1.

(a)

Communator of each component

$$\begin{aligned}
[L_i + g_0 S_i, J_j] =& [L_i + g_0 S_i, L_j + S_j]\\
=& [L_i, L_j] + g_0 [S_i, S_j]\\
=& \mathrm{i}\hbar\varepsilon_{ijk}(L_k + g_0 S_k)\\
\left[L_i + g_0 S_i, \hat{n}\cdot\vec{J}\right] =& \mathrm{i}\hbar\Big(\hat{n}\times\left(\vec{L} + g_0\vec{S}\right)\Big)_i\\
\left[\vec{L} + g_0\vec{S}, \hat{n}\cdot\vec{J}\right] =& \mathrm{i}\hbar\hat{n}\times\left(\vec{L} + g_0\vec{S}\right)
\end{aligned}$$

Therefore for any $\vec{n}$

$$\begin{aligned}
&\mathrm{i}\hbar\hat{n}\times\langle 0|\vec{L} + g_0\vec{S}|0\rangle\\
=&\langle 0|\left[L_i + g_0 S_i, \hat{n}\cdot\vec{J}\right]|0\rangle\\
=&\langle 0|[L_i + g_0 S_i, 0]|0\rangle\\
=&0\\
&\langle 0|\vec{L} + g_0\vec{S}|0\rangle\\
=&0
\end{aligned}$$

This is a special case of the Wigner-Eckart Theorem because the $|0\rangle$ state is spherical symmetric. The physical origin of the factor g_0 is the low energy limit of the Dirac equation of electron (and QED corrections on top of it).

(b)

For $j = \dfrac{1}{2}$, each pure states can be considered the eigenstate of a $\hat{n}\cdot\vec{J}$ (i.e. the direction the state is pointing to). Apply a infinitely small rotation,

$$\begin{aligned}
&\langle\hat{n}|R_{\hat{n}}^\dagger(\mathrm{d}\theta)\left(\vec{L} + g_0\vec{S}\right)R_{\hat{n}}(\mathrm{d}\theta)|\hat{n}\rangle\\
=&\langle\hat{n}|\vec{L} + g_0\vec{S}|\hat{n}\rangle + \langle\hat{n}|\left[\vec{L} + g_0\vec{S}, \hat{n}\cdot\vec{J}\right]|\hat{n}\rangle\mathrm{d}\theta + \mathcal{O}(\mathrm{d}\theta^2)\\
=&\langle\hat{n}|\vec{L} + g_0\vec{S}|\hat{n}\rangle + \mathrm{i}\hbar\hat{n}\times\langle\hat{n}|\vec{L} + g_0\vec{S}|\hat{n}\rangle\mathrm{d}\theta + \mathcal{O}(\mathrm{d}\theta^2)
\end{aligned}$$

However, due to symmetry of the state

$$\begin{aligned}
&\langle\hat{n}|R_{\hat{n}}^\dagger(\mathrm{d}\theta)\left(\vec{L} + g_0\vec{S}\right)R_{\hat{n}}(\mathrm{d}\theta)|\hat{n}\rangle\\
=&\langle\hat{n}|\vec{L} + g_0\vec{S}|\hat{n}\rangle
\end{aligned}$$

Therefore

$$\begin{aligned}
&\hbar\hat{n}\times\langle\hat{n}|\vec{L} + g_0\vec{S}|\hat{n}\rangle\\
=&0
\end{aligned}$$

i.e. $\vec{L}+g_0\vec{S}$ is parallel with $\hat{n}$ for all $\hat{n}$. i.e.

