1.

(a)

Represent the operation using 4×4 matrices that shows the mapping between the nodes.

$$T_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

$$T_4 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \qquad T_5 = \begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & 1 \end{pmatrix} \qquad T_6 = \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{pmatrix}$$

$$T_7 = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \qquad T_8 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{pmatrix} \qquad T_9 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$T_{10} = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{pmatrix} \qquad T_{11} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \qquad T_{12} = \begin{pmatrix} & & & 1 \\ & 1 & & \\ 1 & & & \\ & & 1 & \end{pmatrix}$$

(b)

$$g_{123} = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

$$g_{234} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

$$g_{234}g_{123} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} = T_4$$

(180° rotation around the the axis connecting the middle of 1-2 and 3-4)

$$g_{123}g_{234} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} = T_9 \neq T_4$$

(c)

See (a)

(d)

$$H = \begin{pmatrix} \varepsilon_0 & -t & -t & -t \\ -t & \varepsilon_0 & -t & -t \\ -t & -t & \varepsilon_0 & -t \\ -t & -t & -t & \varepsilon_0 \end{pmatrix}$$

Eigenvalues are $\varepsilon_0 - 3t$ for eigenvector $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\varepsilon_0 + t$ for eigenvectors, $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

2.

(a)

$$H = \begin{pmatrix} \varepsilon_0 & -t & & & & -t & & -t \\ -t & \varepsilon_0 & -t & & -t & & & \\ & -t & \varepsilon_0 & -t & & & & -t \\ & & -t & \varepsilon_0 & -t & & -t & \\ & -t & & -t & \varepsilon_0 & -t & & \\ -t & & & & -t & \varepsilon_0 & -t & \\ & & & -t & & -t & \varepsilon_0 & -t \\ -t & & -t & & & & -t & \varepsilon_0 \end{pmatrix}$$

Eigenvalues and eigenvectors are, $\lambda_1 = \varepsilon_0 - 3t$,

$$v_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\lambda_2 = \varepsilon_0 + 3t$,

$$v_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$\lambda_{3,4,5} = \varepsilon_0 - t$ (3-fold degenerate)

$$v_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$
$$v_4 = \frac{1}{2} \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$
$$v_5 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and $\varepsilon_0 + t$ (3-fold degenerate)

$$v_6 = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}$$
$$v_7 = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix}$$
$$v_8 = \frac{1}{2} \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(b)

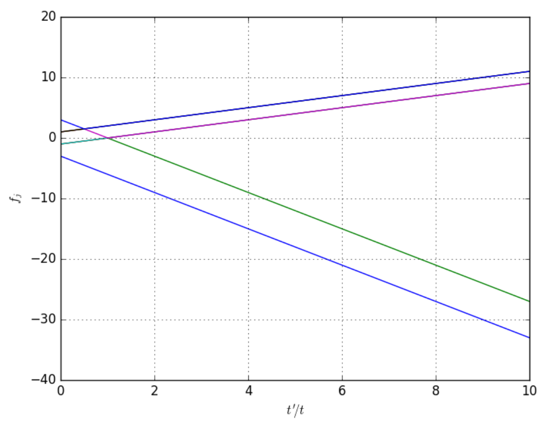
$\varepsilon_0 - 3t$ corresponds to the A_1 representation.
$\varepsilon_0 + 3t$ corresponds to the A_2 representation.
$\varepsilon_0 - t$ corresponds to the T_1 representation.
$\varepsilon_0 + t$ corresponds to the T_2 representation.

(c)

From the form of the Hamiltonian,

$$\begin{aligned}
\lambda_j =& \varepsilon_0 + \lambda_j(0, t, t') \\
\lambda_j(0, kt, kt') =& k\lambda_j(0, t, t') \\
\lambda_j(0, t, t') =& t\lambda_j(0, 1, t'/t) \\
=& f_j(t'/t)
\end{aligned}$$

Plot,



There is accidental degeneracy for certain parameters.

3.

The constrains is that the angular momentum cannot perfectly point in a certain direction and there will always be some fluctuations. This uncertain comes from,

$$\begin{aligned}
\langle \Delta L_x, \Delta L_y \rangle \geqslant& \frac{1}{2\mathrm{i}} \langle [L_x, L_y] \rangle \\
=& \frac{\hbar}{2} \langle L_z \rangle \\
=& \frac{\hbar^2 m}{2}
\end{aligned}$$

Which can only be 0 when $m = 0$.

4.

(a)

Define $g(\phi, \phi_0) \equiv \mathrm{e}^{-\mathrm{i}L_z\phi_0/\hbar} f(\phi)$

$$\begin{aligned}
\frac{\partial g}{\partial \phi_0} =& -\frac{\mathrm{i}L_z}{\hbar}\mathrm{e}^{-\mathrm{i}L_z\phi_0/\hbar} f(\phi)\\
=& -\frac{\partial}{\partial \phi}\mathrm{e}^{-\mathrm{i}L_z\phi_0/\hbar} f(\phi)\\
=& -\frac{\partial g}{\partial \phi}\\
\mathrm{d}g =& \frac{\partial g}{\partial \phi_0}\mathrm{d}\phi_0 + \frac{\partial g}{\partial \phi}\mathrm{d}\phi\\
=& \frac{\partial g}{\partial \phi}(\mathrm{d}\phi - \mathrm{d}\phi_0)
\end{aligned}$$

Therefore $g = g(\phi - \phi_0)$ (since it has 0 gradient in this direction). Since $g(\phi) = f(\phi)$ (when $\phi_0 = 0$), $g(\phi, \phi_0) = f(\phi - \phi_0)$ for all ϕ_0.

