

1.

(a)

Since $i(\lambda a^\dagger - \lambda^* a)$ is Hermitian, $S_\lambda \equiv \exp(\lambda a^\dagger - \lambda^* a)$ is unitary and $|\lambda\rangle \equiv S_\lambda|0\rangle$ is normalized. Since $[a, a^\dagger] = 1$ commutes with both a and a^\dagger

$$\begin{aligned} |\lambda\rangle &= \exp(\lambda a^\dagger - \lambda^* a)|0\rangle \\ &= \exp(\lambda a^\dagger) \exp(-\lambda^* a) \exp\left(-\frac{1}{2}[\lambda a^\dagger, -\lambda^* a]\right)|0\rangle \\ &= \exp(\lambda a^\dagger) \exp(-\lambda^* a) \exp\left(-\frac{|\lambda|^2}{2}\right)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp(\lambda a^\dagger)|0\rangle \end{aligned}$$

$$\begin{aligned} a|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right) a \exp(\lambda a^\dagger)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) (\exp(\lambda a^\dagger)a + [a, \exp(\lambda a^\dagger)])|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) [a, a^\dagger] \lambda \exp(\lambda a^\dagger)|0\rangle \\ &= \lambda|\lambda\rangle \end{aligned}$$

(b)

$$x = z_0(a + a^\dagger), \quad p = i\frac{\hbar}{2z_0}(a^\dagger - a)$$

$$\begin{aligned} \langle x \rangle &= z_0 \langle a + a^\dagger \rangle \\ &= z_0(\lambda + \lambda^*) \\ \langle x^2 \rangle &= z_0^2 \langle (a + a^\dagger)^2 \rangle \\ &= z_0^2 \langle a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a \rangle \\ &= z_0^2 \langle a^2 + a^{\dagger 2} + 2a^\dagger a + 1 \rangle \\ &= z_0^2 \langle (\lambda + \lambda^*)^2 + 1 \rangle \\ \langle \Delta x^2 \rangle &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= z_0^2 \end{aligned}$$

$$\begin{aligned}
 \langle p \rangle &= i \frac{\hbar}{2z_0} \langle a^\dagger - a \rangle \\
 &= i \frac{\hbar}{2z_0} (\lambda^* - \lambda) \\
 \langle p^2 \rangle &= - \frac{\hbar^2}{4z_0^2} \langle (a^\dagger - a)^2 \rangle \\
 &= - \frac{\hbar^2}{4z_0^2} \langle a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a \rangle \\
 &= - \frac{\hbar^2}{4z_0^2} \langle a^2 + a^{\dagger 2} - 2a^\dagger a - 1 \rangle \\
 &= - \frac{\hbar^2}{4z_0^2} \langle (\lambda^* - \lambda)^2 - 1 \rangle \\
 \langle \Delta p^2 \rangle &= \langle p^2 \rangle - \langle p \rangle^2 \\
 &= \frac{\hbar^2}{4z_0^2} \\
 \langle \Delta p^2 \rangle \langle \Delta x^2 \rangle &= \frac{\hbar^2}{4} \\
 &= \frac{1}{4} |\langle [x, p] \rangle|^2
 \end{aligned}$$

(c)

$$\begin{aligned}
 |\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp(\lambda a^\dagger) |0\rangle \\
 &= \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\lambda a^\dagger)^n}{n!} |0\rangle \\
 &= \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 P(n) &= \exp^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!} \\
 [n]_{av} &= \sum_{n=0}^{\infty} n P(n) \\
 &= \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!} \\
 &= |\lambda|^2 \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \\
 &= |\lambda|^2 \\
 [E_n]_{av} &= \hbar\omega \left(|\lambda|^2 + \frac{1}{2} \right)
 \end{aligned}$$

(d)

$$\begin{aligned}
 \langle n^2 \rangle &= \sum_{n=0}^{\infty} n^2 P(n) \\
 &= \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} n \frac{|\lambda|^{2n}}{(n-1)!} \\
 &= \exp^{-|\lambda|^2} \left(\sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-2)!} + \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!} \right) \\
 &= \exp^{-|\lambda|^2} \left(|\lambda|^4 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} + |\lambda|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \right) \\
 &= |\lambda|^4 + |\lambda|^2 \\
 \Delta n &= |\lambda| \\
 \Delta E &= \hbar \omega |\lambda| \\
 \frac{\Delta E}{[E_n]_{av}} &= \frac{1}{|\lambda|}
 \end{aligned}$$

so the relative uncertainty goes to 0 at large n limit.

