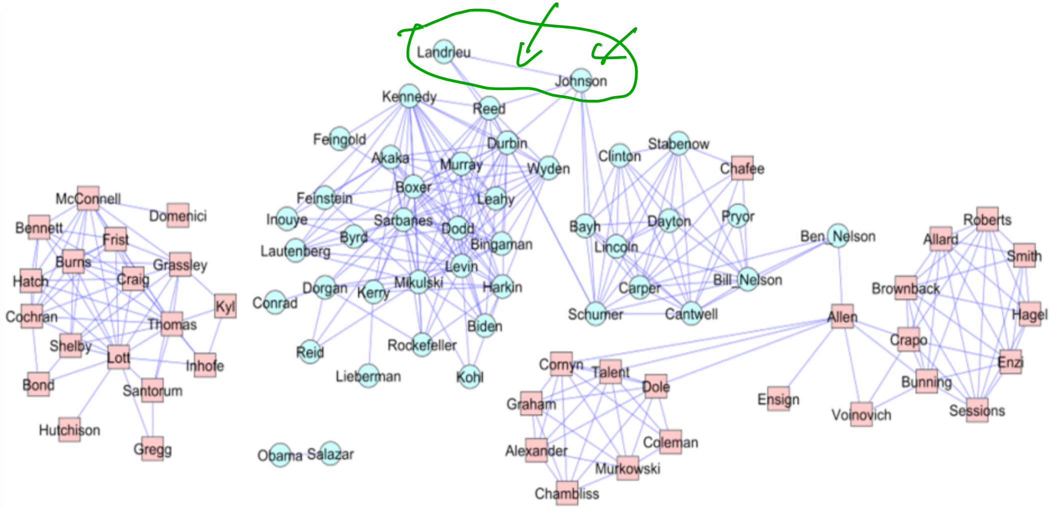
Module 1: Statistical Learning

Lecture 2

Jan 11th, 2023



Banerjee et. al., 2008

- Inputs X are voting records for each Senator
- Output: relationships between Senators
- When training the model, no output is available.
- Unsupervised learning

Unsupervised Learning

- We only have X in the data and want to output something not in the data

	Senate X	Senate Y	Senate XY	Senate Z
Bill 1	0	1	1	1
Bill 2	0	0	1	1
Bill 3	1	1	0	1
Bill 4	1	1	0	1
Bill 5	0	0	1	1
Bill 6	0	0	0	1

Announcement

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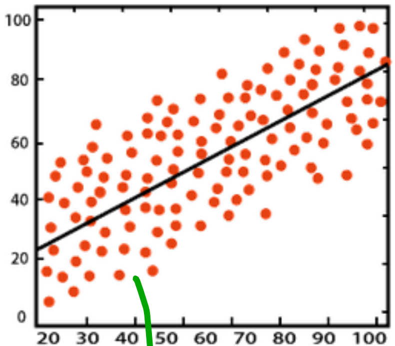
- Go to our **Team** Channel.
  - +1 point on the first HW if you post a gif in the thread!

Recap: Supervised Learning

- Inputs X and output Y both in the data.



Regression

An example data set: Advertising

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X Y

1		TV	Radio	Newspaper	Sales
2	1	230.1	37.8	69.2	22.1
3	2	44.5	39.3	45.1	10.4
4	3	17.2	45.9	69.3	9.3
5	4	151.5	41.3	58.5	18.5
6	5	180.8	10.8	58.4	12.9
7	6	8.7	48.9	75	7.2
8	7	57.5	32.8	23.5	11.8
9	8	120.2	19.6	11.6	13.2
10	9	8.6	2.1	1	4.8
11	10	199.8	2.6	21.2	10.6
12	11	66.1	5.8	24.2	8.6

