1.

Hamiltonian

$$H =\frac{1}{2m}\left(\vec p-\frac{q\vec A}{c}\right)^2$$

Interaction term

$$\begin{aligned}V =&\frac{q}{2mc}\left(-\vec p\cdot\vec A-\vec A\cdot\vec p+\frac{qA^2}{c}\right)\\=&\frac{q}{2mc}\left(\mathrm{i}\hbar\nabla\cdot\vec A-2\vec A\cdot\vec p+\frac{qA^2}{c}\right)\end{aligned}$$

Scattering wavefunction given by Born approximation

$$\begin{aligned}\langle x|\psi^{(+)}\rangle =&\langle x|\vec k\rangle-\frac{2m}{\hbar^2}\int\mathrm{d}^3x'\frac{\mathrm{e}^{\mathrm{i}\vec k\left(\vec x-\vec x'\right)}}{4\pi|\vec x-\vec x'|}\langle x'|V|\vec k\rangle\\=&\langle x|\vec k\rangle-\frac{q}{\hbar^2c}\int\mathrm{d}^3x'\frac{\mathrm{e}^{\mathrm{i}\vec k\left(\vec x-\vec x'\right)}}{4\pi|\vec x-\vec x'|}\langle x'|\left(\mathrm{i}\hbar\nabla\cdot\vec A-2\vec A\cdot\vec p+\frac{qA^2}{c}\right)|\vec k\rangle\\=&\langle x|\vec k\rangle-\frac{q}{\hbar^2c}\int\mathrm{d}^3x'\frac{\mathrm{e}^{\mathrm{i}\vec k\left(\vec x-\vec x'\right)}}{4\pi|\vec x-\vec x'|}\langle x'|\left(\mathrm{i}\hbar\nabla\cdot\vec A'-2\hbar\vec A'\cdot\vec k+\frac{qA'^2}{c}\right)|\vec k\rangle\\\approx&\langle x|\vec k\rangle-\frac{q}{\hbar c}\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\int\mathrm{d}^3x'\mathrm{e}^{-\mathrm{i}\vec k'\cdot\vec x'}\left(\mathrm{i}\nabla\cdot\vec A'-2\vec A'\cdot\vec k+\frac{qA'^2}{\hbar c}\right)\langle x'|\vec k\rangle\\=&\langle x|\vec k\rangle-\frac{q}{\left(2\pi\right)^{3/2}\hbar c}\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\int\mathrm{d}^3x'\mathrm{e}^{\mathrm{i}\left(\vec k-\vec k'\right)\cdot\vec x'}\left(\mathrm{i}\nabla\cdot\vec A'-2\vec A'\cdot\vec k+\frac{qA'^2}{\hbar c}\right)\end{aligned}$$

2.

The plane wave function,

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\vec k\cdot\vec r} =&\mathrm{e}^{\mathrm{i}kr\cos\varphi}\\=&\sum_{m=-\infty}^{\infty}c_n(kr)\mathrm{e}^{\mathrm{i}m\varphi}\end{aligned}$$

Integrate both side with $\mathrm{e}^{-\mathrm{i}m\varphi}$

$$\begin{aligned}c_m(kr) =&\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}kr\cos\varphi}\mathrm{e}^{-\mathrm{i}m\varphi}\mathrm{d}\varphi\\=&\mathrm{i}^mJ_m(kr)\end{aligned}$$

where J_m is the Bessel function.
At $\rho\to\infty$ the wavefunction

$$\psi^{(+)} \approx f(\varphi)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{\rho}}$$

Expanding $f(\varphi)$

$$f(\varphi) = \frac{1}{\sqrt{k}}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\delta_m}\sin\delta_m\mathrm{e}^{\mathrm{i}m\varphi}$$

