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Module 3: Linear Regression

Lecture 7

Jan 25th, 2023

Some Important Questions

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- Is at least one of the predictors useful in predicting the response?

$Y = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p$

- Do all the predictors help to explain Y, or is only a subset of the of the predictors useful?

2^p, Forward / Backward. Shrinkage. F test. BIC, AIC...

- Given a set of predictor values, what response value should we predict, and how accurate is our prediction?

$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_2 X_2 + \hat{\beta}_5 X_5$

Confidence Interval and Prediction

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- ① Confidence interval for a fixed x: a random interval which can cover **f(x)** with probability 95%
- ② Prediction interval for a fixed x : a random interval which can cover a **new** random responds Y with probability 95%

fixed

random

$$\boxed{Y = f(x) + \varepsilon}$$

f

$\mathrm{Var}(\varepsilon) = \sigma^2 \leftarrow$ irreducible variance

special case of ①

C.I. for β_1

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad x^T\beta = (0\ 1\ 0\ 0)\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_x \end{pmatrix} = \beta_1$$

$\beta_2 \quad x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Confidence Interval and Prediction

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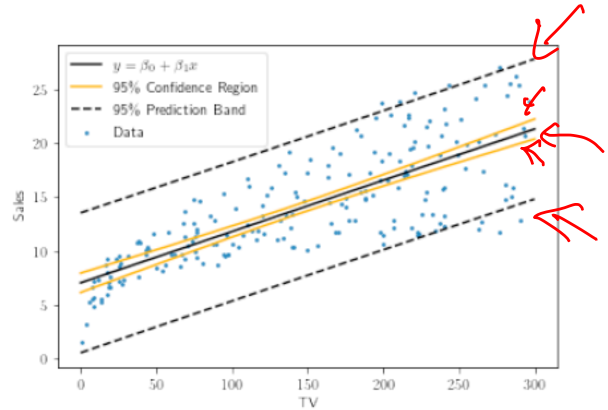
- Confidence interval for a fixed x: a random interval which can cover **f(x)** with probability 95%
- Prediction interval for a fixed x : a random interval which can cover a **new** random responds Y with probability 95%

$Y = f(x) + \epsilon$

Confidence Interval
The range likely to contain the population parameter (mean, standard deviation) of interest.

Prediction Interval
The range that likely contains the value of the dependent variable for a single new observation given specific values of the independent variables.



Other Considerations

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Quantitative Predictors

age

hp

TV

2 4 6

Qualitative

Gender ⇒ # ?

Color

Countries

Blue Red Green

0 2 1

Qualitative Predictors

- Some predictors are not categorical predictors or factor variables.
- For example: gender, student (student status), status (marital status), and ethnicity (Caucasian, African American (AA) or Asian).

Qualitative Predictors

Example: investigate differences in credit card balance between males and females, ignoring the other variables.

Qualitative Predictors

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Example: investigate differences in credit card balance between males and females, ignoring the other variables.

We create a new variable

$$x_i = \begin{cases} 1 & \text{if } i\text{th person is female} \\ 0 & \text{if } i\text{th person is male} \end{cases}$$

Resulting model:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is female} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person is male.} \end{cases}$$

Interpretation?

Credit card data

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Female = $\hat{\beta}_0 + \hat{\beta}_1$

male = $\hat{\beta}_0$

	Coefficient	Std. Error	t-statistic	p-value
Intercept	509.80	33.13	15.389	< 0.0001
gender[Female]	19.73	46.05	0.429	0.6690

More than two levels?

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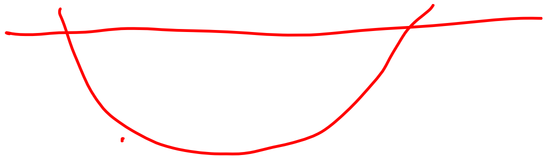
For the ethnicity: Asian, Caucasian, African American

1 — 2 — 3



$$X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0$$

2

$$\text{Asian} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\cancel{AA = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$X \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\hat{\beta} = \boxed{(X^T X)^{-1}} X^T Y$$

More than two levels?

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For the ethnicity: Asian, Caucasian, African American

We will create two dummy variables.

$$x_{i1} = \begin{cases} 1 & \text{if } i\text{th person is Asian} \\ 0 & \text{if } i\text{th person is not Asian,} \end{cases}$$

$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th person is Caucasian} \\ 0 & \text{if } i\text{th person is not Caucasian.} \end{cases}$$

AS.n. C, AA

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \begin{vmatrix}0\\0\end{vmatrix}$$

More than two levels?

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Then we have the following model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is Asian} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if } i\text{th person is Caucasian} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person is AA.} \end{cases}$$

There will always be one fewer dummy variable than the number of levels. The level with no dummy variable African American in this example — is known as the baseline.

More than two levels?

