1.

(a)

Since $\mathrm{i}\left(\lambda a^\dagger - \lambda^* a\right)$ is Hermitian, $S_\lambda \equiv \exp\left(\lambda a^\dagger - \lambda^* a\right)$ is unitary and $|\lambda\rangle \equiv S_\lambda|0\rangle$ is normalized.
Since $[a, a^\dagger] = 1$ commutes with both a and $a^\dagger$

$$\begin{aligned}
|\lambda\rangle =&\exp\left(\lambda a^\dagger - \lambda^* a\right)|0\rangle\\
=&\exp\left(\lambda a^\dagger\right)\exp\left(-\lambda^* a\right)\exp\left(-\frac{1}{2}\left[\lambda a^\dagger, -\lambda^* a\right]\right)|0\rangle\\
=&\exp\left(\lambda a^\dagger\right)\exp\left(-\lambda^* a\right)\exp\left(-\frac{|\lambda|^2}{2}\right)|0\rangle\\
=&\exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(\lambda a^\dagger\right)|0\rangle
\end{aligned}$$

$$\begin{aligned}
a|\lambda\rangle =&\exp\left(-\frac{|\lambda|^2}{2}\right)a\exp\left(\lambda a^\dagger\right)|0\rangle\\
=&\exp\left(-\frac{|\lambda|^2}{2}\right)\left(\exp\left(\lambda a^\dagger\right)a + \left[a, \exp\left(\lambda a^\dagger\right)\right]\right)|0\rangle\\
=&\exp\left(-\frac{|\lambda|^2}{2}\right)\left[a, a^\dagger\right]\lambda\exp\left(\lambda a^\dagger\right)|0\rangle\\
=&\lambda|\lambda\rangle
\end{aligned}$$

(b)

$x = z_0\left(a + a^\dagger\right)$, $p = \mathrm{i}\dfrac{\hbar}{2z_0}\left(a^\dagger - a\right)$

$$\begin{aligned}
\langle x\rangle =&z_0\left\langle a + a^\dagger\right\rangle\\
=&z_0(\lambda + \lambda^*)\\
\left\langle x^2\right\rangle =&z_0^2\left\langle\left(a + a^\dagger\right)^2\right\rangle\\
=&z_0^2\left\langle a^2 + {a^\dagger}^2 + aa^\dagger + a^\dagger a\right\rangle\\
=&z_0^2\left\langle a^2 + {a^\dagger}^2 + 2a^\dagger a + 1\right\rangle\\
=&z_0^2\left\langle\left(\lambda + \lambda^*\right)^2 + 1\right\rangle\\
\left\langle\Delta x^2\right\rangle =&\left\langle x^2\right\rangle - \langle x\rangle^2\\
=&z_0^2
\end{aligned}$$

$$
\begin{aligned}
\langle p\rangle =& \mathrm{i}\frac{\hbar}{2z_0}\left\langle a^\dagger - a\right\rangle\\
=& \mathrm{i}\frac{\hbar}{2z_0}(\lambda^* - \lambda)\\
\left\langle p^2\right\rangle =& -\frac{\hbar^2}{4z_0^2}\left\langle\left(a^\dagger - a\right)^2\right\rangle\\
=& -\frac{\hbar^2}{4z_0^2}\left\langle a^2 + {a^\dagger}^2 - aa^\dagger - a^\dagger a\right\rangle\\
=& -\frac{\hbar^2}{4z_0^2}\left\langle a^2 + {a^\dagger}^2 - 2a^\dagger a - 1\right\rangle\\
=& -\frac{\hbar^2}{4z_0^2}\left\langle(\lambda^* - \lambda)^2 - 1\right\rangle\\
\left\langle\Delta p^2\right\rangle =& \left\langle p^2\right\rangle - \langle p\rangle^2\\
=& \frac{\hbar^2}{4z_0^2}\\
\left\langle\Delta p^2\right\rangle\left\langle\Delta x^2\right\rangle =& \frac{\hbar^2}{4}\\
=& \frac{1}{4}\left|\langle[x,p]\rangle\right|^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
|\lambda\rangle =& \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(\lambda a^\dagger\right)|0\rangle\\
=& \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\lambda a^\dagger\right)^n}{n!}|0\rangle\\
=& \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P(n) =& \exp^{-|\lambda|^2}\frac{|\lambda|^{2n}}{n!}\\
[n]_{av} =& \sum_{n=0}^{\infty} nP(n)\\
=& \exp^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n-1)!}\\
=& |\lambda|^2\exp^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}\\
=& |\lambda|^2\\
[E_n]_{av} =& \hbar\omega\left(|\lambda|^2 + \frac{1}{2}\right)
\end{aligned}
$$

