1.

(a)

Since $i(\lambda a^{\dagger} - \lambda^* a)$ is Hermitian, $S_{\lambda} \equiv \exp(\lambda a^{\dagger} - \lambda^* a)$ is unitary and $|\lambda\rangle \equiv S_{\lambda}|0\rangle$ is normalized. Since $[a, a^{\dagger}] = 1$ commutes with both a and a^{\dagger}

$$\begin{split} |\lambda\rangle &= \exp\left(\lambda a^{\dagger} - \lambda^* a\right) |0\rangle \\ &= \exp\left(\lambda a^{\dagger}\right) \exp\left(-\lambda^* a\right) \exp\left(-\frac{1}{2} \left[\lambda a^{\dagger}, -\lambda^* a\right]\right) |0\rangle \\ &= \exp\left(\lambda a^{\dagger}\right) \exp\left(-\lambda^* a\right) \exp\left(-\frac{|\lambda|^2}{2}\right) |0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp\left(\lambda a^{\dagger}\right) |0\rangle \end{split}$$

$$\begin{aligned} a|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right) a \exp\left(\lambda a^{\dagger}\right) |0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \left(\exp\left(\lambda a^{\dagger}\right) a + \left[a, \exp\left(\lambda a^{\dagger}\right)\right]\right) |0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \left[a, a^{\dagger}\right] \lambda \exp\left(\lambda a^{\dagger}\right) |0\rangle \\ &= \lambda |\lambda\rangle \end{aligned}$$

(b)

$$x = z_0(a + a^{\dagger}), p = i\frac{\hbar}{2z_0}(a^{\dagger} - a)$$

$$\langle x \rangle = z_0\langle a + a^{\dagger} \rangle$$

$$= z_0(\lambda + \lambda^*)$$

$$\langle x^2 \rangle = z_0^2 \langle (a + a^{\dagger})^2 \rangle$$

$$= z_0^2 \langle a^2 + a^{\dagger} + aa^{\dagger} + a^{\dagger} a \rangle$$

$$= z_0^2 \langle a^2 + a^{\dagger} + 2a^{\dagger} a + 1 \rangle$$

$$= z_0^2 \langle (\lambda + \lambda^*)^2 + 1 \rangle$$

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

$$= z_0^2 \langle a^2 + a^2 +$$

$$\langle p \rangle = i \frac{\hbar}{2z_0} \langle a^{\dagger} - a \rangle$$

$$= i \frac{\hbar}{2z_0} (\lambda^* - \lambda)$$

$$\langle p^2 \rangle = -\frac{\hbar^2}{4z_0^2} \langle (a^{\dagger} - a)^2 \rangle$$

$$= -\frac{\hbar^2}{4z_0^2} \langle a^2 + a^{\dagger^2} - aa^{\dagger} - a^{\dagger}a \rangle$$

$$= -\frac{\hbar^2}{4z_0^2} \langle a^2 + a^{\dagger^2} - 2a^{\dagger}a - 1 \rangle$$

$$= -\frac{\hbar^2}{4z_0^2} \langle (\lambda^* - \lambda)^2 - 1 \rangle$$

$$\langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$$

$$= \frac{\hbar^2}{4z_0^2}$$

$$\langle \Delta p^2 \rangle \langle \Delta x^2 \rangle = \frac{\hbar^2}{4}$$

$$= \frac{1}{4} |\langle [x, p] \rangle|^2$$

(c)

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right) \exp\left(\lambda a^{\dagger}\right)|0\rangle$$
$$= \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\lambda a^{\dagger}\right)^n}{n!}|0\rangle$$
$$= \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

Therefore,

$$P(n) = \exp^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!}$$

$$[n]_{av} = \sum_{n=0}^{\infty} nP(n)$$

$$= \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!}$$

$$= |\lambda|^2 \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!}$$

$$= |\lambda|^2$$

$$[E_n]_{av} = \hbar\omega \left(|\lambda|^2 + \frac{1}{2}\right)$$

(d)

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 P(n)$$

$$= \exp^{-|\lambda|^2} \sum_{n=0}^{\infty} n \frac{|\lambda|^{2n}}{(n-1)!}$$

$$= \exp^{-|\lambda|^2} \left(\sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-2)!} + \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(n-1)!} \right)$$

$$= \exp^{-|\lambda|^2} \left(|\lambda|^4 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} + |\lambda|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \right)$$

$$= |\lambda|^4 + |\lambda|^2$$

$$\Delta n = |\lambda|$$

$$\Delta E = \hbar \omega |\lambda|$$

$$\Delta E = \hbar \omega |\lambda|$$

$$\frac{\Delta E}{E_n|_{av}} = \frac{1}{|\lambda|}$$

so the relative uncertainty goes to 0 at large n limit.

