

1.

(a)

Commutator of each component

$$\begin{aligned}
 [L_i + g_0 S_i, J_j] &= [L_i + g_0 S_i, L_j + S_j] \\
 &= [L_i, L_j] + g_0 [S_i, S_j] \\
 &= i\hbar \varepsilon_{ijk} (L_k + g_0 S_k) \\
 [L_i + g_0 S_i, \hat{n} \cdot \vec{J}] &= i\hbar \varepsilon_{ijk} n_j (L_k + g_0 S_k) \\
 &= i\hbar \left(\hat{n} \times (\vec{L} + g_0 \vec{S}) \right)_i \\
 [\vec{L} + g_0 \vec{S}, \hat{n} \cdot \vec{J}] &= i\hbar \varepsilon_{ijk} n_j (L_k + g_0 S_k) \\
 &= i\hbar \hat{n} \times (\vec{L} + g_0 \vec{S})
 \end{aligned}$$

Therefore for any \vec{n}

$$\begin{aligned}
 &i\hbar \hat{n} \times \langle 0 | \vec{L} + g_0 \vec{S} | 0 \rangle \\
 &= \langle 0 | [L_i + g_0 S_i, \hat{n} \cdot \vec{J}] | 0 \rangle \\
 &= \langle 0 | [L_i + g_0 S_i, 0] | 0 \rangle \\
 &= 0 \\
 &\langle 0 | \vec{L} + g_0 \vec{S} | 0 \rangle \\
 &= 0
 \end{aligned}$$

This is a special case of the Wigner-Eckart Theorem because the $|0\rangle$ state is spherical symmetric. The physical origin of the factor g_0 is the low energy limit of the Dirac equation of electron (and QED corrections on top of it).

(b)

2.

3.

(a)

Radial component of \vec{j}

$$\begin{aligned}
 j_r &= \frac{\hbar}{2mi} \left(\psi^* \frac{\partial}{\partial r} \psi - \psi \frac{\partial}{\partial r} \psi^* \right) \\
 &= \frac{\hbar}{m} \Im \left(\psi^* \frac{\partial}{\partial r} \psi \right)
 \end{aligned}$$

The part terms that is due to interference (for $\psi = \psi_1 + \psi_2$)

$$\begin{aligned} j'_r &= j_r - j_{r1} - j_{r2} \\ &= \frac{\hbar}{m} \Im \left(\psi^* \frac{\partial}{\partial r} \psi \right) - \frac{\hbar}{m} \Im \left(\psi_1^* \frac{\partial}{\partial r} \psi_1 \right) - \frac{\hbar}{m} \Im \left(\psi_2^* \frac{\partial}{\partial r} \psi_2 \right) \\ &= \frac{\hbar}{m} \Im \left(\psi_1^* \frac{\partial}{\partial r} \psi_2 \right) + \frac{\hbar}{m} \Im \left(\psi_2^* \frac{\partial}{\partial r} \psi_1 \right) \end{aligned}$$

Scattering wave function

$$\psi = e^{ikr \cos \theta} + f \frac{e^{ikr}}{r}$$

current density

$$\begin{aligned} j'_r &= \frac{\hbar}{m} \Im \left(f e^{-ikr \cos \theta} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} + f^* \frac{e^{-ikr}}{r} \frac{\partial}{\partial r} e^{ikr \cos \theta} \right) \\ &= \frac{\hbar}{m} \Im \left(f e^{-ikr \cos \theta} \frac{rik - 1}{r^2} e^{ikr} + f^* ik \cos \theta \frac{e^{-ikr}}{r} e^{ikr \cos \theta} \right) \end{aligned}$$

Ignoring r^{-2} term for large r

$$\begin{aligned} j'_r &\approx \frac{\hbar k}{m} \frac{1}{r} \Im (i f e^{-ikr \cos \theta} e^{ikr} + i f^* \cos \theta e^{-ikr} e^{ikr \cos \theta}) \\ &= \frac{\hbar k}{m} \frac{1}{r} \Im (i e^{ikr(\cos \theta - 1)} f^* \cos \theta + i e^{ikr(1 - \cos \theta)} f) \end{aligned}$$

(b)

$$\begin{aligned} \int_a^b dx e^{i\lambda x} f &= \int_a^b f d \frac{e^{i\lambda x}}{i\lambda} \\ &= \frac{e^{i\lambda x} f}{i\lambda} \Big|_a^b - \int_a^b dx f' \frac{e^{i\lambda x}}{i\lambda} \end{aligned}$$

Using the same integral by part, we can show that the second term is $O\left(\frac{1}{\lambda^2}\right)$. Therefore,

$$\int_a^b dx e^{i\lambda x} f = \frac{e^{i\lambda b} f(b) - e^{i\lambda a} f(a)}{i\lambda} + O\left(\frac{1}{\lambda^2}\right)$$

(c)

