1.

$$|1,1;2,2\rangle = |1,1\rangle_1|1,1\rangle_2$$
$$|1,1;2,-2\rangle = |1,-1\rangle_1|1,-1\rangle_2$$
$$|1,1;2,1\rangle = \frac{J_-}{2\hbar}|1,1;2,2\rangle$$
$$= \frac{L_{1-} + L_{2-}}{2\hbar}|1,1\rangle_1|1,1\rangle_2$$
$$= \frac{|1,0\rangle_1|1,1\rangle_2 + |1,1\rangle_1|1,0\rangle_2}{\sqrt{2}}$$
$$|1,1;2,-1\rangle = \frac{J_+}{2\hbar}|1,1;2,2\rangle$$
$$= \frac{L_{1+} + L_{2+}}{2\hbar}|1,-1\rangle_1|1,-1\rangle_2$$
$$= \frac{|1,0\rangle_1|1,-1\rangle_2 + |1,-1\rangle_1|1,0\rangle_2}{\sqrt{2}}$$
$$|1,1;2,0\rangle = \frac{J_-}{\sqrt{6}\hbar}|1,1;2,1\rangle$$
$$= \frac{L_{1-} + L_{2-}}{\sqrt{6}\hbar}\frac{|1,0\rangle_1|1,1\rangle_2 + |1,1\rangle_1|1,0\rangle_2}{\sqrt{2}}$$
$$= \frac{|1,-1\rangle_1|1,1\rangle_2 + 2|1,0\rangle_1|1,0\rangle_2 + |1,1\rangle_1|1,-1\rangle_2}{\sqrt{6}}$$

From orthogonality

$$|1,1;1,1\rangle = \frac{|1,0\rangle_1|1,1\rangle_2 - |1,1\rangle_1|1,0\rangle_2}{\sqrt{2}}$$
$$|1,1;1,-1\rangle = \frac{|1,0\rangle_1|1,-1\rangle_2 - |1,-1\rangle_1|1,0\rangle_2}{\sqrt{2}}$$
$$|1,1;1,0\rangle = \frac{J_-}{\sqrt{2}\hbar}|1,1;1,1\rangle$$
$$= \frac{L_{1-} + L_{2-}}{\sqrt{2}\hbar}\frac{|1,0\rangle_1|1,1\rangle_2 - |1,1\rangle_1|1,0\rangle_2}{\sqrt{2}}$$
$$= \frac{|1,-1\rangle_1|1,1\rangle_2 - |1,1\rangle_1|1,-1\rangle_2}{\sqrt{2}}$$

From orthogonality

$$|1,1;0,0\rangle = \frac{|1,-1\rangle_1|1,1\rangle_2 - |1,0\rangle_1|1,0\rangle_2 + |1,1\rangle_1|1,-1\rangle_2}{\sqrt{3}}$$

2.

(a)

Since $J_z = L_z + S_z$ and $[L_z, S_z] = 0$, the J_z eigenstates are sum of L_z and S_z eigenstates that has the same sum of m_l and m_s. (The only variable on the RHS are m_l and m_s among which m_s can only be $\pm\frac{1}{2}$ so the constraint above limit the decomposition to the given form.)

(b)

Hamiltonian

$$
\begin{aligned}
H =& \frac{L^2}{2ma^2} + V_0 + \frac{e^2 \vec{L} \cdot \vec{S}}{2mc^2 a^3} \\
=& \frac{L^2}{2ma^2} + V_0 + \frac{e^2}{4mc^2 a^3}(J^2 - L^2 - S^2) \\
=& \frac{l(l+1)}{2ma^2} + V_0 + \frac{e^2}{4mc^2 a^3}\left(j(j+1) - l(l+1) - \frac{3}{4}\right)
\end{aligned}
$$

When $l = 0$ there's only one manifold instead of two.

3.

$$
\begin{aligned}
e^{\lambda A} B e^{-\lambda A} =& \left(\sum_{m=0}^{\infty} \frac{\lambda^m A^m}{m!}\right) B \left(\sum_{n=0}^{\infty} \frac{\lambda^n (-A)^n}{n!}\right) \\
=& \sum_{m,n=0}^{\infty} \frac{\lambda^m A^m}{m!} B \frac{\lambda^n (-A)^n}{n!} \\
=& \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{\lambda^m A^{m-n} B (-A)^n}{(m-n)!n!} \\
=& \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \sum_{n=0}^{m} \frac{m! A^{m-n} B (-A)^n}{(m-n)!n!}
\end{aligned}
$$

So now we just need to show that

$$
\left[A^{(m)}, B\right] = \sum_{n=0}^{m} \frac{m! A^{m-n} B (-A)^n}{(m-n)!n!}
$$

where $\left[A^{(0)}, B\right] \equiv B$, $\left[A^{(n)}, B\right] \equiv \left[A, \left[A^{(n-1)}, B\right]\right]$. For $m = 0$, this is trivially true. If it is true for $m-1$,

$$
\begin{aligned}
\left[A^{(m)}, B\right] =& \left[A, \sum_{n=0}^{m-1} \frac{(m-1)! A^{m-n-1} B (-A)^n}{(m-n-1)!n!}\right] \\
=& \sum_{n=0}^{m-1} \frac{(m-1)! A^{m-n} B (-A)^n}{(m-n-1)!n!} + \sum_{n=1}^{m} \frac{(m-1)! A^{m-n} B (-A)^n}{(m-n)!(n-1)!}
\end{aligned}
$$