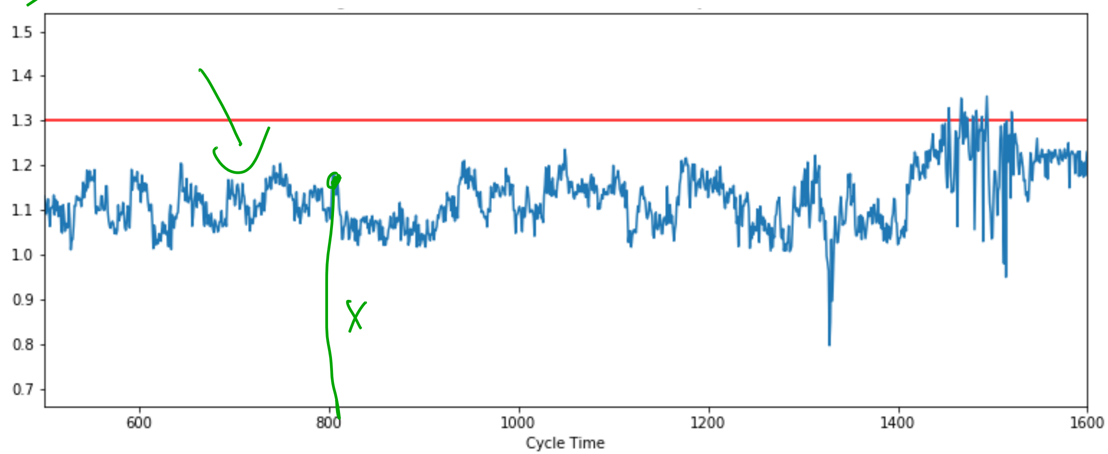
- Sales of a product in 200 markets, along with spent on three type of ad.
- Goal: ? predict Sale

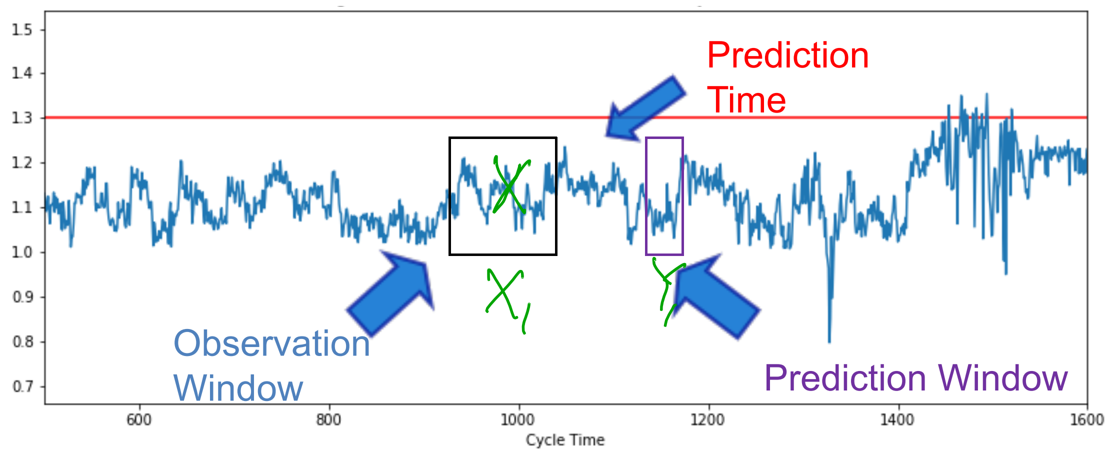
TV, Radio, New

Predicting Failure time for a machine

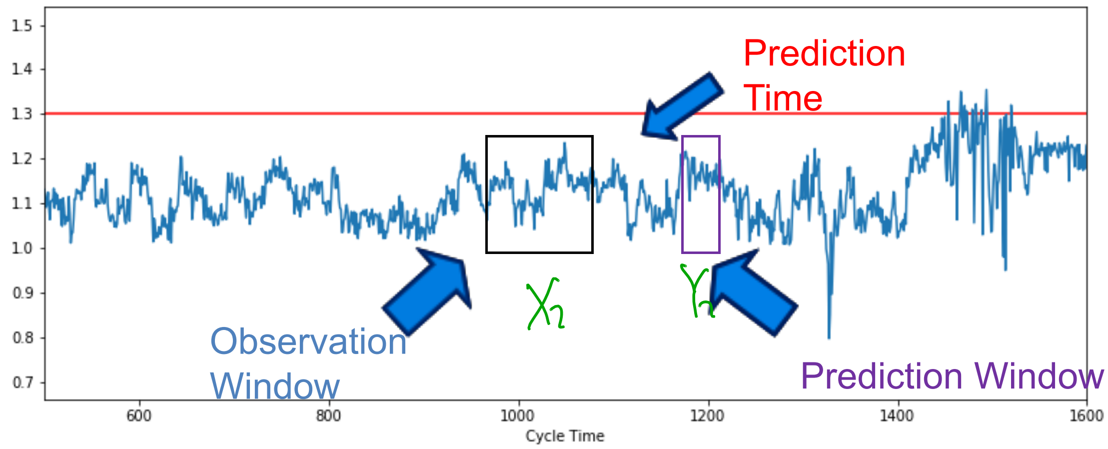
BEET

X





Prediction Time
Observation Window
Prediction Window
X
X1
Y1
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
600
800
1000
1200
1400
1600
Cycle Time