$$
\begin{aligned}
&\langle\hat{n}|\vec{L}+g_0\vec{S}|\hat{n}\rangle\\
=&g'\hat{n}
\end{aligned}
$$

where g' is a scalar c-number constant. Since we also have (from the definition of $\hat{n}$),

$$
\begin{aligned}
&\langle\hat{n}|\vec{J}|\hat{n}\rangle\\
=&g''\hat{n}\\
&\langle\hat{n}|\vec{L}+g_0\vec{S}|\hat{n}\rangle\\
=&g\langle\hat{n}|\vec{J}|\hat{n}\rangle
\end{aligned}
$$

should hold for all $\hat{n}$. (Being a linear relation, it holds try for cross terms as well by decomposing $\hat{n}$ into arbitrary basis.)
Within the J^2 subspace and use $g_0=2$, we then have

$$
\begin{aligned}
&\langle\hat{n}|\vec{J}\cdot\left(\vec{L}+2\vec{S}\right)|\hat{n}\rangle\\
=&g\langle\hat{n}|J^2|\hat{n}\rangle\\
g=&\frac{1}{j(j+1)}\langle\hat{n}|\vec{J}\cdot\left(\vec{L}+2\vec{S}\right)|\hat{n}\rangle\\
=&\frac{1}{j(j+1)}\langle\hat{n}|L^2+2S^2+3\vec{L}\cdot\vec{S}|\hat{n}\rangle\\
=&\frac{1}{j(j+1)}\langle\hat{n}|L^2+2S^2+\frac{3}{2}\left(J^2-L^2-S^2\right)|\hat{n}\rangle\\
=&\frac{3}{2}+\frac{1}{2}\frac{1}{j(j+1)}\langle\hat{n}|S^2-L^2|\hat{n}\rangle\\
=&\frac{3}{2}+\frac{1}{2}\frac{s(s+1)-l(l+1)}{j(j+1)}
\end{aligned}
$$

2.

The derivation of the interaction picture (i.e. expression of V_I) and the Dyson's series (i.e. equation 7) doesn't need to assume anything about the diagonal terms of V. Therefore $f_n=0$. It is of course also fine to split out the diagonal term of V into H_0 in which case it can be said that $f_n=\displaystyle\int_0^t V_{nn}(t')\mathrm{d}t$. However, this also requires substitute V with $V-\displaystyle\sum_n|n\rangle\langle n|V_{nn}$.

3.

(a)

Radial component of $\vec{j}$

$$
\begin{aligned}
j_r=&\frac{\hbar}{2m\mathrm{i}}\left(\psi^*\frac{\partial}{\partial r}\psi-\psi\frac{\partial}{\partial r}\psi^*\right)\\
=&\frac{\hbar}{m}\Im\left(\psi^*\frac{\partial}{\partial r}\psi\right)
\end{aligned}
$$

The part terms that is due to interference (for $\psi = \psi_1 + \psi_2$)

$$\begin{aligned}
j'_r =& j_r - j_{r1} - j_{r2}\\
=& \frac{\hbar}{m}\Im\left(\psi^*\frac{\partial}{\partial r}\psi\right) - \frac{\hbar}{m}\Im\left(\psi_1^*\frac{\partial}{\partial r}\psi_1\right) - \frac{\hbar}{m}\Im\left(\psi_2^*\frac{\partial}{\partial r}\psi_2\right)\\
=& \frac{\hbar}{m}\Im\left(\psi_1^*\frac{\partial}{\partial r}\psi_2\right) + \frac{\hbar}{m}\Im\left(\psi_2^*\frac{\partial}{\partial r}\psi_1\right)
\end{aligned}$$

Scattering wave function

$$\psi = \mathrm{e}^{\mathrm{i}kr\cos\theta} + f\frac{\mathrm{e}^{\mathrm{i}kr}}{r}$$

current density

$$\begin{aligned}
j'_r =& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r} + f^*\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)\\
=& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{r\mathrm{i}k - 1}{r^2}\mathrm{e}^{\mathrm{i}kr} + f^*\mathrm{i}k\cos\theta\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)
\end{aligned}$$

Ignoring r^{-2} term for large r

$$\begin{aligned}
j'_r \approx& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\mathrm{e}^{\mathrm{i}kr} + \mathrm{i}f^*\cos\theta\mathrm{e}^{-\mathrm{i}kr}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)\\
=& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta + \mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f =& \int_a^b f\mathrm{d}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}\\
=& \left.\frac{\mathrm{e}^{\mathrm{i}\lambda x}f}{\mathrm{i}\lambda}\right|_a^b - \int_a^b \mathrm{d}xf'\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}
\end{aligned}$$