2.

In a homogeneous field B_0 the x magnetization is

$$M_x = M_0 \cos \omega_0 t$$

where $\omega_0 = \frac{2\mu_e B_0}{\hbar}$ is the Larmor frequency.

In a non-homogenous field, assuming the initial local magnetization is position independent

$$M_x = M_0 \int \cos \left(\frac{2\mu_e B t}{\hbar} \right) p(B) dB$$

(a)

$$p(B) = \frac{1}{2a}$$

$$\begin{aligned}
 M_x &= \frac{M_0}{2a} \int_{B_0-a}^{B_0+a} \cos \left(\frac{2\mu_e B t}{\hbar} \right) dB \\
 &= \frac{\hbar M_0}{4\mu_e a t} \sin \left(\frac{2\mu_e B t}{\hbar} \right) \Big|_{B_0-a}^{B_0+a} \\
 &= \frac{\hbar M_0}{4\mu_e a t} \left(\sin \left(\frac{2\mu_e t(B_0 + a)}{\hbar} \right) - \sin \left(\frac{2\mu_e t(B_0 - a)}{\hbar} \right) \right) \\
 &= \frac{\hbar M_0}{2\mu_e a t} \cos \left(\frac{2\mu_e t B_0}{\hbar} \right) \sin \left(\frac{2\mu_e t a}{\hbar} \right)
 \end{aligned}$$

(b)

$$p(B) = \frac{1}{\sqrt{\pi}a} e^{-(B-B_0)^2/a^2}$$

$$\begin{aligned} M_x &= \frac{M_0}{\sqrt{\pi}a} \int \cos\left(\frac{2\mu_e B t}{\hbar}\right) e^{-(B-B_0)^2/a^2} dB \\ &= \frac{M_0}{\sqrt{\pi}a} \Re\left(\int \exp\left(-(B-B_0)^2/a^2 + i\frac{2\mu_e B t}{\hbar}\right) dB\right) \\ &= \frac{M_0}{\sqrt{\pi}} \Re\left(\int \exp\left(-x^2 + i\frac{2\mu_e a t}{\hbar}x + i\frac{2\mu_e B_0 t}{\hbar}\right) dx\right) \\ &= \frac{M_0}{\sqrt{\pi}} \Re\left(\int \exp\left(-x^2 + i\frac{2\mu_e a t}{\hbar}x + \left(\frac{\mu_e a t}{\hbar}\right)^2 - \left(\frac{\mu_e a t}{\hbar}\right)^2 + i\frac{2\mu_e B_0 t}{\hbar}\right) dx\right) \\ &= M_0 \Re\left(\exp\left(-\left(\frac{\mu_e a t}{\hbar}\right)^2 + i\frac{2\mu_e B_0 t}{\hbar}\right)\right) \\ &= M_0 \cos\left(\frac{2\mu_e B_0 t}{\hbar}\right) \exp\left(-\left(\frac{\mu_e a t}{\hbar}\right)^2\right) \end{aligned}$$

(c)

$$p(B) = \frac{1}{\pi a} \frac{1}{1 + (B - B_0)^2/a^2}$$

$$\begin{aligned} M_x &= \frac{M_0}{\pi a} \Re\left(\int \exp\left(i\frac{2\mu_e B t}{\hbar}\right) \frac{1}{1 + (B - B_0)^2/a^2} dB\right) \\ &= \frac{M_0}{\pi a} \Re\left(2\pi i \text{Res}_{B=B_0+ia} \left(\exp\left(i\frac{2\mu_e B t}{\hbar}\right) \frac{1}{1 + (B - B_0)^2/a^2}\right)\right) \\ &= M_0 \cos\left(\frac{2\mu_e B_0 t}{\hbar}\right) \exp\left(-\frac{2\mu_e a t}{\hbar}\right) \end{aligned}$$

(d)

$M_x(t)$ is a damped oscillation with a profile corresponds to the Fourier transformation of the B field distribution.

