Light shift and effective B field

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1 Goal

Derive and clarify some effects related to vector and tensor light shifts as well as a few different places they may appear in an experiment. Most, if not all of the discussion will be limited to E1 transitions. I'm not really looking for the most mathematically straight forward derivation, rather trying to see this from different angle for better understanding.

2 Summary of main results[1]

See the linked sections for the quantitative results.

1. Section 3.2.1 checks the Stark shift's dependency on m_F by explicitly compute it using the Clebsch-Gordan coefficients. It confirms that the dependency of the Stark shift on m_F is at most a second order polynomial and it has the expected symmetry when driven with linear or circular polarized light.

2. Section 3.2.2 proves that the second order coupling between to states F and F' is proportional to the Clebsch-Gordan coefficients $\langle F', m'_F|F, k; m_F, p\rangle$ where $k = 0, 1, 2$ denotes scalar, vector and tensor coupling.

3 Vector and tensor light shift

In this section we'll discuss the result of second-order pertubation on dipole transitions. This includes Raman transitions and Stark shifts.

3.1 Generic expression

The effective Hamiltonian from second-order pertubation, [2]

$$H_{\text{eff}} = \sum_e \frac{\vec{d}\cdot\vec{E}|e\rangle\langle e|\vec{d}\cdot\vec{E}^*}{\Delta_e} \tag{1}$$

$$= \left(\vec{E}\vec{E}^*\right)\cdot\sum_e \frac{\vec{d}|e\rangle\langle e|\vec{d}}{\Delta_e} \tag{2}$$

[1] "Results" as in ones that are hard to find elsewhere in a form that I like. I'm sure many people have derived/used these before. This does not include standard ones like Wigner-Eckart theorem since it's easy to find reference for it.

[2] Here we've omitted the counter rotating term. Including such term will not change the qualitative result of this discussion, which only relies on the numerator of each pertubation terms.

Since H_{eff} is a scalar operator for all possible values of $\vec{E}$, the right half of the expression $\sum_e \dfrac{\vec{d}|e\rangle\langle e|\vec{d}}{\Delta_e}$ must be a rank-2 tensor operator.

Therefore, we can always decompose the Hamiltonian into the spherical tensor components H_q^k, where $k = 0, 1, 2$ corresponds to the scalar, vector and tensor parts. The whole effective Hamiltonian can be written as,

$$H_{\text{eff}} = \sum_{k,p} (-1)^p T_{-p}^k H_p^k \tag{3}$$

where $T_p^k = \sum_{q,q'} E_q (E^*)_{q'} \langle 1, 1; q, q' | k, p \rangle$. The matrix element would be of the form,

$$\begin{aligned}
&\langle F, m_F | H_{\text{eff}} | F', m'_F \rangle \\
=& \sum_{k,p} (-1)^p T_{-p}^k \langle F, m_F | H_p^k | F', m'_F \rangle \\
=& \sum_k \langle F || H^k || F' \rangle \sum_p (-1)^p T_{-p}^k \langle F', k; m'_F, p | F, m_F \rangle \\
=& \sum_k (-1)^k \sqrt{\frac{2F+1}{2F'+1}} \langle F || H^k || F' \rangle \sum_p T_p^k \langle k, F; p, m_F | F', m'_F \rangle \\
=& \sum_k H(F, F', k) \sum_p T_p^k \langle k, F; p, m_F | F', m'_F \rangle
\end{aligned} \tag{4}$$

where the $H(F, F', k)$ in the last expression is a generic scalar factor that depends only on F, F' and k but not on p, m_F, or m'_F.

3.1.1 Hamiltonian as a function of angular momentum

Just like the situation that leads to the projection theorem (Ref. A.4.2), we can rewrite the expression slightly differently by replacing the Clebsch-Gordan with angular momentum operators.

We'll define the stark shift Hamiltonian as,

$$H_{\text{stark}} = -\alpha^{(0)} |E|^2 - \mathrm{i}\alpha^{(1)} \left(\vec{E} \times \vec{E}^* \right) \cdot \frac{\vec{F}}{F} - \alpha^{(2)} \left(\frac{3\left\{ \vec{E}, \vec{E}^* \right\}}{2} - |E|^2 \right) : \frac{3\vec{F}\vec{F} - F(F+1)}{2F(2F-1)} \tag{5}$$

where $\left\{ \vec{E}, \vec{E}^* \right\} \equiv \vec{E}\vec{E}^* + \vec{E}^*\vec{E}$.

We can now match each of the We define the scalar polarizability as,

$$-\alpha^{(0)} |E|^2 \langle F, m_F | F, m'_F \rangle = H(F, 0) T_0^0 \langle 0, F; 0, m_F | F, m'_F \rangle \tag{6}$$

$$\alpha^{(0)} = \frac{H(F, 0)}{\sqrt{3}} \tag{7}$$

Vector polarizability,

$$-\mathrm{i}\frac{\alpha^{(1)}}{F} \left(\vec{E} \times \vec{E}^* \right) \cdot \langle F, m_F | \vec{F} | F, m'_F \rangle = H(F, 1) \sum_p T_p^1 \langle 1, F; p, m_F | F, m'_F \rangle \tag{8}$$

$$-\mathrm{i}\frac{\alpha^{(1)}}{F}\sum_p(-1)^p\left(\vec{E}\times\vec{E}^*\right)_p\langle F,m_F|F_{-p}|F,m_F'\rangle = H(F,1)\sum_p T_p^1\langle 1,F;p,m_F|F,m_F'\rangle \tag{9}$$

$$\sqrt{2}\frac{\alpha^{(1)}}{F}\sum_p T_p^1\sqrt{(F+1)F}\langle 1,F;p,m_F|F,m_F'\rangle = H(F,1)\sum_p T_p^1\langle 1,F;p,m_F|F,m_F'\rangle \tag{10}$$

$$\alpha^{(1)} = \sqrt{\frac{F}{2(F+1)}}H(F,1) \tag{11}$$

Tensor polarizability,

$$\begin{aligned}&-\frac{9\alpha^{(2)}}{2F(2F-1)}\left(\frac{\vec{E}\vec{E}^*+\vec{E}\vec{E}^*}{2}-\frac{|E|^2}{3}\right):\langle F,m_F|\vec{F}\vec{F}-\frac{F(F+1)}{3}|F,m_F'\rangle\\ =&H(F,2)\sum_p T_p^2\langle 2,F;p,m_F|F,m_F'\rangle\end{aligned} \tag{12}$$

$$\begin{aligned}&-\frac{9\alpha^{(2)}}{2F(2F-1)}\sum_p(-1)^pT_p^2\langle F,m_F|(\vec{F}\vec{F})^2_{-p}|F,m_F'\rangle\\ =&H(F,2)\sum_p T_p^2\langle 2,F;p,m_F|F,m_F'\rangle\end{aligned} \tag{13}$$

$$\begin{aligned}&-\frac{9\alpha^{(2)}}{2}\sum_p T_p^2\sqrt{\frac{(F+1)(2F+3)}{6F(2F-1)}}\langle 2,F;p,m_F|F,m_F'\rangle\\ =&H(F,2)\sum_p T_p^2\langle 2,F;p,m_F|F,m_F'\rangle\end{aligned} \tag{14}$$

$$\alpha^{(2)} = -\frac{2}{3}\sqrt{\frac{2F(2F-1)}{3(F+1)(2F+3)}}H(F,2) \tag{15}$$

3.2 Direct derivation

The coupling between state $|F,m_F\rangle$ and $|F',m_F'\rangle$

$$\begin{aligned}\langle F,m_F|d_{-q}|F',m_F'\rangle =&\langle F||\mathbf{d}||F'\rangle\langle F,m_F|F',1;m_F',-q\rangle \qquad (16)\\ =&\langle F||\mathbf{d}||F'\rangle(-1)^{F'-1+m_F}\sqrt{2F+1}\begin{pmatrix}F' & 1 & F\\ m_F' & -q & m_F\end{pmatrix} \qquad (17)\end{aligned}$$

where F and m_F (F' and m_F') are the total angular momentum and its projection for the initial (final) state. d is the dipole operator and q is the label for the spherical harmonic component (-1, 0, or 1). $q=\pm1$ corresponds to the $\sigma^{\pm}$ polarization/transition and $q=0$ corresponds to the π polarization/transition.