Using the same integral by part, we can show that the second term is $O\left(\frac{1}{\lambda^2}\right)$. Therefore,

$$\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f = \frac{\mathrm{e}^{\mathrm{i}\lambda b}f(b) - \mathrm{e}^{\mathrm{i}\lambda a}f(a)}{\mathrm{i}\lambda} + O\left(\frac{1}{\lambda^2}\right)$$

(c)

Total interference current

$$\begin{aligned}
J =& r^2\int \mathrm{d}\Omega\frac{\hbar k}{m}\frac{1}{r}\Im\Big(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\\
=&\frac{\hbar kr}{m}\Re\bigg(\int\mathrm{d}\theta\int\mathrm{d}\phi\sin\theta\Big(\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\bigg)\\
=&\frac{2\pi\hbar kr}{m}\Re\bigg(\int_{-1}^{1}\mathrm{d}\cos\theta\Big(\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\bigg)\\
\approx&\frac{2\pi\hbar kr}{m}\Re\bigg(\frac{f^*(0)+\mathrm{e}^{-2\mathrm{i}kr}f^*(\pi)}{\mathrm{i}kr}-\frac{f(0)-\mathrm{e}^{2\mathrm{i}kr}f(\pi)}{\mathrm{i}kr}\bigg)\\
=&\frac{2\pi\hbar}{m}\Im\big(f^*(0)+\mathrm{e}^{-2\mathrm{i}kr}f^*(\pi)-f(0)+\mathrm{e}^{2\mathrm{i}kr}f(\pi)\big)\\
=&-\frac{4\pi\hbar}{m}\Im(f(0))
\end{aligned}$$

(d)

The total current of the incident wave is zero (since it's a plain wave). The total current of the scatterend wave.

$$\begin{aligned}
J_2 =& r^2\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\Big(\psi_r^*\frac{\partial}{\partial r}\psi_2\Big)\\
=& r^2\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\Big(f^*f\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\Big)\\
=& \int\mathrm{d}\Omega\frac{\hbar}{m}\Im\Big(f^*f\frac{r\mathrm{i}k-1}{r}\Big)
\end{aligned}$$

Ignore real term in the integral

$$\begin{aligned}
=&\frac{\hbar k}{m}\int\mathrm{d}\Omega f^*f\\
=&\frac{\hbar k}{m}\sigma_{tot}
\end{aligned}$$

Therefore

$$\begin{aligned}
0 =& \frac{\hbar k}{m}\sigma_{tot}-\frac{4\pi\hbar}{m}\Im(f(0))\\
\sigma_{tot} =& \frac{4\pi}{k}\Im(f(0))
\end{aligned}$$

4.

(a)

In first Born approximation (Use V_r to represent $\partial_r V$)

$$\begin{aligned}
f_\sigma =& -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3 r' \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}'} V(r',\sigma)\chi_\sigma \mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{r}'}\\
=& -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3 r' \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}'}\left(-(1+\mathrm{i}\xi)V(r') + \frac{c}{r'}V_r(r')\vec{\sigma}\cdot\frac{\vec{r}'\times\vec{p}}{\hbar}\right)\mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{r}'}\chi_\sigma\\
=& -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3 r' \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}'}\left(-(1+\mathrm{i}\xi)V(r') - \frac{c}{r'}V_r(r')\vec{\sigma}\cdot\left(\vec{r}'\times\vec{k}'\right)\right)\mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{r}'}\chi_\sigma\\
=&\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3 r' \mathrm{e}^{\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}'}(1+\mathrm{i}\xi)V(r')\chi_\sigma + \frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3 r' \mathrm{e}^{\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}'}\frac{c}{r'}V_r(r')\vec{\sigma}\cdot\left(\vec{r}'\times\vec{k}'\right)\chi_\sigma
\end{aligned}$$

