1.

Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q\vec{A}}{c}\right)^2$$

Interaction term

$$\begin{aligned} V =& \frac{q}{2mc}\left(-\vec{p}\cdot\vec{A} - \vec{A}\cdot\vec{p} + \frac{qA^2}{c}\right) \\ =& \frac{q}{2mc}\left(\mathrm{i}\hbar\nabla\cdot\vec{A} - 2\vec{A}\cdot\vec{p} + \frac{qA^2}{c}\right) \end{aligned}$$

Scattering wavefunction given by Born approximation

$$\begin{aligned} \langle x|\psi^{(+)}\rangle =& \langle x|\vec{k}\rangle - \frac{2m}{\hbar^2}\int \mathrm{d}^3x' \frac{\mathrm{e}^{\mathrm{i}\vec{k}\left(\vec{x}-\vec{x}'\right)}}{4\pi|\vec{x}-\vec{x}'|}\langle x'|V|\vec{k}\rangle \\ =& \langle x|\vec{k}\rangle - \frac{q}{\hbar^2 c}\int \mathrm{d}^3x' \frac{\mathrm{e}^{\mathrm{i}\vec{k}\left(\vec{x}-\vec{x}'\right)}}{4\pi|\vec{x}-\vec{x}'|}\langle x'|\left(\mathrm{i}\hbar\nabla\cdot\vec{A} - 2\vec{A}\cdot\vec{p} + \frac{qA^2}{c}\right)|\vec{k}\rangle \\ =& \langle x|\vec{k}\rangle - \frac{q}{\hbar^2 c}\int \mathrm{d}^3x' \frac{\mathrm{e}^{\mathrm{i}\vec{k}\left(\vec{x}-\vec{x}'\right)}}{4\pi|\vec{x}-\vec{x}'|}\langle x'|\left(\mathrm{i}\hbar\nabla\cdot\vec{A}' - 2\hbar\vec{A}'\cdot\vec{k} + \frac{qA'^2}{c}\right)|\vec{k}\rangle \\ \approx& \langle x|\vec{k}\rangle - \frac{q}{\hbar c}\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\int \mathrm{d}^3x' \mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{x}'}\left(\mathrm{i}\nabla\cdot\vec{A}' - 2\vec{A}'\cdot\vec{k} + \frac{qA'^2}{\hbar c}\right)\langle x'|\vec{k}\rangle \\ =& \langle x|\vec{k}\rangle - \frac{q}{\left(2\pi\right)^{3/2}\hbar c}\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\int \mathrm{d}^3x' \mathrm{e}^{\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{x}'}\left(\mathrm{i}\nabla\cdot\vec{A}' - 2\vec{A}'\cdot\vec{k} + \frac{qA'^2}{\hbar c}\right) \end{aligned}$$

2.

3.

4.

(a)

Since there's only two plain wave solutions.

$$\psi^{(+)} = \frac{1}{\sqrt{2\pi}}\left(\mathrm{e}^{\mathrm{i}kx} + f(sign(x))\mathrm{e}^{\mathrm{i}k|x|}\right)$$

(b)

From the conservation of probability density, the in-coming and out-going wave must have the same amplitude. Therefore, there can only be a phase shift between them.
Using the boundary condition, the wavefunction for $0 < x < a$ is,

$$\psi = A\left(\mathrm{e}^{\mathrm{i}k'x} - \mathrm{e}^{-\mathrm{i}k'x}\right)$$

where $k' = \sqrt{k^2 + 2mV_0}$ and for $x \geqslant a$ is

$$\psi = \mathrm{e}^{\mathrm{i}(kx+\delta)} - \mathrm{e}^{-\mathrm{i}(kx+\delta)}$$

To satisfy the continuity condition at $x = a$

$$\begin{aligned}
A\mathrm{e}^{\mathrm{i}k'a} - A\mathrm{e}^{-\mathrm{i}k'a} =& \mathrm{e}^{\mathrm{i}(ka+\delta)} - \mathrm{e}^{-\mathrm{i}(ka+\delta)}\\
Ak'\mathrm{e}^{\mathrm{i}k'a} + Ak'\mathrm{e}^{-\mathrm{i}k'a} =& k\mathrm{e}^{\mathrm{i}(ka+\delta)} + k\mathrm{e}^{-\mathrm{i}(ka+\delta)}\\
A\sin k'a =& \sin(ka+\delta)\\
Ak'\cos k'a =& k\cos(ka+\delta)\\
\delta =& \arctan\left(\frac{k}{k'}\tan k'a\right) - ka
\end{aligned}$$

5.

6.