Prediction
Time
Observation
Window
X_2
Y_2
Prediction Window
Cycle Time
1.5
1.4
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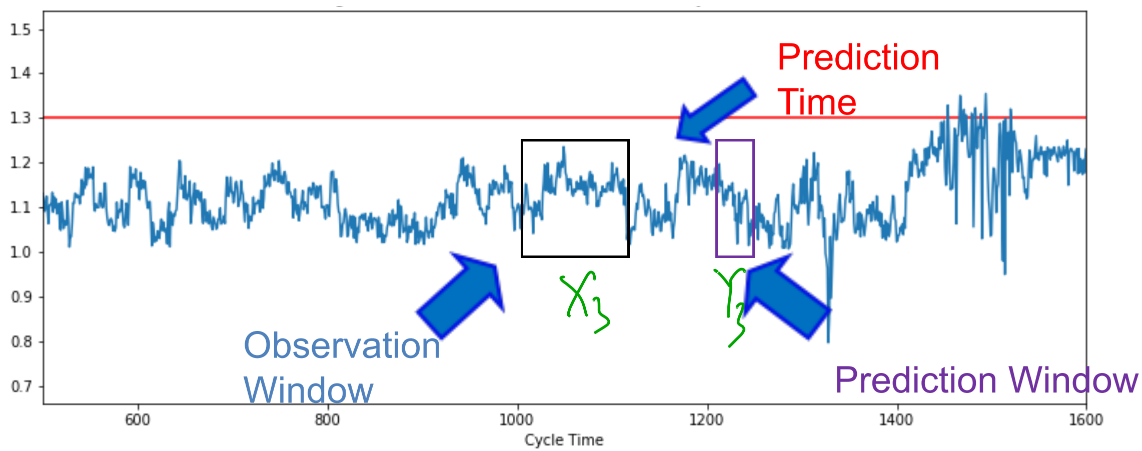
1.5
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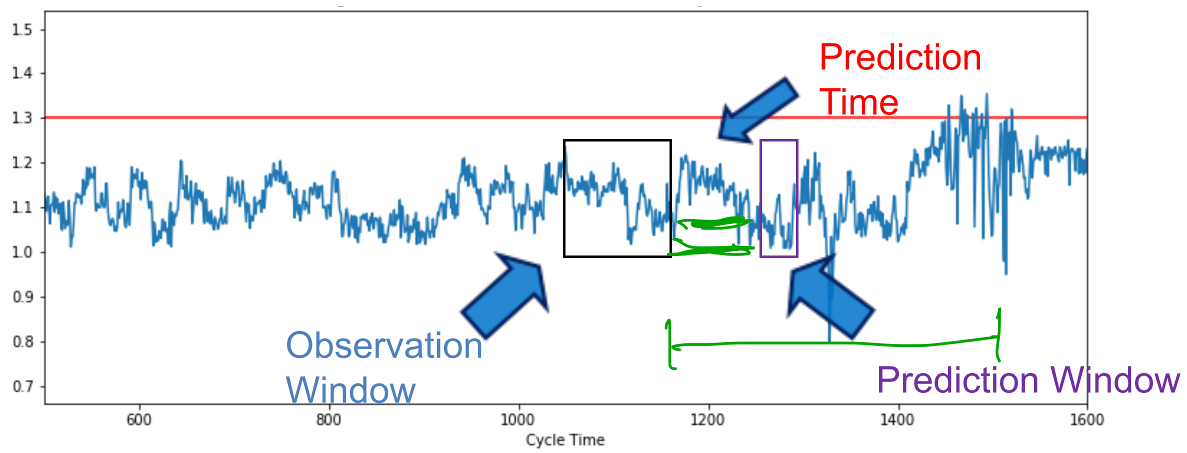
600
800
1000
1200
1400
1600
Cycle Time
Prediction
Time
Observation
Window
Prediction Window
X_3
Y_3