3.2.1 Diagonal terms (Stark shifts) only

We can first calculate the Stark shift for a pure (σ^+, π or σ^-) polarization. This is the case that contains no non-diagonal terms. We should be able to use this to verify the m_F dependency of the final effect. (Scalar, vector and tensor shift should corresponds to 0, 1 and 2 order terms of m_F respectively).

Since we only care about the m_F dependency, we can ignore everything that's m_F independent.

The Stark shift,

$$
\begin{aligned}
\Delta E \propto & \langle F, m_F | d_{-q} | F', m'_F \rangle \langle F', m'_F | d_q | F, m_F \rangle \\
\propto & \left| \langle F, m_F | F', 1; m'_F, -q \rangle \right|^2 \\
\propto & (F+m_F)!(F-m_F)!(F'-m_F-q)!(F'+m_F+q)! \\
& \left| \sum_k \frac{(-1)^k}{k!(1-q-k)!(F'-F+1-k)!(F-F'+q+k)!(F'-q-k-m_F)!(F-1+q+k+m_F)!} \right|^2
\end{aligned}
\tag{18}
$$

The last proportionality relation uses the generic explicit expression for the Clebsch-Gordan coefficients (ignoring m_F independent factors). The sum is over all the k's where the factorials are non-negative. We'll call the last expression $\Delta'(m_F, q)$ in the following part for simplicity.
For $q = -1$

$$
\begin{aligned}
& \Delta'(m_F, -1) \\
= & (F+m_F)!(F-m_F)!(F'-m_F+1)!(F'+m_F-1)! \\
& \left| \sum_k \frac{(-1)^k}{k!(2-k)!(F'-F+1-k)!(F-F'-1+k)!(F'+1-k-m_F)!(F-2+k+m_F)!} \right|^2
\end{aligned}
\tag{19}
$$

Since we have $F'-F+1-k \geqslant 0$ and $F-F'-1+k \geqslant 0$, we have $k = F'-F+1$, (with the explicit condition to make sure $F'+m_F-1 \geqslant 0$)

$$
\begin{aligned}
& \Delta'(m_F, -1) \\
= & \begin{cases} \dfrac{(F+m_F)!(F'-m_F+1)!}{((F'-F+1)!(F-F'+1)!)^2(F-m_F)!(F'+m_F-1)!} & (m_F \geqslant 1-F') \\ 0 & (m_F < 1-F') \end{cases}
\end{aligned}
\tag{20}
$$

To simplify this further, we used the fact that $F' = F-1, F, F+1$

$$
\Delta'(m_F, -1) = \begin{cases} \dfrac{(F+m_F)!(F-m_F)!}{4(F-m_F)!(F+m_F-2)!} & (m_F \geqslant 2-F, F' = F-1) \\ 0 & (m_F < 2-F, F' = F-1) \\ \dfrac{(F+m_F)!(F-m_F+1)!}{(F-m_F)!(F+m_F-1)!} & (m_F \geqslant 1-F, F' = F) \\ 0 & (m_F < 1-F, F' = F) \\ \dfrac{(F+m_F)!(F-m_F+2)!}{4(F-m_F)!(F+m_F)!} & (m_F \geqslant 1-(F+1), F' = F+1) \\ 0 & (m_F < 1-(F+1), F' = F+1) \end{cases}
\tag{21}
$$

$$
\Delta'(m_F, -1) = \begin{cases} \dfrac{(F+m_F)(F+m_F-1)}{4} & (F' = F-1) \\ (F+m_F)(F-m_F+1) & (F' = F) \\ \dfrac{(F-m_F+2)(F-m_F+1)}{4} & (F' = F+1) \end{cases}
\tag{22}
$$

The final simplification uses the fact that $m_F \leqslant F$ and that the expression produces the right value (i.e. 0) even for out-of-bound m_F.

For $q = 0$,

$$\begin{aligned}
&\Delta'(m_F, 0) \\
=&(F + m_F)!(F - m_F)!(F' - m_F)!(F' + m_F)! \\
&\left|\sum_k \frac{(-1)^k}{k!(1-k)!(F'-F+1-k)!(F-F'+k)!(F'-k-m_F)!(F-1+k+m_F)!}\right|^2
\end{aligned} \tag{23}$$

Conditional on the value of F'

$$\begin{aligned}
&\Delta'(m_F, 0) \\
=&\begin{cases}
(F + m_F)!(F - m_F)!(F - 1 - m_F)!(F - 1 + m_F)! & \\
\left|\sum_k \frac{(-1)^k}{k!(1-k)!(-k)!(1+k)!(F-1-k-m_F)!(F-1+k+m_F)!}\right|^2 & (F' = F - 1) \\
(F + m_F)!(F - m_F)!(F - m_F)!(F + m_F)! & \\
\left|\sum_k \frac{(-1)^k}{k!(1-k)!(1-k)!k!(F-k-m_F)!(F-1+k+m_F)!}\right|^2 & (F' = F) \\
(F + m_F)!(F - m_F)!(F' - m_F)!(F' + m_F)! & \\
\left|\sum_k \frac{(-1)^k}{k!(1-k)!(2-k)!(-1+k)!(F+1-k-m_F)!(F-1+k+m_F)!}\right|^2 & (F' = F + 1)
\end{cases}
\end{aligned} \tag{24}$$

For the first and third case, k can only be 0 and 1 respectively. For the second case, k can be either 0 or 1 and we need to sum over both.

$$\Delta'(m_F, 0) = \begin{cases}
\frac{(F + m_F)!(F - m_F)!(F - 1 - m_F)!(F - 1 + m_F)!}{((F-1-m_F)!(F-1+m_F)!)^2} & (F' = F - 1) \\
(F + m_F)!(F - m_F)!(F - m_F)!(F + m_F)! & \\
\left(\frac{1}{(F-m_F)!(F-1+m_F)!} - \frac{1}{(F-1-m_F)!(F+m_F)!}\right)^2 & (F' = F) \\
\frac{(F + m_F)!(F - m_F)!(F + 1 - m_F)!(F + 1 + m_F)!}{((F-m_F)!(F+m_F)!)^2} & (F' = F + 1)
\end{cases} \tag{25}$$

$$\Delta'(m_F, 0) = \begin{cases}
F^2 - m_F^2 & (F' = F - 1) \\
4m_F^2 & (F' = F) \\
(F + 1)^2 - m_F^2 & (F' = F + 1)
\end{cases} \tag{26}$$

For $q = 1$,

$$\begin{aligned}
&\Delta'(m_F, 1) \\
=&(F + m_F)!(F - m_F)!(F' - m_F - 1)!(F' + m_F + 1)! \\
&\left|\sum_k \frac{(-1)^k}{k!(-k)!(F' - F + 1 - k)!(F - F' + 1 + k)!(F' - 1 - k - m_F)!(F + k + m_F)!}\right|^2
\end{aligned} \tag{27}$$

which requires $k = 0$,

$$\begin{aligned}
\Delta'(m_F, 1) =&\frac{(F + m_F)!(F - m_F)!(F' - m_F - 1)!(F' + m_F + 1)!}{((F' - F + 1)!(F - F' + 1)!(F' - 1 - m_F)!(F + m_F)!)^2} \\
=&\frac{(F - m_F)!(F' + m_F + 1)!}{((F' - F + 1)!(F - F' + 1)!)^2(F' - m_F - 1)!(F + m_F)!}
\end{aligned} \tag{28}$$

Here we omitted the check for $F' - m_F - 1 \geqslant 0$ since the final expression would not depend on it. Conditional on the F' values

$$\Delta'(m_F, 1) = \begin{cases} \dfrac{(F - m_F)!(F + m_F)!}{4(F - m_F - 2)!(F + m_F)!} & (F' = F - 1) \\ \dfrac{(F - m_F)!(F + m_F + 1)!}{(F - m_F - 1)!(F + m_F)!} & (F' = F) \\ \dfrac{(F - m_F)!(F + m_F + 2)!}{4(F - m_F)!(F + m_F)!} & (F' = F + 1) \end{cases} \tag{29}$$

$$\Delta'(m_F, 1) = \begin{cases} \dfrac{(F - m_F)(F - m_F - 1)}{4} & (F' = F - 1) \\ (F - m_F)(F + m_F + 1) & (F' = F) \\ \dfrac{(F + m_F + 2)(F + m_F + 1)}{4} & (F' = F + 1) \end{cases} \tag{30}$$

We can see that the expressions for $\Delta'(m_F, q)$ are all second order polynomials of m_F. We can also verify that $\Delta'(-m_F, 1) = \Delta'(m_F, -1)$ as required by symmetry.

