Physics 251b
PROBLEM SET 2

Spring 2016 Due: Monday, Feb. 29

Reading: SN, Secs. 3.8, 3.11 and 2.1-2.4 might be useful

1. Two spinless particles 1 and 2, with identical orbital angular momentum quantum numbers $\ell_1 = 1$ and $\ell_2 = 1$, combine to form a composite object. Construct the towers of states that lead to the normalized eigenfunctions of the composite angular momentum operators L_z and L^2 $(\vec{L} = \vec{L}_1 + \vec{L}_2)$, thus determining the decomposition into irreducible representations of the rotation group SO(3),

$$1 \otimes 1 = 2 \oplus 1 \oplus 0. \tag{1}$$

In this way, find all nonzero Clebsch-Gordon coefficients $c(1m_1, 1m_2; lm)$ in the transformation

$$\left|11, lm\right\rangle = \sum_{m_1=-1}^{+1} \sum_{m_2=-1}^{+1} c(1m_1, 1m_2; lm) \left|11, m_1 m_2\right\rangle \equiv \sum_{m_1=-1}^{+1} \sum_{m_2=-1}^{+1} c(1m_1, 1m_2; lm) \left|1m_1\right\rangle \left|1m_2\right\rangle \tag{2}$$

2.As mentioned in class, it can be useful to reformulate the problem of determining the splitting of the energy levels of the hydrogen atom by spin-orbit interactions in terms of the total angular momentum operator $\vec{J} = \vec{L} + \vec{S}$. The Hamiltonian for an electron with spin $s=½$ is

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r) + W(r)\vec{L}\cdot\vec{S}\right]\begin{pmatrix}\psi_1(\vec{r})\\ \psi_2(\vec{r})\end{pmatrix} = E\begin{pmatrix}\psi_1(\vec{r})\\ \psi_2(\vec{r})\end{pmatrix}, \tag{3}$$

where $V(r) = -e^2/r$ and the spin-orbit interaction potential (a relativistic effect) turns out to be $W(r) = e^2/(2m^2c^2r^3)$. If ℓ is the orbital quantum number, the irreducible representations of $\vec{J}$ are given by $\ell \otimes \frac{1}{2} = \left|\ell - \frac{1}{2}\right| \oplus \left(\ell + \frac{1}{2}\right)$.

(a) Explain why the two towers of states with total angular momentum quantum numbers $j = \ell + 1/2$ and $j = \ell - 1/2$ must take the form

$$\begin{aligned}\left|\ell, \frac{1}{2}; j = \ell + \frac{1}{2}, m\right\rangle &= \alpha\left|\ell, m - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle + \beta\left|\ell, m + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right\rangle \\ \left|\ell, \frac{1}{2}; j = \ell - \frac{1}{2}, m\right\rangle &= \alpha'\left|\ell, m - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle + \beta'\left|\ell, m + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right\rangle\end{aligned} \tag{4}$$

where the C-G coefficients (assume these are real numbers) must satisfy $\alpha^2+\beta^2=\alpha'^2+\beta'^2=1$. Here the <u>reducible</u> representation basis functions on the right side are products of a spherical harmonic and a spinor

$$\left|\ell,m-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle = Y_{\ell,m-1/2}(\theta,\phi)\left|\frac{1}{2},\frac{1}{2}\right\rangle; \quad \left|\ell,m+\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle = Y_{\ell,m+1/2}(\theta,\phi)\left|\frac{1}{2},-\frac{1}{2}\right\rangle \quad (5)$$

(Note that since j is a ½-integer, $m\pm 1/2$ assumes only integer values) By using the orthogonality condition and studying the action of $J^2 = L^2+S^2+2\vec{L}\cdot\vec{S}$ on these states, determine α,β,α' and β'. (Hint: express $\vec{L}\cdot\vec{S}$ in terms of raising and lowering operators.)

(b) Replace $V(r)$ in Eq. (3) above by a spherical delta shell potential $V(r)=V_0\delta(r-a)$ that confines the particle to a spherical shell of radius a (so that we describe the atom as a rigid rotator with a spin orbit coupling) and determine the energy eigenvalues for the two irreducible representations described above. What happens for $\ell=0$?

3. If A and B are operators, prove, using the method sketched in class, that

$$e^{\lambda A}Be^{-\lambda A} = B+\frac{\lambda}{1!}[A,B]+\frac{\lambda^2}{2!}[A,[A,B]]+\frac{\lambda^3}{3!}[A,[A,[A,B]]]+\ldots \quad (6)$$

Use this to show that, if A and B are operators such that $[A,B]=\gamma B$, where γ is a classical c-number, then $e^{\lambda A}Be^{-\lambda A}=e^{\lambda\gamma}B$.

4. If A and B are operators, consider the product $G(\lambda)=e^{\lambda A}e^{\lambda B}$ and prove that

$$\begin{aligned}\frac{dG}{d\lambda} &= \left\{A+B+\frac{\lambda}{1!}[A,B]+\frac{\lambda^2}{2!}[A,[A,B]]+\cdots\right\}G \\ &= G\left\{A+B+\frac{\lambda}{1!}[A,B]+\frac{\lambda^2}{2!}[[A,B],B]+\cdots\right\}\end{aligned}$$

Use this result to show that if A and B are two operators that both commute with their commutator $[A, B]$, then $e^{A}e^{B}=e^{A+B+[A,B]/2}$.

5. Consider a simplified system in which there are just *two* linearly independent states:

$$|1\rangle=\begin{pmatrix}1\\0\end{pmatrix} \text{ and } |2\rangle=\begin{pmatrix}0\\1\end{pmatrix}.$$

The most general state is a normalized linear combination:

$$|\psi\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{with } |a|^2 + |b|^2 = 1.$$

Suppose the Hamiltonian matrix is

$$H = \begin{pmatrix} h & g \\ g^* & h \end{pmatrix},$$

where h is real and g can be a complex number. The time-dependent Schrödinger equation reads

$$H|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle.$$

(a) Find the eigenvalues and (normalized) eigenvectors of this Hamiltonian.

(b) Suppose the system starts out (at $t=0$) in the pure state $|1\rangle$. What is the state at the time t?

6. A linear harmonic oscillator is subjected to a spatially uniform time-dependent external force $F(t)$ that is turned on at $t=0$ and decays exponentially afterwards, i.e., $F(t) = C\theta(t)e^{-t/\tau}$, where C is a constant and $\theta(t)$ is the step function, $\theta(t) = 0,\ t < 0;\ \theta(t) = 1,\ t > 0$.

(a) If the oscillator is in the ground state for t < 0, calculate the probability of finding it at time t in an oscillator eigenstate $|n\rangle$ with quantum number n.

(b) Discuss how your results for the transition probabilities vary with n and with $\omega_0\tau$, where ω_0 is the natural frequency of the harmonic oscillator.