n) = mn - \operatorname{rank} A$.[4]

(d) **Optional.** For the 5×5 grid, find two linearly independent press patterns that leave all of the lights switched off. For the 4×4 grid, find four linearly independent press patterns that leave all of the lights switched off.

(e) **Optional.** Suppose that you play the game on a polyhedron, so that pressing on a face lights up that face, as well as the neighboring faces that share an edge. For the five platonic solids, what is the dimension of the null space of A?

4. **QR factorization using Givens rotations.** By following the steps described in the lecture, write a program that performs the QR factorization using Givens rotations for an arbitrary rectangular $m \times n$ matrix where $m \geq n$.

   To test your program, perform ten trials where a random[5] matrix A of size 11×7 is factored into matrices Q and R. For each of the ten trials, calculate the Frobenius norm $||A - QR||_F$ and check that these are small.

5. **Analysis of a bouncing ball.** In the program files, there is a set of twenty images of a Wham-O Super Ball undergoing two bounces on a table top. Each frame is $\frac{7}{120}$ s apart. The Super Ball is 42.5 mm in diameter. The Super Ball is approximately 43.5 pixels in diameter in the images.

   (a) A text file `super_ypos.txt` is provided that lists the y positions of the ball center (measured in pixels) for each frame f, which were calculated from the images. During this test, the ball follows three parabolic arcs that are separated from each other by the two

[3] If you wish to get creative, your pattern can be on a different-sized grid than 7×7. Any particularly awesome examples will be mentioned in class.

[4] Note that $f(m, n)$ is equal to the number of zero diagonal entries in U, in the LU factorization of A.

[5] Any procedure for constructing random matrices is permissible here. In Python the function `numpy.random.rand` could be used. In MATLAB the function `rand` could be used.

bounces. Using your QR factorization code from Question 4, fit each of the three different arcs to

$$y(f) = \alpha f^2 + \beta f + \gamma. \tag{4}$$

(b) Using your fitted curves from part (a), calculate the following in physical units:

i. the gravitational acceleration g,

ii. the height h above the table top from which the ball is released,[6]

iii. the coefficient of restitution e.

For i and iii, the data provides multiple independent measurements—for example, each bounce provides a seperate measurement of the coefficient of restitution. In these cases you should report the average of the measurements you can make.

(c) Using your measurements from part (b), calculate the time in seconds that a ball released from height h will take before it comes to rest.

6. **Difficult cases for LU factorization.** Matrices of the form

$$G_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 1 \\ -1 & 1 & 0 & 0 & \dots & 1 \\ -1 & -1 & 1 & 0 & \dots & 1 \\ -1 & -1 & -1 & 1 & \dots & 1 \\ \vdots & & & & \ddots & \vdots \\ -1 & -1 & -1 & -1 & \dots & 1 \end{bmatrix} \tag{5}$$

in $\mathbb{R}^{n\times n}$ are examples of the very rare cases in which Gaussian elimination is unstable.

(a) Write a function `generate_g` that returns G_n.

(b) Write a program that measures the time $t(n)$ taken to run `generate_g` as a function of n, for $n = 10, 20, \dots, 1000$. Make a plot of $t(n)$ as a function of n.[7] You should find that $t(n) \sim \alpha n^\beta$. Determine α and β and discuss whether the value of β is reasonable, given the number of operations that `generate_g` does.

(c) For $n = 10, 20, \dots, 200$, set $x = [1, 1, \dots, 1]^\mathsf{T} \in \mathbb{R}^n$ and construct a right-hand side vector $b = G_n x$. Solve the system $G_n \hat{x} = b$ using the LU factorization.[8] Plot the 2-norm relative error as a function of n and explain why we consider Gaussian elimination with partial pivoting to be numerically unstable in this case.

(d) **Optional.** Make a plot that shows that the inequality

$$\frac{\|x - \hat{x}\|_2}{\|\hat{x}\|_2} \leq \kappa(G_n) \frac{\|r(\hat{x})\|_2}{\|G_n\|_2 \|\hat{x}\|_2} \tag{6}$$

is satisfied, where $\kappa(G_n)$ is the condition number of G_n with respect to the Euclidean 2-norm, and $r(\hat{x}) = b - G_n\hat{x}$ is the residual.

[6] Specifically, this should be the distance from the table top to the bottom of the ball.

[7] For examples of how to time functions, see the `lu_time.py` and `chol_time.py` examples from lecture 8.

[8] In NumPy the routine `numpy.linalg.solve` uses the LU factorization. In MATLAB, the "backslash" operator uses the LU factorization.