We can also see that for circular polarization ($q = \pm 1$) the resulting shift always have a non-zero linear term. The slope of this term is $\dfrac{1 - 2F}{4}$, -1, and $\dfrac{2F + 3}{4}$ for σ^+ polarization ($q = 1$) and $F' = F - 1, F, F + 1$ respectively[3]. On the other hand, the expressions for π polarization never have any linear m_F term which is also consistent with symmetry.

It is somewhat interesting that the coefficient for the second order terms are never zero, even for $F = 0, \dfrac{1}{2}$ cases where tensor shift does not exist. Of course since there are not enough "sampling points" on the polynomial the m_F^2 term would just appear at most as a global energy shift in such cases.

3.2.2 Full generic effective Hamiltonian for a single excited state

When the polarization of the light is not one of the pure polarizations, the effect of the second order perturbation would contain off-diagonal terms in addition to the diagonal ones. In such cases, we would need to calculate the full effective Hamiltonian matrix instead of only the Stark shifts.

Let the amplitude of the light be A_q, where $q = -1, 0, 1$ corresponds to the σ^-, π and σ^+ polarizations. The matrix element for the effective Hamiltonian is,

$$\begin{aligned} & \langle F, m_F | H_{\text{eff}} | F', m'_F \rangle \\ =& \frac{1}{4\Delta} \sum_{m''_F, q, q'} \langle F, m_F | (-1)^q A_q d_{-q} | F'', m''_F \rangle (\langle F', m'_F | (-1)^{q'} A_{q'} d_{-q'} | F'', m''_F \rangle)^* \\ =& \frac{1}{4\Delta} \sum_{m''_F, q, q'} (-1)^q A_q (-1)^{q'} A^*_{q'} \langle F, m_F | d_{-q} | F'', m''_F \rangle (\langle F', m'_F | d_{-q'} | F'', m''_F \rangle)^* \\ =& \frac{1}{4\Delta} \sum_{m''_F, q, q'} (-1)^q A_q (A^*)_{-q'} \langle F, m_F | d_{-q} | F'', m''_F \rangle (\langle F', m'_F | d_{-q'} | F'', m''_F \rangle)^* \end{aligned} \tag{31}$$

where $(A^*)_q$ is the spherical component of the complex conjugate of A (ref Eq. 60). Using Wigner-Echart and the spherical decomposition of rank-2 tensor.

[3]For the $F' = F - 1$ expression, $1 - 2F$ may be 0 for $F = \dfrac{1}{2}$ but this cannot happen for $F' = F - 1$.

$$
\begin{aligned}
&\langle F, m_F|H_{\text{eff}}|F', m'_F\rangle \\
=&\frac{\langle F||d||F''\rangle\langle F'||d||F''\rangle^*}{4\Delta}\sum_{m''_F,q,q'}(-1)^q A_q(A^*)_{-q'}\langle F, m_F|F'', 1; m''_F, -q\rangle\langle F'', 1; m''_F, -q'|F', m'_F\rangle \\
=&\frac{\langle F||d||F''\rangle\langle F'||d||F''\rangle^*}{4\Delta}\sum_{k,p}T^k_p \\
&\sum_{m''_F,q,q'}(-1)^q\langle k, p|1, 1; q, -q'\rangle\langle F, m_F|F'', 1; m''_F, -q\rangle\langle F'', 1; m''_F, -q'|F', m'_F\rangle
\end{aligned}
\tag{32}
$$

Rewriting in $3-j$ symbol,

$$
\begin{aligned}
&\sum_{m''_F,q,q'}(-1)^q\langle k, p|1, 1; q, -q'\rangle\langle F, m_F|F'', 1; m''_F, -q\rangle\langle F'', 1; m''_F, -q'|F', m'_F\rangle \\
=&(-1)^{2F''+p-m_F-m'_F}\sqrt{(2k+1)(2F+1)(2F'+1)} \\
&\sum_{m''_F,q,q'}(-1)^q\begin{pmatrix}1 & 1 & k\\ q & -q' & -p\end{pmatrix}\begin{pmatrix}F'' & 1 & F\\ m''_F & -q & -m_F\end{pmatrix}\begin{pmatrix}F'' & 1 & F'\\ m''_F & -q' & -m'_F\end{pmatrix} \\
=&(-1)^{2F''+p-m_F-m'_F}\sqrt{(2k+1)(2F+1)(2F'+1)} \\
&\sum_{m''_F,q,q'}(-1)^q\begin{pmatrix}1 & k & 1\\ q' & -p & -q\end{pmatrix}\begin{pmatrix}1 & F & F''\\ q & -m_F & -m''_F\end{pmatrix}\begin{pmatrix}F'' & F' & 1\\ m''_F & m'_F & -q'\end{pmatrix} \\
=&(-1)^{F''+m_F-p}\sqrt{(2k+1)(2F+1)(2F'+1)} \\
&\sum_{m''_F,q,q'}(-1)^{1+1+F''-q-q'-m''_F}\begin{pmatrix}1 & k & 1\\ q' & -p & -q\end{pmatrix}\begin{pmatrix}1 & F & F''\\ q & -m_F & -m''_F\end{pmatrix}\begin{pmatrix}F'' & F' & 1\\ m''_F & m'_F & -q'\end{pmatrix} \\
=&(-1)^{F''+m_F-p}\sqrt{(2k+1)(2F+1)(2F'+1)}\begin{pmatrix}k & F & F'\\ p & m_F & -m'_F\end{pmatrix}\begin{Bmatrix}k & F & F'\\ F'' & 1 & 1\end{Bmatrix} \\
=&(-1)^{F''+F+k}\sqrt{(2k+1)(2F+1)}\langle k, F; p, m_F|F', m'_F\rangle\begin{Bmatrix}k & F & F'\\ F'' & 1 & 1\end{Bmatrix}
\end{aligned}
\tag{33}
$$

For $F' = F$,

$$
\begin{aligned}
&\sum_{m''_F,q,q'}(-1)^q\langle k, p|1, 1; q, -q'\rangle\langle F, m_F|F'', 1; m''_F, -q\rangle\langle F'', 1; m''_F, -q'|F, m'_F\rangle \\
=&(-1)^{F''+F+k}\sqrt{(2k+1)(2F+1)}\langle k, F; p, m_F|F, m'_F\rangle\begin{Bmatrix}k & F & F\\ F'' & 1 & 1\end{Bmatrix}
\end{aligned}
\tag{34}
$$

4 Vector light shift as effective magnetic field

Electric field, $E_{x,y,z}$. Spherical components of electric field,

$$E_0 = E_z \tag{35}$$

$$E_{\pm 1} = \mp\frac{1}{\sqrt{2}}(E_x \pm \mathrm{i}E_y) \tag{36}$$

Or the inverse conversion,

$$E_x = \frac{1}{\sqrt{2}}(E_{-1} - E_1) \tag{37}$$

$$
\begin{aligned}
E_y =& \frac{\mathrm{i}}{\sqrt{2}}(E_1 + E_{-1}) && (38)\\
E_z =& E_0 && (39)
\end{aligned}
$$

Complex conjugate of the electric field,

$$
\begin{aligned}
(E^*)_0 =& (E^*)_z && (40)\\
(E^*)_{\pm 1} =& \mp \frac{1}{\sqrt{2}}((E^*)_x \pm \mathrm{i}(E^*)_y) && (41)\\
(E^*)_x =& \frac{1}{\sqrt{2}}((E^*)_{-1} - (E^*)_1) && (42)\\
(E^*)_y =& \frac{\mathrm{i}}{\sqrt{2}}((E^*)_1 + (E^*)_{-1}) && (43)\\
(E^*)_z =& (E^*)_0 && (44)
\end{aligned}
$$