2.

In a homogeneous field B_0 the x magnetization is

$$M_x = M_0 \cos \omega_0 t$$

where $\omega_0 = \frac{2\mu_e B_0}{\hbar}$ is the Larmor frequency.

In a non-homogenous field, assuming the initial local magnetization is position independent

$$M_x = M_0 \int \cos\left(\frac{2\mu_e Bt}{\hbar}\right) p(B) dB$$

$$p(B) = \frac{1}{2a}$$

$$\begin{split} M_x &= \frac{M_0}{2a} \int_{B_0 - a}^{B_0 + a} \cos\left(\frac{2\mu_e Bt}{\hbar}\right) \mathrm{d}B \\ &= \frac{\hbar M_0}{4\mu_e at} \sin\left(\frac{2\mu_e Bt}{\hbar}\right) \Big|_{B_0 - a}^{B_0 + a} \\ &= \frac{\hbar M_0}{4\mu_e at} \left(\sin\left(\frac{2\mu_e t (B_0 + a)}{\hbar}\right) - \sin\left(\frac{2\mu_e t (B_0 - a)}{\hbar}\right)\right) \\ &= \frac{\hbar M_0}{2\mu_e at} \cos\left(\frac{2\mu_e t B_0}{\hbar}\right) \sin\left(\frac{2\mu_e t a}{\hbar}\right) \end{split}$$

(b)

$$p(B) = \frac{1}{\sqrt{\pi a}} e^{-(B-B_0)^2/a^2}$$

$$M_x = \frac{M_0}{\sqrt{\pi a}} \int \cos\left(\frac{2\mu_e Bt}{\hbar}\right) e^{-(B-B_0)^2/a^2} dB$$

$$= \frac{M_0}{\sqrt{\pi a}} \Re\left(\int \exp\left(-(B-B_0)^2/a^2 + i\frac{2\mu_e Bt}{\hbar}\right) dB\right)$$

$$= \frac{M_0}{\sqrt{\pi}} \Re\left(\int \exp\left(-x^2 + i\frac{2\mu_e at}{\hbar}x + i\frac{2\mu_e B_0 t}{\hbar}\right) dx\right)$$

$$= \frac{M_0}{\sqrt{\pi}} \Re\left(\int \exp\left(-x^2 + i\frac{2\mu_e at}{\hbar}x + \left(\frac{\mu_e at}{\hbar}\right)^2 - \left(\frac{\mu_e at}{\hbar}\right)^2 + i\frac{2\mu_e B_0 t}{\hbar}\right) dx\right)$$

$$= M_0 \Re\left(\exp\left(-\left(\frac{\mu_e at}{\hbar}\right)^2 + i\frac{2\mu_e B_0 t}{\hbar}\right)\right)$$

(c)
$$p(B) = \frac{1}{\pi a} \frac{1}{1 + (B - B_0)^2 / a^2}$$

$$M_x = \frac{M_0}{\pi a} \Re \left(\int \exp\left(i\frac{2\mu_e Bt}{\hbar}\right) \frac{1}{1 + (B - B_0)^2 / a^2} dB \right)$$

$$= \frac{M_0}{\pi a} \Re \left(2\pi i \mathbf{Res}_{B = B_0 + ia} \left(\exp\left(i\frac{2\mu_e Bt}{\hbar}\right) \frac{1}{1 + (B - B_0)^2 / a^2} \right) \right)$$

$$= M_0 \cos\left(\frac{2\mu_e B_0 t}{\hbar}\right) \exp\left(-\frac{2\mu_e at}{\hbar}\right)$$

 $=M_0\cos\left(\frac{2\mu_e B_0 t}{\hbar}\right)\exp\left(-\left(\frac{\mu_e a t}{\hbar}\right)^2\right)$

(d)

 $M_x(t)$ is a damped oscillation with a profile corresponds to the Fourier transformation of the B field distribution.

