

1.

2.

(a)

Since $J_z = L_z + S_z$ and $[L_z, S_z] = 0$, the J_z eigenstates are sum of L_z and S_z eigenstates that has the same sum of m_l and m_s . (The only variable on the RHS are m_l and m_s among which m_s can only be $\pm \frac{1}{2}$ so the constraint above limit the decomposition to the given form.)

(b)

Hamiltonian

$$\begin{aligned} H &= \frac{L^2}{2ma^2} + V_0 + \frac{e^2 \vec{L} \cdot \vec{S}}{2mc^2 a^3} \\ &= \frac{L^2}{2ma^2} + V_0 + \frac{e^2}{4mc^2 a^3} (J^2 - L^2 - S^2) \\ &= \frac{l(l+1)}{2ma^2} + V_0 + \frac{e^2}{4mc^2 a^3} \left(j(j+1) - l(l+1) - \frac{3}{4} \right) \end{aligned}$$

When $l = 0$ there's only one manifold instead of two.

3.

$$\begin{aligned} e^{\lambda A} B e^{-\lambda A} &= \left(\sum_{m=0}^{\infty} \frac{\lambda^m A^m}{m!} \right) B \left(\sum_{n=0}^{\infty} \frac{\lambda^n (-A)^n}{n!} \right) \\ &= \sum_{m,n=0}^{\infty} \frac{\lambda^m A^m}{m!} B \frac{\lambda^n (-A)^n}{n!} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^m \frac{\lambda^m A^{m-n} B (-A)^n}{(m-n)! n!} \\ &= \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \sum_{n=0}^m \frac{m! A^{m-n} B (-A)^n}{(m-n)! n!} \end{aligned}$$

So now we just need to show that

$$[A^{(m)}, B] = \sum_{n=0}^m \frac{m! A^{m-n} B (-A)^n}{(m-n)! n!}$$

where $[A^{(0)}, B] \equiv B$, $[A^{(n)}, B] \equiv [A, [A^{(n-1)}, B]]$. For $m = 0$, this is trivially true. If it is true for $m - 1$,

$$\begin{aligned} [A^{(m)}, B] &= \left[A, \sum_{n=0}^{m-1} \frac{(m-1)! A^{m-n-1} B (-A)^n}{(m-n-1)! n!} \right] \\ &= \sum_{n=0}^{m-1} \frac{(m-1)! A^{m-n} B (-A)^n}{(m-n-1)! n!} + \sum_{n=1}^m \frac{(m-1)! A^{m-n} B (-A)^n}{(m-n)! (n-1)!} \end{aligned}$$

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Extend the summation taking advantage of $(-1)! = \infty$

$$\begin{aligned} [A^{(m)}, B] &= \sum_{n=0}^m \frac{(m-1)!A^{m-n}B(-A)^n}{(m-n-1)!n!} + \frac{(m-1)!A^{m-n}B(-A)^n}{(m-n)!(n-1)!} \\ &= \sum_{n=0}^m \frac{(m)!A^{m-n}B(-A)^n}{(m-n)!n!} \left(\frac{m-n}{m} + \frac{n}{m} \right) \\ &= \sum_{n=0}^m \frac{(m)!A^{m-n}B(-A)^n}{(m-n)!n!} \end{aligned}$$

Therefore, the equation is true for all $m \geq 0$.

If $[A, B] = \gamma B$, $[A^{(n)}, B] = \gamma^n B$

$$\begin{aligned} e^{\lambda A} B e^{-\lambda A} &= \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} [A^{(m)}, B] \\ &= \sum_{m=0}^{\infty} \frac{\lambda^m \gamma^m}{m!} B \\ &= e^{\lambda \gamma} B \end{aligned}$$

4.

5.

(a)

Eigenvalue λ

$$\begin{aligned} 0 &= (h - \lambda)^2 - |g|^2 \\ h - \lambda &= \pm |g| \\ \lambda &= h \pm |g| \end{aligned}$$

Corresponding eigen vectors are $\frac{1}{\sqrt{2}} \left(1, \pm \frac{g}{|g|} \right)$

(b)

Initial state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

At time t ,

$$\begin{aligned} |\psi_t\rangle &= \frac{1}{\sqrt{2}} \left(\exp\left(-i\frac{h+|g|}{\hbar}t\right)|+\rangle + \exp\left(-i\frac{h-|g|}{\hbar}t\right)|-\rangle \right) \\ &= \frac{e^{-iht/\hbar}}{\sqrt{2}} \left(e^{-i|g|t/\hbar}|+\rangle + e^{i|g|t/\hbar}|-\rangle \right) \\ &= \frac{e^{-iht/\hbar}}{2} \left(e^{-i|g|t/\hbar} \left(|1\rangle + \frac{g}{|g|}|2\rangle \right) + e^{i|g|t/\hbar} \left(|1\rangle - \frac{g}{|g|}|2\rangle \right) \right) \\ &= e^{-iht/\hbar} \left(\cos\left(\frac{|g|t}{\hbar}\right)|1\rangle - i\frac{g}{|g|} \sin\left(\frac{|g|t}{\hbar}\right)|2\rangle \right) \end{aligned}$$

6.

(a)

(b)