Effective magnetic field

$$
\vec{B} = \alpha \vec{E} \times \vec{E}^* \tag{45}
$$

where α is a scalar number.

$$
\begin{aligned}
B_x =& \alpha(E_y(E^*)_z - E_z(E^*)_y) \\
=& \frac{\mathrm{i}\alpha}{\sqrt{2}}((E_1 + E_{-1})(E^*)_0 - E_0((E^*)_1 + (E^*)_{-1})) && (46)\\
B_y =& \alpha(E_z(E^*)_x - E_x(E^*)_z) \\
=& \frac{\alpha}{\sqrt{2}}(E_0((E^*)_{-1} - (E^*)_1) - (E_{-1} - E_1)(E^*)_0) && (47)\\
B_z =& \alpha(E_x(E^*)_y - E_y(E^*)_x) \\
=& \frac{\mathrm{i}\alpha}{2}((E_{-1} - E_1)((E^*)_1 + (E^*)_{-1}) - (E_1 + E_{-1})((E^*)_{-1} - (E^*)_1)) && (48)\\
=& \mathrm{i}\alpha(E_{-1}(E^*)_1 - E_1(E^*)_{-1})
\end{aligned}
$$

Spherical components for the effective magnetic field,

$$
\begin{aligned}
B_{-1} =& \frac{1}{\sqrt{2}}(B_x - \mathrm{i}B_y) \\
=& \mathrm{i}\alpha(E_{-1}(E^*)_0 - E_0(E^*)_{-1}) && (49)\\
B_0 =& B_z \\
=& \mathrm{i}\alpha(E_{-1}(E^*)_1 - E_1(E^*)_{-1}) && (50)\\
B_1 =& -\frac{1}{\sqrt{2}}(B_x + \mathrm{i}B_y) \\
=& \mathrm{i}\alpha(E_0(E^*)_1 - E_1(E^*)_0) && (51)
\end{aligned}
$$

Note that the rank-1 component of the tensor $\vec{E}\vec{E}^*$ is

$$
T^1_p = \sum_{q,q'} \langle 1, 1; q, q' | 1, p \rangle E_q (E^*)_{q'} \tag{52}
$$

Or more explicitly,

$$
T^1_{-1} = \frac{1}{\sqrt{2}}(E_0(E^*)_{-1} - E_{-1}(E^*)_0) \tag{53}
$$

$$T^1_0 = \frac{1}{\sqrt{2}}(E_1(E^*)_{-1} - E_{-1}(E^*)_1) \tag{54}$$

$$T^1_1 = \frac{1}{\sqrt{2}}(E_1(E^*)_0 - E_0(E^*)_1) \tag{55}$$

so we have $B_q = -\mathrm{i}\sqrt{2}\alpha T^1_q$.

The vector shift from the effective magnetic field is,

$$\begin{aligned}
&\langle F, m_F|\vec{\mu}\cdot\vec{B}|F, m'_F\rangle \\
=&\langle F, m_F|\sum_q(-1)^q\mu_{-q}B_q|F, m'_F\rangle \\
=&-\mathrm{i}\sqrt{2}\alpha g\sum_q(-1)^q T^1_q\langle F, m_F|F_{-q}|F, m'_F\rangle \\
=&-\mathrm{i}\sqrt{2}\alpha g\sqrt{(F+1)F}\sum_q(-1)^q T^1_q\langle F, 1; m'_F, -q|F, m_F\rangle
\end{aligned} \tag{56}$$

5 Mitigating the effect of transverse circular polarization in optical tweezers

A Some useful formulas

A.1 Spherical components of vector

Similar to the decomposition of light polarization into $\sigma^\pm$ and π, every 3D vector (operator) can be equivalently expressed as a rank-1 spherical tensor,

$$\begin{aligned}
V_0 =& V_z \\
V_{\pm 1} =& \mp\frac{1}{\sqrt{2}}(V_x \pm \mathrm{i}V_y)
\end{aligned} \tag{57}$$

In particular, when applied to the angular momentum operator,

$$\begin{aligned}
J_0 =& J_z \\
J_{\pm 1} =& \mp\frac{1}{\sqrt{2}}(J_x \pm \mathrm{i}J_y) \\
=& \mp\frac{J_\pm}{\sqrt{2}}
\end{aligned} \tag{58}$$

where $J_\pm$ are the angular momentum raising and lowering operators.

For the complex conjugate of the vector V^*, defined as

$$(V^*)_{x,y,z} = V^*_{x,y,z} \tag{59}$$

The spherical components are

$$\begin{aligned}
(V^*)_0 =& V^*_z \\
(V^*)_{\pm 1} =& \mp\frac{1}{\sqrt{2}}\left(V^*_x \pm \mathrm{i}V^*_y\right)
\end{aligned} \tag{60}$$

Note that in general $(V^*)_{\pm 1} \neq V^*_{\pm 1}$. In fact,

$$\begin{aligned} V^*_{\pm 1} =& \mp \frac{1}{\sqrt{2}}\left(V^*_x \mp \mathrm{i} V^*_y\right) \\ =& -(V^*)_{\mp 1} \end{aligned} \tag{61}$$

$$V^*_q = (-1)^q (V^*)_{-q} \tag{62}$$

i.e. the $+1$ component of V^* is related to the -1 component of V, and the -1 component of V^* is related to the $+1$ component of V.

Dot product of two vector

$$\begin{aligned} \vec{A} \cdot \vec{B} =& \sum_{i=x,y,z} A_i B_i \\ =& -A_{-1} B_{+1} - A_{+1} B_{-1} + A_0 B_0 \\ =& \sum_{q=-1,0,1} (-1)^q A_q B_{-q} \\ =& -\sqrt{3} \sum_{q=-1,0,1} \langle 0, 0 | 1, 1; q, -q \rangle A_q B_{-q} \end{aligned} \tag{63}$$

A.2 Spherical components of rank-2 tensor

A.2.1 Semi-quantitative discussion

A rank-2 tensor is a direct product of two vectors (or at least in the linear space). So to understand how a rank-2 tensor is related to it's spherical components, we can simply study the direct product of two vectors.

Based on the discussion of the spherical components of vectors above, a rank-2 tensor (or the direct product of two vectors) would be equivalent to the product of two rank-1 spherical tensors. Based on group representation theory, or equivalently, angular momentum summation rules, the rank-2 tensor can be written as the direct sum of a rank-0, 1 and 2 spherical tensor.

Without detailed derivation, we can identify the form of the three components based on the following rules,

1. Each element of the pherical tensors is a linear combination of the tensor elements.
2. Each one of the tensors (rank-0, 1 and 2) need to satisfy the correct transformation rules. In particular, the rank-0 component is a scalar and the rank-1 compoment should be equivalent to a vector.
3. Each element should be linearly independent.

Based on these rules, up to a constant factor, the rank-0 spherical tensor must be the dot product of the two vectors and the rank-1 spherical tensor must be equivalent to the cross product of the two vectors.

Although there isn't a direct analogy for the rank-2 spherical tensor part, it must correspond to the remaining elements in the original rank-2 (cartesian) tensor. Since the dot product is the trace of the tensor and the cross product is the anti-symmetric part of the tensor, the remaining part of the original tensor is the zero-trace symmetric part of the tensor. This part has 5 degrees of freedoms, which agrees with the number of elements in the rank-2 spherical tensor that we need. Up to a constant factor, we can write out these elements for a rank-2 cartesian tensor T_{ij} as,

Rank-0 spherical / scalar,

$$E = \sum_i T_{ii} \tag{64}$$

Rank-1 spherical / vector,

$$A_i = \sum_{jk} \varepsilon_{ijk} T_{jk} \tag{65}$$

where ε_{ijk} is the Levi-Civita symbol. Note that for tensor that's the direct product of two vectors, this is just the cross product of the two vectors.