Total interference current

$$\begin{aligned}
J &= r^2 \int d\Omega \frac{\hbar k}{m} \frac{1}{r} \Im \left(i e^{i k r (\cos \theta - 1)} f^* \cos \theta + i e^{i k r (1 - \cos \theta)} f \right) \\
&= \frac{\hbar k r}{m} \Re \left(\int d\theta \int d\phi \sin \theta \left(e^{i k r (\cos \theta - 1)} f^* \cos \theta + e^{i k r (1 - \cos \theta)} f \right) \right) \\
&= \frac{2\pi \hbar k r}{m} \Re \left(\int_{-1}^1 d \cos \theta \left(e^{i k r (\cos \theta - 1)} f^* \cos \theta + e^{i k r (1 - \cos \theta)} f \right) \right) \\
&\approx \frac{2\pi \hbar k r}{m} \Re \left(\frac{f^*(0) + e^{-2i k r} f^*(\pi)}{i k r} - \frac{f(0) - e^{2i k r} f(\pi)}{i k r} \right) \\
&= \frac{2\pi \hbar}{m} \Im (f^*(0) + e^{-2i k r} f^*(\pi) - f(0) + e^{2i k r} f(\pi)) \\
&= -\frac{4\pi \hbar}{m} \Im (f(0))
\end{aligned}$$

(d)

The total current of the incident wave is zero (since it's a plane wave). The total current of the scattered wave.

$$\begin{aligned}
J_2 &= r^2 \int d\Omega \frac{\hbar}{m} \Im \left(\psi_r^* \frac{\partial}{\partial r} \psi_2 \right) \\
&= r^2 \int d\Omega \frac{\hbar}{m} \Im \left(f^* f \frac{e^{-i k r}}{r} \frac{\partial}{\partial r} \frac{e^{i k r}}{r} \right) \\
&= \int d\Omega \frac{\hbar}{m} \Im \left(f^* f \frac{r i k - 1}{r} \right)
\end{aligned}$$

Ignore real term in the integral

$$\begin{aligned}
&= \frac{\hbar k}{m} \int d\Omega f^* f \\
&= \frac{\hbar k}{m} \sigma_{tot}
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &= \frac{\hbar k}{m} \sigma_{tot} - \frac{4\pi \hbar}{m} \Im (f(0)) \\
\sigma_{tot} &= \frac{4\pi}{k} \Im (f(0))
\end{aligned}$$

4.

(a)

In first Born approximation (Use V_r to represent $\partial_r V$)

$$\begin{aligned}
 f_\sigma &= -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3 r' e^{i\vec{k} \cdot \vec{r}'} V(r', \sigma) \chi_\sigma e^{-i\vec{k}' \cdot \vec{r}'} \\
 &= -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3 r' e^{i\vec{k} \cdot \vec{r}'} \left(-(1 + i\xi) V(r') + \frac{c}{r'} V_r(r') \vec{\sigma} \cdot \frac{\vec{r}' \times \vec{p}}{\hbar} \right) e^{-i\vec{k}' \cdot \vec{r}'} \chi_\sigma \\
 &= -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3 r' e^{i\vec{k} \cdot \vec{r}'} \left(-(1 + i\xi) V(r') - \frac{c}{r'} V_r(r') \vec{\sigma} \cdot (\vec{r}' \times \vec{k}') \right) e^{-i\vec{k}' \cdot \vec{r}'} \chi_\sigma \\
 &= \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3 r' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} (1 + i\xi) V(r') \chi_\sigma + \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3 r' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} \frac{c}{r'} V_r(r') \vec{\sigma} \cdot (\vec{r}' \times \vec{k}') \chi_\sigma
 \end{aligned}$$

Let $\vec{q} = \vec{k} - \vec{k}'$, $q = 2k \sin \frac{\theta}{2}$

$$\begin{aligned}
 f_\sigma &= \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d\Omega dr' r'^2 e^{iqr' \cos \theta'} (1 + i\xi) V(r') \chi_\sigma \\
 &\quad + \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d\Omega dr' r'^2 e^{iqr' \cos \theta'} c V_r(r') \vec{\sigma} \cdot (\hat{r}' \times \vec{k}') \chi_\sigma
 \end{aligned}$$