	Coefficient	Std. Error	t-statistic	p-value
`Intercept`	531.00	46.32	11.464	< 0.0001
`ethnicity[Asian]`	-18.69	65.02	-0.287	0.7740
`ethnicity[Caucasian]`	-12.50	56.68	-0.221	0.8260

Interactions

In the advertising data, we assume the effect on sales of increasing one medium is independent of other media.

$$\widehat{\texttt{sales}} = \beta_0 + \beta_1 \times \texttt{TV} + \beta_2 \times \texttt{radio} + \beta_3 \times \texttt{newspaper}$$

But suppose that spending money on radio advertising increases the effectiveness of TV advertising. How to model it?

Extensions of the Linear Model

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Interactions

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$$\widehat{\texttt{sales}} = \beta_0 + \beta_1 \times \texttt{TV} + \beta_2 \times \texttt{radio} + \beta_3 \times \texttt{newspaper}$$

But suppose that spending money on radio advertising increases the effectiveness of TV advertising. How to model it?

x_3 $\quad$ $\text{radio}^2 \cdot TV$

This is called an interaction effect.

$\boxed{x_1 \cdot x_2}$ $\quad$ $\sin TV \ldots$

$$\begin{aligned}
\texttt{sales} &= \beta_0 + \beta_1 \times \overset{x_1}{\texttt{TV}} + \beta_2 \times \overset{x_2}{\texttt{radio}} + \beta_3 \times \boxed{(\cancel{\texttt{radio} \times \texttt{TV}})} + \epsilon \\
&= \beta_0 + \underbrace{\boxed{(\beta_1 + \beta_3 \times \texttt{radio})}}_{\alpha_1} \times \texttt{TV} + \beta_2 \times \texttt{radio} + \epsilon.
\end{aligned}$$

Interpretation

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	Coefficient	Std. Error	t-statistic	p-value
β_0 Intercept	6.7502	0.248	27.23	< 0.0001
β_1 TV	0.0191	0.002	12.70	< 0.0001
β_2 radio	0.0289	0.009	3.24	0.0014
β_3 TV×radio	0.0011	0.000	20.73	< 0.0001

The p-value for the interaction term TV × radio is extremely low, indicating that there is strong evidence for $H_A : \beta_3 \neq 0$.

6%

The R2 for the interaction model is 96.8%, compared to only 89.7% for the model that predicts sales using TV and radio without an interaction term.

This means that (96.8 − 89.7)/(100 − 89.7) = 69% of the variability in sales that remains after fitting the additive model has been explained by the interaction term.

Interpretation

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The coefficient estimates in the table suggest that an increase in TV advertising of \$1, 000 is associated with increased sales of

$$(\hat{\beta}_1 + \hat{\beta}_3 \times \texttt{radio}) \times 1000 = 19 + 1.1 \times \texttt{radio} \text{ units.}$$

An increase in radio advertising of \$1, 000 will be associated with an increase in sales of

$$(\hat{\beta}_2 + \hat{\beta}_3 \times \texttt{TV}) \times 1000 = 29 + 1.1 \times \texttt{TV} \text{ units.}$$

Hierarchy

Sometimes it is the case that an interaction term has a very small p-value, but the associated main effects (in this case, TV and radio) do not.

The hierarchy principle:

If we include the interaction term, we should include the main effects no matter what!

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 (x_1 x_2)$$

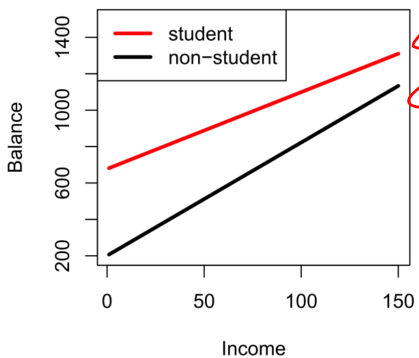
Qualitative and quantitative variables?

Consider the Credit data set, and suppose that we wish to predict balance using income (quantitative) and student (qualitative). With interactions, we have

student$_i$ student$_i$

$$\texttt{balance}_i \approx \beta_0 + \beta_1 \times \texttt{income}_i + \begin{cases} \beta_2 + \beta_3 \times \texttt{income}_i & \text{if student} \\ 0 & \text{if not student} \end{cases}$$

$$= \begin{cases} \boxed{(\beta_0 + \beta_2)} + (\beta_1 + \beta_3) \times \texttt{income}_i & \text{if student} \\ \beta_0 + \beta_1 \times \texttt{income}_i & \text{if not student} \end{cases}$$