3.

The probability is

$$|\langle u|\chi\rangle|^2 = |c_1|^2$$

$$|\langle u|\chi\rangle|^2 = \langle u|\chi\rangle\langle\chi|u\rangle$$

$$= \langle u|\rho|u\rangle$$

$$= \operatorname{Tr}(\langle u|\rho|u\rangle)$$

$$= \operatorname{Tr}(\rho|u\rangle\langle u|)$$

4.

$$\begin{split} |\psi\rangle &= c_a \mathrm{e}^{-\mathrm{i}E_at/\hbar}|a\rangle + c_b \mathrm{e}^{-\mathrm{i}E_bt/\hbar}|b\rangle \\ &\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \mathrm{i}\hbar\frac{\mathrm{d}c_a}{\mathrm{d}t} \mathrm{e}^{-\mathrm{i}E_at/\hbar}|a\rangle + \mathrm{i}\hbar\frac{\mathrm{d}c_b}{\mathrm{d}t} \mathrm{e}^{-\mathrm{i}E_bt/\hbar}|b\rangle + E_a c_a \mathrm{e}^{-\mathrm{i}E_at/\hbar}|a\rangle + E_b c_b \mathrm{e}^{-\mathrm{i}E_bt/\hbar}|b\rangle \\ &H|\psi\rangle = c_a \mathrm{e}^{-\mathrm{i}E_at/\hbar}E_a|a\rangle + c_b \mathrm{e}^{-\mathrm{i}E_bt/\hbar}E_b|b\rangle + c_a \mathrm{e}^{-\mathrm{i}E_at/\hbar}H'|a\rangle + c_b \mathrm{e}^{-\mathrm{i}E_bt/\hbar}H'|b\rangle \\ &\mathrm{i}\hbar\frac{\mathrm{d}c_a}{\mathrm{d}t}|a\rangle + \mathrm{i}\hbar\frac{\mathrm{d}c_b}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\omega_0t}|b\rangle \\ = c_aH'|a\rangle + c_b \mathrm{e}^{-\mathrm{i}\omega_0t}H'|b\rangle \end{split}$$

When $H_{aa}^{\prime}=H_{bb}^{\prime}=0$ we can remove all operators from the equation

$$i\hbar \frac{\mathrm{d}c_a}{\mathrm{d}t}|a\rangle + i\hbar \frac{\mathrm{d}c_b}{\mathrm{d}t} \mathrm{e}^{-\mathrm{i}\omega_0 t}|b\rangle$$
$$= c_a H'_{ba}|b\rangle + c_b \mathrm{e}^{-\mathrm{i}\omega_0 t} H'_{ab}|a\rangle$$

Left multiply by |a| or |b|,

$$\frac{\mathrm{d}c_a}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}c_b \mathrm{e}^{-\mathrm{i}\omega_0 t} H'_{ab}$$
$$\frac{\mathrm{d}c_b}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}c_a \mathrm{e}^{\mathrm{i}\omega_0 t} H'_{ba}$$

(b)

Zeroth order

$$c_a^{(0)} = 1$$
 $c_b^{(0)} = 0$

First order

$$\begin{split} \frac{\mathrm{d}c_{a}^{(1)}}{\mathrm{d}t} &= -\frac{\mathrm{i}}{\hbar}c_{b}^{(0)}\mathrm{e}^{-\mathrm{i}\omega_{0}t}H_{ab}'\\ &= 0\\ \\ \frac{\mathrm{d}c_{b}^{(1)}}{\mathrm{d}t} &= -\frac{\mathrm{i}}{\hbar}c_{a}^{(0)}\mathrm{e}^{\mathrm{i}\omega_{0}t}H_{ba}'\\ &= -\frac{\mathrm{i}}{\hbar}\mathrm{e}^{\mathrm{i}\omega_{0}t}H_{ba}'\\ c_{a}^{(1)} &= 0\\ c_{b}^{(1)} &= \int_{0}^{t} -\frac{\mathrm{i}V_{0}^{*}}{2\hbar}\left(\mathrm{e}^{\mathrm{i}(\omega+\omega_{0})t'} + \mathrm{e}^{\mathrm{i}(\omega_{0}-\omega)t'}\right)\mathrm{d}t'\\ &= -\frac{V_{0}^{*}}{2\hbar}\left(\frac{\mathrm{e}^{\mathrm{i}(\omega+\omega_{0})t} - 1}{\omega+\omega_{0}} + \frac{\mathrm{e}^{\mathrm{i}(\omega_{0}-\omega)t} - 1}{\omega_{0}-\omega}\right) \end{split}$$