Let $\vec{q} = \vec{k} - \vec{k}'$, $q = 2k\sin\dfrac{\theta}{2}$

$$\begin{aligned}
f_\sigma =&\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r' r'^2 \mathrm{e}^{\mathrm{i}qr'\cos\theta'}(1+\mathrm{i}\xi)V(r')\chi_\sigma\\
&+\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r' r'^2 \mathrm{e}^{\mathrm{i}qr'\cos\theta'}cV_r(r')\vec{\sigma}\cdot\left(\hat{r}'\times\vec{k}'\right)\chi_\sigma
\end{aligned}$$

Due to the rotational symmetry around $\vec{q}$

$$\begin{aligned}
f_\sigma =&\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r' r'^2 \mathrm{e}^{\mathrm{i}qr'\cos\theta'}(1+\mathrm{i}\xi)V(r')\chi_\sigma\\
&+\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r' r'^2 \cos\theta' \mathrm{e}^{\mathrm{i}qr'\cos\theta'}cV_r(r')\vec{\sigma}\cdot\left(\hat{q}\times\vec{k}'\right)\chi_\sigma\\
=&\frac{m}{\hbar^2}\int \mathrm{d}r' r'^2(1+\mathrm{i}\xi)V(r')\chi_\sigma\int_{-1}^{1}\mathrm{d}\cos\theta' \mathrm{e}^{\mathrm{i}qr'\cos\theta'}\\
&+\frac{m}{\hbar^2}\int \mathrm{d}r' r'^2 cV_r(r')\vec{\sigma}\cdot\left(\hat{q}\times\vec{k}'\right)\chi_\sigma\int_{-1}^{1}\mathrm{d}\cos\theta'\cos\theta' \mathrm{e}^{\mathrm{i}qr'\cos\theta'}\\
=&\frac{m}{\hbar^2}\int \mathrm{d}r' r'^2(1+\mathrm{i}\xi)V(r')\chi_\sigma\frac{\mathrm{e}^{\mathrm{i}qr'}-\mathrm{e}^{-\mathrm{i}qr'}}{\mathrm{i}qr'}\\
&+\frac{m}{\hbar^2}\int \mathrm{d}r' r'^2 cV_r(r')\vec{\sigma}\cdot\left(\hat{q}\times\vec{k}'\right)\chi_\sigma\left(\mathrm{e}^{\mathrm{i}qr'}\left(\frac{1}{\mathrm{i}qr'}+\frac{1}{q^2r'^2}\right)+\mathrm{e}^{-\mathrm{i}qr'}\left(\frac{1}{\mathrm{i}qr'}-\frac{1}{q^2r'^2}\right)\right)\\
=&\frac{2m}{\hbar^2 q}\int \mathrm{d}r' r'(1+\mathrm{i}\xi)V(r')\sin qr'\chi_\sigma\\
&+\frac{2m\mathrm{i}}{\hbar^2q^2}\int \mathrm{d}r' r' cV_r(r')\vec{\sigma}\cdot\left(\vec{q}\times\vec{k}'\right)\chi_\sigma\left(\frac{\sin qr'}{qr'}-\cos qr'\right)
\end{aligned}$$

Substitute in $\hat{n}$

$$\begin{aligned}
=&\frac{2m}{\hbar^2 q}\int \mathrm{d}r' r'(1+\mathrm{i}\xi)V(r')\sin qr'\chi_\sigma\\
&+\vec{\sigma}\cdot\hat{n}\frac{2\mathrm{i}mk^2c}{\hbar^2q^2}\sin\theta\int \mathrm{d}r' r' V_r(r')\left(\frac{\sin qr'}{qr'}-\cos qr'\right)\chi_\sigma
\end{aligned}$$

Therefore,

$$
\begin{aligned}
A(\theta) =& \frac{2m}{\hbar^2 q}\int \mathrm{d}r' r'(1+\mathrm{i}\xi)V(r')\sin qr'\\
B(\theta) =& \frac{2\mathrm{i}mk^2c}{\hbar^2 q^2}\sin\theta\int \mathrm{d}r' r' V_r(r')\left(\frac{\sin qr'}{qr'} - \cos qr'\right)
\end{aligned}
$$