Non-linear effects of predictors

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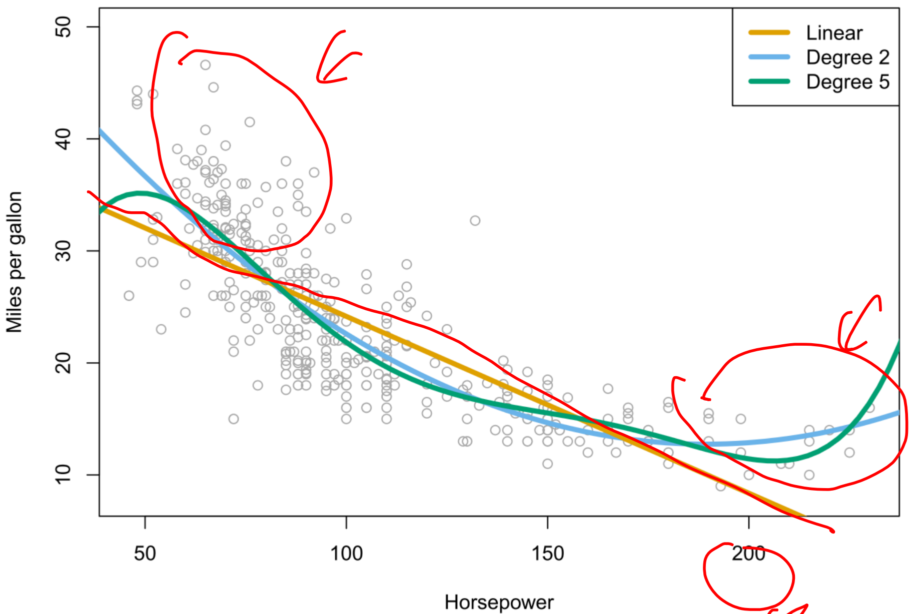
polynomial regression on `Auto` data

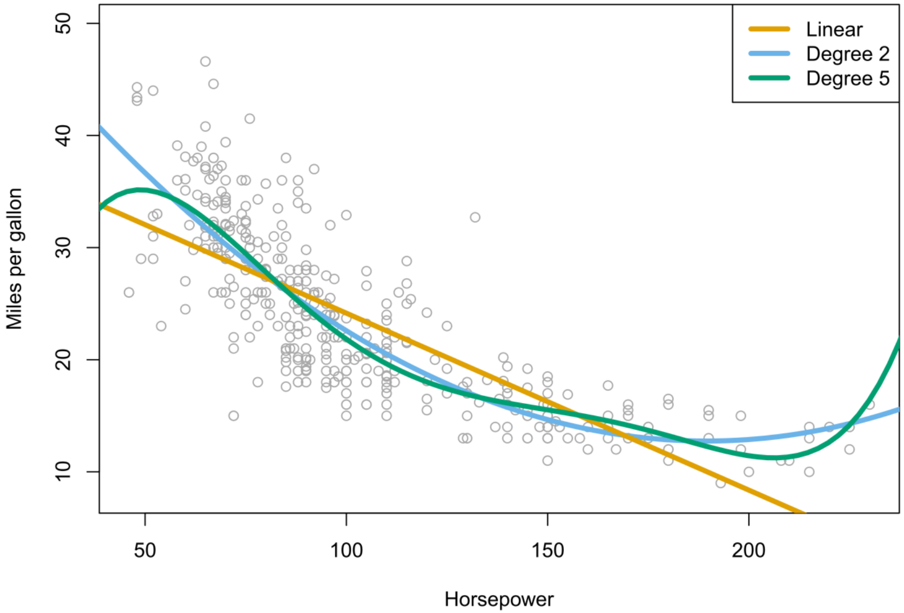


$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \epsilon$

$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \beta_4 X^4 + \beta_5 X^5 + \epsilon$

Non-linear effects of predictors

polynomial regression on `Auto` data



$$\texttt{mpg} = \beta_0 + \beta_1 \times \texttt{horsepower} + \beta_2 \times \texttt{horsepower}^2 + \epsilon$$

Non-linear effects of predictors

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$$\texttt{mpg} = \beta_0 + \beta_1 \times \texttt{horsepower} + \beta_2 \times \texttt{horsepower}^2 + \epsilon$$

	Coefficient	Std. Error	t-statistic	p-value
Intercept	56.9001	1.8004	31.6	< 0.0001
horsepower	-0.4662	0.0311	-15.0	< 0.0001
$\texttt{horsepower}^2$	0.0012	0.0001	10.1	< 0.0001

Bonus Quiz 8

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Assume we have the following model

$$\text{Income} = \beta_0 + \beta_1 X + \epsilon = \beta_0 + \beta_1 \frac{X^*}{100} + \epsilon$$

Here X is height with unit of meter.

$$= \beta_0 + \left(\frac{\beta_1}{100}\right) X^* + \epsilon$$

Now we can have another model

$$\text{Income} = \beta_0 + \beta_1^* X^* + \epsilon \quad = \beta_0 + \beta_1^* X^* + \epsilon$$

$$\Rightarrow \beta_1^* = \frac{\beta_1}{100}.$$

Here X* is also height with unit of centimeter. What is the relationship between β_1 and β_1^*?