1. Gauge invariance and the Lorentz force

(a)

Schroedinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi + q\phi\psi$$

Complex conjugate

$$\begin{aligned}
-\mathrm{i}\hbar\frac{\partial\psi^*}{\partial t} =& -\frac{\hbar^2}{2m}\left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi^* + q\phi\psi^*\\
\mathrm{i}\hbar\psi^*\frac{\partial\psi}{\partial t} =& -\psi^*\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi + q\phi\psi^*\psi\\
-\mathrm{i}\hbar\psi\frac{\partial\psi^*}{\partial t} =& -\frac{\hbar^2}{2m}\psi\left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi^* + q\phi\psi^*\psi\\
\mathrm{i}\hbar\frac{\partial\psi^*\psi}{\partial t} =& -\psi^*\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi + \frac{\hbar^2}{2m}\psi\left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi^*\\
=& -\frac{\hbar^2}{2m}\left(\nabla\left(\psi^*\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi\right) - \left|\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi\right|^2\right)\\
&+ \frac{\hbar^2}{2m}\left(\nabla\left(\psi\left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi^*\right) - \left|\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi\right|^2\right)\\
=&\frac{\hbar^2}{2m}\nabla\left(\psi\left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi^* - \psi^*\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi\right)\\
\frac{\partial\rho}{\partial t} =&\frac{\hbar}{2m\mathrm{i}}\nabla\left(\psi\left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi^* - \psi^*\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)\psi\right)\\
=& -\nabla\cdot\vec{j}\\
0 =&\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j}
\end{aligned}$$

(b)

Under time reversal transformation

$$\begin{aligned}
&-\mathrm{i}\hbar\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\psi^* - q\phi\psi^*\\
=&\frac{\hbar^2}{2m}\left(\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2 - \left(\nabla + \frac{\mathrm{i}q}{\hbar c}\vec{A}\right)^2\right)\psi^*\\
\neq&0
\end{aligned}$$

(c)

Derivatives,

$$\begin{aligned}
\frac{\partial L}{\partial\dot{r}_i} =&m\dot{r}_i + \frac{q}{c}A_i\\
\frac{\partial L}{\partial r_i} =& -q\partial_i\phi + \frac{q}{c}\dot{r}_j\partial_iA_j
\end{aligned}$$

Euler-Lagrange equation

$$
\begin{aligned}
0 =& m\ddot{r}_i + \frac{q}{c}\frac{\mathrm{d}A_i}{\mathrm{d}t} + q\partial_i\phi - \frac{q}{c}\dot{r}_j\partial_iA_j\\
m\ddot{r}_i =& -\frac{q}{c}\frac{\mathrm{d}A_i}{\mathrm{d}t} - q\partial_i\phi + \frac{q}{c}\dot{r}_j\partial_iA_j\\
=& -q\partial_i\phi - \frac{q}{c}\partial_tA_i - \frac{q}{c}\dot{r}_j\partial_jA_i + \frac{q}{c}\dot{r}_j\partial_iA_j\\
=& qE_i + \frac{q}{c}(\delta_{ln}\delta_{im} - \delta_{lm}\delta_{in})\dot{r}_l\partial_mA_n\\
=& qE_i + \frac{q}{c}\varepsilon_{ilj}\dot{r}_l\varepsilon_{jmn}\partial_mA_n\\
=& qE_i + \frac{q}{c}\varepsilon_{ilj}\dot{r}_lB_j\\
=& qE_i + \frac{q}{c}\left(\vec{v}\times\vec{B}\right)_i
\end{aligned}
$$

2.

(a)

(b)

(c)

3.

(a)

(b)

(c)

4.

(a)

(b)

(c)

(d)