When
$$|\omega_0 - \omega| \ll \omega_0 + \omega$$

$$\begin{split} c_b^{(1)} &\approx -\frac{V_0^*}{2\hbar} \frac{\mathrm{e}^{\mathrm{i}(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \\ \left| c_b^{(1)} \right|^2 &\approx \frac{|V_0|^2}{4\hbar^2} \frac{\left| \mathrm{e}^{\mathrm{i}(\omega_0 - \omega)t/2} - \mathrm{e}^{-\mathrm{i}(\omega_0 - \omega)t/2} \right|^2}{(\omega_0 - \omega)^2} \\ &= \frac{|V_0|^2}{\hbar^2} \frac{\sin^2\left((\omega_0 - \omega)t/2\right)}{(\omega_0 - \omega)^2} \end{split}$$

(c)

Right. I guess there isn't anything to solve for this one?

(d)

$$\begin{split} \frac{\mathrm{d}c_a}{\mathrm{d}t} &= -\frac{\mathrm{i}V_0}{2\hbar}c_b\mathrm{e}^{-\mathrm{i}(\omega_0 - \omega)t} \\ &= -\frac{\mathrm{i}\Omega}{2}c_b\mathrm{e}^{-\mathrm{i}(\omega_0 - \omega)t} \\ \frac{\mathrm{d}c_b}{\mathrm{d}t} &= -\frac{\mathrm{i}V_0^*}{2\hbar}c_a\mathrm{e}^{\mathrm{i}(\omega_0 - \omega)t} \\ &= -\frac{\mathrm{i}\Omega^*}{2}c_a\mathrm{e}^{\mathrm{i}(\omega_0 - \omega)t} \end{split}$$

where
$$\Omega = \frac{V_0}{\hbar}$$

$$\begin{split} \frac{\mathrm{d}^2 c_b}{\mathrm{d}t^2} &= -\frac{\mathrm{i}\Omega^*}{2} \frac{\mathrm{d}c_a}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}(\omega_0 - \omega)t} + \frac{\Omega^*(\omega_0 - \omega)}{2} c_a \mathrm{e}^{\mathrm{i}(\omega_0 - \omega)t} \\ &= -\frac{|\Omega|^2}{4} c_b + \mathrm{i}(\omega_0 - \omega) \frac{\mathrm{d}c_b}{\mathrm{d}t} \\ 0 &= \frac{\mathrm{d}^2 c_b}{\mathrm{d}t^2} - \mathrm{i}(\omega_0 - \omega) \frac{\mathrm{d}c_b}{\mathrm{d}t} + \frac{|\Omega|^2}{4} c_b \end{split}$$

For $c_b \propto \mathrm{e}^{\mathrm{i}\omega' t}$

$$\omega' = -\frac{\delta}{2} \pm \omega_R$$

where
$$\delta = \omega - \omega_0$$
. Since $c_b(0) = 0$

$$c_b = c_{b0} e^{-i\delta t/2} \sin \omega_R t$$

$$c_a = \frac{2i}{\Omega^*} e^{i\delta t} \frac{dc_b}{dt}$$

$$= \frac{2ic_{b0}}{\Omega^*} e^{i\delta t/2} \left(-i\frac{\delta}{2} \sin \omega_R t + \omega_R \cos \omega_R t \right)$$

Since $c_a(0) = 1$

$$c_b = -i \frac{\Omega^*}{2\omega_R} e^{-i\delta t/2} \sin \omega_R t$$

$$c_a = e^{i\delta t/2} \left(\cos \omega_R t - i \frac{\delta}{2\omega_R} \sin \omega_R t\right)$$

- (e)
- (f)
- (g)
- **5.**
- (a)
- (b)