Prediction
Time
Observation
Window
Prediction Window
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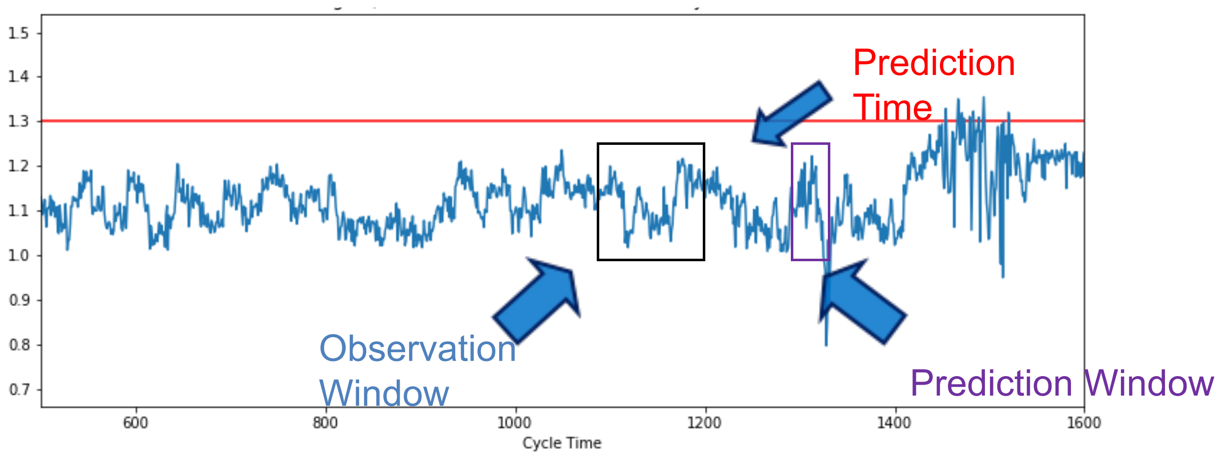
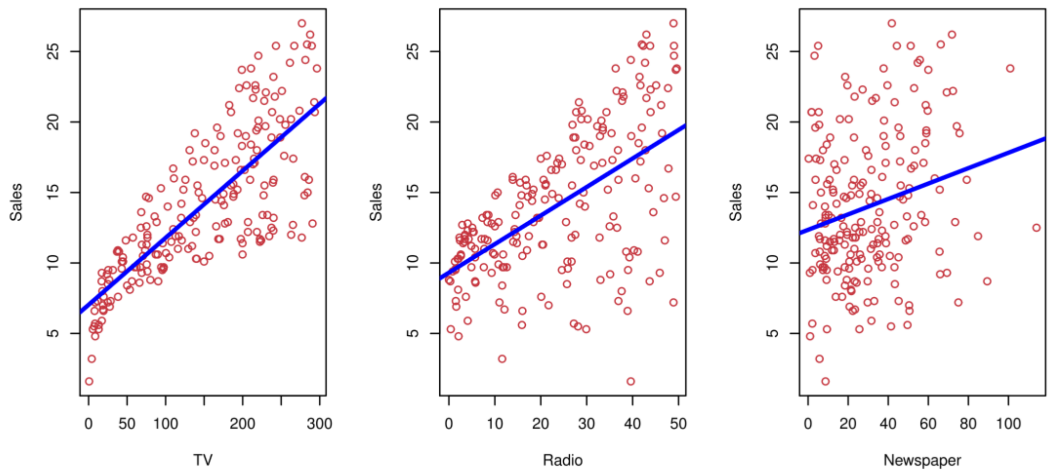
Cycle Time


Prediction
Time
Observation
Window
Prediction Window
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
600
800
1000
1200
1400
1600
Cycle Time

Sale-Advertising

- Sales of a product in 200 different markets
- Expense on TV, radio, and newspaper in these markets

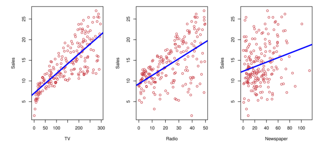


Notation

- We wish to predict sale. We refer it to be the response ***Y***
- TV is a feature (input) which we can control. We denote it **X1**. Similarly, Radio as **X2** ,and so on. We can refer to the input vector collectively as

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

3x1



- Now we can write our model as

(1) $\mathrm{Y} = f(\mathrm{X})$

(2) $\mathrm{Y} = f(\mathrm{X}) + \varepsilon$

$Y = f(X)$

Here f is some fixed but unknown function.

Errors in Machine Learning: World is

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$$Y = f(X) + \epsilon$$

$\varepsilon \sim N(0,$

- World is too complex to model precisely
- Measurement error may not be avoidable
- Many features are not captured
- The error is where the statistics kicks in. Confidence interval, etc….