Rank-2 spherical / tensor,

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\delta_{ij} \sum_k T_{kk} \tag{66}$$

A.2.2 Exact tensor component

The decomposition of the rank-2 tensor into a linear combination of spherical tensors isn't unique, each spherical tensor can always be scaled by an arbitrary non-zero factor. However, for convinience, we would like to pick one that's consistent with how two rank-1 spherical tensors are multiplied together, i.e.

$$T^k_q = \sum_{q_1,q_2} \langle k, q | 1, 1; q_1, q_2 \rangle X^1_{q_1} Y^1_{q_2} \tag{67}$$

First we take the direct product of two vectors X and Y,

$$T_{ij} = X_i Y_j \tag{68}$$

For the rank-0 spherical component,

$$\begin{aligned} T^0_0 &= \sum_{q_1,q_2} \langle 0, 0 | 1, 1; q_1, q_2 \rangle X^1_{q_1} Y^1_{q_2} \\ &= \frac{1}{\sqrt{3}} \left(X^1_{-1} Y^1_1 - X^1_0 Y^1_0 + X^1_1 Y^1_{-1} \right) \\ &= -\frac{1}{\sqrt{3}} \sum_{ii} T_{ii} \\ &= -\frac{\vec{X} \cdot \vec{Y}}{\sqrt{3}} \end{aligned} \tag{69}$$

Rank-1 spherical components,

$$
\begin{aligned}
T_q^1 &= \sum_{q_1,q_2} \langle 1, q | 1, 1; q_1, q_2 \rangle X_{q_1}^1 Y_{q_2}^1 \\
&= \begin{cases} \dfrac{X_0^1 Y_{-1}^1 - X_{-1}^1 Y_0^1}{\sqrt{2}} & (q=-1) \\ \dfrac{X_1^1 Y_{-1}^1 - X_{-1}^1 Y_1^1}{\sqrt{2}} & (q=0) \\ \dfrac{X_1^1 Y_0^1 - X_0^1 Y_1^1}{\sqrt{2}} & (q=1) \end{cases} \\
&= \begin{cases} \dfrac{(T_{zx} - T_{xz}) + \mathrm{i}(T_{yz} - T_{zy})}{2} & (q=-1) \\ \dfrac{\mathrm{i}}{\sqrt{2}}(T_{xy} - T_{yx}) & (q=0) \\ \dfrac{(T_{zx} - T_{xz}) - \mathrm{i}(T_{yz} - T_{zy})}{2} & (q=1) \end{cases} \\
&= \frac{\mathrm{i}}{\sqrt{2}} \begin{cases} \dfrac{\left(\vec{X} \times \vec{Y}\right)_x - \mathrm{i}\left(\vec{X} \times \vec{Y}\right)_y}{\sqrt{2}} & (q=-1) \\ \left(\vec{X} \times \vec{Y}\right)_z & (q=0) \\ -\dfrac{\left(\vec{X} \times \vec{Y}\right)_x + \mathrm{i}\left(\vec{X} \times \vec{Y}\right)_y}{\sqrt{2}} & (q=1) \end{cases} \\
&= \frac{\mathrm{i}}{\sqrt{2}} \left(\vec{X} \times \vec{Y}\right)_q^1
\end{aligned}
\tag{70}
$$

where $\left(\vec{X} \times \vec{Y}\right)_q^1$ represents the spherical tensor components of the $\vec{X} \times \vec{Y}$ vector.

Rank-2 spherical components,

$$
\begin{aligned}
T_q^2 &= \sum_{q_1,q_2} \langle 2,q|1,1;q_1,q_2\rangle X_{q_1}^1 Y_{q_2}^1 \\
&= \begin{cases}
X_{-1}^1 Y_{-1}^1 & (q=-2) \\
\dfrac{X_0^1 Y_{-1}^1 + X_{-1}^1 Y_0^1}{\sqrt{2}} & (q=-1) \\
\dfrac{X_1^1 Y_{-1}^1 + 2X_0^1 Y_0^1 + X_{-1}^1 Y_1^1}{\sqrt{6}} & (q=0) \\
\dfrac{X_1^1 Y_0^1 + X_0^1 Y_1^1}{\sqrt{2}} & (q=1) \\
X_1^1 Y_1^1 & (q=2)
\end{cases} \\
&= \begin{cases}
\dfrac{(T_{xx}-T_{yy}) - \mathrm{i}(T_{xy}+T_{yx})}{2} & (q=-2) \\
\dfrac{(T_{zx}+T_{xz}) - \mathrm{i}(T_{zy}+T_{yz})}{2} & (q=-1) \\
\dfrac{2T_{zz}-T_{xx}-T_{yy}}{\sqrt{6}} & (q=0) \\
-\dfrac{(T_{zx}+T_{xz}) + \mathrm{i}(T_{zy}+T_{yz})}{2} & (q=1) \\
\dfrac{(T_{xx}-T_{yy}) + \mathrm{i}(T_{xy}+T_{yx})}{2} & (q=2)
\end{cases} \\
&= \begin{cases}
\dfrac{S_{xx}-S_{yy}}{2} - \mathrm{i}S_{xy} & (q=-2) \\
S_{zx} - \mathrm{i}S_{yz} & (q=-1) \\
\sqrt{\dfrac{3}{2}} S_{zz} & (q=0) \\
-S_{zx} - \mathrm{i}S_{yz} & (q=1) \\
\dfrac{S_{xx}-S_{yy}}{2} + \mathrm{i}S_{xy} & (q=2)
\end{cases}
\end{aligned} \tag{71}
$$

where the zero-trace symmetric S_{ij} is defined above (Eq. 66).

The inverse transformation,

$$T_{xx} = -\frac{1}{\sqrt{3}}T_0^0 - \frac{1}{\sqrt{6}}T_0^2 + \frac{T_{-2}^2 + T_2^2}{2} \tag{72}$$

$$T_{xy} = -\frac{\mathrm{i}}{\sqrt{2}}T_0^1 + \frac{\mathrm{i}}{2}\left(T_{-2}^2 - T_2^2\right) \tag{73}$$

$$T_{xz} = \frac{T_{-1}^2 - T_1^2}{2} - \frac{T_{-1}^1 + T_1^1}{2} \tag{74}$$

$$T_{yx} = \frac{\mathrm{i}}{\sqrt{2}}T_0^1 + \frac{\mathrm{i}}{2}\left(T_{-2}^2 - T_2^2\right) \tag{75}$$

$$T_{yy} = -\frac{1}{\sqrt{3}}T_0^0 - \frac{1}{\sqrt{6}}T_0^2 - \frac{1}{2}\left(T_{-2}^2 + T_2^2\right) \tag{76}$$

$$T_{yz} = \frac{\mathrm{i}}{2}\left(T_{-1}^2 + T_1^2\right) - \frac{\mathrm{i}}{2}\left(T_{-1}^1 - T_1^1\right) \tag{77}$$

$$T_{zx} = \frac{T_{-1}^2 - T_1^2}{2} + \frac{T_{-1}^1 + T_1^1}{2} \tag{78}$$

$$T_{zy} = \frac{\mathrm{i}}{2}\left(T_{-1}^2 + T_1^2\right) + \frac{\mathrm{i}}{2}\left(T_{-1}^1 - T_1^1\right) \tag{79}$$

$$T_{zz} = -\frac{1}{\sqrt{3}}T_0^0 + \sqrt{\frac{2}{3}}T_0^2 \tag{80}$$

A.2.3 Inner product of tensors

The expressions we'll derive will involve inner product of tensors (since the final energy is a scalar). Here we'll figure out what's the expression for inner product using the spherical harmonic components.

