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Physics 251b Final

1.

(a)

Communator of each component

$$\begin{aligned}
[L_i + g_0 S_i, J_j] =&[L_i + g_0 S_i, L_j + S_j]\\
=&[L_i, L_j] + g_0[S_i, S_j]\\
=&\mathrm{i}\hbar\varepsilon_{ijk}(L_k + g_0 S_k)\\
\left[L_i + g_0 S_i, \hat{n}\cdot\vec{J}\right] =&\mathrm{i}\hbar\varepsilon_{ijk}n_j(L_k + g_0 S_k)\\
=&\mathrm{i}\hbar\left(\hat{n}\times\left(\vec{L} + g_0\vec{S}\right)\right)_i\\
\left[\vec{L} + g_0\vec{S}, \hat{n}\cdot\vec{J}\right] =&\mathrm{i}\hbar\varepsilon_{ijk}n_j(L_k + g_0 S_k)\\
=&\mathrm{i}\hbar\hat{n}\times\left(\vec{L} + g_0\vec{S}\right)
\end{aligned}$$

Therefore for any $\vec{n}$

$$\begin{aligned}
&\mathrm{i}\hbar\hat{n}\times\langle 0|\vec{L} + g_0\vec{S}|0\rangle\\
=&\langle 0|\left[L_i + g_0 S_i, \hat{n}\cdot\vec{J}\right]|0\rangle\\
=&\langle 0|[L_i + g_0 S_i, 0]|0\rangle\\
=&0\\
&\langle 0|\vec{L} + g_0\vec{S}|0\rangle\\
=&0
\end{aligned}$$

This is a special case of the Wigner-Eckart Theorem because the $|0\rangle$ state is spherical symmetric. The physical origin of the factor g_0 is the low energy limit of the Dirac equation of electron (and QED corrections on top of it).

(b)

2.

3.

(a)

Radial component of $\vec{j}$

$$\begin{aligned}
j_r =&\frac{\hbar}{2m\mathrm{i}}\left(\psi^*\frac{\partial}{\partial r}\psi - \psi\frac{\partial}{\partial r}\psi^*\right)\\
=&\frac{\hbar}{m}\Im\left(\psi^*\frac{\partial}{\partial r}\psi\right)
\end{aligned}$$

The part terms that is due to interference (for $\psi = \psi_1 + \psi_2$)

$$
\begin{aligned}
j'_r =& j_r - j_{r1} - j_{r2} \\
=& \frac{\hbar}{m}\Im\left(\psi^* \frac{\partial}{\partial r}\psi\right) - \frac{\hbar}{m}\Im\left(\psi_1^* \frac{\partial}{\partial r}\psi_1\right) - \frac{\hbar}{m}\Im\left(\psi_2^* \frac{\partial}{\partial r}\psi_2\right) \\
=& \frac{\hbar}{m}\Im\left(\psi_1^* \frac{\partial}{\partial r}\psi_2\right) + \frac{\hbar}{m}\Im\left(\psi_2^* \frac{\partial}{\partial r}\psi_1\right)
\end{aligned}
$$

Scattering wave function

$$
\psi = \mathrm{e}^{\mathrm{i}kr\cos\theta} + f\frac{\mathrm{e}^{\mathrm{i}kr}}{r}
$$

current density

$$
\begin{aligned}
j'_r =& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r} + f^*\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right) \\
=& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{r\mathrm{i}k - 1}{r^2}\mathrm{e}^{\mathrm{i}kr} + f^*\mathrm{i}k\cos\theta\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)
\end{aligned}
$$

Ignoring r^{-2} term for large r

$$
\begin{aligned}
j'_r \approx& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\mathrm{e}^{\mathrm{i}kr} + \mathrm{i}f^*\cos\theta\mathrm{e}^{-\mathrm{i}kr}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right) \\
=& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta - 1)}f^*\cos\theta + \mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f =& \int_a^b f\mathrm{d}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda} \\
=& \left.\frac{\mathrm{e}^{\mathrm{i}\lambda x}f}{\mathrm{i}\lambda}\right|_a^b - \int_a^b \mathrm{d}xf'\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}
\end{aligned}
$$

