

**1.**

**(a)**

Represent the operation using  $4 \times 4$  matrices that shows the mapping between the nodes.

$$\begin{aligned}
 T_1 &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} & T_2 &= \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} & T_3 &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\
 T_4 &= \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} & T_5 &= \begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & 1 \end{pmatrix} & T_6 &= \begin{pmatrix} & 1 & & \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \\
 T_7 &= \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} & T_8 &= \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} & T_9 &= \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \\
 T_{10} &= \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} & T_{11} &= \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} & T_{12} &= \begin{pmatrix} & & & 1 \\ & 1 & & \\ 1 & & & \\ & & 1 & \end{pmatrix}
 \end{aligned}$$

**(b)**

$$\begin{aligned}
 g_{123} &= \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} \\
 g_{234} &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\
 g_{234}g_{123} &= \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = T_4
 \end{aligned}$$

( $180^\circ$  rotation around the the axis connecting the middle of 1-2 and 3-4)

$$g_{123}g_{234} = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} = T_9 \neq T_4$$

**(c)**

See (a)

(d)

$$H = \begin{pmatrix} \varepsilon_0 & -t & -t & -t \\ -t & \varepsilon_0 & -t & -t \\ -t & -t & \varepsilon_0 & -t \\ -t & -t & -t & \varepsilon_0 \end{pmatrix}$$

Eigenvalues are  $\varepsilon_0 - 3t$  for eigenvector  $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$  and  $\varepsilon_0 + t$  for eigenvectors,  $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ ,  $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ ,  $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$ .

**2.**

(a)

$$H = \begin{pmatrix} \varepsilon_0 & -t & & & -t & & -t \\ -t & \varepsilon_0 & -t & & -t & & \\ & -t & \varepsilon_0 & -t & & & -t \\ & & -t & \varepsilon_0 & -t & & \\ & -t & & -t & \varepsilon_0 & -t & \\ -t & & & -t & -t & \varepsilon_0 & -t \\ -t & & -t & & -t & -t & \varepsilon_0 \end{pmatrix}$$

(b)

(c)

**3.**

The constrains is that the angular momentum cannot perfectly point in a certain direction and there will always be some fluctuations. This uncertain comes from,

$$\begin{aligned} \langle \Delta L_x, \Delta L_y \rangle &\geq \frac{1}{2i} \langle [L_x, L_y] \rangle \\ &= \frac{\hbar}{2} \langle L_z \rangle \\ &= \frac{\hbar^2 m}{2} \end{aligned}$$

Which can only be 0 when  $m = 0$ .

4.

(a)

Define  $g(\phi, \phi_0) \equiv e^{-iL_z\phi_0/\hbar} f(\phi)$

$$\begin{aligned}\frac{\partial g}{\partial \phi_0} &= -\frac{iL_z}{\hbar} e^{-iL_z\phi_0/\hbar} f(\phi) \\ &= -\frac{\partial}{\partial \phi} e^{-iL_z\phi_0/\hbar} f(\phi) \\ &= -\frac{\partial g}{\partial \phi} \\ dg &= \frac{\partial g}{\partial \phi_0} d\phi_0 + \frac{\partial g}{\partial \phi} d\phi \\ &= \frac{\partial g}{\partial \phi} (d\phi - d\phi_0)\end{aligned}$$

Therefore  $g = g(\phi - \phi_0)$  (since it has 0 gradient in this direction). Since  $g(\phi) = f(\phi)$  (when  $\phi_0 = 0$ ),  $g(\phi, \phi_0) = f(\phi - \phi_0)$  for all  $\phi_0$ .

(b)

Define  $\sigma_n \equiv \sigma \cdot \hat{n}$

$$\begin{aligned}(\sigma \cdot \hat{n})^2 &= n_x^2 + n_y^2 + n_z^2 \\ &= 1\end{aligned}$$

(using the fact that  $\sigma_i$ 's anti-commutes with each other)

$$\begin{aligned}e^{-i\sigma_n\varphi/2} &= \sum_{j=0}^{\infty} \frac{(-i\sigma_n\varphi/2)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{(-i\sigma_n\varphi/2)^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{(-i\sigma_n\varphi/2)^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j(\varphi/2)^{2j}}{(2j)!} - i\sigma_n \sum_{j=0}^{\infty} \frac{(-1)^j(\varphi/2)^{2j+1}}{(2j+1)!} \\ &= \cos \frac{\varphi}{2} - i\sigma_n \sin \frac{\varphi}{2}\end{aligned}$$

(c)

$$\begin{aligned}T_{x180} &= e^{-i\sigma_x\pi/2} \\ &= \cos \frac{\pi}{2} - i\sigma_x \sin \frac{\pi}{2} \\ &= -i\sigma_x\end{aligned}$$

which switches up and down spin with a phase factor. Spining around y-axis gives the same spin flip with a different phase factor.

(d)

$$\begin{aligned} T_{x90} &= e^{-i\sigma_x \pi/4} \\ &= \cos \frac{\pi}{4} - i\sigma_x \sin \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}}(1 - i\sigma_x) \end{aligned}$$

The effect on  $\chi^+$ ,

$$\begin{aligned} T_{x90}\chi^+ &= \frac{1}{\sqrt{2}}(1 - i\sigma_x)\chi^+ \\ &= \frac{1}{\sqrt{2}}(\chi^+ - i\chi^-) \end{aligned}$$

(e)

$$\begin{aligned} T_{z180} &= e^{-i\sigma_z \pi} \\ &= \cos \pi - i\sigma_z \sin \pi \\ &= -1 \end{aligned}$$

The global phase has no observable effect on this system. It could have non-trivial effect if it is possible to interfere this system with another one.

**5.**

(a)

For  $n = 0$ ,  $[A, B^0] = 0$  is true. When the equation is true for  $n - 1$  we have,

$$\begin{aligned} [A, B^n] &= [A, B^{n-1}B] \\ &= [A, B^{n-1}]B + B^{n-1}[A, B] \\ &= (n-1)B^{n-2}[A, B]B + B^{n-1}[A, B] \\ &= (n-1)B^{n-1}[A, B] + B^{n-1}[A, B] \\ &= nB^{n-1}[A, B] \end{aligned}$$

So the equation is true for  $n$  as well. Therefore, the equation is true for all non-negative finite  $n$ .

(b)

(c)