(d)

$$\begin{aligned}
\left\langle n^2\right\rangle =& \sum_{n=0}^{\infty} n^2 P(n)\\
=& \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} n \frac{|\lambda|^{2n}}{(n-1)!}\\
=& \exp^{-|\lambda|^2} \left(\sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-2)!} + \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!}\right)\\
=& \exp^{-|\lambda|^2} \left(|\lambda|^4 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} + |\lambda|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!}\right)\\
=& |\lambda|^4 + |\lambda|^2\\
\Delta n =& |\lambda|\\
\Delta E =& \hbar\omega|\lambda|\\
\frac{\Delta E}{[E_n]_{av}} =& \frac{1}{|\lambda|}
\end{aligned}$$

so the relative uncertainty goes to 0 at large n limit.

2.

In a homogeneous field B_0 the x magnetization is

$$M_x = M_0 \cos\omega_0 t$$

where $\omega_0 = \dfrac{2\mu_e B_0}{\hbar}$ is the Larmor frequency.
In a non-homogenous field, assuming the initial local magnetization is position independent

$$M_x = M_0 \int \cos\left(\frac{2\mu_e Bt}{\hbar}\right) p(B) \mathrm{d}B$$

(a)

$p(B) = \dfrac{1}{2a}$

$$\begin{aligned}
M_x =& \frac{M_0}{2a} \int_{B_0-a}^{B_0+a} \cos\left(\frac{2\mu_e Bt}{\hbar}\right) \mathrm{d}B\\
=& \frac{\hbar M_0}{4\mu_e at} \sin\left(\frac{2\mu_e Bt}{\hbar}\right)\Bigg|_{B_0-a}^{B_0+a}\\
=& \frac{\hbar M_0}{4\mu_e at} \left(\sin\left(\frac{2\mu_e t(B_0+a)}{\hbar}\right) - \sin\left(\frac{2\mu_e t(B_0-a)}{\hbar}\right)\right)\\
=& \frac{\hbar M_0}{2\mu_e at} \cos\left(\frac{2\mu_e tB_0}{\hbar}\right) \sin\left(\frac{2\mu_e ta}{\hbar}\right)
\end{aligned}$$

(b)

$p(B) = \frac{1}{\sqrt{\pi}a}\mathrm{e}^{-(B-B_0)^2/a^2}$

$$\begin{aligned}
M_x =&\frac{M_0}{\sqrt{\pi}a}\int\cos\left(\frac{2\mu_eBt}{\hbar}\right)\mathrm{e}^{-(B-B_0)^2/a^2}\mathrm{d}B\\
=&\frac{M_0}{\sqrt{\pi}a}\Re\left(\int\exp\left(-(B-B_0)^2/a^2+\mathrm{i}\frac{2\mu_eBt}{\hbar}\right)\mathrm{d}B\right)\\
=&\frac{M_0}{\sqrt{\pi}}\Re\left(\int\exp\left(-x^2+\mathrm{i}\frac{2\mu_eat}{\hbar}x+\mathrm{i}\frac{2\mu_eB_0t}{\hbar}\right)\mathrm{d}x\right)\\
=&\frac{M_0}{\sqrt{\pi}}\Re\left(\int\exp\left(-x^2+\mathrm{i}\frac{2\mu_eat}{\hbar}x+\left(\frac{\mu_eat}{\hbar}\right)^2-\left(\frac{\mu_eat}{\hbar}\right)^2+\mathrm{i}\frac{2\mu_eB_0t}{\hbar}\right)\mathrm{d}x\right)\\
=&M_0\Re\left(\exp\left(-\left(\frac{\mu_eat}{\hbar}\right)^2+\mathrm{i}\frac{2\mu_eB_0t}{\hbar}\right)\right)\\
=&M_0\cos\left(\frac{2\mu_eB_0t}{\hbar}\right)\exp\left(-\left(\frac{\mu_eat}{\hbar}\right)^2\right)
\end{aligned}$$

