Physics 251b
PROBLEM SET 3

Spring 2016 Due: Wednesday, March 23

Reading: SN, review Chapter 2. Secs. 5.5, and 5.7-5.9 might also be useful

1. **Coherent states.** A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator *a:*

$$a\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle, \tag{1.1}$$

where λ is, in general, a complex number.

(a) Prove that $\left|\lambda\right\rangle = S_\lambda\left|0\right\rangle \equiv \exp[\lambda a^\dagger - \lambda^* a]\left|0\right\rangle$ and that, furthermore,

$$\left|\lambda\right\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}\left|0\right\rangle \tag{1.2}$$

is a normalized coherent state.

(b) Prove that this state is special, it that it gives the minimum value allowed by the uncertainty principle

$$\left\langle(\Delta x)^2\right\rangle\left\langle(\Delta p)^2\right\rangle \geq (1/4)\left|\left\langle[x,p]\right\rangle\right|^2 . \tag{1.3}$$

(c)Write $\left|\lambda\right\rangle$ as $\left|\lambda\right\rangle = \sum_{n=0}^{\infty} f(n)\left|n\right\rangle$, and show that the probability that the coherent state assume a particular value of n, $P(n) = \left|f(n)\right|^2$, is Poisson distributed. Determine $[n]_{av}$, the value of n averaged over the Poisson probability distribution associated with a coherent state, and hence the Poisson-averaged oscillator energy $[E_n]_{av} = \hbar\omega([n]_{av} + \frac{1}{2})$.

(d) In a similar fashion, compute $\Delta E = \sqrt{\left[\left(E_n - [E_n]_{av}\right)^2\right]_{av}}$. How does the ratio $\Delta E/[E_n]_{av}$ behave in the limit of large n?

For more on coherent states, see R. J. Glauber, *Coherent and incoherent states of the radiation field*, Physical Review **131**, 2766 (1963).

2. **Inhomogeneous magnetic fields.** A magnet has an inhomogeneous static magnetic field $B_0(x,y,z)\hat{z}$ that varies slowly over different spatial regions of the sample. The fraction of spin ½ protons df (averaging over the entire sample) that experience a magnetic field between B_0 and $B_0 + dB_0$ is $df = p(B_0)dB_0$, where the

probability distribution $p(B_0)$ is normalized, $\int_{-\infty}^{\infty} p(B_0)dB_0 = 1$. (We have assumed the inhomogeneity is slight and simply gives a spread in field magnitude with no change in the direction.) Compute the magnetization in the *x*-direction, $M_x(t) = \langle \chi | S_x | \chi \rangle$ perpendicular to the static field above as a function of time, averaged over the entire sample, assuming that at *t=0* the total magnetization was M_0, pointing in the *x*-direction, for the following three forms of $p(B)$:

a. $p(B)$ is a constant for $B_0 - a < B < B_0 + a$ and is zero for all other fields.

b. $p(B) \propto e^{-(B-B_0)^2/a^2}$

c. $p(B) \propto \dfrac{1}{1+(B-B_0)^2/a^2}$

d. Explain, in physical terms, the behavior you find at long times for $M_x(t)$.

3. **Density matrix.** Let A_{op} be a 2×2 Hermitian matrix describing some observable with two eigenspinors $|u\rangle$ and $|v\rangle$, corresponding respectively to two non-degenerate real eigenvalues A_1' and A_2', and show that the probability of finding the value A_1' in a measurement of A_{op} on an arbitrary spinor $|\chi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is given by

$$\left| < u^\dagger | \chi > \right|^2 = Tr\left(\rho \frac{A_2' - A_{op}}{A_2' - A_1'} \right) = Tr(\rho u u^\dagger) \tag{3.1}$$

where ρ is the density matrix associated with $|\chi\rangle$, $\rho = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix}$.

4. **Two level system.** Assume there are just *two* states of an (unperturbed) system, ψ_a and ψ_b. They are eigenstates of some unperturbed Hamiltonian H_0:

$$H_0 \psi_a = E_a \psi_a \text{ and } H_0 \psi_b = E_b \psi_b \tag{4.1}$$

and they are orthonormal: $\langle \psi_a | \psi_b \rangle = \delta_{ab}$. We assume $E_b > E_a$, so E_a is the unperturbed ground state energy. Any initial wave condition can of course be expressed as $\Psi(t=0) = c_a \psi_a + c_b \psi_b$. Now, suppose we turn on a time-dependent 2×2 perturbation matrix $H'(t)$. Since ψ_a and ψ_b constitute a complete set, the wave function $\Psi(t)$ can still be expressed as

$$\Psi(t) = c_a(t) \psi_a e^{-iE_a t/\hbar} + c_b(t) \psi_b e^{-iE_b t/\hbar}, \tag{4.2}$$

where we have factored out the time dependence in the absence of the perturbation.