$$\begin{aligned}
\overline{\overline{T}} : \overline{\overline{U}} &= \sum_{ij} T_{ij}U_{ij} \\
&= \begin{pmatrix}
\left(-\frac{1}{\sqrt{3}}T_0^0 - \frac{1}{\sqrt{6}}T_0^2 + \frac{T_{-2}^2+T_2^2}{2}\right)\left(-\frac{1}{\sqrt{3}}U_0^0 - \frac{1}{\sqrt{6}}U_0^2 + \frac{U_{-2}^2+U_2^2}{2}\right) \\
+\left(-\frac{\mathrm{i}}{\sqrt{2}}T_0^1 + \frac{\mathrm{i}}{2}\left(T_{-2}^2 - T_2^2\right)\right)\left(-\frac{\mathrm{i}}{\sqrt{2}}U_0^1 + \frac{\mathrm{i}}{2}\left(U_{-2}^2 - U_2^2\right)\right) \\
+\left(\frac{T_{-1}^2-T_1^2}{2} - \frac{T_{-1}^1+T_1^1}{2}\right)\left(\frac{U_{-1}^2-U_1^2}{2} - \frac{U_{-1}^1+U_1^1}{2}\right) \\
+\left(\frac{\mathrm{i}}{\sqrt{2}}T_0^1 + \frac{\mathrm{i}}{2}\left(T_{-2}^2 - T_2^2\right)\right)\left(\frac{\mathrm{i}}{\sqrt{2}}U_0^1 + \frac{\mathrm{i}}{2}\left(U_{-2}^2 - U_2^2\right)\right) \\
+\left(-\frac{1}{\sqrt{3}}T_0^0 - \frac{1}{\sqrt{6}}T_0^2 - \frac{1}{2}\left(T_{-2}^2 + T_2^2\right)\right)\left(-\frac{1}{\sqrt{3}}U_0^0 - \frac{1}{\sqrt{6}}U_0^2 - \frac{1}{2}\left(U_{-2}^2 + U_2^2\right)\right) \\
+\left(\frac{\mathrm{i}}{2}\left(T_{-1}^2 + T_1^2\right) - \frac{\mathrm{i}}{2}\left(T_{-1}^1 - T_1^1\right)\right)\left(\frac{\mathrm{i}}{2}\left(U_{-1}^2 + U_1^2\right) - \frac{\mathrm{i}}{2}\left(U_{-1}^1 - U_1^1\right)\right) \\
+\left(\frac{T_{-1}^2-T_1^2}{2} + \frac{T_{-1}^1+T_1^1}{2}\right)\left(\frac{U_{-1}^2-U_1^2}{2} + \frac{U_{-1}^1+U_1^1}{2}\right) \\
+\left(\frac{\mathrm{i}}{2}\left(T_{-1}^2 + T_1^2\right) + \frac{\mathrm{i}}{2}\left(T_{-1}^1 - T_1^1\right)\right)\left(\frac{\mathrm{i}}{2}\left(U_{-1}^2 + U_1^2\right) + \frac{\mathrm{i}}{2}\left(U_{-1}^1 - U_1^1\right)\right) \\
+\left(-\frac{1}{\sqrt{3}}T_0^0 + \sqrt{\frac{2}{3}}T_0^2\right)\left(-\frac{1}{\sqrt{3}}U_0^0 + \sqrt{\frac{2}{3}}U_0^2\right)
\end{pmatrix} \\
&= \begin{pmatrix}
T_0^0U_0^0 + T_{-1}^1U_1^1 - T_0^1U_0^1 + T_1^1U_{-1}^1 \\
+T_{-2}^2U_2^2 - T_{-1}^2U_1^2 + T_0^2U_0^2 - T_1^2U_{-1}^2 + T_2^2U_{-2}^2
\end{pmatrix} \\
&= \sum_{k,q}(-1)^{k+q}T_q^kU_{-q}^k \\
&= \sum_k \sqrt{2k+1}\sum_q \langle 0,0|k,k;q,-q\rangle T_q^kU_{-q}^k
\end{aligned} \tag{81}$$

we can see that each rank of spherical tensor component is only multiplied with that of the same rank and the result is also consistent with the

A.2.4 Cartesian form

For calculation/understanding/visualization, it is sometimes still useful to write the inner product in the cartesian form after decomposing it to scalar/vector/tensor components.

Using the scalar, vector and tensor components from above (Eq. 64,65,66), the corresponding dot products for the three components are,

$$\begin{aligned}
E^TE^U &= \sum_{ij} T_{ii}U_{jj} \\
&= T_{xx}U_{xx} + T_{yy}U_{yy} + T_{zz}U_{zz} + T_{xx}U_{yy} + T_{xx}U_{zz} + T_{yy}U_{xx} + T_{yy}U_{zz} + T_{zz}U_{xx} + T_{zz}U_{yy}
\end{aligned} \tag{82}$$

$$
\begin{aligned}
\vec{A}^T \cdot \vec{A}^U =& \sum_{ijkmm} \varepsilon_{ijk}\varepsilon_{imm} T_{jk} U_{mn} \\
=& \sum_{ijklm} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) T_{jk} U_{mn} \\
=& \sum_{jk} (T_{jk}U_{jk} - T_{jk}U_{kj}) \\
=& T_{xy}U_{xy} + T_{yx}U_{yx} + T_{yz}U_{yz} + T_{zy}U_{zy} + T_{xz}U_{xz} + T_{zx}U_{zx} \\
& - T_{xy}U_{yx} - T_{yx}U_{xy} - T_{yz}U_{zy} - T_{zy}U_{yz} - T_{xz}U_{zx} - T_{zx}U_{xz}
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\overline{\overline{S}}^T : \overline{\overline{S}}^U =& \sum_{ij} \left(\frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\delta_{ij} \sum_k T_{kk} \right) \left(\frac{1}{2}(U_{ij} + U_{ji}) - \frac{1}{3}\delta_{ij} \sum_k U_{kk} \right) \\
=& \frac{2}{3}(T_{xx}U_{xx} + T_{yy}U_{yy} + T_{zz}U_{zz}) \\
& - \frac{1}{3}(T_{xx}U_{yy} + T_{xx}U_{zz} + T_{yy}U_{xx} + T_{yy}U_{zz} + T_{zz}U_{xx} + T_{zz}U_{yy}) \\
& + \frac{1}{2}(T_{xy}U_{xy} + T_{yx}U_{yx} + T_{yz}U_{yz} + T_{zy}U_{zy} + T_{xz}U_{xz} + T_{zx}U_{zx}) \\
& + \frac{1}{2}(T_{xy}U_{yx} + T_{yx}U_{xy} + T_{yz}U_{zy} + T_{zy}U_{yz} + T_{xz}U_{zx} + T_{zx}U_{xz})
\end{aligned}
\tag{84}
$$

Combining the three,

$$
\overline{\overline{T}} : \overline{\overline{U}} = \frac{1}{3} E^T E^U + \frac{1}{2} \vec{A}^T \cdot \vec{A}^U + \overline{\overline{S}}^T : \overline{\overline{S}}^U \tag{85}
$$

By comparing to Eq. 81 and based on the correspondence of the cartesian and spherical expression for the scalar/vector/tensor parts[4], we can conclude that

$$
\begin{aligned}
\frac{1}{3} E^T E^U =& T_0^0 U_0^0 \\
=& \langle 0, 0 | 0, 0; 0, 0 \rangle T_0^0 U_0^0
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
\frac{1}{2} \vec{A}^T \cdot \vec{A}^U =& -\sum_q (-1)^q T_q^1 U_{-q}^1 \\
=& \sqrt{3} \sum_q \langle 0, 0 | 1, 1; q, -q \rangle T_q^1 U_{-q}^1
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
\overline{\overline{S}}^T : \overline{\overline{S}}^U =& \sum_q (-1)^q T_q^2 U_{-q}^2 \\
=& \sqrt{5} \sum_q \langle 0, 0 | 2, 2; q, -q \rangle T_q^2 U_{-q}^2
\end{aligned}
\tag{88}
$$

A.3 Wigner-Eckart theorem

This describes the relation between matrix elements of a vector/tensor operator in the angular momentum basis. The matrix element for different angular momentum states are related to each other with Clebsch-Gordan coefficients.

$$
\langle j, m | T_q^{(k)} | j', m' \rangle = \langle j', k; m', q | j, m \rangle \langle j || T^{(k)} || j' \rangle \tag{89}
$$

[4] i.e. that E, $\vec{A}$, $\overline{\overline{S}}$ contains only terms from T^0, T^1 and T^2 sperical tensors.

where $T_q^{(k)}$ is the q-th component of the spherical tensor operator $T^{(k)}$ of rank k. This is the result of rotation symmetry between all the matrix elements.