3.

The probability is

$$\begin{aligned} |\langle u|\chi\rangle|^2 &= |c_1|^2 \\ |\langle u|\chi\rangle|^2 &= \langle u|\chi\rangle \langle \chi|u\rangle \\ &= \langle u|\rho|u\rangle \\ &= \text{Tr}(\langle u|\rho|u\rangle) \\ &= \text{Tr}(\rho|u\rangle \langle u|) \end{aligned}$$

4.

(a)

$$\begin{aligned}
 |\psi\rangle &= c_a e^{-iE_a t/\hbar} |a\rangle + c_b e^{-iE_b t/\hbar} |b\rangle \\
 i\hbar \frac{d}{dt} |\psi\rangle &= i\hbar \frac{dc_a}{dt} e^{-iE_a t/\hbar} |a\rangle + i\hbar \frac{dc_b}{dt} e^{-iE_b t/\hbar} |b\rangle + E_a c_a e^{-iE_a t/\hbar} |a\rangle + E_b c_b e^{-iE_b t/\hbar} |b\rangle \\
 H|\psi\rangle &= c_a e^{-iE_a t/\hbar} E_a |a\rangle + c_b e^{-iE_b t/\hbar} E_b |b\rangle + c_a e^{-iE_a t/\hbar} H' |a\rangle + c_b e^{-iE_b t/\hbar} H' |b\rangle \\
 &= c_a \frac{dc_a}{dt} |a\rangle + i\hbar \frac{dc_b}{dt} e^{-i\omega_0 t} |b\rangle \\
 &= c_a H' |a\rangle + c_b e^{-i\omega_0 t} H' |b\rangle
 \end{aligned}$$

When $H'_{aa} = H'_{bb} = 0$ we can remove all operators from the equation

$$\begin{aligned}
 &i\hbar \frac{dc_a}{dt} |a\rangle + i\hbar \frac{dc_b}{dt} e^{-i\omega_0 t} |b\rangle \\
 &= c_a H'_{ba} |b\rangle + c_b e^{-i\omega_0 t} H'_{ab} |a\rangle
 \end{aligned}$$

Left multiply by $\langle a|$ or $\langle b|$,

$$\begin{aligned}
 \frac{dc_a}{dt} &= -\frac{i}{\hbar} c_b e^{-i\omega_0 t} H'_{ab} \\
 \frac{dc_b}{dt} &= -\frac{i}{\hbar} c_a e^{i\omega_0 t} H'_{ba}
 \end{aligned}$$

(b)

Zeroth order

$$\begin{aligned}
 c_a^{(0)} &= 1 \\
 c_b^{(0)} &= 0
 \end{aligned}$$

First order

$$\begin{aligned}
 \frac{dc_a^{(1)}}{dt} &= -\frac{i}{\hbar} c_b^{(0)} e^{-i\omega_0 t} H'_{ab} \\
 &= 0 \\
 \frac{dc_b^{(1)}}{dt} &= -\frac{i}{\hbar} c_a^{(0)} e^{i\omega_0 t} H'_{ba} \\
 &= -\frac{i}{\hbar} e^{i\omega_0 t} H'_{ba} \\
 c_a^{(1)} &= 0 \\
 c_b^{(1)} &= \int_0^t -\frac{iV_0^*}{2\hbar} \left(e^{i(\omega+\omega_0)t'} + e^{i(\omega_0-\omega)t'} \right) dt' \\
 &= -\frac{V_0^*}{2\hbar} \left(\frac{e^{i(\omega+\omega_0)t} - 1}{\omega + \omega_0} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega} \right)
 \end{aligned}$$

When $|\omega_0 - \omega| \ll \omega_0 + \omega$

$$\begin{aligned} c_b^{(1)} &\approx -\frac{V_0^*}{2\hbar} \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \\ |c_b^{(1)}|^2 &\approx \frac{|V_0|^2}{4\hbar^2} \frac{|e^{i(\omega_0 - \omega)t/2} - e^{-i(\omega_0 - \omega)t/2}|^2}{(\omega_0 - \omega)^2} \\ &= \frac{|V_0|^2}{\hbar^2} \frac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2} \end{aligned}$$

(c)

Right. I guess there isn't anything to solve for this one?