(a) Assume that the diagonal matrix elements of H' vanish, $H'_{aa} = H'_{bb} = 0$, and show that

$$dc_b(t)/dt = -\frac{i}{\hbar} H'_{ba} e^{-i\omega_0 t} c_a(t); \quad \mathrm{d}c_a(t)/dt = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b(t) \tag{4.3}$$

where $H_{ba}{}' = \left(H_{ab}{}'\right)^*$ and $\omega_0 \equiv \left(E_b - E_a\right)/\hbar \geq 0$.

(b) If the particle starts out in the lower state $c_a(0) = 1,\ c_b(0) = 0,$ the system stays this way forever to *zeroth* order in perturbation theory. i.e., if $H' = 0$) we have $c_a^{(0)}(t) = 1, \quad c_b^{(0)}(t) = 0.$ Now assume that nonzero matrix elements of perturbation take the form $H'_{ab} = V_0 \cos(\omega t);\ \ H'_{ba} = V_0^* \cos(\omega t)$, and show that to first order in perturbation theory,

$$c_b(t) = -\frac{V_0}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t'} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0-\omega)t'} - 1}{\omega_0 - \omega}\right] \tag{4.4}$$

Show that for $\omega \approx \omega_0$, the transition probability – the chance that a particle that started out in the state ψ_a will be found, at time *t*, in the state ψ_b – is

$$P_{a\to b}(t) = \left|c_b(t)\right|^2 \cong \frac{\left|V_0\right|^2}{\hbar^2} \frac{\sin^2\left[\left(\omega_0 - \omega\right)t/2\right]}{\left(\omega_0 - \omega\right)^2} \tag{4.5}$$

(c) The first term in Eq. (4.4) comes from the $e^{i\omega t}/2$ part of $\cos(\omega t)$, and the second from $e^{-i\omega t}/2$. Thus, dropping the first term is *formally* equivalent to writing $H'_{ab} = (V_0/2)e^{-i\omega t}$, where Hermiticity requires $H'_{ba} = (V_0^*/2)e^{i\omega t}$. (This simplification is analogous to the “rotating wave approximation” we used in our discussion of nuclear magnetic resonance.)

(d) Solve Eqs. (4.3) (with initial conditions $c_a(0) = 1, c_b(0) = 0$) in the rotating wave approximation. Express your results for $c_a(t)$ and $c_b(t)$ in terms of the Rabi frequency

$$\omega_R \equiv \frac{1}{2}\sqrt{\left(\omega - \omega_0\right)^2 + \left(\left|V_0\right|/\hbar\right)^2} \tag{4.6}$$

(e) Determine the transition probability, $P_{a\to b}(t)$, and show that it never exceeds unity. Confirm that $\left|c_a(t)\right|^2 + \left|c_b(t)\right|^2 = 1$ in your exact solution.

(f) Show that $P_{a\to b}(t)$ reduces to the perturbation theory result (Eq. 4.5) when the perturbation is "small" and state precisely what small means in this context, as a constraint on *V*.

(g) At what time does the system first return to its initial state?

5. **Forbidden transitions.** A polarizable atom, in the presence of a passing light wave, responds primarily to the electric rather than magnetic field component. If the wavelength is long (compared to the size of the atom), we can ignore the *spatial* variation in the field; the atom is effectively exposed to a sinusoidally oscillating electric field $\vec{E}(t) = \vec{E}_0 \cos(\omega t)$, where $\vec{E}_0 = E_0 \hat{n}$. The actual electric field is of course $\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$. If the atom is centered at the origin, then $\vec{k}\cdot\vec{r} \ll 1$ over the relevant atomic-scale volume $\left(|k| = 2\pi/\lambda\right.$ so $\left.\vec{k}\cdot\vec{r} \sim r/\lambda \ll 1\right)$, thus justifying neglect of the r-dependence. Suppose that we instead keep the first-order correction:

$$\vec{E}(\vec{r},t) \approx \vec{E}_0 \left[\cos(\omega t) + (\vec{k}\cdot\vec{r})\sin(\omega t)\right] \tag{5.1}$$

The first term gives rise to the allowed (electric-dipole) transitions considered, e.g., in SN, Sec. 5.8. The second term gives rise to so-called "forbidden" magnetic dipole and electric quadrupole transitions (higher powers of $\vec{k}\cdot\vec{r}$ lead to even *more* "forbidden" transitions, associated with multipole moments.)

(a) Show that the spontaneous emission rate for forbidden transitions, assuming a fixed polarization direction $\hat{n}$ and propagation direction $\hat{k}$ takes the form

$$R_{b\to a} = A\left|\langle a|(\hat{n}\cdot\vec{r})(\hat{k}\cdot\vec{r})|b\rangle\right|^2 \tag{5.2}$$

and determine the coefficient A.

(b) Show that the $2S \to 1S$ transition in hydrogen is not possible even by a "forbidden" transition. (In fact, no transitions are possible for higher order multipoles as well, and the dominant decay is by a very slow two-photon emission process.)