

1.

(a)

Since $i(\lambda a^\dagger - \lambda^* a)$ is Hermitian, $S_\lambda \equiv \exp(\lambda a^\dagger - \lambda^* a)$ is unitary and $|\lambda\rangle \equiv S_\lambda|0\rangle$ is normalized. Since $[a, a^\dagger] = 1$ commutes with both a and a^\dagger

$$\begin{aligned} |\lambda\rangle &= \exp(\lambda a^\dagger - \lambda^* a)|0\rangle \\ &= \exp(\lambda a^\dagger) \exp(-\lambda^* a) \exp\left(-\frac{1}{2}[\lambda a^\dagger, -\lambda^* a]\right)|0\rangle \\ &= \exp(\lambda a^\dagger) \exp(-\lambda^* a) \exp\left(-\frac{|\lambda|^2}{2}\right)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp(\lambda a^\dagger)|0\rangle \end{aligned}$$

$$\begin{aligned} a|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right) a \exp(\lambda a^\dagger)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) (\exp(\lambda a^\dagger)a + [a, \exp(\lambda a^\dagger)])|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) [a, a^\dagger] \lambda \exp(\lambda a^\dagger)|0\rangle \\ &= \lambda|\lambda\rangle \end{aligned}$$

(b)

$$x = z_0(a + a^\dagger), \quad p = i\frac{\hbar}{2z_0}(a^\dagger - a)$$

$$\begin{aligned} \langle x \rangle &= z_0 \langle a + a^\dagger \rangle \\ &= z_0(\lambda + \lambda^*) \\ \langle x^2 \rangle &= z_0^2 \langle (a + a^\dagger)^2 \rangle \\ &= z_0^2 \langle a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a \rangle \\ &= z_0^2 \langle a^2 + a^{\dagger 2} + 2a^\dagger a + 1 \rangle \\ &= z_0^2 \langle (\lambda + \lambda^*)^2 + 1 \rangle \\ \langle \Delta x^2 \rangle &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= z_0^2 \end{aligned}$$

$$\begin{aligned}
 \langle p \rangle &= i \frac{\hbar}{2z_0} \langle a^\dagger - a \rangle \\
 &= i \frac{\hbar}{2z_0} (\lambda^* - \lambda) \\
 \langle p^2 \rangle &= - \frac{\hbar^2}{4z_0^2} \langle (a^\dagger - a)^2 \rangle \\
 &= - \frac{\hbar^2}{4z_0^2} \langle a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a \rangle \\
 &= - \frac{\hbar^2}{4z_0^2} \langle a^2 + a^{\dagger 2} - 2a^\dagger a - 1 \rangle \\
 &= - \frac{\hbar^2}{4z_0^2} \langle (\lambda^* - \lambda)^2 - 1 \rangle \\
 \langle \Delta p^2 \rangle &= \langle p^2 \rangle - \langle p \rangle^2 \\
 &= \frac{\hbar^2}{4z_0^2} \\
 \langle \Delta p^2 \rangle \langle \Delta x^2 \rangle &= \frac{\hbar^2}{4} \\
 &= \frac{1}{4} |\langle [x, p] \rangle|^2
 \end{aligned}$$

(c)

$$\begin{aligned}
 |\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp(\lambda a^\dagger) |0\rangle \\
 &= \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\lambda a^\dagger)^n}{n!} |0\rangle \\
 &= \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 P(n) &= \exp^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!} \\
 [n]_{av} &= \sum_{n=0}^{\infty} n P(n) \\
 &= \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!} \\
 &= |\lambda|^2 \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \\
 &= |\lambda|^2 \\
 [E_n]_{av} &= \hbar\omega \left(|\lambda|^2 + \frac{1}{2} \right)
 \end{aligned}$$

(d)

$$\begin{aligned}
 \langle n^2 \rangle &= \sum_{n=0}^{\infty} n^2 P(n) \\
 &= \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} n \frac{|\lambda|^{2n}}{(n-1)!} \\
 &= \exp^{-|\lambda|^2} \left(\sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-2)!} + \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!} \right) \\
 &= \exp^{-|\lambda|^2} \left(|\lambda|^4 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} + |\lambda|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \right) \\
 &= |\lambda|^4 + |\lambda|^2 \\
 \Delta n &= |\lambda| \\
 \Delta E &= \hbar \omega |\lambda| \\
 \frac{\Delta E}{[E_n]_{av}} &= \frac{1}{|\lambda|}
 \end{aligned}$$

so the relative uncertainty goes to 0 at large n limit.

2.

(a)

(b)

(c)

(d)

3.

The probability is

$$\begin{aligned}
 |\langle u|\chi \rangle|^2 &= |c_1|^2 \\
 |\langle u|\chi \rangle|^2 &= \langle u|\chi \rangle \langle \chi|u \rangle \\
 &= \langle u|\rho|u \rangle \\
 &= \text{Tr}(\langle u|\rho|u \rangle) \\
 &= \text{Tr}(\rho|u \rangle \langle u|)
 \end{aligned}$$

4.

- (a)
- (b)
- (c)
- (d)
- (e)
- (f)
- (g)

5.

- (a)
- (b)