Equivalently, this also means that no matter what the tensor operator is, it's matrix elements in this (between these) subspace differs from that of a different tensor operator only by a constant factor. (Note that this factor could depend on the j and j' (just not m and m') and it can of course be 0 as well), i.e.

$$\langle j, m|{T_1}_q^{(k)}|j', m'\rangle \propto \langle j, m|{T_2}_q^{(k)}|j', m'\rangle \tag{90}$$

A.4 When $j = j'$

A special case for the Wigner-Echart theorem is when $j = j'$. In this case we can plug in the angular momentum operator J (this would otherwise result in vanishing matrix elements if $j \neq j'$ since J conserves, well, j).

$$\begin{aligned}\langle j, m|J_q|j, m'\rangle =&\langle j, 1; m', q|j, m\rangle\langle j||J||j\rangle\\ \propto&\langle j, 1; m', q|j, m\rangle\end{aligned} \tag{91}$$

This allow us to replace the CG coefficents with the angular momentum operator, i.e.,

$$\langle j, m|V_q|j, m'\rangle \propto \langle j, m|J_q|j, m'\rangle \tag{92}$$

which could make some calculation/expression significantly simpler.

This relation basically states that within the subspace of a single j, we can treat any vector operator as proportional to the angular momentum. The proportionality factor can then be obtained from the dot product with angular momentum, i.e. the projection of the vector onto angular momentum.

A.4.1 $m = 0$ selection rule

The selection rule for $m = m' = 0$ transition directly follows from this relation since,

$$\begin{aligned}\langle j, m|V_0|j, m\rangle \propto&\langle j, m|J_0|j, m\rangle\\ =&\langle j, m|J_z|j, m\rangle\\ =&m\end{aligned} \tag{93}$$

which is 0 when $m = 0$.

A.4.2 Projection theorem

We can use this to derive the projection theorem. Explicitly writing down the proportionality factor in Eq. 92, we have,

$$\langle j, m|V_q|j, m'\rangle =c\langle j, m|J_q|j, m'\rangle \tag{94}$$

Multiply both sides with the angular momentum matrix element and sum over all m' and q

$$\sum_{m',q}\langle j, m|V_q|j, m'\rangle\langle j, m'|J_q^\dagger|j, m''\rangle =c\sum_{m',q}\langle j, m|J_q|j, m'\rangle\langle j, m'|J_q^\dagger|j, m''\rangle \tag{95}$$

$$\sum_q\langle j, m|V_qV_q^\dagger|j, m''\rangle =c\sum_q\langle j, m|J_qJ_q^\dagger|j, m''\rangle \tag{96}$$

$$\begin{aligned}\langle j, m|(\vec{V}\cdot\vec{J})|j, m''\rangle =&c\langle j, m|J^2|j, m''\rangle\\ =&cj(j+1)\end{aligned} \tag{97}$$

Therefore we have

$$c = \frac{\langle j,m|(\vec{V}\cdot\vec{J})|j,m''\rangle}{j(j+1)} \tag{98}$$

$$\langle j,m|V_q|j,m'\rangle = \frac{\langle j,m|(\vec{V}\cdot\vec{J})|j,m''\rangle}{j(j+1)}\langle j,m|J_q|j,m'\rangle \tag{99}$$

A.4.3 Explicit calculation

Just for completeness, we can verify this relation between angular momentum and CG coefficients explicitly. This part can be ignored without affecting the understanding of the rest.
First the expression using angular momentum operators,

$$\begin{aligned}\langle j,m|J_0|j,m'\rangle =&\langle j,m|m'|j,m'\rangle\\ =&m'\delta_{mm'}\end{aligned} \tag{100}$$

$$\begin{aligned}\langle j,m|J_{\pm1}|j,m'\rangle =&\mp\frac{1}{\sqrt{2}}\langle j,m|J_\pm|j,m'\rangle\\ =&\mp\sqrt{\frac{(j\mp m')(j\pm m'+1)}{2}}\langle j,m|j,m'\pm1\rangle\\ =&\mp\sqrt{\frac{(j\mp m')(j\pm m'+1)}{2}}\delta_{m,m'\pm1}\end{aligned} \tag{101}$$

Using the explicit formula for the CG coefficients,

$$\begin{aligned}\langle j,1;m',q|j,m\rangle =&\delta_{m,m'+q}\sqrt{\frac{(2j+1)(j+j-1)!(j-j+1)!(j+1-j)!}{(j+1+j+1)!}}\\ &\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1-q)!(1+q)!}\\ &\sum_k\frac{(-1)^k}{k!(j+1-j-k)!(j-m'-k)!(1+q-k)!(j-1+m+k)!(j-j-q+k)!}\\ =&\delta_{m,m'+q}\frac{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1-q)!(1+q)!}}{2\sqrt{(j+1)j}}\\ &\sum_k\frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1+q-k)!(j-1+m'+k)!(-q+k)!}\end{aligned} \tag{102}$$

For $q=0$

$$\begin{aligned}\langle j,1;m',0|j,m\rangle =&\delta_{mm'}\frac{\sqrt{(j+m)!(j-m)!(j-m)!(j+m)!}}{2\sqrt{(j+1)j}}\\ &\sum_{k=0,1}\frac{(-1)^k}{k!(1-k)!(j-m-k)!(1-k)!(j-1+m+k)!k!}\\ =&\delta_{mm'}\frac{(j-m)!(j+m)!}{2\sqrt{(j+1)j}}\left(\frac{1}{(j-m)!(j-1+m)!}-\frac{1}{(j-m-1)!(j+m)!}\right)\\ =&m\frac{\delta_{mm'}}{\sqrt{j(j+1)}}\end{aligned} \tag{103}$$

For $q = \pm 1$

$$\begin{aligned}
\langle j, 1; m', \pm 1 | j, m\rangle =& \delta_{m,m'\pm 1} \frac{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1\mp 1)!(1\pm 1)!}}{2\sqrt{(j+1)j}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1\pm 1-k)!(j-1+m+k)!(\mp 1+k)!} \\
=& \frac{\delta_{m,m'\pm 1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'\pm 1)!(j-m'\mp 1)!(j-m')!(j+m')!}{2}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1\pm 1-k)!(j-1+m'+k)!(\mp 1+k)!}
\end{aligned} \tag{104}$$

For $q = 1$

$$\begin{aligned}
\langle j, 1; m', 1 | j, m\rangle =& \frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'+1)!(j-m'-1)!(j-m')!(j+m')!}{2}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1+1-k)!(j-1+m'+k)!(-1+k)!} \\
=& - \frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'+1)!(j-m'-1)!(j-m')!(j+m')!}{2(j-m'-1)!(j-m'-1)!(j+m')!(j+m')!}} \\
=& - \sqrt{\frac{(j+m'+1)(j-m')}{2}} \frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}}
\end{aligned} \tag{105}$$

For $q = -1$

$$\begin{aligned}
\langle j, 1; m', -1 | j, m\rangle =& \frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'-1)!(j-m'+1)!(j-m')!(j+m')!}{2}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(-k)!(j-1+m'+k)!(1+k)!} \\
=& \frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'-1)!(j-m'+1)!(j-m')!(j+m')!}{2(j-m')!(j-m')!(j+m'-1)!(j+m'-1)!}} \\
=& \sqrt{\frac{(j-m'+1)(j+m')}{2}} \frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}}
\end{aligned} \tag{106}$$

Comparing the result from the two methods, we can see that the proportionality factor is $\sqrt{(j+1)j}$, or

$$\langle j, m | J_q | j, m'\rangle = \sqrt{(j+1)j} \langle j, 1; m', q | j, m\rangle \tag{107}$$

A.4.4 Generalizing the projection theorem to rank-2 tensor

Following the same procesure for projection theorem, we can also replace the spherical tensor operators with any other spherical tensor operator of the same rank. For rank-2 tensor operators on states with the same angular momentum, we can use the the one constructed from the direct product of two angular momentum operators: $(JJ)^2_p = \sum_{q,q'} \langle 1, 1; q, q' | 2, p\rangle J_q J_{q'}$.