Total cross section

$$\begin{aligned}\sigma_{tot} =& \int \mathrm{d}\varphi\left|\frac{1}{\sqrt{k}}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\delta_m}\sin\delta_m\mathrm{e}^{\mathrm{i}m\varphi}\right|^2\\ =&\frac{1}{k}\int \mathrm{d}\varphi\sum_{m,m'=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\delta_m}\sin\delta_m\mathrm{e}^{\mathrm{i}\delta_m}\sin\delta_{m'}\mathrm{e}^{-\mathrm{i}m'\varphi}\mathrm{e}^{\mathrm{i}m\varphi}\\ =&\frac{2\pi}{k}\sum_{m=-\infty}^{\infty}\sin^2\delta_m\end{aligned}$$

Optical theorem,

$$\begin{aligned}\Im(f(0)) =& \Im\left(\frac{1}{\sqrt{k}}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\delta_m}\sin\delta_m\right)\\ =&\frac{1}{\sqrt{k}}\sum_{m=-\infty}^{\infty}\sin^2\delta_m\\ =&\frac{\sqrt{k}}{2\pi}\sigma_{tot}\end{aligned}$$

3.

Vector potential

$$\begin{aligned}\vec{A} =& \frac{\Phi_0}{2\pi\rho}\hat{\varphi}\\ H =& -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}+\mathrm{i}\frac{q\Phi_0}{2\pi c}\right)^2\right)\end{aligned}$$

Let $\alpha_0 \equiv \frac{q\Phi_0}{2\pi c}$

$$H = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}+\mathrm{i}\alpha_0\right)^2\right)$$

Radial Hamiltonian for state with angular momentum m

$$H_r = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}(m+\alpha_0)^2\right)$$

Therefore the new eigen function is the Bessel function with order $m+\alpha_0$ and (from the asymptotic form) the phase shift is $\frac{\pi\alpha_0}{2}$.
Since the total cross section is the sum of $\sin\delta_m$ which is now a constant. The total cross section is infinite unless α_0 happens to be an even number.
The AB effect changes the phase globally so it is not particularly surprising that the expansion diverges.

4.

(a)

Since there's only two plain wave solutions.

$$\psi^{(+)} = \frac{1}{\sqrt{2\pi}}\left(\mathrm{e}^{\mathrm{i}kx} + f(sign(x))\mathrm{e}^{\mathrm{i}k|x|}\right)$$

(b)

From the conservation of probability density, the in-coming and out-going wave must have the same amplitude. Therefore, there can only be a phase shift between them.
Using the boundary condition, the wavefunction for $0 < x < a$ is,

$$\psi = A\left(\mathrm{e}^{\mathrm{i}k'x} - \mathrm{e}^{-\mathrm{i}k'x}\right)$$

where $k' = \sqrt{k^2 + 2mV_0}$ and for $x \geqslant a$ is

$$\psi = \mathrm{e}^{\mathrm{i}(kx+\delta)} - \mathrm{e}^{-\mathrm{i}(kx+\delta)}$$

To satisfy the continuity condition at $x = a$

$$\begin{aligned} A\mathrm{e}^{\mathrm{i}k'a} - A\mathrm{e}^{-\mathrm{i}k'a} =& \mathrm{e}^{\mathrm{i}(ka+\delta)} - \mathrm{e}^{-\mathrm{i}(ka+\delta)} \\ Ak'\mathrm{e}^{\mathrm{i}k'a} + Ak'\mathrm{e}^{-\mathrm{i}k'a} =& k\mathrm{e}^{\mathrm{i}(ka+\delta)} + k\mathrm{e}^{-\mathrm{i}(ka+\delta)} \\ A\sin k'a =& \sin(ka+\delta) \\ Ak'\cos k'a =& k\cos(ka+\delta) \\ \delta =& \arctan\left(\frac{k}{k'}\tan k'a\right) - ka \end{aligned}$$

Scattering coefficient

$$\left|1 - \mathrm{e}^{2\mathrm{i}\delta}\right|^2 = 4\sin^2\delta$$

Since $\arctan\left(\frac{k}{k'}\tan k'a\right)$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, δ can be smaller than $-\frac{\pi}{2}$ if ka is large, in which case the scattering coefficient will have maxima.

