1.

2.

(a)

Since $J_z = L_z + S_z$ and $[L_z, S_z] = 0$, the J_z eigenstates are sum of L_z and S_z eigenstates that has the same sum of m_l and m_s. (The only variable on the RHS are m_l and m_s among which m_s can only be $\pm\dfrac{1}{2}$ so the constraint above limit the decomposition to the given form.)

(b)

Hamiltonian

$$\begin{aligned}
H =&\frac{L^2}{2ma^2} + V_0 + \frac{e^2\vec{L}\cdot\vec{S}}{2mc^2a^3}\\
=&\frac{L^2}{2ma^2} + V_0 + \frac{e^2}{4mc^2a^3}\left(J^2 - L^2 - S^2\right)\\
=&\frac{l(l+1)}{2ma^2} + V_0 + \frac{e^2}{4mc^2a^3}\left(j(j+1) - l(l+1) - \frac{3}{4}\right)
\end{aligned}$$

When $l = 0$ there's only one manifold instead of two.

3.

$$\begin{aligned}
\mathrm{e}^{\lambda A}B\mathrm{e}^{-\lambda A} =&\left(\sum_{m=0}^{\infty}\frac{\lambda^m A^m}{m!}\right)B\left(\sum_{n=0}^{\infty}\frac{\lambda^n(-A)^n}{n!}\right)\\
=&\sum_{m,n=0}^{\infty}\frac{\lambda^m A^m}{m!}B\frac{\lambda^n(-A)^n}{n!}\\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{\lambda^m A^{m-n}B(-A)^n}{(m-n)!n!}\\
=&\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\sum_{n=0}^{m}\frac{m!A^{m-n}B(-A)^n}{(m-n)!n!}
\end{aligned}$$

So now we just need to show that

$$\left[A^{(m)}, B\right] = \sum_{n=0}^{m}\frac{m!A^{m-n}B(-A)^n}{(m-n)!n!}$$

where $\left[A^{(0)}, B\right] \equiv B$, $\left[A^{(n)}, B\right] \equiv \left[A, \left[A^{(n-1)}, B\right]\right]$. For $m = 0$, this is trivially true. If it is true for $m - 1$,

$$\begin{aligned}
\left[A^{(m)}, B\right] =&\left[A, \sum_{n=0}^{m-1}\frac{(m-1)!A^{m-n-1}B(-A)^n}{(m-n-1)!n!}\right]\\
=&\sum_{n=0}^{m-1}\frac{(m-1)!A^{m-n}B(-A)^n}{(m-n-1)!n!} + \sum_{n=1}^{m}\frac{(m-1)!A^{m-n}B(-A)^n}{(m-n)!(n-1)!}
\end{aligned}$$

Extend the summation taking advantage of $(-1)! = \infty$

$$\begin{aligned}\left[A^{(m)}, B\right] =& \sum_{n=0}^{m} \frac{(m-1)! A^{m-n} B(-A)^n}{(m-n-1)! n!} + \frac{(m-1)! A^{m-n} B(-A)^n}{(m-n)!(n-1)!} \\ =& \sum_{n=0}^{m} \frac{(m)! A^{m-n} B(-A)^n}{(m-n)! n!} \left(\frac{m-n}{m} + \frac{n}{m}\right) \\ =& \sum_{n=0}^{m} \frac{(m)! A^{m-n} B(-A)^n}{(m-n)! n!}\end{aligned}$$

Therefore, the equation is true for all $m \geqslant 0$.
If $[A, B] = \gamma B$, $\left[A^{(n)}, B\right] = \gamma^n B$

$$\begin{aligned}\mathrm{e}^{\lambda A} B \mathrm{e}^{-\lambda A} =& \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[A^{(m)}, B\right] \\ =& \sum_{m=0}^{\infty} \frac{\lambda^m \gamma^m}{m!} B \\ =& \mathrm{e}^{\lambda\gamma} B\end{aligned}$$

4.