Inserting an identity and using equation 107,

$$\begin{aligned}
&\langle j,m|(JJ)^2_p|j,m'\rangle\\
=&\sum_{q,q',m''}\langle 1,1;q,q'|2,p\rangle\langle j,m|J_q|j,m''\rangle\langle j,m''|J_{q'}|j,m'\rangle\\
=&j(j+1)\sum_{q,q',m''}\langle 1,1;q,q'|2,p\rangle\langle j,1;m'',q|j,m\rangle\langle j,1;m',q'|j,m''\rangle\\
=&\sqrt{5}j(j+1)(2j+1)\sum_{q,q',m''}(-1)^{-2j-m-m''-p}\\
&\begin{pmatrix}1 & 2 & 1\\ q' & -p & -q\end{pmatrix}\begin{pmatrix}1 & j & j\\ q & m & -m''\end{pmatrix}\begin{pmatrix}j & j & 1\\ m'' & -m' & -q'\end{pmatrix}\\
=&\sqrt{5}j(j+1)(2j+1)(-1)^{j-m}\sum_{q,q',m''}(-1)^{1+1+j-q'-q-m''}\\
&\begin{pmatrix}1 & 2 & 1\\ q' & -p & -q\end{pmatrix}\begin{pmatrix}1 & j & j\\ q & m & -m''\end{pmatrix}\begin{pmatrix}j & j & 1\\ m'' & -m' & -q'\end{pmatrix}\\
=&\sqrt{5}j(j+1)(2j+1)(-1)^{j-m}\begin{pmatrix}j & 2 & j\\ m' & p & -m\end{pmatrix}\begin{Bmatrix}2 & j & j\\ j & 1 & 1\end{Bmatrix}\\
=&\sqrt{5}j(j+1)\sqrt{2j+1}(-1)^{2j}\begin{Bmatrix}2 & j & j\\ j & 1 & 1\end{Bmatrix}\langle j,2;m',p|j,m\rangle\\
=&\sqrt{\frac{j(j+1)(2j-1)(2j+3)}{6}}\langle j,2;m',p|j,m\rangle
\end{aligned}\tag{108}$$

which is indeed consistent with the Wigner-Echart theorem (the only m or p-dependent term is the Clebsch-Gordan coefficient).

A.4.5 Relation between reduced matrix elements

The exact value reduced matrix element in Eq. 89 does not have a direct physical meaning. In fact, there are multiple convensions that gives different value for their values and the convension we selected is also asymmetric, i.e. $\langle j||T^{(k)}||j'\rangle \neq \langle j'||T^{(k)}||j\rangle$. Nevertheless, it does qualify the coupling strength between different angular momentum manifolds and is an important quality to talk about when coupling angular momentum states together.

For a tensor operator $T^{(k)}_q$ that couples two angular momentum states together, we have

$$\langle j_1,m_1|T^{(k)}_q|j_2,m_2\rangle = \langle j_2,k;m_2,q|j_1,m_1\rangle\langle j_1||T^{(k)}||j_2\rangle \tag{109}$$

Now suppose there's another angular momentum j_0 in the system that does not interact with the operator $T^{(k)}_q$ but does couple to j_1 and j_2 to form j_1' and j_2' respectively[5]. The matric element between the coupled states also satisfy the Wigner-Echart theorem with a different reduced matrix element

$$\langle j_1',m_1'|T^{(k)}_q|j_2',m_2'\rangle = \langle j_2',k;m_2',q|j_1',m_1'\rangle\langle j_1'||T^{(k)}||j_2'\rangle \tag{110}$$

and we must have

$$\langle j_1'||T^{(k)}||j_2'\rangle \propto \langle j_1||T^{(k)}||j_2\rangle \tag{111}$$

[5]This may be the electron spin S or nuclear spin I coupling to the electron orbit for an electric transtion

with some proportionality factor. We can derive this relation by expanding the LHS of Eq. 110.

$$
\begin{aligned}
&\langle j_1', m_1'|T_q^{(k)}|j_2', m_2'\rangle \\
=&\sum_{m_1,m_0,m_2,m_0'} \langle j_1, j_0; m_1, m_0|j_1', m_1'\rangle\langle j_2, j_0; m_2, m_0'|j_2', m_2'\rangle\langle j_0, m_0|\langle j_1, m_1|T_q^{(k)}|j_2, m_2\rangle|j_0, m_0'\rangle \\
=&\sum_{m_1,m_0,m_2} \langle j_1, j_0; m_1, m_0|j_1', m_1'\rangle\langle j_2, j_0; m_2, m_0'|j_2', m_2'\rangle\langle j_2, k; m_2, q|j_1, m_1\rangle\langle j_1||T^{(k)}||j_2\rangle \\
=&\sum_{m_1,m_0,m_2} (-1)^{2j_0-j_1-2j_2+k-m_1'-m_2'-m_1}\sqrt{(2j_1'+1)(2j_2'+1)(2j_1+1)} \\
&\begin{pmatrix} j_1 & j_0 & j_1' \\ m_1 & m_0 & -m_1' \end{pmatrix}\begin{pmatrix} j_2 & j_0 & j_2' \\ m_2 & m_0 & -m_2' \end{pmatrix}\begin{pmatrix} j_2 & k & j_1 \\ m_2 & q & -m_1 \end{pmatrix}\langle j_1||T^{(k)}||j_2\rangle \\
=&\sum_{m_1,m_0,m_2} (-1)^{2j_0-j_2+2k-2m_2'-3m_1'+m_0+m_1+m_2}\sqrt{(2j_1'+1)(2j_2'+1)(2j_1+1)} \\
&\begin{pmatrix} j_1' & j_1 & j_0 \\ -m_1' & m_1 & -m_0 \end{pmatrix}\begin{pmatrix} j_2 & j_2' & j_0 \\ -m_2 & m_2' & m_0 \end{pmatrix}\begin{pmatrix} j_2 & j_1 & k \\ m_2 & -m_1 & q \end{pmatrix}\langle j_1||T^{(k)}||j_2\rangle \\
=&(-1)^{j_0-j_1-2j_2+2k-m_2'+q}\sqrt{(2j_1'+1)(2j_2'+1)(2j_1+1)} \\
&\begin{pmatrix} j_1' & j_2' & k \\ -m_1' & m_2' & q \end{pmatrix}\begin{Bmatrix} j_1' & j_2' & k \\ j_2 & j_1 & j_0 \end{Bmatrix}\langle j_1||T^{(k)}||j_2\rangle \\
=&(-1)^{j_0+j_1+j_2'+k}\sqrt{(2j_1+1)(2j_2'+1)}\langle j_2', k; m_2', q|j_1', m_1'\rangle\begin{Bmatrix} j_1 & j_2 & k \\ j_2' & j_1' & j_0 \end{Bmatrix}\langle j_1||T^{(k)}||j_2\rangle
\end{aligned}
\tag{112}
$$

Combining with the RHS of Eq. 110, we have

$$
\langle j_1'||T^{(k)}||j_2'\rangle =(-1)^{j_0+j_1+j_2'+k}\sqrt{(2j_1+1)(2j_2'+1)}\begin{Bmatrix} j_1 & j_2 & k \\ j_2' & j_1' & j_0 \end{Bmatrix}\langle j_1||T^{(k)}||j_2\rangle \tag{113}
$$

More specifically, for hyperfine structure, we have,

$$
\langle F||T^{(k)}||F'\rangle =(-1)^{I+J+F'+k}\sqrt{(2J+1)(2F'+1)}\begin{Bmatrix} J & J' & k \\ F' & F & I \end{Bmatrix}\langle J||T^{(k)}||J'\rangle \tag{114}
$$

and for LS coupling

$$
\langle J||T^{(k)}||J'\rangle =(-1)^{S+L+J'+k}\sqrt{(2L+1)(2J'+1)}\begin{Bmatrix} L & L' & k \\ J' & J & S \end{Bmatrix}\langle L||T^{(k)}||L'\rangle \tag{115}
$$