1.

(a)

Since $\mathrm{i}\left(\lambda a^\dagger - \lambda^* a\right)$ is Hermitian, $S_\lambda \equiv \exp\left(\lambda a^\dagger - \lambda^* a\right)$ is unitary and $|\lambda\rangle \equiv S_\lambda|0\rangle$ is normalized. Since $[a, a^\dagger] = 1$ commutes with both a and $a^\dagger$

$$\begin{aligned}
|\lambda\rangle =& \exp\left(\lambda a^\dagger - \lambda^* a\right)|0\rangle\\
=& \exp\left(\lambda a^\dagger\right)\exp\left(-\lambda^* a\right)\exp\left(-\frac{1}{2}\left[\lambda a^\dagger, -\lambda^* a\right]\right)|0\rangle\\
=& \exp\left(\lambda a^\dagger\right)\exp\left(-\lambda^* a\right)\exp\left(-\frac{|\lambda|^2}{2}\right)|0\rangle\\
=& \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(\lambda a^\dagger\right)|0\rangle
\end{aligned}$$

$$\begin{aligned}
a|\lambda\rangle =& \exp\left(-\frac{|\lambda|^2}{2}\right) a \exp\left(\lambda a^\dagger\right)|0\rangle\\
=& \exp\left(-\frac{|\lambda|^2}{2}\right)\left(\exp\left(\lambda a^\dagger\right) a + \left[a, \exp\left(\lambda a^\dagger\right)\right]\right)|0\rangle\\
=& \exp\left(-\frac{|\lambda|^2}{2}\right)\left[a, a^\dagger\right]\lambda \exp\left(\lambda a^\dagger\right)|0\rangle\\
=& \lambda|\lambda\rangle
\end{aligned}$$

(b)

$x = z_0\left(a + a^\dagger\right)$, $p = \mathrm{i}\dfrac{\hbar}{2z_0}\left(a^\dagger - a\right)$

$$\begin{aligned}
\langle x\rangle =& z_0\left\langle a + a^\dagger\right\rangle\\
=& z_0(\lambda + \lambda^*)\\
\left\langle x^2\right\rangle =& z_0^2\left\langle\left(a + a^\dagger\right)^2\right\rangle\\
=& z_0^2\left\langle a^2 + {a^\dagger}^2 + aa^\dagger + a^\dagger a\right\rangle\\
=& z_0^2\left\langle a^2 + {a^\dagger}^2 + 2a^\dagger a + 1\right\rangle\\
=& z_0^2\left\langle(\lambda + \lambda^*)^2 + 1\right\rangle\\
\left\langle \Delta x^2\right\rangle =& \left\langle x^2\right\rangle - \langle x\rangle^2\\
=& z_0^2
\end{aligned}$$

$$
\begin{aligned}
\langle p\rangle =& \mathrm{i}\frac{\hbar}{2z_0}\left\langle a^\dagger - a\right\rangle\\
=& \mathrm{i}\frac{\hbar}{2z_0}(\lambda^* - \lambda)\\
\left\langle p^2\right\rangle =& -\frac{\hbar^2}{4z_0^2}\left\langle\left(a^\dagger - a\right)^2\right\rangle\\
=& -\frac{\hbar^2}{4z_0^2}\left\langle a^2 + {a^\dagger}^2 - aa^\dagger - a^\dagger a\right\rangle\\
=& -\frac{\hbar^2}{4z_0^2}\left\langle a^2 + {a^\dagger}^2 - 2a^\dagger a - 1\right\rangle\\
=& -\frac{\hbar^2}{4z_0^2}\left\langle(\lambda^* - \lambda)^2 - 1\right\rangle\\
\left\langle\Delta p^2\right\rangle =& \left\langle p^2\right\rangle - \langle p\rangle^2\\
=& \frac{\hbar^2}{4z_0^2}\\
\left\langle\Delta p^2\right\rangle\left\langle\Delta x^2\right\rangle =& \frac{\hbar^2}{4}\\
=& \frac{1}{4}\left|\langle[x, p]\rangle\right|^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(\lambda a^\dagger\right)|0\rangle\\
&= \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\lambda a^\dagger\right)^n}{n!}|0\rangle\\
&= \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P(n) &= \exp^{-|\lambda|^2}\frac{|\lambda|^{2n}}{n!}\\
[n]_{av} &= \sum_{n=0}^{\infty} nP(n)\\
&= \exp^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n-1)!}\\
&= |\lambda|^2\exp^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}\\
&= |\lambda|^2\\
[E_n]_{av} &= \hbar\omega\left(|\lambda|^2 + \frac{1}{2}\right)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\left\langle n^2\right\rangle =&\sum_{n=0}^{\infty} n^2 P(n)\\
=&\exp^{-|\lambda|^2}\sum_{n=0}^{\infty} n\frac{|\lambda|^{2n}}{(n-1)!}\\
=&\exp^{-|\lambda|^2}\left(\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n-2)!}+\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n-1)!}\right)\\
=&\exp^{-|\lambda|^2}\left(|\lambda|^4\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}+|\lambda|^2\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}\right)\\
=&|\lambda|^4+|\lambda|^2\\
\Delta n =&|\lambda|\\
\Delta E =&\hbar\omega|\lambda|\\
\frac{\Delta E}{[E_n]_{av}} =&\frac{1}{|\lambda|}
\end{aligned}
$$