(b)

Define $\sigma_n \equiv \sigma \cdot \hat{n}$

$$\begin{aligned}
(\sigma \cdot \hat{n})^2 =& n_x^2 + n_y^2 + n_z^2\\
=& 1
\end{aligned}$$

(using the fact that σ_i's anti-commutes with each other)

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}\sigma_n\varphi/2} =& \sum_{j=0}^{\infty}\frac{(-\mathrm{i}\sigma_n\varphi/2)^j}{j!}\\
=& \sum_{j=0}^{\infty}\frac{(-\mathrm{i}\sigma_n\varphi/2)^{2j}}{(2j)!} + \sum_{j=0}^{\infty}\frac{(-\mathrm{i}\sigma_n\varphi/2)^{2j+1}}{(2j+1)!}\\
=& \sum_{j=0}^{\infty}\frac{(-1)^j(\varphi/2)^{2j}}{(2j)!} - \mathrm{i}\sigma_n\sum_{j=0}^{\infty}\frac{(-1)^j(\varphi/2)^{2j+1}}{(2j+1)!}\\
=& \cos\frac{\varphi}{2} - \mathrm{i}\sigma_n\sin\frac{\varphi}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
T_{x180} =& \mathrm{e}^{-\mathrm{i}\sigma_x\pi/2}\\
=& \cos\frac{\pi}{2} - \mathrm{i}\sigma_x\sin\frac{\pi}{2}\\
=& -\mathrm{i}\sigma_x
\end{aligned}$$

which switches up and down spin with a phase factor. Spining around y-axis gives the same spin flip with a different phase factor.

(d)

$$\begin{aligned}
T_{x90} =&\mathrm{e}^{-\mathrm{i}\sigma_x\pi/4}\\
=&\cos\frac{\pi}{4} - \mathrm{i}\sigma_x\sin\frac{\pi}{4}\\
=&\frac{1}{\sqrt{2}}(1-\mathrm{i}\sigma_x)
\end{aligned}$$

The effect on χ^+,

$$\begin{aligned}
T_{x90}\chi^+ =&\frac{1}{\sqrt{2}}(1-\mathrm{i}\sigma_x)\chi^+\\
=&\frac{1}{\sqrt{2}}\left(\chi^+ - \mathrm{i}\chi^-\right)
\end{aligned}$$

(e)

$$\begin{aligned}
T_{z180} =&\mathrm{e}^{-\mathrm{i}\sigma_z\pi}\\
=&\cos\pi - \mathrm{i}\sigma_z\sin\pi\\
=&-1
\end{aligned}$$

The global phase has no observable effect on this system. It could have non-trivial effect if it is possible to interfere this system with another one.

5.

(a)

For $n = 0$, $\left[A, B^0\right] = 0$ is true. When the equation is true for $n - 1$ we have,

$$\begin{aligned}
[A, B^n] =&\left[A, B^{n-1}B\right]\\
=&\left[A, B^{n-1}\right]B + B^{n-1}[A, B]\\
=&(n-1)B^{n-2}[A, B]B + B^{n-1}[A, B]\\
=&(n-1)B^{n-1}[A, B] + B^{n-1}[A, B]\\
=&nB^{n-1}[A, B]
\end{aligned}$$

So the equation is true for n as well. Therefore, the equation is true for all non-negative finite n.

Assume $f(x) \equiv \sum_{n=0}^{\infty} a_n x^n$

$$\begin{aligned}
[p_x, f(x)] =& \left[p_x, \sum_{n=0}^{\infty} a_n x^n\right] \\
=& \sum_{n=0}^{\infty} [p_x, a_n x^n] \\
=& -\mathrm{i}\hbar \sum_{n=0}^{\infty} a_n n x^{n-1} \\
=& -\mathrm{i}\hbar \sum_{n=0}^{\infty} a_n \frac{\partial x^n}{\partial x} \\
=& -\mathrm{i}\hbar \frac{\partial}{\partial x} \sum_{n=0}^{\infty} a_n x^n \\
=& -\mathrm{i}\hbar \frac{\partial f(x)}{\partial x}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{L}_{op} =& \frac{\mathrm{i}}{\hbar}\left[H, \vec{L}\right] \\
=& \frac{\mathrm{i}}{\hbar}\left[\frac{p^2}{2m} + V, \vec{x} \times \vec{p}\right] \\
=& \frac{\mathrm{i}}{\hbar}\left[\frac{p^2}{2m}, \vec{x}\right] \times \vec{p} + \frac{\mathrm{i}}{\hbar}\vec{x} \times [V, \vec{p}] \\
=& \frac{-\mathrm{i}}{2m\hbar} 2\mathrm{i}\hbar\vec{p} \times \vec{p} + \frac{\mathrm{i}}{\hbar}\vec{x} \times \mathrm{i}\hbar\nabla V \\
=& -\vec{x} \times \nabla V \\
=& \vec{N}_{op}
\end{aligned}$$

(c)

If V is spherically symmetric, $\nabla V \parallel \vec{x}$ so $\vec{N}_{op} = -\vec{x} \times \nabla V = 0$ (since everything commutes...)