Dataset:

	Y	X_1	X_2	X_3
Market 1	10	101	20	35
Market 2	20	66	41	85
Market 3	11	101	43	78
Market 4	25	25	10	61
Market 5	5	310	51	11

rows are samples

Supervised Machine Learning Algorithm

- **Input:**

  Training data-set with features (X) and targets (Y)

- **Output:**
  - Prediction function $\boxed{f}$

f f

estimated true funct
known

Prediction vs Inference

Prediction

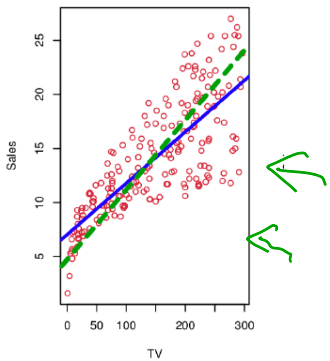
Prediction: Make predictions about future:

Inputs X are readily available, but output Y is hard to obtain

Build a model:

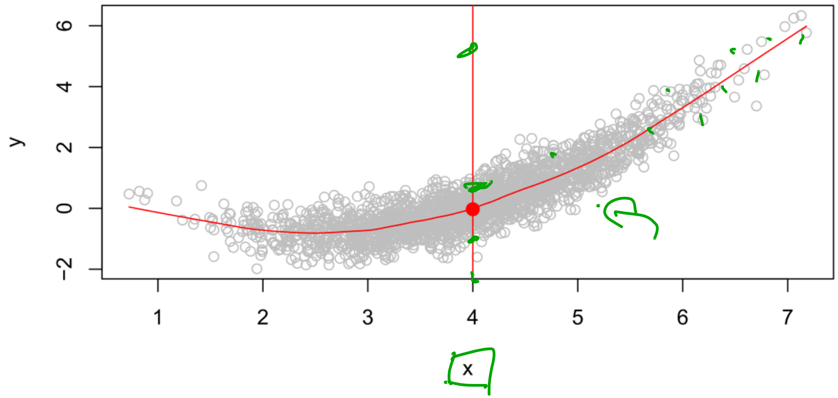
$$\hat{Y} = \hat{f}(X)$$

Example: If we spend $150 on TV advertising, what will we make in sales?



- Want to get a good guess for f, which is unknown blue
- Model is $\hat{f}$ is green dashed lines

Is there an Ideal *f*(X)?



- Given X = 4, what is 'the best' prediction for Y? or what can an Oracle say?

$$f \Rightarrow \min \sum_{i=1}^{n} (f - y_i)^2$$

Is there an Ideal $f(X)$?

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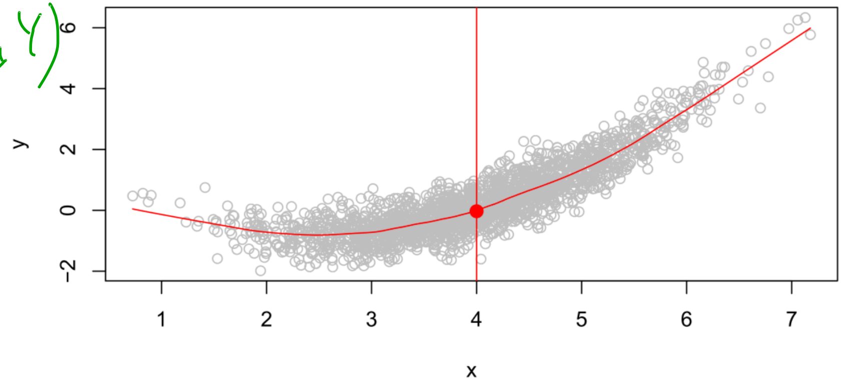
$(X,Y) \sim f(X,Y)$

$E(Y)$



$EY = \int_{\Omega_Y} y \cdot f(y)\, dy$

- Given X = 4, what is 'the best' prediction for Y? or what can an Oracle say?
- A good value is

$f(x) = E(Y|X=x) \Rightarrow E(Y|X)$

$$\boxed{f(4) = E(Y|X=4)} \Leftarrow$$

$E(Y|X=2) = E(Y|X=2)$

where E(Y|X = 4) is the expected value of Y given (condition on) X = 4.

- This $\boxed{f(x) = E(Y|X=x)}$ is called the **regression function or Oracle function**.

$E(Y|X=4) = \int y \cdot f_{Y|X}(y, x=4)\, dy$

The regression function *f(x)*

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- The regression function is also defined for vector X as

$$f(x) = f(x_1, x_2, x_3) = E(Y|X_1 = x_1, X_2 = x_2, X_3 = x_3)$$

- $f(x) = E(Y|X = x)$ is the best predictor of Y given x in what sense?