so the relative uncertainty goes to 0 at large n limit.

2.

In a homogeneous field B_0 the x magnetization is

$$M_x = M_0\cos\omega_0 t$$

where $\omega_0 = \dfrac{2\mu_e B_0}{\hbar}$ is the Larmor frequency.
In a non-homogenous field, assuming the initial local magnetization is position independent

$$M_x = M_0\int\cos\left(\frac{2\mu_e Bt}{\hbar}\right)p(B)\mathrm{d}B$$

(a)

$p(B) = \dfrac{1}{2a}$

$$
\begin{aligned}
M_x =&\frac{M_0}{2a}\int_{B_0-a}^{B_0+a}\cos\left(\frac{2\mu_e Bt}{\hbar}\right)\mathrm{d}B\\
=&\frac{\hbar M_0}{4\mu_e at}\sin\left(\frac{2\mu_e Bt}{\hbar}\right)\bigg|_{B_0-a}^{B_0+a}\\
=&\frac{\hbar M_0}{4\mu_e at}\left(\sin\left(\frac{2\mu_e t(B_0+a)}{\hbar}\right)-\sin\left(\frac{2\mu_e t(B_0-a)}{\hbar}\right)\right)\\
=&\frac{\hbar M_0}{2\mu_e at}\cos\left(\frac{2\mu_e tB_0}{\hbar}\right)\sin\left(\frac{2\mu_e ta}{\hbar}\right)
\end{aligned}
$$

(b)

$p(B) = \frac{1}{\sqrt{\pi}a}\mathrm{e}^{-(B-B_0)^2/a^2}$

$$
\begin{aligned}
M_x =& \frac{M_0}{\sqrt{\pi}a}\int\cos\left(\frac{2\mu_e Bt}{\hbar}\right)\mathrm{e}^{-(B-B_0)^2/a^2}\mathrm{d}B\\
=& \frac{M_0}{\sqrt{\pi}a}\Re\left(\int\exp\left(-(B-B_0)^2/a^2+\mathrm{i}\frac{2\mu_e Bt}{\hbar}\right)\mathrm{d}B\right)\\
=& \frac{M_0}{\sqrt{\pi}}\Re\left(\int\exp\left(-x^2+\mathrm{i}\frac{2\mu_e at}{\hbar}x+\mathrm{i}\frac{2\mu_e B_0t}{\hbar}\right)\mathrm{d}x\right)\\
=& \frac{M_0}{\sqrt{\pi}}\Re\left(\int\exp\left(-x^2+\mathrm{i}\frac{2\mu_e at}{\hbar}x+\left(\frac{\mu_e at}{\hbar}\right)^2-\left(\frac{\mu_e at}{\hbar}\right)^2+\mathrm{i}\frac{2\mu_e B_0t}{\hbar}\right)\mathrm{d}x\right)\\
=& M_0\Re\left(\exp\left(-\left(\frac{\mu_e at}{\hbar}\right)^2+\mathrm{i}\frac{2\mu_e B_0t}{\hbar}\right)\right)\\
=& M_0\cos\left(\frac{2\mu_e B_0t}{\hbar}\right)\exp\left(-\left(\frac{\mu_e at}{\hbar}\right)^2\right)
\end{aligned}
$$

(c)

(d)

3.

The probability is

$$
\begin{aligned}
|\langle u|\chi\rangle|^2 =& |c_1|^2\\
|\langle u|\chi\rangle|^2 =& \langle u|\chi\rangle\langle\chi|u\rangle\\
=& \langle u|\rho|u\rangle\\
=& \mathrm{Tr}(\langle u|\rho|u\rangle)\\
=& \mathrm{Tr}(\rho|u\rangle\langle u|)
\end{aligned}
$$

4.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

5.

(a)

(b)