(b)

For V being a step function,

$$
\begin{aligned}
A(\theta) =& \frac{2mV_0(1+\mathrm{i}\xi)}{\hbar^2 q}\int_0^R \mathrm{d}r' r'\sin qr'\\
=& \frac{2mV_0(1+\mathrm{i}\xi)}{\hbar^2 q^3}(\sin qR - qR\cos qR)\\
B(\theta) =& -\frac{2\mathrm{i}mk^2c}{\hbar^2 q^2}\sin\theta\int \mathrm{d}r' r' V_0\delta(r'-R)\left(\frac{\sin qr'}{qr'} - \cos qr'\right)\\
=& -\frac{2\mathrm{i}mV_0k^2c}{\hbar^2 q^3}\sin\theta(\sin qR - qR\cos qR)
\end{aligned}
$$

Average over input and sum over output to get total scattering cross section

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2}\mathrm{Tr}((A^* + \vec{\sigma}^*\cdot\hat{n}B^*)(A+\vec{\sigma}\cdot\hat{n}B))
$$

Since $\mathrm{Tr}(\sigma_n) = 0$

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} =& \frac{1}{2}\left(|A|^2 + |B|^2\right)\\
=& 2\left|\frac{mV_0(1+\mathrm{i}\xi)}{\hbar^2 q^3}(\sin qR - qR\cos qR)\right|^2 + 2\left|\frac{mV_0k^2c}{\hbar^2 q^3}\sin\theta(\sin qR - qR\cos qR)\right|^2\\
=& 2\left(\frac{mV_0}{\hbar^2 q^3}(\sin qR - qR\cos qR)\right)^2\left(1+\xi^2+k^4c^2\sin^2\theta\right)
\end{aligned}
$$

5.

(a)

Time dependent Schrödinger equation

$$
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = H|\psi\rangle
$$

Expansion,

$$
|\psi\rangle = \sum_n c_n \mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle
$$

where

$$\begin{aligned}
H|\psi_n\rangle =&E_n|\psi_n\rangle\\
0 =&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\sum_n c_n \mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle - H\sum_n c_n \mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle\\
0 =&\mathrm{i}\sum_n c_n\frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta_n}}{\mathrm{d}t}|\psi_n\rangle + \mathrm{i}\sum_n \frac{\mathrm{d}c_n}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle + \mathrm{i}\sum_n c_n \mathrm{e}^{\mathrm{i}\theta_n}\frac{\mathrm{d}}{\mathrm{d}t}|\psi_n\rangle - \sum_n c_n E_n \mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle\\
0 =&\langle\psi_l|\sum_n \frac{\mathrm{d}c_n}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle + \langle\psi_l|\sum_n c_n \mathrm{e}^{\mathrm{i}\theta_n}\frac{\mathrm{d}}{\mathrm{d}t}|\psi_n\rangle\\
\frac{\mathrm{d}c_l}{\mathrm{d}t} =& -\sum_n c_n \mathrm{e}^{\mathrm{i}(\theta_n-\theta_l)}\langle\psi_l|\frac{\mathrm{d}}{\mathrm{d}t}\psi_n\rangle
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}c_l^{(1)}}{\mathrm{d}t} =& -\sum_n c_n^{(0)}\mathrm{e}^{\mathrm{i}(\theta_n-\theta_l)}\langle\psi_l|\frac{\mathrm{d}}{\mathrm{d}t}\psi_n\rangle\\
=& -\sum_n \delta_{nm}\mathrm{e}^{\mathrm{i}\gamma_m t}\mathrm{e}^{\mathrm{i}(\theta_n-\theta_l)}\langle\psi_l|\frac{\mathrm{d}}{\mathrm{d}t}\psi_n\rangle\\
=& -\mathrm{e}^{\mathrm{i}\gamma_m t}\mathrm{e}^{\mathrm{i}(\theta_m-\theta_l)}\langle\psi_l|\frac{\mathrm{d}}{\mathrm{d}t}\psi_m\rangle\\
c_l^{(1)}(t) =&c_l^{(1)}(0) - \int_0^t \mathrm{d}t'\mathrm{e}^{\mathrm{i}\gamma_m t'}\mathrm{e}^{\mathrm{i}\left(\theta_m(t')-\theta_l(t')\right)}\langle\psi_l(t')|\frac{\mathrm{d}}{\mathrm{d}t'}\psi_m(t')\rangle
\end{aligned}$$