Using the same integral by part, we can show that the second term is $O\left(\frac{1}{\lambda^2}\right)$. Therefore,

$$
\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f = \frac{\mathrm{e}^{\mathrm{i}\lambda b}f(b) - \mathrm{e}^{\mathrm{i}\lambda a}f(a)}{\mathrm{i}\lambda} + O\left(\frac{1}{\lambda^2}\right)
$$

(c)

Total interference current

$$
\begin{aligned}
J =& r^2\int \mathrm{d}\Omega\frac{\hbar k}{m}\frac{1}{r}\Im\Big(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\\
=&\frac{\hbar kr}{m}\Re\Big(\int\mathrm{d}\theta\int\mathrm{d}\phi\sin\theta\Big(\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\Big)\\
=&\frac{2\pi\hbar kr}{m}\Re\Big(\int_{-1}^{1}\mathrm{d}\cos\theta\Big(\mathrm{e}^{\mathrm{i}kr(\cos\theta-1)}f^*\cos\theta+\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\Big)\Big)\\
\approx&\frac{2\pi\hbar kr}{m}\Re\Big(\frac{f^*(0)+\mathrm{e}^{-2\mathrm{i}kr}f^*(\pi)}{\mathrm{i}kr}-\frac{f(0)-\mathrm{e}^{2\mathrm{i}kr}f(\pi)}{\mathrm{i}kr}\Big)\\
=&\frac{2\pi\hbar}{m}\Im\big(f^*(0)+\mathrm{e}^{-2\mathrm{i}kr}f^*(\pi)-f(0)+\mathrm{e}^{2\mathrm{i}kr}f(\pi)\big)\\
=&-\frac{4\pi\hbar}{m}\Im(f(0))
\end{aligned}
$$

(d)

The total current of the incident wave is zero (since it's a plain wave). The total current of the scatterend wave.

$$
\begin{aligned}
J_2 =& r^2\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\Big(\psi_r^*\frac{\partial}{\partial r}\psi_2\Big)\\
=& r^2\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\Big(f^*f\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\Big)\\
=&\int\mathrm{d}\Omega\frac{\hbar}{m}\Im\Big(f^*f\frac{r\mathrm{i}k-1}{r}\Big)
\end{aligned}
$$

Ignore real term in the integral

$$
\begin{aligned}
=&\frac{\hbar k}{m}\int\mathrm{d}\Omega f^*f\\
=&\frac{\hbar k}{m}\sigma_{tot}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 =& \frac{\hbar k}{m}\sigma_{tot}-\frac{4\pi\hbar}{m}\Im(f(0))\\
\sigma_{tot} =& \frac{4\pi}{k}\Im(f(0))
\end{aligned}
$$

4.

(a)

In first Born approximation (Use V_r to represent $\partial_r V$)

$$\begin{aligned}
f_\sigma =& -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r'\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}'}V(r',\sigma)\chi_\sigma\mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{r}'}\\
=& -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r'\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}'}\left(-(1+\mathrm{i}\xi)V(r')+\frac{c}{r'}V_r(r')\vec{\sigma}\cdot\frac{\vec{r}'\times\vec{p}}{\hbar}\right)\mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{r}'}\chi_\sigma\\
=& -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r'\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}'}\left(-(1+\mathrm{i}\xi)V(r')-\frac{c}{r'}V_r(r')\vec{\sigma}\cdot\left(\vec{r}'\times\vec{k}'\right)\right)\mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{r}'}\chi_\sigma\\
=&\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r'\mathrm{e}^{\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}'}(1+\mathrm{i}\xi)V(r')\chi_\sigma+\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3r'\mathrm{e}^{\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}'}\frac{c}{r'}V_r(r')\vec{\sigma}\cdot\left(\vec{r}'\times\vec{k}'\right)\chi_\sigma
\end{aligned}$$

