

1.

(a)

Commutator of each component

$$\begin{aligned}
 [L_i + g_0 S_i, J_j] &= [L_i + g_0 S_i, L_j + S_j] \\
 &= [L_i, L_j] + g_0 [S_i, S_j] \\
 &= i\hbar \varepsilon_{ijk} (L_k + g_0 S_k) \\
 [L_i + g_0 S_i, \hat{n} \cdot \vec{J}] &= i\hbar \varepsilon_{ijk} n_j (L_k + g_0 S_k) \\
 &= i\hbar \left(\hat{n} \times (\vec{L} + g_0 \vec{S}) \right)_i \\
 [\vec{L} + g_0 \vec{S}, \hat{n} \cdot \vec{J}] &= i\hbar \varepsilon_{ijk} n_j (L_k + g_0 S_k) \\
 &= i\hbar \hat{n} \times (\vec{L} + g_0 \vec{S})
 \end{aligned}$$

Therefore for any \vec{n}

$$\begin{aligned}
 &i\hbar \hat{n} \times \langle 0 | \vec{L} + g_0 \vec{S} | 0 \rangle \\
 &= \langle 0 | [L_i + g_0 S_i, \hat{n} \cdot \vec{J}] | 0 \rangle \\
 &= \langle 0 | [L_i + g_0 S_i, 0] | 0 \rangle \\
 &= 0 \\
 &\langle 0 | \vec{L} + g_0 \vec{S} | 0 \rangle \\
 &= 0
 \end{aligned}$$

This is a special case of the Wigner-Eckart Theorem because the $|0\rangle$ state is spherical symmetric. The physical origin of the factor g_0 is the low energy limit of the Dirac equation of electron (and QED corrections on top of it).

(b)

2.

3.

(a)

Radial component of \vec{j}

$$\begin{aligned}
 j_r &= \frac{\hbar}{2mi} \left(\psi^* \frac{\partial}{\partial r} \psi - \psi \frac{\partial}{\partial r} \psi^* \right) \\
 &= \frac{\hbar}{m} \Im \left(\psi^* \frac{\partial}{\partial r} \psi \right)
 \end{aligned}$$

The part terms that is due to interference (for $\psi = \psi_1 + \psi_2$)

$$\begin{aligned} j'_r &= j_r - j_{r1} - j_{r2} \\ &= \frac{\hbar}{m} \Im \left(\psi^* \frac{\partial}{\partial r} \psi \right) - \frac{\hbar}{m} \Im \left(\psi_1^* \frac{\partial}{\partial r} \psi_1 \right) - \frac{\hbar}{m} \Im \left(\psi_2^* \frac{\partial}{\partial r} \psi_2 \right) \\ &= \frac{\hbar}{m} \Im \left(\psi_1^* \frac{\partial}{\partial r} \psi_2 \right) + \frac{\hbar}{m} \Im \left(\psi_2^* \frac{\partial}{\partial r} \psi_1 \right) \end{aligned}$$

Scattering wave function

$$\psi = e^{ikr \cos \theta} + f \frac{e^{ikr}}{r}$$

current density

$$\begin{aligned} j'_r &= \frac{\hbar}{m} \Im \left(f e^{-ikr \cos \theta} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} + f^* \frac{e^{-ikr}}{r} \frac{\partial}{\partial r} e^{ikr \cos \theta} \right) \\ &= \frac{\hbar}{m} \Im \left(f e^{-ikr \cos \theta} \frac{rik - 1}{r^2} e^{ikr} + f^* ik \cos \theta \frac{e^{-ikr}}{r} e^{ikr \cos \theta} \right) \end{aligned}$$

Ignoring r^{-2} term for large r

$$\begin{aligned} j'_r &\approx \frac{\hbar k}{m} \frac{1}{r} \Im (i f e^{-ikr \cos \theta} e^{ikr} + i f^* \cos \theta e^{-ikr} e^{ikr \cos \theta}) \\ &= \frac{\hbar k}{m} \frac{1}{r} \Im (i e^{ikr(\cos \theta - 1)} f^* \cos \theta + i e^{ikr(1 - \cos \theta)} f) \end{aligned}$$

(b)

(c)

(d)

4.

(a)

(b)

5.

(a)

(b)

(c)