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Physics 251b Final

1.

(a)

Communator of each component

$$\begin{aligned}
[L_i + g_0 S_i, J_j] =&[L_i + g_0 S_i, L_j + S_j]\\
=&[L_i, L_j] + g_0[S_i, S_j]\\
=&\mathrm{i}\hbar\varepsilon_{ijk}(L_k + g_0 S_k)\\
\left[L_i + g_0 S_i, \hat{n}\cdot\vec{J}\right] =&\mathrm{i}\hbar\varepsilon_{ijk}n_j(L_k + g_0 S_k)\\
=&\mathrm{i}\hbar\Big(\hat{n}\times\left(\vec{L} + g_0\vec{S}\right)\Big)_i\\
\left[\vec{L} + g_0\vec{S}, \hat{n}\cdot\vec{J}\right] =&\mathrm{i}\hbar\varepsilon_{ijk}n_j(L_k + g_0 S_k)\\
=&\mathrm{i}\hbar\hat{n}\times\left(\vec{L} + g_0\vec{S}\right)
\end{aligned}$$

Therefore for any $\vec{n}$

$$\begin{aligned}
&\mathrm{i}\hbar\hat{n}\times\langle 0|\vec{L} + g_0\vec{S}|0\rangle\\
=&\langle 0|\left[L_i + g_0 S_i, \hat{n}\cdot\vec{J}\right]|0\rangle\\
=&\langle 0|[L_i + g_0 S_i, 0]|0\rangle\\
=&0\\
&\langle 0|\vec{L} + g_0\vec{S}|0\rangle\\
=&0
\end{aligned}$$

This is a special case of the Wigner-Eckart Theorem because the $|0\rangle$ state is spherical symmetric. The physical origin of the factor g_0 is the low energy limit of the Dirac equation of electron (and QED corrections on top of it).

(b)

2.

3.

(a)

Radial component of $\vec{j}$

$$\begin{aligned}
j_r =&\frac{\hbar}{2m\mathrm{i}}\left(\psi^*\frac{\partial}{\partial r}\psi - \psi\frac{\partial}{\partial r}\psi^*\right)\\
=&\frac{\hbar}{m}\Im\left(\psi^*\frac{\partial}{\partial r}\psi\right)
\end{aligned}$$

The part terms that is due to interference (for $\psi = \psi_1 + \psi_2$)

$$\begin{aligned}
j_r' =& j_r - j_{r1} - j_{r2}\\
=& \frac{\hbar}{m}\Im\left(\psi^*\frac{\partial}{\partial r}\psi\right) - \frac{\hbar}{m}\Im\left(\psi_1^*\frac{\partial}{\partial r}\psi_1\right) - \frac{\hbar}{m}\Im\left(\psi_2^*\frac{\partial}{\partial r}\psi_2\right)\\
=& \frac{\hbar}{m}\Im\left(\psi_1^*\frac{\partial}{\partial r}\psi_2\right) + \frac{\hbar}{m}\Im\left(\psi_2^*\frac{\partial}{\partial r}\psi_1\right)
\end{aligned}$$

Scattering wave function

$$\psi = \mathrm{e}^{\mathrm{i}kr\cos\theta} + f\frac{\mathrm{e}^{\mathrm{i}kr}}{r}$$

current density

$$\begin{aligned}
j_r' =& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r} + f^*\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)\\
=& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{r\mathrm{i}k - 1}{r^2}\mathrm{e}^{\mathrm{i}kr} + f^*\mathrm{i}k\cos\theta\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)
\end{aligned}$$

Ignoring r^{-2} term for large r

$$\begin{aligned}
j_r' \approx& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\mathrm{e}^{\mathrm{i}kr} + \mathrm{i}f^*\cos\theta\mathrm{e}^{-\mathrm{i}kr}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)\\
=& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta + \mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f =& \int_a^b f\mathrm{d}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}\\
=& \left.\frac{\mathrm{e}^{\mathrm{i}\lambda x}f}{\mathrm{i}\lambda}\right|_a^b - \int_a^b \mathrm{d}xf'\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}
\end{aligned}$$

Using the same integral by part, we can show that the second term is $O\left(\frac{1}{\lambda^2}\right)$. Therefore,

$$\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f = \frac{\mathrm{e}^{\mathrm{i}\lambda b}f(b) - \mathrm{e}^{\mathrm{i}\lambda a}f(a)}{\mathrm{i}\lambda} + O\left(\frac{1}{\lambda^2}\right)$$

(c)

Total interference current

$$
\begin{aligned}
J =& r^2\int \mathrm{d}\Omega\frac{\hbar k}{m}\frac{1}{r}\Im\Big(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\\
=&\frac{\hbar kr}{m}\Re\bigg(\int\mathrm{d}\theta\int\mathrm{d}\phi\sin\theta\Big(\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\bigg)\\
=&\frac{2\pi\hbar kr}{m}\Re\bigg(\int_{-1}^{1}\mathrm{d}\cos\theta\Big(\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\bigg)\\
\approx&\frac{2\pi\hbar kr}{m}\Re\bigg(\frac{f^*(0)+\mathrm{e}^{-2\mathrm{i}kr}f^*(\pi)}{\mathrm{i}kr}-\frac{f(0)-\mathrm{e}^{2\mathrm{i}kr}f(\pi)}{\mathrm{i}kr}\bigg)\\
=&\frac{2\pi\hbar}{m}\Im\big(f^*(0)+\mathrm{e}^{-2\mathrm{i}kr}f^*(\pi)-f(0)+\mathrm{e}^{2\mathrm{i}kr}f(\pi)\big)\\
=&-\frac{4\pi\hbar}{m}\Im(f(0))
\end{aligned}
$$

(d)

The total current of the incident wave is zero (since it's a plain wave). The total current of the scatterend wave.

$$
\begin{aligned}
J_2 =& r^2\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\bigg(\psi_r^*\frac{\partial}{\partial r}\psi_2\bigg)\\
=& r^2\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\bigg(f^*f\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\bigg)\\
=&\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\bigg(f^*f\frac{r\mathrm{i}k-1}{r}\bigg)
\end{aligned}
$$

Ignore real term in the integral

$$
\begin{aligned}
=&\frac{\hbar k}{m}\int\mathrm{d}\Omega f^*f\\
=&\frac{\hbar k}{m}\sigma_{tot}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 =& \frac{\hbar k}{m}\sigma_{tot}-\frac{4\pi\hbar}{m}\Im(f(0))\\
\sigma_{tot} =& \frac{4\pi}{k}\Im(f(0))
\end{aligned}
$$

4.

(a)

(b)

5.

(a)

(b)

(c)