1.

(a)

Represent the operation using 4×4 matrices that shows the mapping between the nodes.

$$T_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}$$

$$T_4 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \qquad T_5 = \begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & 1 \end{pmatrix} \qquad T_6 = \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{pmatrix}$$

$$T_7 = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \qquad T_8 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{pmatrix} \qquad T_9 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$T_{10} = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{pmatrix} \qquad T_{11} = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} \qquad T_{12} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

(b)

$$g_{123} = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

$$g_{234} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

$$g_{234}g_{123} = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \end{pmatrix} = T_4$$

(180° rotation around the the axis connecting the middle of 1-2 and 3-4)

$$g_{123}g_{234} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} = T_9 \neq T_4$$

(c)

See (a)

(d)

$$H = \begin{pmatrix} \varepsilon_0 & -t & -t & -t \\ -t & \varepsilon_0 & -t & -t \\ -t & -t & \varepsilon_0 & -t \\ -t & -t & -t & \varepsilon_0 \end{pmatrix}$$

Eigenvalues are $\varepsilon_0 - 3t$ for eigenvector $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\varepsilon_0 + t$ for eigenvectors, $\left(\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}\right)$, $\left(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}\right)$, $\left(\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}\right)$.

2.

(a)

$$H = \begin{pmatrix} \varepsilon_0 & -t & & & & -t & & -t \\ -t & \varepsilon_0 & -t & & -t & & & \\ & -t & \varepsilon_0 & -t & & & & -t \\ & & -t & \varepsilon_0 & -t & & -t & \\ & -t & & -t & \varepsilon_0 & -t & & \\ -t & & & & -t & \varepsilon_0 & -t & \\ & & & -t & & -t & \varepsilon_0 & -t \\ -t & & -t & & & & -t & \varepsilon_0 \end{pmatrix}$$

(b)

(c)

3.

The constrains is that the angular momentum cannot perfectly point in a certain direction and there will always be some fluctuations. This uncertain comes from,

$$\langle \Delta L_x, \Delta L_y \rangle \geqslant \frac{1}{2i} \langle [L_x, L_y] \rangle$$
$$= \frac{\hbar}{2} \langle L_z \rangle$$
$$= \frac{\hbar^2 m}{2}$$

Which can only be 0 when $m = 0$.

4.

(a)

Define $g(\phi, \phi_0) \equiv e^{-iL_z\phi_0/\hbar} f(\phi)$

$$\frac{\partial g}{\partial \phi_0} = -\frac{iL_z}{\hbar} e^{-iL_z\phi_0/\hbar} f(\phi)$$

$$= -\frac{\partial}{\partial \phi} e^{-iL_z\phi_0/\hbar} f(\phi)$$

$$= -\frac{\partial g}{\partial \phi}$$

$$\mathrm{d}g = \frac{\partial g}{\partial \phi_0} \mathrm{d}\phi_0 + \frac{\partial g}{\partial \phi} \mathrm{d}\phi$$

$$= \frac{\partial g}{\partial \phi} (\mathrm{d}\phi - \mathrm{d}\phi_0)$$

Therefore $g = g(\phi - \phi_0)$ (since it has 0 gradient in this direction). Since $g(\phi) = f(\phi)$ (when $\phi_0 = 0$), $g(\phi, \phi_0) = f(\phi - \phi_0)$ for all ϕ_0.

(b)

Define $\sigma_n \equiv \sigma \cdot \hat{n}$

$$(\sigma \cdot \hat{n})^2 = n_x^2 + n_y^2 + n_z^2$$

$$= 1$$

(using the fact that σ_i's anti-commutes with each other)

$$e^{-i\sigma_n\varphi/2} = \sum_{j=0}^{\infty} \frac{(-i\sigma_n\varphi/2)^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{(-i\sigma_n\varphi/2)^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{(-i\sigma_n\varphi/2)^{2j+1}}{(2j+1)!}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j(\varphi/2)^{2j}}{(2j)!} - i\sigma_n \sum_{j=0}^{\infty} \frac{(-1)^j(\varphi/2)^{2j+1}}{(2j+1)!}$$

$$= \cos\frac{\varphi}{2} - i\sigma_n \sin\frac{\varphi}{2}$$

(c)

$$T_{x180} = e^{-i\sigma_x\pi/2}$$

$$= \cos\frac{\pi}{2} - i\sigma_x \sin\frac{\pi}{2}$$

$$= -i\sigma_x$$

which switches up and down spin with a phase factor. Spining around y-axis gives the same spin flip with a different phase factor.

(d)

(e)

5.

(a)

(b)

(c)