Let $\vec{q}=\vec{k}-\vec{k}'$, $q=2k\sin\dfrac{\theta}{2}$

$$\begin{aligned}
f_\sigma =&\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r'r'^2\mathrm{e}^{\mathrm{i}qr'\cos\theta'}(1+\mathrm{i}\xi)V(r')\chi_\sigma\\
&+\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r'r'^2\mathrm{e}^{\mathrm{i}qr'\cos\theta'}cV_r(r')\vec{\sigma}\cdot\left(\hat{r}'\times\vec{k}'\right)\chi_\sigma
\end{aligned}$$

Due to the rotational symmetry around $\vec{q}$

$$\begin{aligned}
f_\sigma =&\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r'r'^2\mathrm{e}^{\mathrm{i}qr'\cos\theta'}(1+\mathrm{i}\xi)V(r')\chi_\sigma\\
&+\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}\Omega\mathrm{d}r'r'^2\cos\theta'\mathrm{e}^{\mathrm{i}qr'\cos\theta'}cV_r(r')\vec{\sigma}\cdot\left(\hat{q}\times\vec{k}'\right)\chi_\sigma\\
=&\frac{m}{\hbar^2}\int \mathrm{d}r'r'^2(1+\mathrm{i}\xi)V(r')\chi_\sigma\int_{-1}^{1}\mathrm{d}\cos\theta'\mathrm{e}^{\mathrm{i}qr'\cos\theta'}\\
&+\frac{m}{\hbar^2}\int \mathrm{d}r'r'^2cV_r(r')\vec{\sigma}\cdot\left(\hat{q}\times\vec{k}'\right)\chi_\sigma\int_{-1}^{1}\mathrm{d}\cos\theta'\cos\theta'\mathrm{e}^{\mathrm{i}qr'\cos\theta'}\\
=&\frac{m}{\hbar^2}\int \mathrm{d}r'r'^2(1+\mathrm{i}\xi)V(r')\chi_\sigma\frac{\mathrm{e}^{\mathrm{i}qr'}-\mathrm{e}^{-\mathrm{i}qr'}}{\mathrm{i}qr'}\\
&+\frac{m}{\hbar^2}\int \mathrm{d}r'r'^2cV_r(r')\vec{\sigma}\cdot\left(\hat{q}\times\vec{k}'\right)\chi_\sigma\left(\mathrm{e}^{\mathrm{i}qr'}\left(\frac{1}{\mathrm{i}qr'}+\frac{1}{q^2r'^2}\right)+\mathrm{e}^{-\mathrm{i}qr'}\left(\frac{1}{\mathrm{i}qr'}-\frac{1}{q^2r'^2}\right)\right)\\
=&\frac{2m}{\hbar^2q}\int \mathrm{d}r'r'(1+\mathrm{i}\xi)V(r')\sin qr'\chi_\sigma\\
&+\frac{2m\mathrm{i}}{\hbar^2q^2}\int \mathrm{d}r'r'cV_r(r')\vec{\sigma}\cdot\left(\vec{q}\times\vec{k}'\right)\chi_\sigma\left(\frac{\sin qr'}{qr'}-\cos qr'\right)
\end{aligned}$$

Substitute in $\hat{n}$

$$\begin{aligned}
=&\frac{2m}{\hbar^2q}\int \mathrm{d}r'r'(1+\mathrm{i}\xi)V(r')\sin qr'\chi_\sigma\\
&+\vec{\sigma}\cdot\hat{n}\frac{2\mathrm{i}mk^2c}{\hbar^2q^2}\sin\theta\int \mathrm{d}r'r'V_r(r')\left(\frac{\sin qr'}{qr'}-\cos qr'\right)\chi_\sigma
\end{aligned}$$

Therefore,

$$\begin{aligned}
A(\theta) =& \frac{2m}{\hbar^2 q}\int \mathrm{d}r' r'(1+\mathrm{i}\xi)V(r')\sin qr' \\
B(\theta) =& \frac{2\mathrm{i}mk^2c}{\hbar^2 q^2}\sin\theta\int \mathrm{d}r' r' V_r(r')\left(\frac{\sin qr'}{qr'} - \cos qr'\right)
\end{aligned}$$

(b)

5.

(a)

(b)

(c)