Due to the rotational symmetry around \vec{q}

$$\begin{aligned}
 f_\sigma &= \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d\Omega dr' r'^2 e^{iqr' \cos \theta'} (1 + i\xi) V(r') \chi_\sigma \\
 &\quad + \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d\Omega dr' r'^2 \cos \theta' e^{iqr' \cos \theta'} c V_r(r') \vec{\sigma} \cdot (\hat{q} \times \vec{k}') \chi_\sigma \\
 &= \frac{m}{\hbar^2} \int dr' r'^2 (1 + i\xi) V(r') \chi_\sigma \int_{-1}^1 d \cos \theta' e^{iqr' \cos \theta'} \\
 &\quad + \frac{m}{\hbar^2} \int dr' r'^2 c V_r(r') \vec{\sigma} \cdot (\hat{q} \times \vec{k}') \chi_\sigma \int_{-1}^1 d \cos \theta' \cos \theta' e^{iqr' \cos \theta'} \\
 &= \frac{m}{\hbar^2} \int dr' r'^2 (1 + i\xi) V(r') \chi_\sigma \frac{e^{iqr'} - e^{-iqr'}}{iqr'} \\
 &\quad + \frac{m}{\hbar^2} \int dr' r'^2 c V_r(r') \vec{\sigma} \cdot (\hat{q} \times \vec{k}') \chi_\sigma \left(e^{iqr'} \left(\frac{1}{iqr'} + \frac{1}{q^2 r'^2} \right) + e^{-iqr'} \left(\frac{1}{iqr'} - \frac{1}{q^2 r'^2} \right) \right) \\
 &= \frac{2m}{\hbar^2 q} \int dr' r' (1 + i\xi) V(r') \sin qr' \chi_\sigma \\
 &\quad + \frac{2m i}{\hbar^2 q^2} \int dr' r' c V_r(r') \vec{\sigma} \cdot (\vec{q} \times \vec{k}') \chi_\sigma \left(\frac{\sin qr'}{qr'} - \cos qr' \right)
 \end{aligned}$$

Substitute in \hat{n}

$$\begin{aligned}
 &= \frac{2m}{\hbar^2 q} \int dr' r' (1 + i\xi) V(r') \sin qr' \chi_\sigma \\
 &\quad + \vec{\sigma} \cdot \hat{n} \frac{2imk^2 c}{\hbar^2 q^2} \sin \theta \int dr' r' V_r(r') \left(\frac{\sin qr'}{qr'} - \cos qr' \right) \chi_\sigma
 \end{aligned}$$

Therefore,

$$A(\theta) = \frac{2m}{\hbar^2 q} \int dr' r' (1 + i\xi) V(r') \sin qr'$$

$$B(\theta) = \frac{2imk^2 c}{\hbar^2 q^2} \sin \theta \int dr' r' V_r(r') \left(\frac{\sin qr'}{qr'} - \cos qr' \right)$$

(b)

For V being a step function,

$$A(\theta) = \frac{2mV_0(1 + i\xi)}{\hbar^2 q} \int_0^R dr' r' \sin qr'$$

$$= \frac{2mV_0(1 + i\xi)}{\hbar^2 q^3} (\sin qR - qR \cos qR)$$

$$B(\theta) = -\frac{2imk^2 c}{\hbar^2 q^2} \sin \theta \int dr' r' V_0 \delta(r' - R) \left(\frac{\sin qr'}{qr'} - \cos qr' \right)$$

$$= -\frac{2imV_0 k^2 c}{\hbar^2 q^3} \sin \theta (\sin qR - qR \cos qR)$$

Average over input and sum over output to get total scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \text{Tr}((A^* + \vec{\sigma}^* \cdot \hat{n} B^*)(A + \vec{\sigma} \cdot \hat{n} B))$$

Since $\text{Tr}(\sigma_n) = 0$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} (|A|^2 + |B|^2)$$

$$= 2 \left| \frac{mV_0(1 + i\xi)}{\hbar^2 q^3} (\sin qR - qR \cos qR) \right|^2 + 2 \left| \frac{mV_0 k^2 c}{\hbar^2 q^3} \sin \theta (\sin qR - qR \cos qR) \right|^2$$

$$= 2 \left(\frac{mV_0}{\hbar^2 q^3} (\sin qR - qR \cos qR) \right)^2 (1 + \xi^2 + k^4 c^2 \sin^2 \theta)$$

5.

(a)

Time dependent Schrödinger equation

$$i \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

Expansion,

$$|\psi\rangle = \sum_n c_n e^{i\theta_n} |\psi_n\rangle$$

where

$$\begin{aligned}
 H|\psi_n\rangle &= E_n|\psi_n\rangle \\
 0 &= i\frac{d}{dt} \sum_n c_n e^{i\theta_n} |\psi_n\rangle - H \sum_n c_n e^{i\theta_n} |\psi_n\rangle \\
 0 &= i \sum_n c_n \frac{d e^{i\theta_n}}{dt} |\psi_n\rangle + i \sum_n \frac{dc_n}{dt} e^{i\theta_n} |\psi_n\rangle + i \sum_n c_n e^{i\theta_n} \frac{d}{dt} |\psi_n\rangle - \sum_n c_n E_n e^{i\theta_n} |\psi_n\rangle \\
 0 &= \langle\psi_l| \sum_n \frac{dc_n}{dt} e^{i\theta_n} |\psi_n\rangle + \langle\psi_l| \sum_n c_n e^{i\theta_n} \frac{d}{dt} |\psi_n\rangle \\
 \frac{dc_l}{dt} &= - \sum_n c_n e^{i(\theta_n - \theta_l)} \langle\psi_l| \frac{d}{dt} \psi_n\rangle
 \end{aligned}$$

(b)

(c)