5.

For hard sphere, the boundary condition is

$$A_l(R) = 0$$

where A_l is the radial wave function of the lth partial wave.

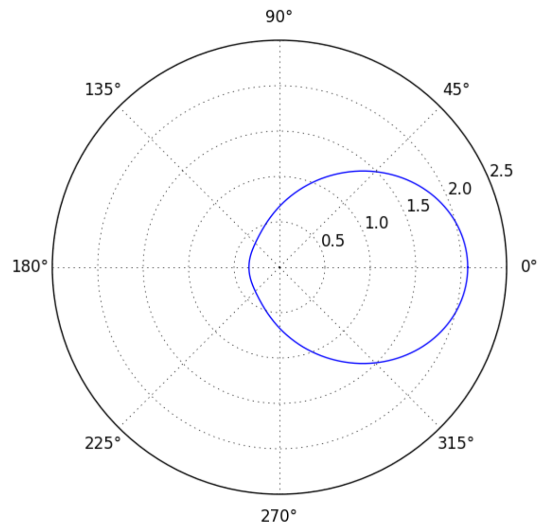
$$A_l(r) = \mathrm{e}^{\mathrm{i}\delta_l} j_l(kr)\cos\delta_l - n_l(kr)\sin\delta_l$$

$$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}$$

For $k = \dfrac{1}{R}$

$$\tan\delta_l = \frac{j_l(1)}{n_l(1)}$$

Differencial cross section



The total cross section is $10.6R^2$. This should be reasonably accurate since higher partial waves doesn't contribute much at low energy. When the wavelength is very large, only the first partial wave is important and the scattering is symmetric.

6.

Let $a \equiv \dfrac{\hbar^2}{mg}$ For incident wave from left

$$\psi = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + r\mathrm{e}^{-\mathrm{i}kx} & x < 0 \\ t\mathrm{e}^{\mathrm{i}kx} & x > 0 \end{cases}$$

$$\begin{aligned} 1 + r =& t \\ \frac{2}{a}t =& \mathrm{i}kt - \mathrm{i}k + \mathrm{i}kr \\ t =& \frac{ka}{ka + \mathrm{i}} \\ r =& \frac{-\mathrm{i}}{ka + \mathrm{i}} \end{aligned}$$

For incident wave from right

$$\psi = \begin{cases} \mathrm{e}^{-\mathrm{i}kx} + r'\mathrm{e}^{\mathrm{i}kx} & x > 0 \\ t'\mathrm{e}^{-\mathrm{i}kx} & x < 0 \end{cases}$$

$$\begin{aligned} 1 + r' =& t' \\ \frac{2}{a}t' =& \mathrm{i}kt' - \mathrm{i}k + \mathrm{i}kr' \\ t' =& \frac{ka}{ka + \mathrm{i}} \\ r' =& \frac{-\mathrm{i}}{ka + \mathrm{i}} \end{aligned}$$

The scattering matrix in the k subspace.

$$
\begin{aligned}
S =& \frac{1}{ka + \mathrm{i}} \begin{pmatrix} ka & -\mathrm{i} \\ -\mathrm{i} & ka \end{pmatrix} \\
SS^\dagger =& \frac{1}{ka + \mathrm{i}} \begin{pmatrix} ka & -\mathrm{i} \\ -\mathrm{i} & ka \end{pmatrix} \frac{1}{ka - \mathrm{i}} \begin{pmatrix} ka & \mathrm{i} \\ \mathrm{i} & ka \end{pmatrix} \\
=& \frac{1}{k^2a^2 + 1} \begin{pmatrix} k^2a^2 + 1 & 0 \\ 0 & k^2a^2 + 1 \end{pmatrix} \\
=& I
\end{aligned}
$$