(c)

$p(B) = \frac{1}{\pi a}\frac{1}{1+(B-B_0)^2/a^2}$

$$\begin{aligned}
M_x =&\frac{M_0}{\pi a}\Re\left(\int\exp\left(\mathrm{i}\frac{2\mu_eBt}{\hbar}\right)\frac{1}{1+(B-B_0)^2/a^2}\mathrm{d}B\right)\\
=&\frac{M_0}{\pi a}\Re\left(2\pi\mathrm{i}\mathbf{Res}_{B=B_0+\mathrm{i}a}\left(\exp\left(\mathrm{i}\frac{2\mu_eBt}{\hbar}\right)\frac{1}{1+(B-B_0)^2/a^2}\right)\right)\\
=&M_0\cos\left(\frac{2\mu_eB_0t}{\hbar}\right)\exp\left(-\frac{2\mu_eat}{\hbar}\right)
\end{aligned}$$

(d)

$M_x(t)$ is a damped oscillation with a profile corresponds to the Fourier transformation of the B field distribution.

3.

The probability is

$$\begin{aligned}
|\langle u|\chi\rangle|^2 =&|c_1|^2\\
|\langle u|\chi\rangle|^2 =&\langle u|\chi\rangle\langle\chi|u\rangle\\
=&\langle u|\rho|u\rangle\\
=&\mathrm{Tr}(\langle u|\rho|u\rangle)\\
=&\mathrm{Tr}(\rho|u\rangle\langle u|)
\end{aligned}$$

4.

(a)

$|\psi\rangle = c_a\mathrm{e}^{-\mathrm{i}E_at/\hbar}|a\rangle + c_b\mathrm{e}^{-\mathrm{i}E_bt/\hbar}|b\rangle$

$$\begin{aligned}
\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle =&\mathrm{i}\hbar\frac{\mathrm{d}c_a}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}E_at/\hbar}|a\rangle + \mathrm{i}\hbar\frac{\mathrm{d}c_b}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}E_bt/\hbar}|b\rangle + E_ac_a\mathrm{e}^{-\mathrm{i}E_at/\hbar}|a\rangle + E_bc_b\mathrm{e}^{-\mathrm{i}E_bt/\hbar}|b\rangle\\
H|\psi\rangle =&c_a\mathrm{e}^{-\mathrm{i}E_at/\hbar}E_a|a\rangle + c_b\mathrm{e}^{-\mathrm{i}E_bt/\hbar}E_b|b\rangle + c_a\mathrm{e}^{-\mathrm{i}E_at/\hbar}H'|a\rangle + c_b\mathrm{e}^{-\mathrm{i}E_bt/\hbar}H'|b\rangle\\
&\mathrm{i}\hbar\frac{\mathrm{d}c_a}{\mathrm{d}t}|a\rangle + \mathrm{i}\hbar\frac{\mathrm{d}c_b}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\omega_0t}|b\rangle\\
=&c_aH'|a\rangle + c_b\mathrm{e}^{-\mathrm{i}\omega_0t}H'|b\rangle
\end{aligned}$$

When $H'_{aa} = H'_{bb} = 0$ we can remove all operators from the equation

$$\begin{aligned}
&\mathrm{i}\hbar\frac{\mathrm{d}c_a}{\mathrm{d}t}|a\rangle + \mathrm{i}\hbar\frac{\mathrm{d}c_b}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\omega_0t}|b\rangle\\
=&c_aH'_{ba}|b\rangle + c_b\mathrm{e}^{-\mathrm{i}\omega_0t}H'_{ab}|a\rangle
\end{aligned}$$

