Physics 251b
PROBLEM SET 4

Spring 2016 Due: Friday, April 8

Reading: SN, review Secs. 6.1-6.3; Secs. 6.4-6.9 might also be useful.

1. In the Born approximation, find a formal expression for the scattering amplitude for a charged particle scattered by arbitrary time-independent but spatially-varying magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Be sure to keep terms in the kinetic energy operator out to order $\left|\vec{A}(\vec{r})\right|^2$ Check explicitly that your result is unchanged by the gauge transformation $\vec{A}(\vec{r},t) \to \vec{A}(\vec{r},t) + \vec{\nabla}\chi(\vec{r},t)$.

2. Formulate the scattering theory, in terms of scattering phase shifts, of the two-dimensional motion of quantum particles of mass m in an axially-symmetrical potential $U = U(\rho)$, $\rho = \sqrt{x^2 + y^2}$. (Imagine that the scatterer is a long cylindrical rod centered on the origin, so that the potential is independent of z, and that the scattering plane is perpendicular to this rod.) What is the asymptotic form of the scattered wave function as $\rho \to \infty$? Obtain an expansion of the scattering amplitude $f(\vec{k}, \vec{k}_0)$, where $\vec{k}_0$ and are $\vec{k}$ are incoming and outgoing wave vectors, respectively, in terms of partial waves. Calculate the total scattering cross section, and derive an analogue to the Optical Theorem.

Hints: (1) Try working in cylindrical coordinates,

$$H = -\frac{\hbar^2}{2M}\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\varphi^2}\right) + U(\rho) \tag{1}$$

(2) Because $\hat{L}_z = -i\dfrac{d}{d\varphi}$ commutes with $\hat{H}$, for both $U(\rho) = 0$ and $U(\rho) \neq 0$, it might be useful to expand the both the incoming plane wave and the outgoing exact solution using the eigenfunctions $\Psi_m = e^{im\varphi}$ of this operator:

$$e^{ik\rho\cos\varphi} = \sum_{m=-\infty}^{\infty} A_m J_{|m|}(k\rho) e^{im\varphi}; \quad \Psi_{\vec{k}}^{+} = \sum_{m=-\infty}^{\infty} B_m R_{km}(\rho) e^{im\varphi} \tag{2}$$

Using your favorite table of integrals, and the completeness of the the functions $\{e^{im\varphi},\ m = 0, \pm 1, \ldots\}$ in the interval $0 \le \varphi < 2\pi$, show that the functions $J_m(x)$ in (2) above are related to Bessel functions , and determine the coefficients A_m.

3. By means of the developments above, calculate the scattering amplitude and differential cross section of a charged particle scattered by an axially-symmetric

magnetic field $\vec{B}(\rho)$. Assume that the field is parallel to $\hat{z}$-axis, and is localized within a thin cylinder along this axis: $\vec{B} = B_0 \hat{z}$ for $\rho \le a$, $\vec{B} = \vec{0}$, otherwise. Here *a* may be considered infinitesimally small, but the total magnetic flux $\Phi_0 = \pi a^2 B_0$ should then be assumed to approach a constant as $a \to 0$. Show that the total scattering cross section is infinite (although it becomes finite in the limit $\hbar \to 0$.) Comment on the relation, if any, with the Aharnov-Bohm effect.

4. Consider a one-dimensional (1d) scattering experiment. The sole possible scattering event (other than transmission though the potential) is back scattering $(\theta = \pi)$.

   (a) By determining the appropriate 1d scattering Green's function, find the one-dimensional analogue of the familiar 3d scattering formula

$$\Psi_k^+ \sim \frac{1}{(2\pi)^{3/2}} \left(e^{ikr\cos\theta} + f_k(\theta) \frac{e^{ikr}}{r} \right) \tag{3}$$

   for an arbitrary one-dimensional potential $V(x)$ within the first Born approximation.

   (b) Now consider a 1d potential given by

$$V(x) = \begin{pmatrix} 0, & x > a \\ -V_0, & 0 < x < a \\ \infty, & x < 0 \end{pmatrix}. \tag{4}$$

   Show that for $x > a$, positive energy scattering solutions take the form

$$e^{i(kx+2\delta)} - e^{-ikx}, \tag{5}$$

   and derive an implicit formula for 1d phase shift $\delta(V_0, a)$. Now calculate the scattering coefficient $\left|1 - e^{2i\delta}\right|^2$, and show that it has maxima (resonances) if the potential is sufficiently deep and broad.

5. Using the first three partial waves, compute and display on a polar graph the differential cross section $d\sigma(\theta)/d\Omega$ for an impenetrable hard sphere (in three dimensions!) when the de Broglie wavelength of the incident particle equals the circumference of the sphere. Evaluate the total cross section and estimate the accuracy of the result. Also discuss what happens if the wavelength becomes very large compared with the size of the sphere.

6. Obtain the "scattering states" (energy eigenstates with $E \ge 0$) for a one-dimensional delta function potential, $g\delta(x)$ where $g > 0$. Calculate the matrix elements $\langle k' | S | k \rangle$ and verify the unitarity of the *S*-matrix.