Since $[\lambda, A] = 0$ and $[A, A] = 0$ (i.e. everything commutes),

$$\frac{\mathrm{d}\mathrm{e}^{\lambda A}}{\mathrm{d}\lambda} = A \mathrm{e}^{\lambda A}$$

We have

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}\lambda} =& \frac{\mathrm{d}\mathrm{e}^{\lambda A} \mathrm{e}^{\lambda B}}{\mathrm{d}\lambda} \\ =& \mathrm{e}^{\lambda A} \frac{\mathrm{d}\mathrm{e}^{\lambda B}}{\mathrm{d}\lambda} + \frac{\mathrm{d}\mathrm{e}^{\lambda A}}{\mathrm{d}\lambda} \mathrm{e}^{\lambda B} \\ =& \mathrm{e}^{\lambda A} B \mathrm{e}^{\lambda B} + A \mathrm{e}^{\lambda A} \mathrm{e}^{\lambda B} \\ =& \mathrm{e}^{\lambda A} B \mathrm{e}^{-\lambda A} \mathrm{e}^{\lambda A} \mathrm{e}^{\lambda B} + A \mathrm{e}^{\lambda A} \mathrm{e}^{\lambda B} \\ =& \left(A + \mathrm{e}^{\lambda A} B \mathrm{e}^{-\lambda A}\right) G \\ =& \left(A + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[A^{(m)}, B\right]\right) G\end{aligned}$$

Using this,

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}\lambda} =& \left(\frac{\mathrm{d}\mathrm{e}^{\lambda B^\dagger} \mathrm{e}^{\lambda A^\dagger}}{\mathrm{d}\lambda}\right)^\dagger \\ =& \left(\left(B^\dagger + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[B^{\dagger(m)}, A^\dagger\right]\right) G^\dagger\right)^\dagger \\ =& G \left(B + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[A, B^{(m)}\right]\right)\end{aligned}$$

If $C \equiv [A, B]$ commutes with both A and B (and therefore $A + B$),

$$\begin{aligned}
\frac{\mathrm{d}G}{\mathrm{d}\lambda} =&(A + B + \lambda C)G\\
G =&\mathrm{e}^{A+B+\lambda^2 C/2}\\
\mathrm{e}^A\mathrm{e}^B =&\mathrm{e}^{A+B+C/2}
\end{aligned}$$

5.

(a)

Eigenvalue λ

$$\begin{aligned}
0 =&(h - \lambda)^2 - |g|^2\\
h - \lambda =&\pm |g|\\
\lambda =&h \pm |g|
\end{aligned}$$

Corresponding eigen vectors are $\dfrac{1}{\sqrt{2}}\left(1, \pm\dfrac{g}{|g|}\right)$

(b)

Initial state

$$|\psi_0\rangle =\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

At time t,

$$\begin{aligned}
|\psi_t\rangle =&\frac{1}{\sqrt{2}}\left(\exp\left(-\mathrm{i}\frac{h + |g|}{\hbar}t\right)|+\rangle + \exp\left(-\mathrm{i}\frac{h - |g|}{\hbar}t\right)|-\rangle\right)\\
=&\frac{\mathrm{e}^{-\mathrm{i}ht/\hbar}}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}|g|t/\hbar}|+\rangle + \mathrm{e}^{\mathrm{i}|g|t/\hbar}|-\rangle\right)\\
=&\frac{\mathrm{e}^{-\mathrm{i}ht/\hbar}}{2}\left(\mathrm{e}^{-\mathrm{i}|g|t/\hbar}\left(|1\rangle + \frac{g}{|g|}|2\rangle\right) + \mathrm{e}^{\mathrm{i}|g|t/\hbar}\left(|1\rangle - \frac{g}{|g|}|2\rangle\right)\right)\\
=&\mathrm{e}^{-\mathrm{i}ht/\hbar}\left(\cos\left(\frac{|g|t}{\hbar}\right)|1\rangle - \mathrm{i}\frac{g}{|g|}\sin\left(\frac{|g|t}{\hbar}\right)|2\rangle\right)
\end{aligned}$$

6.

(a)

(b)