Left multiply by $\rangle a|$ or $\rangle b|$,

$$\begin{aligned}
\frac{\mathrm{d}c_a}{\mathrm{d}t} =& -\frac{\mathrm{i}}{\hbar}c_b\mathrm{e}^{-\mathrm{i}\omega_0t}H'_{ab}\\
\frac{\mathrm{d}c_b}{\mathrm{d}t} =& -\frac{\mathrm{i}}{\hbar}c_a\mathrm{e}^{\mathrm{i}\omega_0t}H'_{ba}
\end{aligned}$$

(b)

Zeroth order

$$\begin{aligned}
c_a^{(0)} =&1\\
c_b^{(0)} =&0
\end{aligned}$$

First order

$$\begin{aligned}
\frac{\mathrm{d}c_a^{(1)}}{\mathrm{d}t} =& -\frac{\mathrm{i}}{\hbar}c_b^{(0)}\mathrm{e}^{-\mathrm{i}\omega_0t}H'_{ab}\\
=&0\\
\frac{\mathrm{d}c_b^{(1)}}{\mathrm{d}t} =& -\frac{\mathrm{i}}{\hbar}c_a^{(0)}\mathrm{e}^{\mathrm{i}\omega_0t}H'_{ba}\\
=& -\frac{\mathrm{i}}{\hbar}\mathrm{e}^{\mathrm{i}\omega_0t}H'_{ba}\\
c_a^{(1)} =&0\\
c_b^{(1)} =&\int_0^t -\frac{\mathrm{i}V_0^*}{2\hbar}\left(\mathrm{e}^{\mathrm{i}(\omega+\omega_0)t'} + \mathrm{e}^{\mathrm{i}(\omega_0-\omega)t'}\right)\mathrm{d}t'\\
=& -\frac{V_0^*}{2\hbar}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega+\omega_0)t} - 1}{\omega+\omega_0} + \frac{\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t} - 1}{\omega_0-\omega}\right)
\end{aligned}$$

When $|\omega_0 - \omega| \ll \omega_0 + \omega$

$$\begin{aligned}
c_b^{(1)} \approx& -\frac{V_0^*}{2\hbar}\frac{\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t} - 1}{\omega_0 - \omega}\\
\left|c_b^{(1)}\right|^2 \approx& \frac{|V_0|^2}{4\hbar^2}\frac{\left|\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t/2} - \mathrm{e}^{-\mathrm{i}(\omega_0-\omega)t/2}\right|^2}{(\omega_0 - \omega)^2}\\
=& \frac{|V_0|^2}{\hbar^2}\frac{\sin^2\left((\omega_0-\omega)t/2\right)}{(\omega_0 - \omega)^2}
\end{aligned}$$

(c)

Right. I guess there isn't anything to solve for this one?

(d)

$$\begin{aligned}
\frac{\mathrm{d}c_a}{\mathrm{d}t} =& -\frac{\mathrm{i}V_0}{2\hbar}c_b\mathrm{e}^{-\mathrm{i}(\omega_0-\omega)t}\\
=& -\frac{\mathrm{i}\Omega}{2}c_b\mathrm{e}^{-\mathrm{i}(\omega_0-\omega)t}\\
\frac{\mathrm{d}c_b}{\mathrm{d}t} =& -\frac{\mathrm{i}V_0^*}{2\hbar}c_a\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t}\\
=& -\frac{\mathrm{i}\Omega^*}{2}c_a\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t}
\end{aligned}$$

where $\Omega = \dfrac{V_0}{\hbar}$

$$\begin{aligned}
\frac{\mathrm{d}^2c_b}{\mathrm{d}t^2} =& -\frac{\mathrm{i}\Omega^*}{2}\frac{\mathrm{d}c_a}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t} + \frac{\Omega^*(\omega_0-\omega)}{2}c_a\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t}\\
=& -\frac{|\Omega|^2}{4}c_b + \mathrm{i}(\omega_0-\omega)\frac{\mathrm{d}c_b}{\mathrm{d}t}\\
0 =& \frac{\mathrm{d}^2c_b}{\mathrm{d}t^2} - \mathrm{i}(\omega_0-\omega)\frac{\mathrm{d}c_b}{\mathrm{d}t} + \frac{|\Omega|^2}{4}c_b
\end{aligned}$$

