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Physics 251b Problem Set 1
Recitation Section 1

1.

(a)

Represent the operation using 4×4 matrices that shows the mapping between the nodes.

$$T_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

$$T_4 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \qquad T_5 = \begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & 1 \end{pmatrix} \qquad T_6 = \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{pmatrix}$$

$$T_7 = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \qquad T_8 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{pmatrix} \qquad T_9 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$T_{10} = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{pmatrix} \qquad T_{11} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \qquad T_{12} = \begin{pmatrix} & & & 1 \\ & 1 & & \\ 1 & & & \\ & & 1 & \end{pmatrix}$$

(b)

$$g_{123} = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

$$g_{234} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

$$g_{234}g_{123} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} = T_4$$

(180° rotation around the the axis connecting the middle of 1-2 and 3-4)

$$g_{123}g_{234} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} = T_9 \neq T_4$$

(c)

See (a)

(d)

$$H = \begin{pmatrix} \varepsilon_0 & -t & -t & -t \\ -t & \varepsilon_0 & -t & -t \\ -t & -t & \varepsilon_0 & -t \\ -t & -t & -t & \varepsilon_0 \end{pmatrix}$$

Eigenvalues are $\varepsilon_0 - 3t$ for eigenvector $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\varepsilon_0 + t$ for eigenvectors, $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

2.

(a)

$$H = \begin{pmatrix} \varepsilon_0 & -t & & & & -t & & -t \\ -t & \varepsilon_0 & -t & & -t & & & \\ & -t & \varepsilon_0 & -t & & & & -t \\ & & -t & \varepsilon_0 & -t & & -t & \\ & -t & & -t & \varepsilon_0 & -t & & \\ -t & & & & -t & \varepsilon_0 & -t & \\ & & & -t & & -t & \varepsilon_0 & -t \\ -t & & -t & & & & -t & \varepsilon_0 \end{pmatrix}$$

(b)

(c)

3.

The constrains is that the angular momentum cannot perfectly point in a certain direction and there will always be some fluctuations. This uncertain comes from,

$$\begin{aligned} \langle \Delta L_x, \Delta L_y \rangle \geqslant & \frac{1}{2\mathrm{i}} \langle [L_x, L_y] \rangle \\ = & \frac{\hbar}{2} \langle L_z \rangle \\ = & \frac{\hbar^2 m}{2} \end{aligned}$$

Which can only be 0 when $m = 0$.

4.

(a)

Define $g(\phi,\phi_0)\equiv \mathrm{e}^{-\mathrm{i}L_z\phi_0/\hbar}f(\phi)$

$$\begin{aligned}\frac{\partial g}{\partial\phi_0}=&-\frac{\mathrm{i}L_z}{\hbar}\mathrm{e}^{-\mathrm{i}L_z\phi_0/\hbar}f(\phi)\\=&-\frac{\partial}{\partial\phi}\mathrm{e}^{-\mathrm{i}L_z\phi_0/\hbar}f(\phi)\\=&-\frac{\partial g}{\partial\phi}\\\mathrm{d}g=&\frac{\partial g}{\partial\phi_0}\mathrm{d}\phi_0+\frac{\partial g}{\partial\phi}\mathrm{d}\phi\\=&\frac{\partial g}{\partial\phi}(\mathrm{d}\phi-\mathrm{d}\phi_0)\end{aligned}$$

Therefore $g=g(\phi-\phi_0)$ (since it has 0 gradient in this direction). Since $g(\phi)=f(\phi)$ (when $\phi_0=0$), $g(\phi,\phi_0)=f(\phi-\phi_0)$ for all ϕ_0.

(b)

Define $\sigma_n\equiv\sigma\cdot\hat{n}$

$$\begin{aligned}(\sigma\cdot\hat{n})^2=&n_x^2+n_y^2+n_z^2\\=&1\end{aligned}$$

(using the fact that σ_i's anti-commutes with each other)

