1.

(a)

Communator of each component

$$
\begin{aligned}
[L_i+g_0S_i,J_j]=&[L_i+g_0S_i,L_j+S_j]\\
=&[L_i,L_j]+g_0[S_i,S_j]\\
=&\mathrm{i}\hbar\varepsilon_{ijk}(L_k+g_0S_k)\\
\left[L_i+g_0S_i,\hat{n}\cdot\vec{J}\right]=&\mathrm{i}\hbar\varepsilon_{ijk}n_j(L_k+g_0S_k)\\
=&\mathrm{i}\hbar\left(\hat{n}\times\left(\vec{L}+g_0\vec{S}\right)\right)_i\\
\left[\vec{L}+g_0\vec{S},\hat{n}\cdot\vec{J}\right]=&\mathrm{i}\hbar\varepsilon_{ijk}n_j(L_k+g_0S_k)\\
=&\mathrm{i}\hbar\hat{n}\times\left(\vec{L}+g_0\vec{S}\right)
\end{aligned}
$$

Therefore for any $\vec{n}$

$$
\begin{aligned}
&\mathrm{i}\hbar\hat{n}\times\langle 0|\vec{L}+g_0\vec{S}|0\rangle\\
=&\langle 0|\left[L_i+g_0S_i,\hat{n}\cdot\vec{J}\right]|0\rangle\\
=&\langle 0|[L_i+g_0S_i,0]|0\rangle\\
=&0\\
&\langle 0|\vec{L}+g_0\vec{S}|0\rangle\\
=&0
\end{aligned}
$$

This is a special case of the Wigner-Eckart Theorem because the $|0\rangle$ state is spherical symmetric. The physical origin of the factor g_0 is the low energy limit of the Dirac equation of electron (and QED corrections on top of it).

(b)

2.

3.

(a)

Radial component of $\vec{j}$

$$
\begin{aligned}
j_r=&\frac{\hbar}{2m\mathrm{i}}\left(\psi^*\frac{\partial}{\partial r}\psi-\psi\frac{\partial}{\partial r}\psi^*\right)\\
=&\frac{\hbar}{m}\Im\left(\psi^*\frac{\partial}{\partial r}\psi\right)
\end{aligned}
$$

The part terms that is due to interference (for $\psi = \psi_1 + \psi_2$)

$$\begin{aligned}
j'_r =& j_r - j_{r1} - j_{r2}\\
=& \frac{\hbar}{m}\Im\left(\psi^* \frac{\partial}{\partial r}\psi\right) - \frac{\hbar}{m}\Im\left(\psi_1^* \frac{\partial}{\partial r}\psi_1\right) - \frac{\hbar}{m}\Im\left(\psi_2^* \frac{\partial}{\partial r}\psi_2\right)\\
=& \frac{\hbar}{m}\Im\left(\psi_1^* \frac{\partial}{\partial r}\psi_2\right) + \frac{\hbar}{m}\Im\left(\psi_2^* \frac{\partial}{\partial r}\psi_1\right)
\end{aligned}$$

Scattering wave function

$$\psi = \mathrm{e}^{\mathrm{i}kr\cos\theta} + f\frac{\mathrm{e}^{\mathrm{i}kr}}{r}$$

current density

$$\begin{aligned}
j'_r =& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{\partial}{\partial r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r} + f^*\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\frac{\partial}{\partial r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)\\
=& \frac{\hbar}{m}\Im\left(f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\frac{r\mathrm{i}k - 1}{r^2}\mathrm{e}^{\mathrm{i}kr} + f^*\mathrm{i}k\cos\theta\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)
\end{aligned}$$

Ignoring r^{-2} term for large r

$$\begin{aligned}
j'_r \approx& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}f\mathrm{e}^{-\mathrm{i}kr\cos\theta}\mathrm{e}^{\mathrm{i}kr} + \mathrm{i}f^*\cos\theta\mathrm{e}^{-\mathrm{i}kr}\mathrm{e}^{\mathrm{i}kr\cos\theta}\right)\\
=& \frac{\hbar k}{m}\frac{1}{r}\Im\left(\mathrm{i}\mathrm{e}^{\mathrm{i}kr(\cos\theta - 1)}f^*\cos\theta + \mathrm{i}\mathrm{e}^{\mathrm{i}kr(1-\cos\theta)}f\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f =& \int_a^b f\mathrm{d}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}\\
=& \left.\frac{\mathrm{e}^{\mathrm{i}\lambda x}f}{\mathrm{i}\lambda}\right|_a^b - \int_a^b \mathrm{d}x f'\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{i}\lambda}
\end{aligned}$$

Using the same integral by part, we can show that the second term is $O\left(\frac{1}{\lambda^2}\right)$

$$\int_a^b \mathrm{d}x\mathrm{e}^{\mathrm{i}\lambda x}f = \frac{\mathrm{e}^{\mathrm{i}\lambda b}f(b) - \mathrm{e}^{\mathrm{i}\lambda a}f(a)}{\mathrm{i}\lambda} + O\left(\frac{1}{\lambda^2}\right)$$

(c)

(d)

4.

(a)

(b)

5.

(a)

(b)

(c)