For $c_b \propto \mathrm{e}^{\mathrm{i}\omega' t}$

$$\omega' = -\frac{\delta}{2} \pm \omega_R$$

where $\delta = \omega - \omega_0$. Since $c_b(0) = 0$

$$\begin{aligned}
c_b =& c_{b0}\mathrm{e}^{-\mathrm{i}\delta t/2}\sin\omega_R t\\
c_a =& \frac{2\mathrm{i}}{\Omega^*}\mathrm{e}^{\mathrm{i}\delta t}\frac{\mathrm{d}c_b}{\mathrm{d}t}\\
=& \frac{2\mathrm{i}c_{b0}}{\Omega^*}\mathrm{e}^{\mathrm{i}\delta t/2}\left(-\mathrm{i}\frac{\delta}{2}\sin\omega_R t + \omega_R\cos\omega_R t\right)
\end{aligned}$$

Since $c_a(0) = 1$

$$\begin{aligned}
c_b =& -\mathrm{i}\frac{\Omega^*}{2\omega_R}\mathrm{e}^{-\mathrm{i}\delta t/2}\sin\omega_R t\\
c_a =& \mathrm{e}^{\mathrm{i}\delta t/2}\left(\cos\omega_R t - \mathrm{i}\frac{\delta}{2\omega_R}\sin\omega_R t\right)
\end{aligned}$$

(e)

Transtion probability

$$\begin{aligned}
P_{a\to b} =& \left|c_b\right|^2\\
=& \frac{\left|\Omega\right|^2}{4\omega_R^2}\sin^2\omega_R t
\end{aligned}$$

Since $|\Omega| \leqslant 2\omega_R$ and $\sin^2\omega_R t \leqslant 1$, $P_{a\to b} \leqslant 1$.

$$\begin{aligned}
\left|c_a\right|^2 + \left|c_b\right|^2 =& \frac{\left|\Omega\right|^2}{4\omega_R^2}\sin^2\omega_R t + \cos^2\omega_R t + \frac{\delta^2}{4\omega_R^2}\sin^2\omega_R t\\
=& \sin^2\omega_R t + \cos^2\omega_R t\\
=& 1
\end{aligned}$$

(f)

For $\omega_R t \ll 1$ (meaning of small)

$$\begin{aligned}
P_{a\to b} \approx& \frac{\left|\Omega\right|^2}{4\omega_R^2}\omega_R^2 t^2\\
=& \frac{\left|\Omega\right|^2 t^2}{4}
\end{aligned}$$

(g)

The system comes back to the original state after $\frac{2\pi}{\omega_R}$.

5.

(a)

Initial state,

$$|i\rangle = |b;0\rangle$$

Final state,

$$|f\rangle = |a;\vec{k},s\rangle$$

where $\vec{k}$ and s are the wave vector and polarization of the photon. Matrix element

$$\begin{aligned}
\langle f|H_{E2}|i\rangle =& -q\langle f|\left(\vec{r}\cdot\vec{E}_0\right)\left(\vec{k}\cdot\vec{r}\right)|i\rangle\\
=& \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}q\langle a;\vec{k},s|(\vec{r}\cdot\vec{n})a^\dagger_{k,s}\left(\vec{k}\cdot\vec{r}\right)|b;0\rangle\\
=& \mathrm{i}q\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\langle a|(\vec{r}\cdot\vec{n})\left(\vec{k}\cdot\vec{r}\right)|b\rangle\\
R_{b\to a} =& \frac{\hbar\omega_k q^2}{2\varepsilon_0 V}\left|\langle a|(\vec{r}\cdot\vec{n})\left(\vec{k}\cdot\vec{r}\right)|b\rangle\right|^2
\end{aligned}$$

Note that the coefficient still depend on the normalization volumn V which should disappear after taking the sum over a finite range of $\vec{k}$.

(b)