It is the best for **the mean-squared prediction error** over all function *g(.)* at all points X = x.

$$f(x) = \underset{g}{\operatorname{argmin}} \mathrm{E}\left[\left(Y - g(X)\right)^2 | X = x\right]$$

$g(x) = E(Y|X)$

- **Q: Let's prove it.**

Recap

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- Given two random variables X and Y with joint probability density function $f_{X,Y}(x,y)$

- E(Y) = $\int_{\Omega_Y} y f_Y(y)dy$

- E(Y|X = x) =?

$$E(Y|X=x) = \int_{\Omega_Y} y f_{Y|X}(y,x)dy$$

- E(*f*(Y) – g(X) | X = x) = ?

$$E(g(X) \mid X = x)$$
$$= E(g(x) \mid X = x)$$
$$= g(x)$$

$$= E(f(Y) \mid X = x) - E(g(X) \mid X = x)$$
$$= E(f(Y) \mid X = x) - g(x)$$

The regression function $f(x)$

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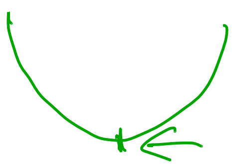
$$g(x) \text{ s.t. } \min E\left(\left(Y - g(X)\right)^2 \mid X = x\right)$$

$$\Rightarrow \min E\left(\left(Y - g(x)\right)^2 \mid X = x\right)$$

$$\Rightarrow \min \int \left(y - g(x)\right)^2 f_{Y|X}(y, x)\, dy$$

$$\Rightarrow \frac{\partial \int (y - g)^2 f_{Y|X}(y, x)\, dy}{\partial g} = 0$$



$$\Rightarrow \int \cancel{2}\,(y - g)\,\cancel{(-1)}\, f_{Y|X}(y, x)\, dy = 0$$

$$\boxed{\int y\, f_{Y|X}(y, x)\, dy} - \int g\, f_{Y|X}(y, x)\, dy = 0$$

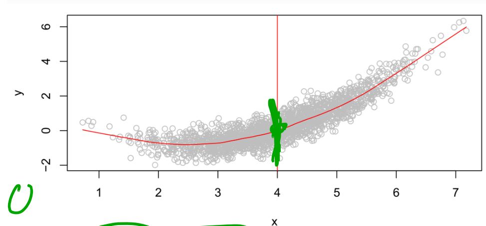
$E(Y|X=x) = g \cdot \left(\int f_{Y|X}(y, x)\, dy\right) = g$

The regression function *f(x)*

- It is the best predictor of Y with regards to mean-squared prediction error over all function g at all points X = x.

$$f(x) = E(Y|X=x) = \underset{f}{\operatorname{argmin}}\, \mathrm{E}\left[(Y - g(X))^2 | X = x\right]$$

- $\epsilon = Y - f(x)$ is the $\boxed{\text{irreducible error}}$. Even if we knew *f(x)*, we will still make errors in prediction. What cause this?



- For any estimate $\hat{f}(x)$ of $f(x)$, we have

$$E[(Y - \hat{f}(X))^2 | X = x] = \underbrace{[f(x) - \hat{f}(x)]^2}_{Reducible} + \underbrace{\boxed{\mathrm{Var}(\epsilon)}}_{Irreducible}$$

Inference

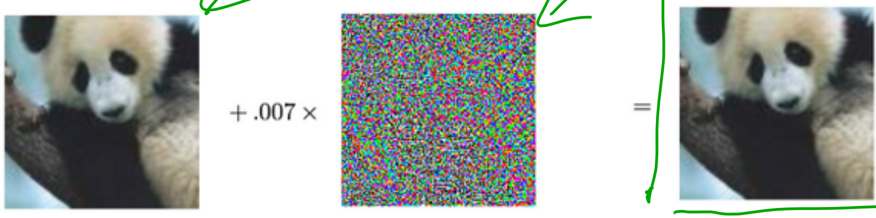
- Inference: Understand the relationship between X and Y within f

  what kind of ads work? Why?

- Which predictors are associated with the response?
- What is the relationship between the response and each predictor?
- Can the relationship between Y and each predictor be adequately summarized using a linear equation? Is it more complicated?

f

Inference is important



"panda"
57.7% confidence

noise

"gibbon"
99.3% confidence

Plan for the lab

- Find a group of 4 or so.
- Download the jupyter notebook and the csv file from github.
- Get started!