(d)

$$\begin{aligned} \frac{dc_a}{dt} &= -\frac{iV_0}{2\hbar} c_b e^{-i(\omega_0 - \omega)t} \\ &= -\frac{i\Omega}{2} c_b e^{-i(\omega_0 - \omega)t} \\ \frac{dc_b}{dt} &= -\frac{iV_0^*}{2\hbar} c_a e^{i(\omega_0 - \omega)t} \\ &= -\frac{i\Omega^*}{2} c_a e^{i(\omega_0 - \omega)t} \end{aligned}$$

where $\Omega = \frac{V_0}{\hbar}$

$$\begin{aligned} \frac{d^2 c_b}{dt^2} &= -\frac{i\Omega^*}{2} \frac{dc_a}{dt} e^{i(\omega_0 - \omega)t} + \frac{\Omega^*(\omega_0 - \omega)}{2} c_a e^{i(\omega_0 - \omega)t} \\ &= -\frac{|\Omega|^2}{4} c_b + i(\omega_0 - \omega) \frac{dc_b}{dt} \\ 0 &= \frac{d^2 c_b}{dt^2} - i(\omega_0 - \omega) \frac{dc_b}{dt} + \frac{|\Omega|^2}{4} c_b \end{aligned}$$

For $c_b \propto e^{i\omega' t}$

$$\omega' = -\frac{\delta}{2} \pm \omega_R$$

where $\delta = \omega - \omega_0$. Since $c_b(0) = 0$

$$\begin{aligned} c_b &= c_{b0} e^{-i\delta t/2} \sin \omega_R t \\ c_a &= \frac{2i}{\Omega^*} e^{i\delta t} \frac{dc_b}{dt} \\ &= \frac{2ic_{b0}}{\Omega^*} e^{i\delta t/2} \left(-i\frac{\delta}{2} \sin \omega_R t + \omega_R \cos \omega_R t \right) \end{aligned}$$

Since $c_a(0) = 1$

$$\begin{aligned} c_b &= -i \frac{\Omega^*}{2\omega_R} e^{-i\delta t/2} \sin \omega_R t \\ c_a &= e^{i\delta t/2} \left(\cos \omega_R t - i \frac{\delta}{2\omega_R} \sin \omega_R t \right) \end{aligned}$$

(e)

Transition probability

$$\begin{aligned} P_{a \rightarrow b} &= |c_b|^2 \\ &= \frac{|\Omega|^2}{4\omega_R^2} \sin^2 \omega_R t \end{aligned}$$

Since $|\Omega| \leq 2\omega_R$ and $\sin^2 \omega_R t \leq 1$, $P_{a \rightarrow b} \leq 1$.

$$\begin{aligned} |c_a|^2 + |c_b|^2 &= \frac{|\Omega|^2}{4\omega_R^2} \sin^2 \omega_R t + \cos^2 \omega_R t + \frac{\delta^2}{4\omega_R^2} \sin^2 \omega_R t \\ &= \sin^2 \omega_R t + \cos^2 \omega_R t \\ &= 1 \end{aligned}$$

(f)

For $\omega_R t \ll 1$ (meaning of small)

$$\begin{aligned} P_{a \rightarrow b} &\approx \frac{|\Omega|^2}{4\omega_R^2} \omega_R^2 t^2 \\ &= \frac{|\Omega|^2 t^2}{4} \end{aligned}$$

(g)

The system comes back to the original state after $\frac{2\pi}{\omega_R}$.

5.

(a)

(b)