(c)

The time dependent Hamiltonian is harmonic oscillator with a shifted center and energy. Center $x_0(t) = f(t)$. Energy spacing isn't time dependent so $\theta_m - \theta_n = (n-m)\omega t$. For $l \neq m$

$$c_l^{(1)}(t) = -\int_0^t \mathrm{d}t'\mathrm{e}^{\mathrm{i}\gamma_m t'}\mathrm{e}^{\mathrm{i}(l-m)\omega t'}\langle\psi_l(t')|\frac{\mathrm{d}}{\mathrm{d}t'}\psi_m(t')\rangle$$

Time derivative of eigenvectors

$$|\frac{\mathrm{d}}{\mathrm{d}t}\psi_m(t)\rangle =\frac{\mathrm{d}}{\mathrm{d}t}\hat{T}(f)|\psi_{m0}\rangle$$

where $\hat{T}$ is the translation operator

$$\begin{aligned}
|\frac{\mathrm{d}}{\mathrm{d}t}\psi_m(t)\rangle =& -\mathrm{i}\frac{\hat{p}}{\hbar}\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\hat{p}f/\hbar}|\psi_{m0}\rangle\\
=& -\mathrm{i}\frac{\hat{p}}{\hbar}\frac{\mathrm{d}f}{\mathrm{d}t}|\psi_m(t)\rangle\\
=&\sqrt{\frac{\mu\omega}{2\hbar}}\left(a^\dagger - a\right)\frac{\mathrm{d}f}{\mathrm{d}t}|\psi_m(t)\rangle
\end{aligned}$$

Therefore $\gamma_m = 0$ and c_l is non-zero only when $l = m \pm 1$

$$\begin{aligned}
c_l^{(1)}(t) &= -\sqrt{\frac{\mu\omega}{2\hbar}} \int_0^t \frac{\mathrm{d}f}{\mathrm{d}t'} \mathrm{e}^{\mathrm{i}(l-m)\omega t'} \langle\psi_l(t')|\big(a^\dagger - a\big)|\psi_m(t')\rangle \mathrm{d}t' \\
c_{m+1}^{(1)}(t) &= -\sqrt{\frac{\mu\omega}{2\hbar}} \int_0^t \frac{\mathrm{d}f}{\mathrm{d}t'} \mathrm{e}^{\mathrm{i}\omega t'} \langle\psi_{m+1}(t')|a^\dagger|\psi_m(t')\rangle \mathrm{d}t' \\
&= -\sqrt{\frac{\mu\omega}{2\hbar}} \sqrt{m+1} \int_0^t \frac{\mathrm{d}f}{\mathrm{d}t'} \mathrm{e}^{\mathrm{i}\omega t'} \mathrm{d}t' \\
c_{m-1}^{(1)}(t) &= -\sqrt{\frac{\mu\omega}{2\hbar}} \int_0^t \frac{\mathrm{d}f}{\mathrm{d}t'} \mathrm{e}^{-\mathrm{i}\omega t'} \langle\psi_{m-1}(t')|a|\psi_m(t')\rangle \mathrm{d}t' \\
&= -\sqrt{\frac{\mu\omega}{2\hbar}} \sqrt{m} \int_0^t \frac{\mathrm{d}f}{\mathrm{d}t'} \mathrm{e}^{-\mathrm{i}\omega t'} \mathrm{d}t'
\end{aligned}$$

Therefore,

$$\begin{aligned}
g &= -\frac{\mathrm{d}f}{\mathrm{d}t} \\
h &= -\frac{\mathrm{d}f}{\mathrm{d}t}
\end{aligned}$$