Extend the summation taking advantage of $(-1)! = \infty$

$$
\begin{aligned}
\left[A^{(m)}, B\right] =& \sum_{n=0}^{m} \frac{(m-1)! A^{m-n} B (-A)^n}{(m-n-1)!n!} + \frac{(m-1)! A^{m-n} B (-A)^n}{(m-n)!(n-1)!} \\
=& \sum_{n=0}^{m} \frac{(m)! A^{m-n} B (-A)^n}{(m-n)!n!} \left(\frac{m-n}{m} + \frac{n}{m}\right) \\
=& \sum_{n=0}^{m} \frac{(m)! A^{m-n} B (-A)^n}{(m-n)!n!}
\end{aligned}
$$

Therefore, the equation is true for all $m \geqslant 0$.
If $[A, B] = \gamma B$, $\left[A^{(n)}, B\right] = \gamma^n B$

$$e^{\lambda A} B e^{-\lambda A} = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[A^{(m)}, B\right]$$
$$= \sum_{m=0}^{\infty} \frac{\lambda^m \gamma^m}{m!} B$$
$$= e^{\lambda \gamma} B$$

4.

Since $[\lambda, A] = 0$ and $[A, A] = 0$ (i.e. everything commutes),

$$\frac{\mathrm{d} e^{\lambda A}}{\mathrm{d}\lambda} = A e^{\lambda A}$$

We have

$$\frac{\mathrm{d} G}{\mathrm{d}\lambda} = \frac{\mathrm{d} e^{\lambda A} e^{\lambda B}}{\mathrm{d}\lambda}$$
$$= e^{\lambda A} \frac{\mathrm{d} e^{\lambda B}}{\mathrm{d}\lambda} + \frac{\mathrm{d} e^{\lambda A}}{\mathrm{d}\lambda} e^{\lambda B}$$
$$= e^{\lambda A} B e^{\lambda B} + A e^{\lambda A} e^{\lambda B}$$
$$= e^{\lambda A} B e^{-\lambda A} e^{\lambda A} e^{\lambda B} + A e^{\lambda A} e^{\lambda B}$$
$$= (A + e^{\lambda A} B e^{-\lambda A}) G$$
$$= \left(A + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[A^{(m)}, B\right]\right) G$$

Using this,

$$\frac{\mathrm{d} G}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d} e^{\lambda B^\dagger} e^{\lambda A^\dagger}}{\mathrm{d}\lambda}\right)^\dagger$$
$$= \left(\left(B^\dagger + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[B^{\dagger(m)}, A^\dagger\right]\right) G^\dagger\right)^\dagger$$
$$= G \left(B + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left[A, B^{(m)}\right]\right)$$

If $C \equiv [A, B]$ commutes with both A and B (and therefore $A + B$),

$$\frac{\mathrm{d} G}{\mathrm{d}\lambda} = (A + B + \lambda C) G$$
$$G = e^{A + B + \lambda^2 C/2}$$
$$e^A e^B = e^{A + B + C/2}$$

5.

(a)

Eigenvalue λ

$$
\begin{aligned}
0 &= (h - \lambda)^2 - |g|^2 \\
h - \lambda &= \pm |g| \\
\lambda &= h \pm |g|
\end{aligned}
$$

Corresponding eigen vectors are $\dfrac{1}{\sqrt{2}}\left(1, \pm\dfrac{g}{|g|}\right)$

(b)

Initial state

$$
|\psi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)
$$

At time t,

$$
\begin{aligned}
|\psi_t\rangle &= \frac{1}{\sqrt{2}}\left(\exp\left(-\mathrm{i}\frac{h + |g|}{\hbar}t\right)|+\rangle + \exp\left(-\mathrm{i}\frac{h - |g|}{\hbar}t\right)|-\rangle\right) \\
&= \frac{\mathrm{e}^{-\mathrm{i}ht/\hbar}}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}|g|t/\hbar}|+\rangle + \mathrm{e}^{\mathrm{i}|g|t/\hbar}|-\rangle\right) \\
&= \frac{\mathrm{e}^{-\mathrm{i}ht/\hbar}}{2}\left(\mathrm{e}^{-\mathrm{i}|g|t/\hbar}\left(|1\rangle + \frac{g}{|g|}|2\rangle\right) + \mathrm{e}^{\mathrm{i}|g|t/\hbar}\left(|1\rangle - \frac{g}{|g|}|2\rangle\right)\right) \\
&= \mathrm{e}^{-\mathrm{i}ht/\hbar}\left(\cos\left(\frac{|g|t}{\hbar}\right)|1\rangle - \mathrm{i}\frac{g}{|g|}\sin\left(\frac{|g|t}{\hbar}\right)|2\rangle\right)
\end{aligned}
$$

6.

(a)

(b)