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\sigma_n\varphi/2}=&\sum_{j=0}^{\infty}\frac{(-\mathrm{i}\sigma_n\varphi/2)^j}{j!}\\=&\sum_{j=0}^{\infty}\frac{(-\mathrm{i}\sigma_n\varphi/2)^{2j}}{(2j)!}+\sum_{j=0}^{\infty}\frac{(-\mathrm{i}\sigma_n\varphi/2)^{2j+1}}{(2j+1)!}\\=&\sum_{j=0}^{\infty}\frac{(-1)^j(\varphi/2)^{2j}}{(2j)!}-\mathrm{i}\sigma_n\sum_{j=0}^{\infty}\frac{(-1)^j(\varphi/2)^{2j+1}}{(2j+1)!}\\=&\cos\frac{\varphi}{2}-\mathrm{i}\sigma_n\sin\frac{\varphi}{2}\end{aligned}$$

(c)

$$\begin{aligned}T_{x180}=&\mathrm{e}^{-\mathrm{i}\sigma_x\pi/2}\\=&\cos\frac{\pi}{2}-\mathrm{i}\sigma_x\sin\frac{\pi}{2}\\=&-\mathrm{i}\sigma_x\end{aligned}$$

which switches up and down spin with a phase factor. Spining around y-axis gives the same spin flip with a different phase factor.

(d)

$$\begin{aligned}
T_{x90} =& e^{-i\sigma_x\pi/4}\\
=& \cos\frac{\pi}{4} - i\sigma_x \sin\frac{\pi}{4}\\
=& \frac{1}{\sqrt{2}}(1 - i\sigma_x)
\end{aligned}$$

The effect on χ^+,

$$\begin{aligned}
T_{x90}\chi^+ =& \frac{1}{\sqrt{2}}(1 - i\sigma_x)\chi^+\\
=& \frac{1}{\sqrt{2}}\left(\chi^+ - i\chi^-\right)
\end{aligned}$$

(e)

$$\begin{aligned}
T_{z180} =& e^{-i\sigma_z\pi}\\
=& \cos\pi - i\sigma_z \sin\pi\\
=& -1
\end{aligned}$$

The global phase has no observable effect on this system. It could have non-trivial effect if it is possible to interfere this system with another one.

5.

(a)

For $n = 0$, $\left[A, B^0\right] = 0$ is true. When the equation is true for $n - 1$ we have,

$$\begin{aligned}
[A, B^n] =& \left[A, B^{n-1}B\right]\\
=& \left[A, B^{n-1}\right]B + B^{n-1}[A, B]\\
=& (n-1)B^{n-2}[A, B]B + B^{n-1}[A, B]\\
=& (n-1)B^{n-1}[A, B] + B^{n-1}[A, B]\\
=& nB^{n-1}[A, B]
\end{aligned}$$

So the equation is true for n as well. Therefore, the equation is true for all non-negative finite n.

Assume $f(x) \equiv \sum_{n=0}^{\infty} a_n x^n$

$$\begin{aligned}
[p_x, f(x)] &= \left[p_x, \sum_{n=0}^{\infty} a_n x^n\right] \\
&= \sum_{n=0}^{\infty} [p_x, a_n x^n] \\
&= -\mathrm{i}\hbar \sum_{n=0}^{\infty} a_n n x^{n-1} \\
&= -\mathrm{i}\hbar \sum_{n=0}^{\infty} a_n \frac{\partial x^n}{\partial x} \\
&= -\mathrm{i}\hbar \frac{\partial}{\partial x} \sum_{n=0}^{\infty} a_n x^n \\
&= -\mathrm{i}\hbar \frac{\partial f(x)}{\partial x}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \vec{L}_{op} =& \frac{\mathrm{i}}{\hbar} \left[H, \vec{L}\right] \\
=& \frac{\mathrm{i}}{\hbar} \left[\frac{p^2}{2m} + V, \vec{x} \times \vec{p}\right] \\
=& \frac{\mathrm{i}}{\hbar} \left[\frac{p^2}{2m}, \vec{x}\right] \times \vec{p} + \frac{\mathrm{i}}{\hbar} \vec{x} \times [V, \vec{p}] \\
=& \frac{-\mathrm{i}}{2m\hbar} 2\mathrm{i}\hbar \vec{p} \times \vec{p} + \frac{\mathrm{i}}{\hbar} \vec{x} \times \mathrm{i}\hbar \nabla V \\
=& -\vec{x} \times \nabla V \\
=& \vec{N}_{op}
\end{aligned}$$

(c)

If V is spherically symmetric, $\nabla V \parallel \vec{x}$ so $\vec{N}_{op} = -\vec{x} \times \nabla V = 0$ (since everything commutes...)